1 Quartic Cost Function

(From Woodrow and Stearns)
Consider the adaptive linear combiner such that

$$y_k = w_0 x_{0k} + w_1 x_{1k} \tag{1}$$
$$\epsilon_k = d_k - y_k. \tag{2}$$

Suppose that we wish to minimize $\mathbb{E}\left[\epsilon_k^4\right]$ rather than $\mathbb{E}\left[\epsilon_k^2\right]$. Assume that the signals x_{0k}, x_{1k}, and d_k are all stationary.

(a) Derive an expression for $\mathbb{E}\left[\epsilon_k^4\right]$.

Your Solution Here!

(b) Is $\mathbb{E}\left[\epsilon_k^4\right]$ quadratic in w_0 and w_1

Your Solution Here!

(c) Is $\mathbb{E}\left[\epsilon_k^4\right]$ a unimodal function in w_0 and w_1?

Your Solution Here!

2 Eigenvectors of Single Input System

(From Woodrow and Stearns)
Show that the eigenvectors of the correlation matrix for any single-input adaptive linear combiner with two weights are given by

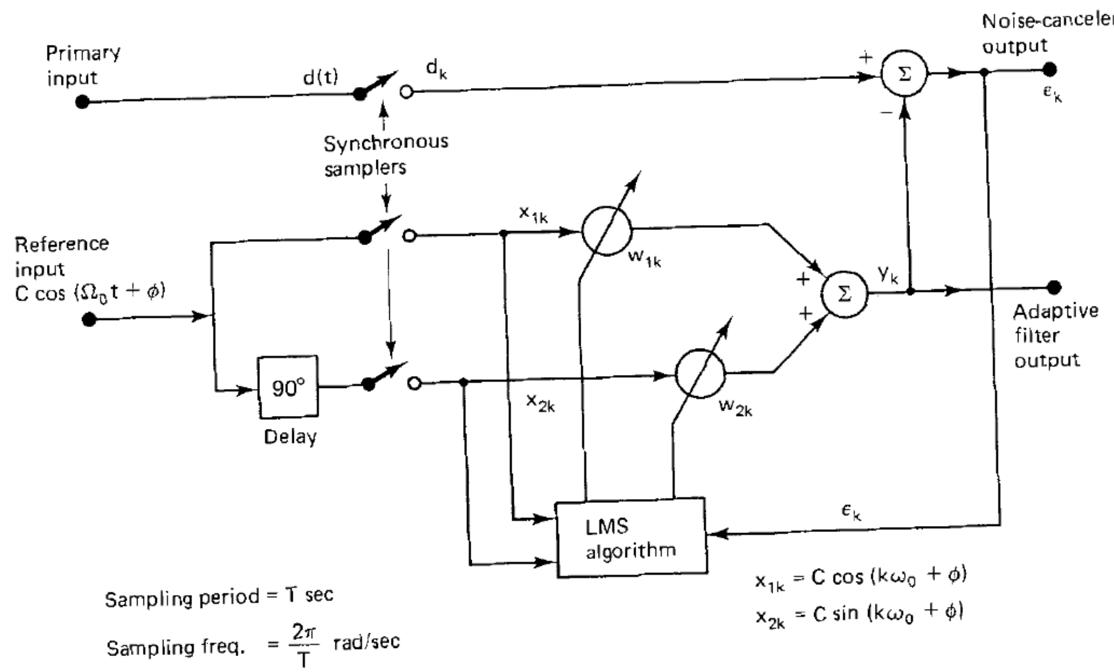
$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{3}$$

Your Solution Here!

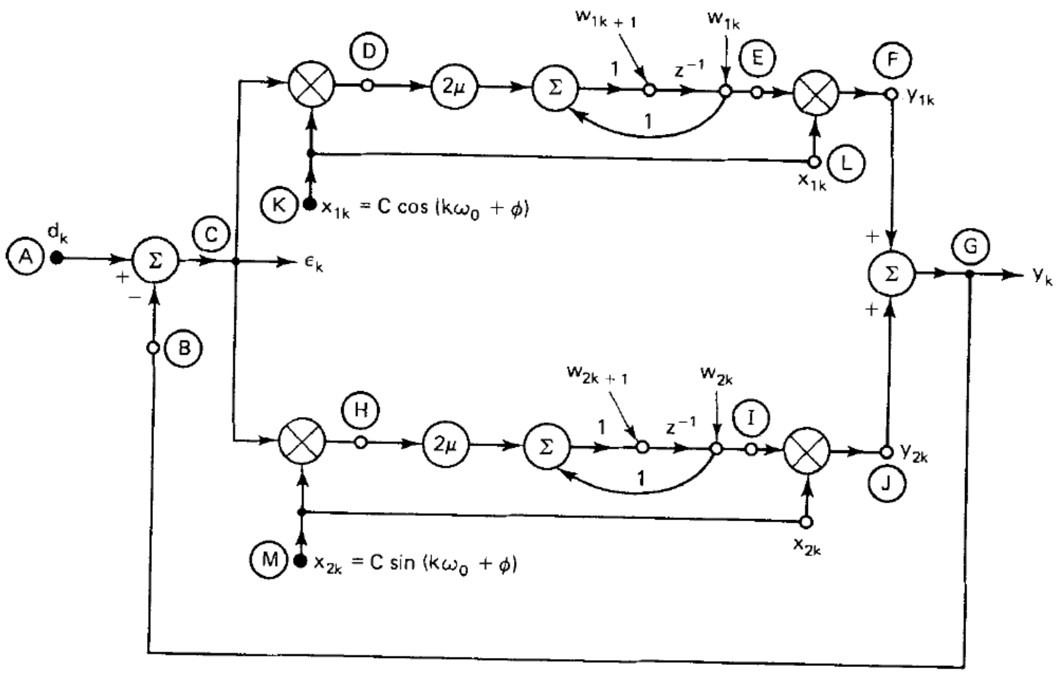
3 Adaptive Construction of a Notch Filter

