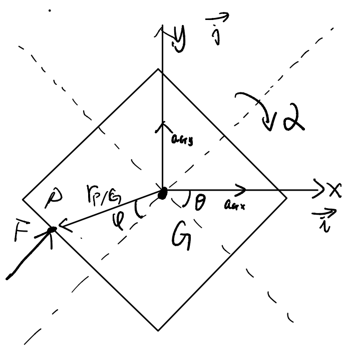
Problem 1

1.



G is the center of mass
The force is exerted on point P
$\vec{r}_{P/G}$ means the vector from G to P.
$\vec{\alpha}$ is the angular acceleration
We can get the following equations

$$\begin{cases} \Sigma\vec{F} = m\vec{a_G} \\ \vec{r}\times\vec{F} = I\vec{\alpha} \end{cases}$$

$$\Rightarrow \begin{cases} F_x = F\sin\theta = m\,a_{Gx} \\ F_y = F\cos\theta = m\,a_{Gy} \end{cases}$$

and

$$\begin{cases} \vec{r}_{P/G}\times\vec{F} = I_G\vec{\omega} \\ \vec{r}_{P/G} = -r\left[\cos(90^\circ-(\theta+\varphi))\vec{i} + \sin(90^\circ-(\theta+\varphi))\vec{j}\right] \\ \vec{F} = F\sin\theta\vec{i} + F\cos\theta\vec{j} \end{cases}$$

$$\Rightarrow \begin{cases} \vec{r}_{P/G} = -r\left(\sin(\theta+\varphi)\vec{i} + \cos(\theta+\varphi)\vec{j}\right) \\ \vec{r}_{P/G}\times\vec{F} = -rF\sin(\theta+\varphi)\cos\theta + rF\cos(\theta+\varphi)\sin\theta \\ \qquad = rF\sin\varphi\ \vec{k} \quad (k \text{ is the axis} \perp \text{planar motions}) \\ I_G = \frac{1}{6}md^2 \end{cases}$$

$$\Rightarrow \alpha = \frac{6Fr\sin\varphi}{md^2} = \frac{d^2\theta}{dt^2}$$

, Hence, the equations are

$$\begin{cases} F\sin\theta = m\,a_{Gx} = m\frac{d^2x_G}{dt^2} \\ F\cos\theta = m\,a_{Gy} = m\frac{d^2y_G}{dt^2} \\ \frac{d\theta^2}{dt^2} = \frac{6Fr\sin\varphi}{md^2} \\ \omega = \frac{d\theta}{dt},\ \alpha = \frac{d\omega}{dt} \\ V_{Gx} = \frac{dx_G}{dt},\ a_{Gx} = \frac{dV_{Gx}}{dt} \ (\text{similar for } V_{Gy} \text{ and } a_{Gy}) \end{cases}$$

, where

$$\begin{cases} r = \|\vec{r}_{P/G}\| = \sqrt{l^2+\frac{1}{4}d^2} \\ \varphi = \tan^{-1}\left(\frac{2l}{d}\right) \end{cases}$$

2.

Using 4th order Runge-kutta to solve the 2nd ODE

For $\frac{d\theta^2}{d^2t} = \frac{6Fr\sin\varphi}{md^2}$, the angular acceleration is always a constant, so by integration, we get $\theta = \frac{6Fr\sin\varphi}{md^2}t^2$

The substitute the θ, the x-axis acceleration is give by

$$m\frac{d^2x_G}{dt^2} = F\sin\left(\frac{6Fr\sin\varphi}{md^2}t^2\right)$$

Let $v_{Gx}(t) = \frac{dx_G}{dt}$, then $\frac{dv(t)}{dt} = \frac{F}{m}\sin\left(\frac{6Fr\sin\varphi}{md^2}t^2\right)$

The RK4 is given by:

