# Module 1: Introduction to Digital Control
## Lecture Note 1

## 1 Digital Control System

A digital control system model can be viewed from different perspectives including control algorithm, computer program, conversion between analog and digital domains, system performance etc. One of the most important aspects is the sampling process level.

In continuous time control systems, all the system variables are continuous signals. Whether the system is linear or nonlinear, all variables are continuously present and therefore known (available) at all times.A typical continuous time control system is shown in Figure 1.

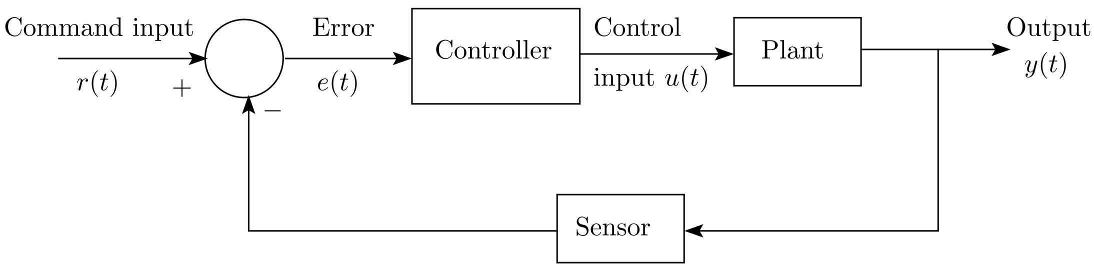



Figure 1: A typical closed loop continuous time control system

In a digital control system, the control algorithm is implemented in a digital computer. The error signal is discretized and fed to the computer by using an A/D (analog to digital) converter. The controller output is again a discrete signal which is applied to the plant after using a D/A (digital to analog) converter. General block diagram of a digital control system is shown in Figure 2.

$e(t)$ is sampled at intervals of $T$. In the context of control and communication, sampling is a process by which a continuous time signal is converted into a sequence of numbers at discrete time intervals. It is a fundamental property of digital control systems because of the discrete nature of operation of digital computer.

Figure 3 shows the structure and operation of a finite pulse width sampler, where (a) represents the basic block diagram and (b) illustrates the function of the same. $T$ is the sampling period and $p$ is the sample duration.

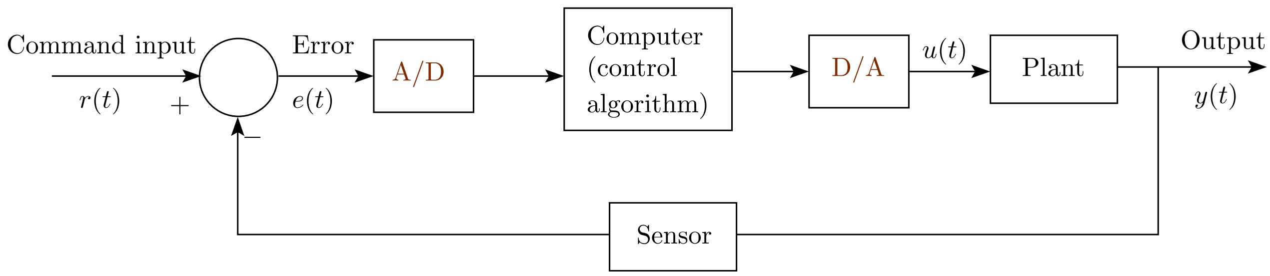



Figure 2: General block diagram of a digital control system

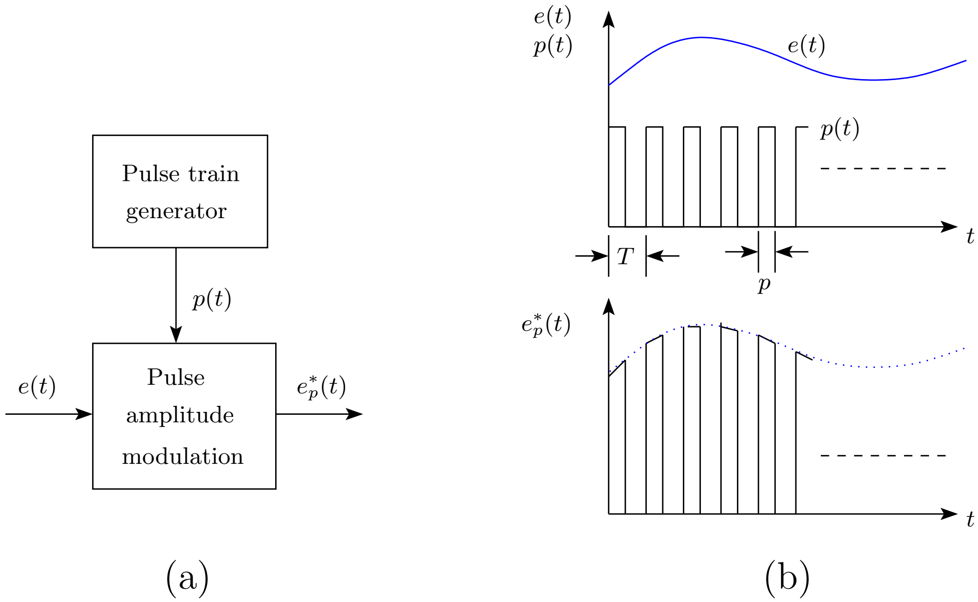



Figure 3: Basic structure and operation of a finite pulse width sampler

In the early development, an analog system, not containing a digital device like computer,in which some of the signals were sampled was referred to as a sampled data system. With the advent of digital computer, the term discrete-time system denoted a system in which all its signals are in a digital coded form. Most practical systems today are of hybrid nature, i.e., contains both analog and digital components.

Before proceeding to any depth of the subject we should first understand the reason behind going for a digital control system. Using computers to implement controllers has a number of advantages. Many of the difficulties involved in analog implementation can be avoided. Few of them are enumerated below.

1. Probability of accuracy or drift can be removed.
2. Easy to implement sophisticated algorithms.
3. Easy to include logic and nonlinear functions.

4. Reconfigurability of the controllers.

### 1.1 A Naive Approach to Digital Control

One may expect that a digital control system behaves like a continuous time system if the sampling period is sufficiently small. This is true under reasonable assumptions. A crude way to obtain digital control algorithms is by writing the continuous time control law as a differential equation and approximating the derivatives by differences and integrations by summations. This will work when the sampling period is very small. However various parameters, like overshoot, settling time will be slightly higher than those of the continuous time control.

**Example: PD controller**
A continuous time PD controller can be discretized as follows:

$$\begin{aligned} u(t) &= K_p e(t) + K_d \frac{de(t)}{dt} \\ \Rightarrow u(kT) &= K_p e(kT) + K_d \frac{[e(kT) - e((k-1)T)]}{T} \end{aligned}$$

where $k$ represents the discrete time instants and $T$ is the discrete time step or the sampling period. We will see later the control strategies with different behaviors, for example deadbeat control, can be obtained with computer control which are not possible with a continuous time control.

### 1.2 Aliasing

Stable linear systems have property that the steady state response to sinusoidal excitations is sinusoidal with same frequency as that of the input. But digital control systems behave in a much more complicated way because sampling will create signals with new frequencies.

Aliasing is an effect of the sampling that causes different signals to become indistinguishable. Due to aliasing, the signal reconstructed from samples may become different than the original continuous signal. This can drastically deteriorate the performance if proper care is not taken.

## 2 Inherently Sampled Systems

Sampled data systems are natural descriptions for many phenomena. In some cases sampling occurs naturally due to the nature of measurement system whereas in some cases it occurs because information is transmitted in pulsed form. The theory of sampled data systems thus has many applications.

1. **Radar:** When a radar antenna rotates, information about range and direction is naturally obtained once per revolution of the antenna.

2. **Economic Systems:** Accounting procedures in economic systems are generally tied to the calendar. Information about important variables is accumulated only at certain times,

e.g., daily, weekly, monthly, quarterly or yearly even if the transactions occur at any point of time.

3. **Biological Systems:** Since the signal transmission in the nervous system occurs in pulsed form, biological systems are inherently sampled.

All these discussions indicate the need for a separate theory for sampled data control systems or digital control systems.

# 3 How Was Theory Developed ?

1. **Sampling Theorem:** Since all computer controlled systems operate at discrete times only, it is important to know the condition under which a signal can be retrieved from its values at discrete points. Nyquist explored the key issue and Shannon gave the complete solution which is known as Shannon's sampling theorem. We will discuss Shannon's sampling theorem in proceeding lectures.

2. **Difference Equations and Numerical Analysis:** The theory of sampled-data system is closely related to numerical analysis. Difference equations replaced the differential equations in continuous time theory. Derivatives and integrals are evaluated numerically by approximating them with differences and sums.

3. **Transform Methods:** Z-transform replaced the role of Laplace transform in continuous domain.

4. **State Space Theory:** In late 1950's, a very important theory in control system was developed which is known as state space theory. The discrete time representation of state models are obtained by considering the systems only at sampling points.

# Module 1: Introduction to Digital Control
## Lecture Note 2

## 1 Discrete time system representations

As mentioned in the previous lecture, discrete time systems are represented by difference equations. We will focus on LTI systems unless mentioned otherwise.

### 1.1 Approximation for numerical differentiation

1. Using backward difference

   (a) First order

$$\begin{aligned}\text{Continuous: } u(t) &= \dot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{e(kT)-e((k-1)T)}{T}\end{aligned}$$

   (b) Second order

$$\begin{aligned}\text{Continuous: } u(t) &= \ddot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{\dot{e}(kT)-\dot{e}((k-1)T)}{T}\\ &= \frac{e(kT)-e((k-1)T)-e((k-1)T)+e((k-2)T)}{T^2}\\ &= \frac{e(kT)-2e((k-1)T)+e((k-2)T)}{T^2}\end{aligned}$$

2. Using forward difference

   (a) First order

$$\begin{aligned}\text{Continuous: } u(t) &= \dot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{e((k+1)T)-e(kT)}{T}\end{aligned}$$

   (b) Second order

$$\begin{aligned}\text{Continuous: } u(t) &= \ddot{e}(t)\\ \text{Discrete: } u(kT) &= \frac{\dot{e}((k+1)T)-\dot{e}(kT)}{T}\\ &= \frac{e((k+2)T)-2e((k+1)T)+e(kT)}{T^2}\end{aligned}$$

## 1.2 Approximation for numerical integration

The numerical integration technique depends on the approximation of the instantaneous continuous time signal. We will describe the process of backward rectangular integration technique.

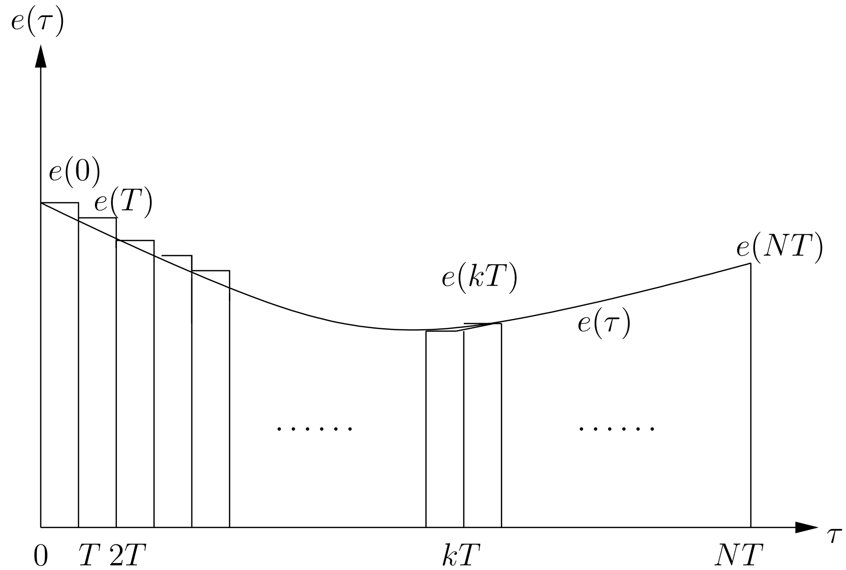



Figure 1: Concept behind Numerical Integration

As shown in Figure 1, the integral function can be approximated by a number of rectangular pulses and the area under the curve can be represented by summation of the areas of all the small rectangles. Thus,

$$\begin{aligned}
\text{if } u(t) &= \int_0^t e(\tau)d\tau \\
\Rightarrow u(NT) &= \int_0^{NT} e(\tau)d\tau \\
&\cong \sum_{k=0}^{N-1} e(kT)\Delta t \\
&= \sum_{k=0}^{N-1} e(kT)T
\end{aligned}$$

where $k = 0, 1, 2, \cdots, N-1$, $\Delta t = T$ and $N > 0$. From the above expression,

$$\begin{aligned}
u((N-1)T) &= \int_0^{(N-1)T} e(\tau)d\tau \\
&= \sum_0^{N-2} e(kT)T \\
\Rightarrow u(NT) - u((N-1)T &= Te((N-1)T) \\
\text{or, } u(NT) &= u((N-1)T) + Te((N-1)T)
\end{aligned}$$

The above expression is a recursive formulation of backward rectangular integration where the expression of a signal at a given time explicitly contains the past values of the signal. Use of this recursive equation to evaluate the present value of $u(NT)$ requires to retain only the immediate past sampled value $e((N-1)T)$ and the immediate past value of the integral $u((N-1)T)$, thus saving the storage space requirement.

In forward rectangular integration, we start approximating the curve from top right corner. Thus the approximation is

$$u(NT) = \sum_{k=1}^{N} e(kT)T$$

The recursive relation of the forward rectangular integration is:

$$u(NT) = u((N-1)T) + Te(NT)$$

Polygonal or trapezoidal integration is another numerical integration technique where the total area is divided into a number of trapezoids and expressed as the sum of areas of individual trapezoids.

**Example 1:** Consider the following continuous time expression of a PID controller:

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)d\tau + K_d \frac{de(t)}{dt}$$

where $u(t)$ is the controller output and $e(t)$ is the input to the controller. Considering $t = NT$, find out the recursive discrete time formulation of $u(NT)$ by approximating the derivative by backward difference and integral by backward rectangular integration technique.

**Solution:** $u(NT)$ can be approximated as

$$u(NT) = K_p e(NT) + K_i \sum_{k=0}^{N-1} e(kT)T + K_d \frac{e(NT) - e((N-1)T)}{T}$$

Similarly $u((N-1)T)$ can be written as

$$u((N-1)T) = K_p e((N-1)T) + K_i \sum_{k=0}^{N-2} e(kT)T + K_d \frac{e((N-1)T) - e((N-2)T)}{T}$$

Subtracting $u((N-1)T)$ from $u(NT)$,

$$
\begin{aligned}
u(NT)-u((N-1)T) &= K_p e(NT)+K_i\sum_{k=0}^{N-1} e(kT)T+K_d\frac{e(NT)-e((N-1)T)}{T}-\\
&\quad K_p e((N-1)T)-K_i\sum_{k=0}^{N-2} e(kT)T-K_d\frac{e((N-1)T)-e((N-2)T)}{T}\\
\Rightarrow u(NT) &= u((N-1)T)+K_p\left[e(NT)-e((N-1)T)\right]+K_iTe((N-1)T)\\
&\quad +K_d\frac{e(NT)-2e((N-1)T)+e((N-2)T)}{T}
\end{aligned}
$$

which is the required recursive relation.

Similarly, if we use forward difference and forward rectangular integration, we would get the recursive relation as

$$
\begin{aligned}
u(NT) &= u((N-1)T)+K_p\left[e(NT)-e((N-1)T)\right]+K_iTe(NT)\\
&\quad +K_d\frac{e((N+1)T)-2e(NT)+e((N-1)T)}{T}
\end{aligned}
$$

## 1.3 Difference Equation Representation

The general linear difference equation of an $n^{th}$ order causal LTI SISO system is:

$$
\begin{aligned}
&y((k+n)T)+a_1y((k+n-1)T)+a_2y\left((k+n-2)T\right)+\cdots+a_ny\left(kT\right)\\
&\qquad = b_0u((k+m)T)+b_1u\left((k+m-1)T\right)+....+b_mu\left(kT\right)
\end{aligned}
$$

where $y$ is the output of the system and $u$ is the input to the system and $m \leq n$. This inequality is required to avoid anticipatory or non-causal model.

**Example 2:** If you express the recursive relation for PID control in general difference equation form, is the system causal?

**Solution:** The output of the PID controller is $u$ and the input is $e$. When approximated with forward difference and forward rectangular integration, $u(NT)$ is found as:

$$
\begin{aligned}
u(NT) &= u((N-1)T)+K_p\left[e(NT)-e((N-1)T)\right]+K_iTe(NT)\\
&\quad +K_d\frac{e((N+1)T)-2e(NT)+e((N-1)T)}{T}
\end{aligned}
$$

By putting $N=k+1$ and comparing with general difference equation, we can say $n=1$ whereas $m=2$. Thus the system is non-causal. However, when the approximation uses backward difference and backward rectangular integration, the approximated model becomes causal.

# Module 1: Introduction to Digital Control
## Lecture Note 3

# 1 Mathematical Modeling of Sampling Process

Sampling operation in sampled data and digital control system is used to model either the sample and hold operation or the fact that the signal is digitally coded. If the sampler is used to represent S/H (Sample and Hold) and A/D (Analog to Digital) operations, it may involve delays, finite sampling duration and quantization errors. On the other hand if the sampler is used to represent digitally coded data the model will be much simpler. Following are two popular sampling operations:

1. Single rate or periodic sampling
2. Multi-rate sampling

We would limit our discussions to periodic sampling only.

## 1.1 Finite pulse width sampler

In general, a sampler is the one which converts a continuous time signal into a pulse modulated or discrete signal. The most common type of modulation in the sampling and hold operation is the pulse amplitude modulation.

The symbolic representation, block digram and operation of a sampler are shown in Figure 1. The pulse duration is $p$ second and sampling period is $T$ second. Uniform rate sampler is a linear device which satisfies the principle of superposition. As in Figure 1, $p(t)$ is a unit pulse train with period $T$.

$$p(t) = \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

where $u_s(t)$ represents unit step function. Assume that leading edge of the pulse at $t = 0$ coincides with $t = 0$. Thus $f_p^*(t)$ can be written as

$$f_p^*(t) = f(t) \sum_{k=-\infty}^{\infty} [u_s(t-kT) - u_s(t-kT-p)]$$

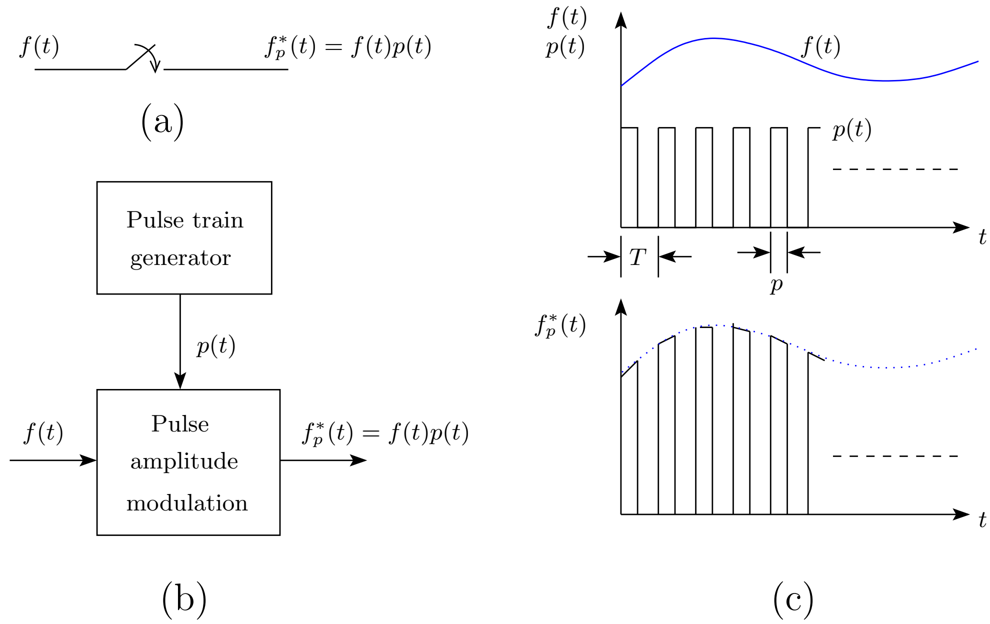



Figure 1: Finite pulse width sampler:(a)Symbolic representation (b)Block diagram (c)Operation

**Frequency domain characteristics:**

Since $p(t)$ is a periodic function, it can be represented by a Fourier series, as

$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_s t}$$

where

$w_s = \dfrac{2\pi}{T}$ is the sampling frequency and $C_n$'s are the complex Fourier series coefficients.

$$C_n = \frac{1}{T}\int_0^T p(t)e^{-jnw_s t}dt$$

Since $p(t) = 1$ for $0 \le t \le p$ and 0 for rest of the period,

$$\begin{aligned} C_n &= \frac{1}{T}\int_0^p e^{-jnw_s t}dt \\ &= \left[\frac{1}{-jnw_s T}.e^{-jnw_s t}\right]_0^p \\ &= \frac{1 - e^{-jnw_s p}}{jnw_s T} \end{aligned}$$

$C_n$ can be rearranged as,

$$\begin{aligned} C_n = &= \frac{e^{-jnw_sp/2}(e^{jnw_sp/2} - e^{-jnw_sp/2})}{jnw_sT} \\ &= \frac{2je^{-jnw_sp/2}\sin(nw_sp/2)}{jnw_sT} \\ &= \frac{p}{T}\frac{\sin(nw_sp/2)}{nw_sp/2}e^{-jnw_sp/2} \end{aligned}$$

Since $f_p^*(t)$ is also periodic, it can be written as

$$f_p^*(t) = \sum_{n=-\infty}^{\infty} C_n f(t) e^{jnw_st}$$

$$\begin{aligned} \Rightarrow F_p^*(jw) &= \mathcal{F}[f_p^*(t)], \quad \text{where } \mathcal{F} \text{ represents Fourier transform} \\ &= \int_{-\infty}^{\infty} f_p^*(t)e^{-jwt}dt \end{aligned}$$

Using complex shifting theorem of Fourier transform

$$\mathcal{F}\left[e^{jnw_st}f(t)\right] = F(jw - jnw_s)$$

$$\Rightarrow F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw - jnw_s)$$

Since $n$ is from $-\infty$ to $\infty$, the above equation can also be written as

$$F_p^*(jw) = \sum_{n=-\infty}^{\infty} C_n F(jw + jnw_s)$$

where,

$$\begin{aligned} C_o &= \lim_{n\to 0} C_n \\ &= \frac{p}{T} \\ F_p^*(jw)|_{n=0} &= C_0F(jw) = \frac{p}{T}F(jw) \end{aligned}$$

The above equation implies that the frequency contents of the original signal $f(t)$ are still present in the sampler output except that the amplitude is multiplied by the factor $\frac{p}{T}$.
For $n \neq 0$, $C_n$ is a complex quantity, the magnitude of which is,

$$|C_n| = \frac{p}{T}\left|\frac{\sin(nw_sp/2)}{nw_sp/2}\right|$$

Magnitude of $F_p^*(jw)$

$$\begin{aligned}\left|F_p^*(jw)\right| &= \left|\sum_{n=-\infty}^{\infty} C_n F(jw+jnw_s)\right| \\ &\leq \sum_{n=-\infty}^{\infty} |C_n||F(jw+jnw_s)|\end{aligned}$$

Sampling operation retains the fundamental frequency but in addition, sampler output also contains the harmonic components.

$$F(jw+jnw_s) \quad \text{for} \quad n = \pm 1, \pm 2, .....$$

According to Shannon's **sampling theorem**, "if a signal contains no frequency higher than $w_c$ rad/sec, it is completely characterized by the values of the signal measured at instants of time separated by $T = \pi/w_c$ sec."

Sampling frequency rate should be greater than the **Nyquist rate** which is twice the highest frequency component of the original signal to avoid aliasing.

If the sampling rate is less than twice the input frequency, the output frequency will be different from the input which is known as **aliasing**. The output frequency in that case is called **alias frequency** and the period is referred to as **alias period**.

The overlapping of the high frequency components with the fundamental component in the frequency spectrum is sometimes referred to as **folding** and the frequency $\frac{w_s}{2}$ is often known as **folding frequency**. The frequency $w_c$ is called **Nyquist frequency**.

A low sampling rate normally has an adverse effect on the closed loop stability. Thus, often we might have to select a sampling rate much higher than the theoretical minimum.

## 1.2 Flat-top approximation of finite-pulsewidth sampling

The Laplace transform of $f_p^*(t)$ can be written as

$$F_p^*(s) = \sum_{n=-\infty}^{\infty} \frac{1-e^{-jnw_s p}}{jnw_s T} F(s+jnw_s)$$

If the sampling duration $p$ is much smaller than the sampling period $T$ and the smallest time constant of the signal $f(t)$, the sampler output can be approximated by a sequence of rectangular pulses since the variation of $f(t)$ in the sampling duration will be less significant. Thus for $k = 0, 1, 2, ..........$, $f_p^*(t)$ can be expressed as an infinite series

$$f_p^*(t) = \sum_{k=0}^{\infty} f(kT)\left[u_s(t-kT) - u_s(t-kT-p)\right]$$

Taking Laplace transform,

$$F_p^*(s) = \sum_{k=0}^{\infty} f(kT) \left[\frac{1-e^{-ps}}{s}\right] e^{-kTs}$$

Since $p$ is very small, $e^{-ps}$ can be approximated by taking only the first 2 terms, as

$$\begin{aligned} 1-e^{-ps} &= 1-[1-ps+\frac{(ps)^2}{2!}.......] \\ &\cong ps \\ \text{Thus, } \; F_p^*(s) &\cong p\sum_{k=0}^{\infty} f(kT)e^{-kTs} \end{aligned}$$

In time domain,

$$f_p^*(t) \;=\; p\sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

where, $\delta(t)$ represents the unit impulse function. Thus the finite pulse width sampler can be viewed as an impulse modulator or an ideal sampler connected in series with an attenuator with attenuation $p$.

## 1.3 The ideal sampler

In case of an ideal sampler, the carrier signal is replaced by a train of unit impulse as shown in Figure 2. The sampling duration $p$ approaches 0, i.e., its operation is instantaneous.

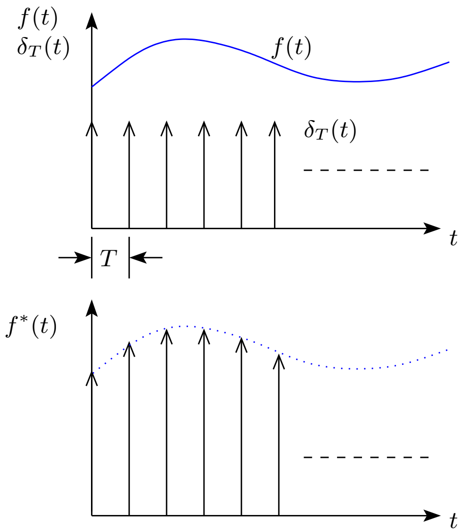



Figure 2: Ideal sampler operation

The output of an ideal sampler can be expressed as

$$\begin{aligned} f^*(t) &= \sum_{k=0}^{\infty} f(kT)\delta(t-kT) \\ \Rightarrow F^*(s) &= \sum_{k=0}^{\infty} f(kT)e^{-kTs} \end{aligned}$$

One should remember that practically the output of a sampler is always followed by a hold device which is the reason behind the name sample and hold device. Now, the output of a hold device will be the same regardless the nature of the sampler and the attenuation factor $p$ can be dropped in that case. Thus the sampling process can be be always approximated by an ideal sampler or impulse modulator.

# Module 1: Introduction to Digital Control
## Lecture Note 4

## 1 Data Reconstruction

Most of the control systems have analog controlled processes which are inherently driven by analog inputs. Thus the outputs of a digital controller should first be converted into analog signals before being applied to the systems. Another way to look at the problem is that the high frequency components of $f(t)$ should be removed before applying to analog devices. A low pass filter or a data reconstruction device is necessary to perform this operation.

In control system, hold operation becomes the most popular way of reconstruction due to its simplicity and low cost. Problem of data reconstruction can be formulated as: **" Given a sequence of numbers,** $f(0), f(T), f(2T), \cdots, f(kt), \cdots$**, a continuous time signal** $f(t),\ t \geq 0$**, is to be reconstructed from the information contained in the sequence."**

Data reconstruction process may be regarded as an extrapolation process since the continuous data signal has to be formed based on the information available at past sampling instants. Suppose the original signal $f(t)$ between two consecutive sampling instants $kT$ and $(k+1)T$ is to be estimated based on the values of $f(t)$ at previous instants of $kT$, i.e., $(k-1)T$, $(k-2)T$, $\cdots 0$.

Power series expansion is a well known method of generating the desired approximation which yields

$$\begin{aligned}
f_k(t) &= f(kT) + f^{(1)}(kT)(t-kT) + \frac{f^{(2)}(kT)}{2!}(t-kT)^2 + \ldots.. \\
\text{where, } f_k(t) &= f(t) \ \text{ for } \ kT \leq t \leq (k+1)T \ \text{ and} \\
f^{(n)}(kT) &= \frac{d^n f(t)}{dt^n}\Big|_{t=kT} \quad \text{for } \ n = 1, 2, \ldots
\end{aligned}$$

Since the only available information about $f(t)$ is its magnitude at the sampling instants, the derivatives of $f(t)$ must be estimated from the values of $f(kT)$, as

$$\begin{aligned}
f^{(1)}(kT) &\cong \frac{1}{T}[f(kT) - f((k-1)T)] \\
\text{Similarly, } \quad f^{(2)}(kT) &\cong \frac{1}{T}[f^{(1)}(kT) - f^{(1)}((k-1)T)] \\
\text{where, } \quad f^{(1)}((k-1)T) &\cong \frac{1}{T}[f((k-1)T) - f((k-2)T)]
\end{aligned}$$

## 1.1 Zero Order Hold

Higher the order of the derivatives to be estimated is, larger will be the number of delayed pulses required. Since time delay degrades the stability of a closed loop control system, using higher order derivatives of $f(t)$ for more accurate reconstruction often causes serious stability problem. Moreover a high order extrapolation requires complex circuitry and results in high cost.

For the above reasons, use of only the first term in the power series to approximate $f(t)$ during the time interval $kT \leq t < (k+1)T$ is very popular and the device for this type of extrapolation is known as zero-order extrapolator or zero order hold. It holds the value of $f(kT)$ for $kT \leq t < (k+1)T$ until the next sample $f((k+1)T)$ arrives. Figure 1 illustrates the operation of a ZOH where the green line represents the original continuous signal and brown line represents the reconstructed signal from ZOH.

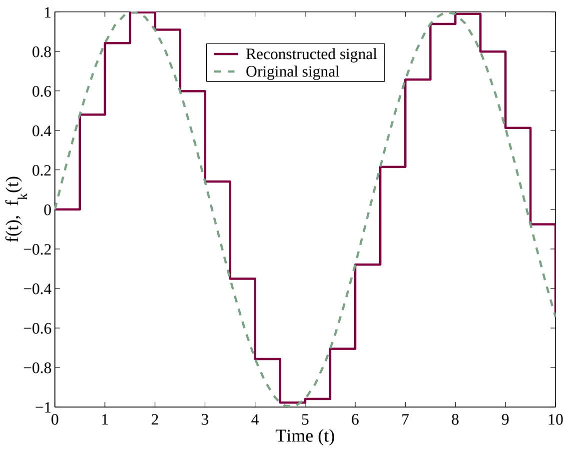



Figure 1: Zero order hold operation

The accuracy of zero order hold (ZOH) depends on the sampling frequency. When $T \to 0$, the output of ZOH approaches the continuous time signal. Zero order hold is again a linear device which satisfies the principle of superposition.

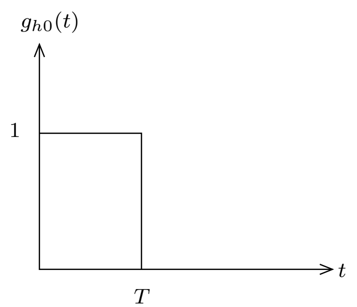



Figure 2: Impulse response of ZOH

The **impulse response** of a ZOH, as shown in Figure 2, can be written as

$$\begin{aligned}
g_{ho}(t) &= u_s(t) - u_s(t-T) \\
\Rightarrow G_{ho}(s) &= \frac{1-e^{-TS}}{s} \\
G_{ho}(jw) &= \frac{1-e^{-jwT}}{jw} = T\,\frac{\sin(wT/2)}{wT/2}e^{-jwT/2}
\end{aligned}$$

Since $T = \dfrac{2\pi}{w_s}$, we can write

$$G_{ho}(jw) = \frac{2\pi}{w_s}\frac{\sin(\pi w/w_s)}{\pi w/w_s}\,e^{-j\pi w/w_s}$$

Magnitude of $G_{ho}(jw)$:

$$|G_{ho}(jw)| = \frac{2\pi}{w_s}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|$$

Phase of $G_{ho}(jw)$:

$$\angle G_{h0}(jw) = \angle\sin(\pi w/w_s) - \frac{\pi w}{w_s} \quad \text{rad}$$

The sign of $\angle\sin(\pi w/w_s)$ changes at every integral value of $\dfrac{\pi w}{w_s}$. The change of sign from $+$ to $-$ can be regarded as a phase change of $-180^0$. Thus the phase characteristics of ZOH is linear with jump discontinuities of $-180^0$ at integral multiple of $w_s$. The magnitude and phase characteristics of ZOH are shown in Figure 3.

At the cut off frequency $w_c = \dfrac{w_s}{2}$, magnitude is 0.636. When compared with an ideal low pass filter, we see that instead of cutting of sharply at $w = \frac{w_s}{2}$, the amplitude characteristics of $G_{ho}(jw)$ is zero at $\dfrac{w_s}{2}$ and integral multiples of $w_s$.

## 1.2 First Order Hold

When the $1^{st}$ two terms of the power series are used to extrapolate $f(t)$, over the time interval $kT < t < (k+1)T$, the device is called a first order hold (FOH). Thus

$$\begin{aligned}
f_k(t) &= f(kT) + f^1(kT)(t-kT) \\
\text{where, } f^1(kT) &= \frac{f(kT) - f((k-1)T)}{T} \\
\Rightarrow f_k(t) &= f(kT) + \frac{f(kT)-f((k-1)T)}{T}(t-kT)
\end{aligned}$$

Impulse response of FOH is obtained by applying a unit impulse at $t = 0$, the corresponding output is obtained by setting $k = 0, 1, 2, \ldots..$

$$\text{for } k=0,\ \text{when } 0 \le t < T, \quad f_0(t) = f(0) + \frac{f(0)-f(-T)}{T}t$$

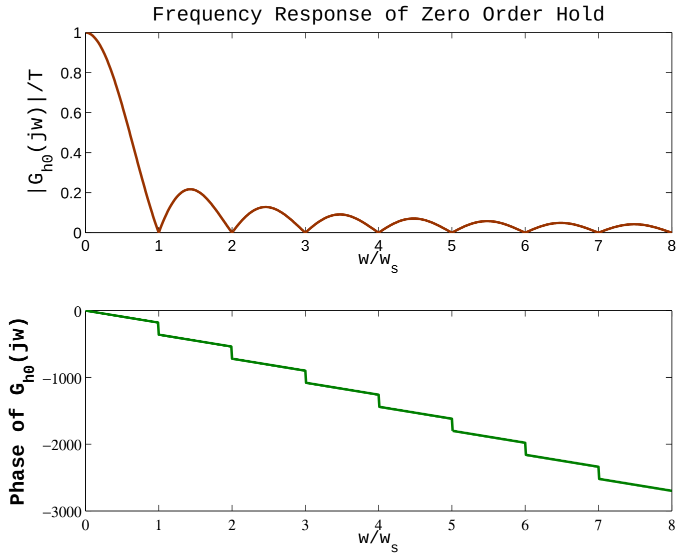



Figure 3: Frequency response of ZOH

$f(0) = 1$ [impulse unit] $f(-T) = 0$ $f_{h1}(t) = 1 + \dfrac{t}{T}$ in this region. When $T \leq t < 2T$

$$f_1(t) = f(T) + \frac{f(T) - f(0)}{T}(t - T)$$

Since, $f(T) = 0$ and $f(0) = 1$, $f_{h1}(t) = 1 - \dfrac{t}{T}$ in this region. $f_{h1}(t)$ is 0 for $t \geq 2T$, since $f(t) = 0$ for $t \geq 2T$. Figure 4 shows the impulse response of first order hold.

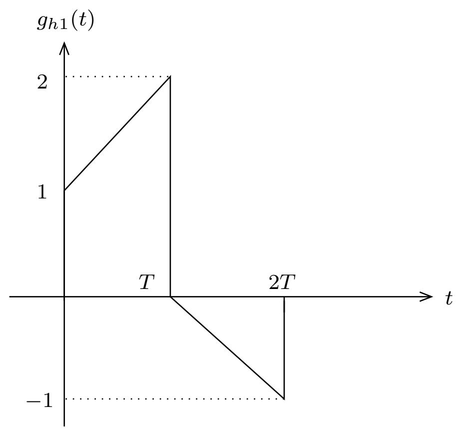



Figure 4: Impulse response of First Order Hold

If we combine all three regions, we can write the impulse response of a first order hold as,

$$\begin{aligned} g_{h1}(t) &= (1 + \frac{t}{T})u_s(t) + (1 - \frac{t}{T})u_s(t - T) - (1 + \frac{t}{T})u_s(t - T) - (1 - \frac{t}{T})u_s(t - 2T) \\ &= (1 + \frac{t}{T})u_s(t) - 2\frac{t}{T}u_s(t - T) - (1 - \frac{t}{T})u_s(t - 2T) \end{aligned}$$

One can verify that according to the above expression, when $0 \le t < T$, only the first term produces a nonzero value which is nothing but $(1 + t/T)$. Similarly, when $T \le t < 2T$, first two terms produce nonzero values and the resultant is $(1 - t/T)$. In case of $t \ge 2T$, all three terms produce nonzero values and the resultant is 0.

The transfer function of a first order hold is:

$$G_{h1}(s) = \frac{1+Ts}{T}\left[\frac{1-e^{-Ts}}{s}\right]^2$$

**Frequency Response** $\quad G_{h1}(jw) = \dfrac{1+jwT}{T}\left[\dfrac{1-e^{-jwT}}{s}\right]^2$

$$\begin{aligned}
\text{Magnitude: } |G_{h1}(jw)| &= \left|\frac{1+jwT}{T}\right| |G_{h0}(jw)|^2 \\
&= \frac{2\pi}{w_s}\sqrt{1+\frac{4\pi^2 w^2}{w_s^2}}\left|\frac{\sin(\pi w/w_s)}{\pi w/w_s}\right|^2 \\
\text{Phase: } \angle G_{h1}(jw) &= \tan^{-1}(2\pi w/w_s) - 2\pi w/w_s \quad \text{rad}
\end{aligned}$$

The frequency response is shown in Figure 5.

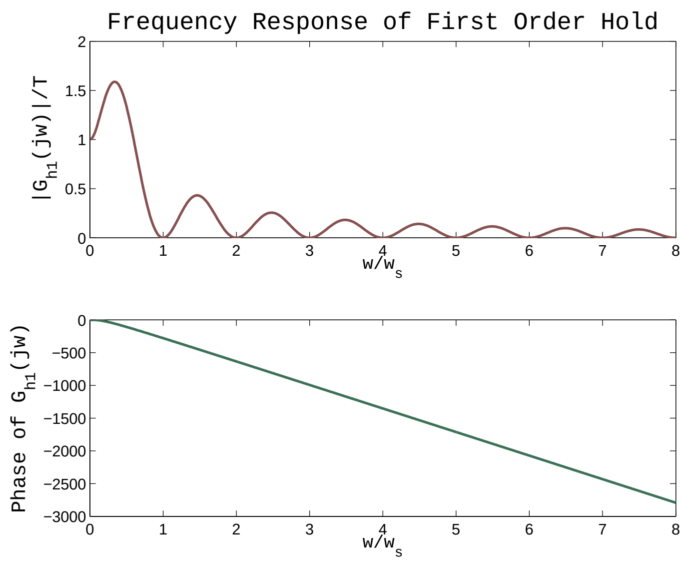



Figure 5: Frequency response of FOH

Figure 6 shows a comparison of the reconstructed outputs of ZOH and FOH.

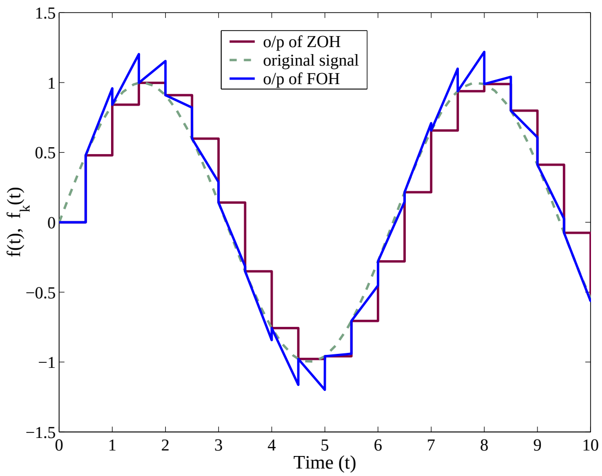



Figure 6: Operation of ZOH and FOH

# Module 2: Modeling Discrete Time Systems by Pulse Transfer Function
# Lecture Note 1

## 1 Motivation for using Z-transform

In general, control system design methods can be classified as:

**conventional or classical control techniques**

**modern control techniques**

Classical methods use transform techniques and are based on transfer function models, whereas modern techniques are based on modeling of system by state variable methods.

Laplace transform is the basic tool of the classical methods in continuous domain. In principle, it can also be used for modeling digital control systems. However typical Laplace transform expressions of systems with digital or sampled signals contain exponential terms in the form of $e^{Ts}$ which makes the manipulation in the Laplace domain difficult. This can be regarded as a motivation of using Z-transform.

Let the output of an ideal sampler be denoted by $f^*(t)$.

$$\begin{aligned} L[f^*(t)] &= F^*(s) \\ &= \sum_{k=0}^{\infty} f(kT)e^{-kTs} \end{aligned}$$

Since $F^*(s)$ contains the term $e^{-kTs}$, it is not a rational function of $s$. When terms with $e^{-Ts}$ appear in a transfer function other than a multiplying factor, difficulties arise while taking the inverse Laplace. It is desirable to transfer the irrational function $F^*(s)$ to a rational function for which one obvious choice is:

$$\begin{aligned} z &= e^{Ts} \\ \Rightarrow s &= \frac{1}{T} ln\ z \end{aligned}$$

If, $s = \sigma + jw$,

$$\begin{aligned} Re\ z &= e^{T\sigma} \cos wT \\ Im\ z &= e^{T\sigma} \sin wT \end{aligned}$$

Z-transform:

$$\begin{aligned} F^*\left[s = \frac{1}{T} ln\ z\right] &= F(z) \\ &= \sum_{k=0}^{\infty} f(kT) z^{-k} \end{aligned}$$

$F(z)$ is the Z-transform of $f(t)$ at the sampling instants $k$.

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

In general, we can say that if $f(t)$ is Laplace transformable then it also has a Z-transform.

$$\begin{aligned} L[f(t)] &= \int_0^{\infty} f(t) e^{-st} dt \\ Z[f(t)] &= \sum_{k=0}^{\infty} f(kT) z^{-k} \end{aligned}$$

## 2 Revisiting Z-Transforms

Z-transform is a powerful operation method to deal with discrete time systems. In considering Z-transform of a time function $x(t)$, we consider only the sampled values of $x(t)$, i.e., $x(0), x(T), x(2T)$....... where $T$ is the sampling period.

$$\begin{aligned} X(z) = Z[x(t)] &= Z[x(kT)] \\ &= \sum_{k=0}^{\infty} x(kT) z^{-k} \end{aligned}$$

For a sequence of numbers $x(k)$

$$\begin{aligned} X(z) &= Z[x(k)] \\ &= \sum_{k=0}^{\infty} x(k) z^{-k} \end{aligned}$$

The above transforms are referred to as one sided z-transform. In one sided z-transform, we assume that $x(t) = 0$ for $t < 0$ or $x(k) = 0$ for $k < 0$. In two sided z-transform, we assume that $-\infty < t < \infty$ or $k =, \pm 1, \pm 2, \pm 3, \ldots\ldots..$

$$\begin{aligned} X(z) &= Z[x(kT)] \\ &= \sum_{k=-\infty}^{\infty} x(kT) z^{-k} \end{aligned}$$

or for $x(k)$

$$\begin{aligned} X(z) &= Z[x(k)] \\ &= \sum_{k=-\infty}^{\infty} x(k) z^{-k} \end{aligned}$$

The one sided z-transform has a convenient closed form solution in its region of convergence (ROC) for most engineering applications. Whenever $X(z)$, an infinite series in $z^{-1}$, converges outside the circle $|z| = R$, where $R$ is the radius of absolute convergence, it is not needed each time to specify the values of $z$ over which $X(z)$ is convergent.

$$|z| > R \Rightarrow convergent$$
$$|z| < R \Rightarrow divergent.$$

In one sided z-transform theory, while sampling a discontinuous function $x(t)$, we assume that the function is continuous from the right, i.e., if discontinuity occurs at 0 we assume that $x(0) = x(0+)$.

## 2.1 Z-Transforms of some elementary functions

**Unit step function** is defined as:

$$\begin{aligned} u_s(t) &= 1, \quad \text{for} \quad t \geq 0 \\ &= 0, \quad \text{for} \quad t < 0 \end{aligned}$$

Assuming that the function is continuous from right

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty} u_s(kT) z^{-k} \\ &= \sum_{k=0}^{\infty} z^{-k} \\ &= 1 + z^{-1} + z^{-2} + z^{-3} + \ldots\ldots\ldots \\ &= \frac{1}{1 - z^{-1}} \\ &= \frac{z}{z - 1} \end{aligned}$$

The above series converges if $|z| > 1$.
One should note that the Unit step sequence is defined as

$$\begin{aligned} u_s(k) &= 1, \quad \text{for} \quad k = 0, 1, 2 \cdots \\ &= 0, \quad \text{for} \quad k < 0 \end{aligned}$$

with a same Z-transform.

**Unit ramp function** is defined as:

$$\begin{aligned} u_r(t) &= t, \quad \text{for} \quad t \geq 0 \\ &= 0, \quad \text{for} \quad t < 0 \end{aligned}$$

The Z-transform is:

$$\begin{aligned} U_r(z) &= \frac{Tz}{(z-1)^2} \\ &= T\frac{z^{-1}}{(1-z^{-1})^2} \end{aligned}$$

with ROC $|z| > 1$.

For a **polynomial function** $x(k) = a^k$, the Z-transform is:

$$\begin{aligned} X(z) &= \frac{1}{1-a.z^{-1}} \\ &= \frac{z}{z-a} \end{aligned}$$

where ROC is $|z| > a$.

**Exponential function** is defined as:

$$\begin{aligned} x(t) &= e^{-at}, \quad \text{for} \quad t \geq 0 \\ &= 0, \quad \text{for} \quad t < 0 \end{aligned}$$

We have $x(kT) = e^{-akT}$ for $k = 0, 1, 2 \cdots$. Thus,

$$\begin{aligned} X(z) &= \frac{1}{1-e^{-aT}z^{-1}} \\ &= \frac{z}{z-e^{-aT}} \end{aligned}$$

Similarly Z-transforms can be computed for sinusoidal and other compound functions. One should refer the Z-transform table provided in the appendix.

## 2.2 Properties of Z- transform

1. **Multiplication by a constant:** $Z[ax(t)] = aX(z)$, where $X(z) = Z[x(t)]$.
2. **Linearity:** If $x(k) = \alpha f(k) \pm \beta g(k)$, then $X(z) = \alpha F(z) \pm \beta G(z)$.
3. **Multiplication by $a^k$:** $Z[a^k x(k)] = X(a^{-1}z)$
4. **Real shifting:** $Z[x(t-nT)] = z^{-n}X(z)$ and $z[x(t+nT)] = z^n\left[X(z) - \sum_{k=0}^{n-1} x(kT)z^{-k}\right]$

5. **Complex shifting:** $Z[e^{\pm at}x(t)] = X(ze^{\mp aT})$

6. **Initial value theorem:**

$$x(0) = \lim_{z\to\infty} X(z)$$

7. **Final value theorem:**

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}[(1-z^{-1})X(z)]$$

## 2.3 Inverse Z-transforms

Single sided Laplace transform and its inverse make a unique pair,i.e. if $F(s)$ is the Laplace transform of $f(t)$, then $f(t)$ is the inverse Laplace transform of $F(s)$. But the same is not true for Z-transform. Say $f(t)$ is the continuous time function whose Z-transform is $F(z)$ then the inverse transform is not necessarily equal to $f(t)$, rather it is equal to $f(kT)$ which is equal to $f(t)$ only at the sampling instants. Once $f(t)$ is sampled by an the ideal sampler, the information between the sampling instants is totally lost and we cannot recover actual $f(t)$ from $F(z)$,

$$\Rightarrow f(kT) = Z^{-1}[F(z)]$$

The transform can be obtained by using
→ Partial fraction expansion
→ Power series
→ Inverse formula.
Inverse Z-transform formula:

$$f(kT) = \frac{1}{2\pi j}\oint_{\Gamma} F(z)z^{k-1}dz$$

## 2.4 Other Z-transform properties

**Partial differentiation theorem:**

$$Z\left[\frac{\partial}{\partial a}[f(t,a)]\right] = \frac{\partial}{\partial a}F(z,a)$$

**Real convolution theorem:**
If $f_1(t)$ and $f_2(t)$ have z-transforms $F_1(z)$ and $F_2(z)$ and $f_1(t) = 0 = f_2(t)$ for $t < 0$, then

$$F_1(z)F_2(z) = Z\left[\sum_{n=0}^{k} f_1(nT)f_2(kT-nT)\right]$$

**Complex convolution:**

$$Z[f_1(t)f_2(t)] = \frac{1}{2\pi j}\oint_{\Gamma}\frac{F_1(\xi)F_2(z\xi^{-1})}{\xi}d\xi$$

$\Gamma$ : circle / closed path in z-plane which lie in the region $\sigma < |\xi| < \dfrac{|z|}{\sigma_2}$

$\sigma_1$: radius of convergence of$F_1(\xi)$
$\sigma_1$: radius of convergence of $F_2(\xi)$

## 2.5 Limitation of Z-transform method

1. Ideal sampler assumption
   $\Rightarrow$ z-transform represents the function only at sampling instants.

2. Non uniqueness of z-transform.

3. Accuracy depends on the magnitude of the sampling frequency $w_s$ relative to the highest frequency component contained in the function $f(t)$.

4. A good approximation of $f(t)$ can only be interpolated from $f(kT)$, the inverse z-transform of $F(z)$ by connecting $f(kT)$ with a smooth curve.

## 2.6 Application of Z-transform in solving Difference Equation

One of the most important applications of Z-transform is in the solution of linear difference equations. Let us consider that a discrete time system is described by the following difference equation.

$$y(k+2) + 0.5y(k+1) + 0.06y(k) = -(0.5)^{k+1}$$

with the initial conditions $y(0) = 0,\ y(1) = 0$. We have to find the solution $y(k)$ for $k > 0$. Taking z-transform on both sides of the above equation:

$$\begin{aligned} z^2Y(z) + 0.5zY(z) + 0.06Y(z) &= -0.5\frac{z}{z-0.5} \\ \text{or, } Y(z) &= -\frac{0.5z}{(z-0.5)(z^2+0.5z+0.06)} \\ &= -\frac{0.5z}{(z-0.5)(z+0.2)(z+0.3)} \end{aligned}$$

Using partial fraction expansion:

$$\begin{aligned} Y(z) &= -\frac{0.8937z}{z-0.5} + \frac{7.143z}{z+0.2} - \frac{6.25z}{z+0.3} \\ \text{Taking Inverse Laplace: } y(k) &= -0.893(0.5)^k + 7.143(-0.2)^k - 6.25(-0.3)^k \end{aligned}$$

To emphasize the fact that $y(k) = 0$ for $k < 0$, it is a common practice to write the solution as:

$$y(k) = -0.893(0.5)^k u_s(k) + 7.143(-0.2)^k u_s(k) - 6.25(-0.3)^k u_s(k)$$

where $u_s(k)$ is the unit step sequence.

