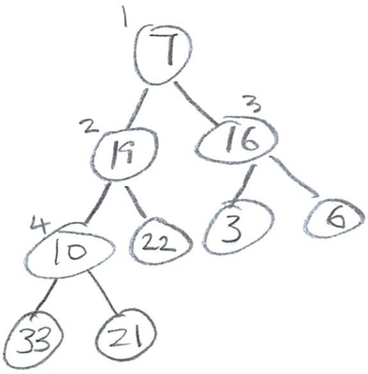
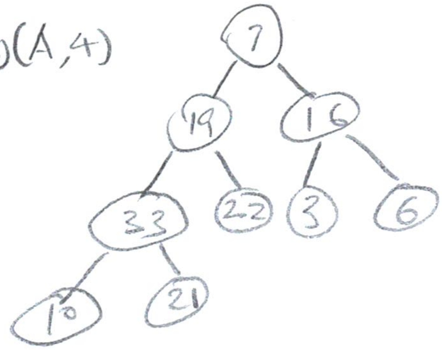
Xueying Chen
ID: 113366600
DSA - Fall 2019
Homework #4

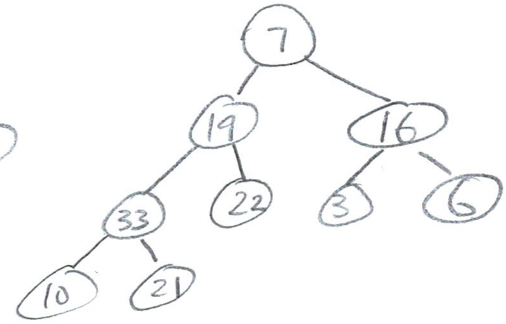
1 $A = \langle 7, 19, 16, 10, 22, 3, 6, 33, 21 \rangle$ $A.length/2 = 4$



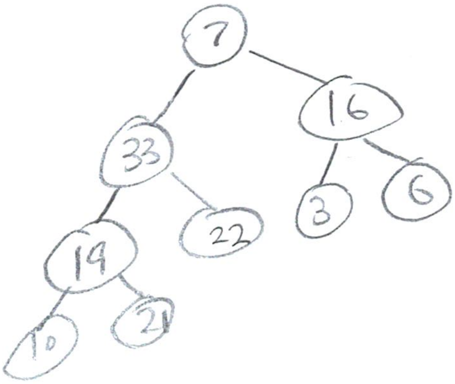
Max-heapify(A,4)

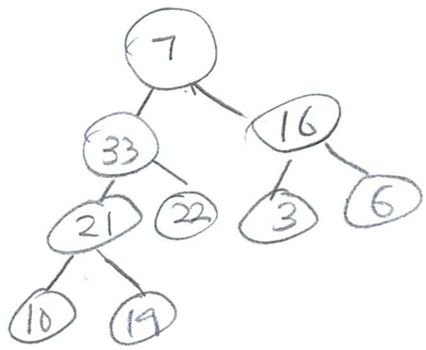


max-heapify(A, 3)

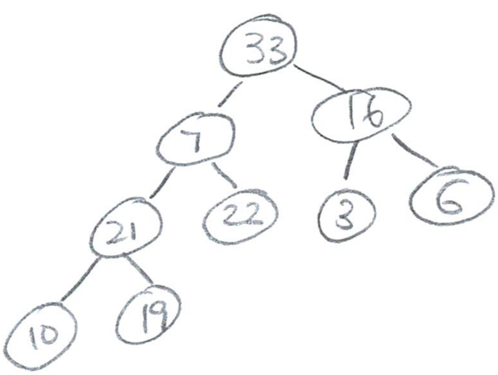


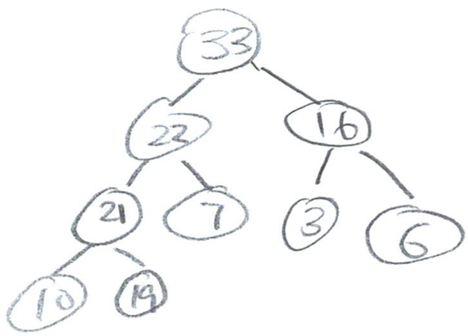
max-heapify(A, 2)





