7^{η} Ομάδα Ασκήσεων

φροντιστήρια πουκαμισάς

43. (Ακέραιο Μέρος)

$$f(x)=\begin{cases} c+\frac{x}{4}, & 0\leq x<2 \\ 0, & \text{αλλού} \end{cases}$$

(α') Ισχύει ότι $\int_{-\infty}^{\infty} f(x)dx=1 \Rightarrow \int_0^2 f(x)dx=1 \Rightarrow \int_0^2\left(c+\frac{x}{4}\right)dx=1 \Rightarrow \int_0^2 c\,dx+\int_0^2\frac{x}{4}dx=1\Rightarrow$

$\Rightarrow 2c+\frac{1}{4}\int_0^2\left(\frac{x^2}{2}\right)'dx=1 \Rightarrow 2c+\frac{1}{8}\int_0^2(x^2)'dx=1\Rightarrow 2c+\frac{1}{\cancel{8}_2}(\cancel{4})=1\Rightarrow 2c+\frac{1}{2}=1\Rightarrow 2c=\frac{1}{2}\Rightarrow$

$\Rightarrow c=\frac{1}{4}$

(β') $E(X)=\int_{-\infty}^{+\infty}x f(x)\,dx=\int_0^2 x\left(\frac{1}{4}+\frac{x}{4}\right)dx=\int_0^2\left(\frac{x}{4}+\frac{x^2}{4}\right)dx=\int_0^2\frac{x}{4}dx+\int_0^2\frac{x^2}{4}dx=$

$=\frac{1}{4}\left(\int_0^2\left(\frac{x^2}{2}\right)'dx+\int_0^2\left(\frac{x^3}{3}\right)'dx\right)=\frac{1}{\cancel{4}}\left(\frac{4}{2}+\frac{8}{3}\right)=\frac{1}{2}+\frac{8}{12}=\frac{14}{12}=\frac{7}{6}$

(γ') Όπως διαπιστώνουμε από την πυκνότητα, αν $S=(-\infty,0)\cup[2,+\infty)$ τότε $P(X\in S)=\int_S f(x)dx=\int_S 0\,dx=0$. Άρα το X μπορεί να λαμβάνει μόνο τιμές στο $[0,2)$.

Έχουμε λοιπόν: Για $0\leq X<1$ θα είναι $Y=0$ και $p_Y(0)=P(Y=0)=$

$=P(0\leq X<1)=\int_0^1\left(\frac{1}{4}+\frac{x}{4}\right)dx=\frac{1}{4}+\int_0^1\left(\frac{x^2}{8}\right)'dx=\frac{1}{4}+\frac{1}{8}=\frac{3}{8}$

Για $1<X<2$ θα είναι $Y=1$ και $p_Y(1)=P(Y=1)=$

$=P(1<X<2)=\int_1^2\left(\frac{1}{4}+\frac{x}{4}\right)dx=\frac{1}{\cancel{4}_2}\cdot 2-\frac{1}{4}+\int_1^2\left(\frac{x^2}{8}\right)'dx=$

$=\frac{1}{2}-\frac{1}{4}+\frac{4}{8}-\frac{1}{8}=\frac{1}{4}+\frac{3}{8}=\frac{5}{8}$

44. (Πατάτες)

$$f(x)=\begin{cases} ax+\beta x^2, & 0<x<2 \\ 0, & \text{αλλού} \end{cases} \qquad E(X)=1$$

(α΄) Ισχύει ότι $\int_{-\infty}^{+\infty} f(x)\,dx=1 \Rightarrow \int_0^2 (ax+bx^2)\,dx=1 \Rightarrow \int_0^2 \left(a\frac{x^2}{2}+b\frac{x^3}{3}\right)'dx=1 \Rightarrow$

$\Rightarrow a\frac{\cancel{4}^{2}}{\cancel{2}}+b\cdot\frac{8}{3}=1 \Rightarrow 6a+8b=3 \Rightarrow a=\frac{3-8b}{6}$ ①

Επίσης δίνεται ότι $E(X)=1 \Rightarrow \int_{-\infty}^{+\infty} x f(x)\,dx=1 \Rightarrow \int_0^2 x(ax+\beta x^2)\,dx=1$