# Module 2: Modeling Discrete Time Systems by Pulse Transfer Function
# Lecture Note 2

## 1 Relationship between s-plane and z-plane

In the analysis and design of continuous time control systems, the pole-zero configuration of the transfer function in s-plane is often referred. We know that:

- Left half of s-plane $\Rightarrow$ Stable region.
- Right half of s-plane $\Rightarrow$ Unstable region.

For relative stability again the left half is divided into regions where the control loop transfer function poles should preferably be located.

Similarly the poles and zeros of a transfer function in z-domain govern the performance characteristics of a digital system.

One of the properties of $F^*(s)$ is that it has an infinite number of poles, located periodically with intervals of $\pm m w_s$ with $m = 0, 1, 2, .....$, in the s-plane where $w_s$ is the sampling frequency in rad/sec.

If the primary strip is considered, the path, as shown in Figure 1, will be mapped into a unit circle in the z-plane, centered at the origin. The mapping is shown in Figure 2.

Since

$$\begin{aligned} e^{(s+jmw_s)T} &= e^{Ts} e^{j2\pi m} \\ &= e^{Ts} \\ &= z \end{aligned}$$

where $m$ is an integer, all the complementary strips will also map into the unit circle.

### 1.1 Mapping guidelines

1. All the points in the left half s-plane correspond to points inside the unit circle in z-plane.
2. All the points in the right half of the s-plane correspond to points outside the unit circle.

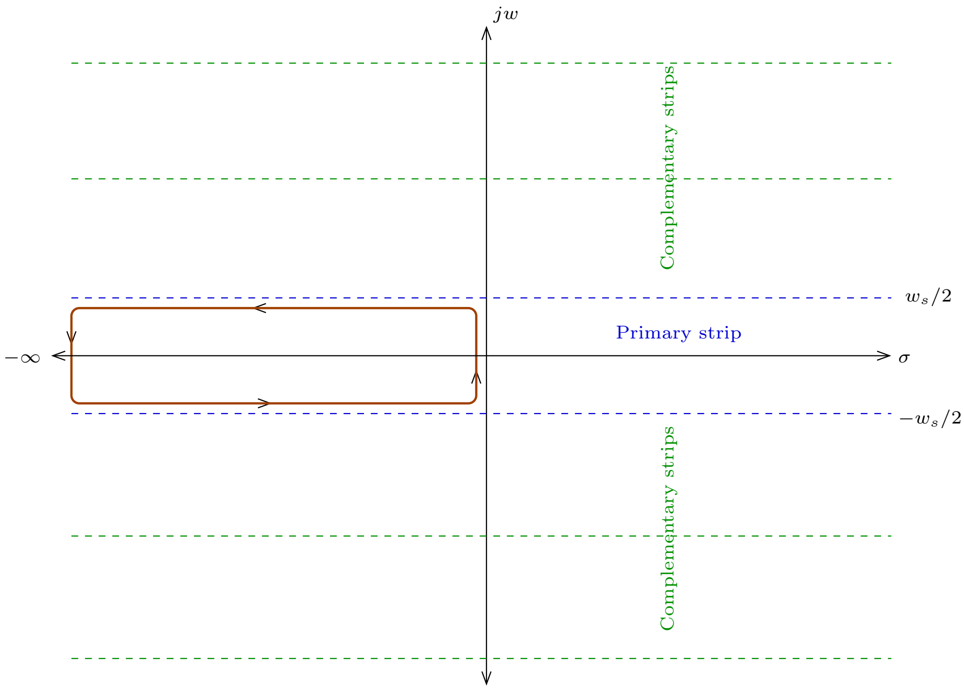



Figure 1: Primary and complementary strips in s-plane

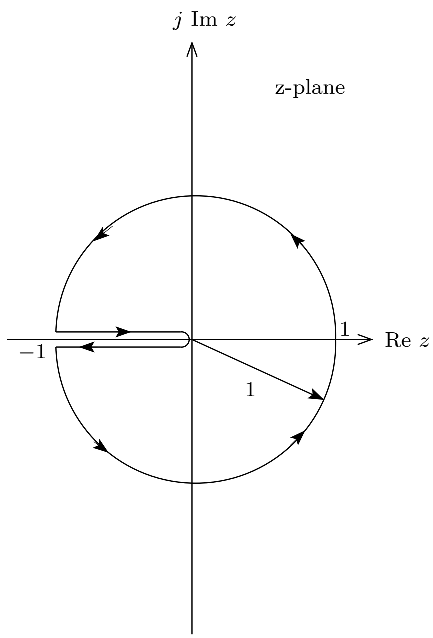



Figure 2: Mapping of primary strip in z-plane

3. Points on the $jw$ axis in the s-plane correspond to points on the unit circle $|z| = 1$ in the

z-plane.

$$\begin{aligned} s &= jw \\ z &= e^{Ts} \\ &= e^{jwT} \Rightarrow \text{ magnitude } = 1 \end{aligned}$$

### 1.2 Constant damping loci, constant frequency loci and constant damping ratio loci

**Constant damping loci:** The real part $\sigma$ of a pole, $s = \sigma + jw$, of a transfer function in s-domain, determines the damping factor which represents the rate of rise or decay of time response of the system.

- Large $\sigma$ represents small time constant and thus a faster decay or rise and vice versa.
- The loci in the left half s-plane (vertical line parallel to $jw$ axis as in Figure 3(a)) denote positive damping since the system is stable
- The loci in the right half s-plane denote negative damping.
- Constant damping loci in the z-plane are concentric circles with the center at $z = 0$, as shown in Figure 3(b).
- Negative damping loci map to circles with radii $> 1$ and positive damping loci map to circles with radii $< 1$.

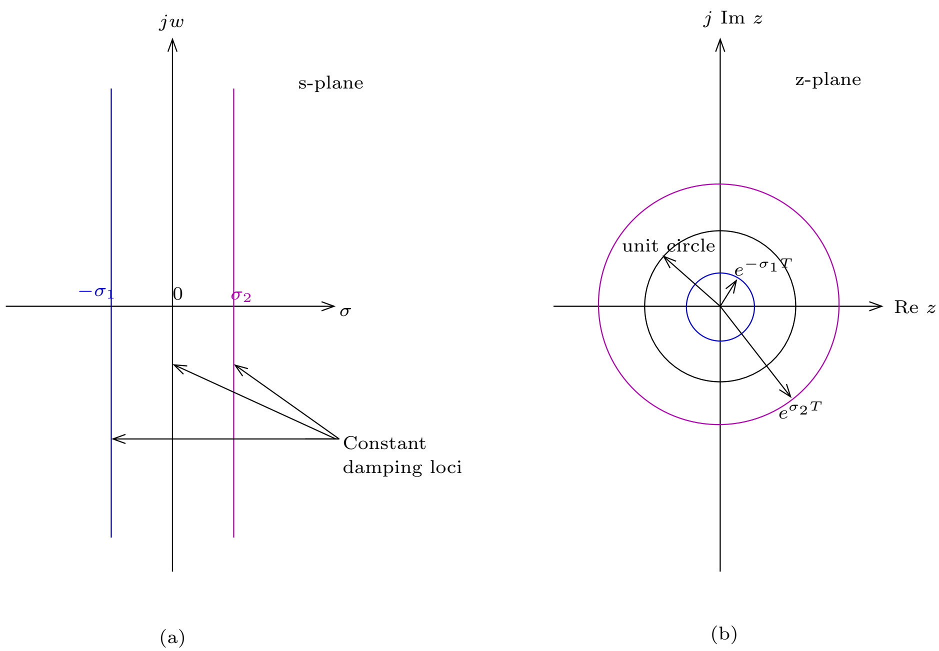



Figure 3: Constant damping loci in (a) s-plane and (b) z-plane

**Constant frequency loci:** These are horizontal lines in s-plane, parallel to the real axis as shown in Figure 4(a).

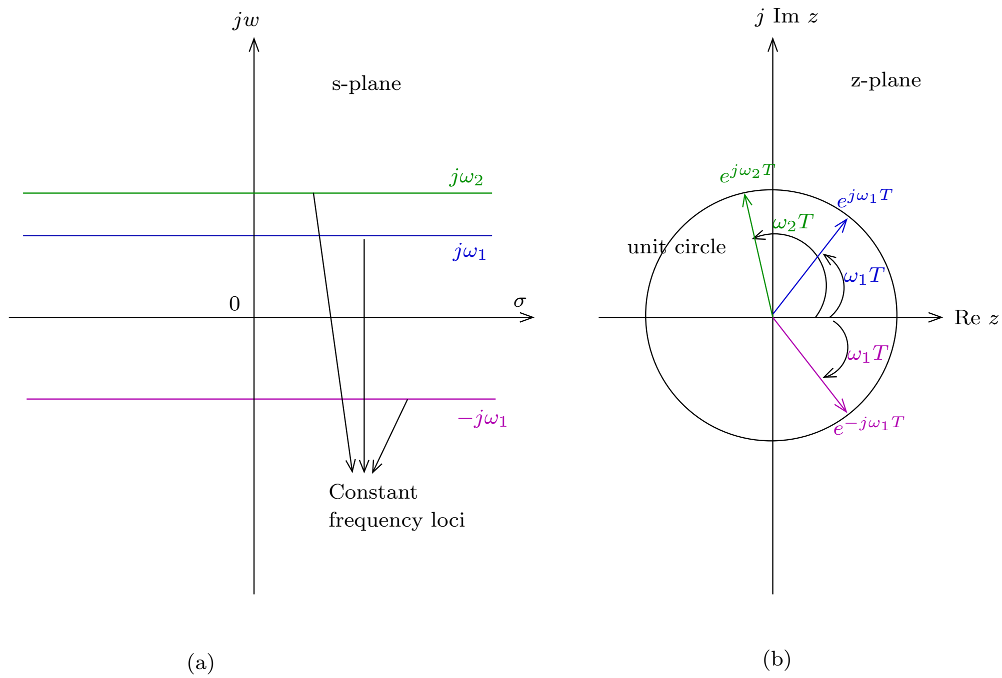



Figure 4: Constant frequency loci in (a) s-plane and (b) z-plane

Corresponding Z-transform:

$$\begin{aligned} z &= e^{Ts} \\ &= e^{jwT} \end{aligned}$$

When $w =$ constant, it represents a straight line from the origin at an angle of $\theta = wT$ rad, measured from positive real axis as shown in Figure 4(b).

**Constant damping ratio loci:** If $\xi$ denotes the damping ratio:

$$\begin{aligned} s &= -\xi w_n \pm j w_n \sqrt{1-\xi^2} \\ &= -\frac{w}{\sqrt{1-\xi^2}}\xi \pm jw \\ &= -w\tan\beta \pm jw \end{aligned}$$

where $w_n$ is the natural undamped frequency and $\beta = \sin^{-1}\xi$. If we take Z-transform

$$\begin{aligned} z &= e^{T(-w tan\beta + jw)} \\ &= e^{-2\pi w \tan\beta / w_s} \angle(2\pi w/w_s) \end{aligned}$$

For a given $\xi$ or $\beta$, the locus in s-plane is shown in Figure 5(a). In z-plane, the corresponding locus will be a logarithmic spiral as shown in Figure 5(b), except for $\xi = 0$ or $\beta = 0^o$ and $\xi = 1$ or $\beta = 90^o$.

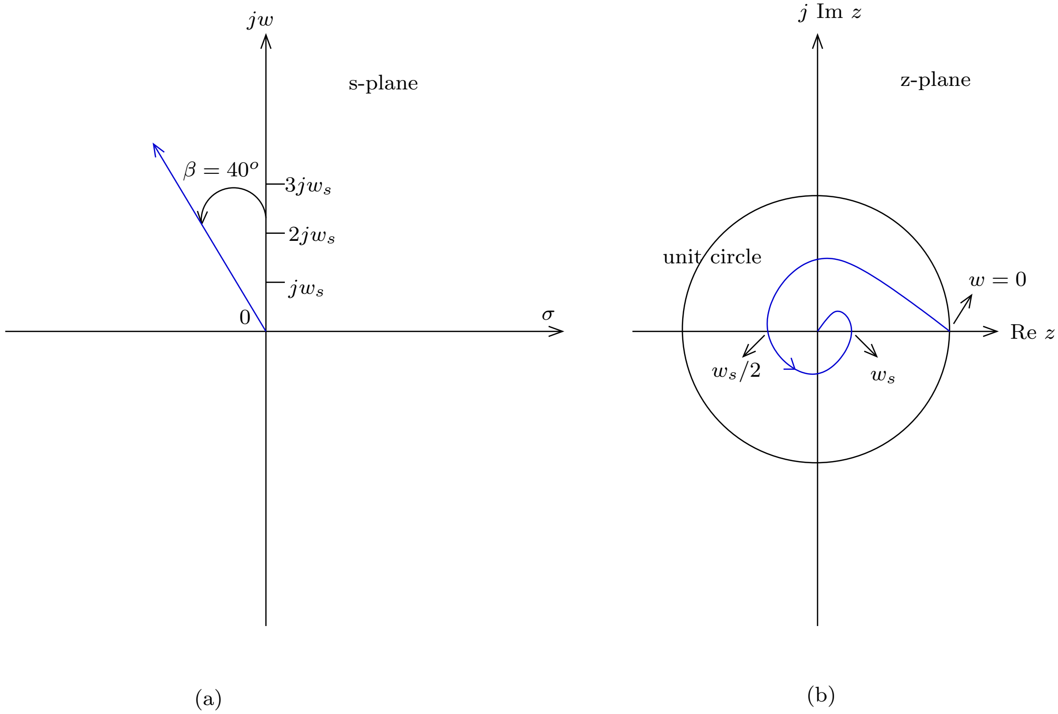



Figure 5: Constant damping ratio locus in (a) s-plane and (b) z-plane

# Module 2: Modeling Discrete Time Systems by Pulse Transfer Function
# Lecture Note 3

## 1 Pulse Transfer Function

Transfer function of an LTI (Linear Time Invariant) continuous time system is defined as

$$G(s) = \frac{C(s)}{R(s)}$$

where $R(s)$ and $C(s)$ are Laplace transforms of input $r(t)$ and output $c(t)$. We assume that initial condition are zero.

Pulse transfer function relates z-transform of the output at the sampling instants to the Z-transform of the sampled input. When the same system is subject to a sampled data or digital signal $r^*(t)$, the corresponding block diagram is given in Figure 1.

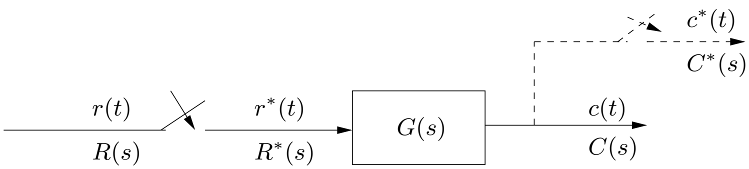



Figure 1: Block diagram of a system subject to a sampled input

The output of the system is $C(s) = G(s)R^*(s)$. The transfer function of the above system is difficult to manipulate because it contains a mixture of analog and digital components. Thus, it is desirable to express the system characteristics by a transfer function that relates $r^*(t)$ to $c^*(t)$, a fictitious sampler output as shown in Figure 1. One can then write:

$$\begin{aligned} C^*(s) &= \sum_{k=0}^{\infty} c(kT)e^{-kTs} \\ \text{Since } c(kT) \text{ is periodic, } \ C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jnw_s) \ \text{ with } c(0)=0 \end{aligned}$$

The detailed derivation of the above expression is omitted. Similarly,

$$\begin{aligned} R^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s+jnw_s) \\ \text{Again, } C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} C(s+jnw_s) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s+jnw_s)G(s+jnw_s) \end{aligned}$$

Since $R^*(s)$ is periodic $R^*(s+jnW_s) = R^*(s)$. Thus

$$\begin{aligned} C^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} R^*(s)G(s+jnw_s) \\ &= R^*(s)\frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s) \end{aligned}$$

If we define $G^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s)$, then $C^*(s) = R^*(s)G^*(s)$.

$$G^*(s) = \frac{C^*(s)}{R^*(s)}$$

is known as **pulse transfer function**. Sometimes it is also referred to as the **starred transfer function**. If we now substitute $z = e^{Ts}$ in the previous expression we will directly get the **z-transfer function** $G(z)$ as

$$G(z) = \frac{C(z)}{R(z)}$$

$G(z)$ can also be defined as

$$G(z) = \sum_{k=0}^{\infty} g(kT)z^{-k}$$

where $g(kT)$ denotes the sequence of the impulse response $g(t)$ of the system of transfer function $G(s)$. The sequence $g(kT), k = 0, 1, 2, ..$ is also known as impulse sequence.

**Overall Conclusion**

1. Pulse transfer function or z-transfer function characterizes the discrete data system responses only at sampling instants. The output information between the sampling instants is lost.

2. Since the input of discrete data system is described by output of the sampler, for all practical purposes the samplers can be simply ignored and the input can be regarded as $r^*(t)$.

**Alternate way to arrive at $G(z) = \dfrac{C(z)}{R(z)}$:**

$$\begin{aligned} c^*(t) &= g^*(t)\big|_{\text{when } r^*(t) \text{ is an impulse function}} \\ &= \sum_{k=0}^{\infty} g(kT)\delta(t-kT) \end{aligned}$$

When the input is $r^*(t)$,

$$\begin{aligned} c(t) &= r(0)g(t) + r(T)g(t-T) + ... \\ \Rightarrow c(kT) &= r(0)g(kT) + r(T)g((k-1)T) + ... \\ \Rightarrow c(kT) &= \sum_{n=0}^{k} r(nT)g(kT-nT) \\ \Rightarrow C(z) &= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{k} r(nT)g(kT-nT)z^{-k} \end{aligned}$$

Using real convolution theorem

$$\begin{aligned} C(z) = R(z)G(z) \\ \Rightarrow G(z) = \frac{C(z)}{R(z)} \end{aligned}$$

## 1.1 Pulse transfer of discrete data systems with cascaded elements

Care must be taken when the discrete data system has cascaded elements. Following two cases will be considered here.

- Cascaded elements are separated by a sampler
- Cascaded elements are not separated by a sampler

The block diagram for the first case is shown in Figure 2.

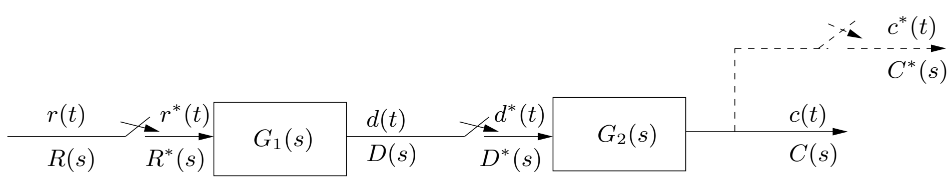



Figure 2: Discrete data system with cascaded elements, separated by a sampler

The input-output relations of the two systems $G_1$ and $G_2$ are described by

$$D(z) = G_1(z)R(z)$$

and

$$C(z) = G_2(z)D(z)$$

Thus the input-output relation of the overall system is

$$C(z) = G_1(z)G_2(z)R(z)$$

We can therefore conclude that the z-transfer function of two linear system separated by a sampler are the products of the individual z-transfer functions.

Figure 3 shows the block diagram for the second case. The continuous output $C(s)$ can be

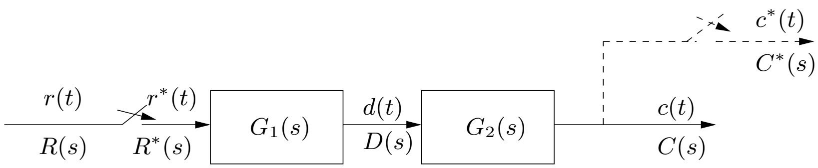



Figure 3: Discrete data system with cascaded elements, not separated by a sampler

written as

$$C(s) = G_1(s)G_2(s)R^*(s)$$

The output of the fictitious sampler is

$$C(z) = Z\left[G_1(s)G_2(s)\right]R(z)$$

z-transform of the product $G_1(s)G_2(s)$ is denoted as

$$Z\left[G_1(s)G_2(s)\right] = G_1G_2(z) = G_2G_1(z)$$

One should note that in general $G_1G_2(z) \neq G_1(z)G_2(z)$, except for some special cases. The overall output is thus,

$$C(z) = G_1G_2(z)R(z)$$

## 1.2 Pulse transfer function of ZOH

As derived in lecture 4 of module 1, transfer function of zero order hold is

$$G_{ho}(s) = \frac{1 - e^{-Ts}}{s}$$

$$\begin{aligned} \Rightarrow \text{Pulse transfer function } G_{ho}(z) &= Z\left[\frac{1 - e^{-Ts}}{s}\right] \\ &= (1 - z^{-1})Z\left[\frac{1}{s}\right] \\ &= (1 - z^{-1})\frac{z}{z-1} \\ &= 1 \end{aligned}$$

This result is expected because zero order hold simply holds the discrete signal for one sampling period, thus taking z-transform of ZOH would revert back its original sampled signal.

A common situation in discrete data system is that a sample and hold (S/H) device precedes a linear system with transfer function $G(s)$ as shown in Figure 4. We are interested in finding the transform relation between $r^*(t)$ and $c^*(t)$.

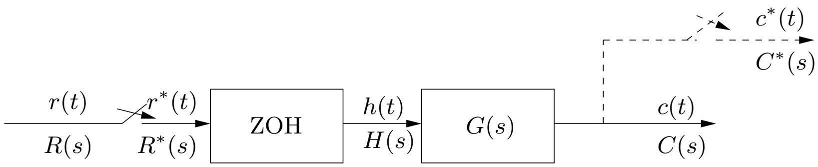



Figure 4: Block diagram of a system subject to a sample and hold process

z-transform of output $c(t)$ is

$$
\begin{aligned}
C(z) &= Z\left[G_{ho}(s)G(s)\right]R(z) \\
&= Z\left[\frac{1-e^{-Ts}}{s}G(s)\right]R(z) \\
&= (1-z^{-1})Z\left[\frac{G(s)}{s}\right]R(z)
\end{aligned}
$$

where $(1-z^{-1})Z\left[\frac{G(s)}{s}\right]$ is the z-transfer function of an S/H device and a linear system.

It was mentioned earlier that when sampling frequency reaches infinity a discrete data system may be regarded as a continuous data system. However, this does not mean that if the signal $r(t)$ is sampled by an ideal sampler then $r^*(t)$ can be reverted to $r(t)$ by setting the sampling time $T$ to zero. This simply bunches all the samples together. Rather, if the output of the sampled signal is passed through a hold device then setting the sampling time $T$ to zero the original signal $r(t)$ can be recovered. In relation with Figure 4,

$$\lim_{T\to 0} H(s) = R(s)$$

**Example**
Consider that the input is $r(t) = e^{-at}u_s(t)$, where $u_s(t)$ is the unit step function.

$$\Rightarrow R(s) = \frac{1}{s+a}$$

Laplace transform of sampled signal $r^*(t)$ is

$$R^*(s) = \frac{e^{Ts}}{e^{Ts} - e^{-aT}}$$

Laplace transform of the output after the ZOH is

$$\begin{aligned} H(s) &= G_{ho}(s)R^*(s) \\ &= \frac{1-e^{-Ts}}{s} \cdot \frac{e^{Ts}}{e^{Ts}-e^{-aT}} \end{aligned}$$

When $T \to 0$,

$$\lim_{T\to 0} H(s) = \lim_{T\to 0} \frac{1-e^{-Ts}}{s} \frac{e^{Ts}}{e^{Ts}-e^{-aT}}$$

The limit can be calculated using **L' hospital's rule**. It says that:

*If* $\lim_{x\to a} f(x) = 0/\infty$ *and if* $\lim_{x\to a} g(x) = 0/\infty$, *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

For the given example, $x = T$, $f(T) = \dfrac{1-e^{-Ts}}{s}$ and $g(T) = \dfrac{e^{Ts}-e^{-aT}}{e^{Ts}}$. Both the expressions approach zero as $T \to 0$. So,

$$\begin{aligned} H(s) &= \lim_{T\to 0} \frac{f(T)}{g(T)} \\ &= \lim_{T\to 0} \frac{f'(T)}{g'(T)} \\ &= \lim_{T\to 0} \frac{e^{Ts}}{(s+a)e^{-T(s+a)}} \\ &= \frac{1}{s+a} \\ &= R(s) \end{aligned}$$

which implies that the original signal can be recovered from the output of the **sample and hold** device if the sampling period approaches zero.

# Module 2: Modeling Discrete Time Systems by Pulse Transfer Function
## Lecture Note 4

## 1 Pulse Transfer Functions of Closed Loop Systems

We know that various advantages of feedback make most of the control systems closed loop nature. A simple single loop system with a sampler in the forward path is shown in Figure 1.

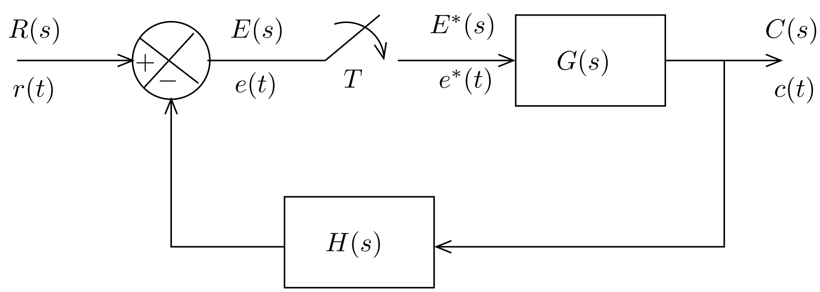



Figure 1: Block diagram of a closed loop system with a sampler in the forward path

The objective is to establish the input-output relationship. For the above system, the output of the sampler is regarded as an input to the system. The input to the sampler is regarded as another output. Thus the input-output relations can be formulated as

$$E(s) = R(s) - G(s)H(s)E^*(s) \tag{1}$$
$$C(s) = G(s)E^*(s) \tag{2}$$

Taking pulse transform on both sides of (1),

$$E^*(s) = R^*(s) - GH^*(s)E^*(s) \tag{3}$$

where

$$\begin{aligned} GH^*(s) &= [G(s)H(s)]^* \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} G(s+jnw_s)H(s+jnw_s) \end{aligned}$$

We can write from equation (3),

$$\begin{aligned} E^*(s) &= \frac{R^*(s)}{1+GH^*(s)} \\ \Rightarrow C(s) &= G(s)E^*(s) \\ &= \frac{G(s)R^*(s)}{1+GH^*(s)} \end{aligned}$$

Taking pulse transformation on both sides of (2)

$$\begin{aligned} C^*(s) &= [G(s)E^*(s)]^* \\ &= G^*(s)E^*(s) \\ &= \frac{G^*(s)R^*(s)}{1+GH^*(s)} \\ \therefore \frac{C^*(s)}{R^*(s)} &= \frac{G^*(s)}{1+GH^*(s)} \\ \Rightarrow \frac{C(z)}{R(z)} &= \frac{G(z)}{1+GH(z)} \end{aligned}$$

where $GH(z) = Z[G(s)H(s)]$.

Now, if we place the sampler in the feedback path, the block diagram will look like the Figure 2.

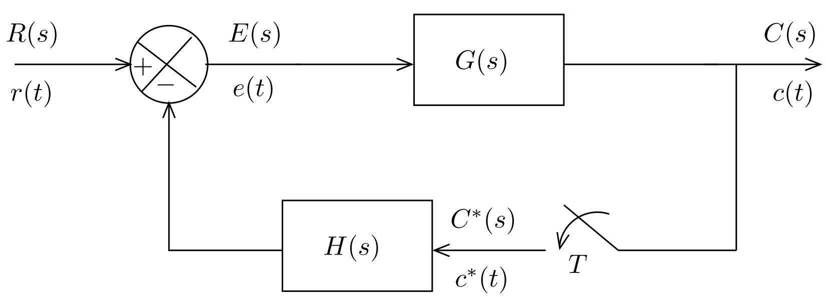



Figure 2: Block diagram of a closed loop system with a sampler in the feedback path

The corresponding input output relations can be written as:

$$E(s) = R(s) - H(S)C^*(s) \tag{4}$$

$$C(s) = G(s)E(s) = G(s)R(s) - G(s)H(s)C^*(s) \tag{5}$$

Taking pulse transformation of equations (4) and (5)

$$\begin{aligned}
E^*(s) &= R^*(s) - H^*(s)C^*(s) \\
C^*(s) &= GR^*(s) - GH^*(s)C^*(s) \\
\text{where, } GR^*(s) &= [G(s)R(s)]^* \\
GH^*(s) &= [G(s)H(s)]^*
\end{aligned}$$

$C^*(s)$ can be written as

$$\begin{aligned}
C^*(s) &= \frac{GR^*(s)}{1 + GH^*(s)} \\
\Rightarrow C(z) &= \frac{GR(z)}{1 + GH(z)}
\end{aligned}$$

We can no longer define the input output transfer function of this system by either $\dfrac{C^*(s)}{R^*(s)}$ or $\dfrac{C(z)}{R(z)}$. Since the input $r(t)$ is not sampled, the sampled signal $r^*(t)$ does not exist. The continuous-data output $C(s)$ can be expressed in terms of input as.

$$C(s) = G(s)R(s) - \frac{G(s)H(s)}{1 + GH^*(s)} GR^*(s)$$

## 1.1 Characteristics Equation

Characteristics equation plays an important role in the study of linear systems. As said earlier, an $n^{th}$ order LTI discrete data system can be represented by an $n^{th}$ order difference equation,

$$\begin{aligned}
c(k+n) + a_{n-1}c(k+n-1) + a_{n-2}c(k+n-2) + ... + a_1 c(k+1) + a_0 c(k) \\
= b_m r(k+m) + b_{m-1} r(k+m-1) + ... + b_0 r(k)
\end{aligned}$$

where $r(k)$ and $c(k)$ denote input and output sequences respectively. The input output relation can be obtained by taking Z-transformation on both sides, with zero initial conditions, as

$$\begin{aligned}
G(z) &= \frac{C(z)}{R(z)} \\
&= \frac{b_m z^m + b_{m-1} z^{m-1} + ... + b_0}{z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0}
\end{aligned} \tag{6}$$

The characteristics equation is obtained by equating the denominator of $G(z)$ to 0, as

$$z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 = 0$$

**Example**

Consider the forward path transfer function as $G(s) = \dfrac{2}{s(s+2)}$ and the feedback transfer function as 1. If the sampler is placed in the forward path, find out the characteristics equation of the overall system for a sampling period $T = 0.1$ sec.

**Solution:**

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

Since the feedback transfer function is 1,

$$\begin{aligned}
G(z) = GH(z) &= z\left[\frac{2}{s(s+2)}\right] \\
&= \frac{2}{2}\frac{(1 - e^{-2T})z}{(z-1)(z - e^{-2T})} \\
&= \frac{0.18z}{z^2 - 1.82z + 0.82} \\
\Rightarrow \frac{C(z)}{R(z)} &= \frac{0.18z}{z^2 - 1.64z + 0.82}
\end{aligned}$$

So, the characteristics equation of the system is $z^2 - 1.64z + 0.82 = 0$.

## 1.2 Causality and Physical Realizability

- In a causal system, the output does not precede the input. In other words, in a causal system, the output depends only on the past and present inputs, not on the future ones.
- The transfer function of a causal system is physically realizable, i.e., the system can be realized by using physical elements.
- For a causal discrete data system, the power series expansion of its transfer function must not contain any positive power in $z$. Positive power in $z$ indicates prediction. Therefore, in the transfer function (6), $n$ must be greater than or equal to $m$.

  $m = n \quad \Rightarrow$ proper transfer function
  $m < n \quad \Rightarrow$ strictly proper Transfer function

# Module 2: Modeling Discrete Time Systems by Pulse Transfer Function

## Lecture Note 5

## 1 Sampled Signal Flow Graph

It is known fact that the transfer functions of linear continuous time data systems can be determined from signal flow graphs using Mason's gain formula.

Since most discrete data control systems contain both analog and digital signals, Mason's gain formula cannot be applied to the original signal flow graph or block diagram of the system.

The first step in applying signal flow graph to discrete data systems is to express the system's equation in terms of discrete data variables only.

**Example 1:** Let us consider the block diagram of a sampled data system as shown in Figure 1(a). We can write:

$$
\begin{aligned}
E(s) &= R(s) - G(s)H(s)E^*(s) && (1)\\
C(s) &= G(s)E^*(s) && (2)
\end{aligned}
$$

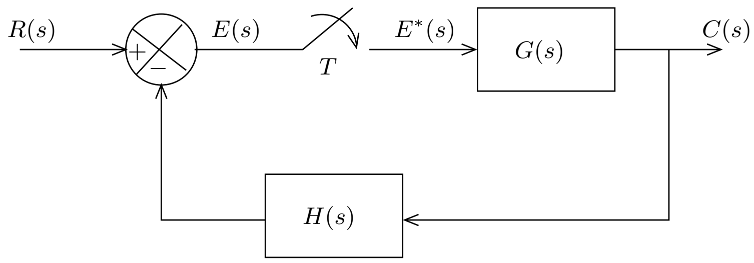
The sampled data signal flow graph (SFG) is shown in Figure 1(b).

Taking pulse transform on both sides of equations (1) and (2), we get:

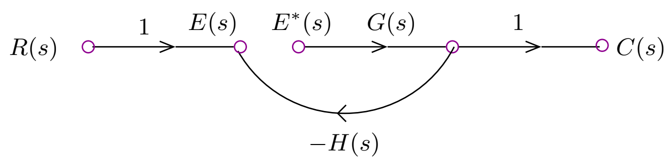
$$
\begin{aligned}
E^*(s) &= R^*(s) - GH^*(s)E^*(s)\\
C^*(s) &= G^*(s)E^*(s)
\end{aligned}
$$

The above equations contain only discrete data variables for which the equivalent SFG will take a form as shown in Figure 1(c). If we apply Mason's gain formula, we will get the following transfer functions.

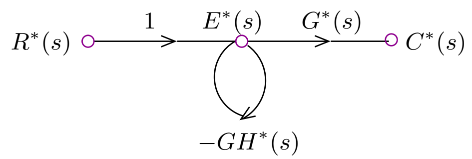
$$
\begin{aligned}
\frac{C^*(s)}{R^*(s)} &= \frac{1}{1+GH^*(s)}(G^*(s) \times 1)\\
&= \frac{G^*(s)}{1+GH^*(s)}\\
\frac{E^*(s)}{R^*(s)} &= \frac{1}{1+GH^*(s)}
\end{aligned}
$$




(a)




(b)




(c)

Figure 1: (a) Block diagram, (b) sampled signal flow graph and (c) equivalent signal flow graph for Example 1

The composite signal flow graph is formed by combining the equivalent and the original sampled signal flow graphs as shown in Figure 2.

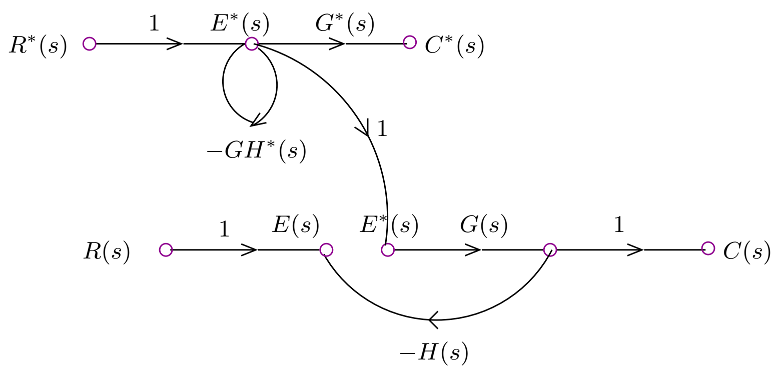



Figure 2: Composite signal flow graph for Example 1

The transfer function between the inputs and continuous data outputs are obtained from composite SFG using Mason's gain formula.

In the composite SFG, the output nodes of the sampler on the sampled SFG are connected to the same nodes on equivalent SFG with unity gain. If we apply Mason's gain formula to the composite SFG:

$$\begin{aligned}\frac{C^*(s)}{R^*(s)} &= \frac{G^*(s)}{1+GH^*(s)} \\ E(s) &= R(s) - \frac{G(s)H(s)}{1+GH^*(s)}R^*(s)\end{aligned}$$

**Example 2:** Consider the block diagram as shown in Figure 3(a).

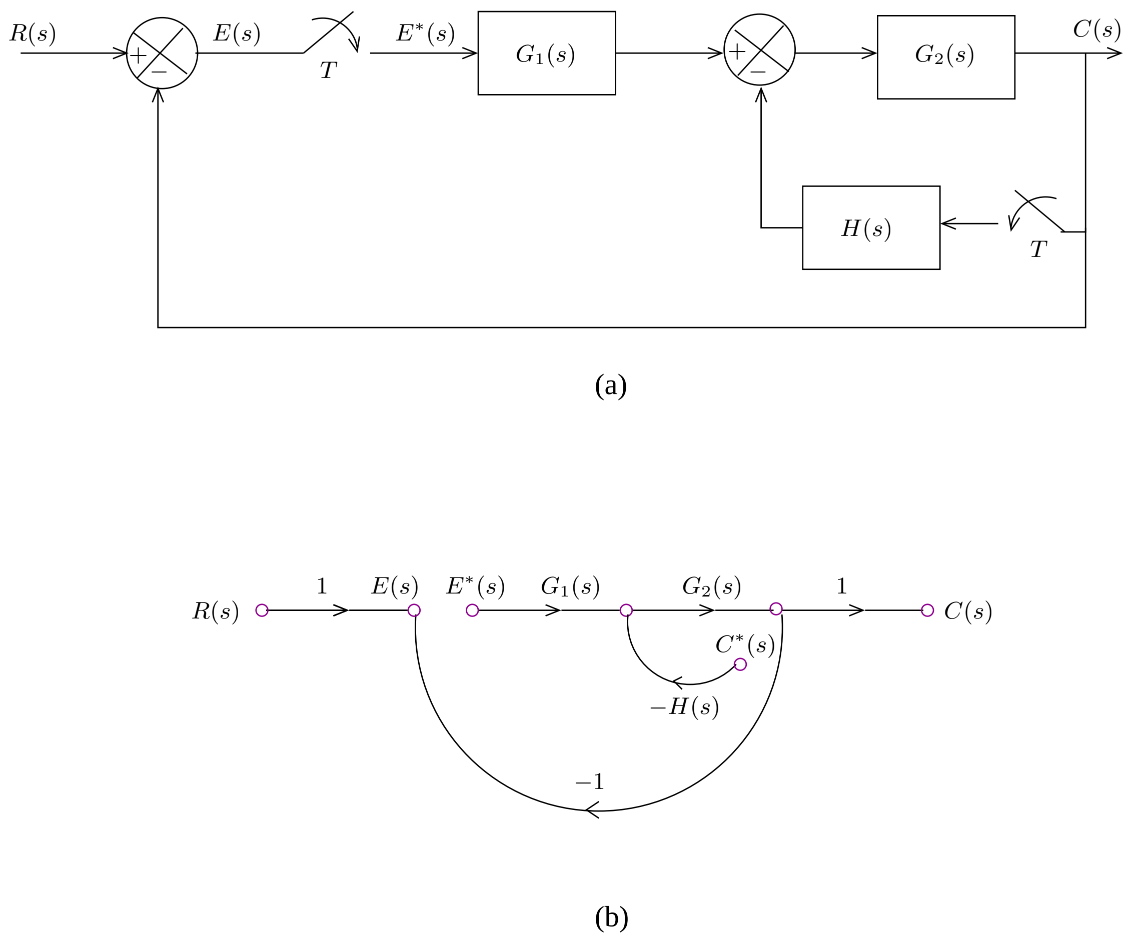



Figure 3: (a) Block diagram and (b) sampled signal flow graph for Example 2

The input output relations:

$$\begin{aligned}E(s) &= R(s) - C(s) &(3)\\ C(s) &= (G_1(s)E^*(s) - H(s)C^*(s))G_2(s) &(4)\\ &= G_1(s)G_2(s)E^*(s) - G_2(s)H(s)C^*(s) &(5)\end{aligned}$$

The sampled SFG is shown in Figure 3(b).

To find out the composite SFG, we take pulse transform on equations (5) and (3):

$$\begin{aligned}
C^*(s) &= G_1G_2^*(s)E^*(s) - G_2H^*(s)C^*(s) \\
E^*(s) &= R^*(s) - C^*(s) \\
&= R^*(s) - G_1G_2^*(s)E^*(s) + G_2H^*(s)C^*(s)
\end{aligned}$$

The composite SFG is shown in Figure 4.

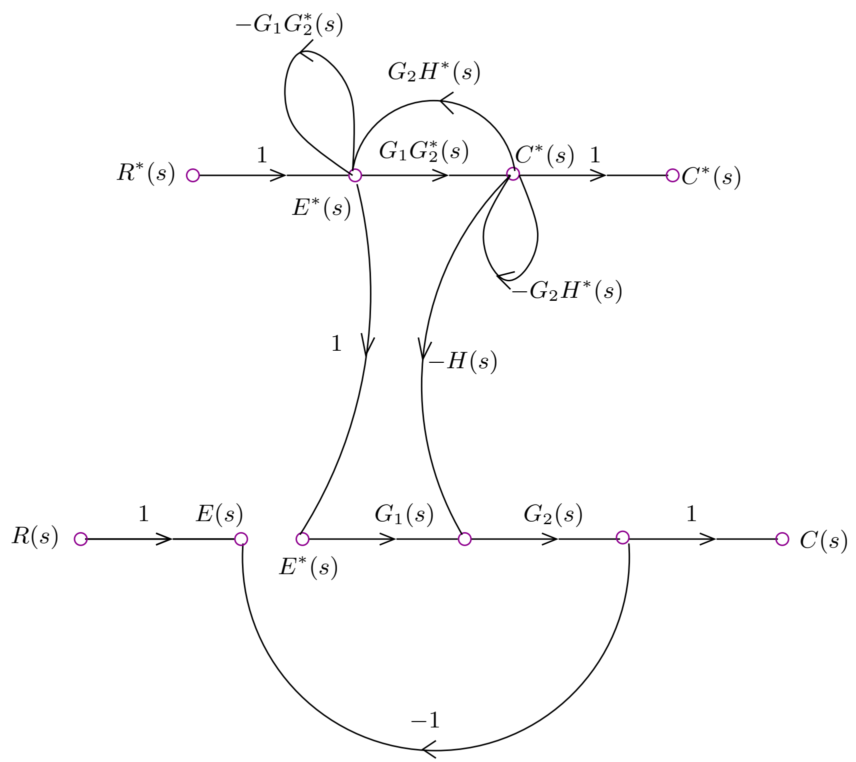



Figure 4: Composite signal flow graph for Example 2

$\dfrac{C^*(s)}{R^*(s)}$, $\dfrac{E^*(s)}{R^*(s)}$, $\dfrac{C(s)}{R^*(s)}$ can be computed from Mason's gain formula, as:

$$\frac{C^*(s)}{R^*(s)} = \frac{G_1G_2^*(s)}{1 + G_1G_2^*(s) + G_2H^*(s)}$$

$$\begin{aligned}
\frac{E^*(s)}{R^*(s)} &= \frac{1 \times (1 - (-G_2H^*(s)))}{1 + G_1G_2^*(s) + G_2H^*(s)} \\
&= \frac{1 + G_2H^*(s)}{1 + G_1G_2^*(s) + G_2H^*(s)}
\end{aligned}$$

To derive $\frac{C(s)}{R^*(s)}$: Number of forward paths = 2 and the corresponding gains are

$$\Rightarrow \quad 1 \times 1 \times G_1(s) \times G_2(s) = G_1(s)G_2(s)$$
$$\Rightarrow \quad 1 \times G_1G_2^*(s) \times (-H(s)) \times G_2(s) = -G_2(s)H(s)G_1G_2^*(s)$$

$\Delta_1 = 1 + G_2H^*(s)$ and $\Delta_2 = 1$.

$$\therefore \frac{C(s)}{R^*(s)} = \frac{G_1(s)G_2(s)[1 + G_2H^*(s)] - G_2(s)H(s)G_1G_2^*(s)}{1 + G_1G_2^*(s) + G_2H^*(s)}$$

In Z-domain,

$$\frac{E(z)}{R(z)} = \frac{1 + G_2H(z)}{1 + G_1G_2(z) + G_2H(z)}$$
$$\frac{C(z)}{R(z)} = \frac{G_1G_2(z)}{1 + G_1G_2(z) + G_2H(z)}$$

The sampled signal flow graph is not the only signal flow graph method available for discrete-data systems. The direct signal flow graph is an alternate method which allows the evaluation of the input-output transfer function of discrete data systems by inspection. This method depends on an entirely different set of terminologies and definitions than those of Mason's signal flow graph and will be omitted in this course.

## Practice Problem

1. Draw the composite signal flow graph of the system represented by the block diagram shown in Figure 5.

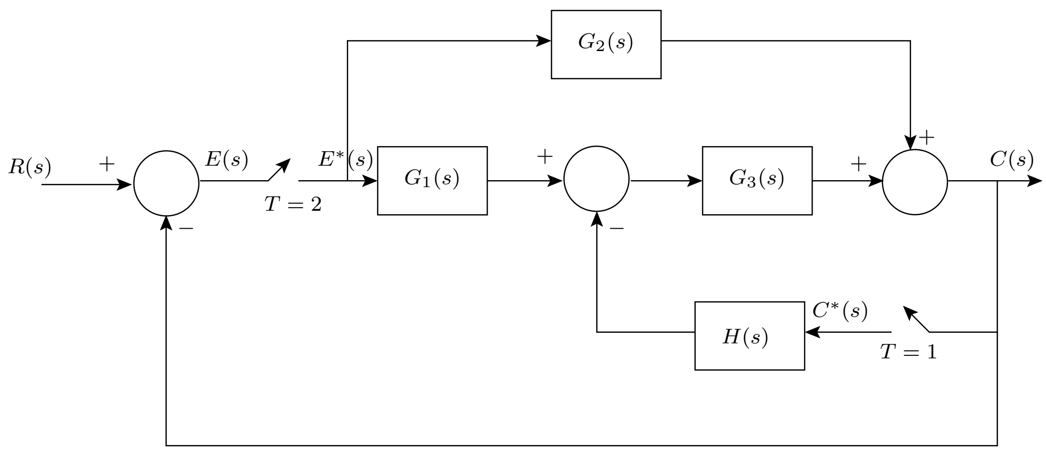



Figure 5: Block diagram for Exercise 1

Find out the closed loop discrete transfer function $\frac{C(z)}{R(z)}$ if

$$G_1(s) = 2G_{h0}(s); \quad G_2(s) = \frac{G_{h0}(s)}{s+2}; \quad G_3(s) = \frac{1}{s+1}; \quad H(s) = 5G_{h0}(s);$$

where $G_{h0}(s)$ represents zero order hold.

# Module 3: Stability Analysis of Discrete Time Systems
## Lecture Note 1

## 1 Stability Analysis of closed loop system in z-plane

Stability is the most important issue in control system design. Before discussing the stability test let us first introduce the following notions of stability for a linear time invariant (LTI) system.

1. BIBO stability or zero state stbaility

2. Internal stability or zero input stability

Since we have not introduced the concept of state variables yet, as of now, we will limit our discussion to BIBO stability only.

An initially relaxed (all the initial conditions of the system are zero) LTI system is said to be BIBO stable if for every bounded input, the output is also bounded.

However, the stability of the following closed loop system

$$\frac{C(z)}{R(Z)} = \frac{G(z)}{1 + GH(z)}$$

can be determined from the location of closed loop poles in z-plane which are the roots of the characteristic equation

$$1 + GH(z) = 0$$

1. For the system to be stable, the closed loop poles or the roots of the characteristic equation must lie within the unit circle in z-plane. Otherwise the system would be unstable.

2. If a simple pole lies at $|z| = 1$, the system becomes marginally stable. Similarly if a pair of complex conjugate poles lie on the $|z| = 1$ circle, the system is marginally stable. Multiple poles on unit circle make the system unstable.

**Example 1:**
Determine the closed loop stability of the system shown in Figure 1 when $K = 1$.

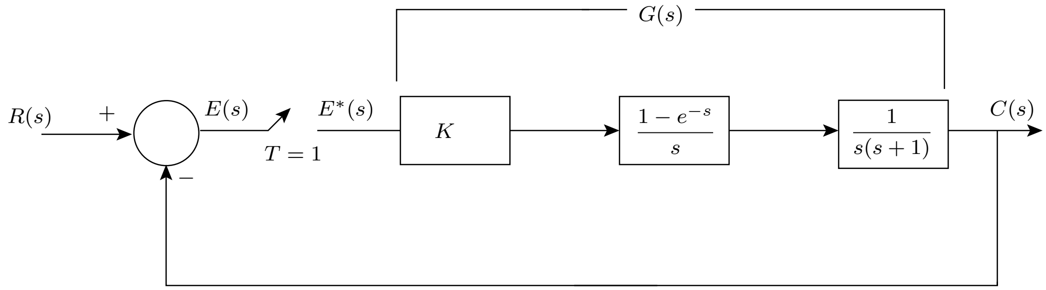



Figure 1: Example 1

**Solution:**

$$\begin{aligned} G(z) &= Z\left[\frac{1-e^{-s}}{s}.\frac{1}{s(s+1)}\right] \\ &= (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right] \end{aligned}$$

Since $H(s) = 1$,

$$\begin{aligned} \therefore \frac{C(z)}{E(z)} &= \frac{z-1}{z}.\left[\frac{z}{(z-1)^2} - \frac{(1-e^{-1})z}{(z-1)(z-e^{-1})}\right] \\ &= \frac{(z-e^{-1})-(1-e^{-1})(z-1)}{(z-1)(z-e^{-1})} \\ &= \frac{z-0.368-0.632z+0.632}{(z-1)(z-0.368)} \\ G(z) &= \frac{0.368z+0.264}{(z-1)(z-0.368)} \end{aligned}$$

We know that the characteristics equation is $\Rightarrow 1 + G(z) = 0$

$$\begin{aligned} &\Rightarrow\ (z-1)(z-0.368)+0.368z+0.264=0 \\ &\Rightarrow\ z^2-z+0.632=0 \\ &\Rightarrow\ z_1 = 0.5+0.618j \\ &\Rightarrow\ z_2 = 0.5-0.618j \end{aligned}$$

Since $|z_1| = |z_2| \quad < 1$, the system is stable.

Three stability tests can be applied directly to the characteristic equation without solving for the roots.
→ Schur-Cohn stability test
→ Jury Stability test
→ Routh stability coupled with bi-linear transformation.

Other stability tests like Lyapunov stability analysis are applicable for state space system models which will be discussed later. Computation requirements in Jury test is simpler than Schur-Cohn when the co-efficients are real which is always true for physical systems.

## 1.1 Jury Stability Test

Assume that the characteristic equation is as follows,

$$P(z) = a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n$$

where $a_0 > 0$.

