Algorithm Theory

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November 2, 2022

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Chapter 1

Graph Matrix

Let G(V, E) be a graph, where $|V| = n$ and $|E| = m$. Assume G is unweighted and undirected, with no self loops or multiple edges.

1.1 Laplacian

1.1.1 Adjacency Matrix

For graph G, let the adjacency matrix $A = A_G$ be defined as

$$A_G = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$$

1.1.2 Laplacian

The Laplacian of the graph, $L = L_G$ is defined as

$$L_G = \begin{cases} -1 & (i,j) \in E \\ d_i & i = j \\ 0 & otherwise \end{cases}$$

$$L_G = D_G - A_G$$

Let some vector $\mathbf{v}$ be thought of as a function that maps to the index to some value. ie.

$$v_i = X(i)$$

$L_G\mathbf{v}$ can then be though as:

$$[L_G\mathbf{v}]_i = d_i[X(i) - (\overline{X(j)} \mid (i,j) \in E)]$$

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of L_G with corresponding eigenvectors $\mathbf{v_1}, \ldots \mathbf{v_n}$. Then

- $\mathbf{v_1} = \mathbf{1}$, $\lambda_1 = 0$
- All $\lambda_i \geq 0$
- Much information can be obtained by the first few non-trival eigenvectors.

1.1.3 Properties of the Laplacian

Edge Union

If G and H share the same vertex set have **disjoint** edge sets,

$$L_{G \cup H} = L_G + L_H$$

Isolated Vertex

If $d_i = 0$, then $[L_G]_{i,j} = 0$ and $[L_G]_{j,i} = 0$

Disjoint Union

If G and H have disjoint vertex sets, then

$$L_{G \sqcup H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$$

If L_G had eigenvectors $\mathbf{v_i}$ with eigenvalues λ_i and L_H had eigenvectors $\mathbf{u_i}$ with eigenvalues μ_i, then $L_{G \sqcup H}$ has eigenvectors

$$\mathbf{v_i} \oplus \mathbf{0}, \mathbf{0} \oplus \mathbf{u_i}$$

with the eigenvalues λ_i and μ_i each.

Laplacian of Edge

The Laplacian of an edge L_e has the form $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix}$.

The graph can be written as $\sum L_e$. Hence for vector $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathbf{x^T} L_e \mathbf{x} = (x_1 - x_2)^2 \geq 0$. Hence the Laplacian is **positive semidefinite**. This implies that all eigenvalues $\lambda_i \geq 0$ and that the matrix can be diagonalized as $L_e = Q^T \Lambda Q$. In addition, $L_e = A^T A$.

Factoring the Laplacian

The **Incidence Matrix** $\nabla_G = \nabla_{e,v} = \begin{cases} 1 & e = (v,w), v < w \\ -1 & e = (v,w), v > w \\ 0 & otherwise \end{cases}$

Then $L_G = \nabla^T \nabla$.

Dimension of Nullspace

If G is connected, then the nullspace is $span\{\mathbf{1}\}$.

Thus, for connected graphs, $\lambda_2 \geq 0$

The dimensions of the nullspace is equal to the number of connected components of G.

Spectra of Common Graphs

Complete Graph has eigenvalue 0 with multiplicity 1 and eigenvalue n with multiplicity (n - 1)

Ring Graph with n vertices has eigenvectors

$$x_k(u) = \sin \frac{2\pi k u}{n}$$

$$y_k(u) = \cos \frac{2\pi k u}{n}$$

where the eigenvalue x_k and y_k is $2 - 2\cos\frac{2\pi k}{n}$

Path Graph with n vertices has the eigenvectors

$$v_k(u) = \sin(\frac{\pi k u}{n} + \frac{\pi}{2n})$$

That has the same Eigenvalues as R_{2n}

Graph Products

Definition Let G=(V,E) and H=(W,F). The Graph $G \times H$ has vertex set $V \times W$ and edge set

$$((v_1, w), (v_2, w)), \forall (v_1, v_2) \in E, w \in W$$
$$((v, w_1), (v, w_2)), \forall v \in V, (w_1, w_2) \in F$$

The product of 2 paths graph is the grid graph.
Eigenvales If L_G has the eigenvalues λ_i with the eigenvectors v_i and L_H has the eigenvalues μ_i with the eigenvectors w_i, then the eigenvectors of $L_{G\times H}$ are listed

$$z_{ij}(v, w) = x_i(v) y_j(w)$$

with eigenvalues $\lambda_i + \mu_u$.

