Sixth TA Session: Growth Model

Macroeconomics 2

Alvin U. Lumbanraja

Highlights

- Solow Growth Model
  - Formulation of the model
  - Dynamics of the state variable k
  - Balanced Growth Path
  - Model implication
- Solow-Swan vs Neoclassical (Ramsey-Cass-Koopman) growth model
- Endogenous Growth Model
- Some stylized facts about economic growth

Introduction

- Why are some countries richer than other countries? This question has eluded economists for centuries, and we still have lots of questions to answer when it comes to economic growth
- Solow-Swan growth model: an extension to Harrod-Domar model
- Basic setup of Solow-Swan growth model
  - Focused on production (i.e. production is what drives economic growth in the long run)
  - Production function is homogenous of degree 1 (i.e. $f(c\mathbf{x}) = cf(\mathbf{x})$) $\longrightarrow$ constant return to scale
  - Technological progress is labor-augmenting (i.e. A and L enters the production function multiplicatively
  - Consumption choice is not explicitly microfounded; savings rate is exogenous

Model Setup (1)

- Throughout this presentation, lowercase letter means "per unit of effective labor". The production function is as follows:

$$\begin{aligned} Y &= F(K, AL) \\ &= K^{\alpha}(AL)^{1-\alpha} \end{aligned} \tag{1}$$

- Production function, as shown above, follows CRS Cobb-Douglas functional form. Let us then restate the production function in terms of unit of effective labor

$$\begin{aligned} y &= \left(\frac{K}{AL}\right)^{\alpha}\left(\frac{K}{AL}\right)^{1-\alpha} \\ &= k^{\alpha} \end{aligned} \tag{2}$$

Model Setup (2): Time Evolution of A and L

- We assume that initial level of capital, labor, and knowledge as given. We also assume that the rate of growth of labor and knowledge is constant over time

$$\frac{\dot{L}}{L(t)} = n \tag{3}$$

$$\frac{\dot{G}}{G(t)} = n \tag{4}$$

- However, $L(t)$ and $G(t)$ is a function of time so how do we know the evolution of labor and knowledge, with no further information?

Model Setup (3): Time Evolution of A and L

- Notice that (3) and (4) are basically simple first-order ODE, such that we can solve (3) and (4) by recalling that

$$
\begin{aligned}
\frac{d\ln L(t)}{dt} &= \frac{d\ln L(t)}{dL}\frac{dL(t))}{dt} \\
&= \frac{1}{L(t)}\dot{L} \\
\therefore \frac{d\ln L(t)}{dt} &= n \\
\int \frac{d\ln L(t)}{dt}\,dt &= \int n\;dt \\
\ln L(t) &= nt + C = nt + \ln L(0) \\
L(t) &= L(0)e^{nt} \qquad (5) \\
A(t) &= A(0)e^{gt} \qquad (6)
\end{aligned}
$$

Model Setup (4): Time Evolution of K

- Unlike labor and technological progress (which we take as given), we keep track of K as our state variable. This means that we are interested to solve for K evolves

$$\begin{aligned} \dot{K} &= [Y(t) - C(t)] - \delta K(t) \\ &= sY(t) - \delta K(t) \end{aligned} \tag{7}$$

- Equation (7) is the key differentiating point between Solow-Swan growth model and later Neoclassical (Ramsey-Cass-Koopman) growth model
- This model assumes the consumer decision of allocating output between savings and consumption, and thus savings rate, to be taken exogenously

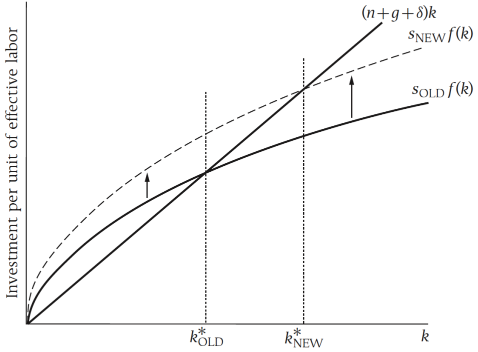
The Dynamics of K

- How does capital per unit of effective labor (k) evolves over time? We can use the chain rule to find out:

$$\begin{aligned}
\dot{k} = \frac{d\left(\frac{K}{AL}\right)}{dt} &= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)[A(t)\dot{L}(t) + \dot{A}L(t)]}{[A(t)L(t)]^2} \\
&= \frac{\dot{K}(t)}{A(t)L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{L}}{L(t)} - \frac{K(t)}{A(t)L(t)}\frac{\dot{A}}{A(t)} \\
&= \frac{sY(t) - \delta K(t)}{A(t)L(t)} - nk(t) - gk(t) \\
&= s\frac{Y(t)}{A(t)L(t)} - \delta k(t) - nk(t) - gk(t) \\
&= sf(k(t)) - (\delta + n + g)k(t) \qquad (8)
\end{aligned}$$

Balanced Growth Path (1)

