1 Linear State Observer

Consider a linear state-space system that evolves according to

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$
$$y_t = Cx_t + Dv_t, \tag{2}$$

where $u_t \sim \mathcal{N}(0, \sigma_u^2 I)$, $v_t \sim \mathcal{N}(0, \sigma_v^2 I)$, and the states and observations are both vectors of length N and M respectively. Define an associated observer system as

$$\hat{x}_{t+1} = A\hat{x}_t - L(\hat{y}_t - y_t) \tag{3}$$
$$\hat{y}_t = C\hat{x}_t. \tag{4}$$

Question: Define the error dynamics and give a set of non-trivial sufficient conditions on A, C, D, L such that the error is stable in some sense. By non-trivial, I just mean that you should not require A to be zero, for instance, but you can require some relationship between the matrices.

It will be useful to remember that a linear system is stable when all eigenvalues of the state-transition operator have magnitude less than 1. Start by giving a condition for error stability with a random unknown initial state with finite mean and variance, and no input or measurement noise.

Your Solution Here!

2 Antenna Spacing

In lecture, we discussed the relationship between array processing, sampling, and aliasing. In this problem, we want to solidify the reasoning around these ideas.

Suppose we have three antennas numbered $-1, 0, 1$, that is to say we observe

$$y_k(t) = x(t)e^{jk\frac{2\pi d \sin(\theta)}{\lambda}} \qquad \forall k \in \{-1, 0, 1\}, \tag{5}$$

where we have previously defined

$$\nu = \frac{2\pi d \sin(\theta)}{\lambda}. \tag{6}$$

We used the belief that the mapping from θ to ν is invertible. What is the largest closed domain D containing 0 for which this is true?

Your Solution Here!

Now suppose that $d = \lambda/2$ such that $\nu = \pi \sin(\theta)$.
Sketch the radiation pattern for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Your Solution Here!

Now, keeping $\sin(\theta)$ in the original range of $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, introduce a factor $\gamma \in \mathbb{R}^+$ such that $d = \gamma\frac{\lambda}{2}$. This γ represents the number of half-wavelengths that the antennas are spaced. Thinking of the antennas as sampling a frequency domain representation as motivation, sketch the radiation pattern for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and $\gamma \in \{\frac{1}{2}, 2, 3\}$.

Your Solution Here!

Suppose you have instead a linear array of N antennas each spaced by $\lambda/2$. While you don't encounter any of the aliasing-style effects from improper spacing, you do still have a potentially large ripple in the rejected regions. Each of these ripples can be referred to as sidelobes. What is the height of tallest sidelobe when simply adding all y_k together?

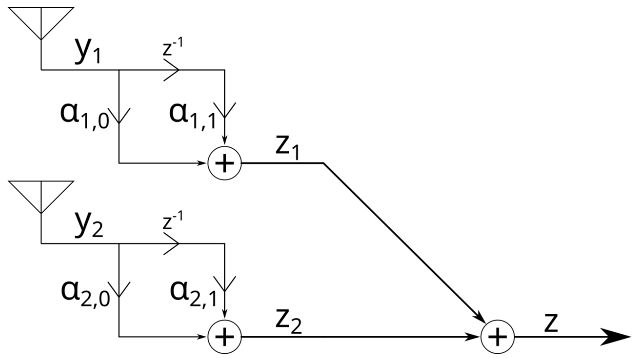
Your Solution Here!

Now, consider applying multiplicative real-valued weights to each antenna. How would you reduce the magnitude of the sidelobes? Give one possible set of weights and the corresponding maximum sidelobe height. Feel free to use any reference you need regarding Fourier transforms. No need to compute the value by hand.

Your Solution Here!

3 Space-Time Processing

The figure below illustrates a system with two antennas, each with their own filter involving a single delay indicated by z^{-1}.



As seen in lecture, we can represent the physical delay as multiplication by $e^{-j\nu}$, where ν is a parameter determined from the geometry of the setup. If x is the original signal then the following relationships hold:

$$
\begin{aligned}
y_1[n] &= x_0[n] \\
y_2[n] &= e^{-j\nu_0} x_0[n] \\
z_i[n] &= \alpha_{i,0} y_i[n] + \alpha_{i,1} y_i[n-1] \\
z[n] &= z_1[n] + z_2[n].
\end{aligned}
$$

Choose the filter values to place the central lobe at the ω_0, ν_0 pair of frequency and spatial location. Plot the filter response as a 2D image with one axis corresponding to spatial coordinate, and the other corresponding to frequency. This can either be a sketch or a computer generated plot.

Your Solution Here!

4 MUSIC Algorithm (Computational)

MUSIC can be used to identify individual frequency elements in a sparse signal.

In this situation, the system model is formulated in the following manner

$$y = Ax + w, \tag{7}$$

where the columns of A correspond to the complex sinusoids, the rows of X correspond to the amplitude of the sinusoids

Please implement the MUSIC algorithm and apply it in this situation. Assume that you know the number of complex exponentials ahead of time. Feel free to use any available linear algebra functions that you need, in particular, a function to compute the SVD or eigendecomposition.

Below are 8 samples of a complex valued signal comprised of two complex exponentials and additive noise. Please report the frequencies found of the two terms and include a plot of $P_{MU}(\theta)$, sometimes referred to as a pseudo-spectrum.

```
 2.3294301243940634 - 0.2296259966276833im
-0.4134361204474038 + 1.0082105392432457im
-0.3972955429876873 - 0.3077140046653479im
 -0.328835156792498 + 0.3412442337892796im
 0.5276539575334861 - 2.6968518609509466im
 1.4068435099399519 + 1.609400966603145im
-1.4358532259748622 + 0.4052265092075893im
-0.6869999254086611 + 0.22150298448055664im
```

The true exponentials are of the form $e^{j0.429\pi n}$ and $e^{j0.858\pi n}$.