Module 4
Upper and Lower Bounds

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Upper and Lower Bounds on T(n)

- Consider $T(n) = \frac{3}{4}n^2 - 3n \quad \rightarrow ①$
- Since T(n) is defined only for n ≥ 0, it follows that

$$T(n) = \frac{3}{4}n^2 - 3n \leq \frac{3}{4}n^2 \quad \rightarrow ②$$

- That is, $\frac{3}{4}n^2$ is an **upper bound** on T(n), for all n ≥ 0

Upper and Lower Bounds on T(n)

- The question is, does there exist a constant $c_1 > 0$ and $n_0 > 0$ such that

$$c_1 n^2 \le T(n) \quad \rightarrow ③$$

for all $n \ge n_0$?

Upper and Lower Bounds on T(n)

- To answer this, consider

$$c_1 n^2 \leq \frac{3}{4} n^2 - 3n$$

- Dividing both sides by n^2

$$c_1 n^2 \leq \frac{3}{4} n^2 - 3n$$

Upper and Lower Bounds on T(n)

- Notice that the right hand side is not positive for $0 < n \leq 4$. Hence, the least value of n = 2 when used,

$$c_1 \leq \frac{3}{4} - \frac{3}{5} = \frac{3}{20}$$

- Thus, for all $n \geq n_0 = 5$, there is a constant $c_1 = \frac{3}{20}$ such that

$$\frac{3}{20}n^2 \leq \frac{3}{4}n^2 - 3n \quad \rightarrow 4$$

Upper and Lower Bounds on T(n)

- Combining ② and ④:

$$\frac{3}{20}n^2 \le T(n) = \frac{3}{4}n^2 - 3n \le \frac{3}{4}n^2 \quad \rightarrow ⑤$$

where the left inequality holds for all $n \ge n_0 = 5$ and the right inequality holds for all $n \ge 0$

- In other words, both inequalities are true for all $n \ge n_0 = 5$

Upper and Lower Bounds on T(n)

- Generalizing, let $T(n) = an^2 + bn + c$ with a>0
- Then there exists constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 n^2 \leq T(n) = an^2 + bn + c \leq \ c_2 n^2$$

for all $n \geq n_0$

Upper and Lower Bounds on T(n)

- Consider

$$an^2 + bn + c \le c_2 n^2$$

- Dividing by n^2

$$a + \frac{b}{n} + \frac{c}{n^2} \le \; c_2 \quad \rightarrow 6$$

Upper and Lower Bounds on T(n)

- Since a > 0, there exists at least a value of n_1 such that for all $n \geq n_1$, inequality ⑥ would be true. Then,

$$c_2 = a + \frac{b}{n_1} + \frac{c}{{n_1}^2} > 0$$

Upper and Lower Bounds on T(n)

- Similarly, from

$$c_1 \le a + \frac{b}{n} + \frac{c}{n^2}$$

- It follows that there exists a least value $n=n_2$ such that

$$0 < \ c_1 = a + \frac{b}{n_2} + \frac{c}{{n_2}^2}$$

Upper and Lower Bounds on T(n)

- Now let $n_0 = \max\{n_1, n_2\}$. Then, combining we get

$$c_1 n^1 \le an^2 + bn + c \le c_2 n^2$$

for all $n \ge n_0$

Upper and Lower Bounds on T(n)

- Stated in other words: For all $n \geq n_0$, $T(n)$ is simultaneously bounded above by quadratic function c_2n^2 and bounded below by a quadratic function c_1n^2.