$\Rightarrow \int_0^2 (ax^2+\beta x^3)\,dx=1 \Rightarrow \int_0^2 \left(\frac{ax^3}{3}+\frac{\beta x^4}{4}\right)'dx=1 \Rightarrow \frac{a\cdot 8}{3}+\frac{\beta\cdot \cancel{16}^{4}}{\cancel{4}}=1 \Rightarrow$

$\Rightarrow 8a+12b=3 \overset{①}{\Rightarrow} \cancel{8}^{4}\left(\frac{3-8b}{\cancel{6}_3}\right)+12b=3 \Rightarrow \frac{12-32b}{3}+12b=3 \Rightarrow$

$\Rightarrow 12-32b+36b=9 \Rightarrow 4b=-3 \Rightarrow b=-\frac{3}{4}$

① $\Rightarrow a=\frac{3-\cancel{8}^{2}\left(\frac{-3}{\cancel{4}}\right)}{6}=\frac{3+6}{6}=\frac{\cancel{9}^{3}}{\cancel{6}_2}=\frac{3}{2}$

(β΄) • Για την $P(X<1)$ έχουμε $P(X<1)=\int_{-\infty}^{1} f(x)\,dx=\int_0^1 \left(\frac{3}{2}x-\frac{3}{4}x^2\right)dx=$

$=\int_0^1 \left(\frac{3}{4}x^2-\frac{3}{12}x^3\right)'dx=\frac{3}{4}-\frac{\cancel{3}}{\cancel{12}_4}=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

• Επίσης $VAR(x)=E(X^2)-(E(X))^2$ ②

Είναι $E(X^2)=\int_{-\infty}^{\infty} x^2\left(\frac{3}{2}x-\frac{3}{4}x^2\right)dx=\int_0^2\left(\frac{3}{2}x^3-\frac{3}{4}x^4\right)dx=$

$=\int_0^2\left(\frac{3}{8}x^4-\frac{3}{20}x^5\right)'dx=\frac{3}{\cancel{8}}\cdot\cancel{16}^{2}-\frac{3}{\cancel{20}_5}\cdot\cancel{32}^{8}=6-\frac{24}{5}=\frac{30-24}{5}=\frac{6}{5}$

Οπότε ② $\overset{(α΄)}{\Rightarrow} VAR(X)=\frac{6}{5}-(1)^2=\frac{6}{5}-\frac{5}{5}=\frac{1}{5}$

45. (Μη φραγμένη πυκνότητα)

$$F(x)=\begin{cases}0, & x\le 0\\ 2\sqrt{x}, & 0<x<1/4\\ 1, & x\ge 1/4\end{cases}$$

• Είναι $P\left(X>\frac{1}{9}\,\middle|\,X>\frac{1}{16}\right)=\dfrac{P\left(X>\frac{1}{9}\text{ και } X>\frac{1}{16}\right)}{P\left(X>\frac{1}{16}\right)}=\dfrac{P\left(X>\frac{1}{9}\right)}{P\left(X>\frac{1}{16}\right)}=$

$$=\frac{1-P\left(X\le\frac{1}{9}\right)}{1-P\left(X\le\frac{1}{16}\right)}=\frac{1-F(1/9)}{1-F(1/16)}=\frac{1-2\sqrt{1/9}}{1-2\sqrt{1/16}}=\frac{1-2\cdot\frac{1}{3}}{1-\cancel{2}\cdot\frac{1}{\cancel{4}_2}}=\frac{1}{3}\cdot 2=\frac{2}{3}$$

• Για τον υπολογισμό των $E(X)$, $E\left(\left|X-\frac{1}{8}\right|\right)$ υπολογίζουμε κατ' αρχάς την πυκνότητα ($f(x)$).