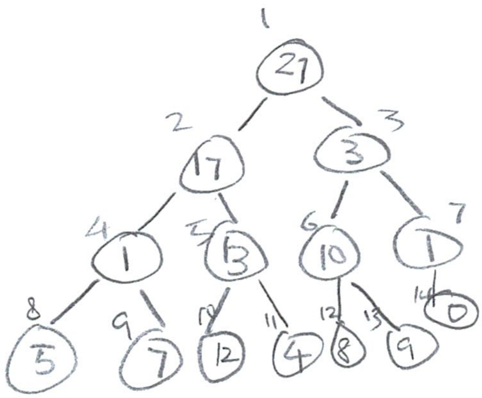
max-heapify(A,1)



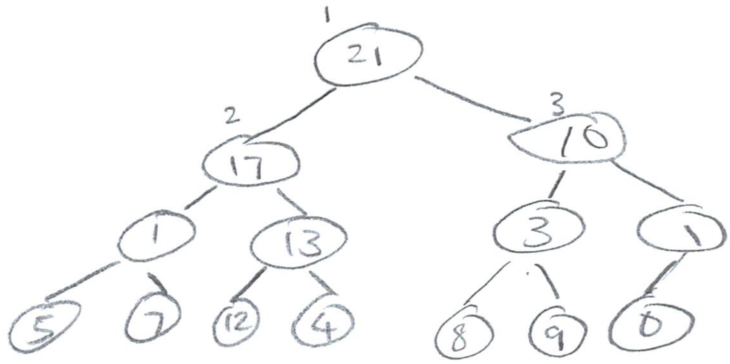


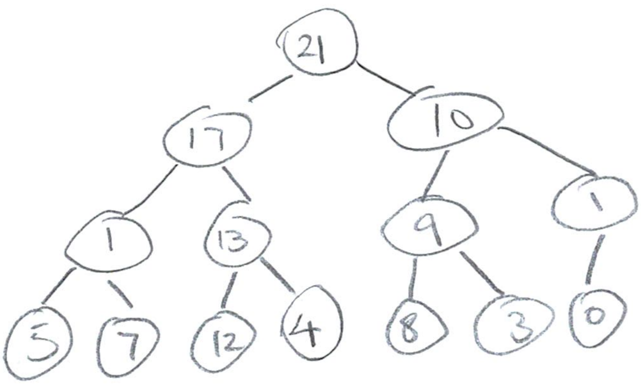
$A = \langle 33, 22, 16, 21, 7, 3, 6, 10, 19 \rangle$

② $A = \langle 27, 17, 3, 1, 13, 10, 1, 5, 7, 12, 4, 8, 9, 0 \rangle$



Max-heapify(A, 3)





③ Quick Sort average case analysis.

Assume the pivot occupies in any of the n locations with probality $P = 1/n$.

The reccurrent function $T_a(n)$ for average case

$$T_a(n) = (n-1) + \frac{1}{n}\sum_{j=1}^{n}\Big(\underbrace{T_a(j-1)}_{\text{average sorting on LHS}} + \underbrace{T(n-j)}_{\text{average sorting on RHS}}\Big)]$$

$$\sum_{j=1}^{n}\big(T_a(j-1) + T(n-j)\big) =$$

$$= T_a(0) + T(n-1) + T_a(1) + T(n-2) + T_a(2) + T(n-3) \cdots T_a(n-1) + T_a(0)$$

$$= 2T_a(0) + 2T(1) + 2T(2) \cdots 2T(n-3) + 2T(n-2) + 2T(n-1)$$

$$= 2\big(T_a(0) + T(1) + T(2) \cdots + T(n-3) + T(n-2) + T(n-1)\big]$$

$$= 2\sum_{j=0}^{n} T_a(j)\big)$$

$$T_a(n) = (n-1) + \frac{2}{n}\sum_{j=0}^{n} T_a(j)$$

$$= (n-1) + \frac{2}{n}\sum_{j=2}^{n} T_a(j)$$

$T_a(0) = 0$

$T_1(1) = 0$

$$T_a(n) = O(n\log n)$$

4)