1.1.4 Why it is Called the Laplacian

If one is to discrete the derivative on graphs, as in treat the vertex as a function, then the "Laplacian" would be $-L_G$

Laplacian Eigenvalues

Sum of Eigenvalues Given an n-vertex graph, where $d_{max} = \max d_i$,

$$\sum \lambda_i = \sum d_i \leq d_{max} n$$

This is obtained from the trace.

$$\lambda_2 \leq \frac{\sum d_i}{n-1}$$
$$\lambda_n \geq \frac{\sum d_i}{n-1}$$

Bounds of λ_2 and λ_{max}

Courant-Fischer Formula For any $n \times n$ symmetric matrix A,

$$\lambda_1 = \min_{||x||=1} \mathbf{x^T} A \mathbf{x} = \min_{||x||\neq 0} \frac{\mathbf{x^T} A \mathbf{x}}{\mathbf{x^T x}}$$

$$\lambda_{max} = \max_{||x||=1} \mathbf{x^T} A \mathbf{x} = \max_{x \neq \vec{0}} \frac{\mathbf{x^T} A \mathbf{x}}{\mathbf{x^T x}}$$

Let S_k denote the span of $v_1 \cdots v_k$, and let $S_k^\perp$ denote the orthogonal complement of S_k

$$\lambda_k = \min_{|x|=1, x \in S_k^\perp} x^T A x = \min_{x \neq 0, x \in S_k^\perp} \frac{x^T A x}{x^T x}$$

Rayleigh Quotient Let G=(V,E) be a graph with Laplacian L_G, then

$$\lambda_1 = 0 \qquad \mathbf{v_1} = \mathbf{1}$$

$$\lambda_2 = \min_{x \perp v_1, x \neq 0} \frac{x^T L_G x}{x^T x} = \min_{\sum x = 0, x \neq 0} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

$$\lambda_{max} = \max_{x \neq 0} \frac{x^T L_G x}{x^T x} = \max_{x \neq 0} \frac{\sum_{(i,j)\in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

One can use this to bound the eigenvalues of graphs eaasily.

1.2 Cut Graphs

1.2.1 Cut Ratio

Let $e(S)$ of a cut $S - \overline{S}$ denote the number of edges between S and $\overline{S}$
Cut Ratio ϕ of a cut $S - \overline{S}$ is $\phi = \frac{e(S)}{\min(|S, |\overline{S}|)}$
The **cut of minimum ratio** is the cut that minimizes $\phi(S)$.
The **isoperimetric number** of a graph is the value of minimum cut.

Integer Function of the Cut Ratio

Associate every cut $S - \overline{S}$ with a vector $x \in \{-1, 1\}^n$, such that

$$x_i = \begin{cases} 1 & i \in S \\ -1 & i \in \overline{S} \end{cases}$$

Thus,

$$e(S) = \frac{1}{4} \sum_{(i,j)\in E} (x_i - x_j)^2$$

$$|S||\overline{S}| = \sum_{i,j\in V} [i \in S, j \in \overline{S}] = \frac{1}{4} \sum_{i<j} (x_i - x_j)^2$$
$$\frac{n}{2} \min(|S|, |\overline{S}|) \leq |S||\overline{S}| \leq n \min(|S|, |\overline{S}|)$$

Hence:

$$\frac{1}{n}\phi(G) \leq \min_{x\in\{1,-1\}^n} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2} \leq \frac{2}{n}\phi(G)$$

This allows approximation of $\phi(G)$ to a factor of 2, but NP-hard. Can change the range of x to be $[-1, 1]^n$

To find the min of function $f(x)$ on C, relax the constraints $x \in C' \supseteq C$, let $f(p)$ and $f(q)$ be the min with constraints C and C'. $f(p) \geq f(q)$. Round q to $q' \in C$ such that $f(q') \leq \gamma f(q)$. Thus $f(q') \leq \gamma f(q) \leq \gamma f(p)$

Relax $x \in \{-1, 1\}$ to $x \in R^n, \sum x_i = 0$

$$\sum_{i<j} (x_i - x_j)^2 = n \sum x_i^2$$

$$\min_{x\in R^n} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{\sum_{i<j}(x_i - x_j)^2} = \min_{x\in R^n, x\perp 1} \frac{\sum_{(i,j)\in E}(x_i - x_j)^2}{n \sum x_i^2} = \frac{\lambda_2}{n}$$

By definition of Courant-Fischer formula of the second eigenvalue. So,

$$\phi(G) \geq \frac{\lambda_2}{2}$$

1.2.2 Cheeger's Inequality

Upper bound on the minimum cut. These bounds are tight.

$$\frac{\phi(G)^2}{2d_{max}} \leq \lambda_2 \leq 2\phi(G)$$

Proof

For any $x \perp \mathbf{1}, x_1 \leq x_2 \cdots x_n$, $\exists i$

$$\frac{x^T L x}{x^T x} \geq \frac{\phi(\{1 \cdots i\})}{2d_{max}}$$

Let

$$m = \frac{n+1}{2}$$
$$y_i = x_i - x_m$$

Then

$$\frac{x^T L x}{x^T x} \geq \frac{y^T L y}{y^T y}$$

Next, if $(i, j) \in E, i < m < j$, then replace (i, j) with (i, m) and (m, j) in E'.

