First TA Session: Mini Math Camp

Macroeconomics 2

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Highlights

- Matrix Operation
  - Addition and Subtraction
  - Matrix Multiplication
  - Matrix Transpose
  - Determinant and Matrix Inverse
- Comparative Statics
  - Partial derivative
  - Total derivative

Background

- Matrices are often used to represent system of equations
- Very useful especially if we are dealing with more complex system (as matrix form can be solved analytically by software such as Matlab)
- Example of system of equations:

$$2x + y = 5 \tag{1}$$
$$x + 2y = 3 \tag{2}$$

- Example of the corresponding system in a matrix form:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \tag{3}$$

Addition and Subtraction

- Addition and subtraction should be pretty straightforward operations
- Note: you can only add or subtract matrix of the same size (e.g. $(m \times n)$ matrix can only be added/subtracted by another $(m \times n)$ matrix)
- Example of addition and subtraction:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} \zeta & \eta \\ \theta & \iota \end{pmatrix} = \begin{pmatrix} \alpha + \zeta & \beta + \eta \\ \gamma + \theta & \delta + \iota \end{pmatrix} \tag{4}$$

Matrix addition properties

- Commutative: $A + B = B + A$
- Associative: $A + (B + C) = (A + B) + C = (A + C) + B$

Matrix Multiplication

- Only a tad bit more complicated than matrix addition and subtraction
- You can only multiply matrices if the number of columns for the first matrix is **equal** to the second matrix. For example, multiplying $(m \times n)$ matrix with $(n \times m)$ matrix will result in $(m \times m)$ matrix
- Example of addition and subtraction:

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \times \begin{bmatrix} \zeta & \eta \\ \theta & \iota \end{bmatrix} = \begin{bmatrix} (\alpha\zeta + \beta\delta) & (\alpha\eta + \beta\iota) \\ (\gamma\zeta + \delta\theta) & (\gamma\eta + \delta\iota) \end{bmatrix} \tag{5}$$

Matrix multiplication properties

- NOT Commutative: $A \times B \neq B \times A$
- Associative: $A \times (B \times C) = (A \times B) \times C$

Matrix Determinant

- Determinant is a scalar value of a **square** matrix that contains useful properties for its linear transformation
- Example of a 3×3 square matrix named matrix A below:

$$\begin{array}{ccccc} + & + & + & & \\ \left| a \right. & b & c & \left| a \right. & b \\ \left| e \right. & f & g & \left| e \right. & f \\ \left| h \right. & i & j & \left| h \right. & i \\ - & - & - & & \end{array}$$

$$\det(A) = (afj + bgh + cei) - (hfc + iga + jeb)$$

Matrix Inverse

- Matrix "division" is non-existent in the traditional sense, but what we have is matrix inversion
- In order for a matrix to be invertible, it needs to be **square** and **non-singular** (basically just that the determinant of the matrix should be nonzero ($\det(A) \neq 0$)
- How is matrix inversion useful in our case? Consider the following problem where we want to find the value of dependent variables in our linear system

$$\begin{aligned} Ax &= B \\ A^{-1}Ax &= A^{-1}B \\ x &= A^{-1}B \end{aligned} \tag{6}$$

Computing Matrix Inverse

- Realistically, most people won't even bother with computing matrix by hand as it's very tedious and error-prone
- But you need to do it for the test so let's just suck it up and do it
- Let's take for example our 3x3 matrix (2x2 matrix inversion is trivial and left to you as an exercise)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \tag{7}$$

Step 1: Compute the Determinant

- The determinant of the matrix is straightforward to compute as follows:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \det A = {} & [(1 \times 1 \times 1) + (0 \times 0 \times 2) + (1 \times 0 \times 0)] - \\ & [(2 \times 1 \times 1) + (0 \times 0 \times 1) + (1 \times 0 \times 0)] \\ = {} & -1 \end{aligned} \tag{8}$$