- We have so far covered how capital per unit of effective labor k evolves over time
- Logically, we always expect for well-behaved system (especially the ones that obey Inada condition) to converge towards certain level of steady-state k, where $\dot{k} = 0$



Balanced Growth Path (2)

- This "steady-state" level of k is what we call balanced growth path

$$
\begin{aligned}
sf(k(t)) &= (\delta + n + g)k(t) \\
k_{BGP} &= f(k_{BGP}(s, n, g, \delta)) \qquad (9) \\
K_{BGP} &= k_{BGP}AL \\
y_L = \frac{Y_{BGP}}{L(t)} &= (A \times k_{BGP})^{\alpha} (A \times 1)^{1-\alpha} \\
g_y = \frac{\dot{y}_L}{y_L} &= g \qquad (10)
\end{aligned}
$$

- Looking at equation (10), we know that k_{BGP} is constant and that output per worker (g_y) grows at the rate of g. It can also be trivially shown that total output grows at the rate of $n + g$

Balanced Growth Path (3)

- What does BGP tell us?
- BGP is the rate at which all variables in the economy grow at a constant rate
- Observational data from developed economies indeed show that economies grow broadly according to this BGP trend
- However, note that even if economy is growing along BGP in Solow model, this does not necessarily insure that the economy optimize the level of consumption (so-called Golden Rule)
  - This is precisely because we do not endogenize savings rate into consumer decision, unlike in neoclassical (RCK) model
  - To obtain Golden Rule level , we need to explicitly optimize the Solow-Swan growth model to optimize the consumption level

Golden Rule of Capital

- We would like to maximize the following

$$\max_{k \in \Lambda} y(t) - sy(t)$$

- Since we know that along BGP, $sy(t) = (\delta + n + g)k(t)$, we can rearrange that equation to obtain the Golden Rule

$$\begin{aligned} c = y(t) - sy(t) &= f(k(t)) - (\delta + n + g)k(t) \\ \frac{dc}{dk} &= f'(k) - (\delta + n + g) = 0 \\ f'(k) &= \delta + n + g \end{aligned} \tag{11}$$

Quantitative Implications: Level of Capital

- Recall that

$$sf(k_{BGP}(s,n,g,\delta)) = (n+g+\delta)k_{BGP}(s,n,g,\delta) \tag{12}$$

- How does increase in savings rate affect the level of capital along the balanced growth path? We can solve for (12) by applying implicit differentiation to both sides w.r.t. savings rate (s)

$$\underbrace{sf'(k_{BGP})\frac{\partial k_{BGP}}{\partial s} + f(k_{BGP})}_{\text{Using the product rule u'v+uv'}} = (n+g+\delta)\frac{\partial k_{BGP}}{\partial s}$$

$$\frac{\partial k_{BGP}}{\partial s} = \frac{f(k_{BGP})}{(n+g+\delta) - sf'(k_{BGP})} \tag{13}$$

Quantitative Implications: Level of Output

- To determine the effect of increase in savings to output level, it follows that:

$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial k_{BGP}} \frac{\partial k_{BGP}}{\partial s} = \frac{f'(k_{BGP}) f(k_{BGP})}{(n+g+\delta) - s f'(k_{BGP})} \tag{14}$$

- For both (13) and (14), we can safely infer that increase in savings rate increases both capital stock per effective labor and output per effective labor along the BGP
- Does increase in savings, however, necessarily increases consumption? Not necessarily!
- Again, this is because the BGP solution is not explicitly microfounded

Ramsey-Cass-Koopman

- Neoclassical (Ramsey-Cass-Koopman) model differs significantly with Solow-Swan on one key point: endogenizing the savings rate
- How does neoclassical model put microfoundation underpinning on the savings rate?
  - Household is assumed to be infinitely lived and maximize their present value of consumption
  - Household owns firms' assets, such that the investment decision of firms is rooted in shareholder's profit maximization $\longrightarrow$ households' utility maximization
  - By linking firms' decision with household's profitability, the BGP found by this model is always consumption-maximizing

Model Construction: Some Snippets

- How do we know that the model is explicitly microfounded? Consider the household problem in neoclassical model[1]

$$\max_{c(t),I(t),n(t)} \int_0^\infty e^{-\rho t} U[\underbrace{c(t)}_{\text{Consumption}}, \underbrace{1-n(t)}_{\text{Leisure}}]\, dt \tag{15}$$

- This model assumes that representative household provide labor and capital to firms (who rent both), such that we maximize (15) subject to the following constraint (law of motion for capital stock)

$$\dot{k}(t) = \underbrace{[(1-\tau_\kappa)R(t)-\delta]k(t)}_{\text{Rental Income}} + \underbrace{(1-\tau_n)w(t)n(t)}_{\text{Wage Income}} - \underbrace{(1+\tau_c)c(t)}_{\text{Consumption}} \tag{16}$$

[1] I provided Econ 330 note from Prof. Nancy Stokey's class if you wish to read up on proper treatment of neoclassical growth model