Είναι $f(x)=F'(x)=\begin{cases}0, & x\le 0\\ \frac{1}{\sqrt{x}}, & 0<x<\frac{1}{4}\\ 0, & x\ge\frac{1}{4}\end{cases}$ $\xrightarrow{\text{γιατί}}\left((2\sqrt{x})'=\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x}}=\frac{1}{\sqrt{x}}\right)$

• $E(X)=\displaystyle\int_{-\infty}^{+\infty}x\,f(x)\,dx=\int_0^{1/4}x\cdot\frac{1}{\sqrt{x}}\,dx=\int_0^{1/4}(\sqrt{x})^{\cancel{2}}\cdot\frac{1}{\cancel{\sqrt{x}}}\,dx=\int_0^{1/4}x^{\frac{1}{2}}\,dx=\int_0^{1/4}\left(\frac{2}{3}x^{\frac{3}{2}}\right)'dx=$

$$=\frac{2}{3}\cdot\left(\frac{1}{4}\right)^{\frac{3}{2}}=\frac{2}{3}\cdot\frac{1}{\sqrt{4^3}}=\frac{2}{3}\cdot\frac{1}{8}=\frac{2}{24}=\frac{1}{12}$$

• $E\left(\left|X-\frac{1}{8}\right|\right)=\displaystyle\int_{-\infty}^{+\infty}\left|x-\frac{1}{8}\right|f(x)\,dx=\int_0^{1/4}\left|x-\frac{1}{8}\right|f(x)\,dx=\int_0^{1/8}\left(\frac{1}{8}-x\right)\frac{1}{\sqrt{x}}\,dx+\int_{1/8}^{1/4}\left(x-\frac{1}{8}\right)\cdot\frac{1}{\sqrt{x}}\,dx$

$$=\int_0^{1/8}\left(\frac{1}{2\cdot 4\sqrt{x}}-\sqrt{x}\right)dx+\int_{1/8}^{1/4}\left(\sqrt{x}-\frac{1}{2\cdot 4\sqrt{x}}\right)dx=\int_0^{1/8}\left(\frac{1}{4}\sqrt{x}-\frac{2}{3}x^{\frac{3}{2}}\right)'dx+\int_{1/8}^{1/4}\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{4}\sqrt{x}\right)'dx$$

$$=\frac{1}{4}\cdot\frac{1}{2\sqrt{2}}-\frac{2}{3}\cdot\left(\frac{1}{8}\right)^{\frac{3}{2}}+\left(\frac{2}{3}\cdot\left(\frac{1}{4}\right)^{\frac{3}{2}}-\frac{1}{4}\cdot\frac{1}{2}-\frac{2}{3}\cdot\left(\frac{1}{8}\right)^{\frac{3}{2}}+\frac{1}{4}\cdot\frac{1}{2\sqrt{2}}\right)=$$

$$=\frac{1}{8\sqrt{2}}-\frac{2}{3}\cdot\frac{1}{16\sqrt{2}}+\frac{\cancel{2}}{3}\cdot\frac{1}{4\cdot\cancel{2}}-\frac{1}{8}-\frac{2}{3}\cdot\frac{1}{16\sqrt{2}}+\frac{1}{4\cdot 2\sqrt{2}}=$$

$$=\frac{2}{8\sqrt{2}}-\frac{\sqrt{2}}{48}+\frac{1}{12}-\frac{1}{8}-\frac{\sqrt{2}}{48}=\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{24}+\frac{1}{12}-\frac{1}{8}=\frac{3\sqrt{2}-\sqrt{2}+2-3}{24}=$$

$$=\frac{2\sqrt{2}-1}{24}$$

46. (Πυκνότητα Πιθανότητας 1)

$$f_X(x) = \begin{cases} c(1-x^2), & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}$$

• Ισχύει ότι $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Rightarrow \int_{-1}^{1} c(1-x^2)\,dx = 1 \Rightarrow c\int_{-1}^{1}\left(x-\frac{x^3}{3}\right)'dx = 1 \Rightarrow$

$\Rightarrow c\left(1-\frac{1}{3}\right) - c\left(-1-\left(-\frac{1}{3}\right)\right) = 1 \Rightarrow c - \frac{c}{3} + c - \frac{c}{3} = 1 \Rightarrow \frac{3c-c+3c-c}{3} = 1 \Rightarrow$

$\Rightarrow 4c = 3 \Rightarrow c = \frac{3}{4}$

• Έχουμε $E(X) = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-1}^{1} x\cdot c(1-x^2)\,dx = \frac{3}{4}\int_{-1}^{1}(x-x^3)\,dx = \frac{3}{4}\int_{-1}^{1}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)'dx =$