Step 2: Compute the Cofactor Matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 3: Compute the Matrix Inverse

$$A^{-1} = \frac{1}{\det(A)}(A) = \frac{1}{\det(A)}C^T$$
$$= \frac{1}{-1}\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^T$$
$$= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Hint for Homework (1)

- No. 3: Solve the following national income model, using either matrix inversion or Cramer's rule

$$Y = C + I + G \tag{9}$$
$$C = \alpha + \beta(Y - T) \tag{10}$$
$$T = \tau_y Y \tag{11}$$

- Rearrange the system of equations; independent variables on the left-hand side and the exogenous parameters on the right-hand side

$$Y - C = (I + G)$$
$$-\beta Y + C + \beta T = \alpha$$
$$-\tau_y Y + T = 0$$

Hint for Homework (2)

- Rearrange the equation into matrices form; independent variable in the left-hand side and the dependent variable in the right-hand side. Then solve for (Y, C, T) using matrix inverse

$$\underbrace{\begin{pmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\tau_y & 0 & 1 \end{pmatrix}}_{A} \times \underbrace{\begin{pmatrix} Y \\ C \\ T \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} (I+G) \\ \alpha \\ 0 \end{pmatrix}}_{b}$$

Primers

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- I will presume that you have some degree of familiarity with the rules of differentiation as you have presumably passed advanced micro
- Instead, what I will present today is a serviceable refresher, particularly for partial and total derivative
- The working assumption: you should actually remember the basics of derivative. May God help you if you don't
- I also will not delve too long on the math proof (if you're interested to learn about the maths, consult Rudin's real analysis book); most of the solutions to optimization problems in this class can be shown to be sufficient just by taking the second derivative of the function

Rules of Differentiation

- Sum Rules:

$$(u+v)' = u' + v'$$

- Product Rules:

$$(uv)' = u'v + uv'$$

- Quotient Rules:

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

- Chain Rules:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Partial Derivative

- Consider the following function:

$$y = f(x_1, x_2, ..., x_n) \tag{12}$$

- We can rephrase the problem in equation (12) into a more concrete economic example; what would happen to total output if we experience positive technological shock?
- Enter the partial derivative:

$$\frac{\partial y}{\partial x_1} = \frac{\partial f(x_1, x_2, ..., x_n)}{\partial x_1} \tag{13}$$

Partial Derivative: Example

- Consider the following production function:

$$y = Ak^{\alpha}m^{\gamma}n^{1-\alpha-\gamma} \tag{14}$$

- What is the equilibrium price of oil (m) if we assume a competitive economy in its steady state?

$$\begin{aligned} p_m = \frac{\partial y}{\partial m} &= Ak^{\alpha}m^{\gamma}n^{1-\alpha-\gamma} \\ &= \gamma Ak^{\alpha}m^{\gamma-1}n^{1-\alpha-\gamma} \end{aligned} \tag{15}$$

Total Derivative

- Consider again function (12):

$$y = f(x_1, x_2, ..., x_n)$$

- Unlike partial derivative (where we assume other variables remain constant), we can also check how the change in one variable affect the other variables using total derivative

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 + ... + \frac{\partial y}{\partial x_n} dx_n \\ &= \sum_{i=1}^{n} \frac{\partial y}{\partial x_i} dx_i \end{aligned} \tag{16}$$

Total Derivative: Example (1)

- Find the total derivative of y w.r.t its arguments:

$$y = 3x_1^2 + x_1 x_2 - 2x_2^3$$

- Answer

$$\begin{aligned} dy &= \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2 \\ &= (6x_1 + x_2) dx_1 + (x_1 - 6x_2^2) dx_2 \end{aligned}$$

Total Derivative: Example (2)

- Find the total derivative of y w.r.t its arguments:

$$y = \frac{x_1}{x_1 + x_2}$$

- Answer: use the product rules!

$$dy = \frac{x_2 dx_1 - x_1 dx_2}{(x_1 + x_2)^2} \tag{17}$$