Why Endogenous Growth

- Basic conclusion of Solow-Swan & Ramsey-Cass-Koopman models shows that only growth in A can sustain growth in per-capita output along the BGP (i.e. in the long run)
- But we treat knowledge so far as residual, not something embedded in our model explicitly
- Need formal treatment on how knowledge is produced in an economy and how it affects various economic aggregates
- Knowledge accumulation should be endogeneized in our model

Variants of Endogenous Growth Models

- Ideally, it is preferable to simplify our model depending on our needs by focusing only on one state variable (usually achieved by assume away physical capital as our input)
- Sustained, increasing return to scale enters the model through accumulated knowledge (A) (Romer, 1990)
- Microfoundation of innovation: growth is driven by monopolistic firms who engage in Schumpeterian creative destruction (Grossman & Helpman, 1991; Aghion & Howitt, 1992; Klette & Kortum, 2004)
- Growth through human capital revisited: better education begets more inventor (Akcigit, Nicholas, Grisby, 2017)

Dynamics of Knowledge Accumulation

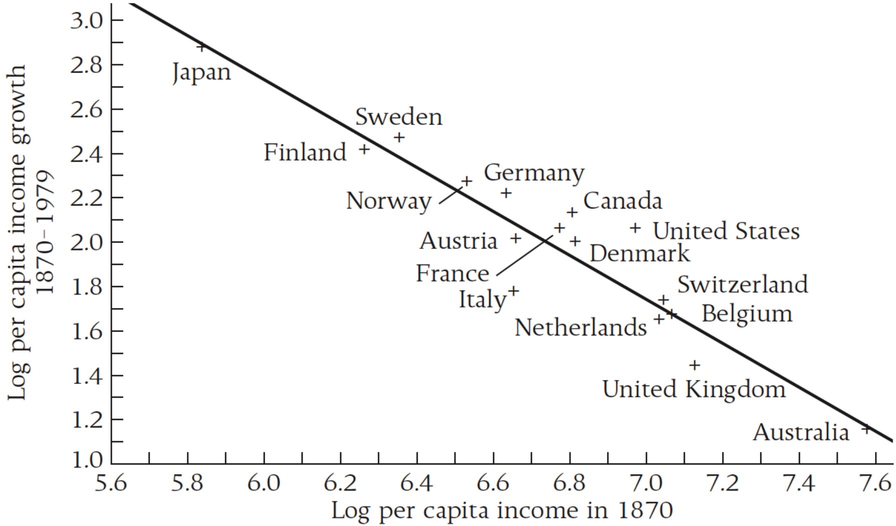
- How do we attain increasing return to scale? Romer (David the textbook, not Paul) model

$$Y(t) = [(1 - a_K)K(t)]^{\alpha}[A(t)(1 - a_L)L(t)^{1-\alpha}, \text{ s.t.} \quad (17)$$

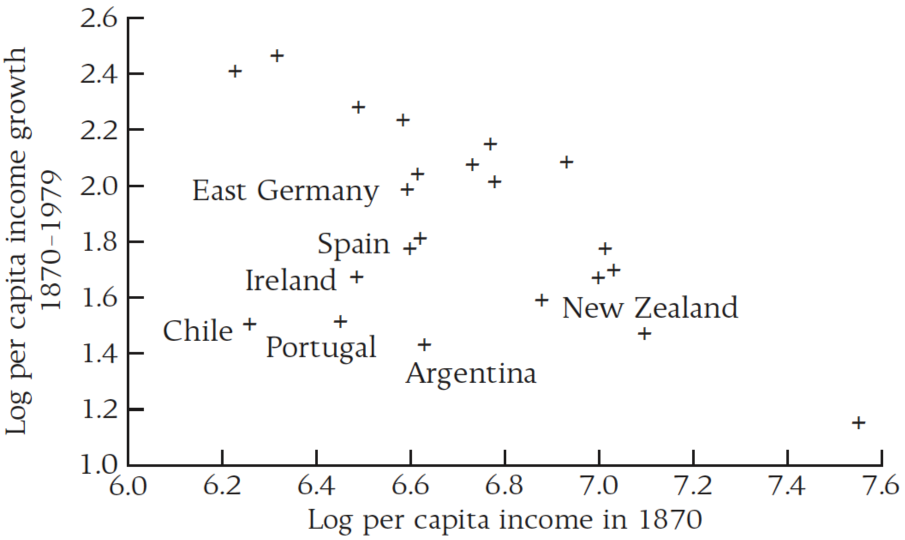
$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta} \quad (18)$$

- Important conclusion of (18): knowledge production may follow increasing return to scale (depending on the parameters β,γ, and θ)

Baumol: Balanced Growth and Convergence



DeLong: Baumol's Data Have Selection Bias?



Contemporary Data

UNIVERSITAS INDONESIA
Veritas, Probitas, Iustitia | Est. 1849