(a) $T(n) = 2T(\frac{n}{4}) + 1$, $T(1) = 1$ $\quad k=1$

$k=2$, $T(\frac{n}{4}) = 2T(\frac{n}{16}) + 1 \Rightarrow T(n) = 2(2T(\frac{n}{16}) + 1) + 1$

$= 2 \cdot 2\, T(\frac{n}{16}) + 2 + 1$

$k=3$, $T(\frac{n}{16}) = 2T(\frac{n}{64}) + 1 \Rightarrow T(n) = 2(2(2T(\frac{n}{64}) + 1) + 1) + 1$

$= 2(2 \cdot 2T(\frac{n}{64}) + 2) + 1) + 1$

$= 2 \cdot 2 \cdot 2\, T(\frac{n}{64}) + 4 + 2 + 1$

$$T(n) = 2^k\, T\left(\frac{n}{2^{2k}}\right) + \sum_{k=1}^{n} 2^k$$

$$\frac{n}{2^{2k}} = 1 \Rightarrow \log n = 2k$$

$$k = \frac{\log n}{2}$$

$$T(n) = 2^{\log n/2} + \sum_{k=1}^{n} 2^{k-1}$$

$$T(n) = 2^{(\log n)/2} + \sum_{k=1}^{n} 2^{k-1} = 2^{(\log n)/2} + \sum_{k=1}^{n} 2^{k-1}$$

$$\sum_{k=1}^{n} 2^{k-1} = \frac{1}{2}\sum_{k=1}^{n} 2^k = \frac{1}{2}\left(\frac{2^{n+1}-1}{2-1}\right)$$

$$\boxed{T(n) = 2^{(\log n)/2} + \frac{1}{2}(2^{n+1} - 1)}$$

(b) $T(n) = 2(n/4) + n^2$

$T(n/4) = 2T(\frac{n}{16}) + (\frac{n}{4})^2 \Rightarrow T(n) = 2[2T(\frac{n}{16}) + (\frac{n}{4})^2] + n^2$

$= 2 \cdot 2\, T(\frac{n}{16}) + 2(\frac{n}{4})^2 + n^2$

$T(\frac{n}{16}) = 2T(\frac{n}{64}) + (\frac{n}{16})^2 \Rightarrow T(n) = 2 \cdot 2 \cdot [2T(\frac{n}{64}) + (\frac{n}{16})^2] + 2(\frac{n}{4})^2 + n^2$

$= 2 \cdot 2 \cdot 2\, T(\frac{n}{64}) + 2 \cdot 2(\frac{n}{16})^2 + 2(\frac{n}{4})^2 + n^2$

$$T(n) = 2^k\, T\left(\frac{n}{4^k}\right) + \sum_{k=1}^{n} 2^{k-1} \frac{n^2}{4^{k-1}}$$

$$T(n) = 2^k T\left(\frac{n}{4^k}\right) + n^2 \sum_{k=1}^{n} \frac{\cancel{2^{k-1}}}{\cancel{2^{k-1}} \cdot 2^{k-1}}$$

$$T(n) = 2^k\, T\left(\frac{n}{4^k}\right) + n^2 \sum_{k=1}^{n} \frac{1}{2^{k-1}}$$

$\frac{n}{4^k}=1 = \log_4 n = k$ $\qquad \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 2\sum_{k=1}^{n} \left(\frac{1}{2}\right)^k$

$$= \frac{1-\left(\frac{1}{2}\right)^{n+1}}{\frac{1}{2}}$$

$$= 2\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)$$

$$T(n) = 2^{\log_4 n} + (\log_4 n)^2 \cdot 2\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)$$

$$\boxed{T(n) = 2^{\log_4 n} + 2(\log_4 n)^2\left(1-\left(\tfrac{1}{2}\right)^{n+1}\right)}$$