**Jury Table**

| Row | $z^0$ | $z^1$ | $z^2$ | $z^3$ | $z^4$ | ... | $z^n$ |
|---|---|---|---|---|---|---|---|
| 1 | $a_n$ | $a_{n-1}$ | $a_{n-2}\dots$ | ... | ... | | $a_0$ |
| 2 | $a_0$ | $a_1$ | $a_2\dots$ | ... | ... | ... | $a_n$ |
| 3 | $b_{n-1}$ | $b_{n-2}$ | ... | ... | $b_0$ | | |
| 4 | $b_0$ | $b_1$ | ... | ... | ... | $b_{n-1}$ | |
| 5 | $c_{n-2}$ | $c_{n-3}\dots$ | $c_0$ | | | | |
| 6 | $c_0$ | $c_1\dots$ | ... | $c_{n-2}$ | | | |
| . | ........................... | | | | | | |
| . | ........................... | | | | | | |
| . | ........................... | | | | | | |
| $2n-3$ | $q_2$ | $q_1$ | $q_0$ | | | | |

where,

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ...., n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix}$$

$$k = 0, 1, 2, 3, ...., n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix}$$

This system will be stable if:

1. $|a_n| \quad < a_0$

2. $P(z)|_{z=1} \quad > 0$

3. $P(z)|_{z=-1} \quad > 0$ for $n$ even and $P(z)|_{z=-1} < 0$ for $n$ odd

4.

$$\begin{array}{cl} |b_{n-1}| & > |b_0| \\ |c_{n-2}| & > |c_0| \\ . & \ldots\ldots\ldots\ldots \\ . & \ldots\ldots\ldots\ldots \\ |q_2| & > |q_0| \end{array}$$

**Example 2**: The characteristic equation: $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$
Thus, $a_0 = 1 \quad a_1 = -1.2 \quad a_2 = 0.07 \quad a_3 = 0.3 \quad a_4 = -0.08$

We will now check the stability conditions.

1. $|a_n| = |a_4| = 0.08 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0 \Rightarrow$ Second condition is satisfied.

3. $P(-1) = 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0 \Rightarrow$ Third condition is satisfied.

Jury Table

$$b_3 = \begin{vmatrix} a_n & a_0 \\ a_0 & a_n \end{vmatrix} = \quad 0.0064 - 1 = -0.9936$$

$$b_2 = \begin{vmatrix} a_n & a_1 \\ a_0 & a_3 \end{vmatrix} = \quad -0.08 \times 0.3 + 1.2 = 1.176$$

Rest of the elements are also calculated in a similar fashion. The elements are $b_1 = -0.0756$ $b_0 = -0.204 \quad c_2 = 0.946 \quad c_1 = -1.184 \quad c_0 = 0.315$. One can see
$|b_3| = 0.9936 > |b_0| = 0.204$
$|c_2| = 0.946 > |c_0| = 0.315$

All criteria are satisfied. Thus the system is stable.

**Example 3** The characteristic equation: $P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$
Thus $a_0 = 1 \quad a_1 = -1.3 \quad a_2 = -0.08 \quad a_3 = 0.24$.

Stability conditions are:
1. $|a_3| = 0.24 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 - 1.3 - 0.08 + 0.24 = -0.14 < 0 \Rightarrow$ Second condition is not satisfied.

Since one of the criteria is violated, we may stop the test here and conclude that the system is unstable. $P(1) = 0$ or $P(-1) = 0$ indicates the presence of a root on the unit circle and in that case the system can at the most become marginally stable if rest of the conditions are satisfied.

The stability range of a parameter can also be found from Jury's test which we will see in the next example.

**Example 4**: Consider the system shown in Figure 1. Find out the range of $K$ for which the system is stable.

**Solution:**

$$G(z) = \frac{K(0.368z + 0.264)}{(z-1)(z-0.368)}$$

The closed loop transfer function: $$\frac{C(z)}{R(z)} = \frac{K(0.368z + 0.264)}{z^2 + (0.368K - 1.368)z + 0.368 + 0.264K}$$

Characteristic equation: $P(z) = z^2 + (0.368K - 1.368)z + 0.368 + 0.264K = 0$
Since it is a second order system only 3 stability conditions will be there.

1. $|a_2| < a_0$
2. $P(1) > 0$
3. $P(-1) > 0$ since $n = 2$=even. This implies:

1. $|0.368 + 0.264K| < 1 \quad \Rightarrow 2.39 > K > -5.18$
2. $P(1) = 1 + (0.368K - 1.368) + 0.368 + 0.264K = 0.632K > 0 \Rightarrow K > 0$
3. $P(-1) = 1 - (0.368K - 1.368) + 0.368 + 0.264K = 2.736 - 0.104K > 0 \Rightarrow 26.38 > K$

Combining all, the range of $K$ is found to be $0 < K < 2.39$.

If $K = 2.39$, system becomes critically stable.
The characteristics equation becomes:

$$z^2 - 0.49z + 1 = 0 \quad \Rightarrow z = 0.244 \pm j0.97$$

Sampling period $T = 1$ sec.

$$w_d = \frac{w_s}{2\pi}\angle z = \frac{2\pi}{2\pi}\tan^{-1}\frac{0.97}{0.244} \cong 1.324 \text{ rad/sec}$$

The above frequency is the frequency of sustained oscillation.

## 1.2 Singular Cases

When some or all of the elements of a row in the Jury table are zero, the tabulation ends prematurely. This situation is referred to as a singular case. It can be avoided by expanding or contracting unit circle infinitesimally by an amount $\epsilon$ which is equivalent to move the roots of $P(z)$ off the unit circle. The transformation is:

$$z_1 = (1+\epsilon)z$$

where $\epsilon$ is a very small number. When $\epsilon$ is positive the unit circle is expanded and when $\epsilon$ is negative the unit circle is contracted. The difference between the number of zeros found inside or outside the unit circle when the unit circle is expanded or contracted is the number of zeros on the unit circle. Since $(1+\epsilon)^n z^n \cong (1+n\epsilon)z^n$ for both positive and negative $\epsilon$, the transformation requires the coefficient of the $z^n$ term be multiplied by $(1+n\epsilon)$.

**Example 5**: The characteristic equation: $P(z) = z^3 + 0.25z^2 + z + 0.25 = 0$
Thus, $a_0 = 1 \quad a_1 = 0.25 \quad a_2 = 1 \quad a_3 = 0.25$.

We will now check the stability conditions.

1. $|a_n| = |a_3| = 0.25 < a_0 = 1 \Rightarrow$ First condition is satisfied.

2. $P(1) = 1 + 0.25 + 1 + 0.25 = 2.5 > 0 \Rightarrow$ Second condition is satisfied.

3. $P(-1) = -1 + 0.25 - 1 + 0.25 = -1.5 < 0 \Rightarrow$ Third condition is satisfied.

Jury Table

$$b_2 = \begin{vmatrix} a_3 & a_0 \\ a_0 & a_3 \end{vmatrix} = \quad 0.0625 - 1 = -0.9375$$

$$b_1 = \begin{vmatrix} a_3 & a_1 \\ a_0 & a_2 \end{vmatrix} = \quad 0.25 - 0.25 = 0$$

Thus the tabulation ends here and we know that some of the roots lie on the unit circle. If we replace $z$ by $(1+\epsilon)z$, the characteristic equation would become:

$$(1+3\epsilon)z^3 + 0.25(1+2\epsilon)z^2 + (1+\epsilon)z + 0.25 = 0$$

First three stability conditions are satisfied.

### Jury Table

| Row | $z^0$ | $z^1$ | $z^2$ | $z^3$ |
|---|---|---|---|---|
| 1 | $0.25$ | $1+\epsilon$ | $0.25(1+2\epsilon)$ | $1+3\epsilon$ |
| 2 | $1+3\epsilon$ | $0.25(1+2\epsilon)$ | $1+\epsilon$ | $0.25$ |
| 3 | $0.25^2 - (1+3\epsilon)^2$ | $0.25(1+\epsilon) - 0.25(1+2\epsilon)(1+3\epsilon)$ | $-(1+3\epsilon)(1+\epsilon) + 0.25^2(1+2\epsilon)$ | |

$|b_2| = |0.0625 - (1 + 6\epsilon + 9\epsilon^2)|$ and $|b_0| = |1 + 3.875\epsilon + 3\epsilon^2 - 0.0625|$. Since, when $\epsilon \to 0^+$, $1 + 6\epsilon + 9\epsilon^2 > 1 + 3.875\epsilon + 3\epsilon^2$, thus $|b_2| > |b_0|$ which implies that the roots which are not on the unit circle are actually inside it and the system is marginally stable.

# Module 3: Stability Analysis of Discrete Time Systems
## Lecture Note 2

## 1 Stability Analysis using Bilinear Transformation and Routh Stability Criterion

Another frequently used method in stability analysis of discrete time system is the bilinear transformation coupled with Routh stability criterion. This requires transformation from $z$-plane to another plane called $w$-plane.

The bilinear transformation has the following form.

$$z = \frac{aw+b}{cw+d}$$

where $a$, $b$, $c$, $d$ are real constants. If we consider $a = b = c = 1$ and $d = -1$, then the transformation takes a form

$$z = \frac{w+1}{w-1}$$
$$\text{or, } w = \frac{z+1}{z-1}$$

This transformation maps the inside of the unit circle in the $z$-plane into the left half of the $w$-plane. Let the real part of $w$ be $\alpha$ and imaginary part be $\beta \Rightarrow w = \alpha + j\beta$. The inside of the unit circle in $z$-plane can be represented by:

$$|z| = \left|\frac{w+1}{w-1}\right| = \left|\frac{\alpha+j\beta+1}{\alpha+j\beta-1}\right| < 1$$

$$\Rightarrow \frac{(\alpha+1)^2+\beta^2}{(\alpha-1)^2+\beta^2} < 1 \Rightarrow (\alpha+1)^2+\beta^2 < (\alpha-1)^2+\beta^2 \Rightarrow \alpha < 0$$

Thus inside of the unit circle in $z$-plane maps into the left half of $w$-plane and outside of the unit circle in $z$-plane maps into the right half of $w$-plane. Although $w$-plane seems to be similar to $s$-plane, quantitatively it is not same.

In the stability analysis using bilinear transformation, we first substitute $z = \dfrac{w+1}{w-1}$ in the characteristics equation $P(z) = 0$ and simplify it to get the characteristic equation in $w$-plane as $Q(w) = 0$. Once the characteristics equation is transformed as $Q(w) = 0$, Routh stability criterion is directly used in the same manner as in a continuous time system.

We will now solve the same examples which were used to understand the Jury's test.

**Example 1** The characteristic equation: $P(z) = z^3 - 1.3z^2 - 0.08z + 0.24 = 0$

Transforming $P(z)$ into $w$-domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 - 1.3\left[\frac{w+1}{w-1}\right]^2 - 0.08\left[\frac{w+1}{w-1}\right] + 0.24 = 0$$

$$\text{or,}\ \ Q(w) = 0.14w^3 - 1.06w^2 - 5.1w - 1.98 = 0$$

We can now construct the Routh array as

$$\begin{array}{lcc} w^3 & 0.14 & -5.1 \\ w^2 & -1.06 & -1.98 \\ w^1 & -5.36 & \\ w^0 & -1.98 & \end{array}$$

There is one sign change in the first column of the Routh array. Thus the system is unstable with one pole at right hand side of the $w$-plane or outside the unit circle of $z$-plane.

**Example 2**: The characteristic equation: $P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08 = 0$

Transforming $P(z)$ into $w$-domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^4 - 1.2\left[\frac{w+1}{w-1}\right]^3 + 0.07\left[\frac{w+1}{w-1}\right]^2 + 0.3\left[\frac{w+1}{w-1}\right] - 0.08 = 0$$

$$\text{or,}\ \ Q(w) = 0.09w^4 + 1.32w^3 + 5.38w^2 + 7.32w + 1.89 = 0$$

We can now construct the Routh array as

$$\begin{array}{lccc} w^4 & 0.09 & 5.38 & 1.89 \\ w^3 & 1.32 & 7.32 & \\ w^2 & 4.88 & 1.89 & \\ w^1 & 6.81 & & \\ w^0 & 1.89 & & \end{array}$$

All elements in the first column of Routh array are positive. Thus the system is stable.

**Example 3**: Consider the system shown in Figure 1. Find out the range of $K$ for which the system is stable.

**Solution:**

$$G(z) = \frac{K(0.084z^2 + 0.17z + 0.019)}{(z^3 - 1.5z^2 + 0.553z - 0.05)}$$

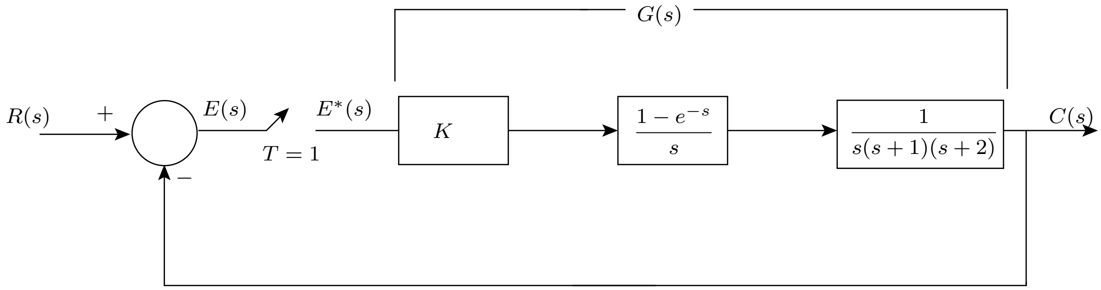



Figure 1: Figure for Example 3

Characteristic equation:

$$1 + \frac{K(0.084z^2 + 0.17z + 0.019)}{(z^3 - 1.5z^2 + 0.553z - 0.05)} = 0$$

or,

$$P(z) = z^3 + (0.084K - 1.5)z^2 + (0.17K + 0.553)z + (0.019K - 0.05) = 0$$

Transforming $P(z)$ into $w$-domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 + (0.084K - 1.5)\left[\frac{w+1}{w-1}\right]^2 + (0.17K + 0.553)\left[\frac{w+1}{w-1}\right] + (0.019K - 0.05) = 0$$

or, $Q(w) = (0.003 + 0.27K)w^3 + (1.1 - 0.11K)w^2 + (3.8 - 0.27K)w + (3.1 + 0.07K) = 0$

We can now construct the Routh array as

$$\begin{array}{lcc}
w^3 & 0.003 + 0.27K & 3.8 - 0.27K \\
w^2 & 1.1 - 0.11K & 3.1 + 0.07K \\
w^1 & \dfrac{0.01K^2 - 1.55K + 4.17}{1.1 - 0.11K} & \\
w^0 & 3.1 + 0.07K &
\end{array}$$

The system will be stable if all the elements in the first column have same sign. Thus the conditions for stability, in terms of $K$, are

$$\begin{aligned}
0.003 + 0.27K > 0 &\Rightarrow K > -0.011 \\
1.1 - 0.11K > 0 &\Rightarrow K < 10 \\
0.01K^2 - 1.55K + 4.17 > 0 &\Rightarrow K < 2.74 \quad \text{or,} \quad K > 140.98 \\
3.1 + 0.07K > 0 &\Rightarrow K > -44.3
\end{aligned}$$

Combining above four constraints, the stable range of $K$ can be found as

$$-0.011 < K < 2.74$$

---

## 1.1 Singular Cases

In Routh array, tabulation may end in occurance with any of the following conditions.

- The first element in any row is zero.
- All the elements in a single row are zero.

The remedy of the first case is replacing zero by a small number $\epsilon$ and then proceeding with the tabulation. Stability can be checked for the limiting case. Second singular case indicates one or more of the following cases.

- Pairs of real roots with opposite signs.
- Pairs of imaginary roots.
- Pairs of complex conjugate roots which are equidistant from the origin.

When a row of all zeros occurs, an auxiliary equation $A(w) = 0$ is formed by using the coefficients of the row just above the row of all zeros. The roots of the auxiliary equation are also the roots of the characteristic equation. The tabulation is continued by replacing the row of zeros by the coefficients of $\dfrac{dA(w)}{dw}$.

Looking at the correspondence between $w$-plane and $z$-plane, when an all zero row occurs, we can conclude that following two scenarios are likely to happen.

- Pairs of real roots in the $z$-plane that are inverse of each other.
- Pairs of roots on the unit circle simultaneously.

**Example 4:** Consider the characteristic equation

$$P(z) = z^3 - 1.7z^2 - z + 0.8 = 0$$

Transforming $P(z)$ into $w$-domain:

$$Q(w) = \left[\frac{w+1}{w-1}\right]^3 - 1.7\left[\frac{w+1}{w-1}\right]^2 - \left[\frac{w+1}{w-1}\right] + 0.8 = 0$$

$$\text{or,}\quad Q(w) = 0.9w^3 + 0.1w^2 - 8.1w - 0.9 = 0$$

The Routh array:

$$\begin{array}{ccc} w^3 & 0.9 & -8.1 \\ w^2 & 0.1 & -0.9 \\ w^1 & 0 & 0 \end{array}$$

The tabulation ends here. The auxiliary equation is formed by using the coefficients of $w^2$ row, as:

$$A(w) = 0.1w^2 - 0.9 = 0$$

Taking the derivative,

$$\frac{dA(w)}{dw} = 0.2w$$

Thus the Routh tabulation is continued as

$$\begin{array}{lcc}
w^3 & 0.9 & -8.1 \\
w^2 & 0.1 & -0.9 \\
w^1 & 0.2 & 0 \\
w^0 & -0.9 &
\end{array}$$

As there is one sign change in the first row, one of the roots lie in $w$-plane is on the right hand side of the $w$-plane. This implies that one root in $z$-plane lies outside the unit circle.

To verify our conclusion, the roots of the polynomial $z^3 - 1.7z^2 - z + 0.8 = 0$, are found out to be $z = 0.5$, $z = -0.8$ and $z = 2$. Thus one can see that $z = 2$ lies outside the unit circle and it is inverse of $z = 0.5$ which caused the all zero row in $w$-plane.

# Module 4: Time Response of discrete time systems
## Lecture Note 1

# 1 Time Response of discrete time systems

Absolute stability is a basic requirement of all control systems. Apart from that, good relative stability and steady state accuracy are also required in any control system, whether continuous time or discrete time. **Transient response** corresponds to the system closed loop poles and **steady state response** corresponds to the excitation poles or poles of the input function.

## 1.1 Transient response specifications

In many practical control systems, the desired performance characteristics are specified in terms of time domain quantities. Unit step input is most commonly used in analysis of a system since it is easy to generate and represent a sufficiently drastic change thus providing useful information on both transient and steady state responses.

The transient response of a system depends on the initial conditions. It is a common practice to consider the system initially at rest.
Consider the digital control system shown in Figure1.

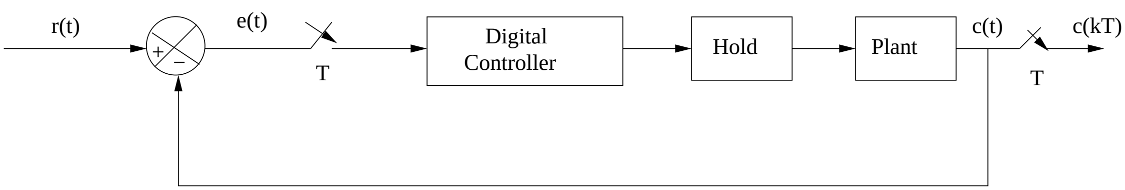



Figure 1: Block Diagram of a closed loop digital system

Similar to the continuous time case, transient response of a digital control system can also be characterized by the following.

1. Rise time ($t_r$): Time required for the unit step response to rise from 0% to 100% of its final value in case of underdamped system or 10% to 90% of its final value in case of overdamped system.

2. Delay time ($t_d$): Time required for the the unit step response to reach 50% of its final value.

3. Peak time ($t_p$): Time at which maximum peak occurs.

4. Peak overshoot ($M_p$): The difference between the maximum peak and the steady state value of the unit step response.

5. Settling time ($t_s$): Time required for the unit step response to reach and stay within 2% or 5% of its steady state value.

However since the output response is discrete the calculated performance measures may be slightly different from the actual values. Figure 2 illustrates this. The output has a maximum value $c_{\max}$ whereas the maximum value of the discrete output is $c^*_{\max}$ which is always less than or equal to $c_{\max}$. If the sampling period is small enough compared to the oscillations of the response then this difference will be small otherwise $c^*_{\max}$ may be completely erroneous.

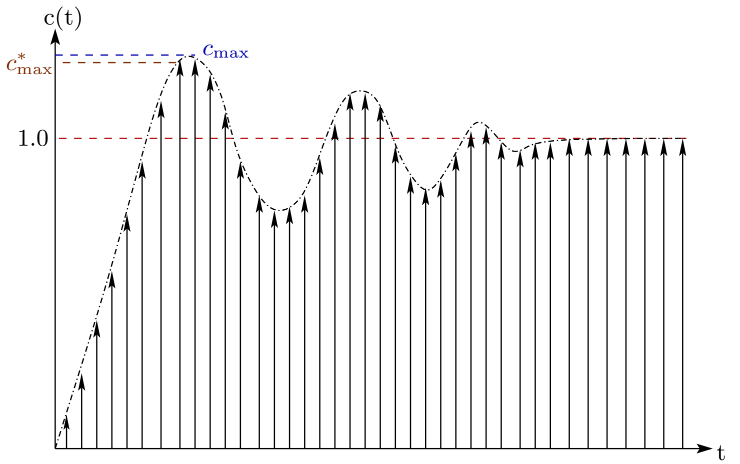



Figure 2: Unit step response of a discrete time system

## 1.2 Steady state error

The steady state performance of a stable control system is measured by the steady error due to step, ramp or parabolic inputs depending on the system type. Consider the discrete time system as shown in Figure 3.

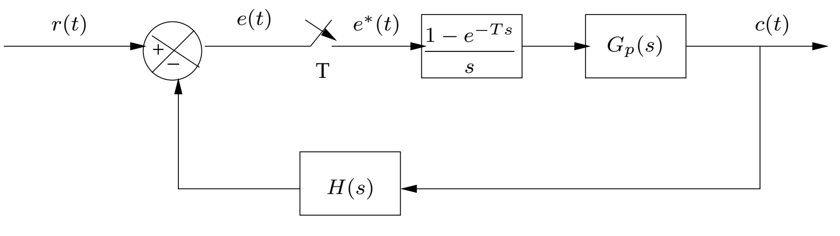



Figure 3: Block Diagram 2

From Figure 2, we can write

$$E(s) = R(s) - H(s)C(s)$$

We will consider the steady state error at the sampling instants.
From final value theorem

$$\begin{aligned}
\lim_{k\to\infty} e(kT) &= \lim_{z\to 1}\left[(1-z^{-1})E(z)\right] \\
G(z) &= (1-z^{-1})Z\left[\frac{G_p(s)}{s}\right] \\
GH(z) &= (1-z^{-1})Z\left[\frac{G_p(s)H(s)}{s}\right] \\
\frac{C(z)}{R(z)} &= \frac{G(z)}{1+GH(z)} \\
\text{Again, } E(z) &= R(z) - GH(z)E(z) \\
\text{or, } E(z) &= \frac{1}{1+GH(z)}R(z) \\
\Rightarrow e_{ss} &= \lim_{z\to 1}\left[(1-z^{-1})\frac{1}{1+GH(z)}R(z)\right]
\end{aligned}$$

The steady state error of a system with feedback thus depends on the input signal $R(z)$ and the loop transfer function $GH(z)$.

### 1.2.1 Type-0 system and position error constant

Systems having a finite nonzero steady state error with a zero order polynomial input (step input) are called **Type-0** systems. The position error constant for a system is defined for a step input.

$$\begin{aligned}
r(t) &= u_s(t) \quad \text{unit step input} \\
R(z) &= \frac{1}{1-z^{-1}} \\
e_{ss} &= \lim_{z\to 1}\frac{1}{1+GH(z)} = \frac{1}{1+K_p}
\end{aligned}$$

where $K_p = \lim_{z\to 1} GH(z)$ is known as the **position error constant**.

### 1.2.2 Type-1 system and velocity error constant

Systems having a finite nonzero steady state error with a first order polynomial input (ramp input) are called **Type-1** systems. The velocity error constant for a system is defined for a ramp input.

$$\begin{aligned}
r(t) &= u_r(t) \quad \text{unit ramp} \\
R(z) &= \frac{Tz}{(z-1)^2} = \frac{TZ^{-1}}{(1-Z^{-1})^2} \\
e_{ss} &= \lim_{z\to 1}\frac{T}{(z-1)GH(z)} = \frac{1}{K_v}
\end{aligned}$$

where $K_v = \frac{1}{T}\lim_{z\to 1}\left[(z-1)GH(z)\right]$ is known as the **velocity error constant**.

### 1.2.3 Type-2 system and acceleration error constant

Systems having a finite nonzero steady state error with a second order polynomial input (parabolic input) are called **Type-2** systems. The acceleration error constant for a system is defined for a parabolic input.

$$
\begin{aligned}
R(z) &= \frac{T^2 z(z+1)}{2(z-1)^3} \quad = \frac{T^2(1+z^{-1})z^{-1}}{2(1-z^{-1})^3} \\
e_{ss} &= \frac{T^2}{2}\lim_{z\to 1}\frac{(z+1)}{(z-1)^2\left[1+GH(z)\right]} \quad = \frac{1}{\lim_{z\to 1}\frac{(z-1)^2}{T^2}GH(z)} = \frac{1}{K_a}
\end{aligned}
$$

where $K_a = \lim_{z\to 1}\dfrac{(z-1)^2}{T^2}GH(z)$ is known as the **acceleration error constant**.

Table 1 shows the steady state errors for different types of systems for different inputs.

Table 1: Steady state errors

| System | Step input | Ramp input | Parabolic input |
|---|---|---|---|
| Type-0 | $\frac{1}{1+K_p}$ | $\infty$ | $\infty$ |
| Type-1 | 0 | $\frac{1}{K_v}$ | $\infty$ |
| Type-2 | 0 | 0 | $\frac{1}{K_a}$ |

**Example 1**: Calculate the steady state errors for unit step, unit ramp and unit parabolic inputs for the system shown in Figure 4.

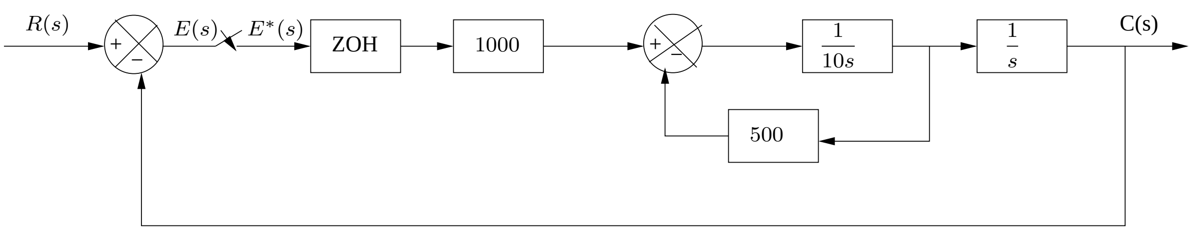



Figure 4: Block Diagram for Example 1

**Solution**: The open loop transfer function is:

$$
\begin{aligned}
G(s) &= \frac{C(s)}{E^*(s)} = G_{ho}(s)G_p(s) \\
&= \frac{1-e^{-Ts}}{s}\,\frac{1000/10}{s(s+500/10)}
\end{aligned}
$$

Taking Z-transform

$$\begin{aligned}
G(z) &= 2(1-z^{-1})\ \mathrm{Z}\left[\frac{1}{s^2} - \frac{10}{500s} + \frac{10}{500(s+5000)}\right] \\
&= 2(1-z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{10z}{500(z-1)} + \frac{10z}{500(z-e^{-50T})}\right] \\
&= \frac{1}{250}\left[\frac{(500T-10+10e^{-50T})z-(500T+10)e^{-50T}+10}{(z-1)(z-e^{-50T})}\right]
\end{aligned}$$

Steady state error for step input $= \dfrac{1}{1+K_p}$ where $K_p = \lim_{z\to 1} G(z) = \infty$. $\Rightarrow e_{ss}^{step} = 0$.

Steady state error for ramp input $= \dfrac{1}{K_v}$ where $K_v = \frac{1}{T}\lim_{z\to 1}\left[(z-1)G(z)\right] = 2$. $\Rightarrow e_{ss}^{ramp} = 0.5$.

Steady state error for parabolic input $= \dfrac{1}{K_a}$ where $K_a = \frac{1}{T^2}\lim_{z\to 1}\left[(z-1)^2G(z)\right] = 0$. $\Rightarrow$ $e_{ss}^{para} = \infty$.

# Module 4: Time Response of discrete time systems
## Lecture Note 2

## 1 Prototype second order system

The study of a second order system is important because many higher order system can be approximated by a second order model if the higher order poles are located so that their contributions to transient response are negligible. A standard second order continuous time system is shown in Figure 1. We can write

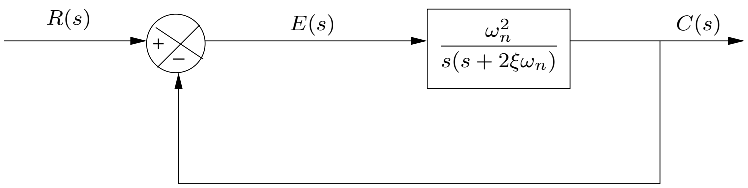



Figure 1: Block Diagram of a second order continuous time system

$$
\begin{aligned}
G(s) &= \frac{\omega_n^2}{s(s+2\xi\omega_n)} \\
\text{Closed loop: } G_c(s) &= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} \\
\text{where,} \quad \xi &= \text{damping ratio} \\
\omega_n &= \text{natural undamped frequency} \\
\text{Roots:} \quad & -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}
\end{aligned}
$$

### 1.1 Comparison between continuous time and discrete time systems

The simplified block diagram of a space vehicle control system is shown in Figure 2. The objective is to control the attitude in one dimension, say in pitch. For simplicity vehicle body is considered as a rigid body.

Position $c(t)$ and velocity $v(t)$ are fedback. The open loop transfer function can be calculated

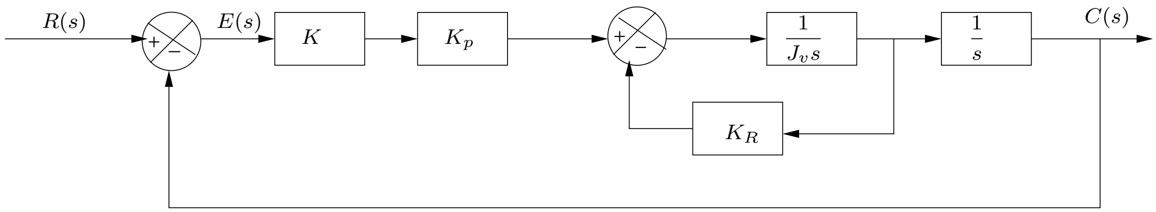



Figure 2: Space vehicle attitude control

as

$$\begin{aligned} G(s) &= \frac{C(s)}{E(s)} \\ &= KK_P \times \frac{1}{K_R + J_v s} \times \frac{1}{s} \\ &= \frac{KK_P}{s(J_v s + K_R)} \end{aligned}$$

Closed loop transfer function is

$$\begin{aligned} G_c s &= \frac{G(s)}{1+G(s)} \\ &= \frac{KK_p}{J_v s^2 + K_R s + KK_P} \end{aligned}$$

$$\begin{aligned} K_P &= \text{Position Sensor gain} = 1.65 \times 10^6 \\ K_R &= \text{Rate sensor gain} = 3.71 \times 10^5 \\ K &= \text{Amplifier gain which is a variable} \\ J_v &= \text{Moment of inertia} = 41822 \end{aligned}$$

With the above parameters,

$$\begin{aligned} G(s) &= \frac{39.45K}{s(s+8.87)} \\ \frac{C(s)}{R(s)} &= \frac{39.45K}{s^2 + 8.87s + 39.45K} \\ \text{Characteristics equation} &\Rightarrow s^2 + 8.87s + 39.45K = 0 \\ \omega_n &= \sqrt{39.45K}\ rad/sec, \quad \xi = \frac{8.87}{2\omega_n} \end{aligned}$$

Since the system is of $2^{nd}$ order, the continuous time system will always be stable if $K_P,\ K_R,\ K,\ J_v$ are all positive.

Now, consider that the continuous data system is subject to sampled data control as shown in Figure 3.

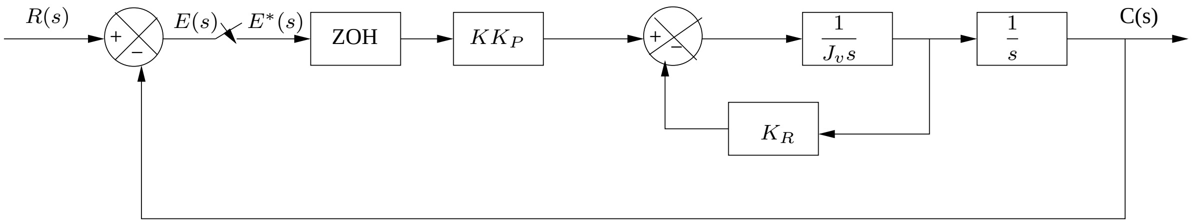



Figure 3: Discrete representation of space vehicle attitude control

For comparison purpose, we assume that the system parameters are same as that of the continuous data system.

$$\begin{aligned}
G(s) &= \frac{C(s)}{E^*(s)} \\
&= G_{ho}(s)G_p(s) \\
&= \frac{1-e^{-Ts}}{s}.\frac{KKp/J_v}{s(s+K_R/J_v)} \\
G(z) &= (1-z^{-1})\frac{KKp}{K_R}\ \mathrm{Z}\left[\frac{1}{s^2}-\frac{J_v}{K_R s}+\frac{J_v}{K_R(s+K_R/J_v)}\right] \\
&= (1-z^{-1})\frac{KKp}{K_R}\left[\frac{Tz}{(z-1)^2}-\frac{J_v z}{K_R(z-1)}+\frac{J_v z}{K_R(z-e^{-K_R T/Jv})}\right] \\
&= \frac{KKp}{K_R^2}\left[\frac{(TK_R-J_v+J_v e^{-K_R T/Jv})z-(TK_R+J_v)e^{-K_R T/Jv}+J_v}{(z-1)(z-e^{-K_R T/Jv})}\right]
\end{aligned}$$

Characteristic equation of the closed loop system: $z^2+\alpha_1 z+\alpha_0=0$, where

$$\alpha_1 = f_1(K, Kp, K_R, Jv)$$

$$\alpha_0 = f_0(K, Kp, K_R, Jv)$$

Substituting the known parameters:

$$\begin{aligned}
\alpha_1 &= 0.000012K(3.71\times 10^5 T-41822+41822e^{-8.87T})-1-e^{-8.87T} \\
\alpha_0 &= e^{-8.87T}+0.000012K\left[41822-(3.71\times 10^5 T+41822)e^{-8.87T}\right]
\end{aligned}$$

For stability

$$\begin{aligned}
&(1) & |\alpha_0| &< 1 \\
&(2) & P(1) &= 1+\alpha_1+\alpha_0>0 \\
& & &= 1-e^{-8.87T}>0 \quad \text{always satisfied since } T \text{ is positive} \\
&(3) & P(-1) &= 1-\alpha_1+\alpha_0>0
\end{aligned}$$

**Choice of $K$ and $T$**: If we plot $K$ versus $T$ then according to conditions (1) and (3) the stable region is shown in Figure 4. Pink region represents the situation when condition (1) is satisfied but the (3) is not. Red region depicts the situation when condition (1) is satisfied, not the (3). Yellow is the stable region where both the conditions are satisfied. If we

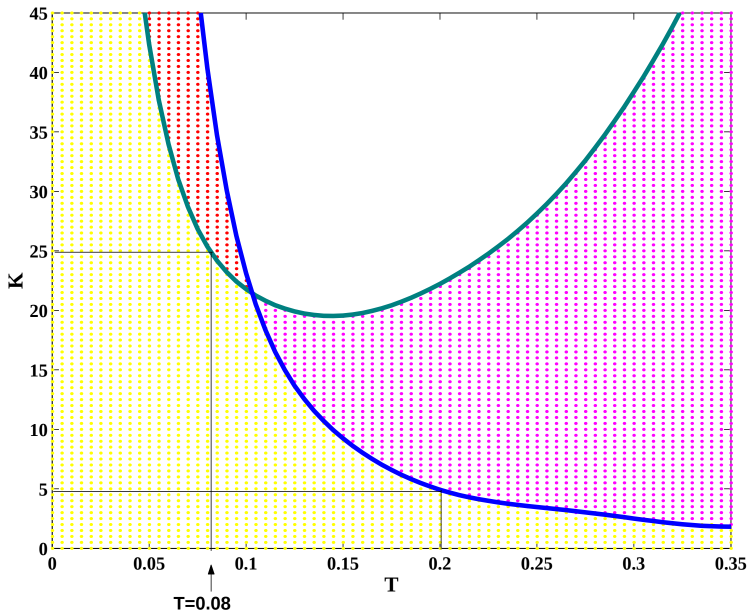



Figure 4: K vs. T for space vehicle attitude control system

want a comparatively large $T$, such as 0.2, the gain $K$ is limited by the range $K < 5$. Similarly if we want a comparatively high gain such as 25, we have to go for $T$ as small as 0.08 or even less.

From studies of continuous time systems it is well known that increasing the value of $K$ generally reduces the damping ratio, increases peak overshoot, bandwidth and reduces the steady state error if it is finite and nonzero.

# 2 Correlation between time response and root locations in s-plane and z-pane

The mapping between s-plane and z-plane was discussed earlier. For continuous time systems, the correlation between root location in s-plane and time response is well established and known.

- A root in negative real axis of s-plane produces an output exponentially decaying with time.
- Complex conjugate pole pairs in negative s-plane produce damped oscillations.
- Imaginary axis conjugate poles produce undamped oscillations.
- Complex conjugate pole pairs in positive s-plane produce growing oscillations.

Digital control systems should be given special attention due to sampling operation. For example, if the sampling theorem is not satisfied, the folding effect can entirely change the true response of the system.

The pole-zero map and natural response of a continuous time second order system is shown in Figure 5.

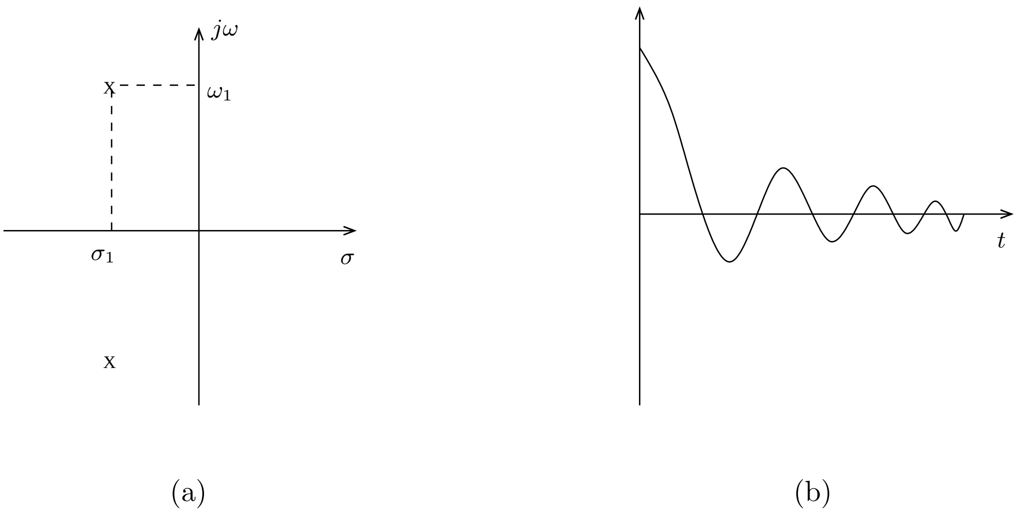



Figure 5: Pole zero map and natural response of a second order system

If the system is subject to sampling with frequency $\omega_s < 2\omega_1$, it will generate an infinite number of poles in the s-plane at $s = \sigma_1 \pm j\omega_1 + jn\omega_s$ for $n = \pm 1, \pm 2, ........$ The sampling operation will fold the poles back into the primary strip where $-\omega_s/2 < \omega < \omega_s/2$. The net effect is equivalent to having a system with poles at $s = \sigma_1 \pm j(\omega_s - \omega_1)$. The corresponding plot is shown in Figure 6.

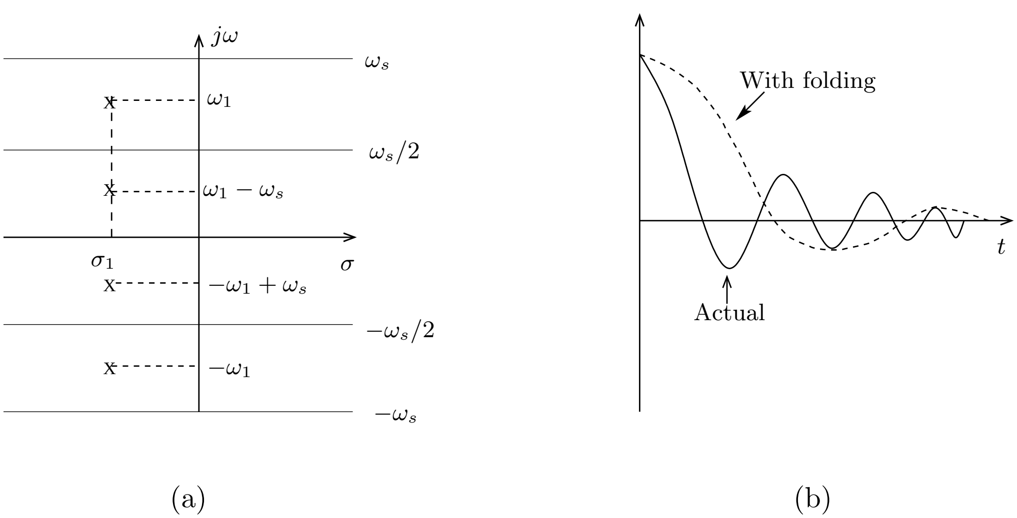



Figure 6: Pole zero map and natural response of a second order system

Root locations in z-plane and the corresponding time responses are shown in Figure 7.

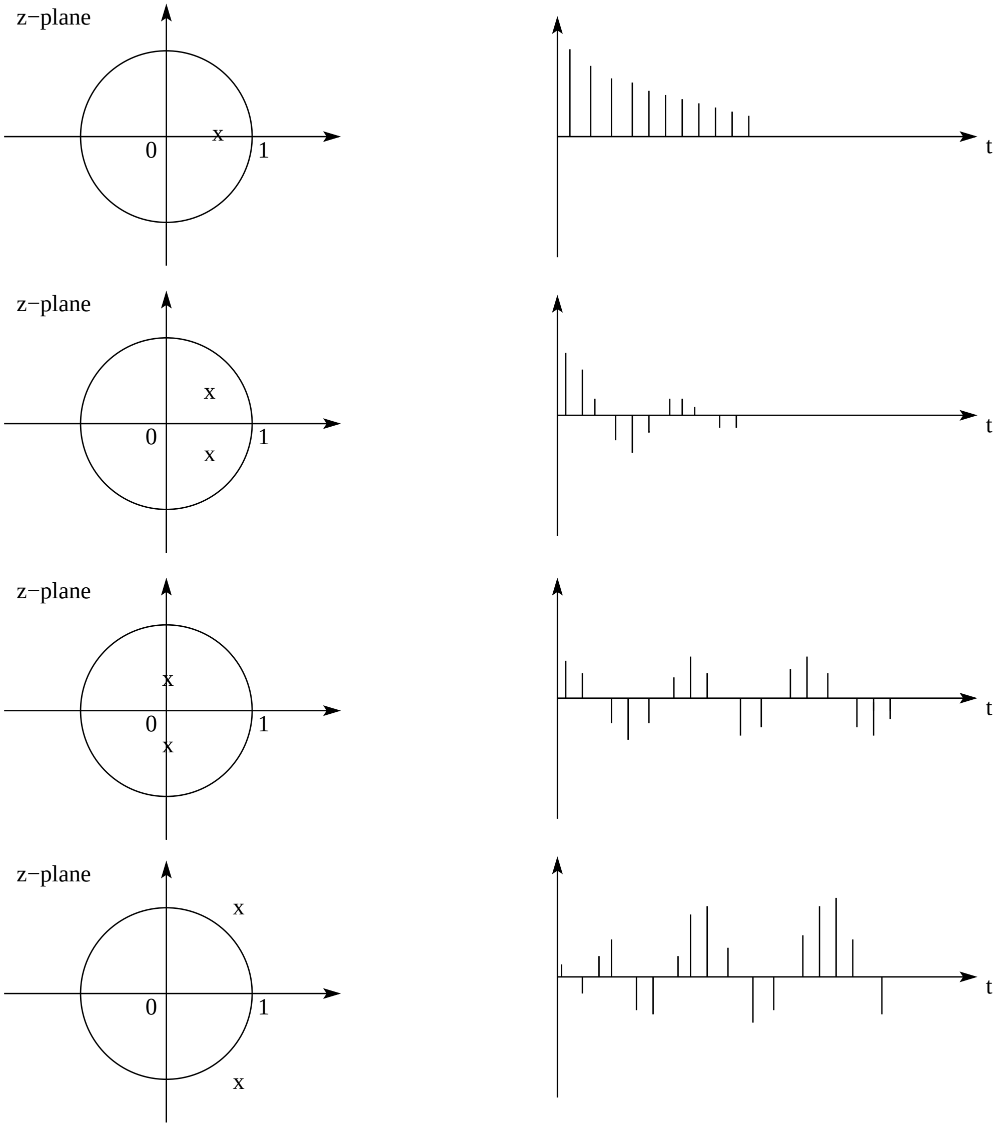



Figure 7: Pole zero map and natural response of a second order system

# 3 Dominant Closed Loop Pole Pairs

As in case of s-plane, some of the roots in z-plane have more effects on the system response than the others. It is important for design purpose to separate out those roots and give them the name dominant roots.
In s-plane, the roots that are closest to $j\omega$ axis in the left plane are the dominant roots because the corresponding time response has slowest decay. Roots that are for away from $j\omega$ axis correspond to fast decaying response.

- In Z-plane dominant roots are those which are inside and closest to the unit circle whereas insignificant region is near the origin.

- The negative real axis is generally avoided since the corresponding time response is oscillatory in nature with alternate signs.

Figure 8 shows the regions of dominant and insignificant roots in s-plane and z-plane.

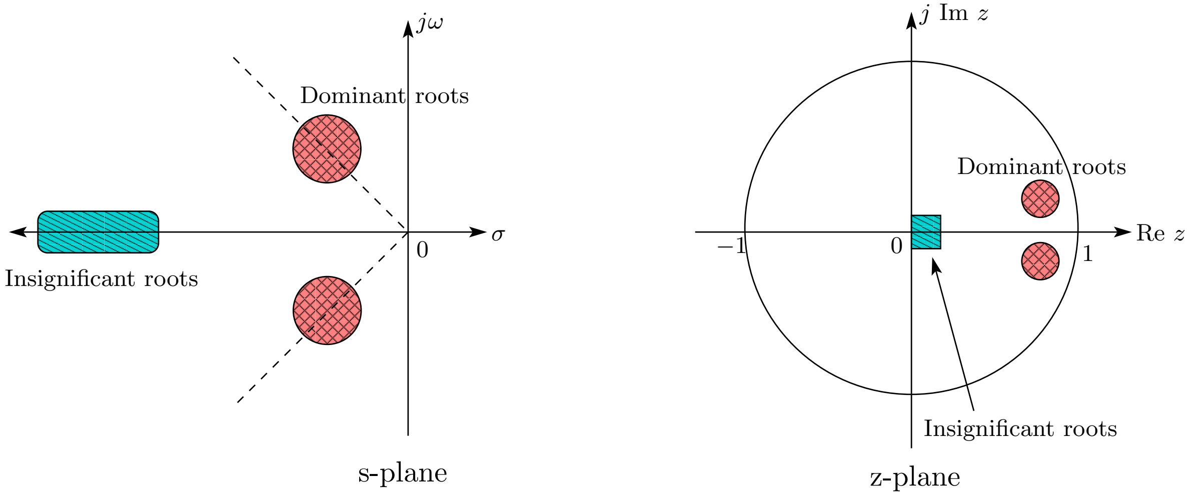



Figure 8: Pole zero map of a second order system

In s-plane the insignificant roots can be neglected provided the dc-gain (0 frequency gain) of the system is adjusted. For example,

$$\frac{10}{(s^2+2s+2)(s+10)} \approx \frac{1}{(s^2+2s+2)}$$

In z-plane, roots near the origin are less significant from the maximum overshoot and damping point of view.

However these roots cannot be completely discarded since the excess number of poles over zeros has a delay effect in the initial region of the time response, e.g., adding a pole at $z = 0$ would not effect the maximum overshoot or damping but the time response would have an additional delay of one sampling period.

The proper way of simplifying a higher order system in z-domain is to replace the poles near origin by Poles at $z = 0$ which will simplify the analysis since the Poles at $z = 0$ correspond to pure time delays.

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 1

So far we have discussed about the modelling of a discrete time system by pulse transfer function, various stability tests and time domain performance criteria. The main objective of a control system is to design a controller either in forward or in feedback path so that the closed loop system is stable with some desired performance. Two most popular design techniques for continuous time LTI systems are using root locus and frequency domain methods.

# 1 Design based on root locus method

- The effect of system gain and/or sampling period on the absolute and relative stability of the closed loop system should be investigated in addition to the transient response characteristics. Root locus method is very useful in this regard.
- The root locus method for continuous time systems can be extended to discrete time systems without much modifications since the characteristic equation of a discrete control system is of the same form as that of a continuous time control system.

In many LTI discrete time control systems, the characteristics equation may have either of the following two forms.

$$\begin{aligned} 1 + G(z)H(z) &= 0 \\ 1 + GH(z) &= 0 \end{aligned}$$

To combine both, let us define the characteristics equation as:

$$1 + L(z) = 0 \tag{1}$$

where,$L(z) = G(z)H(z)$ or $L(z) = GH(z)$. $L(z)$ is popularly known as the loop pulse transfer function. From equation (1), we can write

$$L(z) = -1$$

Since $L(z)$ is a complex quantity it can be split into two equations by equating angles and magnitudes of two sides. This gives us the angle and magnitude criteria as

$$\text{Angle Criterion:} \qquad \angle L(z) = \pm 180^0(2k+1), \quad k = 0, 1, 2....$$

$$\text{Magnitude Criterion:} \qquad |L(z)| = 1$$

The values of $z$ that satisfy both criteria are the roots of the characteristics equation or close loop poles. Before constructing the root locus, the characteristics equation $1 + L(z) = 0$ should be rearranged in the following form

$$1 + K\frac{(z+z_1)(z+z_2)....(z+z_m)}{(z+p_1)(z+p_2)....(z+p_n)} = 0$$

where $z_i$'s and $p_i$'s are zeros and poles of open loop transfer function, $m$ is the number of zeros $n$ is the number of poles.

## 1.1 Construction Rules for Root Locus

Root locus construction rules for digital systems are same as that of continuous time systems.

1. The root locus is symmetric about real axis. Number of root locus branches equals the number of open loop poles.

2. The root locus branches start from the open loop poles at $K = 0$ and ends at open loop zeros at $K = \infty$. In absence of open loop zeros, the locus tends to $\infty$ when $K \to \infty$. Number of branches that tend to $\infty$ is equal to difference between the number of poles and number of zeros.

3. A portion of the real axis will be a part of the root locus if the number of poles plus number of zeros to the right of that portion is odd.

4. If there are $n$ open loop poles and $m$ open loop zeros then $n - m$ root locus branches tend to $\infty$ along the straight line asymptotes drawn from a single point $s = \sigma$ which is called centroid of the loci.

   $$\sigma = \frac{\sum \text{real parts of the open loop poles} - \sum \text{real parts of the open loop zeros}}{n - m}$$

   Angle of asymptotes

   $$\phi_q = \frac{180^o(2q+1)}{n-m}, \quad q = 0, 1, \ldots, n-m-1$$

5. Breakaway (Break in) points or the points of multiple roots are the solution of the following equation:

   $$\frac{dK}{dz} = 0$$

   where $K$ is expressed as a function of $z$ from the characteristic equation. This is a necessary but not sufficient condition. One has to check if the solutions lie on the root locus.

6. The intersection (if any) of the root locus with the unit circle can be determined from the Routh array.

7. The angle of departure from a complex open loop pole is given by

$$\phi_p = 180^o + \phi$$

where $\phi$ is the net angle contribution of all other open loop poles and zeros to that pole.

$$\phi = \sum_i \psi_i - \sum_{j \neq p} \gamma_j$$

$\psi_i$'s are the angles contributed by zeros and $\gamma_j$'s are the angles contributed by the poles.

