1 LTI Filtering - Strength of Fourier

Suppose you have N samples of a band-limited signal $x[n]$ such that, if $X = \mathcal{F}(\mathbf{x})$ is the Fourier transform of $\mathbf{x}$, $X(\omega) = 0$ for all $\omega \geq B$. You have reason to believe that the signal has roughly constant magnitude within the passband, such that, in some sense, $|X(\omega)| \approx P$ where $P \in \mathbb{R}^+$ for all $\omega < B$.

The signal is corrupted with a noise vector $\mathbf{n}$ of i.i.d. standard normal random variables such that $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. The measurement available is $\mathbf{y} = (\mathbf{x} + \mathbf{n}) \sim \mathcal{N}(\mathbf{x}, \mathbf{I})$.

You remember that the law of large numbers says that if you average enough well-behaved random variables, they will eventually converge to the mean, in this case, the exact signal. Unfortunately, you only have one realization of the signal, so you decide to try using a moving average to capture some of the effect.

Question: How many points would you average? Explain your decision.
How much of the noise is rejected? How much is the signal distorted?
Hint: Recall that expectation is linear and that the signal and noise are independent.
There isn't necessarily one correct answer.
Recall the Fourier transform of a rectangular pulse is

$$x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{o.w} \end{cases} \quad \overset{\mathcal{F}}{\Leftrightarrow} \quad X(\omega) = \left(\frac{\sin[\omega(M+1)/2]}{\sin(\omega/2)} \right) e^{-j\omega M/2} \tag{1}$$

Your Solution Here!

2 Signal Analysis - Limitations of Fourier

One of the nice properties of the Fourier transform is the ability to interpret the transform domain. We have some sense of "slow" vs "fast" signals, hence why moving averages can be used as a heuristic to reduce the noise on certain signals.

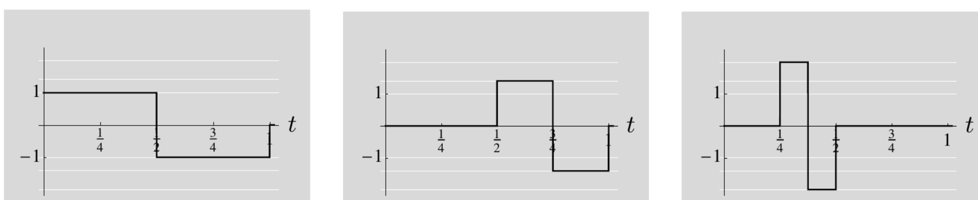
In this problem, we'll look at two limitations of a Fourier Transform for signal analysis, in comparison to the Haar wavelet. We will describe the Haar wavelet basis having a base length L through the following equation.

$$\varphi(t) = \begin{cases} 1, & \text{for } 0 \leq t < \frac{L}{2} \\ -1, & \text{for } \frac{L}{2} \leq t < L \\ 0, & \text{otherwise} \end{cases}. \tag{2}$$

The additional basis vectors are defined as scaled and shifted versions by

$$\varphi_{m,n} = 2^{-m/2}\varphi\left(\frac{t - n2^m}{2^m}\right). \tag{3}$$

Some examples for base length 1 are included in the following figure from the text.



(a) $\psi_{0,0}(t)$. (b) $\psi_{-1,1}(t)$. (c) $\psi_{-2,1}(t)$.

Finally, we add a constant vector $\varphi_0(t) = 1$.

(Problems on next page to avoid an unfortunate page break)

(a) (Multirate) Given that the Haar Wavelet basis appears structurally similar to the Fourier basis and acquires an ability to localize high frequency changes, we will briefly consider the following set of functions

$$\phi(t) = \begin{cases} \sin(2\pi t), & \text{for } 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

where the additional scaling and shifting is defined in the same way as the Haar wavelets

$$\phi_{m,n} = 2^{-m/2}\phi\left(\frac{t - n2^m}{2^m}\right). \tag{5}$$

Question: How would you represent

$$f(t) = \begin{cases} \sin(8\pi t) + \sin(4\pi t) + \sin(2\pi t), & 0 \le t < \frac{1}{4} \\ \sin(4\pi t) + \sin(2\pi t), & \frac{1}{4} \le t < \frac{1}{2} \\ \sin(2\pi t) & \frac{1}{2} \le t < 1 \end{cases} \tag{6}$$

in the set of functions $\{\phi_{m,n}\}$? What makes the analysis of a signal into the set of functions $\{\phi_{m,n}\}$ difficult?

Your Solution Here!

(b) (Piecewise-Constant Signals)

In some sense, piecewise-constant signals can be very *sparse*, loosely meaning that you can fully describe the signal using very few numbers. Unfortunately, such signals do not illustrate their sparsity in the frequency domain. Consider the following function:

$$f(t) = \begin{cases} 1 & 0 \le t - 2k \le \frac{1}{2} \text{ for any } k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Question: Describe what happens to the discontinuities as you add terms in a Fourier series. Assume a periodic extension of the Haar wavelet described earlier with a base length of 2. How would you represent the signal in the periodic Haar basis?