$= \frac{3}{4}\left(\cancel{\frac{1}{2}-\frac{1}{4}}\right) - \frac{3}{4}\left(\cancel{\frac{1}{2}-\frac{1}{4}}\right) = 0$ (αναμενόμενο αφού η συνάρτηση $xf(x)$ είναι περιττή και το όριο ολοκλήρωσης της μορφής $(-a,a)$)

• Για τη διασπορά $VAR(X) = E(X^2) - (E(X))^2$ ①

Είναι $E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = c\int_{-1}^{1} x^2(1-x^2)\,dx = \frac{3}{4}\int_{-1}^{1}(x^2-x^4)\,dx =$

$= \frac{3}{4}\int_{-1}^{1}\left(\frac{x^3}{3}-\frac{x^5}{5}\right)'dx = \frac{3}{4}\left(\frac{1}{3}-\frac{1}{5}\right) - \frac{3}{4}\left(-\frac{1}{3}+\frac{1}{5}\right) = \cancel{2}\cdot\frac{3}{\cancel{4}_2}\left(\frac{5-3}{15}\right) =$

$= \frac{\cancel{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{\cancel{15}_5} = \frac{1}{5}$

Άρα ① $\Rightarrow VAR(X) = \frac{1}{5} - 0^2 = \frac{1}{5}$

• Για την κατανομή $F_X(x)$ έχουμε:

- Αν $x \leq -1$ τότε $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$

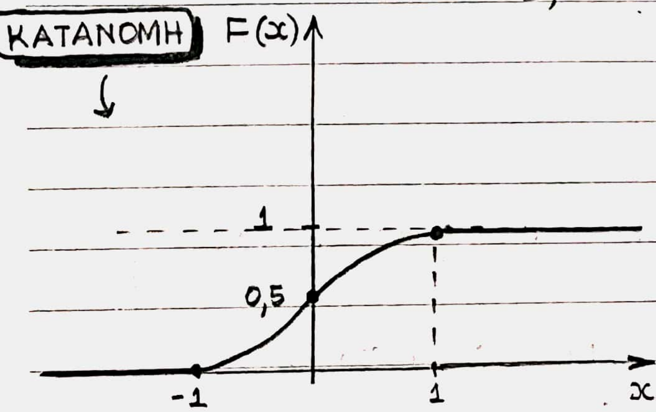
- Αν $x \in (-1,1)$ τότε $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-1}^{x}\frac{3}{4}(1-t^2)\,dt =$

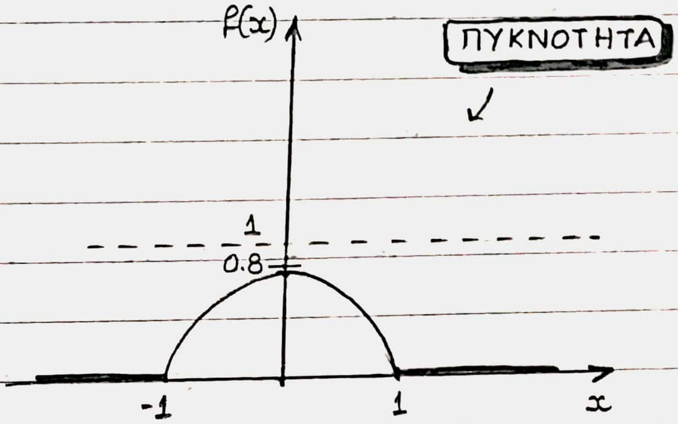
$= \frac{3}{4}\int_{-1}^{x}\left(t-\frac{t^3}{3}\right)'dt = \frac{3}{4}\left(x-\frac{x^3}{3}\right) - \frac{3}{4}\left(-1+\frac{1}{3}\right) = \frac{3x-x^3+2}{4}$

- Αν $x > -1$ τότε $F_X(x) = P(X \le x) = \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx =$

$$= \frac{3}{4}\int_{-1}^{1}\left(x - \frac{x^3}{3}\right)' dx = \frac{3}{4}\left(1-\frac{1}{3}\right) - \frac{3}{4}\left(-1+\frac{1}{3}\right) = \frac{3}{4} - \frac{1}{4} + \frac{3}{4} - \frac{1}{4} = \frac{2}{4} + \frac{2}{4} = 1$$