8. The angle of arrival at a complex zero is given by

$$\phi_z = 180^o - \phi$$

where $\phi$ is same as in the above rule.

9. The gain at any point $z_0$ on the root locus is given by

$$K = \frac{\prod_{j=1}^{n} |z_0 + p_j|}{\prod_{i=1}^{m} |z_0 + z_i|}$$

## 1.2 Root locus diagram of digital control systems

We will first investigate the effect of controller gain $K$ and sampling time $T$ on the relative stability of the closed loop system as shown in Figure 1.

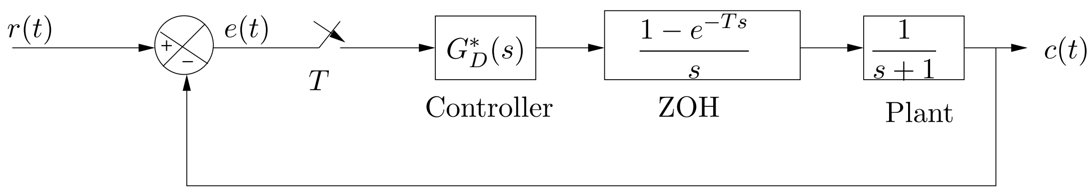



Figure 1: A discrete time control system

Let us first take T=0.5 sec.

$$\begin{aligned}
Z\left[G_{ho}(s)G_p(s)\right] &= Z\left[\frac{1-e^{-Ts}}{s}.\frac{1}{s+1}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s(s+1)}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s} - \frac{1}{s+1}\right] \\
&= \frac{z-1}{z}\left[\frac{z}{z-1} - \frac{z}{z-e^{-T}}\right] \\
&= \frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}$$

Let us assume that the controller is an integral controller, i.e., $G_D(z) = \dfrac{Kz}{z-1}$. Thus,

$$\begin{aligned}
G(z) &= G_D(z) \cdot G_h G_p(z) \\
&= \frac{Kz}{z-1} \cdot \frac{1-e^{-T}}{z-e^{-T}}
\end{aligned}$$

The characteristic equation can be written as

$$\begin{aligned}
& 1 + G(z) = 0 \\
\Rightarrow \quad & 1 + \frac{Kz(1-e^{-T})}{(z-1)(z-e^{-T})} = 0 \\
\text{when } T = 0.5sec, \quad & L(z) = \frac{0.3935Kz}{(z-1)(z-0.6065)}
\end{aligned}$$

$L(z)$ has poles at $z = 1$ and $z = 0.605$ and zero at $z = 0$.

Break away/ break in points are calculated by putting $\dfrac{dK}{dz} = 0$.

$$\begin{aligned}
K &= -\frac{(z-1)(z-0.6065)}{0.3935z} \\
\frac{dK}{dz} &= -\frac{z^2-0.6065}{0.3935z^2} = 0 \\
\Rightarrow z^2 = 0.6065 &\Rightarrow z_1 = 0.7788 \text{ and } z_2 = -0.7788
\end{aligned}$$

Critical value of $K$ can be found out from the magnitude criterion.

$$\left| \frac{0.3935z}{(z-1)(z-0.6065)} \right| = \frac{1}{K}$$

Critical gain corresponds to point $z = -1$. Thus

$$\begin{aligned}
\left| \frac{-0.03935}{(-2)(-1.6065)} \right| &= \frac{1}{K} \\
\text{or,} \quad K &= 8.165
\end{aligned}$$

Figure 2 shows the root locus of the system for $K = 0$ to $K = 10$. Two root locus branches start from two open loop poles at $K = 0$. If we further increase $K$ one branch will go towards the zero and the other one will tend to infinity. The blue circle represents the unit circle. Thus the stable range of $K$ is $0 < K < 8.165$.

If $T = 1$ sec,

$$G(z) = \frac{0.6321Kz}{(z-1)(z-0.3679)}$$

Break away/ break in points:
$z^2 = 0.3679 \Rightarrow z_1 = 0.6065$ and $z_2 = -0.6065$ Critical gain $(K_c) = 4.328$ Figure 3 shows

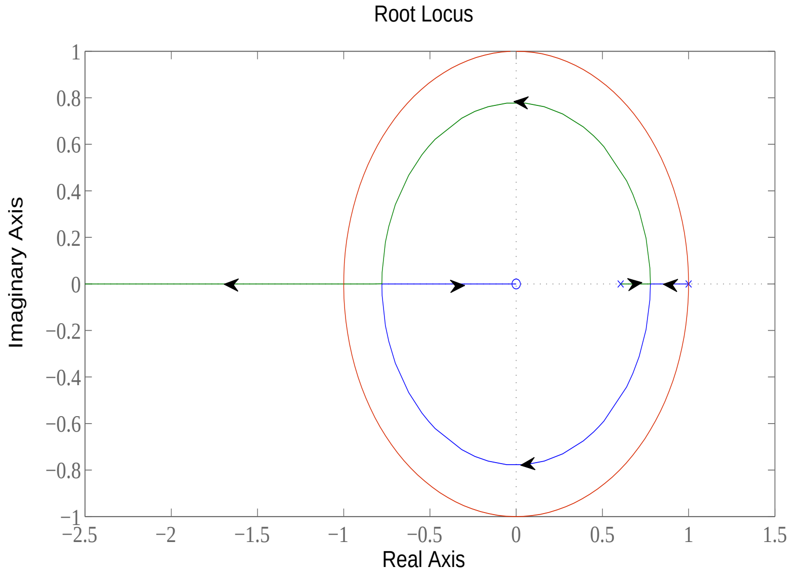



Figure 2: Root Locus when T=0.5 sec

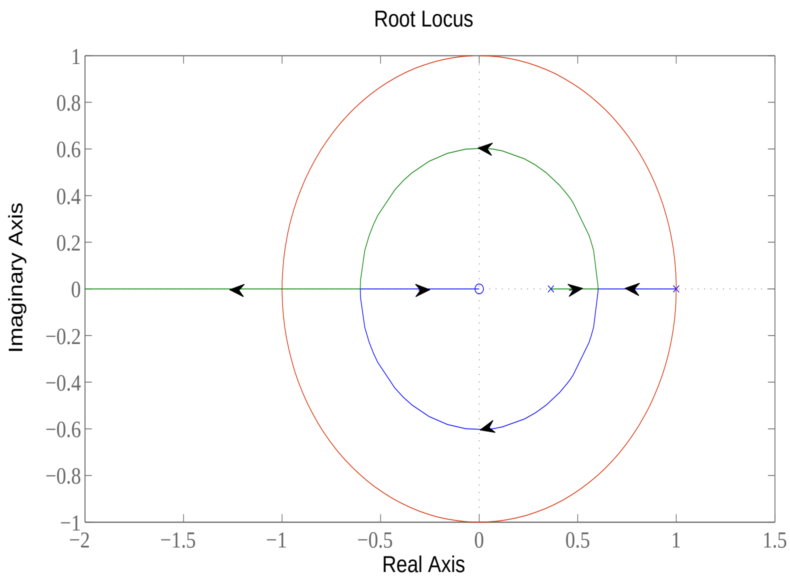



Figure 3: Root Locus when T=1 sec

the root locus for $K = 0$ to $K = 10$. It can be seen from the figure that the radius of the inside circle reduces and the maximum value of stable $K$ also decreases to $K = 4.328$.

Similarly if $T = 2$ sec,

$$G(z) = \frac{0.8647Kz}{(z-1)(z-0.1353)}$$

One can find that the critical gain in this case further reduces to 2.626.

---

### 1.2.1 Effect of sampling period T

As can be seen from the previous example, large $T$ has detrimental effect on relative stability. A thumb rule is to sample eight to ten times during a cycle of the damped sinusoidal oscillation of the output if it is underdamped. If overdamped 8/10 times during rise time.
As seen from the example making the sampling period smaller allows the critical gain to be larger, i.e., maximum allowable gain can be made larger by increasing sampling frequency /rate. It seems from the example that damping ratio decreases with the decrease in $T$. But one should take a note that damping ratio of the closed loop poles of a digital control system indicates the relative stability only if the sampling frequency is sufficiently high (8 to 10 times). If it is not the case, prediction of overshoot from the damping ratio will be erroneous and in practice the overshoot will be much higher than the predicted one.

Next, we may investigate the effect of $T$ on the steady state error. Let us take a fixed gain $K = 2$.

When $T = 0.5$ sec. and $K = 2$,

$$G(z) = \frac{0.787z}{(z-1)(z-0.6065)}$$

Since this is a second order system, velocity error constant will be a non zero finite quantity.

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 4$$

$$\text{Thus,} \quad e_{ss} = \frac{1}{4} = 0.25$$

When $T = 1$ sec. and $K = 2$

$$G(z) = \frac{1.2642z}{(z-1)(z-0.3679)}$$

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 2$$

$$e_{ss} = \frac{1}{2} = 0.5$$

When $T = 2$ sec. and $K = 2$

$$G(z) = \frac{1.7294z}{(z-1)(z-0.1353)}$$

$$Kv = \lim_{z \to 1} \frac{(1-z^{-1})G(z)}{T} = 1$$

$$e_{ss} = \frac{1}{1} = 1$$

Thus, increasing sampling period (decreasing sampling frequency) has an adverse effect on the steady state error as well.

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 2

If we remember the controller design in continuous domain using root locus, we see that the design is based on the approximation that the closed loop system has a complex conjugate pole pair which dominates the system behavior. Similarly for a discrete time case also the controller will be designed based on the concept of a dominant pole pair.

**Controller types:** We have already studied different variants of controllers such as PI, PD, PID etc. We know that PI controller is generally used to improve steady state performance whereas PD controller is used to improve the relative stability or transient response. Similarly a phase lead compensator improves the dynamic performance whereas a lag compensator improves the steady state response.

**Pole-Zero cancellation** A common practice in designing controllers in s-plane or z-plane is to cancel the undesired poles or zeros of plant transfer function by the zeros and poles of controller. New poles and zeros can also be added in some advantageous locations. However, one has to keep in mind that pole-zero cancellation scheme does not always provide satisfactory solution. Moreover, if the undesired poles are near $j\omega$ axis, inexact cancellation, which is almost inevitable in practice, may lead to a marginally stable or even unstable closed loop system. For this reason one should never try to cancel an unstable pole.

**Design Procedure:** Consider a compensator of the form $K\dfrac{z+a}{z+b}$. It will be a lead compensator if the zero lies on the right of the pole.

1. Calculate the desired closed loop pole pairs based on design criteria.
2. Map the s-domain poles to z-domain.
3. Check if the sampling frequency is $8-10$ times the desired damped frequency of oscillation.
4. Calculate the angle contributions of all open loop poles and zeros to the desired closed loop pole.
5. Compute the required contribution by the controller transfer function to satisfy angle criterion.
6. Place the controller zero in a suitable location and calculate the required angle contribution of the controller pole.

7. Compute the location of the controller pole to provide the required angle.
8. Find out the gain $K$ from the magnitude criterion.

The following example will illustrate the design procedure.

### An Example on Controller Design

Consider the closed loop discrete control system as shown in Figure 1. Design a digital controller

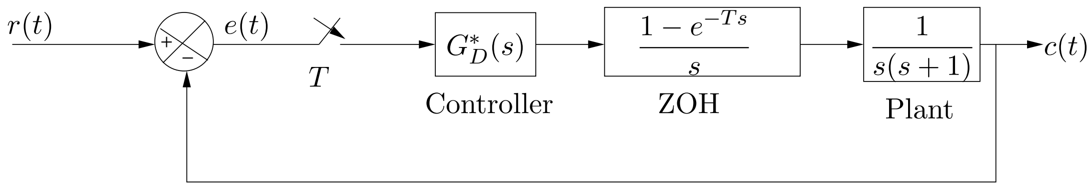



Figure 1: A discrete time control system

such that the dominant closed loop poles have a damping ratio $\xi = 0.5$ and settling time $t_s = 2$ sec for 2% tolerance band. Take the sampling period as $T = 0.2$ sec. The dominant pole pair in continuous domain is $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ where $\omega_n$ is the natural undamped frequency.

$$
\begin{aligned}
\text{Given that settling time } t_s &= \frac{4}{\xi\omega_n} = \frac{4}{0.5\omega_n} = 2 \text{ sec.} \\
\text{Thus, } \omega_n &= 4 \\
\text{Damped frequency } \omega_d &= 4\sqrt{1-0.5^2} = 3.46 \\
\text{Sampling frequency } \omega_s &= \frac{2\pi}{T} = \frac{2\pi}{0.2} = 31.4
\end{aligned}
$$

Since $\frac{31.4}{3.46} = 9.07$, we get approximately 9 samples per cycle of the damped oscillation.
The closed loop poles in s-plane

$$
\begin{aligned}
s_{1,2} &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \\
&= -2 \pm j3.46
\end{aligned}
$$

Thus the closed loop poles in z-plane

$$
\begin{aligned}
z_{1,2} &= exp(T(-2 \pm j3.46)) \\
|z| &= e^{-T\xi\omega_n} = exp(-0.4) = 0.67 \\
\angle z &= T\omega_d = 0.2 \times 3.464 = 0.69 \text{ rad} = 39.69^0 \\
\text{Thus, } z_{1,2} &= 0.67\angle 39.7^0 \cong 0.52 \pm j0.43
\end{aligned}
$$

$$
\begin{aligned}
G(z) &= Z\left[\frac{1-e^{-0.2s}}{s}.\frac{1}{s(s+1)}\right] \\
&= (1-z^{-1})Z\left[\frac{1}{s^2(s+1)}\right] \\
&\cong \frac{0.02(z+0.93)}{(z-1)(z-0.82)}
\end{aligned}
$$

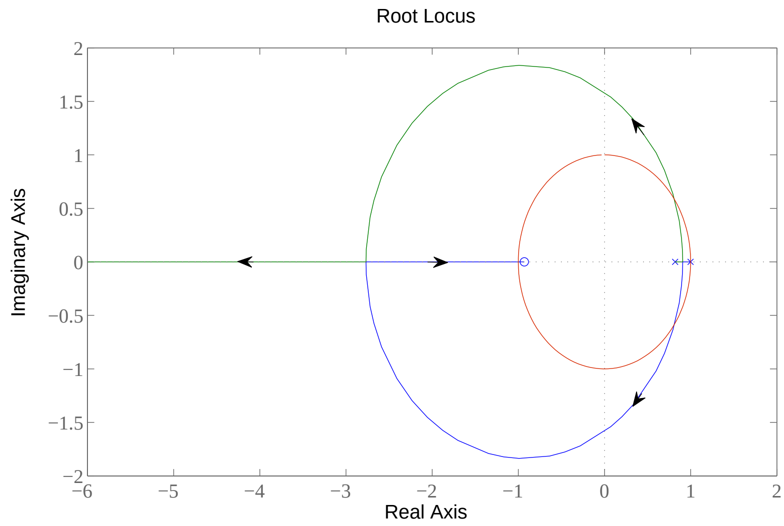



Figure 2: Root locus of uncompensated system

The root locus of the uncompensated system (without controller) is shown in Figure 2. It is clear from the root locus plot that the uncompensated system is stable for a very small range of $K$.

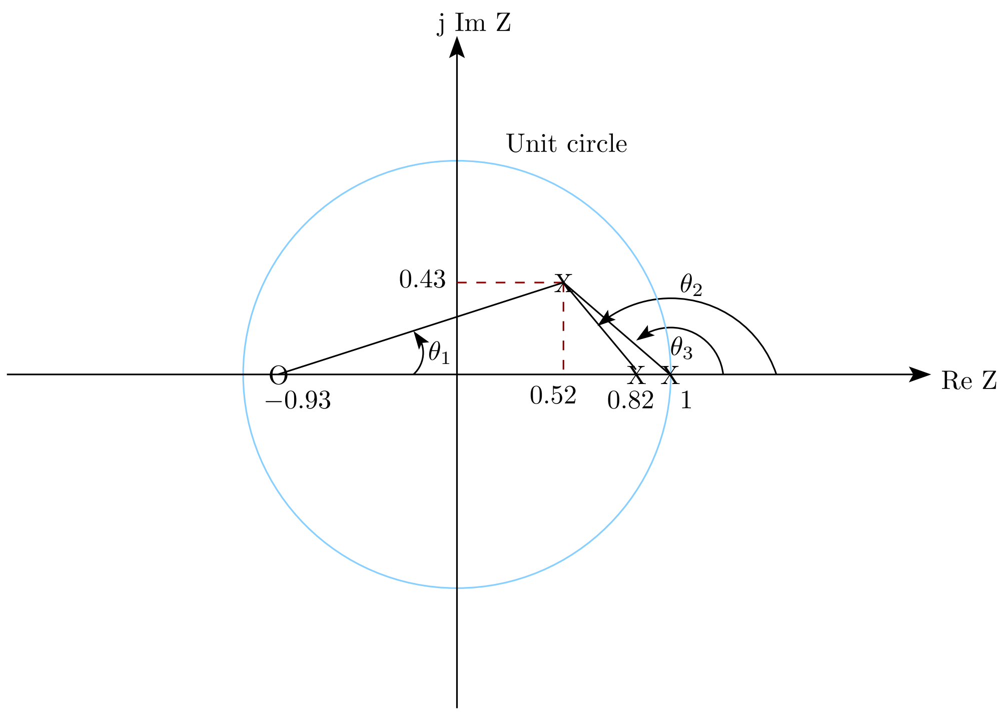



Figure 3: Pole zero map to compute angle contributions

Pole zero map of the uncompensated system is shown in Figure 3. Sum of angle contributions at the desired pole is $A = \theta_1 - \theta_2 - \theta_3$, where $\theta_1$ is the angle by the zero, $-0.93$, and $\theta_2$ and $\theta_3$ are the angles contributed by the two poles, 0.82 and 1 respectively.

From the pole zero map as shown in Figure 3, the angles can be calculated as $\theta_1 = 16.5^o$, $\theta_2 = 124.9^o$ and $\theta_3 = 138.1^o$.

Net angle contribution is $A = 16.5^o - 124.9^o - 138.1^o = -246.5^o$. But from angle criterion a point will lie on root locus if the total angle contribution at that point is $\pm 180^o$. Angle deficiency is $-246.5^o + 180^o = -66.5^o$

Controller pulse transfer function must provide an angle of $66.5^o$. Thus we need a Lead Compensator. Let us consider the following compensator.

$$G_D(z) = K\frac{z+a}{z+b}$$

If we place controller zero at $z = 0.82$ to cancel the pole there, we can avoid some of the calculations involved in the design. Then the controller pole should provide an angle of $124.9^o - 66.5^o = 58.4^o$.

Once we know the required angle contribution of the controller pole, we can easily calculate the pole location as follows.

The pole location is already assumed at $z = -b$. Since the required angle is greater than $\tan^{-1}(0.43/0.52) = 39.6^o$ we can easily say that the pole must lie on the right half of the unit circle. Thus $b$ should be negative. To satisfy angle criterion,

$$\begin{aligned}
\tan^{-1}\frac{0.43}{0.52 - |b|} &= 58.4^o \\
\text{or,}\ \frac{0.43}{0.52 - |b|} &= \tan(58.4^o) = 1.625 \\
\text{or,}\ 0.52 - |b| &= \frac{0.43}{1.56} = 0.267 \\
\text{or,}\ |b| &= 0.52 - 0.267 = 0.253 \\
\text{Thus,}\ b &= -0.253
\end{aligned}$$

The controller is then written as $G_D(z) = K\dfrac{z - 0.82}{z - 0.253}$. The root locus of the compensated system (with controller) is shown in Figure 4.

If we compare Figure 4 with Figure 2, it is evident that stable region of $K$ is much larger for the compensated system than the uncompensated system. Next we need to calculate $K$ from the magnitude criterion.

$$\text{Magnitude criterion:} \qquad \left|\frac{0.02K(z+0.93)}{(z-0.253)(z-1)}\right|_{z=0.52+j0.43} = 1$$

$$\begin{aligned}
\text{or,}\ K &= \left|\frac{(z-0.253)(z-1)}{0.02(z+0.93)}\right|_{z=0.52+j0.43} \\
&= \frac{|0.52+j0.43-0.253||0.52+j0.43-1|}{0.02|0.52+j0.43+0.93|} = 10.75
\end{aligned}$$

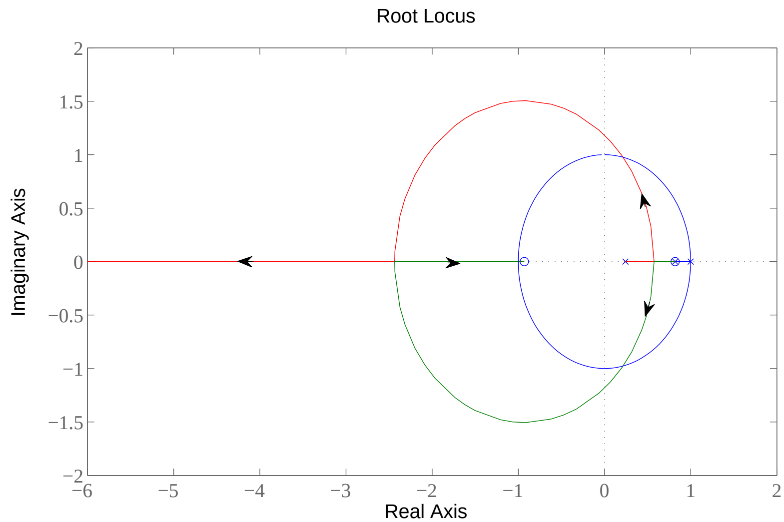



Figure 4: Root locus of the compensated system

Thus the required controller is $G_D(z) = 10.75\dfrac{z-0.82}{z-0.253}$. The SIMULINK block to compute the output response is shown in Figure 5. All discrete blocks in the SIMULINK model should have same sampling period which is 0.2 sec in this example. The scope output is shown in Figure 6.

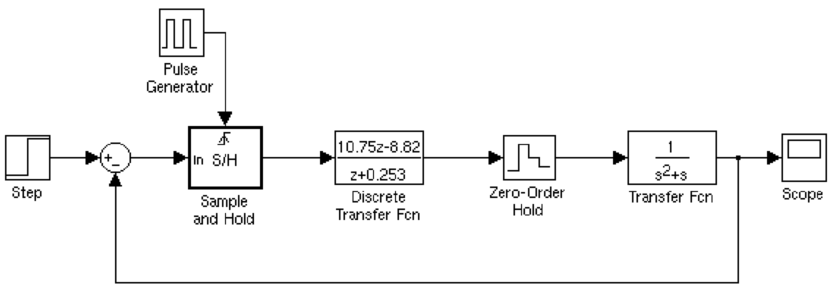



Figure 5: Simulink diagram of the closed loop system

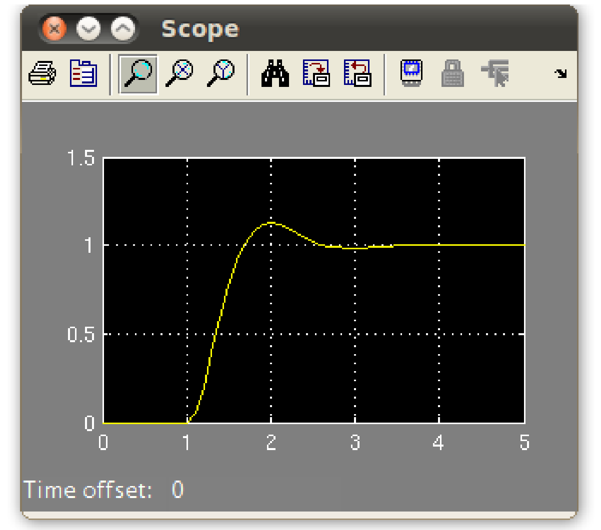



Figure 6: Output response of the closed loop system

# Module 5: Design of Sampled Data Control Systems
# Lecture Note 3

## 1 Root Locus Based Controller Design Using MATLAB

In this lecture we will show how the MATLAB platform can be utilized to design a controller using root locus technique.

Consider the closed loop discrete control system as shown in Figure 1. Design a digital controller such that the closed loop system has zero steady state error to step input with a reasonable dynamic performance. Velocity error constant of the system should at least be 5.

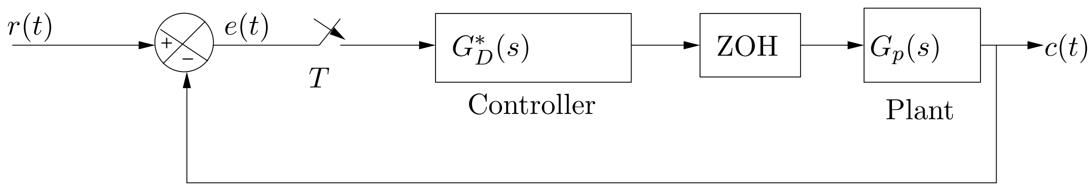



Figure 1: A discrete time control system

$$G_p(s) = \frac{10}{(s+1)(s+2)}, \qquad T = 0.1 \text{ sec}$$

$$\begin{aligned} G_{h0}G_p(z) &= Z\left[\frac{1-e^{-0.1s}}{s}\frac{10}{(s+1)(s+2)}\right] \\ &= (1-z^{-1})Z\left[\frac{10}{s(s+1)(s+2)}\right] \\ &\cong \frac{0.04528(z+0.9048)}{(z-0.9048)(z-0.8187)} \end{aligned}$$

The MATLAB script to find out $G_{h0}G_p(z)$ is as follows.

```
>> s=tf('s');
>> Gp=10/((s+1)*(s+2));
>> GhGp=c2d(Gp,0.1,'zoh');
```

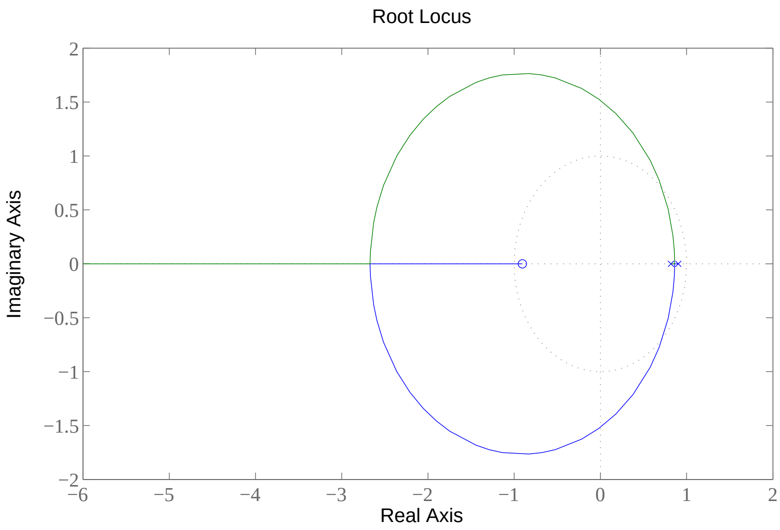



Figure 2: Root locus of the uncompensated system

The root locus of the uncompensated system (without controller) is shown in Figure 2 for which the MATLAB command is

```
>> rlocus(GhGp)
```

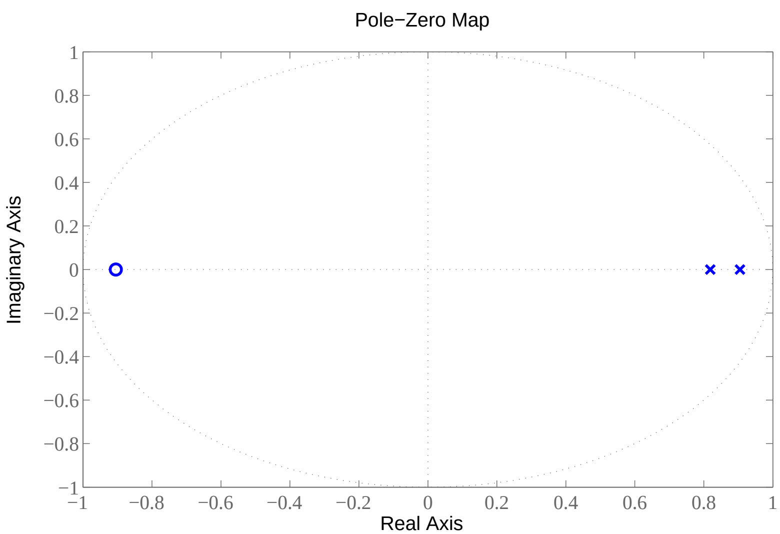



Figure 3: Pole zero map of the uncompensated system

Pole zero map of the uncompensated system is shown in Figure 3 which can be generated using the MATLAB command

```
>> pzplot(GhGp)
```

One of the design criteria is that the closed loop system should have a zero steady state error for unit step input. Thus a PI controller is required which has the following transfer function in z-domain when backward rectangular integration is used.

$$G_D(z) = K_p + \frac{K_i T}{z-1} = \frac{K_p z - (K_p - K_i T)}{z-1}$$

The parameter $K_i$ can be designed using the velocity error constant requirement.

$$k_v = \frac{1}{T}\lim_{z\to 1}(z-1)G_D(z)G_{h0}G_p(z) = 5K_i \geq 5$$

Above condition will be satisfied if $K_i \geq 1$. Let us take $K_i = 1$. With $K_i = 1$, the characteristic equation becomes

$$(z-1)(z-0.9048)(z-0.8187) + 0.004528(z+0.9048) + 0.04528K_p(z-1)(z+0.9048) = 0$$

$$\text{or,}\;\; 1 + \frac{0.04528K_p(z-1)(z+0.9048)}{z^3 - 2.724z^2 + 2.469z - 0.7367} = 0$$

Now, we can plot the root locus of the compensated system with $K_p$ as the variable parameter. The MATLAB script to plot the root locus is as follows.

```
>> z=tf('z',0.1);
>> Gcomp=0.04528*(z-1)*(z+0.9048)/(z^3 - 2.724*z^2 + 2.469*z - 0.7367);
>> zero(Gcomp);
>> pole(Gcomp);
>> rlocus(Gcomp)
```

The zeros of the system are 1 and $-0.9048$ and the poles of the system are $1.0114 \pm 0.1663i$ and 0.7013 respectively. The root locus plot is shown in Figure 4.

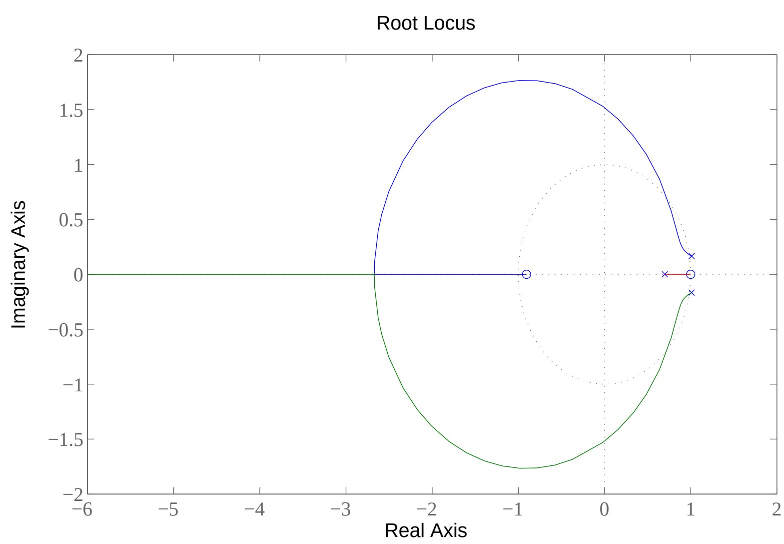



Figure 4: Root locus of the system with PI controller

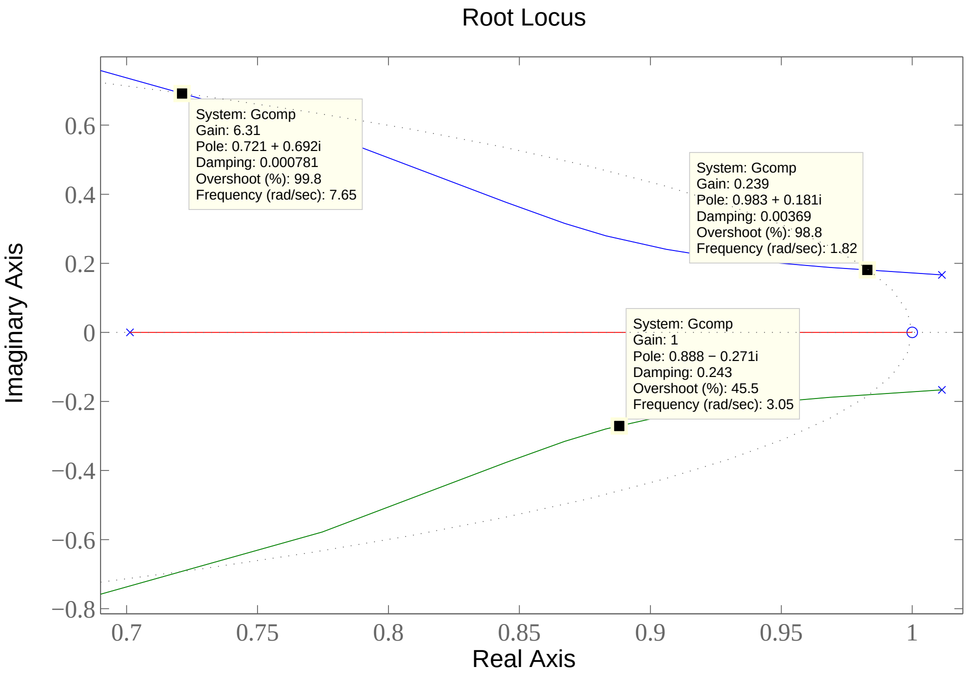



Figure 5: Root locus of the system with PI controller

It is clear from the figure that the system is stable for a very small range of $K_p$. The stable portion of the root locus is zoomed in Figure 5. The figure shows that the stable range of $K_p$ is $0.239 < K_p < 6.31$. The best achievable overshoot is 45.5%, for $K_p = 1$, which is very high for any practical system. To improve the relative stability, we need to introduce D action. Let us modify the controller to a PID controller for which the transfer function in z-domain is given as below.

$$G_D(z) = \frac{(K_pT + K_d)z^2 + (K_iT^2 - K_pT - 2K_d)z + Kd}{Tz(z-1)}$$

To satisfy velocity error constant, $K_i \geq 1$. If we assume 15% overshoot (corresponding to $\xi \cong 0.5$) and 2 sec settling time (corresponding to $\omega_n \cong 4$), the desired dominant poles can be calculated as,

$$\begin{aligned} s_{1,2} &= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} \\ &= -2 \pm j3.46 \end{aligned}$$

Thus the closed loop poles in z-plane

$$\begin{aligned} z_{1,2} &= exp(T(-2 \pm j3.46)) \\ &\cong 0.77 \pm j0.28 \end{aligned}$$

The pole zero map including the poles of the PID controller is shown in Figure 6 where the red cross denotes the desired poles.

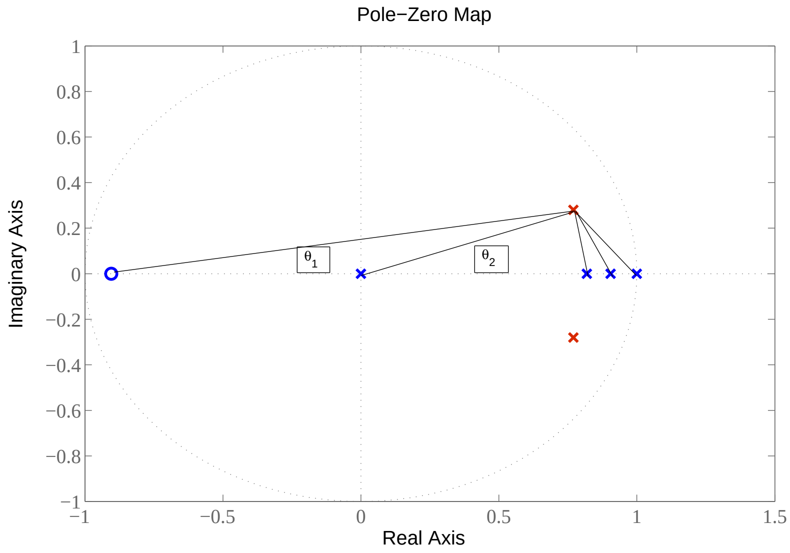



Figure 6: Pole zero map including poles of the PID controller

Let us denote the angle contribution starting from the zero to the right most pole as $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ and $\theta_5$ respectively. The angles can be calculated as $\theta_1 = 9.5^o$, $\theta_2 = 20^o$, $\theta_3 = 99.9^o$, $\theta_4 = 115.7^o$ and $\theta_5 = 129.4^o$.

Net angle contribution is $A = 9.5^o - 20^o - 99.9^o - 115.7^o - 129.4^o = -355.5^o$. Angle deficiency is $-355.5^o + 180^o = -175.5^o$

Thus the two zeros of PID controller must provide an angle of $175.5^o$. Let us place the two zeros at the same location, $z_{pid}$.

Since the required angle by individual zero is $87.75^o$, we can easily say that the zeros must lie on the left of the desired closed loop pole.

$$\begin{aligned}
\tan^{-1}\frac{0.28}{0.77 - z_{pid}} &= 87.75^o \\
\text{or,}\ \frac{0.28}{0.77 - z_{pid}} &= \tan(87.75^o) = 25.45 \\
\text{or,}\ 0.77 - z_{pid} &= \frac{0.28}{25.45} = 0.011 \\
\text{or,}\ z_{pid} &= 0.77 - 0.011 = 0.759
\end{aligned}$$

The controller is then written as $G_D(z) = K\dfrac{(z-0.759)^2}{z(z-1)}$. The root locus of the compensated system (with PID controller) is shown in Figure 7. This figure shows that the desired closed loop pole corresponds to $K = 4.33$.

Thus the required controller is $G_D(z) = 4.33\dfrac{z^2 - 1.518z + 0.5761}{z(z-1)}$. If we compare the above

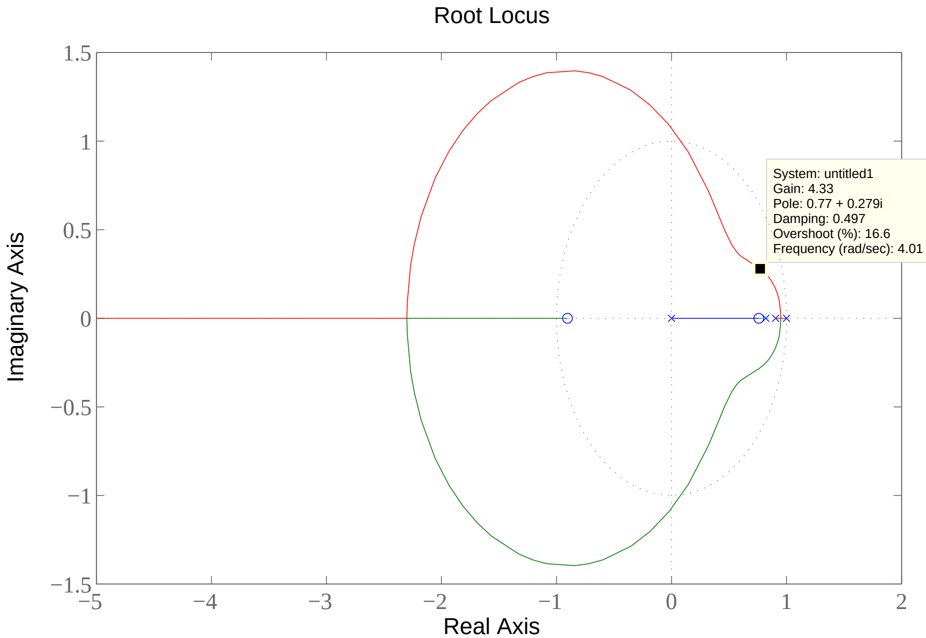



Figure 7: Root locus of compensated system

transfer function with the general PID controller, $K_p$ and $K_d$ can be computed as follows.

$$\begin{aligned} K_d/T = 0.5761 * 4.33 \quad &\Rightarrow \quad K_d = 0.2495 \\ K_p + K_d/T = 4.33 \quad &\Rightarrow \quad K_p = 1.835 \\ K_iT - K_p - 2K_d/T = -1.518 * 4.33 \quad &\Rightarrow \quad K_i = 2.521 \end{aligned}$$

Note that the above $K_i$ satisfies the constraint $K_i \geq 1$. One should keep in mind that the design is based on second order dominant pole pair approximation. But, in practice, there will be other poles and zeros of the closed loop system which might not be insignificant compared to the desired poles. Thus the actual overshoot of the system may differ from the designed one.

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 4

# 1 Frequency Domain Analysis

When a sinusoidal input is given to a stable LTI system it produces a sinusoidal output of same frequency but with different magnitude and phase.

The variation of output magnitude and phase with input frequency is known as frequency response of the system.

Frequency domain analysis provides a good design in presence of uncertainty in plant model. Experimental results can be used to construct frequency response even if the plant model is unknown.

Analysis of digital control systems in frequency domain depends on the extension of the existing techniques in continuous time case.

Two most popular graphical representations in frequency domain are **Nyquist plot** and **Bode diagram**.

## 1.1 Nyquist plot

The Nyquist plot of a transfer function, usually the loop transfer function $GH(z)$, is a mapping of Nyquist contour in z-plane onto $GH(z)$ plane which is in polar coordinates. Thus it is sometimes known as polar plot.

Absolute and relative stabilities can be determined from the Nyquist plot using Nyquist stability criterion.

Given the loop transfer function $GH(z)$ of a digital control system, the polar plot of $GH(z)$ is obtained by setting $z = e^{j\omega T}$ and varying $\omega$ from 0 to $\infty$.

**Nyquist stability criterion:** The closed loop transfer function of single input single output digital control system is described by

$$M(z) = \frac{G(z)}{1 + GH(z)}$$

Characteristic equation $1 + GH(z) = 0$

The stability of the system depends on the roots of the characteristic equation or poles of the system. All the roots of the characteristic equation must lie inside the unit circle for the system to be stable.

Before discussing Nyquist stability criterion for the digital system, following steps are necessary.

1. Defining the Nyquist path in the $z$-plane that encloses the exterior of the unit circle. Here the region to the left of a closed path is considered to be enclosed by that path when the direction of the path is taken anticlockwise.

2. Mapping the Nyquist path in $z$-plane onto the $GH(z)$ plane which results in Nyquist plot of $GH(z)$.

3. Stability of the closed loop system is investigated by studying the behavior of Nyquist plot with respect to the critical point $(-1,\ j0)$ in the $GH(z)$ plane.

Two Nyquist paths are defined. The Nyquist path **z1** as shown in Figure 1 does not enclose poles on the unit circle whereas the Nyquist path **z2** as shown in Figure 2 encloses poles on the unit circle.

These figures are the mapping of the Nyquist contours in s-plane where the entire right half of the s-plane, without or with the imaginary axis poles, is enclosed by the contours.

Let us now define the following parameters.

$Z_{-1}$ = number of zeros of $1 + GH(z)$ outside the unit circle in the $z$-plane.

$P_{-1}$ = number of poles of $1 + GH(z)$ outside the unit circle in the $z$-plane.

$P_0$ = number of poles of $GH(z)$ (same as number of poles of $1+GH(z)$) that are on the unit circle.

$N_1$ = number of times the $(-1,\ j0)$ point is encircled by the Nyquist plot of $GH(z)$ corresponding to **z1**.

$N_2$ = number of times $(-1,\ j0)$ point is encircled by Nyquist plot of $GH(z)$ corresponding to **z2**.

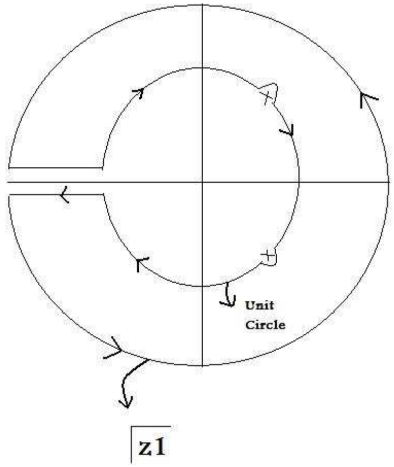



Figure 1: Nyquist path that does not enclose poles on the unit circle

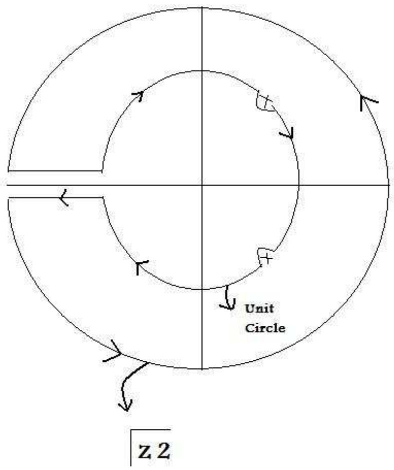



Figure 2: Nyquist path that encloses poles on the unit circle

According to the principle of argument in complex variable theory

$$N_1 = Z_{-1} - P_{-1}$$

$$N_2 = Z_{-1} - P_{-1} - P_0$$

Now let us denote the angle traversed by the phasor drawn from $(-1,\ j0)$ point to the Nyquist plot of $GH(z)$ as $\omega$ varies from $\frac{\omega_s}{2}$ to 0, on the unit circle of **z1** excluding the small indentations, by $\phi$.

It can be shown that

$$\phi = (Z_{-1} - P_{-1} - 0.5P_0)180^0 \tag{1}$$

For the closed loop digital control system to be stable, $Z_{-1}$ should be equal to zero. Thus the Nyquist criterion for stability of the closed loop digital control systems is

$$\phi = -(P_{-1} + 0.5P_0)180^0 \tag{2}$$

Hence, we can conclude that ***for the closed loop digital control system to be stable, the angle, traversed by the phasor drawn to the*** $GH(z)$ ***plot from (-1, j0) point as*** $\omega$ ***varies from*** $\frac{\omega_s}{2}$ ***to*** $0$***, must satisfy equation*** $(2)$***.***

**Example 1:** Consider a digital control system for which the loop transfer function is given as

$$GH(z) = \frac{0.095Kz}{(z-1)(z-0.9)}$$

where $K$ is a gain parameter. The sampling time $T = 0.1$ sec.

Since $GH(z)$ has one pole on the unit circle and does not have any pole outside the unit circle,

$$P_{-1} = 0 \quad \text{and} \quad P_0 = 1$$

Nyquist path has a small indentation at $z = 1$ on the unit circle.

Nyquist plot of $GH(z)$, as shown in Figure 3, intersects the negative real axis at $-0.025K$ when $\omega = \frac{\omega_s}{2} = 31.4$ rad/sec.

$\phi$ can be computed as

$$\phi = -(0 + 0.5 \times 1)180^0 = -90^0$$

It can be seen from Figure 3 that for $\phi$ to be $-90^0$, $(-1,\ j0)$ point should be located at the left of $-0.025K$ point. Thus for stability

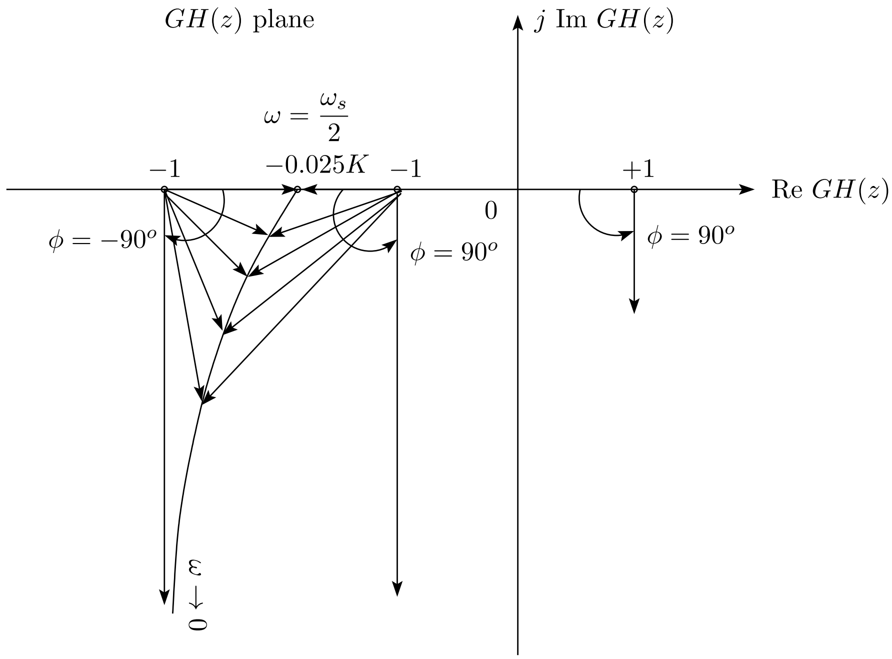



Figure 3: Nyquist plot for Example 1

$$-1 < -0.025K$$

$$\Rightarrow K < 40$$

If $K > 40$, $(-1,\ j0)$ will be at the right of $-0.025K$ point, hence making $\phi = 90^o$.

If $\phi = 90^o$, we get from (1)

$$Z_{-1} = \frac{\phi}{180^o} + 0 + 0.5 = 1$$

Thus for $K > 40$, one of the closed loop poles will be outside the unit circle.

If $K$ is negative we can still use the same Nyquist plot but refer $(+1,\ j0)$ point as the critical point. $\phi$ in this case still equals $+90^o$ and the system is unstable. Hence the stable range of $K$ is

$$0 \leq K < 40$$

More details can be found in **Digital Control Systems** by B. C. Kuo.

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 5

## 1 Bode Plot

Bode plot is the graphical tool for drawing the frequency response of a system.

It is represented by two separate plots, one is the magnitude vs frequency and the other one is phase vs frequency.The magnitude is expressed in dB and the frequency is generally plotted in log scale.

One of the advantages of the Bode plot in s-domain is that the magnitude curve can be approximated by straight lines which allows the sketching of the magnitude plot without exact computation.

This feature is lost when we plot Bode diagram in z-domain . To incorporate this feature we use bi-linear transformation to transform unit circle of the z-plane into the imaginary axis of another complex plane, $w$ plane, where

$$w = \frac{1}{T}\ln(z)$$

From the power series expansion

$$w = \frac{2}{T}\frac{(z-1)}{(z+1)}$$

$$\Rightarrow z = \frac{\frac{2}{T}+w}{\frac{2}{T}-w} = \frac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$$

For frequency domain analysis the above bi-linear transformation may be used to convert $GH(z)$ to $GH(w)$ and then construct the Bode plot.

**Example 1:** Let us consider a digital control system for which the loop transfer function is given by

$$GH(z) = \frac{0.095z}{(z-1)(z-0.9)}$$

where sampling time $T = 0.1$ sec. Putting $z = \dfrac{1+\frac{wT}{2}}{1-\frac{wT}{2}}$, we get the transfer function in $w$ plane as

$$\begin{aligned} GH(w) &= \frac{10.02(1-0.0025w^2)}{w(1+1.0026w)} \qquad (T = 0.1\,\text{sec}) \\ &= \frac{10.02(1-0.05w)(1+0.05w)}{w(1+1.0026w)} \\ &= \frac{10.02(1-0.05j\omega_w)(1+0.05j\omega_w)}{j\omega_w(1+j1.0026\omega_w)} \end{aligned}$$

where $\omega_w$ is the frequency in $w$ plane. Corner frequencies are $1/1.0026 = 0.997$ rad/sec and $1/0.05 = 20$ rad/sec.

The straight line asymptotes of the Bode plot can be drawn using the following.

- Up to $\omega_w = 0.997$ rad/sec, the magnitude plot is a straight line with slope $-20$ dB/decade. At $\omega_w = 0.01$ rad/sec, the magnitude is $20\log_{10}(10.02) - 20\log_{10}(0.01) = 60$ dB.
- From $\omega_w = 0.997$ rad/sec to $\omega_w = 20$ rad/sec, the magnitude plot is a straight line with slope $-20-20 = -40$ dB/decade.
- Since both of the zeros will contribute same to the magnitude plot, after $\omega_w = 20$ rad/sec, the slope of the straight line will be $-40+20+20 = 0$dB/decade.

The asymptotic magnitude plot is shown in Figure 1.

