1 Uncertainty relation

From [1]
Consider the uncertainty relation $\Delta_f \Delta_t \geq 1/2$

1. Show that scaling does not change $\Delta_f \Delta_t$. Either use scaling that conserves the $\mathcal{L}_2$ norm ($f'(t) = \sqrt{a} f(at)$) or be sure to renormalize Δ_f, Δ_t.
2. Can you give the time-bandwidth product of a rectangular pulse, $p(t) = 1, -1/2 \leq t \leq 1/2$, and 0 otherwise?
3. Same as above, but for a triangular pulse.
4. What can you say about the time-bandwidth product as the time-domain function is obtained from convolving more and more rectangular pulses with themselves?

Your Solution Here!

2 Sinc Expansion Completeness

Adapted from [1].

One method for constructing an orthogonal basis is to take a complementary pair of ideal low and high pass filters, sinc functions of the form

$$\begin{aligned} g_0[n] &= \frac{1}{\sqrt{2}} \cdot \frac{\sin((\pi/2)n))}{(\pi/2)n} \\ g_1[n] &= (-1)^n g_0[-n+1], \end{aligned}$$

and, combined with an offset, form a basis using all the even shifted versions. Show that without the offset "+1", the basis is not complete in $\ell_2(\mathbb{Z})$. That is to say, find a sequence that cannot be represented by

$$\begin{aligned} g_0[n] &= \frac{1}{\sqrt{2}} \cdot \frac{\sin((\pi/2)n))}{(\pi/2)n} \\ g_1[n] &= (-1)^n g_0[-n] \end{aligned}$$

and all even shifts.

Hint: Consider trying to represent an odd function such as $y[n] = \sin((\pi/2)n)$

3 Truncation as an Orthogonal Projection

Fall 2018
Let $I \in \mathbb{Z}$ be an index subset. Define the truncation operator $T_I : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by $(T_I h)[t]$, where w is the window indicator function of I.

1. For $h \in \ell_2(\mathbb{Z})$, use the projection theorem to show that $\hat{h} := T_I h$ is the least-squares approximation (of h) whos support is limited to I.

2. Show that T_I is an orthogonal projection in $\ell_2(\mathbb{Z})$.

3. Filter truncation is a lossy operation: the output $\hat{h} * x$ is perturbed from $h * x$. But by how much ? Show that the deviation sequence $D = \hat{h} * x - h * x$ is bounded by the bound below:

$$|D[n]| \leq \|x\| \|\hat{h} - h\| \tag{1}$$

   and conclude that the ratio $M := \frac{\|D\|_\infty}{\|x\|}$ is bounded. You may assume that x and h are absolutely summable (and bounded). **Optional:** determine whether M attains its maximum.

4 Parseval's relation for Non-orthogonal Bases

From [1]
Consider the space $V = \mathbb{R}^n$ and a biorthogonal basis, that is, two sets $\{\alpha_i\}$, $\{\beta_i\}$ such that

$$\langle \alpha_i, \beta_j \rangle = \delta_{i,j} \qquad i, j = 0, 1, ..., n-1 \tag{2}$$

1. Show that any vector $v \in V$ can be written in the following two ways:

$$v = \sum_{i=0}^{n-1} \langle \alpha_i, v \rangle \beta_i = \sum_{i=0}^{n-1} \langle \beta_i, v \rangle \alpha_i \tag{3}$$

2. Call v_α the vector with entries $\langle \alpha_i, v \rangle$ and similarly v_β with entries $\langle \beta_i, v \rangle$. Given $\|v\|$, what can you say about $\|v_\alpha\|$ and $\|v_\beta\|$?

3. Show that the generalization of Parseval's identity to biorthogonal systems is

$$\|v\|^2 = \langle v, v \rangle = \langle v_\alpha, v_\beta \rangle \tag{4}$$

and

$$\langle v, g \rangle = \langle v_\alpha, g_\beta \rangle \tag{5}$$

5 Range-Finding Pulses (Computational)

In the previous homework, you observed that finding the delay between two signals is fairly straightforward using the cross-correlation. In this problem, you will explore different waveforms for a range-finding application. Frequency shifts from the Doppler effect introduces significant additional complexity (frequency shifts and time shifts don't commute), for the purpose of most of this problem, we will ignore the issue and assume no Doppler shifts.

In a radar environment, there may be multiple targets that reflect a given pulse, resulting in a system generating a multipath response represented by

$$H = \sum_{i=0}^{N-1} \alpha_i \sigma_{t_i} \qquad t_i \in \mathbb{R}^+,\ \alpha_i \in (-1, 1), \tag{6}$$

where σ_d is a delay by d, and α_i represents some attenuation of the signal. The set of $\{\alpha_i\}$ simply lump together the different losses from both propagation and the imperfect reflection from the targets.

Goal: Identify each element of $\{\sigma_i\}$ in the system.

Q1: Suppose $N = 1$ (representing a single reflection), how would you apply your algorithm from homework 3 to identify σ_0? What happens when $t_i \notin \mathbb{Z}$?

Your Solution Here!

Q2: Now assume $N > 1$, how would you extend your algorithm to identify all t_i? This doesn't need to be optimal in any sense, just pick something that seems reasonable.

Your Solution Here!

Q3: Implement your algorithm from Q2 and apply it to data generated by the provided function. How well does it work? What seem to be the issues? What happens as you adjust the distribution of t_i such that the delays are clustered more tightly?

Your Solution Here!

There are actually a couple of issues at play. While we haven't formally covered noise yet, we can see that the noise seems to blur together the peaks of the cross-correlation.

Q4: Given no physical constraints, what is the optimal waveform for time localization (assuming operation in continuous time)? What is the associated time-frequency trade-off?

Your Solution Here!

Q5: Ultra-wide bandwidth electronics are very complicated, can be very expensive to produce, and can interfere with other items sharing the spectrum. For this reason, consider limiting your bandwidth to a fixed width B. How would you adjust your answer to Q4 to account for this constraint?

Your Solution Here!

Q6: Now that we have a finite bandwidth, we may as well operate in discrete time. Fix the amount of energy in the signal, and simulate the application of your algorithm from Q2 applied to pulses of the form of your response to Q5.

Your Solution Here!

Unfortunately, you may additionally have some form of instantaneous power constraint, or, related, a maximum voltage constraint on the electronics.

Q7: How would you preserve the desired time-bandwidth trade-off while satisfying a maximum amplitude constraint (i.e. $x(t) \leq P$ for all t)? Hint: What happens to the phase when we compute an autocorrelation of a signal? Alternatively, think about the relationship between an STFT and the Fourier Transform of the entire sequence.

Your Solution Here!

Q8: Write a script that generates pulses of the form described in Q7, include a plot of the waveform.

Your Solution Here!

Q9: Finally, apply the algorithm to the pulses in your response. Compare the behavior to sinusoids with rectangular, triangular, and Hanning windows. Include plots of the cross-correlation in simulation for all four.

Your Solution Here!

References

[1] Martin Vetterli and Jelena Kovačević, *Wavelets and Subband Coding*, Prentice Hall, Englewood Cliffs, NJ, 1995.