(Adapted From Woodrow and Stearns)
In lecture, we briefly mentioned the ability to construct a notch filter in an adaptive manner by weighting a sine and cosine term at a particular frequency.

The visually, the system is illustrated in the following way from the text by Widrow and Stearns.



A flow diagram for the discrete-time component is shown below, also from the same book.



The above diagram labels many useful nodes, and describes the full LMS algorithm. Our goal is to find the impulse response of the system.

(a) Suppose we apply an impulse to node C in the above diagram of the form $\epsilon_k = \alpha\delta[k-m]$. What is the response at point D?

Your Solution Here!

(b) From D to E, we have a digital integrator of the form

$$E[n] = E[n-1] + 2\mu D[n-1]. \tag{4}$$

What is the impulse response from D to E and from C to E?

Your Solution Here!

(c) What is the impulse response from C to F?

Your Solution Here!

(d) The bottom pathway follows very similarly. What is the impulse response from C to G when $m = 0$?

Your Solution Here!

(e) Now note the feedback loop and conclude the transfer function from A to G. You may need to remember back to an undergraduate course to recall the formula.

Your Solution Here!

(f) Plot the frequency response of this adaptive system.

Your Solution Here!

(g) Simulate the system applied to a vector of 512 i.i.d. Gaussian random variables. Plot the magnitude of the DFT of the result.

4 Leaky LMS

From 2007
The Leaky-LMS algorithm is an adaptation strategy which attempts to minimize the following cost function:

$$\mathcal{E}_L = \mathbb{E}\left[e_n^2 + \lambda w_n^\top w\right] \tag{5}$$

where $\lambda > 0$ and w_n is the vector of coefficients of the usual adaptive filter problem, i.e.

$$y_n = w_n^\top x_n \tag{6}$$
$$x_n = \begin{bmatrix} x[n] & x[n-1] & x[n-p+1] \end{bmatrix} \tag{7}$$
$$e_n = d_n - y_n, \tag{8}$$

where d_n is the desired response, and x_n is the vector of inputs to the filter. Assume that x_n and d_n are zero-mean, wide-sense stationary random processes. The autocorrelation function of x_n is given by $\mathbb{E}\left[x[n]x[n-m]\right] = R_{xx}[m]$, and the cross correlation between $d[n]$ and $x[n]$ is given by $\mathbb{E}\left[d[n]x[n-m]\right] = r_{dx}[m]$.

(a) Determine w_{opt}, the set of filter coefficients which minimize $\mathcal{E}_L$ Clearly define all terms in your expression.

Your Solution Here!

(b) Determine $\nabla_w \mathcal{E}_L$, the gradient of the error with respect to the filter coefficients. By analogy to the LMS algorithm, determine the leaky-LMS update equations. Note that as with the LMS algorithm, this should not depend on knowledge of $R_{xx}[m]$ or $r_{dx}[m]$.

Your Solution Here!

(c) If $\mathbb{E}\left[w[n]\right]$ converges, to what does it converge? Make appropriate assumptions, and you may (falsely) assume independence between x and w.

Your Solution Here!

(d) Under what conditions does the leaky-LMS algorithm converge in the mean, i.e., $\mathbb{E}\left[w\right] \to w^*$ for some w^* ? Make appropriate assumptions

5 Computational Problem

Implement LMS and RLS to be applied to an FIR filter with 10 parameters, then run the following experiment.

1. Choose an arbitrary FIR filter with 10 coefficients
2. Generate a sequence if i.i.d. Gaussian data $\mathbf{x}$
3. Use $x[n]$ as the input for RLS, LMS, and the original FIR filter. Optimize over MSE between the output of the adaptive system and the true filter.
4. Plot the convergence in MSE
5. Choose a pair of eigenvectors of the system, plot the trajectory of the weights associated with those eigenvectors.