Your Solution Here!

3 Inner Product Spaces Proofs

Problem from Fall 2018 (See page 29 of VKG if you need hints)

(a) Prove Pythagorean Theorem

Your Solution Here!

(b) Prove Parallellogram Law

Your Solution Here!

(c) Prove Cauchy-Schwartz

Your Solution Here!

4 A Different Type of Addition...

Problem from Fall 2018

Let $\mathbb{R}^+$ denote the positive reals, and consider the collection

$$V = \left\{(x_1, x_2, ..., x_n) \mid x_k \in \mathbb{R}^+ \text{ for } k = 1, ..., n\right\}$$

for $\mathbf{x}, \mathbf{y} \in V$ define the following summation and scaling rules:

$$\mathbf{x} + \mathbf{y} := (x_1 y_1, ..., x_n y_n) \qquad \text{(summation)}$$
$$\alpha \mathbf{x} := (x_1^\lambda, ..., x_n^\lambda) \qquad \text{(scaling)}$$

(a) Show that V is a linear space over $\mathbb{R}$ with respect to the above rules. Specify the neutral vector in V, and check each of the axioms individually.

Your Solution Here!

(b) Check whether the mapping $T : V \to \mathbb{R}^N$ defined below is linear. Explain your answer. **Note:** $\mathbb{R}^N$ is the Euclidean space with standard sum and scaling.

$$T(x) := (\log(x_1), ..., \log(x_n))$$

Your Solution Here!

(c) Find an inner product on V.
In the next homework, you will be asked to find an orthonormal basis for the space.

Your Solution Here!

5 Computational Problems - Setup

Every problem set will have some computational problem. This assignment is meant to make sure you have some familiarity with Python before the later assignments.

5.1 Setup Environment

The tools used in this assignment are

- Python 3 [https://www.python.org/]
- IPython and Jupyter Notebook [https://ipython.org/]
- Numpy [https://numpy.org/]
- Matplotlib [https://matplotlib.org/]
- Numba (Optional, but recommended) [http://numba.pydata.org/]
- IPywidgets (Optional, but recommended) [https://ipywidgets.readthedocs.io/en/latest/]

IPython, Numpy, Matplotlib, Numba, and IPywidgets are all libraries for Python.

Numba allows Just-In-Time (JIT) compilation of functions, and will dramatically speed up the execution of one part of the assignment, though it will run without it.

We include one interactive plot at the end of the notebook which requires ipywidgets to use, though it's not needed to complete the assignment.

Please familiarize yourself with the basics of Python and Numpy, as they will be used throughout the semester. There are many tutorials available online, but here are the links to the official tutorials from the documentation.

Python: https://docs.python.org/3/tutorial/
Numpy: https://numpy.org/devdocs/user/quickstart.html

5.2 Window Functions

For similar reasons to the effects that appear when representing discontinuities in a Fourier series, when taking some form of moving average it can sometimes be advisable to use a different **window** function. In this problem, you will implement three different window functions. At the end of the assignment, you will qualitatively see the different behavior in when filtering and perhaps get a sense for when a more smooth window may be preferable vs. a standard rectangular window.

In `HW1.py` please implement three functions which construct three different windows, a rectangular window, a Bartlett window, and a Hann window. The windows should have no padding and take the entire target length, and the unnormalized forms are the following

$$\text{Rectangular}[n] = 1/N \quad \forall\, 0 \leq n < N \tag{8}$$

$$\text{Bartlett}[n] = 1 - \left|\frac{2n - (N-1)}{N-1}\right| \quad \forall\, 0 \leq n < N \tag{9}$$

$$\text{Hann}[n] = \sin^2\left(\frac{n\pi}{N-1}\right) \quad \forall\, 0 \leq n < N \tag{10}$$

where N is the width of the window function. Please normalize the windows such that the sum of the entire window is equal to 1.

The file `HW1_test.py` contains some tests for both the window functions and the convolutions. Use the flags at the top of the file to disable irrelevant tests while you're debugging.

5.3 Circular Convolution

While many expressions of convolution imply an infinite sequence, one method of finite-length convolution is called circular convolution. A circular convolution is a convolution which can be thought of as convolution under periodic boundary conditions, and can be written as

$$(\mathbf{x} \circledast \mathbf{y})[n] = \sum_{i=0}^{N-1} x[i]y[N + n - i \bmod N]. \tag{11}$$

This form of convolution is important because it arises from the same periodicity assumption that is implicit in a Discrete Fourier Transform, and thus, while you have likely learned that convolution equates to multiplication in the frequency domain, it is in fact circular convolution through a periodic extension of the signal.
In `HW1.py` please implement two different methods of doing a circular convolution

- A direct form implementing equation **??**.
- Convolution through the frequency domain using `np.fft.fft` and `np.fft.ifft`.

5.4 Jupyter Notebook

Finally, there is a Jupyter Notebook that generates some plots using your code, and includes an interactive component at the end that relates to problem 1 of the homework. Please attach the generated plots (not including the interactive one) in your homework write-up. Nothing else needs to be implemented, just read through the notebook.