Big-O Notation
Asymptotic Upper Bound

- Let $g:N \rightarrow R^+$ be the set of all functions from non-negative integers to the positive real numbers
- Thus, g(n) could be a complexity function of an algorithm

Big-O Notation
Asymptotic Upper Bound

- Given g(n), we define
- $O(\, g(n)\,) = \{\, f(n)$: there exists a real constant $c > 0$ and an integer n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0$

Big-O Notation
Asymptotic Upper Bound

- Thus, O(g(n)) denotes the set of all functions f(n) that are upper bounded by a constant multiple of g(n) for all $n \geq n_0$
- This set is said to be asymptotically upper bounded by g(n)
- Consequently, if T(n) = $\frac{3}{4}n^2 - 3n$, then

  $T(n) \in O(n^2)$ where g(n) = n^2

Big-O Notation
Asymptotic Upper Bound

- By abuse of notation, we generally say

$$T(n) = O(n^2)$$

when in fact we mean the inclusion

Big- Ω Notation
Asymptotic Lower Bound

- Let g(n) be as defined before.
- Define Ω(g(n)) = { $f(n)$: there exists a real constant
  $\mathrm{c} > 0$ and an integer n_0 such that
  $0 \leq cg(n) \leq f(n)$
  for all $\mathrm{n} \geq \mathrm{n}_0$

Big- Ω Notation
Asymptotic Lower Bound

- $\Omega(\ g(n)\ )$ denotes the set of all functions f(n) that are lower bounded by a constant multiple of g(n) for all $n \geq n_0$
- This set is asymptotically lower bounded by g(n)
- Consequently, if $T(n) = \frac{3}{4}n^2 - 3n$, then

  $T(n) \in \Omega(n^2)$ where $g(n) = n^2$

Big- Ω Notation
Asymptotic Lower Bound

- We say that T(n) is lower bounded by n^2 by writing

$$T(n) = \Omega(n^2)$$

when in fact we mean the inclusion

Big-θ Notation
Simultaneous Upper and Lower Bound

- $\Theta(\ g(n)\ ) = \{\ f(n) \mid c_1 g(n) \le f(n) \le\ c_2 g(n)$

  $for\ all\ n \ge\ n_0, a\ positive\ integer$

  $and\ positive\ constants\ c_1\ and\ c_2\ \}$

- It is a class of functions that are upper and lower bounded by g(n)

Big-θ Notation
Simultaneous Upper and Lower Bound

- Thus,

$$T(n) = \frac{3}{4}n^2 - 3n$$

is such that $T(n) \in \theta(n^2)$

Big-θ Notation
Simultaneous Upper and Lower Bound

- Let $p(n) = \sum_{i=0}^{k} a_i n^i = a_0 + a_1 n + a_2 n^2 + \ ... + a_k n^k$

  be a k^{th} degree polynomial with $a_k > 0$

- Then, it can be verified

$$\left.\begin{aligned} p(n) &= O(n^k) \\ p(n) &= \Omega(n^k) \end{aligned}\right\} p(n) = \theta(n^k)$$

Big-θ Notation
Simultaneous Upper and Lower Bound

- By convention: O(1) represents an arbitrary positive and finite constant
- If $f(n) = O(n^{O(1)})$, then f(n) is said to be polynomially bounded

Homework

1. Show that

   a) $\log_2 n = O(n)$

   b) $n \log_2 n = O(n^2)$

2. Find an upper and lower bound on

$$T(n) = n \log_2 n - n + 1$$

Homework

3. $T(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2} = \theta(n^2)$

4. $T(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \theta(n^3)$

5. $T(n) = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 = \theta(n^4)$

6. $T(n) = \sum_{i=0}^{n-1} 2^i = 2^n - 1 = \theta(2^n)$

Homework

7. Let $T(n) = n!$ Then,

$$\left(\frac{n}{2}\right)^n \leq n! = 1 \ * 2 \ * 3 \ * \ \ldots * n \leq n^n$$

Taking logarithm →

$$n[\log_2 n - 1] \leq \ \log_2 n! \leq n \log_2 n$$

Verify that $\log_2 n! = O(n \log n)$

$$\log_2 n! = \Omega(n \log n)$$

Homework

7. $T(n) = \sum_{i=0}^{n} \frac{1}{2^i} = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}$

8. $\leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - {}^1/_2} = 2 = O(1)$