(c) $T(n) = 3T(n/2) + 5n,\ T(1)=1$ $\qquad k=1$

$k=2$ $\quad T(n/2) = 3T\left(\frac{n}{4}\right) + 5\left(\frac{n}{2}\right)$

$$T(n) = 3\left(3T\left(\tfrac{n}{4}\right)+5\left(\tfrac{n}{2}\right)\right) + 5n$$

$$= 3\cdot 3T\left(\tfrac{n}{4}\right) + 3\cdot 5\left(\tfrac{n}{2}\right) + 5n$$

$k=3$ $\quad T\left(\frac{n}{4}\right) = 3T\left(\frac{n}{8}\right) + 5\left(\frac{n}{4}\right)$

$$T(n) = 3\cdot 3\left(3T\left(\tfrac{n}{8}\right)+5\left(\tfrac{n}{4}\right)\right) + 3\cdot 5\left(\tfrac{n}{2}\right) + 5n$$

$$T(n) = 3\cdot 3\cdot 3\,T\left(\tfrac{n}{8}\right) + 3\cdot 3\cdot 5\left(\tfrac{n}{4}\right) + 3\cdot 5\left(\tfrac{n}{2}\right) + 5n$$

$$T(n) = 3^k\, T\left(\tfrac{n}{2^k}\right) + 5n\sum_{k=1}^{n} \frac{3^{k-1}}{2^{k-1}}$$

$$\frac{n}{2^k} = 1 \Rightarrow k = \log n$$

$$T(n) = 3^{\log n} + 5n\sum_{k=1}^{n}\left(\tfrac{3}{2}\right)^{k-1}$$

$$T(n) = n^{\log 3} + 5n\left(\tfrac{2}{3}\right)\sum_{k=1}^{n}\left(\tfrac{3}{2}\right)^k$$

$$= n^{\log 3} + \frac{10n}{3}\left(\frac{(3/2)^{n+1}-1}{3/2-1}\right)$$

$$\boxed{T(n) = n^{\log 3} + \frac{20n}{3}\left((3/2)^{n+1}-1\right)}$$

(d) $T(n) = 2T(n/2) + (n-1)$ $\quad k=1$

$k=2$ $\quad T(n/2) = 2T(\frac{n}{4}) + (\frac{n}{2}-1)$

$$T(n) = 2\left(2T(\tfrac{n}{4}) + (\tfrac{n}{2}-1)\right) + (n-1)$$

$$T(n) = 2\cdot 2\, T(\tfrac{n}{4}) + 2(\tfrac{n}{2}-1) + n - 1$$

$$= 2\cdot 2\, T(\tfrac{n}{4}) + \tfrac{2n}{2} - 2 + n - 1$$

$$= 2\cdot 2\, T(\tfrac{n}{4}) + \tfrac{2n}{2} + n - 2 - 1$$

$k=3$ $\quad T(\frac{n}{4}) = 2T(\frac{n}{8}) + (\frac{n}{4}-1)$

$$T(n) = 2\cdot 2\cdot\left(2T(\tfrac{n}{8}) + (\tfrac{n}{4}-1)\right) + \tfrac{2}{2}n + n - 2 - 1$$

$$= 2\cdot 2\cdot 2\, T(\tfrac{n}{8}) + 2\cdot 2\tfrac{n}{4} - 2\cdot 2 + \tfrac{2}{2}n + n - 2 - 1$$

$$= 2\cdot 2\cdot 2\, T(\tfrac{n}{8}) + 2\cdot 2\tfrac{n}{4} + \tfrac{2}{2}n + n - 4 - 2 - 1$$

$$= 2^k T(\tfrac{n}{2^k}) + kn - \sum_{k=1}^{n} 2^{k-1}$$

$$k = \log n$$

$$T(n) = n + n\log n - \sum_{k=1}^{n} 2^{k-1}$$

$$\boxed{T(n) = n + n\log n - \tfrac{1}{2}(2^{n+1} - 1)}$$