One should remember that the actual plot will be slightly different from the asymptotic plot. In the actual plot, errors due to straight line assumptions is compensated.

Phase plot is drawn by varying the frequency from 0.01 to 100 rad/sec at regular intervals. The phase angle contributed by one zero will be canceled by the other. Thus the phase will vary from $-90^o(270^o)$ to $-180^o(180^o)$.

Figure 2 shows the actual magnitude and phase plot as drawn in MATLAB.

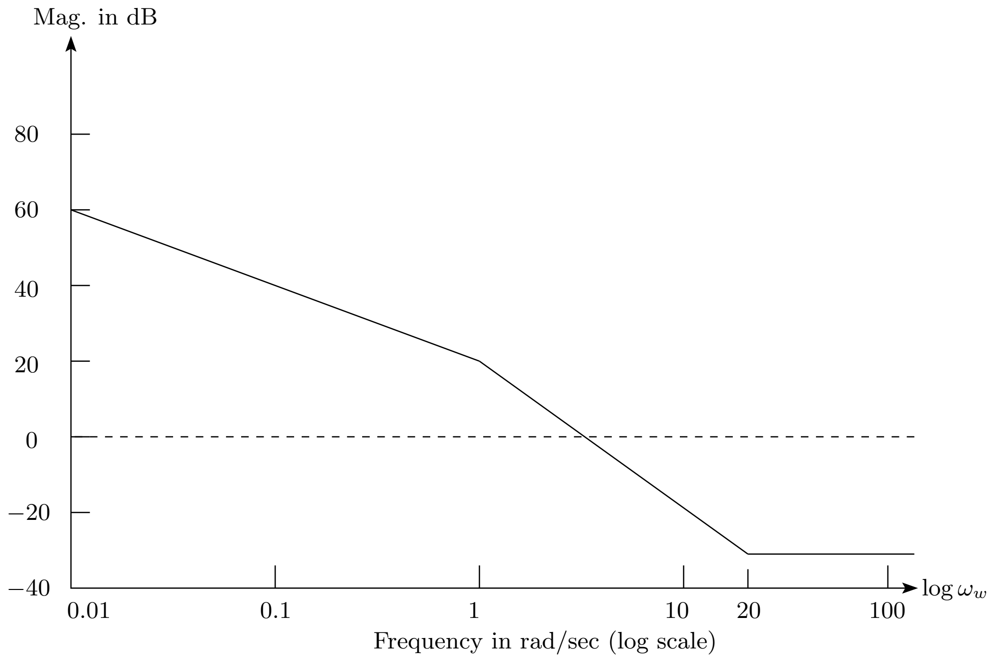



Figure 1: Bode asymptotic magnitude plot for Example 1

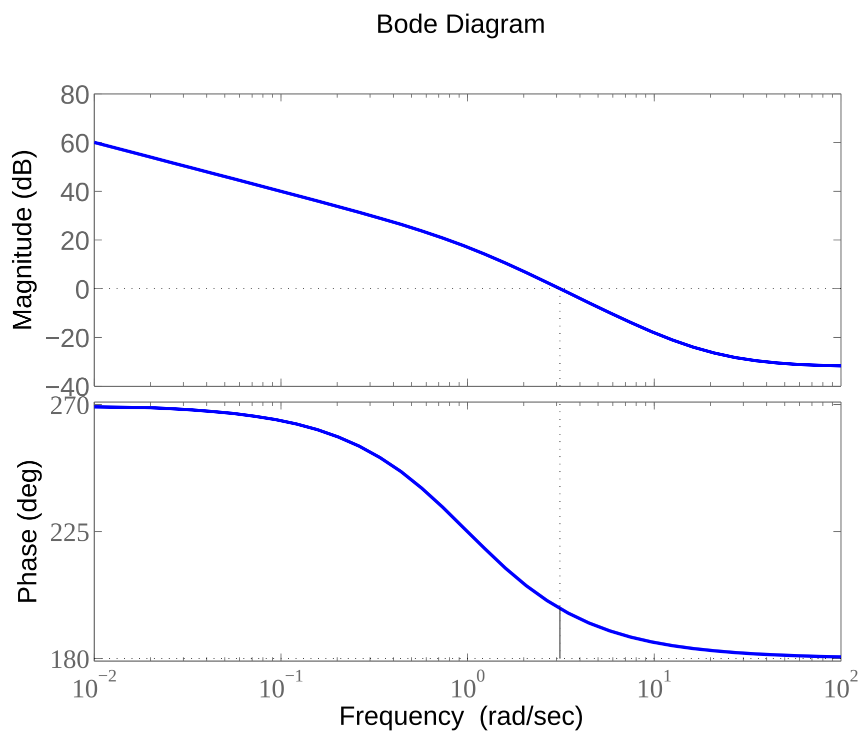



Figure 2: Bode magnitude and phase plot for Example 1

## 1.1 Gain margin and Phase margin

Gain margin and phase margins are the measures of relative stability of a system.

Similar to continuous time case, we have to first define phase and gain cross over frequencies before defining gain margin and phase margin.

Gain margin is the safety factor by which the open loop gain of a system can be increased before the system becomes unstable. It is measured as

$$GM = 20\log_{10}\left|\frac{1}{GH(e^{i\omega_p T})}\right| dB$$

where $\omega_p$ is the phase crossover frequency which is defined as the frequency where the phase of the loop transfer function $GH(e^{i\omega T})$ is $180^o$.

Similarly Phase margin (PM) is defined as

$$PM = 180^o + \angle GH(e^{i\omega_g T})$$

where $\omega_g$ is the gain crossover frequency which is defined as the frequency where the loop gain magnitude of the system becomes one.

## 1.2 Compensator design using Bode plot

A compensator or controller is added to a system to improve its steady state as well as dynamic responses.

Nyquist plot is difficult to modify after introducing controller.

Instead Bode plot is used since two important design criteria, phase margin and gain crossover frequency are visible from the Bode plot along with gain margin.

**Points to remember**

- Low frequency asymptote of the magnitude curve is indicative of one of the error constants $K_p, K_v, K_a$ depending on the system types.
- Specifications on the transient response can be translated into phase margin (PM), gain margin (GM), gain crossover frequency, bandwidth etc.
- Design using bode plot is simple and straight forward.
- Reconstruction of Bode plot is not a difficult task.

## 1.3 Phase lead, Phase lag and Lag-lead compensators

Phase lead, phase lag and lag-lead compensators are widely used in frequency domain design.

Before going into the details of the design procedure, we must remember the following.

- Phase lead compensation is used to improve stability margins. It increases system bandwidth thus improving the spread of the response.
- Phase lag compensation reduces the system gain at high frequencies with out reducing low frequency gain. Thus the total gain/low frequency gain can be increased which in turn will improve the steady state accuracy. High frequency noise can also be attenuated. But stability margin and bandwidth reduce.
- Using a lag lead compensator, where a lag compensator is cascaded with a lead compensator, both steady state and transient responses can be improved.

Bi-linear transformation transfers the loop transfer function in $z$-plane to $w$-plane.

Since qualitatively $w$-plane is similar to $s$-plane, design technique used in $s$-plane can be employed to design a controller in $w$-plane.

Once the design is done in $w$-plane, controller in $z$-plane can be determined by using the inverse transformation from $w$-plane to $z$-plane.

In the next two lectures we will discuss compensator design in $s$-plane and solve examples to design digital controllers using the same concept.

# Module 5: Design of Sampled Data Control Systems
# Lecture Note 6

## 1 Compensator Design Using Bode Plot

In this lecture we would revisit the continuous time design techniques using frequency domain since these can be directly applied to design for digital control system by transferring the loop transfer function in $z$-plane to $w$-plane.

### 1.1 Phase lead compensator

If we look at the frequency response of a simple PD controller, it is evident that the magnitude of the compensator continuously grows with the increase in frequency.

The above feature is undesirable because it amplifies high frequency noise that is typically present in any real system.

In lead compensator, a first order pole is added to the denominator of the PD controller at frequencies well higher than the corner frequency of the PD controller.

A typical lead compensator has the following transfer function.

$$C(s) = K\frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha < 1$$

$\dfrac{1}{\alpha}$ is the ratio between the pole zero break point (corner) frequencies.

Magnitude of the lead compensator is $K\dfrac{\sqrt{1+\omega^2\tau^2}}{\sqrt{1+\alpha^2\omega^2\tau^2}}$. And the phase contributed by the lead compensator is given by

$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\alpha\omega\tau$$

Thus a significant amount of phase is still provided with much less amplitude at high frequencies.

The frequency response of a typical lead compensator is shown in Figure 1 where the magnitude varies from $20\log_{10} K$ to $20\log_{10}\dfrac{K}{\alpha}$ and maximum phase is always less than $90^o$ (around $60^o$ in general).

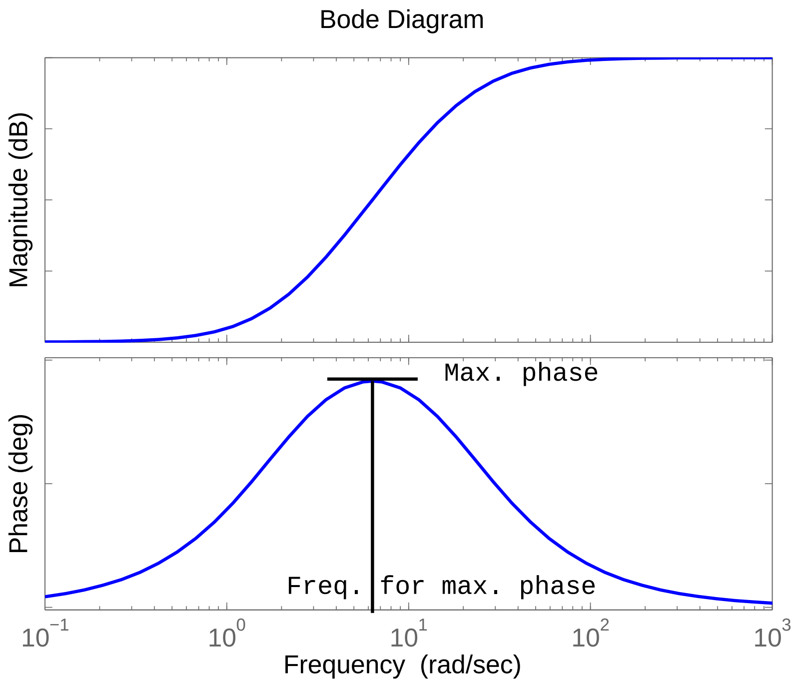



Figure 1: Frequency response of a lead compensator

It can be shown that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

The maximum phase corresponds to

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

$$\Rightarrow \alpha = \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right)$$

The magnitude of $C(s)$ at $\omega_{\max}$ is $\dfrac{K}{\sqrt{\alpha}}$.

**Example 1:** Consider the following system

$$G(s) = \frac{1}{s(s+1)}, \quad H(s) = 1$$

Design a cascade lead compensator so that the phase margin (PM) is at least $45^o$ and steady state error for a unit ramp input is $\leq 0.1$.

The lead compensator is

$$C(s) = K\frac{\tau s + 1}{\alpha\tau s + 1}, \quad \text{where, } \alpha < 1$$

When $s \to 0$, $C(s) \to K$.

Steady state error for unit ramp input is

$$\frac{1}{\lim_{s\to 0} sC(s)G(s)} = \frac{1}{C(0)} = \frac{1}{K}$$

Thus $\dfrac{1}{K} = 0.1$, or $K = 10$.

PM of the closed loop system should be $45^o$. Let the gain crossover frequency of the uncompensated system with $K$ be $\omega_g$.

$$\begin{aligned}
G(j\omega) &= \frac{1}{j\omega(j\omega+1)} \\
Mag. &= \frac{1}{\omega\sqrt{1+\omega^2}} \\
Phase &= -90^o - \tan^{-1}\omega \\
\Rightarrow \frac{10}{\omega_g\sqrt{1+\omega_g^2}} &= 1 \\
\frac{100}{\omega_g^2(1+\omega_g^2)} &= 1 \\
\Rightarrow \omega_g &= 3.1
\end{aligned}$$

Phase angle at $\omega_g = 3.1$ is $-90 - \tan^{-1} 3.1 = -162^o$. Thus the PM of the uncompensated system with $K$ is $18^o$.

If it was possible to add a phase without altering the magnitude, the additional phase lead required to maintain PM=$45^o$ is $45^o - 18^o = 27^o$ at $\omega_g = 3.1$ rad/sec.

However, maintaining same low frequency gain and adding a compensator would increase the crossover frequency. As a result of this, the actual phase margin will deviate from the designed one. Thus it is safe to add a safety margin of $\epsilon$ to the required phase lead so that if it devaites also, still the phase requirement is met. In general $\epsilon$ is chosen between $5^o$ to $15^o$.

So the additional phase requirement is $27^o + 10^o = 37^o$. The lead part of the compensator will provide this additional phase at $\omega_{\max}$.

Thus

$$\begin{aligned}
\phi_{\max} &= 37^o \\
\Rightarrow \alpha &= \left(\frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})}\right) = 0.25
\end{aligned}$$

The only parameter left to be designed is $\tau$. To find $\tau$, one should locate the frequency at which the uncompensated system has a logarithmic magnitude of $-20\log_{10}\dfrac{1}{\sqrt{\alpha}}$.

Select this frequency as the new gain crossover frequency since the compensator provides a gain of $20\log_{10}\dfrac{1}{\sqrt{\alpha}}$ at $\omega_{\max}$. Thus

$$\omega_{\max} = \omega_{g_{new}} = \frac{1}{\tau\sqrt{\alpha}}$$

In this case $\omega_{\max} = \omega_{g_{new}} = 4.41$. Thus

$$\tau = \frac{1}{4.41\sqrt{\alpha}} = 0.4535$$

The lead compensator is thus

$$C(s) = 10\,\frac{0.4535s+1}{0.1134s+1}$$

With this compensator actual phase margin of the system becomes $49.6^o$ which meets the design criteria. The corresponding Bode plot is shown in Figure 2

**Example 2:**

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time $T = 0.2$ sec. Thus

$$\begin{aligned}
G_z(z) &= (1-z^{-1})\mathrm{Z}\left[\frac{1}{s^2(s+1)}\right] \\
&= \frac{0.0187z+0.0175}{z^2-1.8187z+0.8187}
\end{aligned}$$

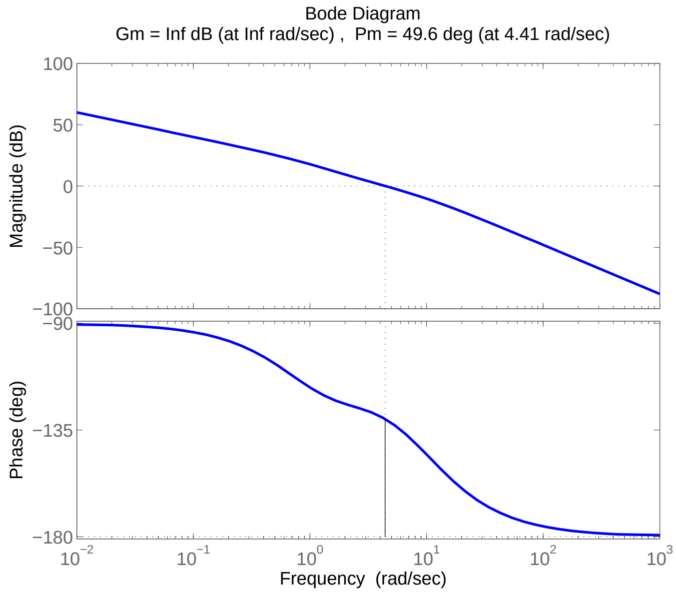



Figure 2: Bode plot of the compensated system for Example 1

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.1w)}{(1 - 0.1w)}$$

will transfer $G_z(z)$ into $w$-plane, as

$$G_w(w) = \frac{\left(1 + \frac{w}{300}\right)\left(1 - \frac{w}{10}\right)}{w(w+1)} \quad \text{[please try the simplification]}$$

We need first design a phase lead compensator so that PM of the compensated system is at least $50^0$ with $K_v = 2$ . The compensator in $w$-plane is

$$C(w) = K\frac{1 + \tau w}{1 + \alpha\tau w} \qquad 0 < \alpha < 1$$

Design steps are as follows.

- $K$ has to be found out from the $K_v$ requirement.
- Compute the gain crossover frequency $\omega_g$ and phase margin of the uncompensated system after introducing $K$ in the system.
- At $\omega_g$ check the additional/required phase lead, add safety margin, find out $\phi_{\max}$. Calculate $\alpha$ from the required $\phi_{\max}$.

- Since the lead part of the compensator provides a gain of $20\log_{10}\frac{1}{\sqrt{\alpha}}$, find out the frequency of the uncompensated system where the logarithmic magnitude is $-20\log_{10}\frac{1}{\sqrt{\alpha}}$. This will be the new gain crossover frequency where the maximum phase lead should occur.
- Make $\omega_{\max} = \omega_{g_{new}}$.
- Calculate $\tau$ from the relation

$$\omega_{g_{new}} = \omega_{\max} = \frac{1}{\tau\sqrt{\alpha}}$$

Now,

$$K_v = \lim_{w\to 0} wC(w)G_w(w) = 2$$

$$\Rightarrow K = 2$$

Using MATLAB command "margin", phase margin of the system with $K = 2$ is computed as $31.6^0$ with $\omega_g = 1.26$ rad/sec, as shown in Figure 3.

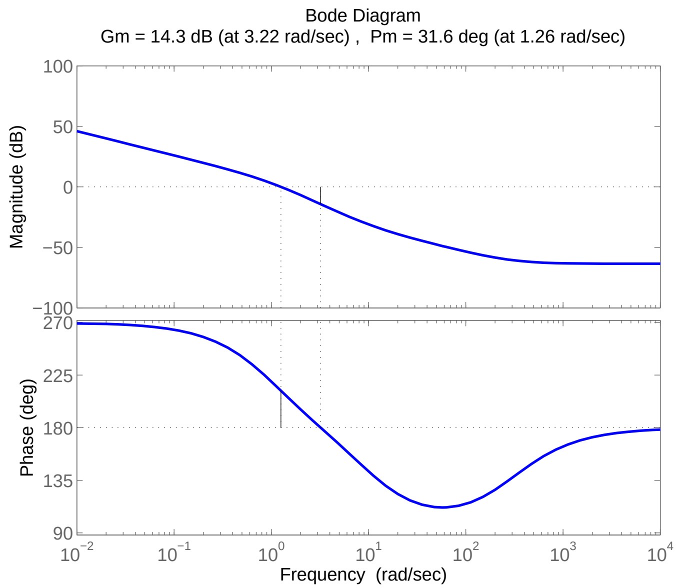



Figure 3: Bode plot of the uncompensated system for Example 2

Thus the required phase lead is $50^o - 31.6^o = 18.4^o$. After adding a safety margin of $11.6^o$, $\phi_{\max}$ becomes $30^o$. Hence

$$\alpha = \left(\frac{1-\sin(30^o)}{1+\sin(30^o)}\right) = 0.33$$

From the frequency response of the system it can be found out that at $\omega = 1.75$ rad/sec, the magnitude of the system is $-20\log_{10}\dfrac{1}{\sqrt{\alpha}}$. Thus $\omega_{g_{new}} = \omega_{\max} = 1.75$ rad/sec. This gives

$$1.75 = \frac{1}{\tau\sqrt{\alpha}}$$

Or,

$$\tau = \frac{1}{1.75\sqrt{0.33}} = 0.99$$

Thus the controller in $w$-plane is

$$C(w) = 2\frac{1 + 0.99w}{1 + 0.327w}$$

The Bode plot of the compensated system is shown in Figure 4.

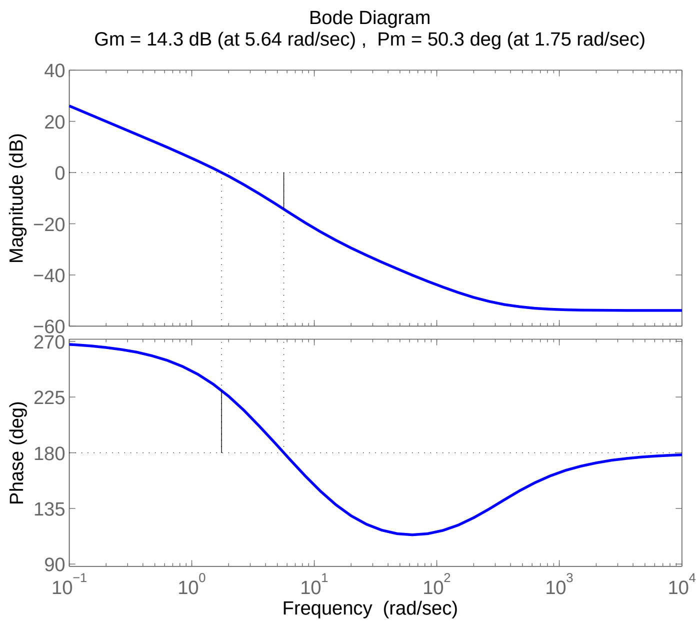



Figure 4: Bode plot of the compensated system for Example 2

Re-transforming the above controller into $z$-plane using the relation $w = 10\dfrac{z-1}{z+1}$, we get the controller in $z$-plane, as

$$C_z(z) \cong 2\,\frac{2.55z - 2.08}{z - 0.53}$$

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 7

# 1 Lag Compensator Design

In the previous lecture we discussed lead compensator design. In this lecture we would see how to design a phase lag compensator

## 1.1 Phase lag compensator

The essential feature of a lag compensator is to provide an increased low frequency gain, thus decreasing the steady state error, without changing the transient response significantly.

For frequency response design it is convenient to use the following transfer function of a lag compensator.

$$C_{lag}(s) = \alpha \frac{\tau s + 1}{\alpha \tau s + 1}, \quad \text{where, } \alpha > 1$$

The above expression is only the lag part of the compensator. The overall compensator is

$$C(s) = KC_{lag}(s)$$

$$\begin{aligned} \text{when, } s \to 0, \; & C_{lag}(s) \to \alpha \\ \text{when, } s \to \infty, \; & C_{lag}(s) \to 1 \end{aligned}$$

Typical objective of lag compensator design is to provide an additional gain of $\alpha$ in the low frequency region and to leave the system with sufficient phase margin.

The frequency response of a lag compensator, with $\alpha = 4$ and $\tau = 3$, is shown in Figure 1 where the magnitude varies from $20 \log_{10} \alpha$ dB to 0 dB.

Since the lag compensator provides the maximum lag near the two corner frequencies, to maintain the PM of the system, zero of the compensator should be chosen such that $\omega = 1/\tau$ is much lower than the gain crossover frequency of the uncompensated system.

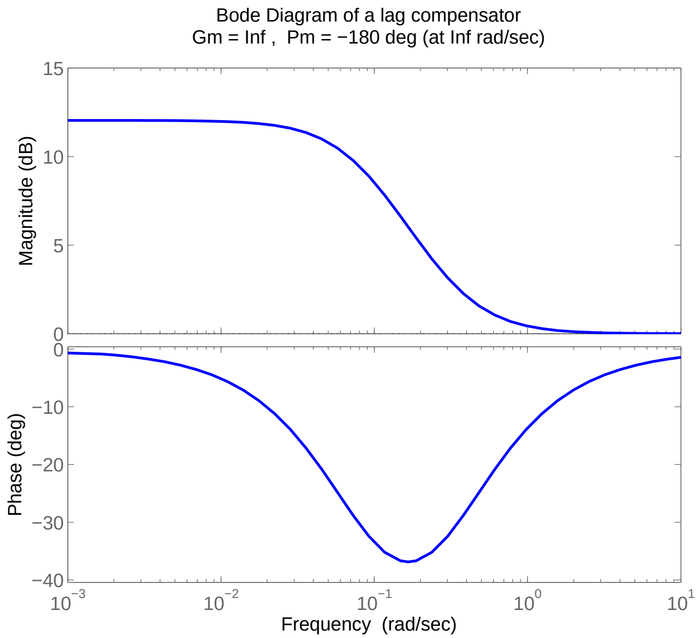



Figure 1: Frequency response of a lag compensator

In general, $\tau$ is designed such that $1/\tau$ is at least one decade below the gain crossover frequency of the uncompensated system. Following example will be comprehensive to understand the design procedure.

**Example 1:** Consider the following system

$$G(s) = \frac{1}{(s+1)(0.5s+1)}, \quad H(s) = 1$$

Design a lag compensator so that the phase margin (PM) is at least $50^o$ and steady state error to a unit step input is $\leq 0.1$.

The overall compensator is

$$C(s) = KC_{lag}(s) = K\alpha\frac{\tau s+1}{\alpha\tau s+1}, \quad \text{where, } \alpha > 1$$

When $s \to 0$, $C(s) \to K\alpha$.

Steady state error for unit step input is

$$\frac{1}{1+\lim_{s\to 0} C(s)G(s)} = \frac{1}{1+C(0)} = \frac{1}{1+K\alpha}$$

Thus, $\dfrac{1}{1+K\alpha} = 0.1$, or, $K\alpha = 9$.

Now let us modify the system transfer function by introducing $K$ with the original system. Thus the modified system becomes

$$G_m(s) = \frac{K}{(s+1)(0.5s+1)}$$

PM of the closed loop system should be $50^o$. Let the gain crossover frequency of the uncompensated system with $K$ be $\omega_g$.

$$\begin{aligned}
G_m(j\omega) &= \frac{K}{(j\omega+1)(0.5j\omega+1)}\\
Mag. &= \frac{K}{\sqrt{1+\omega^2}\sqrt{1+0.25\omega^2}}\\
Phase &= -\tan^{-1}\omega - \tan^{-1}0.5\omega
\end{aligned}$$

Required PM is $50^o$. Since the PM is achieved only by selecting $K$, it might be deviated from this value when the other parameters are also designed. Thus we put a safety margin of $5^o$ to the PM which makes the required PM to be $55^o$.

$$\begin{aligned}
\Rightarrow 180^o - \tan^{-1}\omega_g - \tan^{-1}0.5\omega_g &= 55^o\\
\text{or, } \tan^{-1}\frac{\omega_g + 0.5\omega_g}{1-0.5\omega_g^2} &= 125^o\\
\text{or, } \tan^{-1}\frac{1.5\omega_g}{1-0.5\omega_g^2} &= \tan 125^o = -1.43\\
\text{or, } 0.715\omega_g^2 - 1.5\omega_g - 1.43 &= 0\\
\Rightarrow \omega_g &= 2.8 \text{ rad/sec}
\end{aligned}$$

To make $\omega_g = 2.8$ rad/sec, the gain crossover frequency of the modified system, magnitude at $\omega_g$ should be 1. Thus

$$\frac{K}{\sqrt{1+\omega_g^2}\sqrt{1+0.25\omega_g^2}} = 1$$

Putting the value of $\omega_g$ in the last equation, we get $K = 5.1$.

Thus,

$$\alpha = \frac{9}{K} = 1.76$$

The only parameter left to be designed is $\tau$.

Since the desired PM is already achieved with gain $K$, we should place $\omega = 1/\tau$ such that it does not much effect the PM of the modified system with $K$. If we place $1/\tau$ one decade below the gain crossover frequency, then

$$\frac{1}{\tau} = \frac{2.8}{10}, \quad \text{or,} \quad \tau = 3.57$$

The overall compensator is

$$C(s) = 9\,\frac{3.57s + 1}{6.3s + 1}$$

With this compensator actual phase margin of the system becomes $52.7^o$, as shown in Figure 2, which meets the design criteria.

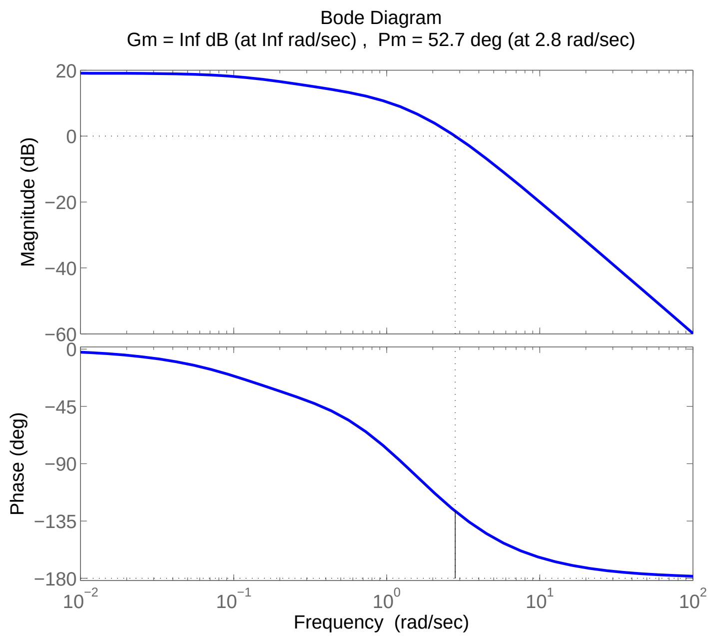



Figure 2: Bode plot of the compensated system for Example 1

**Example 2:**

Now let us consider that the system as described in the previous example is subject to a sampled data control system with sampling time $T = 0.1$ sec. We would use MATLAB to derive the plant transfer function $w$-plane.

Use the below commands.

```
>> s=tf('s');
>> gc=1/((s+1)*(0.5*s+1));
>> gz=c2d(Gp,0.1,'zoh');
```

You would get

$$G_z(z) = \frac{0.009z + 0.0008}{z^2 - 1.724z + 0.741}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.05w)}{(1 - 0.05w)}$$

will transfer $G_z(z)$ into $w$-plane. Use the below commands

```
>> aug=[0.1,1];
>> gwss = bilin(ss(gz),-1,'S_Tust',aug)
>> gw=tf(gwss)
```

to find out the transfer function in $w$-plane, as

$$\begin{aligned} G_w(w) &= \frac{1.992 - 0.09461w - 0.00023w^2}{w^2 + 2.993w + 1.992} \\ &\cong \frac{-0.00025(w - 20)(w + 400)}{(w + 1)(w + 2)} \end{aligned}$$

The Bode plot of the uncompensated system is shown in Figure 3.

We need to design a phase lag compensator so that PM of the compensated system is at least $50^0$ and steady state error to a unit step input is $\leq 0.1$. The compensator in $w$-plane is

$$C(w) = K\alpha\frac{1 + \tau w}{1 + \alpha\tau w} \qquad \alpha > 1$$

where,

$$C(0) = K\alpha$$

Since $G_w(0) = 1$, $K\alpha = 9$ for 0.1 steady state error.

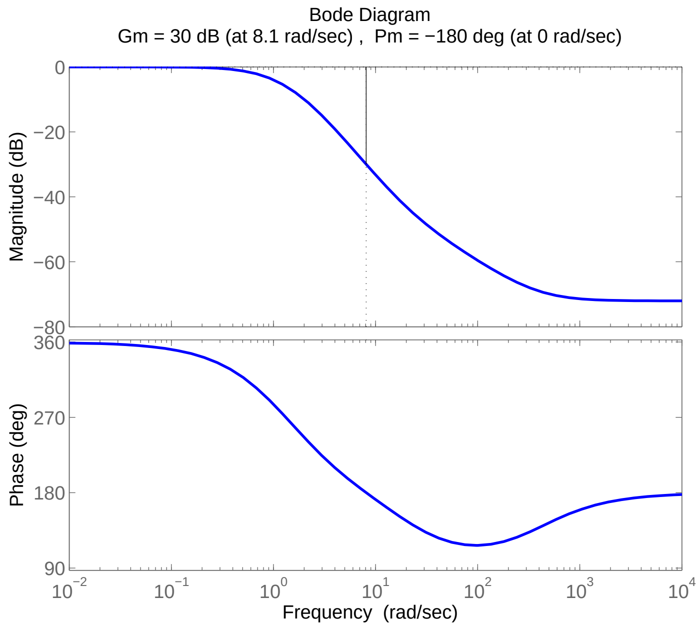



Figure 3: Bode plot of the uncompensated system for Example 2

Now let us modify the system transfer function by introducing $K$ to the original system. Thus the modified system becomes

$$G_m(w) = \frac{-0.00025K(w-20)(w+400)}{(w+1)(w+2)}$$

PM of the closed loop system should be $50^o$. Let the gain crossover frequency of the uncompensated system with $K$ be $\omega_g$. Then,

$$\begin{aligned} Mag.(G_m) &= \frac{0.00025K\sqrt{400+\omega^2}\sqrt{160000+\omega^2}}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}} \\ Phase(G_m) &= -\tan^{-1}\omega - \tan^{-1}0.5\omega - \tan^{-1}0.05\omega + \tan^{-1}0.0025\omega \end{aligned}$$

Required PM is $50^o$. Let us put a safety margin of $5^o$. Thus the PM of the system modified with $K$ should be $55^o$.

$$\begin{aligned} \Rightarrow 180^o - \tan^{-1}\omega_g - \tan^{-1}0.5\omega_g - \tan^{-1}0.05\omega_g + \tan^{-1}0.0025\omega_g &= 55^o \\ \text{or,}\ \tan^{-1}\frac{\omega_g + 0.5\omega_g}{1-0.5\omega_g^2} + \tan^{-1}\frac{0.05\omega_g - 0.0025\omega_g}{1+0.000125\omega_g^2} &= 125^o \end{aligned}$$

By solving the above, $\omega_g = 2.44$ rad/sec. Thus the magnitude at $\omega_g$ should be 1.

$$\Rightarrow \frac{0.00025K\sqrt{400+\omega_g^2}\sqrt{160000+\omega_g^2}}{\sqrt{1+\omega_g^2}\sqrt{4+\omega_g^2}} = 1$$

Putting the value of $\omega_g$ in the last equation, we get $K = 4.13$. Thus,

$$\alpha = \frac{9}{K} = 2.18$$

If we place $1/\tau$ one decade below the gain crossover frequency, then

$$\frac{1}{\tau} = \frac{2.44}{10}, \quad \text{or,} \quad \tau = 4.1$$

Thus the controller in $w$-plane is

$$C(w) = 9\frac{1 + 4.1w}{1 + 8.9w}$$

Re-transforming the above controller into $z$-plane using the relation $w = 20\dfrac{z-1}{z+1}$, we get

$$\begin{aligned} C_z(z) &= 9\,\frac{1 + 20 \times 4.1 \times \dfrac{z-1}{z+1}}{1 + 20 \times 8.9 \times \dfrac{z-1}{z+1}} \\ &= 9\,\frac{83z - 81}{179z - 177} \end{aligned}$$

The Bode plot of the uncompensated system is shown in Figure 3.

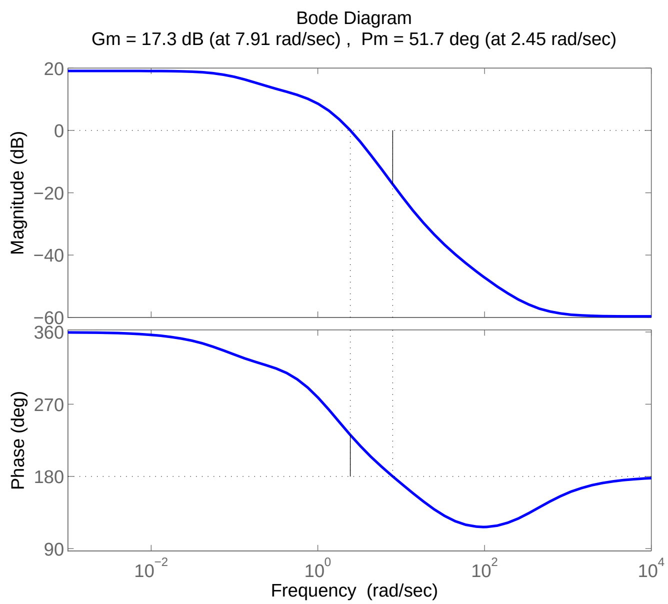



Figure 4: Bode plot of the compensated system for Example 2

In the next lecture, we would discuss lag-lead and PID controllers and conclude the topic of compensator design.

# Module 5: Design of Sampled Data Control Systems
## Lecture Note 8

## 1 Lag-lead Compensator

When a single lead or lag compensator cannot guarantee the specified design criteria, a lag-lead compensator is used.

In lag-lead compensator the lag part precedes the lead part. A continuous time lag-lead compensator is given by

$$C(s) = K\,\frac{1+\tau_1 s}{1+\alpha_1\tau_1 s}\,\frac{1+\tau_2 s}{1+\alpha_2\tau_2 s} \quad \text{where,}\ \ \alpha_1 > 1,\ \ \alpha_2 < 1$$

The corner frequencies are $\dfrac{1}{\alpha_1\tau_1}, \dfrac{1}{\tau_1}, \dfrac{1}{\tau_2}, \dfrac{1}{\alpha_2\tau_2}$. The frequency response is shown in Figure 1.

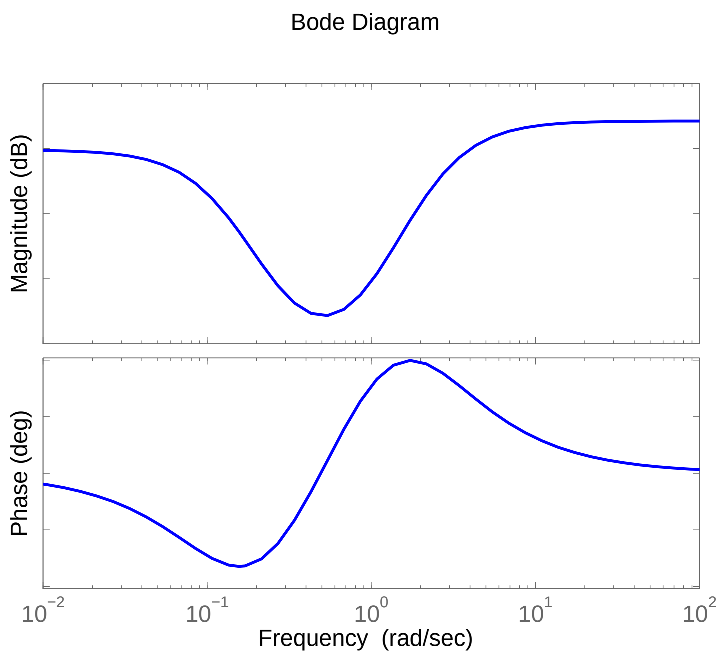



Figure 1: Frequency response of a lag-lead compensator

In a nutshell,

- If it is not specified which type of compensator has to be designed, one should first check the PM and BW of the uncompensated system with adjustable gain $K$.
- If the BW is smaller than the acceptable BW one may go for lead compensator. If the BW is large, lead compensator may not be useful since it provides high frequency amplification.
- One may go for a lag compensator when BW is large provided the open loop system is stable.
- If the lag compensator results in a too low BW (slow speed of response), a lag-lead compensator may be used.

## 1.1 Lag-lead compensator design

Consider the following system with transfer function

$$G(s) = \frac{1}{s(1+0.1s)(1+0.2s)}$$

Design a lag-lead compensator $C(s)$ such that the phase margin of the compensated system is at least $45^o$ at gain crossover frequency around 10 rad/sec and the velocity error constant $K_v$ is 30.

The lag-lead compensator is given by

$$C(s) = K\,\frac{1+\tau_1 s}{1+\alpha_1\tau_1 s}\,\frac{1+\tau_2 s}{1+\alpha_2\tau_2 s} \quad \text{where,}\ \ \alpha_1 > 1,\ \ \alpha_2 < 1$$

When $s \to 0$, $C(s) \to K$.

$$K_v = \lim_{s\to 0} sG(s)C(s) = C(0) = 30$$

Thus $K = 30$. Bode plot of the modified system $KG(s)$ is shown in Figure 2. The gain crossover frequency and phase margin of $KG(s)$ are found out to be 9.77 rad/sec and $-17.2^o$ respectively.

Since the PM of the uncompensated system with $K$ is negative, we need a lead compensator to compensate for the negative PM and achieve the desired phase margin.

However, we know that introduction of a lead compensator will eventually increase the gain crossover frequency to maintain the low frequency gain.

Thus the gain crossover frequency of the system cascaded with a lead compensator is likely to be much above the specified one, since the gain crossover frequency of the uncompensated

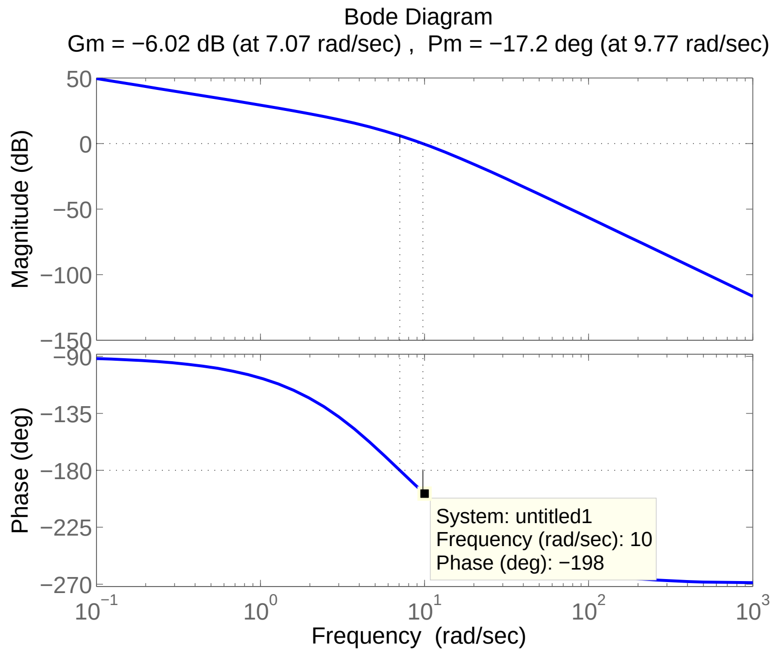



Figure 2: Frequency response of the uncompensated system of Example 1

system with $K$ is already 9.77 rad/sec.

Thus a lag-lead compensator is required to compensate for both.

We design the lead part first.

From Figure 2, it is seen that at 10 rad/sec the phase angle of the system is $-198^o$.

Since the new $\omega_g$ should be 10 rad/sec, the required additional phase at $\omega_g$, to maintain the specified PM, is $45 - (180 - 198) = 63^o$. With safety margin $2^o$,

$$\alpha_2 = \left(\frac{1 - \sin(65^o)}{1 + \sin(65^o)}\right) = 0.05$$

And

$$10 = \frac{1}{\tau_2 \sqrt{\alpha_2}}$$

which gives $\tau_2 = 0.45$. However, introducing this compensator will actually increase the gain crossover frequency where the phase characteristic will be different than the designed one. This can be seen from Figure 3.

The gain crossover frequency is increased to 23.2 rad/sec. At 10 rad/sec, the phase angle is $-134^o$ and gain is 12.6 dB. To make this as the actual gain crossover frequency, lag part

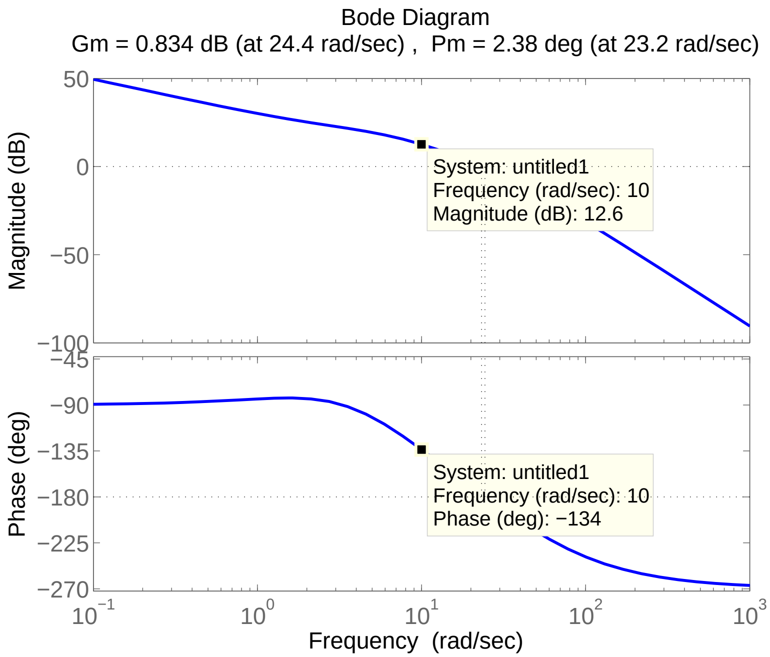



Figure 3: Frequency response of the system in Example 1 with only a lead compensator

should provide an attenuation of $-12.6$ dB at high frequencies.

At high frequencies the magnitude of the lag compensator part is $1/\alpha_1$. Thus ,

$$20\log_{10}\alpha_1 = 12.6$$

which gives $\alpha_1 = 4.27$. Now, $1/\tau_1$ should be placed much below the new gain crossover frequency to retain the desired PM. Let $1/\tau_1$ be 0.25. Thus

$$\tau_1 = 4$$

The overall compensator is

$$C(s) = 30\,\frac{1+4s}{1+17.08s}\,\frac{1+0.45s}{1+0.0225s}$$

The frequency response of the system after introducing the above compensator is shown in Figure 4, which shows that the desired performance criteria are met.

**Example 2:**

Now let us consider that the system as described in the previous example is subject to a

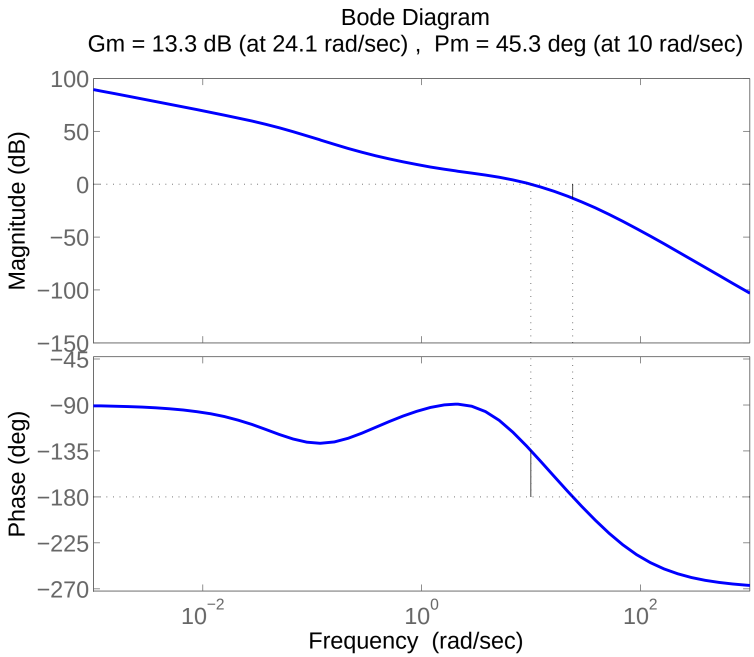



Figure 4: Frequency response of the system in Example 1 with a lag-lead compensator

sampled data control system with sampling time $T = 0.1$ sec. We would use MATLAB to derive the plant transfer function $w$-plane.

Use the below commands.

```
>> s=tf('s');
>> gc=1/(s*(1+0.1*s)*(1+0.2*s));
>> gz=c2d(gc,0.1,'zoh');
```

You would get

$$G_z(z) = \frac{0.005824z^2 + 0.01629z + 0.002753}{z^3 - 1.974z^2 + 1.198z - 0.2231}$$

The bi-linear transformation

$$z = \frac{1 + wT/2}{1 - wT/2} = \frac{(1 + 0.05w)}{(1 - 0.05w)}$$

will transfer $G_z(z)$ into $w$-plane. Use the below commands

```
>> aug=[0.1,1];
>> gwss = bilin(ss(gz),-1,'S_Tust',aug)
>> gw=tf(gwss)
```

to find out the transfer function in $w$-plane, as

$$G_w(w) = \frac{0.001756w^3 - 0.06306w^2 - 1.705w + 45.27}{w^3 + 14.14w^2 + 45.27w - 5.629 \times 10^{-13}}$$

Since the velocity error constant criterion will produce the same controller dcgain $K$, the gain of the lag-lead compensator is designed to be 30.

The Bode plot of the uncompensated system with $K = 30$ is shown in Figure 5.

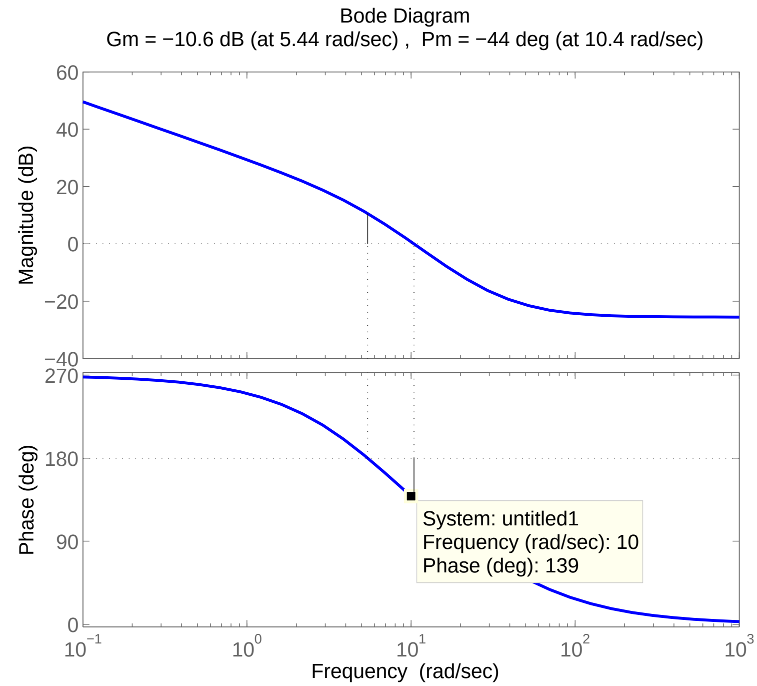



Figure 5: Bode plot of the uncompensated system for Example 2

From Figure 5, it is seen that at 10 rad/sec the phase angle of the system is $139 = -221^o$.

Thus a huge phase lead ($86^o$) is required if we want to acieve a PM of $45^o$ which is not possible with a single lead compensator. Let us lower the PM requirement to a minimum of $20^o$ at $\omega_g = 10$ rad/sec.

Since the new $\omega_g$ should be 10 rad/sec, the required additional phase at $\omega_g$, to maintain the specified PM, is $20 - (180 - 221) = 61^o$. With safety margin $5^o$,

$$\alpha_2 = \left(\frac{1 - \sin(66^o)}{1 + \sin(66^o)}\right) = 0.045$$

And

$$10 = \frac{1}{\tau_2 \sqrt{\alpha_2}}$$

which gives $\tau_2 = 0.47$. However, introducing this compensator will actually increase the gain crossover frequency where the phase characteristic will be different than the designed one. This can be seen from Figure 6.

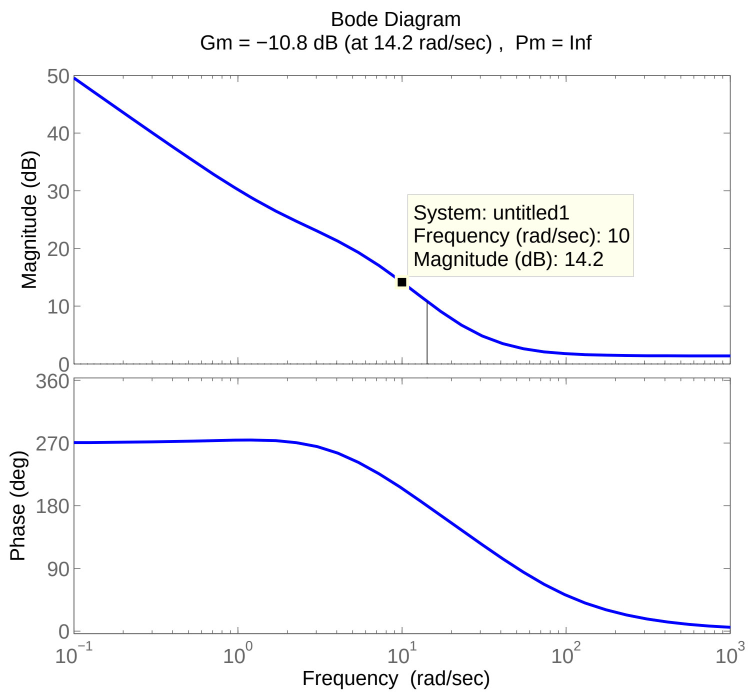



Figure 6: Frequency response of the system in Example 2 with only a lead compensator

Also, as seen from Figure 6, the GM of the system is negative. Thus we need a lag compensator to lower the magnitude at 10 rad/sec. At 10 rad/sec, the magnitude is 14.2 dB. To make this as the actual gain crossover frequency, lag part should provide an attenuation of $-14.2$ dB at high frequencies.