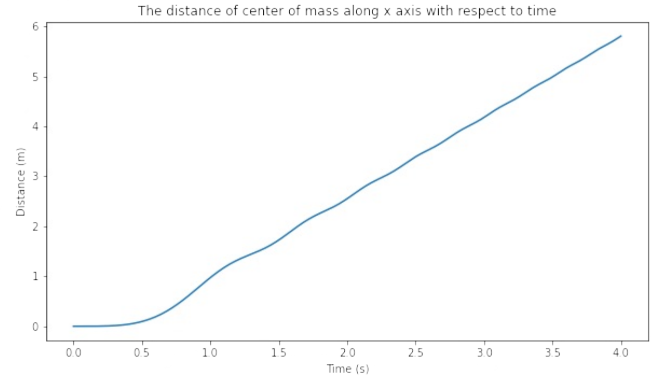
$$\begin{bmatrix} x_{G\,n+1} \\ v_{Gx\,n+1} \end{bmatrix} = \begin{bmatrix} x_{G\,n} + \frac{1}{6}(k_{1x} + 2k_{2x} + 2k_{3x} + k_{4x}) \\ v_{Gx\,n} + \frac{1}{6}(k_{1v} + 2k_{2v} + 2k_{3v} + k_{4v}) \end{bmatrix}$$

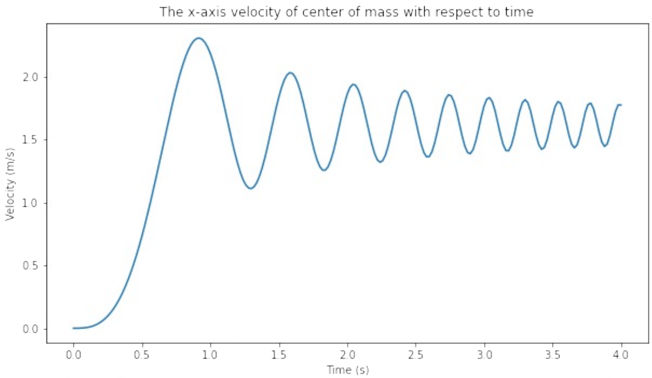
We shall figure out the initial value for x and v

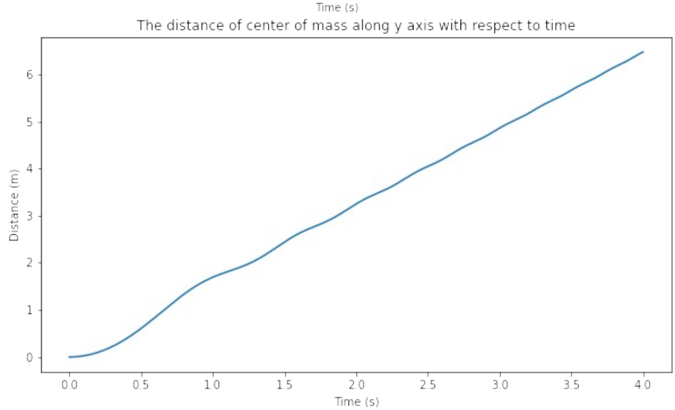
when $t=0$ $x_G = 0$ $v_{Gx} = 0$

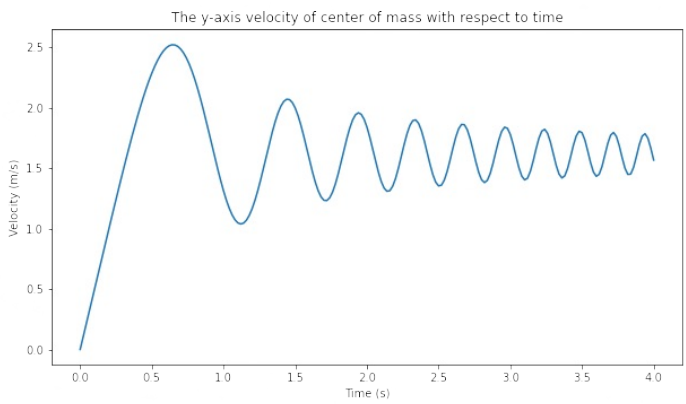
The methods above apply to v_{Gy} and y_G

The results are plotted in the following graphs.


The distance of center of mass along x axis with respect to time
Distance (m)
Time (s)
0 1 2 3 4 5 6
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0

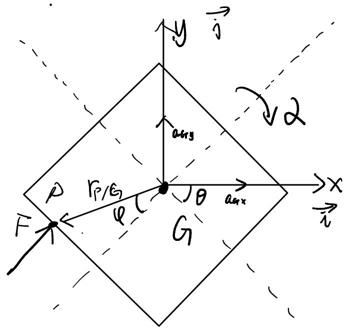

The x-axis velocity of center of mass with respect to time
Velocity (m/s)
Time (s)
0.0 0.5 1.0 1.5 2.0
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0


The distance of center of mass along y axis with respect to time
Distance (m)
Time (s)
0 1 2 3 4 5 6
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0


The y-axis velocity of center of mass with respect to time
Velocity (m/s)
Time (s)
0.0 0.5 1.0 1.5 2.0 2.5
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0

3. In the previous question, we have already got the center's distance on x and y axises, Next step is to get the contact point trajectory

First, convert the contact to the general frame.