$$\frac{\sum_{(i,j)\in E}(y_i - y_j)^2}{\sum_{i\in V} y_i^2} \geq \frac{\sum_{(i,j)\in E'}(y_i - y_j)^2}{\sum_{i\in V} y_i^2}$$

Split the latter on m,

$$\frac{\sum_{(i,j)\in E'_-}(y_i - y_j)^2 + \sum_{(i,j)\in E'_+}(y_i - y_j)^2}{\sum_{i\in[1,m]} y_i^2 + \sum_{i\in[m,n]} y_i^2}$$

which has lower bound

$$\frac{\sum_{(i,j)\in E'_-}(y_i - y_j)^2}{\sum_{i\in[1,m]} y_i^2}$$

Lemma for $z_1 \leq z_2 \cdots \leq z_n = 0$,

$$\sum_{(i,j)\in E'_-} |z_i - z_j| \geq \phi \sum_{i=1}^{m} |z_i|$$

Put it together

1. Normalize: $\sum_{i=1}^m y_i^2 = 1$

2. Let $z_i = -y_i^2$, apply lemma to gather

$$\sum_{(i,j)\in E'_-} |y_i^2 - y_j^2| \geq \phi \sum_{i=1}^{m} |y_i|^2 = \phi$$

3. Cauchy-Schwartz:

$$\sum_{(i,j)\in E'_-} |y_i^2 - y_j^2| \leq \Big(\sum_{(i,j)\in E'_-} (y_i - y_j)^2 \Big)^{1/2} \Big(\sum_{(i,j)\in E'_-} (y_i + y_j)^2 \Big)^{1/2}$$

4.

$$\sum_{(i,j)\in E'_-} (y_i + y_j)^2 \leq 2 \sum_{(i,j)\in E'_-} (y_i^2 + y_j^2) \leq 2 \sum_{i=1}^{m} d_{max} y_i^2 \leq 2d_{max}$$

5.

$$\frac{\sum_{(i,j)\in E'_-} (y_i - y_j)^2}{\sum_{i=0}^{m} y_i^2} \geq \frac{\big(\sum_{(i,j)\in E'_-} |y_i^2 - y_j^2|\big)^2}{\sum_{(i,j)\in E'_-} (y_i + y_j)^2} \geq \frac{\phi^2}{2d_{max}}$$

6.

$$\frac{x^T L x}{x^T x} \geq \frac{y^T L y}{y^T y} \geq \frac{\phi^2}{2d_{max}}$$

Chapter 2

Combinatorics

2.1 Expanders and Eigenvalues

2.1.1 Expander Graphs

The **edge expansion** or **Cheeger Constant** is

$$h(G) = \min_{|S| \leq n/2} \frac{e(S, \overline{S})}{|S|} \tag{2.1}$$

A graph is **(d, ϵ)-expander** if it is d-regular and $h(G) \geq \epsilon$

2.1.2 Random Graphs

A bipartite graph with $n + n$ vertices $L \cup R$ is a **(d, β)-expander** if the degrees in L are d and any set of vertex $S \subset L$ of size $|S| \leq n/d$ has at least $\beta|S|$ neighbors in R.
Let $d \geq 4$ and G be a bipartite graph obtained by randomly choosing d edges for each vertex in L, then G is a (d, d/4)-expander with constant positive probability.

$$Pr[\exists S, T; |S| \leq n/d, T \leq \beta|S|] < \sum_{s=1}^{n/d} \binom{n}{s} \binom{n}{\beta s} (\frac{\beta s}{n})^{ds}$$

2.1.3 Eigenvalue Bounds on Expansion

$$h(G) \geq \frac{1}{2}(d - \lambda_2)$$

Proof For any subset of the vertices S let $\mathbf{x} = (n-s)\mathbf{1}_S - s\mathbf{1}_{\overline{S}}$. Then $\mathbf{x^T x} = (n-s)^2 s + s^2(n-s) = s(n-s)n$.

$$\begin{aligned}
\mathbf{x^T} A\mathbf{x} &= 2\sum_{(i,j)\in E} x_i x_j = 2(n-s)^2 e(S) - 2s(n-s)e(S,\overline{S}) + 2s^2 e(\overline{S}) \\
&= (n-s)^2(ds - e(S,\overline{S})) - 2s(n-s)e(S,\overline{S}) + s^2(d(n-s) - e(S,\overline{S})) \\
&= dns(n-s) - n^2 e(S,\overline{S})
\end{aligned}$$

$$\lambda_2 \geq \frac{x^T A x}{x^T x} = \frac{dns(n-s) - n^2 e(S,\overline{S})}{s(n-s)n} = d - \frac{ne(S,\overline{S})}{s(n-s)}$$

For $s \leq \frac{n}{2}$,

$$\frac{e(S,\overline{S})}{|S|} \geq \frac{n-s}{n}(d - \lambda_2) \geq \frac{1}{2}(d - \lambda_2)$$

$(d - \lambda_2)$ is called the **spectral gap**

$$h(G) \leq \sqrt{d(d - \lambda_2)}$$