1 Unmatched Sampling (Theory)

Modified from Fall 2018
Let U, V be finite dimensional signal spaces, and let $T : U \to V$ be some (linear, perhaps) mapping. This problem is about restricting the domain of a mapping to allow invertibility.

For a subset $R \subset U$, we denote the restriction of T to R by $T|_R$ (same map, whose domain is now restricted to the subset R). As an example, you may think of sampling bandlimited signals. The sampling operation itself is not inherently invertible, but given the constraint of strictly bandlimited signals, it becomes invertible.

(a) For this part only assume $U = \mathbb{C}^3$. Find a linear map $T : U \to \mathbb{C}^2$ (naturally non-invertible) and some subspace $R \subset U$ such that the restriction is $T|_R$ invertible. Prove that your chosen T is indeed invertible on R.

Your Solution Here!

(b) For your given $T : U \to V$, determine what are the subspace $R \subset U$ on which T is invertible. **Hint:** Avoid the nullspace of T (the set of vectors v such that $Tv = 0$).

Your Solution Here!

Let Φ and $\tilde{\Phi}$ be synthesis and sampling operators, both of full rank

$$\Phi : \mathbb{C}^M \to U \qquad \text{(Synthesis)}$$
$$\Phi : U \to \mathbb{C}^N \qquad \text{(Sampling)}$$

and assume that $R = \text{rng}(\Phi)$ and $S = \text{rng}(\tilde{\Phi})$ are the ranges of the respective operators. We look for conditions under which signals in R can be fully recovered when sampled by $\tilde{\Phi}$, that is, there exists Ψ such that

$$\Psi\tilde{\Phi}^* x = x \qquad \text{for all } x \in \mathbb{R}, \tag{1}$$

or in other words, the restriction $\Psi\tilde{\Phi}^*|_R$ is an identity.

(c) Show that the condition below is necessary for such recovery:

$$(\tilde{\Phi}^*) \cap R = \{0\}, \tag{2}$$

which means that the only $x \in R$ such that $\tilde{\Phi}^* x = 0$ is $x = 0$.
What is the implication (in terms of sampling) of this condition ?

Optional: Show that the condition is also sufficient.

Your Solution Here!

(d) Let $\tilde{\Phi}^* = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and let $\Phi = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Compute the matrix represnetation of the recovery system Ψ (with respect to the standard basis), and $\Psi\tilde{\Phi}^*$

Your Solution Here!

(e) Show that $\Psi\tilde{\Phi}^*$ is a projection from U to R.
This problem refers specifically to the matrix in part (d).

Your Solution Here!

2 Radon Transform (Theory)

In this section, you will formalize many notions regarding the Radon transform.

Q1:

Recall that in lecture, we gave the definition

$$(Rx)[n,\theta] = \int_{\ell_{n,\theta}} f(l)dl = \int_{-\infty}^{\infty} f\left(G_\theta \begin{bmatrix} n \\ t \end{bmatrix}\right) dt \tag{3}$$

for the forward operator. Give one additional form for the operator: express it as a double integral over x, y using a delta function.

Your Solution Here!

Q2:

Now, we loosely talked about how R^* should spread the measurements back across the image. Please define the operator mathematically.

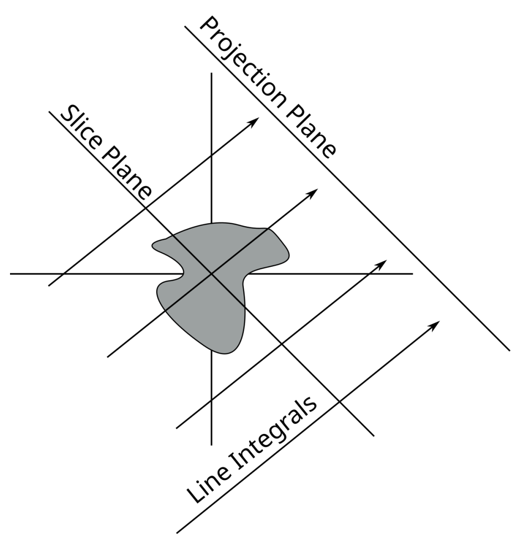
Your Solution Here!

Q3:

There is a well-known theorem relating the Radon Transform and the Fourier Transform. It is very helpful for understanding the pseudoinverse operator. The following terminology may be confusing because the forward operator is referred to as a projection.

The Fourier transform of a projection of an image (applying R for a fixed angle θ) is equal to the slice of the center of the 2D Fourier transform of the image parallel to the projection plane.

The relevant planes are illustrated in the below figure.



If we use $\mathcal{F}_k$ to denote a k dimensional Fourier Transform, then

$$[\mathcal{F}_2(f)]\left(G_\theta \begin{bmatrix} \omega \\ 0 \end{bmatrix}\right) = \mathcal{F}_1\left(\int_{-\infty}^{\infty} f\left(G_\theta \begin{bmatrix} x \\ t \end{bmatrix}\right) dt\right), \tag{4}$$

where in the above expression, ω is being used to indicate the frequency index, x is being used to indicate the spatial index, and θ is setting the direction of the projection. Note that this gives us a single scalar characterizing the indexing of both Fourier transforms.

Please prove the above theorem.

Hint: First, think intuitively about the meaning of this theorem, there is a nice reason it holds. Then use your expressions from Q1 and Q2.

As a side note from the TA, the description in words and the picture takes priority over equation 4. If the orientation of the slice is incorrect or anything, let me know. I'm fairly confident in the equation, but it's still possible I messed up the rotation.

Your Solution Here!