Thus,

$$20 \log_{10} \alpha_1 = 14.2$$

which gives $\alpha_1 = 5.11$. Now, $1/\tau_1$ should be placed much below the new gain crossover frequency to retain the desired PM. Let $1/\tau_1$ be $10/10 = 1$. Thus

$$\tau_1 = 1$$

The overall compensator is

$$C(w) = 30\left(\frac{1+w}{1+5.11w}\right)\left(\frac{1+0.47w}{1+0.02115w}\right)$$

The frequency response of the system after introducing the above compensator is shown in Figure 7, which shows that the desired performance criteria are met.

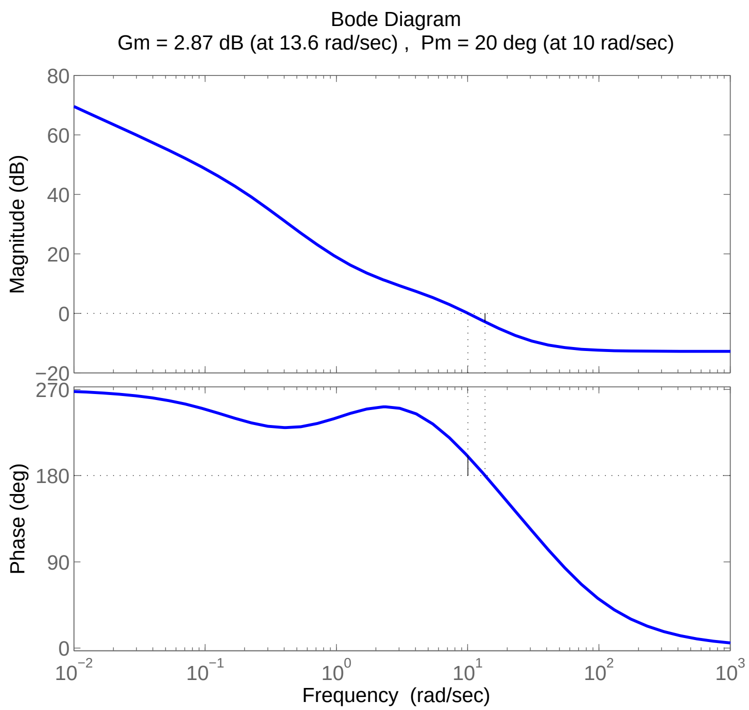



Figure 7: Frequency response of the system in Example 2 with a lag-lead compensator

Re-converting the controller in z-domain, we get

$$C(z) = 30\left(\frac{0.2035z - 0.1841}{z - 0.9806}\right)\left(\frac{7.309z - 5.903}{z + 0.4055}\right)$$

# Module 6: Deadbeat Response Design

## Lecture Note 1

## 1 Design of digital control systems with dead beat response

So far we have discussed the design methods which are extensions of continuous time design techniques.

We will now deal with the dead beat response design of digital control system.

We must distinguish between the designs of deadbeat response for a digital control system, where all the components are subject to only digital data, and a sampled data control system, where both continuous and discrete components are present.

An all digital control system is shown in Figure 1.

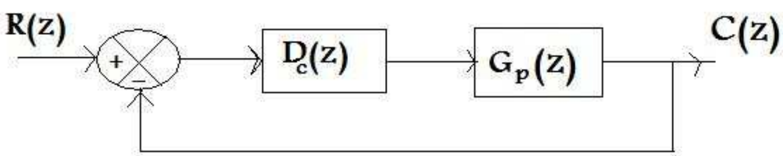



Figure 1: An all digital control system

The transfer function of the digital plant is given by

$$G_p(z) = \frac{z + 0.6}{3z^2 - z - 1}$$

Let us assume that the transfer function of the cascade digital controller is

$$D_c(z) = \frac{3z^2 - z - 1}{(z-1)(z+0.6)}$$

Thus the open loop transfer function becomes

$$G(z) = D_c(z)G_p(z) = \frac{1}{z-1}$$

And we get the closed loop transfer function as

$$M(z) = \frac{G(z)}{1+G(z)} = \frac{1}{z}$$

Thus for unit step input, the output comes out to be

$$C(z) = \frac{1}{z}\frac{z}{(z-1)} = z^{-1} + z^{-2} + ......$$

Thus, the output $c(k)$ represents a unit step response where $k$ starts from 1, that is one sample later.

In other words, $c(k)$ reaches the desired steady state value 1, in one sampling period with out any overshoot and stays there for ever.

This type of response is known as dead beat response.

One should note that if $G_p(z)$ was the result of sampling a continuous data system, the $D_c(z)$ does not guarantee that no ripples occur between two sampling instants in constant output $c(t)$.

## 1.1 Deadbeat response design when the system poles and zeros are inside the unit circle

Design criteria:

1. The system must have a zero steady state error at sampling instants.

2. The time to reach final output must be finite and minimum.

3. The controller should be physically realizable, i.e., it should be causal.

We can write from Figure 1,

$$M(z) = \frac{C(z)}{R(z)} = \frac{D_c(z)G_p(z)}{1+D_c(z)G_p(z)}$$

Thus

$$D_c(z) = \frac{1}{G_p(z)} \frac{M(z)}{1 - M(z)}$$

The error signal

$$E(z) = R(z) - C(z) = \frac{R(z)}{1 + D_c(z)G_p(z)}$$

Let us assume

$$R(z) = \frac{A(z)}{(1 - z^{-1})^N}$$

where,
$N$: positive integer
$A(z)$: polynomial in $z^{-1}$ with no zeros at $z = 1$.

For unit step signal $A(z) = 1$ and $N = 1$.
For unit ramp signal $A(z) = Tz^{-1}$ and $N = 2$.

To achieve zero steady state error

$$\lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1})E(z)$$

$$= \lim_{z \to 1} \frac{(1 - z^{-1})A(z)(1 - M(z))}{(1 - z^{-1})^N} = 0$$

Since $A(z)$ does not contain any zero at $z = 1$, necessary condition for zero steady state error is that $1 - M(z)$ should contain $(1 - z^{-1})^N$ as a factor, i.e.,

$$1 - M(z) = (1 - z^{-1})^N F(z)$$

Or,

$$M(z) = 1 - (1 - z^{-1})^N F(z) = \frac{Q(z)}{z^p} \quad p > N$$

where,
$F(z)$ is a polynomial in $z^{-1}$.
$Q(z)$ is a polynomial in $z$.

Substituting $M(z)$ in the expression of $E(z)$, $E(z) = A(z)F(z)$. Since $A(z)$ and $F(z)$ are both polynomials of $z^{-1}$, $E(z)$ will have a finite number of terms in the power series in the inverse power of $z$, i.e., the error will go to zero in a finite number of sampling periods.

**Physical realizability of $D_c(z)$ :**

Physical realizability condition on $D_c(z)$ imposes constraints on the form of $M(z)$. Let

$$G_p(z) = g_n z^{-n} + g_{n+1} z^{-n-1} + \dots\dots\dots$$

$$\text{and,} \quad M(z) = m_k z^{-k} + m_{k+1} z^{-k-1} + \dots\dots\dots$$

where, $n$ and $k$ are the excess poles over zeros of $G_p(z)$ and $M(z)$ respectively. This implies

$$D_c(z) = d_{k-n} z^{-(k-n)} + d_{k-n+1} z^{-(k-n+1)} + \dots\dots\dots$$

For $D_c(z)$ to be realizable, $k \geq n$, i.e., excess of poles over zeros for $M(z)$ must be at least equal to excess of poles over zeros for $G_p(z)$.

Thus, if $G_p(z)$ does not have poles or zeros outside the unit circle, then $M(z)$ should have the following forms.

1. Step input :

$$R(z) = \frac{z}{z-1}$$

$$M(z) = \frac{1}{z^n}$$

2. Ramp input:

$$M(z) = \frac{(n+1)z - n}{z^{n+1}}$$

Try to prove the above as an **exercise** problem.

**Example 1:** Let us consider the earlier example where

$$G_p(z) = \frac{z + 0.6}{3z^2 - z - 1}$$

When the input is a step function, $M(z) = z^{-1}$

$$\Rightarrow D_c(z) = \frac{1}{G_p(z)} \frac{M(z)}{1 - M(z)}$$

Thus

$$D_c(z) = \frac{3z^2 - z - 1}{z + 0.6} \frac{\frac{1}{z}}{1 - \frac{1}{z}}$$

$$= \frac{3z^2 - z - 1}{(z + 0.6)(z - 1)}$$

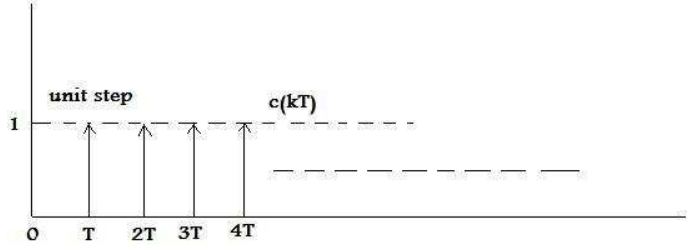



Figure 2: Deadbeat response of an all digital control system for unit step input

Hence the output sequence follows the input after one sampling instant. Figure 2 shows the output response.

When the input is a ramp function

$$M(z) = \frac{2z-1}{z^2}$$

Thus

$$D_c(z) = \frac{3z^2 - z - 1}{z + 0.6} \frac{\frac{2z-1}{z^2}}{1 - \frac{2z-1}{z^2}}$$

$$= \frac{(3z^2 - z - 1)(2z - 1)}{(z^2 - 2z + 1)(z + 0.6)}$$

$$\Rightarrow C(z) = \frac{T(z - 0.5)}{z(z-1)^2}$$

$$= T(2z^{-2} + 3z^{-3} + ........)$$

We can conclude from the above expression that the output sequence follows the input after 2 sampling periods which is shown in Figure 3.

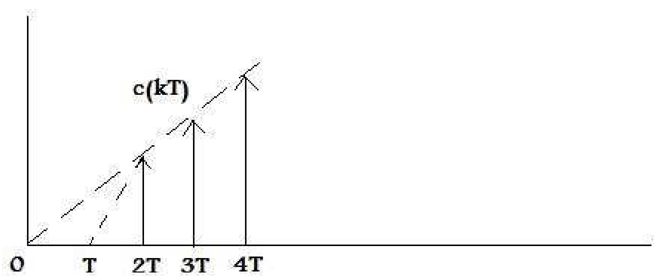



Figure 3: Deadbeat response of an all digital control system for unit ramp input

**Example 2:**

$$G_p(z) = \frac{0.05(z+0.5)}{(z-0.9)(z-0.8)(z-0.3)}$$

When the input is a step function, $M(z) = z^{-2}$

$$\Rightarrow D_c(z) = \frac{1}{G_p(z)} \frac{M(z)}{1-M(z)}$$

$$= \frac{(z-0.9)(z-0.8)(z-0.3)}{0.05(z+0.5)} \frac{\frac{1}{z^2}}{1-\frac{1}{z^2}}$$

$$= \frac{20(z-0.9)(z-0.8)(z-0.3)}{(z+0.5)(z+1)(z-1)}$$

Thus

$$\begin{aligned} C(z) &= M(z)R(z) \\ &= \frac{1}{z^2}\frac{z}{z-1} = \frac{1}{z^2-z} \\ &= z^{-2} + z^{-3} + z^{-3} + \cdots \end{aligned}$$

Hence the output sequence follows the input after two sampling instants.

# Module 6: Deadbeat Response Design
## Lecture Note 2

There are some practical issues in deadbeat response design.

Dead beat response design depends on the cancellation of poles and zeros of the plant transfer function.

If the poles are on or outside the unit circle, imperfect cancellation may lead to instability.

Thus, for practical constraints, one should not attempt to cancel poles which are on or outside the unit circle.

# 1 Deadbeat response design when some of the poles and zeros are on or outside the unit circle

Let the plant transfer function be

$$G_p(z) = \frac{\prod_{i=1}^{K}(1 - z_i z^{-1})}{\prod_{j=1}^{L}(1 - p_j z^{-1})} B(z)$$

where, $K$ and $L$ are the number of zeros and poles on or outside the unit circle and $B(z)$ is a rational transfer function in $z^{-1}$ with poles and zeros inside the unit circle. This implies

$$D_c(z) = \frac{\prod_{j=1}^{L}(1 - p_j z^{-1})}{\prod_{i=1}^{K}(1 - z_i z^{-1})} \frac{M(z)}{B(z)(1 - M(z))}$$

Since we should not cancel poles or zeros which are on or outside unit circle by the controller $D_c(z)$, we have to choose $M(z)$ such that these get canceled out.

Thus $M(z)$ must contain the factors

$$\prod_{i=1}^{K}(1 - z_i z^{-1})$$

and $(1 - M(z))$ must contain the factors

$$\prod_{j=1}^{L}(1 - p_j z^{-1})$$

So,

$$M(z) = \prod_{i=1}^{K}(1 - z_i z^{-1})(m_k z^{-k} + m_{k+1} z^{-k-1} + \ldots\ldots..), \qquad k \geq n$$

and

$$1 - M(z) = \prod_{j=1}^{L}(1 - p_j z^{-1})(1 - z^{-1})^P(1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots\ldots\ldots\ldots\ldots)$$

$P$ equals either the order of the poles of $R(z)$ or the order of poles of $G_p(z)$ at $z = 1$ which ever is greater. Truncation depends on the following.

1. The order of poles of $M(z)$ and $(1 - M(z))$ must be equal.

2. Total number of unknowns must be equal to the order of $M(z)$ so that they can be solved independently.

**Example 1:**

Let us consider the plant transfer function as

$$G_p(z) = \frac{0.01(z + 0.2)(z + 2.8)}{z(z - 1)(z - 0.4)(z - 0.8)}$$

$$= \frac{0.01z^{-2}(1 + 0.2z^{-1})(1 + 2.8z^{-1})}{(1 - z^{-1})(1 - 0.4z^{-1})(1 - 0.8z^{-1})}$$

**For Unit Step Input:**

$G_p(z)$ has a zero at $-2.8$ and pole at $z = 1$. Therefore $M(z)$ must contain the term $1 + 2.8z^{-1}$ and $(1 - M(z))$ should contain $1 - z^{-1}$.

$G_p(z)$ has two more poles than zeros. This implies

$$M(z) = (1 + 2.8z^{-1})m_2 z^{-2}$$

$$1 - M(z) = (1 - z^{-1})(1 + a_1 z^{-1} + a_2 z^{-2})$$

Since minimum order of $M(z)$ is 3, we have 3 unknowns in total. Combining the 2 equations

$$a_1 = 1$$

$$a_1 - a_2 = m_2$$

$$a_2 = 2.8m_2$$

$$\Rightarrow 1 - 2.8m_2 = m_2$$

$$m_2 = \frac{1}{3.8} = 0.26$$

$$a_2 = 2.8 \times 0.26 = 0.73$$

Thus

$$M(z) = 0.26z^{-2}(1 + 2.8z^{-1})$$

and

$$1 - M(z) = (1 - z^{-1})(1 + z^{-1} + 0.73z^{-2})$$

Putting the expressions of $M(z)$ and $1 - M(z)$ in the controller equation

$$\begin{aligned} D_c(z) &= \frac{(1 - z^{-1})}{(1 + 2.8z^{-1})} \frac{(1 + 2.8z^{-1})0.26z^{-2}}{(1 - z^{-1})(1 + z^{-1} + 0.73z^{-2})} \frac{(1 - 0.4z^{-1})(1 - 0.8z^{-1})}{0.01z^{-2}(1 + 0.2z^{-1})} \\ &= \frac{0.26z^{-2}(1 - 0.4z^{-1})(1 - 0.8z^{-1})}{0.01z^{-2}(1 + 0.2z^{-1})(1 + z^{-1} + 0.73z^{-2})} \\ &= \frac{26z(z - 0.4)(z - 0.8)}{(z + 0.2)(z^2 + z + 0.73)} \end{aligned}$$

Thus

$$\begin{aligned} C(z) &= \frac{0.26(z + 2.8)}{z^2(z - 1)} \\ &= 0.26z^{-2} + z^{-3} + z^{-4} + \ldots\ldots\ldots \end{aligned}$$

One should note that poles on or outside the unit circle are still present in the output expression.

$c(kT)$ tracks the unit step perfectly after 3 sampling periods (first term has a non unity coefficient). The output response is shown in Figure 1.

If $G_p(z)$ did not have any poles or zeros on or outside the unit circle it would take two sampling

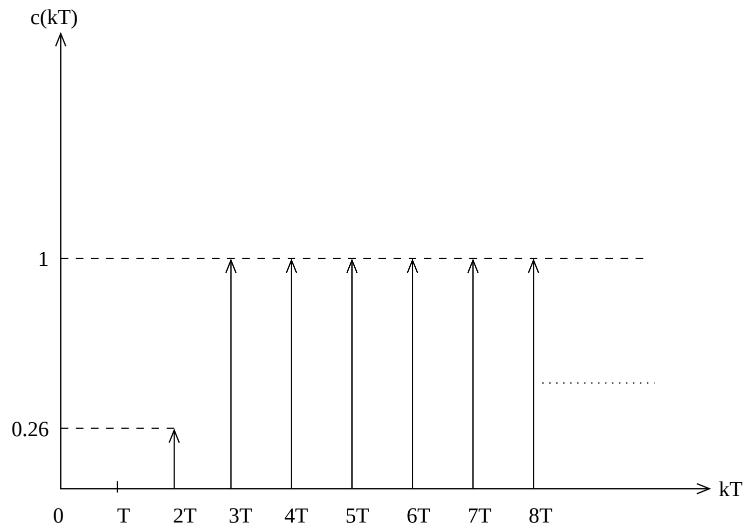



Figure 1: Deadbeat Response of The System in Example 1

periods to track the step input when $G_p(z)$ has two more poles than zeros.

**Example 2:**

Let us consider the plant transfer function as

$$\begin{aligned} G_p(z) &= \frac{0.0004(z+0.2)(z+2.8)}{(z-1)^2(z-0.28)} \\ &= \frac{0.0004z^{-1}(1+0.2z^{-1})(1+2.8z^{-1})}{(1-z^{-1})^2(1-0.28z^{-1})} \end{aligned}$$

**For Unit Step Input:**

$G_p(z)$ has a zero at $-2.8$ and two poles at $z = 1$. The number of poles exceeds the number of zeros by one.

$M(z)$ must contain the term $1 + 2.8z^{-1}$ and $1 - M(z)$ should contain $(1 - z^{-1})^2$.

This implies

$$M(z) = (1 + 2.8z^{-1})(m_1 z^{-1} + m_2 z^{-2})$$

$$1 - M(z) = (1 - z^{-1})^2(1 + a_1 z^{-1})$$

Combining the 2 equations and equating the like powers of $z^{-1}$,

$$m_1 = 2 - a_1$$

$$2.8m_1 + m_2 = 2a_1 - 1$$

$$2.8m_2 = -a_1$$

The solutions of the above equations are $m_1 = 0.72$, $m_2 = -0.457$ and $a_1 = 1.28$. Thus

$$M(z) = (0.72z^{-1} - 0.457z^{-2})(1 + 2.8z^{-1})$$

and

$$1 - M(z) = (1 - z^{-1})^2(1 + 1.28z^{-1})$$

Putting the expressions of $M(z)$ and $1 - M(z)$ in the controller equation

$$\begin{aligned}
D_c(z) &= \frac{(1-z^{-1})^2}{(1+2.8z^{-1})} \frac{(0.72z^{-1} - 0.457z^{-2})(1+2.8z^{-1})}{(1-z^{-1})^2(1+1.28z^{-1})} \frac{(1-0.28z^{-1})}{0.0004z^{-1}(1+0.2z^{-1})} \\
&= \frac{(0.72z^{-1} - 0.457z^{-2})(1-0.28z^{-1})}{0.0004z^{-1}(1+0.2z^{-1})(1+1.28z^{-1})} \\
&= \frac{1800(z-0.635)(z-0.28)}{(z+0.2)(z+1.28)}
\end{aligned}$$

Thus

$$\begin{aligned}
C(z) = M(z)R(z) &= \frac{z(0.72z^2 + 1.56z - 1.28)}{z^3(z-1)} \\
&= 0.72z^{-1} + 2.28z^{-2} + z^{-3} + z^{-4} + \ldots\ldots\ldots
\end{aligned}$$

The output response is plotted in Figure 2.
Note that although $c(kT)$ tracks the unit step perfectly after 3 sampling periods, the maximum overshoot is 128 percent.

This is because of the fact that the digital plant is a type 2 system, hence a deadbeat response without overshoot cannot be obtained for a unit step input.

Thus one can conclude that it is not always possible to design dead beat response without any over shoot.

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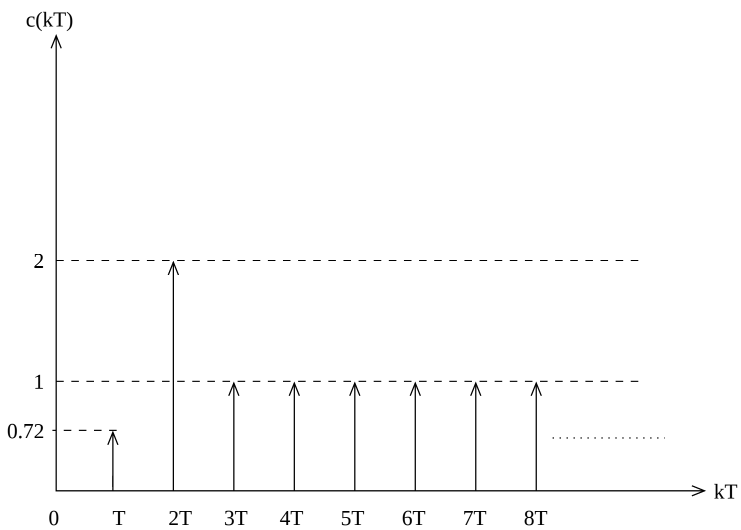


Figure 2: Deadbeat Response of The System in Example 2

# Module 6: Deadbeat Response Design
## Lecture Note 3

## 1 Sampled data control systems with Dead beat response

In case of a continuous time controlled process, the output $c(t)$ is a function of time $t$ and the dead beat response design, based on cancellation of stable poles and zeros, may lead to inter sampling ripples in the output.

The reason behind this is since the process zeros are canceled by controller poles, the continuous dynamics are excited by the input and are not affected by feed back.

The strategy of designing dead beat response for a sampled data system with the process plant transfer function $G_{h0}G_p(z)$ having at least one zero is not to cancel the zeros, whether they are inside or outside the unit circle.

H.P Sirisena gave a mathematical formulation and analysis to dead beat response.

If $G_{h0}G_p(z^{-1}) = \dfrac{Q(z^{-1})}{P(z^{-1})}$, then according to Sirisena the digital controller for ripple free dead beat response to step input is

$$D_c(z) = \frac{P(z^{-1})}{Q(1) - Q(z^{-1})}$$

The design of ripple free dead beat response can still be done using similar approach as discussed in the previous chapters except for an added constraint which will increase the response time of the system.

Following example will illustrate the design procedure

**Example 1:**

Let us consider a sampled data system as shown in Figure 1, where,

$$G_p(s) = \frac{2}{s(s+2)}$$

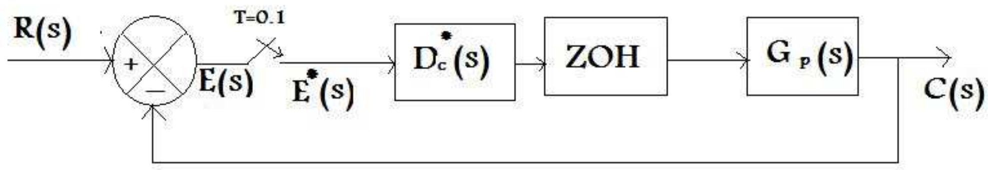


Figure 1: A sampled data control system

Thus

$$G_{h0}G_p(z) = \frac{0.01(z+0.9)}{(z-1)(z-0.8)}$$

If we design $D_c(z)$ without bothering about the inter sample ripples then

$$M(z) = z^{-1}, \qquad 1 - M(z) = 1 - z^{-1}$$

$$D_c(z) = \frac{100(z-0.8)}{(z+0.9)}$$

$$C(z) = \frac{1}{z-1} = z^{-1} + z^{-2} + \ldots\ldots\ldots$$

This implies that the output response is deadbeat only at sampling instants. However, the true output $c(t)$ has inter sampling ripples which makes the system response as shown in Figure 2.

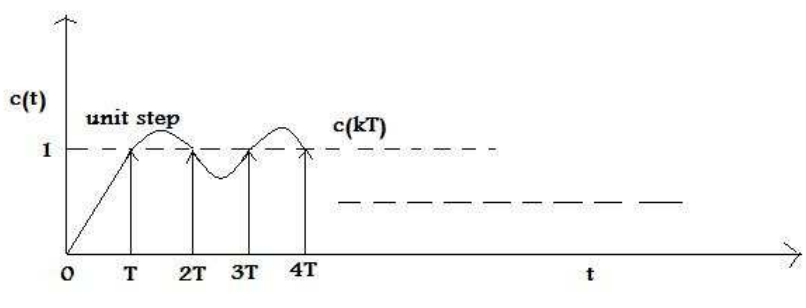


Figure 2: Rippled output response for Example 1

Thus the system takes forever to reach its steady state. The necessary and sufficient condition for $c(t)$ to track a unit step input in finite time is

$$c(NT) = 1 \qquad\qquad \left.\frac{dc(t)}{d(t)}\right|_{t=NT} = 0$$

for finite $N$ and all the higher derivatives should equal to zero. Let

$$w(t) = \frac{dc(t)}{d(t)}$$

Taking $Z$-transform,

$$W(z) = \frac{D_c(z)(1-z^{-1})Z[G_p(s)]}{1 + D_c(z)(1-z^{-1})Z\left[\frac{G_p(s)}{s}\right]}R(z)$$

$$= \frac{A_1(z-1)}{z(z+0.9)}R(z)$$

where $A_1$ is a constant. Unit step response of $W(z)$ will not go to zero in finite time since poles of $\dfrac{W(z)}{R(z)}$ are not all at $z = 0$.

If we now apply the condition that zero of $G_{h0}G_p(z)$ at $z = -0.9$ should not be canceled by $D_c(z)$, then

$$M(z) = (1 + 0.9z^{-1})m_1 z^{-1}$$

$$1 - M(z) = (1 - z^{-1})(1 + a_1 z^{-1})$$

Solving

$$\Rightarrow m_1 = 0.53, \qquad\qquad a_1 = 0.47$$

Thus

$$M(z) = \frac{0.53(z+0.9)}{z^2}$$

$$D_c(z) = \frac{A_2(z-0.8)}{z+0.47}$$

$$C(z) = A_3 z^{-1} + z^{-2} + z^{-3} + \ldots\ldots$$

where $A_2$ and $A_3$ are constants. This implies that the dead beat response reaches the steady state after two sampling periods.

---

To show that the output response is indeed deadbeat, we derive the z-transform of $w(t)$ as

$$W(z) = 2z^{-1}$$

Thus $c(t)$ will actually reach its steady state in two sampling periods with no inter sample ripples which is shown in Figure 3.

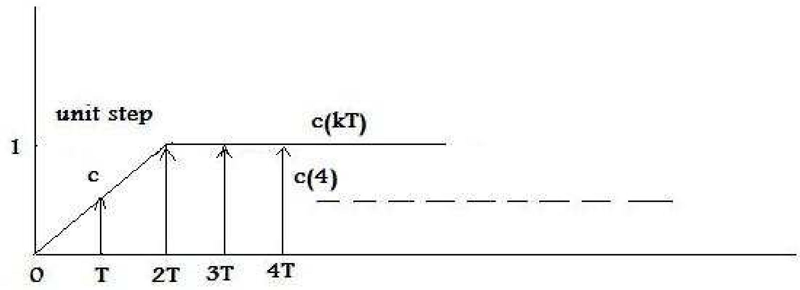


Figure 3: Ripple free deadbeat response for Example 1

**Example 2:** Consider the plant transfer function as

$$G_{h0}G_p(z) = \frac{0.01(z+0.2)(z+2.8)}{z(z-1)(z-0.4)(z-0.8)}$$

If we apply the condition that zeros of $G_{h0}G_p(z)$ at $z = -0.2$ and $z = -2.8$ should not be canceled by $D_c(z)$, then

$$M(z) = (1+0.2z^{-1})(1+2.8z^{-1})m_1 z^{-2}$$

$$1 - M(z) = (1-z^{-1})(1+a_1z^{-1}+a_2z^{-2}+a_3z^{-3})$$

While considering $M(z)$ and $1-M(z)$, following points should be kept in mind

1. $M(z)$ should contain all the zeros of $G_{h0}G_p(z)$.
2. The number of poles over zeros of $M(z)$ should be at least equal to that of $G_{h0}G_p(z)$ which is 2 in this case.
3. $1-M(z)$ must include the term $1-z^{-1}$.
4. The orders of $M(z)$ and $1-M(z)$ should be same and should equal the number of unknown coefficients.

Solving for the coefficients of $M(z)$ and $1 - M(z)$, we get

$$1 - a_1 = 0$$

$$m_1 = a_1 - a_2$$
$$3m_1 = a_2 - a_3$$
$$0.56m_1 = a_3$$

The solutions of the above are $m_1 = 0.219$, $a_1 = 1$, $a_2 = 0.781$ and $a_3 = 0.123$. The closed loop transfer function is

$$M(z) = \frac{0.219z^2 + 0.657z + 0.123}{z^4}$$

The transfer function of the digital controller is obtained as

$$D_c(z) = \frac{21.9z(z-0.4)(z-0.8)}{z^3 + z^2 + 0.781z + 0.123}$$

The output for a unit step input is written as

$$\begin{aligned} C(z) &= \frac{0.219z^2 + 0.657z + 0.123}{z^3(z-1)} \\ &= 0.219z^{-2} + 0.876z^{-3} + z^{-4} + z^{-5}\ldots\ldots \end{aligned}$$

Thus the output response $c(kT)$ reaches the steady state in 4 sampling instants. This is one more sampling instant than the previous example where we considered the plant to be all digital.

This implies that for sampled data control system, the dead beat response $c(t)$ reaches the steady state after three sampling periods but inter sample ripples occur. After four sampling instants the inter sample ripples disappear.

To show that the output response is indeed deadbeat, we derive the z-transform of $w(t)$ which will come out to be

$$W(z) = A_1 z^{-2} + A_2 z^{-3}$$

where $A_1$, $A_2$ are constants.

Thus the derivative of $c(t)$ is zero for $kT \geq 4T$, which implies that the step response reaches the steady state in 4 sampling instants with no inter sample ripples, as shown in Figure 4.

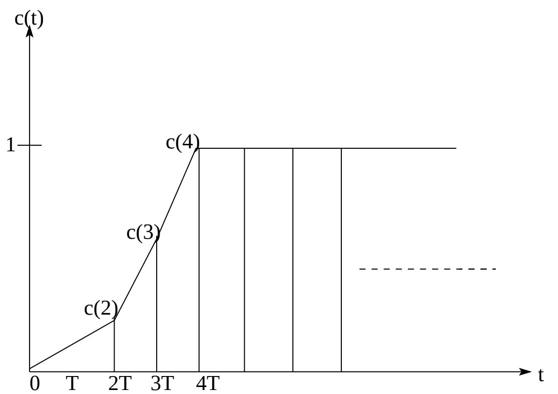


Figure 4: Ripple free deadbeat response for Example 2

# Module 7: Discrete State Space Models

## Lecture Note 1

## 1 Introduction to State Variable Model

In the preceding lectures, we have learned how to design a sampled data control system or a digital system using the transfer function of the system to be controlled. Transfer function approach of system modeling provides final relation between output variable and input variable. However, a system may have other internal variables of importance. State variable representation takes into account of all such internal variables. Moreover, controller design using classical methods, e.g., root locus or frequency domain method are limited to only LTI systems, particularly SISO (single input single output) systems since for MIMO (multi input multi output) systems controller design using classical approach becomes more complex. These limitations of classical approach led to the development of state variable approach of system modeling and control which formed a basis of modern control theory.

State variable models are basically time domain models where we are interested in the dynamics of some characterizing variables called state variables which along with the input represent the state of a system at a given time.

- State: The state of a dynamic system is the smallest set of variables, $\boldsymbol{x} \in R^n$, such that given $\boldsymbol{x}(t_0)$ and $u(t)$, $t > t_0$, $\boldsymbol{x}(t)$, $t > t_0$ can be uniquely determined.
- Usually a system governed by a $n^{th}$ order differential equation or $n^{th}$ order transfer function is expressed in terms of $n$ state variables: $x_1, x_2, \ldots, x_n$.
- The generic structure of a state-space model of a $n^{th}$ order continuous time dynamical system with $m$ input and $p$ output is given by:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \quad &: \text{State Equation} \\ \boldsymbol{y}(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \quad &: \text{Output Equation} \end{aligned} \tag{1}$$

where, $\boldsymbol{x}(t)$ is the $n$ dimensional state vector, $\boldsymbol{u}(t)$ is the $m$ dimensional input vector, $\boldsymbol{y}(t)$ is the $p$ dimensional output vector and $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

**Example**

Consider a nth order differential equation

$$\frac{d^n y}{dt^n} + a_1\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_n y = u$$

Define following variables,

$$
\begin{aligned}
y &= x_1 \\
\frac{dy}{dt} &= x_2 \\
\vdots &= \vdots \\
\frac{d^{n-1}y}{dt^{n-1}} &= x_n \\
\frac{d^n y}{dt^n} &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}
$$

The nth order differential equation may be written in the form of $n$ first order differential equations as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\vdots &= \vdots \\
\dot{x}_n &= -a_1 x_{n-1} - a_2 x_{n-2} - \ldots - a_n x_1 + u
\end{aligned}
$$

or in matrix form as,

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + Bu$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The output can be one of states or a combination of many states. Since, $y = x_1$,

$$y = [1\,0\,0\,0\ \ldots\ 0]\boldsymbol{x}$$

## 1.1 Correlation between state variable and transfer functions models

The transfer function corresponding to state variable model (1), when $u$ and $y$ are scalars, is:

$$
\begin{aligned}
G(s) &= \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \qquad (2) \\
&= \frac{Q(s)}{|sI - A|}
\end{aligned}
$$

where $|sI - A|$ is the characteristic polynomial of the system.

## 1.2 Solution of Continuous Time State Equation

The solution of state equation (1) is given as

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

where $e^{At} = \Phi(t)$ is known as the state transition matrix and $\boldsymbol{x}(t_0)$ is the initial state of the system.

# 2 State Variable Analysis of Digital Control Systems

The discrete time systems, as discussed earlier, can be classified in two types.

1. Systems that result from sampling the continuous time system output at discrete instants only, i.e., sampled data systems.

2. Systems which are inherently discrete where the system states are defined only at discrete time instants and what happens in between is of no concern to us.

## 2.1 State Equations of Sampled Data Systems

Let us assume that the following continuous time system is subject to sampling process with an interval of $T$.

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + Bu(t) \quad : \text{State Equation} \\ y(t) &= C\boldsymbol{x}(t) + Du(t) \quad : \text{Output Equation} \end{aligned} \tag{3}$$

We know that the solution to above state equation is:

$$\boldsymbol{x}(t) = \Phi(t - t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \Phi(t - \tau)Bu(\tau)d\tau$$

Since the inputs are constants in between two sampling instants, one can write:

$$u(\tau) = u(kT) \quad \text{for,} \quad kT \leq \tau \leq (k+1)T$$

which implies that the following expression is valid within the interval $kT \leq \tau \leq (k+1)T$ if we consider $t_0 = kT$:

$$\boldsymbol{x}(t) = \Phi(t - kT)\boldsymbol{x}(kT) + \int_{kT}^{t} \Phi(t - \tau)Bu(kT)d\tau$$

Let us denote $\int_{kT}^{t} \Phi(t-\tau)Bd\tau$ by $\theta(t - KT)$. Then we can write:

$$\boldsymbol{x}(t) = \Phi(t - kT)\boldsymbol{x}(kT) + \theta(t - KT)u(kT)$$

If $t = (k+1)T$,

$$\boldsymbol{x}((k+1)T) = \Phi(T)\boldsymbol{x}(kT) + \theta(T)u(kT) \tag{4}$$

where $\Phi(T) = e^{AT}$ and $\theta(T) = \int_{kT}^{(k+1)T} \Phi((k+1)T - \tau)Bd\tau$. If $t' = \tau - kT$, we can rewrite $\theta(T)$ as $\theta(T) = \int_{0}^{T} \Phi(T - t')Bdt'$. Equation (4) has a similar form as that of equation (3) if we consider $\phi(T) = \bar{A}$ and $\theta(T) = \bar{B}$. Similarly by setting $t = kT$, one can show that the output equation also has a similar form as that of the continuous time one.

When $T = 1$,

$$\begin{aligned} \boldsymbol{x}(k+1) &= \Phi(1)\boldsymbol{x}(k) + \theta(1)u(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

## 2.2 State Equations of Inherently Discrete Systems

When a discrete system is composed of all digital signals, the state and output equations can be described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

## 2.3 Discrete Time Approximation of A Continuous Time State Space Model

Let us consider the dynamical system described by the state space model (3). By approximating the derivative at $t = kT$ using forward difference, we can write:

$$\begin{aligned} \dot{\boldsymbol{x}}(t)\Big|_{t=kT} &= \frac{1}{T}[\boldsymbol{x}((k+1)T) - \boldsymbol{x}(kT)] \\ \Rightarrow \frac{1}{T}[\boldsymbol{x}((k+1)T) - \boldsymbol{x}(kT)] &= A\boldsymbol{x}(kT) + Bu(kT) \\ \text{and,} \quad y(kT) &= C\boldsymbol{x}(kT) + Du(kT) \end{aligned}$$

Rearranging the above equations,

$$\begin{aligned} \boldsymbol{x}((k+1)T) &= (I + TA)\boldsymbol{x}(kT) + TBu(kT) \\ \text{If, } T = 1 \Rightarrow \boldsymbol{x}(k+1) &= (I + A)\boldsymbol{x}(k) + Bu(k) \\ \text{and } y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

We can thus conclude from the discussions so far that the discrete time state variable model of a system can be described by

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned}$$

where $A$, $B$ are either the descriptions of an all digital system or obtained by sampling the continuous time process.

# Module 7: Discrete State Space Models
## Lecture Note 2

In this lecture we will discuss about the relation between transfer function and state space model for a discrete time system and various standard or canonical state variable models.

## 1 State Space Model to Transfer Function

Consider a discrete state variable model

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k) \\ y(k) &= C\boldsymbol{x}(k) + Du(k) \end{aligned} \tag{1}$$

Taking the Z-transform on both sides of Eqn. (1), we get

$$\begin{aligned} zX(z) - z\boldsymbol{x}_0 &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z) \end{aligned}$$

where $\boldsymbol{x}_0$ is the initial state of the system.

$$\begin{aligned} &\Rightarrow \qquad (zI - A)X(z) = z\boldsymbol{x}_0 + BU(z) \\ &\text{or,} \quad X(z) = (zI - A)^{-1}z\boldsymbol{x}_0 + (zI - A)^{-1}BU(z) \end{aligned}$$

To find out the transfer function, we assume that the initial conditions are zero, i.e., $x_0 = 0$, thus

$$Y(z) = \Big(C(zI - A)^{-1}B + D\Big)U(z)$$

Therefore, the transfer function becomes

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D \tag{2}$$

which has the same form as that of a continuous time system.

# 2 Various Canonical Forms

We have seen that transform domain analysis of a digital control system yields a transfer function of the following form.

$$G(z) = \frac{Y(z)}{U(z)} = \frac{\beta_0 z^m + \beta_1 z^{m-1} + \ldots + \beta_m}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \quad m \leq n \tag{3}$$

Various canonical state variable models can be derived from the above transfer function model.

## 2.1 Controllable canonical form

Consider the transfer function as given in Eqn. (3). Without loss of generality, let us consider the case when $m = n$. Let

$$\frac{\bar{X}(z)}{U(z)} = \frac{1}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n}$$

In time domain, the above equation may be written as

$$\bar{x}(k+n) + \alpha_1 \bar{x}(k+n-1) + \ldots + \alpha_n \bar{x}(k) = u(k)$$

Now, the output $Y(z)$ may be written in terms of $\bar{X}(z)$ as

$$Y(z) = \left(\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n\right) \bar{X}(z)$$

or in time domain as

$$y(k) = \beta_0 \bar{x}(k+n) + \beta_1 \bar{x}(k+n-1) + \ldots + \beta_n \bar{x}(k)$$

The block diagram representation of above equations is shown in Figure 1. State variables are selected as shown in Figure 1.

The state equations are then written as:

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= x_3(k) \\
\vdots &= \vdots \\
x_n(k+1) &= -\alpha_n x_1(k) - \alpha_{n-1} x_2(k) - \ldots - \alpha_1 x_n(k) + u(k)
\end{aligned}$$

Output equation can be written as by following the Figure 1.

$$y(k) = (\beta_n - \alpha_n \beta_0) x_1(k) + (\beta_{n-1} - \alpha_{n-1}\beta_0) x_2(k) + \ldots + (\beta_1 - \alpha_1 \beta_0) x_n(k) + \beta_0 u(k)$$

In state space form, we have

$$\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\
y(k) &= C\mathbf{x}(k) + Du(k)
\end{aligned} \tag{4}$$

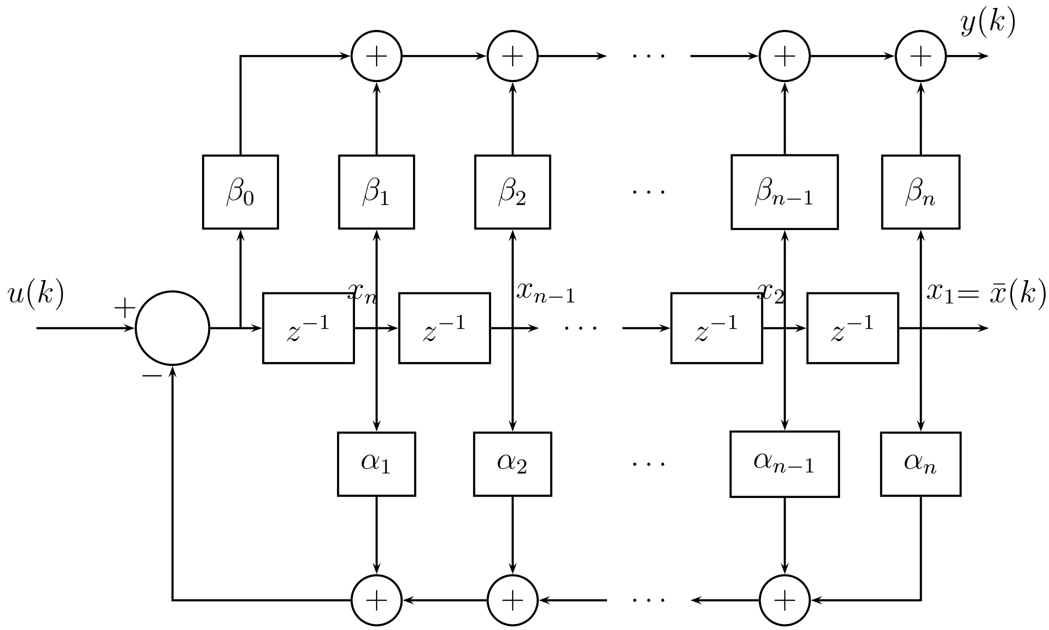


Figure 1: Block Diagram representation of controllable canonical form

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & \dots & -\alpha_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n\beta_0 \;\; \beta_{n-1} - \alpha_{n-1}\beta_0 \;\; \dots \;\; \beta_1 - \alpha_1\beta_0] \qquad D = \beta_0$$

## 2.2 Observable Canonical Form

Equation (3) may be rewritten as

$$\begin{aligned} & \left(z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n\right) Y(z) = \left(\beta_0 z^n + \beta_1 z^{n-1} + \ldots + \beta_n\right) U(z) \\ \text{or,} \quad & z^n[Y(z) - \beta_0 U(z)] + z^{n-1}[\alpha_1 Y(z) - \beta_1 U(z)] + \ldots + [\alpha_n Y(z) - \beta_n U(z)] = 0 \\ \text{or,} \quad & Y(z) = \beta_0 U(z) - z^{-1}[\alpha_1 Y(z) - \beta_1 U(z)] - \ldots - z^{-n}[\alpha_n Y(z) - \beta_n U(z)] \end{aligned}$$

The corresponding block diagram is shown in Figure 2. Choosing the outputs of the delay blocks

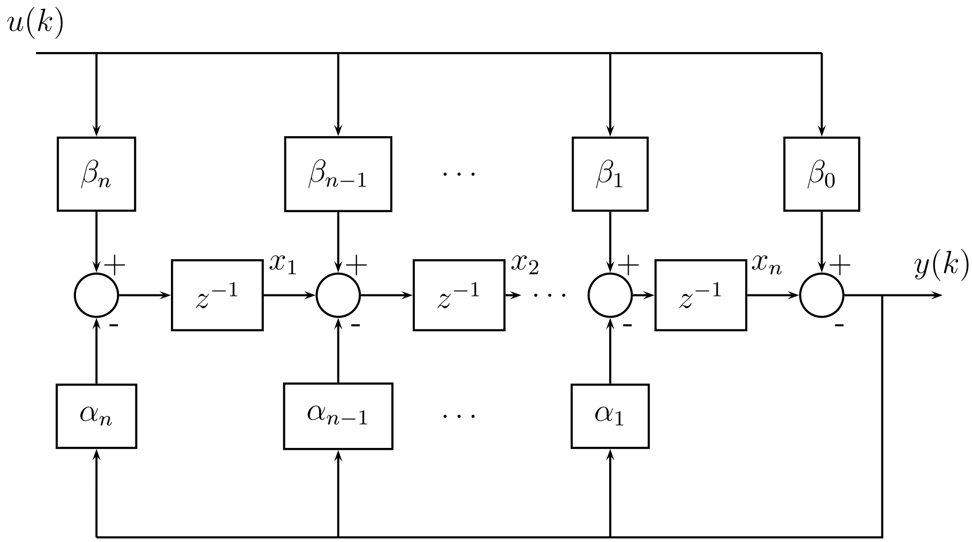


Figure 2: Block Diagram representation of observable canonical form

as the state variables, we have following state equations

$$
\begin{aligned}
x_n(k+1) &= x_{n-1}(k) - \alpha_1(x_n(k) + \beta_0 u(k)) + \beta_1 u(k) \\
x_{n-1}(k+1) &= x_{n-2}(k) - \alpha_2(x_n(k) + \beta_0 u(k)) + \beta_2 u(k) \\
\vdots &= \vdots \\
x_1(k+1) &= -\alpha_n(x_n(k) + \beta_0 u(k)) + \beta_n u(k) \\
y(k) &= x_n(k) + \beta_0 u(k)
\end{aligned}
$$

This can be rewritten in matrix form (4) with

$$
A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -\alpha_n \\ 1 & 0 & 0 & \dots & 0 & -\alpha_{n-1} \\ 0 & 1 & 0 & \dots & 0 & -\alpha_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & 1 & -\alpha_1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_n - \alpha_n\beta_0 \\ \beta_{n-1} - \alpha_{n-1}\beta_0 \\ \vdots \\ \beta_1 - \alpha_1\beta_0 \end{bmatrix} \quad C = [0 \;\; 0 \;\; \dots \;\; 1] \quad D = \beta_0
$$

## 2.3 Duality

In previous two sections we observed that the system matrix $A$ in observable canonical form is transpose of the system matrix in controllable canonical form. Similarly, control matrix $B$ in observable canonical form is transpose of output matrix $C$ in controllable canonical form. So also output matrix $C$ in observable canonical form is transpose of control matrix $B$ in controllable canonical form.

## 2.4 Jordan Canonical Form

In Jordan canonical form, the system matrix $A$ represents a diagonal matrix for distinct poles which basically form the diagonal elements of $A$.

Assume that $z = \lambda_i, \quad i = 1, 2, \ldots, n$ are the distinct poles of the given transfer function (3). Then partial fraction expansion of the transfer function yields

$$
\begin{aligned}
\frac{Y(z)}{U(z)} &= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \ldots + \bar{\beta}_n}{z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n} \\
&= \beta_0 + \frac{\bar{\beta}_1 z^{n-1} + \bar{\beta}_2 z^{n-2} + \ldots + \bar{\beta}_n}{(z-\lambda_1)(z-\lambda_2)\ldots(z-\lambda_n)} \\
&= \beta_0 + \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \ldots + \frac{r_n}{z-\lambda_n} \qquad (5)
\end{aligned}
$$

A parallel realization of the transfer function (5) is shown in Figure 3.

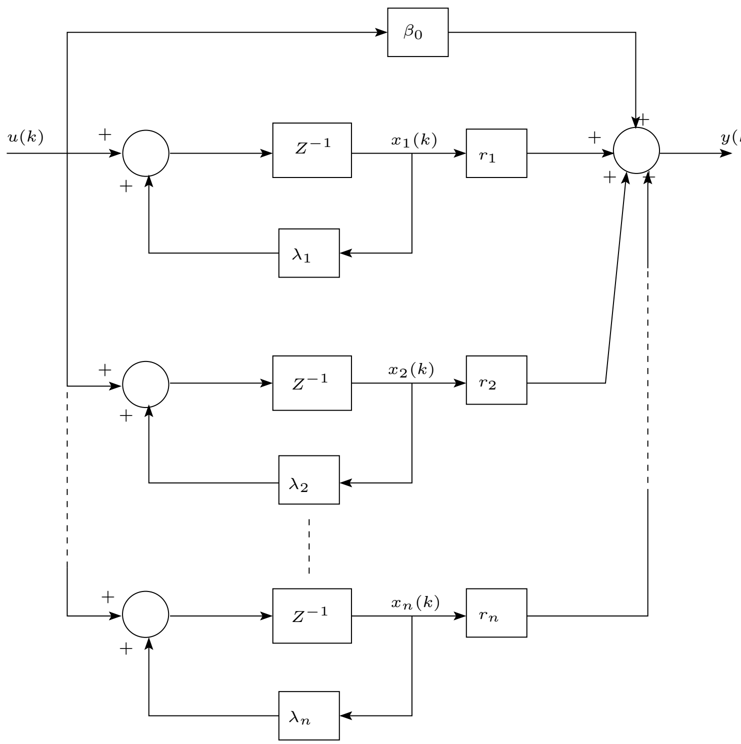


Figure 3: Block Diagram representation of Jordan canonical form

Considering the outputs of the delay blocks as the state variables, we can construct the state model in matrix form (4), with

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} r_1 & r_2 & \dots & r_n \end{bmatrix} \quad D = \beta_0$$

When the matrix $A$ has repeated eigenvalues, it cannot be expressed in a proper diagonal form. However, it can be expressed in a Jordan canonical form which is nearly a diagonal matrix. Let us consider that the system has eigenvalues, $\lambda_1$, $\lambda_1$, $\lambda_2$ and $\lambda_3$. In that case, $A$ matrix in Jordan canonical form will be

$$A = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}$$

1. The diagonal elements of the matrix $A$ are eigenvalues of the same.
2. The elements below the principal diagonal are zero.
3. Some of the elements just above the principal diagonal are one.
4. The matrix can be divided into a number of blocks, called Jordan blocks, along the diagonal. Each block depends on the multiplicity of the eigenvalue associated with it. For example Jordan block associated with a eigenvalue $z_1$ of multiplicity 4 can be written as

$$A = \begin{bmatrix} z_1 & 1 & 0 & 0 \\ 0 & z_1 & 1 & 0 \\ 0 & 0 & z_1 & 1 \\ 0 & 0 & 0 & z_1 \end{bmatrix}$$

**Example:** Consider the following discrete transfer function.

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24}$$

Find out the state variable model in 3 different canonical forms.