Συνεπώς $F_X(x) = \begin{cases} 0, & x \le -1 \\ \frac{-x^3+3x+2}{4}, & x \in (-1,1) \\ 1, & x \ge 1 \end{cases}$





47. (Πυκνότητα Πιθανότητας ΙΙ)

$$f_X(x) = \begin{cases} Cx + e^{-x}, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

(α') Ισχύει ότι $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \Rightarrow \int_0^1 (Cx + e^{-x})\,dx = 1 \Rightarrow \int_0^1 \left(C\frac{x^2}{2}\right)' dx + \int_0^1 (-e^{-x})'\,dx = 1$

$$\Rightarrow C\cdot\frac{1}{2} + (-e^{-1}) - (-e^{0}) = 1 \Rightarrow \frac{C}{2} - e^{-1} + \cancel{1} = \cancel{1} \Rightarrow \frac{C}{2} = \frac{1}{e} \Rightarrow C = \frac{2}{e}$$

(β') Είναι $E(X) = \int_{-\infty}^{+\infty} x \cdot f_X(x)\,dx = \int_0^1 x\cdot(Cx + e^{-x})\,dx = \int_0^1 \left(\frac{2}{e}x^2 + xe^{-x}\right) dx =$

$$= \int_0^1 \left(\frac{2}{e}\cdot\frac{x^3}{3}\right)' dx + \int_0^1 xe^{-x}\,dx = \frac{2}{e}\cdot\frac{1}{3} + \int_0^1 x(-e^{-x})'\,dx = \frac{2}{e}\cdot\frac{1}{3} + 1(-e^{-1}) - \int_0^1 -e^{-x}\,dx$$

$$= \frac{2}{e}\cdot\frac{1}{3} - e^{-1} + \int_0^1 (-e^{-x})'\,dx = \frac{2}{e}\cdot\frac{1}{3} - e^{-1} + (-e^{-1}) - (-e^{0}) =$$

$$= \frac{2}{3e} - \frac{2}{e} + 1$$

(γ') $P(X<\frac{1}{2})=\int_{-\infty}^{1/2}(Cx+e^{-x})\,dx=\int_{0}^{1/2}(\frac{2}{e}x+e^{-x})\,dx=\int_{0}^{1/2}(\frac{2}{e}\frac{x^2}{2}-e^{-x})'\,dx=$

$$=\frac{\cancel{2}}{e}\frac{(\frac{1}{2})^2}{\cancel{2}}-e^{-1/2}+e^{0}=\frac{\frac{1}{4}}{e}-e^{-1/2}=\frac{1}{4e}-e^{-1/2}+1$$

(δ') Για την κατανομή έχουμε $F_X(x)$:

- Αν $x<0$ τότε $F_X(x)=P(X\le x)=\int_{-\infty}^{x}f(t)\,dt=\int_{-\infty}^{x}0\,dt=0$
- Αν $x\in[0,1]$ τότε $F_X(x)=P(X\le x)=\int_{-\infty}^{x}f(t)\,dt=\int_{0}^{x}(Ct+e^{-t})\,dt=$

$$=\int_{0}^{x}(\frac{2}{e}\frac{t^2}{2}-e^{-t})'\,dt=\frac{\cancel{2}}{e}\cdot\frac{x^2}{\cancel{2}}-e^{-x}+e^{0}=\frac{x^2}{e}-e^{-x}+1$$

- Αν $x>1$ τότε $F_X(x)=P(X\le x)=\int_{-\infty}^{+\infty}(\frac{2}{e}\cdot\frac{x^2}{2}-e^{-x})'\,dx=$

$$=\int_{0}^{1}(\frac{2}{e}\frac{x^2}{2}-e^{-x})\,dx=\frac{\cancel{2}}{e}\cdot\frac{1}{\cancel{2}}-e^{-1}+e^{0}=\cancel{\frac{1}{e}}-\cancel{\frac{1}{e}}+1=1$$

Συνεπώς $F_X(x)=\begin{cases}0, & x<0\\ \frac{x^2}{e}-e^{-x}+1, & x\in[0,1]\\ 1, & x>1\end{cases}$

$\boxed{\text{ΚΑΤΑΝΟΜΗ}}$