1 Some DFT properties

Fall 2018
Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ be signals with corresponding DFTs $\mathbf{X}, \mathbf{Y} \in \mathbb{C}^N$, and let $C_x \in \mathbb{C}^{N\times N}$ be a circulant matrix, whose first column is $\mathbf{x}$. Prove the following

1. Time reversal: $x[-n \mod N] \overset{\text{DFT}}{\longleftrightarrow} X[-k \mod N]$
2. Circular convolution theroem (in time and frequency):
$$
\begin{aligned}
(\mathbf{x} \circledast \mathbf{y})[n] &\overset{\text{DFT}}{\longleftrightarrow} X[k]Y[k] \\
x[n]y[n] &\overset{\text{DFT}}{\longleftrightarrow} \frac{1}{N}(\mathbf{X} \circledast \mathbf{Y})[k]
\end{aligned}
$$
3. If $\mathbf{x}$ is a real symmetric signal, namely $x[n] = x[-n \mod N]$, then $\mathbf{X}$ is real.
4. If $\mathbf{x}$ is a real antisymmetric signal, $x[n] = -x[-n \mod N]$, then X is imaginary.
5. The eigenvectors of C_x are $w_k[m] = \exp(\frac{2\pi i k}{N} m)$, with eigenvalues $\lambda_k = X[k]$.

Your Solution Here!

2 Z-Transform of Downsampled Signals

Fall 2018
Let $y[n] = x[Nn]$. Show that the z-transform of this downsampled signal satisfies

$$y[n] = x[Nn] \overset{\text{ZT}}{\longleftrightarrow} Y(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(W^k z^{1/N}) \tag{1}$$

here $X(z)$ is the z-transform of $\mathbf{x}$, and W is a primitive N^{th} root of unity, namely $W^N = 1$ and $W^k \neq 1$ for all $0 < k < N$.

Use that result to argue that the DTFT transform downsampling relation is

$$y[n] = x[Nn] \overset{\text{DTFT}}{\longleftrightarrow} Y(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{\omega - 2\pi k}{N}\right) \tag{2}$$

Your Solution Here!

3 Deterministic Correlation and Delay Detection

Fall 2018
Denote the unit delay operator by $\sigma : \mathbb{C}^{\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}}$ by $()\sigma \mathbf{x})[n] := x[n-1]$. For $\mathbf{x}, \mathbf{y} \in \ell_2^{\mathbb{Z}}$, define the associated deterministic autocorrelation and crosscorrelation sequences as

$$a_x[k] := \langle \mathbf{x}, \sigma^k \mathbf{x} \rangle = \sum_{n \in \mathbb{Z}} x[n]x[n-k]^* \tag{3}$$

$$c_{x,y}[k] := \langle \mathbf{x}, \sigma^k \mathbf{y} \rangle = \sum_{n \in \mathbb{Z}} x[n]y[n-k]^* \tag{4}$$

(a) Prove or counter the following statements (note the underlying field is $\mathbb{C}$)

1. $a_x[k] = a_x[-k]^*$
2. $|a_x[k]| \leq a_x[0]$ for every $k \in \mathbb{Z}$
3. $c_{x,y}[k] = c_{y,x}[-k]^*$
4. $c_{x,y}[k] = c_{x,y}[-k]^*$
5. $C_{x,y}(\omega) = X(\omega)Y(\omega)^*$, where $\mathbf{X}, \mathbf{Y}, \mathbf{C}_{x,y}$ are the DTFTs of $\mathbf{x}, \mathbf{y}$ and $\mathbf{cv}_{x,y}$.

Your Solution Here!

A signal $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ whose support is bounded to $[0, ..., N-1]$ is received by two antennas, each introduces a different gain and delay. The received signals $\mathbf{x}_1, \mathbf{x}_2$ are given by

$$x_1[n] = \alpha_1 x_1[n - n_1], \qquad x_2[n] = \alpha_2 x[n - n_2]. \tag{5}$$

The constants $\alpha_1, \alpha_2 \in \mathbb{R}$ are gain coefficients and $n_1, n_2 \in \mathbb{Z}$ are delays, all unknown.

(b) Based on the result from part a.ii, derive an algorithm to determine the time delay $\Delta := n_2 - n_1$ and the gain ratio $\rho = \frac{\alpha_1}{\alpha_2}$ given inputs $\mathbf{x}_1$ and $\mathbf{x}_2$.

Your Solution Here!

(c) Explain why the explicit delay values n_1 and n_2 cannot be determined, but only their difference Δ. Is the same true for the gains α_1, α_2 ?

Your Solution Here!

4 Interchange of Multirate Operations and Filtering

Fall 2018
Consider the system given by the input output relation

$$\mathbf{y} = D_2 A D_2 A D_2 A \mathbf{x} \tag{6}$$

where A is a convolution filter and D_2 downsamples by a factor of 2.

(a) Using the multirate identities, find the simplest equivalent system of the form $\mathbf{y} = D_N H \mathbf{x}$, where D_N is downsampling by N and H is a convolutional filter. Specify the downsampling factor N, and write H in the z-transform and Fourier domains.

Your Solution Here!

(b) If A is an ideal half-band lowpass filter, draw the DTFT $H(\omega)$, clearly specifying the cutoff frequencies.

Your Solution Here!

(c) If A is an ideal half-band highpass filter, draw the DTFT $H(\omega)$, clearly specifying the cutoff frequencies.

Your Solution Here!

5 Computational Problem

Fall 2018
The provided code in `HW3.py` generates two samples $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^{100}$ of some randomly shifted/scaled waveform based on your UIN (or any other number you input of the same length).

Implement the algorithm you suggested in problem 3 to detect the shift and scale ratio between $\mathbf{x}_1$ and $\mathbf{x}_2$. Plot the input signals, their crosscorrelation, and estimated values, and comment on the results. Compare your estimated parameters with the true ones.

Your Solution Here!