**Solution:**

The state variable model in controllable canonical form can directly be derived from the transfer function, where the $A$, $B$, $C$ and $D$ matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ -0.24 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.04 & 0.17 \end{bmatrix}, \quad D = 0$$

The matrices in state model corresponding to observable canonical form are obtained as,

$$A = \begin{bmatrix} 0 & -0.24 \\ 1 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.04 \\ 0.17 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0$$

To find out the state model in Jordan canonical form, we need to fact expand the transfer function using partial fraction, as

$$G(z) = \frac{0.17z + 0.04}{z^2 - 1.1z + 0.24} = \frac{0.352}{z - 0.8} + \frac{-0.182}{z - 0.3}$$

Thus the $A$, $B$, $C$ and $D$ matrices will be:

$$A = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.352 & -0.182 \end{bmatrix}, \quad D = 0$$

# Module 7: Discrete State Space Models
## Lecture Note 3

## 1 Characteristic Equation, eigenvalues and eigen vectors

For a discrete state space model, the characteristic equation is defined as

$$|zI - A| = 0$$

The roots of the characteristic equation are the eigenvalues of matrix $A$.

1. If $det(A) \neq 0$, i.e., $A$ is nonsingular and $\lambda_1$, $\lambda_2$, $\cdots$, $\lambda_n$ are the eigenvalues of $A$, then, $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, $\cdots$, $\frac{1}{\lambda_n}$ will be the eigenvalues of $A^{-1}$.
2. Eigenvalues of $A$ and $A^T$ are same when $A$ is a real matrix.
3. If $A$ is a real symmetric matrix then all its eigenvalues are real.

The $n \times 1$ vector $v_i$ which satisfies the matrix equation

$$Av_i = \lambda_i v_i \tag{1}$$

where $\lambda_i$, $i = 1, 2, \cdots, n$ denotes the $i^{th}$ eigenvalue, is called the eigen vector of $A$ associated with the eigenvalue $\lambda_i$. If eigenvalues are distinct, they can be solved directly from equation (1).

### Properties of eigen vectors

1. An eigen vector cannot be a null vector.
2. If $v_i$ is an eigen vector of $A$ then $mv_i$ is also an eigen vector of $A$ where $m$ is a scalar.
3. If $A$ has $n$ distinct eigenvalues, then the $n$ eigen vectors are linearly independent.

### Eigen vectors of multiple order eigenvalues

When the matrix $A$ an eigenvalue $\lambda$ of multiplicity $m$, a full set of linearly independent may not exist. The number of linearly independent eigen vectors is equal to the degeneracy $d$ of $\lambda I - A$. The degeneracy is defined as

$$d = n - r$$

where $n$ is the dimension of $A$ and $r$ is the rank of $\lambda I - A$. Furthermore,

$$1 \leq d \leq m$$

# 2 Similarity Transformation and Diagonalization

Square matrices $A$ and $\bar{A}$ are similar if

$$\begin{aligned} AP &= P\bar{A} \\ \text{or,} \quad \bar{A} &= P^{-1}AP \\ \text{and,} \quad A &= P\bar{A}P^{-1} \end{aligned}$$

The non-singular matrix $P$ is called similarity transformation matrix. It should be noted that eigenvalues of a square matrix $A$ are not altered by similarity transformation.

**Diagonalization:**

If the system matrix $A$ of a state variable model is diagonal then the state dynamics are decoupled from each other and solving the state equations become much more simpler.

In general, if $A$ has distinct eigenvalues, it can be diagonalized using similarity transformation. Consider a square matrix $A$ which has distinct eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$. It is required to find a transformation matrix $P$ which will convert $A$ into a diagonal form

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

through similarity transformation $AP = P\Lambda$. If $v_1,\ v_2,\ \ldots,\ v_n$ are the eigenvectors of matrix $A$ corresponding to eigenvalues $\lambda_1,\ \lambda_2,\ \ldots\ \lambda_n$, then we know $Av_i = \lambda_i v_i$. This gives

$$A\,[v_1\ v_2\ \ldots\ v_n] = [v_1\ v_2\ \ldots\ v_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Thus $P = [v_1\ v_2\ \ldots\ v_n]$. Consider the following state model.

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

If $P$ transforms the state vector $\boldsymbol{x}(k)$ to $\boldsymbol{z}(k)$ through the relation

$$\boldsymbol{x}(k) = P\boldsymbol{z}(k), \text{or,} \quad \boldsymbol{z}(k) = P^{-1}\boldsymbol{x}(k)$$

then the modified state space model becomes

$$\boldsymbol{z}(k+1) = P^{-1}AP\boldsymbol{z}(k) + P^{-1}Bu(k)$$

where $P^{-1}AP = \Lambda$.

# 3 Computation of $\Phi(t)$

We have seen that to derive the state space model of a sampled data system, we need to know the continuous time state transition matrix $\Phi(t) = e^{At}$.

### 3.1 Using Inverse Laplace Transform

For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, the state transition matrix $e^{At}$ can be computed as,

$$e^{At} = \mathscr{L}^{-1}\{(sI - A)^{-1}\}$$

### 3.2 Using Similarity Transformation

If $\Lambda$ is the diagonal representation of the matrix $A$, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

Given $e^{\Lambda t}$, we can show that

$$e^{At} = Pe^{\Lambda t}P^{-1}$$

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \dots \\ \Rightarrow P^{-1}e^{At}P &= P^{-1}\left[I + At + \frac{1}{2!}A^2t^2 + \dots\right]P \\ &= I + P^{-1}APt + \frac{1}{2!}P^{-1}APP^{-1}APt^2 + \dots \\ &= I + \Lambda t + \frac{1}{2!}\Lambda^2t^2 + \dots \\ &= e^{\Lambda t} \\ \Rightarrow e^{At} &= Pe^{\Lambda t}P^{-1} \end{aligned}$$

### 3.3 Using Caley Hamilton Theorem

Every square matrix $A$ satisfies its own characteristic equation. If the characteristic equation is

$$\triangle(\lambda) = |\lambda I - A| = \lambda^n + \alpha_1\lambda^{n-1} + \dots + \alpha_n = 0$$

then,

$$\triangle(A) = A^n + \alpha_1 A^{n-1} + \dots + \alpha_n I = 0$$

Application: Evaluation of any function $f(\lambda)$ and $f(A)$

$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n + \dots \quad \text{order } \infty$$

$$\frac{f(\lambda)}{\triangle(\lambda)} = q(\lambda) + \frac{g(\lambda)}{\triangle(\lambda)}$$

$$\begin{aligned} f(\lambda) &= q(\lambda)\triangle(\lambda) + g(\lambda) \\ &= g(\lambda) \\ &= \beta_0 + \beta_1\lambda + \dots + \beta_{n-1}\lambda^{n-1} \quad \text{order } n-1 \end{aligned}$$

If $A$ has distinct eigenvalues $\lambda_1, \cdots, \lambda_n$, then,

$$f(\lambda_i) = g(\lambda_i), \quad i = 1, \cdots, n$$

The solution will give rise to $\beta_0, \beta_1, \cdots, \beta_{n-1}$, then

$$f(A) = \beta_0 I + \beta_1 A + \cdots + \beta_{n-1} A^{n-1}$$

If there are multiple roots (multiplicity = 2), then

$$f(\lambda_i) = g(\lambda_i) \tag{2}$$

$$\frac{\partial}{\partial \lambda_i} f(\lambda_i) = \frac{\partial}{\partial \lambda_i} g(\lambda_i) \tag{3}$$

**Example 1:**

$$\text{If } A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

then compute the state transition matrix using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1 \text{ (with multiplicity 2)}, \ \lambda_2 = 2$$

$$\text{Let } f(\lambda) = e^{\lambda t} \text{ and } g(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$

Then using (2) and (3), we can write

$$f(\lambda_1) = g(\lambda_1)$$

$$\frac{\partial}{\partial \lambda_1} f(\lambda_1) = \frac{\partial}{\partial \lambda_1} g(\lambda_1)$$

$$f(\lambda_2) = g(\lambda_2)$$

This implies

$$\begin{aligned} e^t &= \beta_0 + \beta_1 + \beta_2 \quad (\lambda_1 = 1) \\ te^t &= \beta_1 + 2\beta_2 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 + 4\beta_2 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_0 = e^{2t} - 2te^t, \qquad \beta_1 = 3te^t + 2e^t - 2e^{2t}, \qquad \beta_2 = e^{2t} - e^t - te^t$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A + \beta_2 A^2 \\ &= \begin{bmatrix} 2e^t - e^{2t} & 0 & 2e^t - 2e^{2t} \\ 0 & e^t & 0 \\ e^{2t} - e^t & 0 & 2e^{2t} - e^t \end{bmatrix} \end{aligned}$$

**Example 2** For the system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$, where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. compute $e^{At}$ using 3 different techniques.

**Solution:** Eigenvalues of matrix $A$ are $1 \pm j1$.

**Method 1**

$$\begin{aligned}
e^{At} &= \mathscr{L}^{-1}(sI - A)^{-1} = \mathscr{L}^{-1} = \begin{bmatrix} s-1 & -1 \\ 1 & s-1 \end{bmatrix}^{-1} \\
&= \mathscr{L}^{-1} \frac{1}{s^2 - 2s + 2} \begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix} \\
&= \mathscr{L}^{-1} \begin{bmatrix} \frac{s-1}{(s-1)^2+1} & \frac{1}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos\, t & e^t \sin\, t \\ -e^t \sin\, t & e^t \cos\, t \end{bmatrix}
\end{aligned}$$

**Method 2**

$e^{At} = Pe^{\Lambda t}P^{-1}$ where $e^{\Lambda t} = \begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix}$. Eigen values are $1 \pm j$. The corresponding eigenvectors are found by using equation $Av_i = \lambda_i v_i$ as follows:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (1+j) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Taking $v_1 = 1$, we get $v_2 = j$. So, the eigenvector corresponding to $1+j$ is $\begin{bmatrix} 1 \\ j \end{bmatrix}$ and the one corresponding to $1-j$ is $\begin{bmatrix} 1 \\ -j \end{bmatrix}$. The transformation matrix is given by

$$P = [v_1 \;\; v_2] = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix}$$

Now,

$$\begin{aligned}
e^{At} &= Pe^{\Lambda t}P^{-1} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{(1+j)t} & 0 \\ 0 & e^{(1-j)t} \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{(1+j)t} & e^{(1-j)t} \\ je^{(1+j)t} & -je^{(1-j)t} \end{bmatrix} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{(1+j)t} + e^{(1-j)t} & -j\left(e^{(1+j)t} - e^{(1-j)t}\right) \\ j\left(e^{(1+j)t} - e^{(1-j)t}\right) & e^{(1+j)t} + e^{(1-j)t} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2e^t \cos\, t & -j(j)e^t 2 \sin\, t \\ e^t(j)(j)2 \sin\, t & 2e^t \cos\, t \end{bmatrix} \\
&= \begin{bmatrix} e^t \cos\, t & e^t \sin\, t \\ -e^t \sin\, t & e^t \cos\, t \end{bmatrix}
\end{aligned}$$

**Method 3: Caley Hamilton Theorem**

The eigenvalues are $\lambda_{1,2} = 1 \pm j$.

$$\begin{aligned} e^{\lambda_1 t} &= \beta_0 + \beta_1 \lambda_1 \\ e^{\lambda_2 t} &= \beta_0 + \beta_1 \lambda_2 \end{aligned}$$

Solving,

$$\begin{aligned} \beta_0 &= \frac{1}{2}(1+j)e^{(1+j)t} + \frac{1}{2}(1-j)e^{(1-j)t} \\ \beta_1 &= \frac{1}{2j}\left(e^{(1+j)^t} - e^{(1-j)^t}\right) \end{aligned}$$

Hence,

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^t \cos\ t & e^t \sin t \\ -e^t \sin t & e^t \cos\ t \end{bmatrix} \end{aligned}$$

We will now show through an example how to derive discrete state equation from a continuous one.

**Example:** Consider the following state model of a continuous time system.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) &= x_1(t) \end{aligned}$$

If the system is under a sampling process with period $T$, derive the discrete state model of the system.

To derive the discrete state space model, let us first compute the state transition matrix of the continuous time system using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$$

$$\text{Let } f(\lambda) = e^{\lambda t}$$

This implies

$$\begin{aligned} e^t &= \beta_0 + \beta_1 \quad (\lambda_1 = 1) \\ e^{2t} &= \beta_0 + 2\beta_1 \quad (\lambda_2 = 2) \end{aligned}$$

Solving the above equations

$$\beta_1 = e^{2t} - e^t \qquad \beta_0 = 2e^t - e^{2t}$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{bmatrix} \end{aligned}$$

Thus the discrete state matrix $A$ is given as

$$A = \Phi(T) = \begin{bmatrix} e^T & e^{2T} - e^T \\ 0 & e^{2T} \end{bmatrix}$$

The discrete input matrix $B$ can be computed as

$$\begin{aligned} B &= \Theta(T) = \int_0^T \Phi(T - t') \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \int_0^T \begin{bmatrix} e^T.e^{-t'} & e^{2T}.e^{-2t'} - e^T.e^{-t'} \\ 0 & e^{2T}.e^{-2t'} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \begin{bmatrix} e^T - 1 & 0.5e^{2T} - e^T + 0.5 \\ 0 & 0.5e^{2T} - 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} \end{aligned}$$

The discrete state equation is thus described by

$$\begin{aligned} \mathbf{x}((k+1)T) &= \begin{bmatrix} e^T & e^{2T} - e^T \\ 0 & e^{2T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} 0.5e^{2T} - e^T + 0.5 \\ 0.5e^{2T} - 0.5 \end{bmatrix} u(kT) \\ y(kT) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(kT) \end{aligned}$$

When $T = 1$, the state equations become

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 2.72 & 4.67 \\ 0 & 7.39 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1.48 \\ 3.19 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

# Module 7: Discrete State Space Models
## Lecture Note 4

In this lecture we would discuss about the solution of discrete state equation, computation of discrete state transition matrix and state diagram.

## 1 Solution to Discrete State Equation

Consider the following state model of a discrete time system:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$$

where the initial conditions are $\boldsymbol{x}(0)$ and $u(0)$. Putting $k = 0$ in the above equation, we get

$$\boldsymbol{x}(1) = A\boldsymbol{x}(0) + Bu(0)$$

Similarly if we put $k = 1$, we would get

$$\begin{aligned} \boldsymbol{x}(2) &= A\boldsymbol{x}(1) + Bu(1) \\ \text{Putting the expression of } \boldsymbol{x}(1) \Rightarrow \boldsymbol{x}(2) &= A^2\boldsymbol{x}(0) + ABu(0) + Bu(1) \end{aligned}$$

For $k = 2$,

$$\begin{aligned} \boldsymbol{x}(3) &= A\boldsymbol{x}(2) + Bu(2) \\ &= A^3\boldsymbol{x}(0) + A^2Bu(0) + ABu(1) + Bu(2) \end{aligned}$$

and so on. If we combine all these equations, we would get the following expression as a general solution:

$$\boldsymbol{x}(k) = A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i)$$

As seen in the above expression, $\boldsymbol{x}(k)$ has two parts. One is the contribution due to the initial state $\boldsymbol{x}(0)$ and the other one is the contribution of the external input $u(i)$ for $i = 0, 1, 2, \cdots, k-1$.

When the input is zero, solution of the homogeneous state equation $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ can be written as

$$\boldsymbol{x}(k) = A^k \boldsymbol{x}(0)$$

where $A^k = \phi(k)$ is the state transition matrix.

## 2 Evaluation of $\phi(k)$

Similar to the continuous time systems, the state transition matrix of a discrete state model can be evaluated using the following different techniques.

1. **Using Inverse Z-transform**:

$$\phi(k) = \mathscr{Z}^{-1}\{(zI - A)^{-1}\}$$

2. **Using Similarity Transformation** If $\Lambda$ is the diagonal representation of the matrix $A$, then $\Lambda = P^{-1}AP$. When a matrix is in diagonal form, computation of state transition matrix is straight forward:

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

   Given $\Lambda^k$, we can compute $A^k = P\Lambda^k P^{-1}$

3. **Using Caley Hamilton Theorem**

**Example** Compute $A^k$ for the following system using three different techniques and hence find $y(k)$ for $k \geq 0$.

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1)^k; \quad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ y(k) &= x_2(k) \end{aligned}$$

**Solution:** $A = \begin{bmatrix} 0 & 1 \\ -0.21 & -1 \end{bmatrix}$ and eigenvalues of $A$ are $-0.3$ and $-0.7$.

**Method 1**

$$A^k = \mathscr{Z}^{-1}(zI - A)^{-1} = \mathscr{Z}^{-1}\left\{\begin{bmatrix} z-1 & -1 \\ 1 & z-1 \end{bmatrix}^{-1}\right\}$$

$$
\begin{aligned}
A^k &= \mathscr{Z}^{-1}\begin{bmatrix} \frac{z+1}{z^2+z+0.21} & \frac{1}{z^2+z+0.21} \\ \\ \frac{-0.21}{z^2+z+0.21} & \frac{z}{z^2+z+0.21} \end{bmatrix} \\
&= \mathscr{Z}^{-1}\begin{bmatrix} \frac{1.75}{z+0.3} - \frac{0.75}{z+0.7} & \frac{2.5}{z+0.3} - \frac{2.5}{z+0.7} \\ \\ \frac{-0.525}{z+0.3} + \frac{0.525}{z+0.7} & \frac{-0.75}{z+0.3} + \frac{1.75}{z+0.7} \end{bmatrix} \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

**Method 2**

$A^k = P\Lambda^k P^{-1}$ where $\Lambda^k = \begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix}$. Eigen values are $-0.3$ and $-0.7$. The corresponding eigenvectors are found, by using equation $Av_i = \lambda_i v_i$, as $\begin{bmatrix} 1 \\ -0.3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -0.7 \end{bmatrix}$ respectively. The transformation matrix is given by

$$
P = \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix}
$$

Thus,

$$
\begin{aligned}
A^k &= P\Lambda^k P^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -0.3 & -0.7 \end{bmatrix} \begin{bmatrix} (-0.3)^k & 0 \\ 0 & (-0.7)^k \end{bmatrix} \begin{bmatrix} 1.75 & 2.5 \\ -0.75 & -2.5 \end{bmatrix} \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

**Method 3: Caley Hamilton Theorem**

The eigenvalues are $-0.3$ and $-0.7$.

$$
\begin{aligned}
(-0.3)^k &= \beta_0 - 0.3\beta_1 \\
(-0.7)^k &= \beta_0 - 0.7\beta_1
\end{aligned}
$$

Solving,

$$
\begin{aligned}
\beta_0 &= 1.75(-0.3)^k - 0.75(-0.7)^k \\
\beta_1 &= 2.5(-0.3)^k - 2.5(-0.7)^k
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\phi(k) = A^k &= \beta_0 I + \beta_1 A \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k & 2.5(-0.3)^k - 2.5(-0.7)^k \\ \\ -0.525(-0.3)^k + 0.525(-0.7)^k & -0.75(-0.3)^k + 1.75(-0.7)^k \end{bmatrix}
\end{aligned}
$$

---

The solution $\boldsymbol{x}(k)$ is

$$\begin{aligned}
\boldsymbol{x}(k) &= A^k\boldsymbol{x}(0) + \sum_{i=0}^{k-1} A^{k-1-i}Bu(i) \\
&= \begin{bmatrix} 1.75(-0.3)^k - 0.75(-0.7)^k \\ -0.525(-0.3)^k + 0.525(-0.7)^k \end{bmatrix} + \sum_{i=0}^{k-1} \begin{bmatrix} 2.5(-0.3)^{k-1-i} - 2.5(-0.7)^{k-1-i} \\ -0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i} \end{bmatrix} (-1)^i
\end{aligned}$$

Since $y(k) = x_2(k)$, we can write

$$\begin{aligned}
y(k) &= -0.525(-0.3)^k + 0.525(-0.7)^k + \sum_{i=0}^{k-1}[-0.75(-0.3)^{k-1-i} + 1.75(-0.7)^{k-1-i}](-1)^i \\
&= -0.525(-0.3)^k + 0.525(-0.7)^k - 0.75(-0.3)^{k-1}\sum_{i=0}^{k-1}(1/0.3)^i + 1.75(-0.7)^{k-1}\sum_{i=0}^{k-1}(1/0.7)^i
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{i=0}^{k-1}(1/0.3)^i &= \sum_{i=0}^{k-1}(3.33)^i = \frac{1-(3.33)^k}{1-3.33} = -0.43[1-(3.33)^k] \\
\sum_{i=0}^{k-1}(1/0.7)^i &= \sum_{i=0}^{k-1}(1.43)^i = \frac{1-(1.43)^k}{1-1.43} = -2.33[1-(1.43)^k]
\end{aligned}$$

Putting the above expression in $y(k)$

$$y(k) = 0.475(-0.3)^k - 5.3(-0.7)^k + (-0.3)^k(3.33)^k + 5.825(-0.7)^{k-1}(1.43)^k$$

# 3 State Diagram

Conventional signal flow graph method was meant for only algebraic equation, thus these are generally used for the derivation of input output relation in a transformed domain.

State diagram or state transition signal flow graph is an extension of conventional signal flow graph which can be applied to represent differential and difference equations as well.

**Example 1**: Draw the state diagram for the following differential equation.

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = u(t)$$

Considering the state variables as $x_1(t) = y(t)$ and $x_2(t) = \dot{y}(t)$, we can write

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -x_1(t) - 2x_2(t) + u(t)
\end{aligned}$$

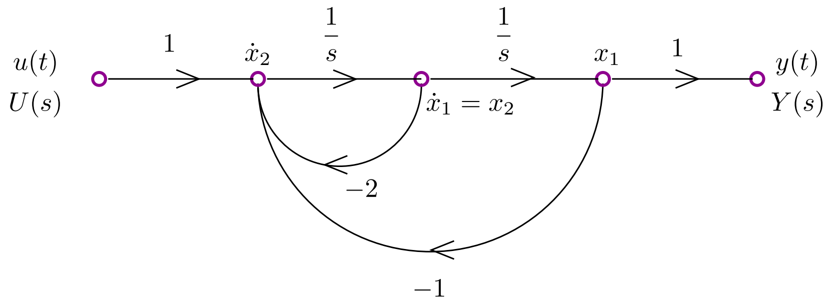


Figure 1: State Diagram of Example 1

The state diagram is shown in Figure 1.

**Example 2**: Consider a discrete time system described by the following state difference equations.

$$\begin{aligned} x_1(k+1) &= -x_1(k) + x_2(k) \\ x_2(k+1) &= -x_1(k) + u(k) \\ y(k) &= x_1(k) + x_2(k) \end{aligned}$$

Draw the state diagram.

The state diagram is shown in Figure 2.

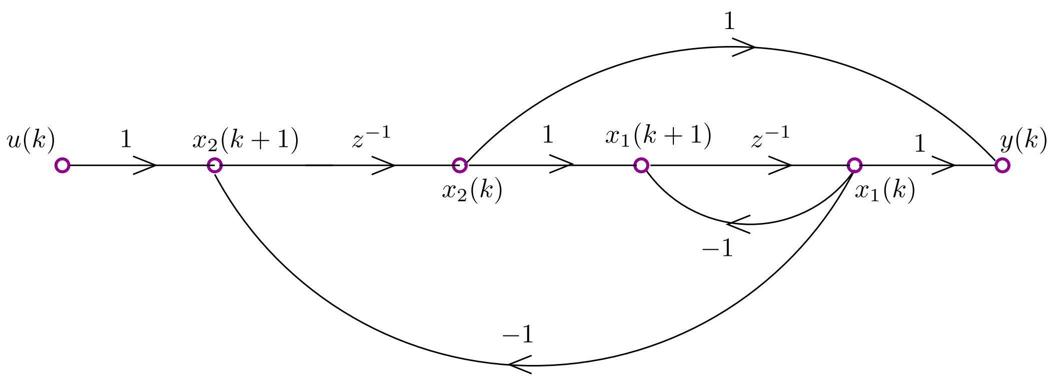


Figure 2: State Diagram of Example 2

## 3.1 State Diagram of Zero Order Hold

State diagram of zero order hold is important for sampled date control systems. Let the input to and output of a ZOH is $e^*(t)$ and $h(t)$ respectively. Then, for the inetrval $kT \leq t \leq (k+1)T$,

$$h(t)e(kT)$$

Or,

$$H(s) = \frac{e(kT)}{s}$$

Therefore, the state diagram, as shown in Figure 3, consists of a single branch with gain $s^{-1}$.

$$e(kT) \circ \xrightarrow{s^{-1}} \circ\ H(s)$$

Figure 3: State Diagram of Zero Order Hold

# 4 System Response between Sampling Instants

State variable method is a convenient way to evaluate the system response between the sampling instants of a sampled data system. State transition equation is given as:

$$\boldsymbol{x}(t) = \phi(t - t_0)\boldsymbol{x}(t_0) + u(t_0)\int_{t_0}^{t} \phi(t - \tau)Bd\tau$$

where $\boldsymbol{x}(t_0)$ is the initial state of the system and $u(t)$ is the external input.

$$\text{when } t_0 = kT, \qquad \boldsymbol{x}(t) = \phi(t - kT)\boldsymbol{x}(kT) + u(kT)\int_{kT}^{t} \phi(t - \tau)Bd\tau$$

Since we are interested in response between the sampling instants, let us consider $t = (k + \Delta)T$ where $k = 0, 1, 2, \cdots$ and $0 \leq \Delta \leq 1$. This implies

$$\boldsymbol{x}((k + \Delta)T) = \phi(\Delta T)\boldsymbol{x}(kT) + \theta(\Delta T)u(kT)$$

where $\theta(\Delta T) = \int_{kT}^{(k+\Delta)T} \phi((k + \Delta)T - \tau)Bd\tau$. By varying the value of $\Delta$ between 0 and 1 all information on $\boldsymbol{x}(t)$ for all $t$ can be obtained.

**Example 3:** Consider the following state model of a continuous time system.

$$\begin{aligned}
\dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \\
y(t) &= x_1(t)
\end{aligned}$$

which undergoes through a sampling process with period $T$. To derive the discrete state space model, let us first compute the state transition matrix of the continuous time system using Caley Hamilton Theorem.

$$\triangle(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda + 1 & 1 \\ 0 & \lambda + 2 \end{vmatrix} = (\lambda + 1)(\lambda + 2) = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$

$$\text{Let } f(\lambda) = e^{\lambda t}$$

This implies

$$\begin{aligned}
e^{-t} &= \beta_0 - \beta_1 \quad (\lambda_1 = -1) \\
e^{-2t} &= \beta_0 - 2\beta_1 \quad (\lambda_2 = -2)
\end{aligned}$$

Solving the above equations

$$\beta_1 = e^{-t} - e^{-2t} \qquad \beta_0 = 2e^{-t} - e^{-2t}$$

Then

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^{-t} & e^{-2t} - e^{-t} \\ 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Thus the discrete state matrix $A$ is given as

$$A = \phi(T) = \begin{bmatrix} e^{-T} & e^{-2T} - e^{-T} \\ 0 & e^{-2T} \end{bmatrix}$$

The discrete input matrix $B$ can be computed as

$$\begin{aligned} B &= \theta(T) = \int_0^T \Phi(T - t') \begin{bmatrix} 0 \\ 1 \end{bmatrix} dt' \\ &= \begin{bmatrix} e^{-T} - 0.5e^{-2T} - 0.5 \\ 0.5 - 0.5e^{-2T} \end{bmatrix} \end{aligned}$$

When $t = (k+1)T$, the discrete state equation is described by

$$\mathbf{x}((k+1)T) = \begin{bmatrix} e^{-T} & e^{-2T} - e^{-T} \\ 0 & e^{-2T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} e^{-T} - 0.5e^{-2T} - 0.5 \\ 0.5 - 0.5e^{-2T} \end{bmatrix} u(kT)$$

When $t = (k + \Delta)T$,

$$\mathbf{x}(kT + \Delta T) = \begin{bmatrix} e^{-\Delta T} & e^{-2\Delta T} - e^{-\Delta T} \\ 0 & e^{-2\Delta T} \end{bmatrix} \mathbf{x}(kT) + \begin{bmatrix} e^{-\Delta T} - 0.5e^{-2\Delta T} - 0.5 \\ 0.5 - 0.5e^{-2\Delta T} \end{bmatrix} u(kT)$$

If the sampling period $T = 1$,

$$\mathbf{x}(k+1) = \begin{bmatrix} 0.37 & -0.23 \\ 0 & 0.14 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.2 \\ 0.43 \end{bmatrix} u(k)$$

At the sampling instants we can find $\mathbf{x}(k)$ by putting $k = 0, 1, 2 \cdots$. If $\Delta = 0.5$, then between the sampling instants,

$$\mathbf{x}(k+0.5) = \begin{bmatrix} 0.61 & -0.24 \\ 0 & 0.37 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} -0.08 \\ 0.32 \end{bmatrix} u(k)$$

The responses in between the sampling instants, i.e., $\mathbf{x}(0.5)$, $\mathbf{x}(1.5)$, $\mathbf{x}(2.5)$ etc., can be found by putting $k = 0, 1, 2 \cdots$.

# Module 8: Controllability, Observability and Stability of Discrete Time Systems
## Lecture Note 1

Controllability and observability are two important properties of state models which are to be studied prior to designing a controller.

Controllability deals with the possibility of forcing the system to a particular state by application of a control input. If a state is uncontrollable then no input will be able to control that state. On the other hand whether or not the initial states can be observed from the output is determined using observability property. Thus if a state is not observable then the controller will not be able to determine its behavior from the system output and hence not be able to use that state to stabilize the system.

## 1 Controllability

Before going to any details, we would first formally define controllability. Consider a dynamical system

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k) \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k) + D\boldsymbol{u}(k) \end{aligned} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$.

**Definition 1. Complete State Controllability:** *The state equation* (1) *(or the pair* $(A, B)$ *) is said to be completely state controllable or simply state controllable if for any initial state* $\boldsymbol{x}(0)$ *and any final state* $\boldsymbol{x}(N)$*, there exists an input sequence* $\boldsymbol{u}(k)$*,* $k = 0, 1, 2, \cdots, N$*, which transfers* $\boldsymbol{x}(0)$ *to* $\boldsymbol{x}(N)$ *for some finite* $N$*. Otherwise the state equation* (1) *is state uncontrollable.*

**Definition 2. Complete Output Controllability:** *The system given in equation* (1) *is said to be completely output controllable or simply output controllable if any final output* $\boldsymbol{y}(N)$ *can be reached from any initial state* $\boldsymbol{x}(0)$ *by applying an unconstrained input sequence* $\boldsymbol{u}(k)$*,* $k = 0, 1, 2, \cdots, N$*, for some finite* $N$*. Otherwise* (1) *is not output controllable.*

## 1.1 Theorems on controllability

**State Controllability:**

1. The state equation (1) or the pair $(A, B)$ is state controllable if and only if the $n \times nm$ state controllability matrix

$$U_C = \begin{bmatrix} B & AB & A^2B & \ldots\ldots & A^{n-1}B \end{bmatrix}$$

   has rank $n$, i.e., full row rank.

2. The state equation (1) is controllable if the $n \times n$ controllability grammian matrix

$$W_c = \sum_{i=0}^{N-1} A^i BB^T (A^i)^T = \sum_{i=0}^{N-1} A^{N-1-i} BB^T (A^{N-1-i})^T$$

   is non-singular for any nonzero finite $N$.

3. If the system has a single input and the state model is in controllable canonical form then the system is controllable.

4. When $A$ has distinct eigenvalues and in Jordan/Diagonal canonical form, the state model is controllable if and only if all the rows of $B$ are nonzero.

5. When $A$ has multiple order eigenvalues and in Jordan canonical form, then the state model is controllable if and only if

   i. each Jordan block corresponds to one distinct eigenvalue and

   ii. the elements of $B$ that correspond to last row of each Jordan block are not all zero.

**Output Controllability:** The system in equation (1) is completely output controllable if and only if the $p \times (n+1)m$ output controllability matrix

$$U_{OC} = \begin{bmatrix} D & CB & CAB & CA^2B & \ldots\ldots & CA^{n-1}B \end{bmatrix}$$

has rank $p$, i.e., full row rank.

## 1.2 Controllability to the origin and Reachability

There exist three different definitions of state controllability in the literature:

1. Input transfers any state to any state. This definition is adopted in this course.

2. Input transfers any state to zero state. This is called controllability to the origin.

3. Input transfers zero state to any state. This is referred as controllability from the origin or reachability.

Above three definitions are equivalent for continuous time system. For discrete time systems definitions (1) and (3) are equivalent but not the second one.

**Example:** Consider the system $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + Bu(k)$, $y(k) = Cx(k)$. where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } C = [0 \;\; 1]$$

Show if the system is controllable. Find the transfer function $\dfrac{Y(z)}{U(z)}$. Can you see any connection between controllability and the transfer function?
***Solution:*** The controllability matrix is given by

$$U_C = [B \;\; AB] = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Its determinant $\|U_C\| = 0 \Rightarrow U_C$ has a rank 1 which is less than the order of the matrix, i.e., 2. Thus the system is not controllable. The transfer function

$$G(z) = \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B \qquad = [0\; 1] \begin{bmatrix} z+2 & -1 \\ -1 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{z+1}$$

Although state model is of order 2, the transfer function has order 1. The eigenvalues of $A$ are $\lambda_1 = -1$ and $\lambda_2 = -3$. This implies that the transfer function is associated with pole-zero cancellation for the pole at $-3$. Since one of the dynamic modes is cancelled, the system became uncontrollable.

## 2 Observability

**Definition 3.** *The state model* (1) *(or the pair* $(A, C)$ *) is said to be observable if any initial state* $\boldsymbol{x}(0)$ *can be uniquely determined from the knowldge of output* $y(k)$ *and input sequence* $u(k)$*, for* $k = 0, 1, 2, \cdots, N$*, where* $N$ *is some finite time. Otherwise the state model* (1) *is unobservable.*

### 2.1 Theorems on observability

1. The state model (1) or the pair $(A, C)$ is observable if the $np \times n$ observability matrix

$$U_O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank $n$, i.e., full column rank.

2. The state model (1) is observable if the $n \times n$ observability grammian matrix

$$W_O = \sum_{i=0}^{N-1} (A^i)^T C^T C A^i = \sum_{i=0}^{N-1} (A^{N-1-i})^T C^T C A^{N-1-i}$$

is non-singular for any nonzero finite $N$.

3. If the state model is in observable canonical form then the system is observable.

4. When $A$ has distinct eigenvalues and in Jordan/Diagonal canonical form, the state model is observable if and only if none of the columns of $C$ contain zeros.

5. When $A$ has multiple order eigenvalues and in Jordan canonical form, then the state model is observable if and only if

    i. each Jordan block corresponds to one distinct eigenvalue and

    ii. the elements of $C$ that correspond to first column of each Jordan block are not all zero.

### 2.2 Theorem of Duality

*The pair $(A, B)$ is controllable if and only if the pair $(A^T, B^T)$ is observable.*

**Exercise:** Prove the theorem of duality.

## 3 Loss of controllability or observability due to pole-zero cancellation

We have already seen through an example that a system becomes uncontrollable when one of the modes is cancelled. Let us take another example.

**Example:**

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [1 \quad 1]\boldsymbol{x}(k) \end{aligned}$$

The controllability matrix

$$U_C = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

implies that the state model is controllable. On the other hand, the observability matrix

$$U_O = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

has a rank 1 which implies that the state model is unobservable. Now, if we take a different set of state variables so that, $\bar{x}_1(k) = y(k)$, then the state variable model will be:

$$
\begin{aligned}
\bar{x}_1(k+1) &= y(k+1) \\
\bar{x}_1(k+2) &= y(k+2) = -y(k) - 2y(k+1) + u(k+1) + u(k)
\end{aligned}
$$

Lets us take $\bar{x}_2(k) = y(k+1) - u(k)$. The new state variable model is:

$$
\begin{aligned}
\bar{x}_1(k+1) &= \bar{x}_2(k) + u(k) \\
\bar{x}_2(k+1) &= -\bar{x}_1(k) - 2\bar{x}_2(k) - u(k)
\end{aligned}
$$

which implies

$$
\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = [1 \quad 0]
$$

The controllability matrix

$$
\bar{U}_C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
$$

implies that the state model is uncontrollable. The observability matrix

$$
\bar{U}_O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

implies that the state model is observable. The system difference equation will result in a transfer function which would involve pole-zero cancellation. Whenever there is a pole zero cancellation, the state space model will be either uncontrollable or unobservable or both.

# 4 Controllability/Observability after sampling

Question: If a continuous time system is undergone a sampling process will its controllability or observability property be maintained?

The answer to the question depends on the sampling period $T$ and the location of the eigenvalues of $A$.

- Loss of controllability and/or observability occurs only in presence of oscillatory modes of the system.
- A sufficient condition for the discrete model with sampling period $T$ to be controllable is that whenever $Re[\lambda_i - \lambda_j] = 0$, $|Im[\lambda_i - \lambda_j]| \neq 2\pi m/T$ for $m = 1, 2, 3, ...$
- The above is also a necessary condition for a single input case.

**Note:** *If a continuous time system is not controllable or observable, then its discrete time version, with any sampling period, is not controllable or observable.*

# Module 8: Controllability, Observability and Stability of Discrete Time Systems
## Lecture Note 2

The utmost important requirement in control system design is the stability. We would revisit some of the definitions related to stability of a system.

## 1 Revisiting the basics

Let us consider the following system.

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + Bu(k), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y(k) &= C\boldsymbol{x}(k) \end{aligned} \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times 1}$, $C \in R^{1\times n}$.

**Zero State Response:** The output response of system (1) that is due to the input only (initial states are zero) is called zero state response.

**Zero Input Response:** The output response of system (1) that is driven by the initial states only (in absence of any external input) is called zero input response.

**BIBO Stability:** If for any bounded input $u(k)$, the output $y(k)$ is also bounded, then the system is said to be BIBO stable.

**Bounded Input Bounded State Stability:** If for any bounded input $u(k)$, the states are also bounded, then the system is said to be Bounded Input Bounded State stable.

**$L^2$ Norm:** $L^2$ norm of a state vector $\boldsymbol{x}(k)$ is defined as

$$||\boldsymbol{x}(k)||_2 = \left[\sum_{i=1}^{n} x_i^2(k)\right]^{\frac{1}{2}}$$

$\boldsymbol{x}(k)$ is said to be bounded if $||\boldsymbol{x}(k)|| < M$ for all $k$ where $M$ is finite.

**Zero Input or Internal Stability:** If the zero input response of a system subject to a finite initial condition is bounded and reaches zero as $k \to \infty$, then the system is said to be internally stable.

The above condition can be formulated as

$$\begin{array}{ll} 1. & |y(k)| \leq M < \infty \\ 2. & \lim_{k \to \infty} |y(k)| = 0 \end{array}$$

The above conditions are also requirements for **asymptotic stability**.

To ensure all possible stability for an LTI system, the only requirement is that the roots of the characteristic equations are inside the unit circle.

# 2 Definitions Related to Stability for A Generic System

We know that a general time invariant system (linear or nonlinear) with no external input can be modeled by the following equation

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k)) \tag{2}$$

**Equilibrium Point:** The equilibrium point or equilibrium state of a system is that point in the state space where the dynamics of the system is zero which implies that the states will remain there forever once brought.

Thus the equilibrium points are the solutions of the following equation.

$$\boldsymbol{f}(\boldsymbol{x}(k)) = \mathbf{0} \tag{3}$$

One should note that since an LTI system with no external input can be modeled by $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$, $\boldsymbol{x}(k) = \mathbf{0}$ is the only equilibrium point for such a system.

Nonlinear systems can have multiple equilibrium points. Thus when we talk about the stability of a nonlinear system, we do so with respect to the equilibrium points.

For convenience, we state all definitions for the case when the equilibrium point is at the origin. There is no loss of generality if we do so because any equilibrium point can be shifted to origin via a change of variables.

**Example:**

Find out the equilibrium points of the following nonlinear system.

$$\begin{aligned} x_1(k+1) &= x_1(k) - x_1^3(k) \\ x_2(k+1) &= -x_2(k) \end{aligned}$$

Equating $x_1(k) - x_1^3(k)$ and $-x_2(k)$ to 0, we get $x_2 = 0$ always. $x_1$ can take 3 values, which are 0, 1 and $-1$ respectively. Thus the sytem has three equilibrium points, located at $(0,\ 0)$, $(1,\ 0)$ and $(-1,\ 0)$ respectively.

**Stability in the sense of Lyapunov:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is stable if, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, k_0) > 0$ such that

$$||\boldsymbol{x}(k_0)|| < \delta \quad \Rightarrow \quad ||\boldsymbol{x}(k)|| < \epsilon,\ \forall\ k \geq k_0 \geq 0$$

The above condition is illustrated in Figure 1.

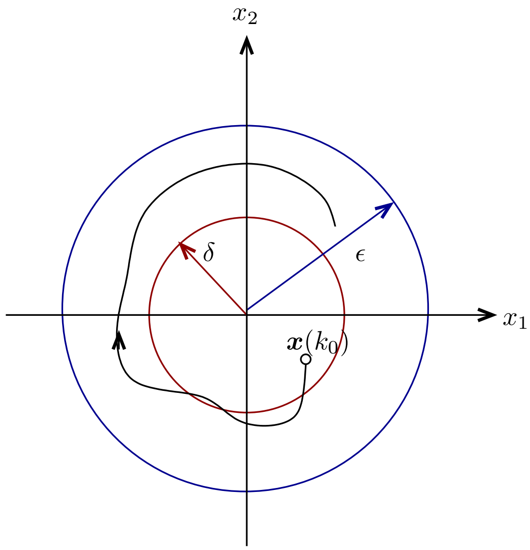


Figure 1: Illustration of stable equilibrium in the sense of Lyapunov in two dimension

**Asymptotic Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is asymptotically stable if it is stable and there is a positive constant $c = c(k_0)$ such that

$$\boldsymbol{x}(k) \to \mathbf{0} \text{ as } k \to \infty, \quad \forall\ ||\boldsymbol{x}(k_0)|| < c$$

The above condition is illustrated in Figure 2.

**Instability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is unstable if it is not stable.

The above condition is illustrated in Figure 3.

**Uniform Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is uniformly stable if, for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$, independent of $k_0$, such that

$$||\boldsymbol{x}(k_0)|| < \delta \quad \Rightarrow \quad ||\boldsymbol{x}(k)|| < \epsilon,\ \forall\ k \geq k_0 \geq 0$$

**Uniform Asymptotic Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is uniformly

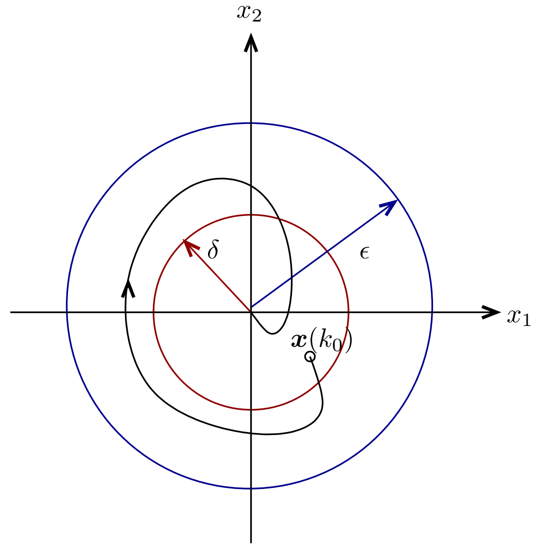


Figure 2: Illustration of asymptotically stable equilibrium in two dimension

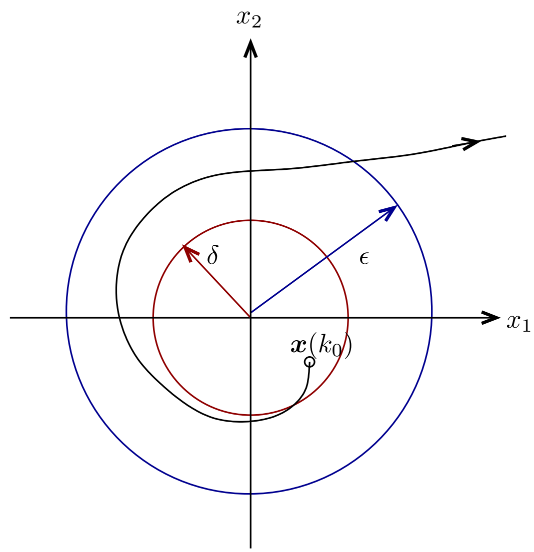


Figure 3: Illustration of unstable equilibrium in two dimension

asymptotically stable if it is uniformly stable and there is a positive constant $c$, independent of $k_0$, such that for all $||\boldsymbol{x}(k_0)|| < c$, $\boldsymbol{x}(k) \to \mathbf{0}$ as $k \to \infty$ uniformly in $k_0$.

**Global Uniform Asymptotic Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is globally uniformly asymptotically stable if it is uniformly asymptotically stable for such a $\delta$ when $\delta(\epsilon)$ can be chosen to satisfy the following condition

$$\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$$

**Exponential Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is exponentially stable if there exist positive constants $c$, $\gamma$ and $\lambda$ such that

$$||\boldsymbol{x}(k)|| \leq \gamma ||\boldsymbol{x}(k_0)|| e^{-\lambda(k-k_0)}, \;\; \forall \; ||\boldsymbol{x}(k_0)|| < c$$

**Global Exponential Stability:** The equilibrium point $\boldsymbol{x} = \mathbf{0}$ of (2) is globally exponentially stable if it is exponentially stable for any initial state $\boldsymbol{x}(k_0)$.

# Module 8: Controllability, Observability and Stability of Discrete Time Systems
## Lecture Note 3

In this lecture we would discuss Lyapunov stability theorem and derive the Lyapunov Matrix Equation for discrete time systems.

## 1 Revisiting the basics

**Linearization of A Nonlinear System** Consider a system

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k), \boldsymbol{u}(k))$$

where functions $f_i(.)$ are continuously differentiable. The equilibrium point $(\boldsymbol{x}_e, \boldsymbol{u}_e)$ for this system is defined as

$$\boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$$

- *What is linearization?*
  Linearization is the process of replacing the nonlinear system model by its linear counterpart in a small region about its equilibrium point.

- *Why do we need it?*
  We have well stabilised tools to analyze and stabilize linear systems.

**The method:** Let us write the general form of nonlinear system $\dot{x} = f(x, u)$ as:

$$\begin{aligned}
x_1(k+1) &= f_1(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k)) \\
x_2(k+1) &= f_2(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k)) \\
\vdots &= \vdots \\
x_n(k+1) &= f_n(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k))
\end{aligned}$$

Let $\boldsymbol{u}_e = [u_{1e} \quad u_{2e} \quad \ldots \quad u_{me}]^T$ be a constant input that forces the system to settle into a constant equilibrium state $\boldsymbol{x}_e = [x_{1e} \quad x_{2e} \quad \ldots \quad x_{ne}]^T$ such that $\boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$ holds true.

We now perturb the equilibrium state by allowing: $\boldsymbol{x} = \boldsymbol{x}_e + \Delta\boldsymbol{x}$ and $\boldsymbol{u} = \boldsymbol{u}_e + \Delta\boldsymbol{u}$. ***Taylor's expansion*** yields

$$\Delta\boldsymbol{x}(k+1) = f(\boldsymbol{x}_e + \Delta\boldsymbol{x}, \boldsymbol{u}_e + \Delta\boldsymbol{u}) = f(\boldsymbol{x}_e, \boldsymbol{u}_e) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e, \boldsymbol{u}_e)\Delta\boldsymbol{x} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}_e, \boldsymbol{u}_e)\Delta\boldsymbol{u} + \ldots \quad (1)$$

where

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e,\boldsymbol{u}_e) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x_e,u_e} \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}_e,\boldsymbol{u}_e) = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{x_e,u_e} \tag{2}$$

are the ***Jacobian*** matrices of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ and $\boldsymbol{u}$, evaluated at the equilibrium point, $[\boldsymbol{x}_e^T \quad \boldsymbol{u}_e^T]^T$.

Note that $\boldsymbol{f}(\boldsymbol{x}_e,\boldsymbol{u}_e) = 0$. Let

$$A = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}_e,\boldsymbol{u}_e) \text{ and } B = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}_e,\boldsymbol{u}_e) \tag{3}$$

Neglecting higher order terms, we arrive at the *linear approximation*

$$\Delta\boldsymbol{x}(k+1) = A\Delta\boldsymbol{x}(k) + B\Delta\boldsymbol{u}(k) \tag{4}$$

Similarly, if the *outputs* of the nonlinear system model are of the form

$$\begin{aligned} y_1(k) &= h_1(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k)) \\ y_2(k) &= h_2(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k)) \\ \vdots &= \vdots \\ y_p(k) &= h_p(x_1(k), x_2(k), \ldots, x_n(k), u_1(k), u_2(k), \ldots, u_m(k)) \end{aligned}$$

or in vector notation

$$\boldsymbol{y}(k) = \boldsymbol{h}(\boldsymbol{x}(k), \boldsymbol{u}(k)) \tag{5}$$

then Taylor's series expansion can again be used to yield the linear approximation of the above output equations. Indeed, if we let

$$\boldsymbol{y} = \boldsymbol{y}_e + \Delta\boldsymbol{y} \tag{6}$$

then we obtain

$$\Delta\boldsymbol{y}(k) = C\Delta\boldsymbol{x}(k) + D\Delta\boldsymbol{u}(k) \tag{7}$$

**Example:** Consider a nonlinear system

$$x_1(k+1) = -x_1(k) + x_2(k) + x_1(k)(x_1^2(k) + x_2^2(k)) \tag{8a}$$

$$x_2(k+1) = -x_1(k) - x_2(k) + x_2(k)(x_1^2(k) + x_2^2(k)) \tag{8b}$$

Linearize the system about origin which is an equilibrium point.

Evaluating the coefficients of Eqn. (3), we get

$$\frac{\partial f_1(k)}{\partial x_1(k)} = -1 + 3x_1^2(k) + x_2^2(k) \quad \frac{\partial f_1(k)}{\partial x_2(k)} = 1 + 2x_1(k)x_2(k)$$

$$\frac{\partial f_2(k)}{\partial x_1(k)} = -1 + 2x_1(k)x_2(k) \quad \frac{\partial f_2(k)}{\partial x_2(k)} = -1 + x_1^2(k) + 3x_2^2(k)$$

Thus $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$. Hence, the linearized system around origin is given by

$$\Delta \boldsymbol{x}(k+1) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \Delta \boldsymbol{x}(k) \tag{9}$$

## Sign definiteness of functions and matrices

**Positive Definite Function:** A continuously differentiable function $f : R^n \to R^+$ is said to be positive definite in a region $S \in R^n$ that contains the origin if

1. $f(0) = 0$
2. $f(x) > 0; \quad x \in S \quad and \quad x \neq 0$

The function $f(x)$ is said to be positive semi-definite

1. $f(0) = 0$
2. $f(x) \geq 0; \quad x \in S \quad and \quad x \neq 0$

If the condition (2) becomes $f(x) < 0$, the function is negative definite and if it becomes $f(x) \leq 0$ it is negative semi-definite.

**Example:** Is the function $f(x_1, x_2) = x_1^4 + x_2^4$ positive definite?
Answer: $f(0,0) = 0$ shows that the first condition is satisfied. $f(x_1, x_2) > 0$ for $x_1, x_2 \neq 0$. Second condition is also satisfied. Hence the function is positive definite.

