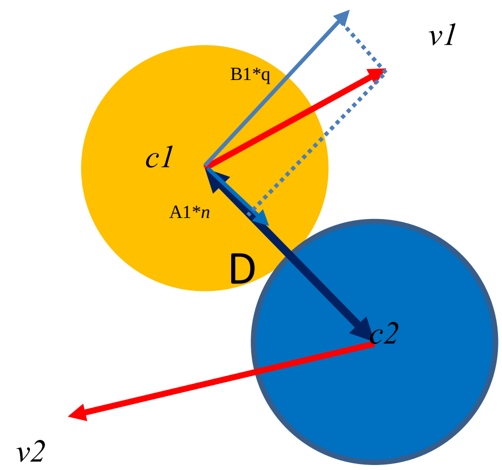
Physics of Moving Ball Collisions

1. Background

When two balls collide the collision is nearly elastic. An elastic collision is one in which the kinetic energy of the system is conserved before and after impact. For simplicity, we assume that the ball collision is perfectly elastic. Therefore, momentum is always conserved.

Conservation of Momentum states that the total momentum before the collision is equal to the total momentum after the collision. What this means is that as objects come in contact with each other, momentum is transferred from one item to the next without a net gain or loss in momentum.

The figure below shows a collision between two balls 1 and 2.



v_1 and v_2 (velocity of each ball before collision)
c_1 and c_2 (center locations of two balls when two balls collide)
M1 and M2 (mass of each ball)

If we represent the momentum of a ball to be $P = M * v$, where M is the mass and *v* is its velocity vector, then we can derive the equation based on Conservation of Momentum:

$$(M1 * v1) + (M2 * v2) = (M1 * v1') + (M2 * v2')$$

where *v*1'and *v*2' are the velocity vectors of ball 1 and ball 2 respectively after the collision

Since the second ball gains any momentum lost by the first, we can represent the difference between the momentums of the balls before and after by the same vector, ***deltaP***.

$$M1 * v1' = M1 * v1 - deltaP$$
$$M2 * v2' = M2 * v2 + deltaP$$

We can break **deltaP** = **P *n*** where a unit vector *n* representing the direction which momentum is exchanged and a scalar **P** representing the magnitude of ***deltaP***. Apply this to the equations above and solve for the new movement vectors of the balls:

$$v1' = v1 - \frac{P}{M1}n \qquad\qquad v2' = v2 + \frac{P}{M2}n$$

If we solve for **P**, we can calculate the new movement vectors.

As shown in the figure, ***v1*** and ***v2*** can be represented by the sum of two vectors: ***n*** and ***q*** (perpendicular to ***n)***. Using this information, we can represent ***v1, v1', v2***, and ***v2***' by:

$$v1 = (A1 * n) + (B1 * q)$$
$$v2 = (A2 * n) + (B2 * q)$$
$$v1' = (A1' * n) + (B1' * q)$$
$$v2' = (A1' * n) + (B2' * q)$$

where **A1, A2, B1**, and **B2** are scalars, ***n*** is the same ***n*** as mentioned before,and ***q*** is the normalized vector perpendicular to the line along which momentum is exchanged and on the same plane as ***n*** and the velocity vector.

Substituting ***v1*** and ***v2*** in equation, we get:

$$v1' = v1 - \frac{P}{M1}n = (A1 * n) + (B1 * q) - \frac{P}{M1}n = \left(A1 - \frac{P}{M1}\right) * n + B1 * q$$
$$v2' = v2 + \frac{P}{M2}n = (A2 * n) + (B2 * q) - \frac{P}{M2}n = \left(A2 + \frac{P}{M2}\right) * n + B2 * q$$

And since $v1' = (A1' * n) + (B1' * q)$ and $v2' = (A1' * n) + (B2' * q)$, we get

$$A1' = A1 - \frac{P}{M1}$$
$$B1' = B1$$
$$A2' = A2 + \frac{P}{M2}$$
$$B2' = B2$$

Now we can use the Conservation of Energy to solve for P. The equation for kinetic energy is:

$$Energy = \frac{Mass}{2} * velocity^2$$

Since energy is conserved, the total energy before the collision must equal the total energy after the collision:

$$\frac{M1}{2}\|v1\|^2 + \frac{M2}{2}\|v2\|^2 = \frac{M1}{2}\|v1'\|^2 + \frac{M2}{2}\|v2'\|^2$$

Using the movement vector as the hypotenuse of a right triangle, we can substitute:

$$\frac{M1}{2}(A1^2 + B1^2) + \frac{M2}{2}(A2^2 + B2^2) = \frac{M1}{2}(A1'^2 + B1'^2) + \frac{M2}{2}(A2'^2 + B2'^2)$$

Then, we get:

$$\frac{M1}{2}(A1^2 + B1^2) + \frac{M2}{2}(A2^2 + B2^2) = \frac{M1}{2}((A1 - \frac{P}{M1})^2 + B1^2) + \frac{M2}{2}((A2 - \frac{P}{M2})^2 + B2^2)$$

Note that the B1 and B2 terms in equation drop out of the equation.

With an equation in terms of ***M1, M2, A1, A2,*** and ***P***, solve for **P** and then use ***P*** to calculate the new movement vectors.

After solving for **P**, we get

$$P = \frac{2 * M1 * M2 * (A1 - A2)}{M1 + M2}$$

So, plugging this into Equations:

$$v1' = v1 - \frac{2(A1 - A2)}{M1 + M2} M2 * n$$

$$v2' = v2 + \frac{2(A1 - A2)}{M1 + M2} M1 * n$$

Let $K = \frac{2(A1-A2)}{(M1+M2)}$

$$v1' = v1 - K * M2 * n$$

$$v2' = v2 + K * M1 * n$$

2. Algorithm

Input: v_1 and v_2 (velocity of each ball before collision), c_1 and c_2 (center locations of two balls when two balls collide), and M1 and M2 (mass of each ball)
Output: *v1'* and *v2*' (new velocity of each ball after collision)

1. Calculate the distance $D = |c_1c_2|$ between the two points of collision where c_1 and c_2 are center locations of two balls.

2. Find the unit vector *n* from the point of collision for the first ball and the point of collision of the second ball.

$$n = \frac{(c2 - c1)}{|C1C2|} = \frac{(c2 - c1)}{D}$$

3. Calculate the K-value that takes into account the velocities of both balls.

$$K = \frac{2(v1 \cdot n - v2 \cdot n)}{M1 + M2}$$

where M1 and M2 are masses of two balls
A1= $v1 \bullet n$ and A2 = $v2 \bullet n$

4. Calculate the new velocity of each ball using K-value.

$$v1' = v1 - K * M2 * n$$

$$v2' = v2 + K * M1 * n$$

Update position of each ball using the new velocity.