Koch's Snowflake

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1 Proof of Infinite Perimeter

Firstly, it needs to be found how many segments are created with each iteration of Koch's Snowflake. Starting with an equilateral triangle, the first iteration adds 3 new triangles with $\frac{1}{3}$ the perimeter of the original. Since only equilateral triangles are made in Koch's Snowflake, any new segments made would be the same length as all other segments in the figure and $\frac{1}{3}$ the length of a segment from the last iteration. Looking at the 1st iteration, each equilateral triangle "protrusion" from a line segment of the original equilateral triangle creates 4 new line segments that are $\frac{1}{3}$ the length of the original. This applies for all iterations of Koch's Snowflake. The total length generated from the original segment is $\frac{4}{3}a$.
Let the side length of the original equilateral triangle be a. Since each iteration generates a length $\frac{4}{3}$ the length of the original, the length of the section after n iterations will be $a(\frac{4}{3})^n$. The perimeter will be $3a(\frac{4}{3})^n$. Since Koch's Snowflake is defined with infinite iterations of that pattern, the perimeter can be defined as

$$\lim_{n\to\infty} 3a(\frac{4}{3})^n$$

which diverges for all a. Therefore, the perimeter of Koch's Snowflake is infinite. $\square$

2 Area of Koch's Snowflake

By definition, with each iteration of the Koch's Snowflake a new equilateral triangle is generated on each preexisting segment, with a side length $\frac{1}{3}$ of the segment length. Therefore, given that the nth iteration has k segments, the $n+1$th iteration will generate k new triangles. It is known from the perimeter proof that each segment separates into 4 segments after an iteration to create the triangle, and thus in the iteration after that 4 times the triangles more are created than in the last iteration. The 1st iteration creates 3 new triangles, the 2nd iteration creates 12, and so on. It can be concluded that the number of triangles created in the nth iteration is $3 \times 4^{n-1}$. The area of the triangle in each iteration can also be found; it is known that the side length of the triangle is $\frac{1}{3}$ the segment length of the last iteration, and the area of an equilateral triangle of side length s is $\frac{s^2\sqrt{3}}{4}$. If the side length of the original equilateral triangle is a, then the area of each triangle generated in the nth iteration is $\frac{(\frac{a}{3^n})^2\sqrt{3}}{4}$. Multiplying the number of triangles and the area of the triangles in each iteration gives the total area of Koch's Snowflake, which is

$$\sum_{n=1}^{\infty} \frac{(\frac{a}{3^n})^2\sqrt{3}}{4} \times 3 \times 4^{n-1} = \frac{3\sqrt{3}a^2}{16} \sum_{n=1}^{\infty} (\frac{4^n}{3^{2n}})$$

not counting iteration 0, which is the original triangle. That results in the sum of an infinite geometric sequence.

$$\sum_{n=1}^{\infty} \frac{4^n}{3^{2n}} = \sum_{n=1}^{\infty} (\frac{4}{9})^n = \frac{\frac{4}{9}}{1-\frac{4}{9}} = \frac{4}{5}$$

Therefore the total area of Koch's Snowflake given original side length a is

$$\frac{3\sqrt{3}a^2}{16} \times \frac{4}{5} + \frac{a^2\sqrt{3}}{4} = \boxed{\frac{2\sqrt{3}}{5}a^2}$$

$\square$