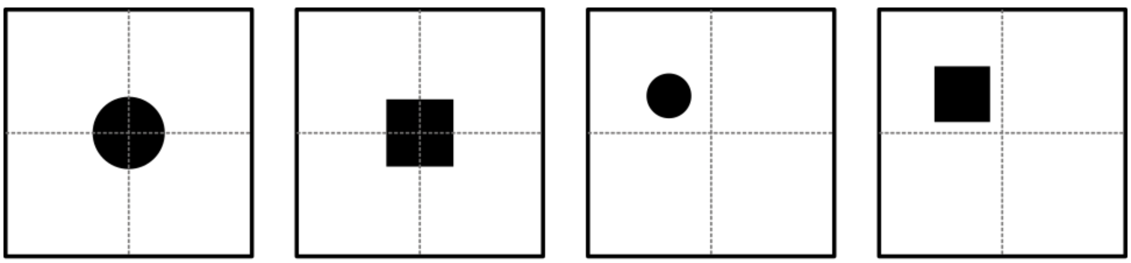
Q4:

Show that in the under-determined case $(RR^*)^{-1}$ is equivalent to applying an LTI filter.

Your Solution Here!

Q5:

By using the x-axis as the projection angle, the y-axis as the projection position, and the intensity as the value, sketch the Radon transforms of the following images.



(a) (b) (c) (d)

Your Solution Here!

3 Multichannel Sampling

From Spring 2018 Boston University ENG EC 416/516

Let $x \in BL[-\pi, \pi]$ and T=1. Consider an N-channel system with sampling prefilters $\tilde{g}_i, i = 0, 1, ..., N-1$ followed by uniform sampling with period NT.

(a) Let $N = 3$, and let the sampling prefilters be derivative filters: $\tilde{G}_i(\omega) = (j\omega)^i, i = 0, 1, 2$. Show that the determinant of the matrix $\tilde{G}(\omega)$ is a nonzero constant.

Your Solution Here!

(b) Let $N \in \mathbb{Z}^+$, and let the sampling prefilters be derivative filters: $\tilde{G}_i(\omega) = (j\omega)^i, i = 0, 1, ..., N-1$. Show that the determinant of the matrix $\tilde{G}(\omega)$ is a nonzero constant. *Hint: Use the concept of a Vandermonde matrix, defined on page 151 equation 2.248 in the book*

Your Solution Here!

(c) Let $N = 3$, and let the sampling prefilters be delay filters: $\tilde{G}_0(\omega) = 1, \tilde{G}_1(\omega) = e^{-j\omega(1+\alpha)}, \tilde{G}_2(\omega) = e^{-j\omega(1+\beta)}$. For which values of $\alpha, \beta \in [-1, 1]$ is the matrix $\tilde{G}(\omega)$ singular? Numerically compute the condition number $\kappa(\tilde{G}(\omega))$; you should obtain that it does not depend on ω. Plot $\tilde{G}(\omega)$ for $\alpha = 1$ and $\beta \in [-\frac{1}{2}, \frac{1}{2})$. Comment on the results.

Your Solution Here!

4 Radon Transform (Computation)

We discussed that in the Radon transform of an image, we take values of the pixels that the particular line passes through.

(a) Consider taking the image x and vectorizing it by stacking columns of the image vertically. To reason about applying the Radon transform in a discrete fashion, we may consider representing the transform as a matrix-vector multiplication such that

$$\text{vec}(y) = R\,\text{vec}(x)$$

. Let the i'th row of R represent a line at angle θ with offset n. The $R_{i,j}$ element takes value 1 if line i intersects pixel j, and takes the value 0 otherwise. Treating the Radon transform as a matrix-vector multiplication, where the vector is the result of $\text{vec}(x)$ and x is the original image, what is an explicit expression for $R_{i,j}$?

Your Solution Here!

(b) Sample and interopolate the Shepp-Logan phantom using the Radon transform and backprojection, both with and without the ramp filtering. Feel free to use

- `skimage.transform.rescale` (To downsample the image)
- `skimage.transform.radon`
- `skimage.transform.iradon`
- `skimage.data.shepp_logan_phanton`

Vary the number of N equally spaced angles of projection from 0 to 180 degrees, and provide images for $N \in \{4, 16, 64, 256\}$. Describe the quality of the reconstruction. Feel free to use the rescale function to shrink the image if the code takes too long to run.

Your Solution Here!

5 Unmatched Sampling (Computational)

From Fall 2018

Consider the signal space $\mathbb{R}^M$ with the synthesis operator $\Phi : \mathbb{R}^K \to \mathbb{R}^M$

$$(\Phi u)[m] = \sum_{k=1}^{K} u_k \cos(2\pi k \frac{m}{M}), \quad 0 \leq m \leq M-1, \quad 0 \leq k \leq K-1. \tag{5}$$

Define an analysis operator $\tilde{\Phi}^* : \mathbb{R}^M \to \mathbb{R}^N$, where $N \leq M-2$, by

$$(\tilde{\Phi}^* x)[n] = \sum_{m=n}^{n+1} x[m], where 0 \leq n \leq N-1 \tag{6}$$

(a) Construct the synthesis and analysis operators as NumPy matrices of appropriate dimensions, named `Phi` and `tPhi`

Your Solution Here!

```
# Your Code Here
```

(b) Is matrix multiplication the most efficient way of implementing the analysis operator? If so, explain. Otherwise, implement something faster and compare with the matrix-multiplication implementation in terms of complexity and performance.

Your Solution Here!

(c) Compute a matrix `Psi` that recovers a vector $x \in \text{rng}(\Phi)$ from its samples $\tilde{\Phi}^* x$ (assuming that is possible), and show that `Psi` is indeed the (left) inverse of `tPhi`. You may use any `SciPy/NumPy` function (`numpy.linalg.pinv()` might be useful in particular)

Your Solution Here!