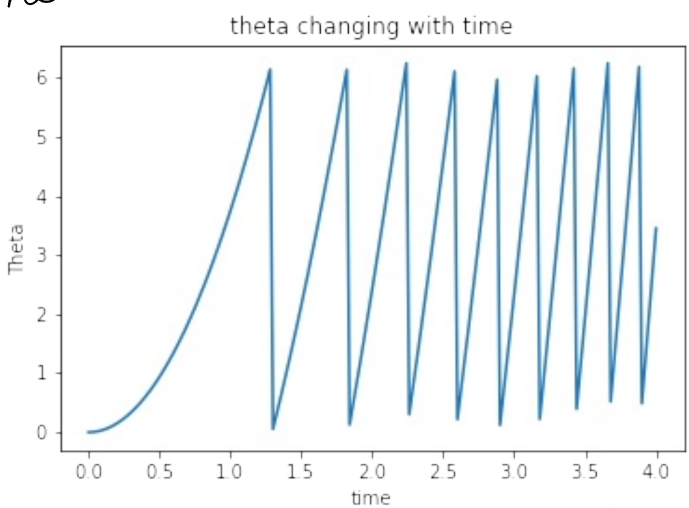
$$T_{OG} = \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix}$$

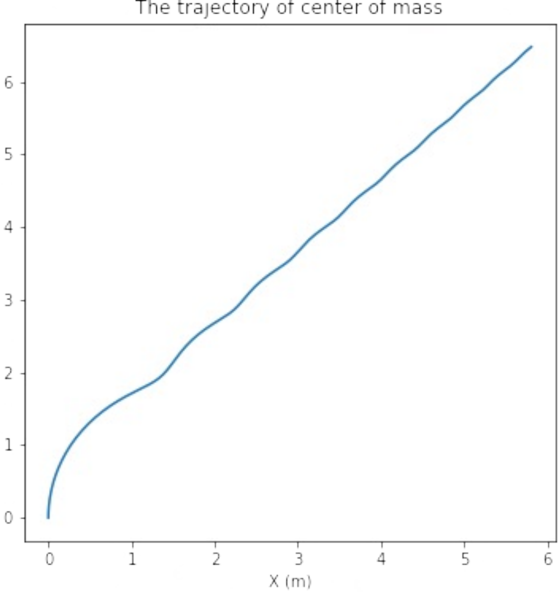
The point p in G frame is $\begin{bmatrix} -L \\ -\frac{d}{2} \\ 1 \end{bmatrix}$

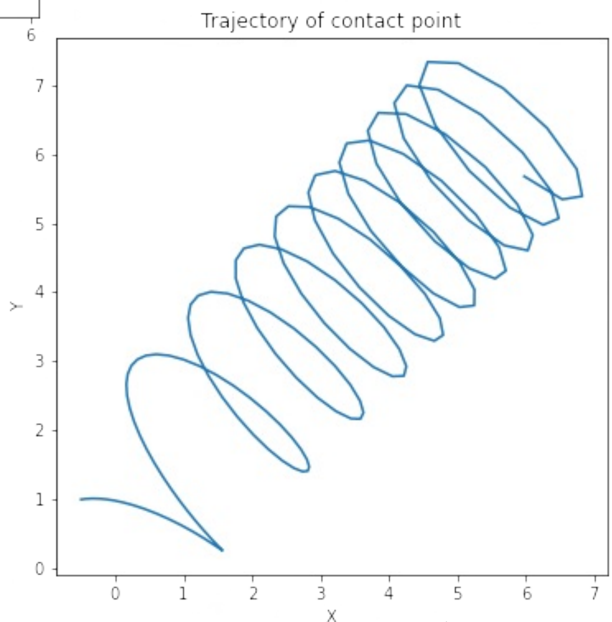
$$P' = \begin{bmatrix} \cos\theta & -\sin\theta & dx \\ \sin\theta & \cos\theta & dy \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -L \\ -\frac{d}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$x' = -L\cos\theta + \frac{d}{2}\sin\theta + dx \qquad y' = -L\sin\theta - \frac{d}{2}\cos\theta + dy$$

The results are shown below







X Y

4. The final position of center is (5.807, 6.477)

The final position of contact point is (5.978, 5.677)

Problem 2

1. $\mathcal{L}(x,\lambda) = \frac{1}{2}\|Ax-b\|_2^2 + \lambda(x^Tx - \epsilon)$

$$\mathcal{L} = \frac{1}{2}[x^TA^TAx - x^TA^Tb - b^TAx - b^Tb] + \lambda(x^Tx - \epsilon)$$

$$\nabla_x \mathcal{L} = A^T(Ax-b) + 2\lambda x$$

2. If $\lambda=0$, $\nabla_x \mathcal{L} = A^T(Ax-b) = A^TAx - A^Tb = 0$ to satisfy the KKT condition

Then $A^TAx = A^Tb \Rightarrow x = (A^TA)^{-1}A^Tb$ is the closed-form solution to the unconstrained least-square equation.

3. (a) $\nabla_x \mathcal{L} = 0 \Rightarrow A^T(Ax-b) + 2\lambda x = 0$

$$\Rightarrow (A^TA + 2I\lambda)x = A^Tb$$

$$x = (A^TA + 2I\lambda)^{-1}A^Tb = h(\lambda)$$

(b) $A^TA = U\Lambda U^T = U\Lambda U^{-1}$ by eigen-decomposition properties

Hence $(A^TA + 2I\lambda)^{-1} = (U\Lambda U^T + 2I\lambda)^{-1} = U(2\lambda I + \Lambda)^{-1}U^{-1}$

Then, $h(\lambda) = U(2\lambda I + \Lambda)^{-1}U^{-1}A^Tb$

$$h^T(\lambda)\, h(\lambda) = b^T A U [(2\lambda I + \Lambda)^{-1}]^T U^{-1} \cdot U (2\lambda I + \Lambda)^{-1} U^T A^T b$$

$$= b^T A U \left[(2\lambda I + \Lambda)^{-2} \right] U^T A^T b$$

let $U^T A^T b = C = \left[c_1, c_2, \cdots c_n \right]$, all elements are constant

$$(2\lambda I + \Lambda)^{-2} = \begin{bmatrix} \frac{1}{(2\lambda+\varphi_1)^2} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{(2\lambda+\varphi_n)^2} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_n \end{bmatrix}$$

$h^T(\lambda) h(\lambda) =$

$$C^T \begin{bmatrix} \frac{1}{(2\lambda+\varphi_1)^2} & & \\ & \ddots & \\ & & \frac{1}{(2\lambda+\varphi_n)^2} \end{bmatrix} C$$

$$= \frac{c_1^2}{(2\lambda+\varphi_1)^2} + \frac{c_2^2}{(2\lambda+\varphi_2)^2} + \cdots + \frac{c_n^2}{(2\lambda+\varphi_n)^2}$$

all elements C_n and φ_n are constant for determined A, b

Hence when λ is increasing $(\lambda > 0)$ the sum above is decreasing. $h(\lambda)^T h(\lambda)$ is monotonically decreasing.

4. We can optimize the problem by gradient descent. Because we proved the monotone property of $h(x)^T h(x)$ and it is also positive semi-definite. So the Newton method can solve this minization problem

The results are shown below. ($x^T x \approx 0.5$)

The codes are attached behind.

```
[[ 0.06373495]
 [-0.18785585]
 [-0.03770888]
 [ 0.00965131]
 [ 0.11506622]
 [ 0.15070863]
 [-0.18593416]
 [ 0.08699669]
 [ 0.30243714]
 [-0.0531146 ]
 [ 0.15369267]
 [ 0.05346432]
 [-0.04253699]
 [-0.09098992]
 [ 0.07908604]
 [-0.07419913]
 [ 0.13569381]
 [-0.07162559]
 [-0.08790902]
 [ 0.14271374]
 [ 0.06964736]
 [ 0.10583163]
 [-0.02759693]
 [ 0.00333129]
 [ 0.00687458]
 [-0.30006567]
 [-0.04840911]
 [-0.0735365 ]
 [ 0.18392214]
 [-0.18943771]]
```

Problem 3.