A square matrix $P$ is symmetric if $P = P^T$. A scalar function has a quadratic form if it can be written as $\boldsymbol{x}^T P \boldsymbol{x}$ where $P = P^T$ and $\boldsymbol{x}$ is any real vector of dimension $n \times 1$.

**Positive Definite Matrix:** A real symmetric matrix $P$ is positive definite, i.e. $P > 0$ if

1. $\boldsymbol{x}^T P \boldsymbol{x} > 0$ for every non-zero $\boldsymbol{x}$.
2. $\boldsymbol{x}^T P \boldsymbol{x} = 0$ only if $\boldsymbol{x} = 0$.

A real symmetric matrix $P$ is positive semi-definite, i.e. $P \geq 0$ if $\boldsymbol{x}^T P \boldsymbol{x} \geq 0$ for every non-zero $\boldsymbol{x}$. This implies that $\boldsymbol{x}^T P \boldsymbol{x} = 0$ for some $\boldsymbol{x} \neq 0$.

**Theorem:** A symmetric square matrix $P$ is positive definite if and only if any one of the following conditions holds.

1. Every eigenvalue of $P$ is positive.
2. All the leading principal minors of $P$ are positive.
3. There exists an $n \times n$ non-singular matrix $Q$ such that $P = Q^T Q$.

Similarly, a matrix $P$ is said to be negative definite if $-P$ is positive definite. When none of these two conditions satisfies, the definiteness of the matrix cannot be calculated or in other words it is said to be sign indefinite.

**Example:** Consider the following third order matrices. Determine the sign definiteness of them.

$$A_1 = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 3 & 4 \\ 1 & 2 & 5 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & -1 \\ 0 & 0 & -3 \end{bmatrix}$$

The leading principal minors of the matrix $A_1$ are 2, 1 and 2, hence the matrix is positive definite.

The eigenvalues of the matrix $A_2$ can be straightaway computed as 2, 5 and $-3$, i.e., all the eigenvalues are not positive. Again, the eigenvalues of the matrix $-A_2$ are $-2$, $-5$ and 3 and hence the matrix $A_2$ is sign indefinite.

# 2 Lyapunov Stability Theorems

In the last section we have discussed various stability definitions. But the big question is how do we determine or check the stability or instability of an equilibrium point?

Lyapunov introduced two methods.

The first is called **Lyapunov's *first* or *indirect* method**: we have already seen it as the linearization technique. Start with a nonlinear system

$$\boldsymbol{x}(k+1) = \boldsymbol{f}(\boldsymbol{x}(k)) \tag{10}$$

Expanding in Taylor series around $\boldsymbol{x}_e$ and neglecting higher order terms.

$$\Delta\boldsymbol{x}(k+1) = A\Delta\boldsymbol{x}(k) \tag{11}$$

where

$$A = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_e} \tag{12}$$

Then the nonlinear system (10) is asymptotically stable around $\boldsymbol{x}_e$ if and only if the linear system (11) is; i.e., if all eigenvalues of $A$ are inside the unit circle.

The above method is very popular because it is easy to apply and it works well for most systems, all we need to do is to be able to evaluate partial derivatives.

One disadvantage of the method is that if some eigenvalues of $A$ are on the unit circle and the rest are inside the unit circle, then we cannot draw any conclusions, the equilibrium can be either stable or unstable.

The major drawback, however, is that since it involves linearization it is applied for situations when the initial conditions are “close” to the equilibrium. The method provides no indication as to how close is “close”, which may be extremely important in practical applications.

The second method is **Lyapunov's** *second* **or** *direct* **method:** this is a generalization of Lagrange's concept of stability of minimum potential energy.

Consider the nonlinear system (10). Without loss of generality, we assume origin as the equilibrium point of the system. Suppose that there exists a function, called ‘Lyapunov function’, $V(x)$ with the following properties:

1. $V(\mathbf{0}) = 0$
2. $V(\boldsymbol{x}) > 0$, for $\boldsymbol{x} \neq \mathbf{0}$
3. $\Delta V(\boldsymbol{x}) < 0$ along trajectories of (10).

Then, origin is asymptotically stable.

We can see that the method hinges on the existence of a Lyapunov function, which is an energy-like function, zero at equilibrium, positive definite everywhere else, and continuously decreasing as we approach the equilibrium.

The method is very powerful and it has several advantages:

- answers questions of stability of nonlinear systems
- can easily handle time varying systems
- determines asymptotic stability as well as normal stability
- determines the region of asymptotic stability or the domain of attraction of an equilibrium

The main drawback of the method is that there is no systematic way of obtaining Lyapunov functions, this is more of an art than science.

## Lyapunov Matrix Equation

It is also possible to find a Lyapunov function for a linear system. For a linear system of the form $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ we choose as Lyapunov function the quadratic form

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k)P\boldsymbol{x}(k) \tag{13}$$

where $P$ is a symmetric positive definite matrix. Thus

$$\Delta V(\boldsymbol{x}(k)) = V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) = \boldsymbol{x}(k+1)^T P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) \tag{14}$$

Simplifying the above equation and omitting $k$

$$\begin{aligned} \Delta V(\boldsymbol{x}) &= (A\boldsymbol{x})^T PA\boldsymbol{x} - \boldsymbol{x}^T P\boldsymbol{x} \\ &= \boldsymbol{x}^T A^T PA\boldsymbol{x} - \boldsymbol{x}^T P\boldsymbol{x} \\ &= \boldsymbol{x}^T (A^T PA - P)\boldsymbol{x} \\ &= -\boldsymbol{x}^T Q\boldsymbol{x} \end{aligned} \tag{15}$$

where

$$A^T PA - P = -Q \tag{16}$$

If the matrix $Q$ is positive definite, then the system is asymptotically stable. Therefore, we could pick $Q = I$, the identity matrix and solve

$$A^T PA - P = -I$$

for $P$ and see if $P$ is positive definite.

The equation (16) is called **Lyapunov's matrix equation** for discrete time systems and can be solved through MATLAB by using the command *dlyap*.

**Example:** Determine the stability of the following system by solving Lyapunov matrix equation.

$$\boldsymbol{x}(k+1) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \boldsymbol{x}(k)$$

Let us take $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix}$. Putting these into Lyapunov matrix equation,

$$\begin{aligned} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} &= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -p_1 - p_2 & p_1 - p_2 \\ -p_2 - p_4 & p_2 - p_4 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_4 \end{bmatrix} &= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2p_2 + p_4 & -p_1 + p_4 - p_2 \\ -p_1 + p_4 - p_2 & p_1 - 2p_2 \end{bmatrix} &= -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} 2p_2 + p_4 &= -1 \\ -p_1 + p_4 - p_2 &= 0 \\ p_1 - 2p_2 &= -1 \end{aligned}$$

Solving

$$p_1 = -1 \quad p_2 = 0 \quad p_4 = -1$$

which shows $P$ is a negative definite matrix. Hence the system is unstable. To verify the result, if you compute the eigenvalues of $A$ you would find out that they are outside the unit circle.

# Module 9: State Feedback Control Design
## Lecture Note 1

The design techniques described in the preceding lectures are based on the transfer function of a system. In this lecture we would discuss the state variable methods of designing controllers.

The advantages of state variable method will be apparent when we design controllers for multi input multi output systems. Moreover, transfer function methods are applicable only for linear time invariant and initially relaxed systems.

# 1 State Feedback Controller

Consider the state-space model of a SISO system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned} \tag{1}$$

where $\mathbf{x}(k) \in R^n$, $u(k)$ and $y(k)$ are scalar. In state feedback design, the states are fedback to the input side to place the closed poles at desired locations.

Regulation Problem: When we want the states to approach zero starting from any arbitrary initial state, the design problem is known as regulation where the internal stability of the system, with desired transients, is achieved. Control input:

$$u(k) = -K\mathbf{x}(k) \tag{2}$$

Tracking Problem: When the output has to track a reference signal, the design problem is known as tracking problem. Control input:

$$u(k) = -K\mathbf{x}(k) + Nr(k)$$

where $r(k)$ is the reference signal.

First we will discuss designing a state feedback control law using pole placement technique for regulation problem.

By substituting the control law (2) in the system state model (1), the closed loop system becomes $\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$. If $K$ can be designed such that eigenvalues of $A - BK$ are within the unit circle then the problem of regulation will be solved.

The control problem can thus be defined as: Design a state feedback gain matrix $K$ such that the control law given by equation (2) places poles of the closed loop system $\mathbf{x}(k+1) = (A-BK)\mathbf{x}(k)$ in desired locations.

- A necessary and sufficient condition for arbitrary pole placement is that the pair $(A, B)$ must be controllable.
- Since the states are fedback to the input side, we assume that all the states are measurable.

## 1.1 Designing $K$ by transforming the state model into controllable canonical form

The problem is first solved for the controllable canonical form. Let us denote the controllability matrix by $U_C$ and consider a transformation matrix $T$ as

$$T = U_C W$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$a_i$'s are the coefficients of the characteristic polynomial $|zI - A| = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$.

Define a new state vector $\mathbf{x} = T\bar{\mathbf{x}}$. This will transform the system given by (1) into controllable canonical form, as

$$\bar{\mathbf{x}}(k+1) = \bar{A}\bar{\mathbf{x}}(k) + \bar{B}u(k) \tag{3}$$

You should verify that

$$\bar{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix} \text{ and } \bar{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We first find $\bar{K}$ such that $u(k) = -\bar{K}\bar{\mathbf{x}}(k)$ places poles in desired locations. Since eigenvalues remain unaffected under similarity transformation, $u(k) = -\bar{K}T^{-1}\mathbf{x}(k)$ will also place the poles of the original system in desired locations.

If poles are placed at $z_1, z_2, ...., z_n$, the desired characteristic equation can be expressed as:

$$\begin{aligned} (z-z_1)(z-z_2)\dots(z-z_n) &= 0 \\ \text{or,}\quad z^n+\alpha_1 z^{n-1}+\dots+\alpha_{n-1}z+\alpha_n &= 0 \end{aligned} \tag{4}$$

Since the pair $(\bar{A}, \bar{B})$ are in controllable-companion form then, we have

$$\bar{A}-\bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -(a_n-\bar{k}_1) & -(a_{n-1}-\bar{k}_2) & & \dots & -(a_1-\bar{k}_n) \end{bmatrix}$$

Please note that the characteristic equation of both original and canonical form is expressed as: $|zI-A|$=$|zI-\bar{A}|$=$z^n+a_1z^{n-1}+\dots+a_n$=0.

The characteristic equation of the closed loop system with $u=-\bar{K}\bar{\mathbf{x}}$ is given as:

$$z^n+(a_1+\bar{k}_n)z^{n-1}+(a_2+\bar{k}_{n-1})z^{n-2}+\dots+(a_n+\bar{k}_1)=0 \tag{5}$$

Comparing Eqs. (4) and (5), we get

$$\bar{k}_n=\alpha_1-a_1,\ \bar{k}_{n-1}=\alpha_2-a_2,\ \bar{k}_1=\alpha_n-a_n \tag{6}$$

We need to compute the transformation matrix $T$ to find the actual gain matrix $K=\bar{K}T^{-1}$ where $\bar{K}=[\bar{k}_1, \bar{k}_2, \dots, \bar{k}_n]$.

## 1.2 Designing $K$ by Ackermann's Formula

Consider the state-space model of a SISO system given by equation (1). The control input is

$$u(k)=-K\mathbf{x}(k) \tag{7}$$

Thus the closed loop system will be

$$\mathbf{x}(k+1)=(A-BK)\mathbf{x}(k)=\hat{A}\mathbf{x}(k) \tag{8}$$

Desired characteristic Equation:

$$\begin{aligned} |zI-A+BK|=|zI-\hat{A}| &= 0 \\ \text{or,}\quad (z-z_1)(z-z_2)\cdots(z-z_n) &= 0 \\ \text{or,}\quad z^n+\alpha_1 z^{n-1}+\dots+\alpha_{n-1}z+\alpha_n &= 0 \end{aligned}$$

Using Cayley-Hamilton Theorem

$$\hat{A}^n+\alpha_1\hat{A}^{n-1}+\dots+\alpha_{n-1}\hat{A}+\alpha_n I=0$$

Consider the case when $n = 3$.

$$\begin{aligned}
\hat{A} &= A - BK \\
\hat{A}^2 = (A - BK)^2 = A^2 - ABK - BKA - BKBK &= A^2 - ABK - BK\hat{A} \\
\hat{A}^3 = (A - BK)^3 &= A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2
\end{aligned}$$

We can then write

$$\begin{aligned}
\alpha_3 I + \alpha_2 \hat{A} + \alpha_1 \hat{A}^2 + \hat{A}^3 &= 0 \\
\text{or,} \quad \alpha_3 I + \alpha_2(A - BK) + \alpha_1(A^2 - ABK - BK\hat{A}) + A^3 - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0 \\
\text{or,}\ \alpha_3 I + \alpha_2 A + \alpha_1 A^2 + A^3 - \alpha_2 BK - \alpha_1 ABK - \alpha_1 BK\hat{A} - A^2BK - ABK\hat{A} - BK\hat{A}^2 &= 0
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(A) &= B(\alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2) + AB(\alpha_1 K + K\hat{A}) + A^2BK \\
&= \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
&= U_C \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix}
\end{aligned}$$

where $\phi(.)$ is the closed loop characteristic polynomial and $U_C$ is the controllability matrix. Since $U_C$ is nonsingular

$$\begin{aligned}
U_C^{-1}\phi(A) &= \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or,} \ \ [0 \ \ 0 \ \ 1]\, U_C^{-1}\phi(A) &= [0 \ \ 0 \ \ 1] \begin{bmatrix} \alpha_2 K + \alpha_1 K\hat{A} + K\hat{A}^2 \\ \alpha_1 K + K\hat{A} \\ K \end{bmatrix} \\
\text{or,} \ \ K &= [0 \ \ 0 \ \ 1]\, U_C^{-1}\phi(A)
\end{aligned}$$

Extending the above for any $n$,

$$K = [0 \ \ 0 \ \ \ldots \ \ 1]\, U_C^{-1}\phi(A) \qquad \text{where } U_C = [B \ \ AB \ \ A^2B \ \ \ldots \ \ A^{n-1}B]$$

The above equation is popularly known as Ackermann's formula.

**Example 1:** Find out the state feedback gain matrix $K$ for the following system using two different methods such that the closed loop poles are located at 0.5, 0.6 and 0.7.

$$\mathbf{x}(k+1) \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

Solution:

$$U_C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

The above matrix has rank 3, so the system is controllable.

Open loop characteristic equation:

$$\text{or,}\ \ z^3 + 3z^2 + 2z + 1 = 0$$

Desired characteristic equation:

$$\begin{aligned} (z-0.5)(z-0.6)(z-0.7) &= 0 \\ \text{or,}\ \ z^3 - 1.8z^2 + 1.07z - 0.21 &= 0 \end{aligned}$$

Since the open loop system is already in controllable canonical form, $T = I$.

$$K = [\alpha_3 - a_3\ \ \alpha_2 - a_2\ \ \alpha_1 - a_1]$$

where, $\alpha_3 = -0.21,\ \alpha_2 = 1.07,\ \alpha_1 = -1.8$ and $a_3 = 1,\ a_2 = 2,\ a_1 = 3$. Thus

$$K = [-1.21 \quad -0.93 \quad -4.8]$$

Using Ackermann's formula:

$$U_C^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & -3 & -1 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \phi(A) &= A^3 - 1.8A^2 + 1.07A - 0.21I \\ &= \begin{bmatrix} -1 & -2 & -3 \\ 3 & 5 & 7 \\ -7 & -11 & -10 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1.8 \\ 1.8 & 3.6 & 5.4 \\ -5.4 & -9 & -10.6 \end{bmatrix} + \begin{bmatrix} 0 & 1.07 & 0 \\ 0 & 0 & 1.07 \\ -1.07 & -2.14 & -3.21 \end{bmatrix} \\ &\quad + \begin{bmatrix} -0.21 & 0 & 0 \\ 0 & -0.21 & 0 \\ 0 & 0 & -0.21 \end{bmatrix} \\ &= \begin{bmatrix} -1.21 & -0.93 & -4.8 \\ 4.8 & 8.39 & 13.47 \\ -13.47 & -22.14 & -30.02 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} K &= [0\ 0\ 1]U_C^{-1}\phi(A) \\ &= [0\ 0\ 1]\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\phi(A) \\ &= [1\ 0\ 0]\phi(A) = [-1.21 \quad -0.93 \quad -4.8] \end{aligned}$$

**Example 2:** Find out the state feedback gain matrix $K$ for the following system by converting the system into controllable canonical form such that the closed loop poles are located at 0.5 and 0.6.

$$\mathbf{x}(k+1) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

Solution:

$$U_C = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

The above matrix has rank 2, so the system is controllable.

Open loop characteristic equation:

$$\text{or,} \;\; z^2 + 3z + 2 = 0$$

Desired characteristic equation:

$$\begin{aligned} (z-0.5)(z-0.6) &= 0 \\ \text{or,} \;\; z^2 - 1.1z^2 + 0.3 &= 0 \end{aligned}$$

To convert into controllable canonical form:

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

The transformation matrix:

$$T = U_C W = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

Check:

$$T^{-1}AT = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad T^{-1}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now,

$$\alpha_1 = -1.1, \qquad \alpha_2 = 0.3, \qquad a_1 = 3, \qquad a_2 = 2$$

Thus

$$\bar{K} = [\alpha_2 - a_2 \quad \alpha_1 - a_1] = [-1.7 \quad -4.1]$$

We can then write

$$K = \bar{K}T^{-1} = [-1.7 \quad -4.1] \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = [-2.4 \quad -4.1]$$

Taking the initial state to be $\boldsymbol{x}(0) = [2 \quad 1]^T$, the plots for state variables and control variable are shown in Figure 1.

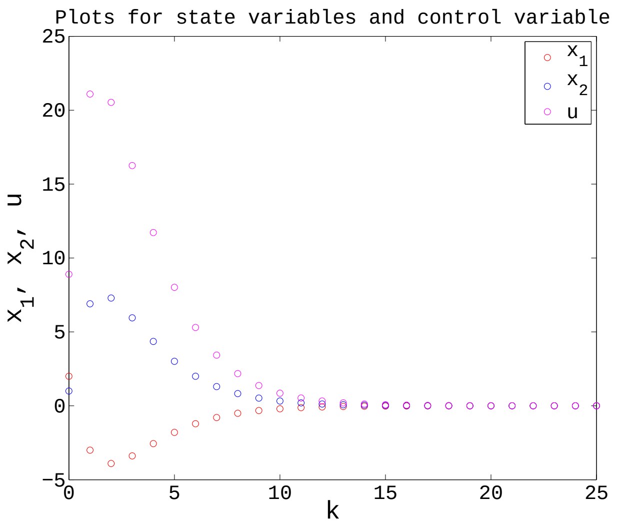


Figure 1: Example 2: Plots for state variables and control variable

# Module 9: State Feedback Control Design
## Lecture Note 2

# 1 Set Point Tracking

We discussed in the last lecture that a state feedback design can be done to place poles such that the system is stable. However, the tracking is not guaranteed.

## 1.1 Feed Forward Gain Design

Consider the state space model

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k)\\ y(k) &= C\mathbf{x}(k)\end{aligned}$$

A control law is selected

$$u(k) = -K\mathbf{x}(k) + Nr(k)$$

as shown in figure 1 so that output can track any step reference command signal $r$. The closed loop dynamics of this configuration becomes

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k) + BNr(k)$$

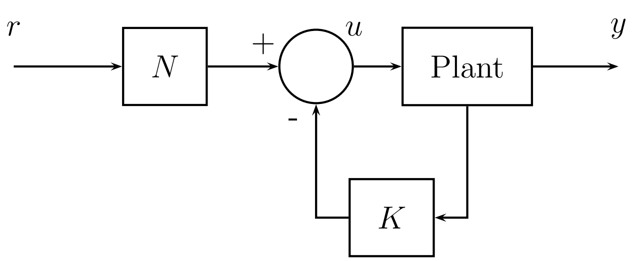


Figure 1: State feedback controller with feed forward gain for set point tracking

At steady state, say $\mathbf{x} = \mathbf{x}_{ss}$, $y = C\mathbf{x}_{ss} = r$ and $u = u_{ss}$. Since the states or the output do not change with time in steady state, we can write

$$\mathbf{x}_{ss} = A\mathbf{x}_{ss} + Bu_{ss}$$

Let us assume

$$\tilde{u}(k) = u(k) - u_{ss}, \quad \tilde{\mathbf{x}}(k) = \mathbf{x}(k) - \mathbf{x}_{ss}, \quad \tilde{y}(k) = y(k) - r$$

Thus in shifted domain,

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= A\mathbf{x}(k) + Bu(k) - A\mathbf{x}_{ss} - Bu_{ss} \\ &= A\tilde{\mathbf{x}}(k) + B\tilde{u}(k) \\ \tilde{y}(k) &= C\tilde{\mathbf{x}}(k) \end{aligned}$$

If we design a stable control $\tilde{u}(k) = -K\tilde{\mathbf{x}}(k)$ in the shifted domain, it will drive the state variables in shifted domain to 0.

$$\tilde{\mathbf{x}}(k) \to 0 \Rightarrow \mathbf{x}(k) \to \mathbf{x}_{ss} \Rightarrow y(k) \to r$$

The problem of tracking is thus converted into a simple regulator problem.

$$\begin{aligned} u(k) - u_{ss} &= -K\mathbf{x}(k) + K\mathbf{x}_{ss} \\ \Rightarrow u(k) &= -K\mathbf{x}(k) + u_{ss} + K\mathbf{x}_{ss} \end{aligned}$$

We know that $\mathbf{x}_{ss} = A\mathbf{x}_{ss} + Bu_{ss}$. Thus

$$\begin{aligned} 0 &= (A - I)\mathbf{x}_{ss} + Bu_{ss} \\ &= (A - I)\mathbf{x}_{ss} + BK\mathbf{x}_{ss} - BK\mathbf{x}_{ss} + Bu_{ss} \\ & \phantom{=} (A - BK - I)\mathbf{x}_{ss} + B(u_{ss} + K\mathbf{x}_{ss}) \\ \Rightarrow \mathbf{x}_{ss} &= -(A - BK - I)^{-1}B(u_{ss} + K\mathbf{x}_{ss}) \end{aligned}$$

Now,

$$C\mathbf{x}_{ss} = -C(A - BK - I)^{-1}B(u_{ss} + K\mathbf{x}_{ss}) = r$$

Thus the possible solution for $N$ is $u_{ss} + K\mathbf{x}_{ss} = Nr$ where

$$N^{-1} = -C(A - BK - I)^{-1}B$$

and the control input is

$$u(k) = -K\mathbf{x}(k) + Nr$$

**Example 1:** Consider the following system

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.47 & 1.47 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [0.007 \quad 0.009]\, \mathbf{x}(k) \end{aligned}$$

Design a state feedback controller such that the output follows a step input with the desired closed loop poles at 0.5 and 0.6.

Solution: Desired characteristic equation:

$$z^2 - 1.1z + 0.3 = 0$$

Open loop characteristic equation:

$$z^2 - 1.47z + 0.47 = 0$$

The controller can be designed using the following expression

$$u(k) = -K\mathbf{x}(k) + Nr$$

Since the system is in controllable canonical form, where the controllability matrix is $U_C = \begin{bmatrix} 0 & 1 \\ 1 & 1.47 \end{bmatrix}$ (non singular), the state feedback gain can be straight away designed as

$$K = [0.3 - 0.47 \quad -1.1 + 1.47] = [-0.17 \quad 0.37]$$

$N$ can be designed as

$$N^{-1} = -C(A - BK - I)^{-1}B = 0.08$$

Thus

$$u(k) = -[-0.17 \quad 0.37]\ \mathbf{x}(k) + 12.5\ r$$

## 1.2 State Feedback with Integral Control

Calculation of feed forward gain requires exact knowledge of the system parameters. Change in parameters will effect the steady state error.

In this scheme we feedback the states as well as the integral of the output error which will eventually make the actual output follow the desired one.

One way to introduce the integrator is to augment the integral state $v$ with the plant state vector $\mathbf{x}$.

$v$ integrates the difference between the output $y(k)$ and reference $r$. By using backward rectangular integration, it can be defined as

$$v(k) = v(k-1) + y(k) - r$$

Thus

$$\begin{aligned} v(k+1) &= v(k) + y(k+1) - r \\ &= v(k) + C[A\mathbf{x}(k) + Bu(k)] - r \end{aligned}$$

If we augment the above with the state equation,

$$\begin{bmatrix} \mathbf{x}(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} B \\ CB \end{bmatrix} u(k) + \begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r$$

Since $r$ is constant, if the system is stable, then $\mathbf{x}(k+1) = \mathbf{x}(k)$ and $v(k+1) = v(k)$ in the steady state. So, in steady state,

$$\begin{bmatrix} \mathbf{0} \\ -1 \end{bmatrix} r = \begin{bmatrix} \mathbf{x}_{ss} \\ v_{ss} \end{bmatrix} - \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{ss} \\ v_{ss} \end{bmatrix} - \begin{bmatrix} B \\ CB \end{bmatrix} u_{ss}$$

Let us define $\tilde{\mathbf{x}}_a = [\mathbf{x} - \mathbf{x}_{ss} \quad v - v_{ss}]^T$ and $\tilde{u} = u - u_{ss}$. This implies

$$\tilde{\mathbf{x}}_a(k+1) = \bar{A}\tilde{\mathbf{x}}_a(k) + \bar{B}\tilde{u}(k)$$

where $\bar{A} = \begin{bmatrix} A & \mathbf{0} \\ CA & 1 \end{bmatrix}$ and $\bar{B} = \begin{bmatrix} B \\ CB \end{bmatrix}$

The design problem is now converted to a standard regulation problem. We need to design

$$\tilde{u}(k) = -K\tilde{\mathbf{x}}_a(k)$$

where $K = [K_p \quad K_i]$, $\quad K_i$ is the integral gain. Now,

$$\begin{aligned} u - u_{ss} &= -[K_p \quad K_i] \begin{bmatrix} \mathbf{x} - \mathbf{x}_{ss} \\ v - v_{ss} \end{bmatrix} \\ &= -K_p(\mathbf{x} - \mathbf{x}_{ss}) - K_i(v - v_{ss}) \end{aligned}$$

The steady state terms in the above expression must balance, which implies

$$u(k) = -K_p\mathbf{x}(k) - K_i v(k)$$

At steady state,

$$\begin{aligned} \tilde{\mathbf{x}}_a(k+1) - \tilde{\mathbf{x}}_a(k) &= 0 \\ v(k+1) - v(k) &= = 0 \end{aligned}$$

We can write from the above expression $y(k) - r = 0$ at steady state. In other words, $y(k)$ follows $r$ at steady state. The block diagram of state feedback with integral control is shown in Figure 2.

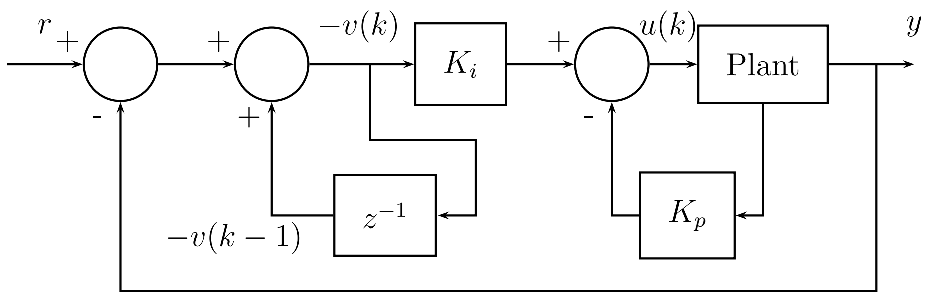


Figure 2: State feedback controller with integral control for set point tracking

**Example:** Consider the problem of digital control of the following plant

$$G(s) = \frac{1}{s(s+2)}$$

for a sampling period $T = 0.1$ sec using a state feedback with integral control such that the plant output follows a step input. The closed loop continuous poles of the system must be located at $-1 \pm j$ and $-5$ respectively.

Solution:

Discretization of the plant model gives

$$\begin{aligned} G(z) &= Z\left[\left(\frac{1-e^{-Ts}}{s}\right)\left(\frac{1}{s(s+2)}\right)\right] \\ &= \frac{0.005z + 0.004}{z^2 - 1.82z + 0.82} \end{aligned}$$

The discrete state space model of the plant is:

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ -0.82 & 1.82 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [0.004 \quad 0.005] \, \mathbf{x}(k) \end{aligned}$$

Augmenting the plant state vector with the integral state $v(k)$, defined by

$$v(k) = v(k-1) + y(k) - r$$

we obtain

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ v(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0.82 & 1.82 & 0 \\ -0.004 & 0.013 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ v(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0.005 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} r$$

In terms of the state variables representing deviations from the steady state,

$$\tilde{\mathbf{x}}_a(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ -0.82 & 1.82 & 0 \\ -0.004 & 0.013 & 1 \end{bmatrix} \tilde{\mathbf{x}}_a(k) + \begin{bmatrix} 0 \\ 1 \\ 0.005 \end{bmatrix} \tilde{u}(k)$$

where $\tilde{\mathbf{x}}_a = [\mathbf{x} - \mathbf{x}_{ss} \quad v - v_{ss}]^T$ and $\tilde{u} = u - u_{ss}$. $\tilde{u}(k)$ can be designed for the desired pole locations, $0.9 \pm 0.1j$ and $0.6$ (in discrete domain) using Ackermann's formula, as,

$$\tilde{u}(k) = -K\tilde{\mathbf{x}}_a(k) = -[-0.328 \quad 0.416 \quad 0.889]\,\tilde{\mathbf{x}}_a(k)$$

Thus

$$u(k) = -[-0.328 \quad 0.416]\,\mathbf{x}(k) - 0.889\,v(k)$$

# Module 9: State Feedback Control Design
## Lecture Note 3

# 1 State Estimators or Observers

- One should note that although state feedback control is very attractive because of precise computation of the gain matrix $K$, implementation of a state feedback controller is possible only when all state variables are directly measurable with help of some kind of sensors.
- Due to the excess number of required sensors or unavailability of states for measurement, in most of the practical situations this requirement is not met.
- Only a subset of state variables or their combinations may be available for measurements. Sometimes only output $y$ is available for measurement.
- Hence the need for an estimator or observer is obvious which estimates all state variables while observing input and output.

**Full Order Observer**: If the state observer estimates all the state variables, regardless of whether some are available for direct measurements or not, it is called a full order observer.

**Reduced Order Observer**: An observer that estimates fewer than "n" states of the system is called reduced order observer.

**Minimum Order Observer**: If the order of the observer is minimum possible then it is called minimum order observer.

# 2 Full Order Observers

Consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

where $\mathbf{x} \in R^{n\times 1}$, $\mathbf{u} \in R^{m\times 1}$ and $\mathbf{y} \in R^{p\times 1}$.

**Assumption**: The pair $(A,\ C)$ is observable.

**Goal**: To construct a dynamic system that will estimate the state vector based on the information of the plant input $\mathbf{u}$ and output $\mathbf{y}$.

## 2.1 Open Loop Estimator

The schematic of an open loop estimator is shown in Figure 1.

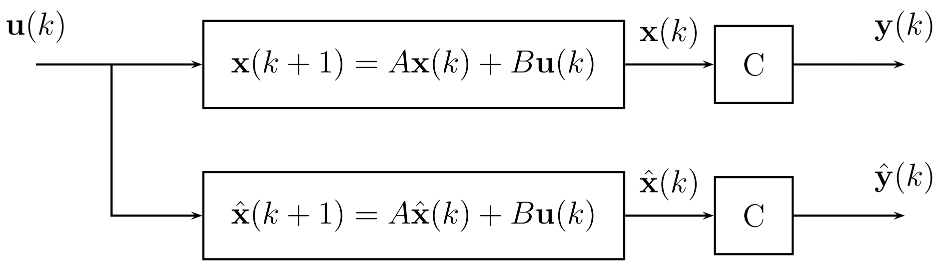


Figure 1: Open Loop Observer

The dynamics of this estimator are described by the following

$$\begin{aligned} \hat{\mathbf{x}}(k+1) &= A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) \\ \hat{\mathbf{y}}(k) &= C\hat{\mathbf{x}}(k) \end{aligned}$$

where $\hat{\mathbf{x}}$ is the estimate of $\mathbf{x}$ and $\hat{\mathbf{y}}$ is the estimate of $\mathbf{y}$.

Let $\tilde{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}$ be the estimation error. Then the error dynamics are defined by

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= \hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1) \\ &= A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) - A\mathbf{x}(k) - B\mathbf{u}(k) \\ &= A\tilde{\mathbf{x}}(k) \end{aligned}$$

with the initial estimation error as

$$\tilde{\mathbf{x}}(0) = \hat{\mathbf{x}}(0) - \mathbf{x}(0)$$

If the eigenvalues of $A$ are inside the unit circle then $\tilde{\mathbf{x}}$ will converge to 0. But we have no control over the convergence rate.

Moreover, $A$ may have eigenvalues outside the unit circle. In that case $\tilde{\mathbf{x}}$ will diverge from 0. Thus the open loop estimator is impractical.

## 2.2 Luenberger State Observer

Consider the system $\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$. Luenberger observer is shown in Figure 2. The observer dynamics can be expressed as:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + L(\mathbf{y}(k) - \hat{\mathbf{y}}(k)) \tag{1}$$

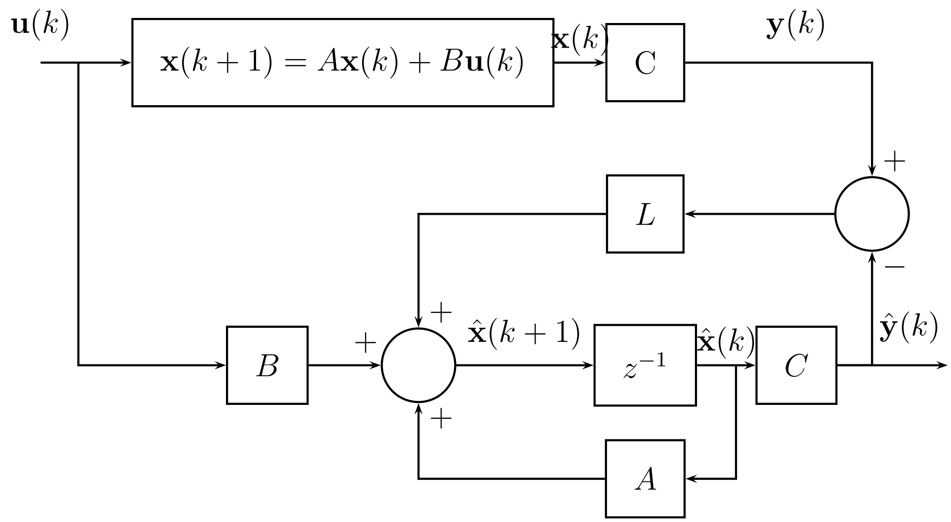


Figure 2: Luenberger observer

The closed loop error dynamics can be derived as:

$$\begin{aligned}
\tilde{\mathbf{x}}(k) &= \hat{\mathbf{x}}(k) - \mathbf{x}(k) \\
\hat{\mathbf{x}}(k+1) - \mathbf{x}(k+1) &= A(\hat{\mathbf{x}}(k) - \mathbf{x}(k)) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k)) \\
\tilde{\mathbf{x}}(k+1) &= (A - LC)\tilde{\mathbf{x}}(k)
\end{aligned}$$

It can be seen that $\tilde{\mathbf{x}} \to 0$ if $L$ can be designed such that $(A - LC)$ has eigenvalues inside the unit circle of $z$-plane.

The convergence rate can also be controlled by properly choosing the closed loop eigenvalues.

### Computation of Observer gain matrix $L$

The task is to place the poles of $|A - LC|$. Necessary and sufficient condition for arbitrary pole placement is that the pair should be controllable.

Assumption: The pair $(A,\ C)$ is observable. Thus, from the theorem of duality, the pair $(A^T,\ C^T)$ is controllable.

You should note that the eigenvalues of $A^T - C^T L^T$ are same as that of $A - LC$. It is same as a hypothetical pole placement problem for the system $\bar{\mathbf{x}}(k+1) = A^T\bar{\mathbf{x}}(k) + C^T\bar{\mathbf{u}}(k)$, using a control law $\bar{u}(k) = -L^T\bar{\mathbf{x}}(k)$.

**Example:**

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= [1 \ \ 0]\mathbf{x}(k) \end{aligned}$$

The observability matrix

$$U_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is non singular. Thus the pair $(A,\ C)$ is observable. The observer dynamics are

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + Bu(k) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k))$$

$L$ should be designed such that the observer poles are at 0.2 and 0.3.

We design $L^T$ such that $A^T - C^T L^T$ has eigenvalues at 0.2 and 0.3.

$$A^T = \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using Ackermann's formula, $L^T = [-0.5 \quad 20.06]$. Thus

$$L = \begin{bmatrix} -0.5 \\ 20.06 \end{bmatrix}$$

## 2.3 Controller with Observer

The observer dynamics:

$$\hat{\mathbf{x}}(k+1) = A\hat{\mathbf{x}}(k) + B\mathbf{u}(k) + LC(\mathbf{x}(k) - \hat{\mathbf{x}}(k))$$

Combining with the system dynamics

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(k+1) \\ \hat{\mathbf{x}}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ LC & A - LC \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} \mathbf{u}(k) \\ \mathbf{y}(k) &= [C \ \ 0] \begin{bmatrix} \mathbf{x}(k) \\ \hat{\mathbf{x}}(k) \end{bmatrix} \end{aligned}$$

Since the states are unavailable for measurements, the control input is

$$\mathbf{u}(k) = -K\hat{\mathbf{x}}(k)$$

Putting the control law in the augmented equation

$$\begin{aligned}
\begin{bmatrix}\mathbf{x}(k+1)\\ \hat{\mathbf{x}}(k+1)\end{bmatrix} &= \begin{bmatrix}A & -BK\\ LC & A-LC-BK\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix}\\
\mathbf{y}(k) &= [C \;\; 0]\begin{bmatrix}\mathbf{x}(k)\\ \hat{\mathbf{x}}(k)\end{bmatrix}
\end{aligned}$$

The error dynamics is

$$\tilde{\mathbf{x}}(k+1) = (A-LC)\tilde{\mathbf{x}}(k)$$

If we augment the above with the system dynamics, we get

$$\begin{aligned}
\begin{bmatrix}\mathbf{x}(k+1)\\ \tilde{\mathbf{x}}(k+1)\end{bmatrix} &= \begin{bmatrix}A-BK & -BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}\\
\mathbf{y}(k) &= [C \;\; 0]\begin{bmatrix}\mathbf{x}(k)\\ \tilde{\mathbf{x}}(k)\end{bmatrix}
\end{aligned}$$

where the dimension of the augmented system matrix is $R^{2n\times 2n}$. Looking at the matrix one can easily understand that $2n$ eigenvalues of the augmented matrix are equal to the individual eigenvalues of $A - BK$ and $A - LC$.

**Conclusion:** We can reach to a conclusion from the above fact is the design of control law, i.e., $A - BK$ is separated from the design of the observer, i.e., $A - LC$.

The above conclusion is commonly referred to as **separation principle**.

The block diagram of controller with observer is shown in Figure 3.

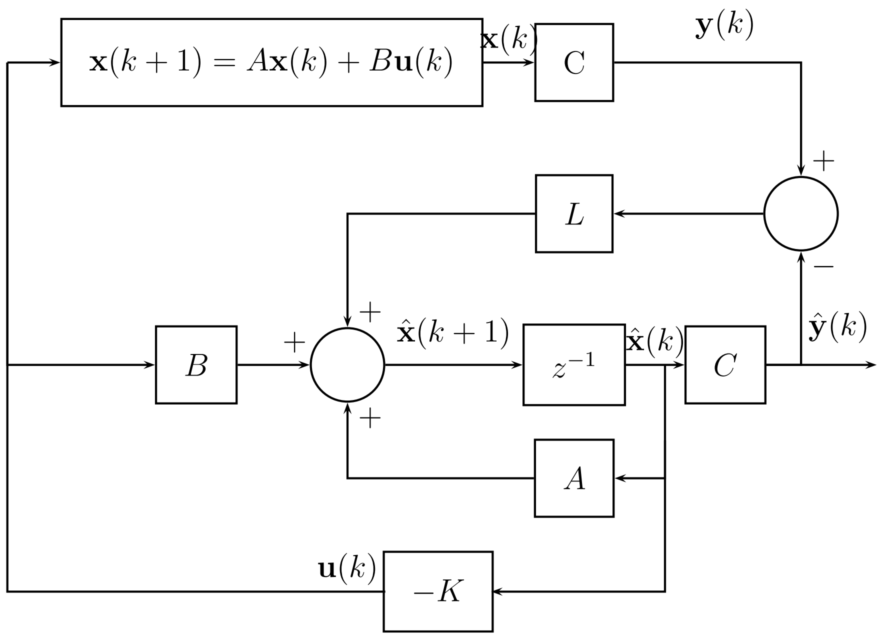


Figure 3: Controller with observer

# Module 9: State Feedback Control Design
## Lecture Note 4

In the last lecture we have already acquired some idea about observation and learned how a full order observer can be designed.

In this lecture we will discuss reduced order observers.

# 1 Reduced Order Observers

We know that an observer that estimates fewer than "n" states of the system is called reduced order observer.

Consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

where $\mathbf{x} \in R^{n\times 1}$, $\mathbf{u} \in R^{m\times 1}$ and $\mathbf{y} \in R^{p\times 1}$.

Since the output $\mathbf{y}$ is a vector with dimension $p$ where $p < n$, we would like to use these $p$ outputs to determine $p$ states of the state vector and design an estimator of order $n-p$ to estimate the rest.

If rank$(C) = p$ then $\mathbf{y}(k) = C\mathbf{x}(k)$ can be used to solve for $p$ of the $x_i$'s in terms of $y_i$'s and remaining $n-p$ state variables $x_k$'s will be estimated.

Let us assume that the dynamics of the observer are given by

$$\bar{\mathbf{x}}(k+1) = D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + G\mathbf{y}(k) \tag{1}$$

Let us take a transformation $P$ such that

$$\bar{\mathbf{x}} = P\mathbf{x}$$

Applying this transformation on the system dynamics,

$$P\mathbf{x}(k+1) = PA\mathbf{x}(k) + PB\mathbf{u}(k) \tag{2}$$

Subtracting (2) from (1),

$$(PA - DP - GC)\mathbf{x}(k) + (PB - E)\mathbf{u}(k) = 0$$

The above relation will be true for all $k$ and any arbitrary input $\mathbf{u}(k)$, if

$$\begin{aligned} PA - DP &= GC \\ \text{and, } E &= PB \end{aligned}$$

If $\bar{\mathbf{x}} \neq P\mathbf{x}$ but the above equation holds true, then we can write

$$\begin{aligned} \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1) &= D\bar{\mathbf{x}}(k) - PA\mathbf{x}(k) + GC\mathbf{x}(k) \\ &= D\bar{\mathbf{x}}(k) - DP\mathbf{x}(k) \\ &= D(\bar{\mathbf{x}}(k) - P\mathbf{x}(k)) \end{aligned}$$

If $D$ has eigenvalues inside the unit circle, then we can write

$$\bar{\mathbf{x}}(k) \to P\mathbf{x}(k) \;\text{ as, } k \to \infty$$

If we try $P = I_{n\times n}$ where $I_{n\times n}$ is the identity matrix with dimension $n \times n$, and $G = L$, we get

$$\begin{aligned} A - LC &= D \\ \text{and, } E &= B \end{aligned}$$

The resulting estimator is the Luenberger full order estimator. $A - LC = D$ can be solved for $L$ such that $D$ has eigenvalues at the prescribed locations.

The above is possible if and only if the pair $(A,\ C)$ is observable which was the only assumption in observer design.

The dimensions of $P$, $D$ and $G$ are as follows

$$P \in R^{(n-p)\times n}, \quad D \in R^{(n-p)\times(n-p)}, \quad G \in R^{(n-p)\times p}$$

Now we have

$$\begin{aligned} \mathbf{y}(k) &= C\mathbf{x}(k) \\ \text{and, } \bar{\mathbf{x}}(k) &= P\mathbf{x}(k) \quad (\text{as } \; k \to \infty) \\ \Rightarrow \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{x}} \end{bmatrix} &= \begin{bmatrix} C \\ P \end{bmatrix} \mathbf{x} \end{aligned}$$

Thus the estimated state vector will be

$$\hat{\mathbf{x}} = \begin{bmatrix} C \\ P \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y} \\ \bar{\mathbf{x}} \end{bmatrix}$$

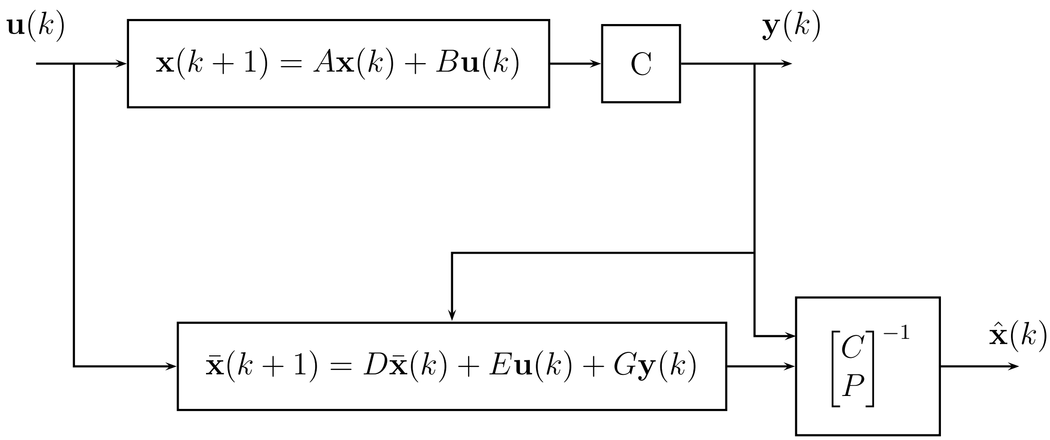


Figure 1: Reduced Order observer

Thus rank of $\begin{bmatrix} C \\ P \end{bmatrix}$ should be equal to $n$.

$PA - DP = GC$ can be uniquely solved if no eigenvalues of $D$ is an eigenvalue of $A$.

Figure 1 shows the block diagram of a reduced order observer.

While choosing $D$ and $G$, we have to make sure that $\begin{bmatrix} C \\ P \end{bmatrix}$ has rank $n$. The following example will illustrate the observer design.

**Example 1:** Let us take the following discrete time system

$$\begin{aligned} \mathbf{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(k) \end{aligned}$$

Thus, $D \in R^{1\times 1}$. Let us take $D = 0.5$. We know,

$$PA - DP = GC$$

Let us assume $P = [p_1 \quad p_2] \in R^{1\times 2}$. Putting this in the above equation,

$$\begin{aligned} &[p_1 \quad p_2] \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix} - 0.5[p_1 \quad p_2] = GC \\ &\text{or, } [20p_2 - 0.5p_1 \quad p_1 - 0.5p_2] = GC \end{aligned}$$

If we take $G = 20$,

$$GC = [20 \quad 0]$$

Thus, we get

$$\begin{aligned} p_1 &= 0.5p_2 \\ \text{and, } 20p_2 - 0.5p_1 &= 20 \end{aligned}$$

Solving the above equations, $p_1 = 0.51$ and $p_2 = 1.01$ and $E = [0.51 \quad 1.01] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1.01$.

## 1.1 Controller with Reduced Order Observer

The observer dynamics:

$$\bar{\mathbf{x}}(k+1) = D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + GC\mathbf{x}(k)$$

The state feedback control

$$\mathbf{u}(k) = -K\hat{\mathbf{x}}(k)$$

where

$$\hat{\mathbf{x}}(k) = \begin{bmatrix} C \\ P \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix}$$

and $\mathbf{y}(k) = C\mathbf{x}(k)$. Let's assume $\begin{bmatrix} C \\ P \end{bmatrix}^{-1} = [Q_1 \;\; Q_2]$. Thus

$$\begin{aligned} \mathbf{u}(k) &= -K[Q_1 \;\; Q_2] \begin{bmatrix} \mathbf{y}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ &= -KQ_1\mathbf{y}(k) - KQ_2\bar{\mathbf{x}}(k) \end{aligned}$$

Combining the observer with the system dynamics

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) \end{bmatrix} &= \begin{bmatrix} A & 0 \\ GC & D \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} B \\ PB \end{bmatrix} (-KQ_1\mathbf{y}(k) - KQ_2\bar{\mathbf{x}}(k)) \\ &= \begin{bmatrix} A - BKQ_1C & -BKQ_2 \\ GC - PBKQ_1C & D - PBKQ_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \\ \mathbf{y}(k) &= [C \;\; 0] \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix} \end{aligned}$$

Let us define

$$\begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ -P & I_{n-p} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \bar{\mathbf{x}}(k) \end{bmatrix}$$

We can write

$$\begin{bmatrix} I_n & 0 \\ -P & I_{n-p} \end{bmatrix}^{-1} = \begin{bmatrix} I_n & 0 \\ P & I_{n-p} \end{bmatrix}$$

Again we can write

$$\begin{aligned}\mathbf{x}(k+1) &= (A - BKQ_1C)\mathbf{x}(k) - BKQ_2\bar{\mathbf{x}}(k) \\ &= (A - BKQ_1C)\mathbf{x}(k) - BKQ_2(\bar{\mathbf{x}}(k) - P\mathbf{x}(k)) - BKQ_2P\mathbf{x}(k) \\ &= (A - BKQ_1C - BKQ_2P)\mathbf{x}(k) - BKQ_2(\bar{\mathbf{x}}(k) - P\mathbf{x}(k))\end{aligned}$$

and

$$\begin{aligned}\bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1) &= D\bar{\mathbf{x}}(k) + E\mathbf{u}(k) + GC\mathbf{x}(k) - PA\mathbf{x}(k) - PB\mathbf{u}(k) \\ &= (DP - PA + GC)\mathbf{x}(k) + D(\bar{\mathbf{x}}(k) - P\mathbf{x}(k))\end{aligned}$$

Thus

$$\begin{bmatrix}\mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1)\end{bmatrix} = \begin{bmatrix}A - BKQ_1C - BKQ_2P & -BKQ_2 \\ DP - PA + GC & D\end{bmatrix}\begin{bmatrix}\mathbf{x}(k) \\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k)\end{bmatrix}$$

Since $PA - DP - GC = 0$ and $Q_2P = I_n - Q_1C$, we can write

$$\begin{bmatrix}\mathbf{x}(k+1) \\ \bar{\mathbf{x}}(k+1) - P\mathbf{x}(k+1)\end{bmatrix} = \begin{bmatrix}A - BK & -BKQ_2 \\ 0 & D\end{bmatrix}\begin{bmatrix}\mathbf{x}(k) \\ \bar{\mathbf{x}}(k) - P\mathbf{x}(k)\end{bmatrix}$$

From the above we can say that if $(A - BK)$ and $D$ have eigenvalues inside the unit circle then $\mathbf{x}(k) \to 0$ and $\bar{\mathbf{x}}(k) \to P\mathbf{x}(k)$

Again, the eigenvalues of $\begin{bmatrix}A - BK & -BKQ_2 \\ 0 & D\end{bmatrix}$ are the eigenvalues of $A - BK$ together with eigenvalues of $D$.

Thus $K$ and $D$ can be separately designed to ensure that both $(A-BK)$ and $D$ have eigenvalues inside the unit circle

Thus **separation principle** is valid for reduced order observer too.

Figure 2 shows the block diagram of controller with reduced order observer.

## Points to remember

1. Separation principle assumes that the observer uses an exact dynamics of the plant. In reality, the precise dynamic model is hardly known.
2. The information known about the real process is often too complicated to be used in the observer.
3. The above points indicate that separation principle is not good enough for observer design, robustness of the observer must be checked as well.
4. $K$ should be designed such that the resulting $\mathbf{u}$ is not much high because of hardware limitation. Also, large control increases the possibility of entering the system into nonlinear region.
5. Dynamics of the observer poles should be much faster than the controller poles.

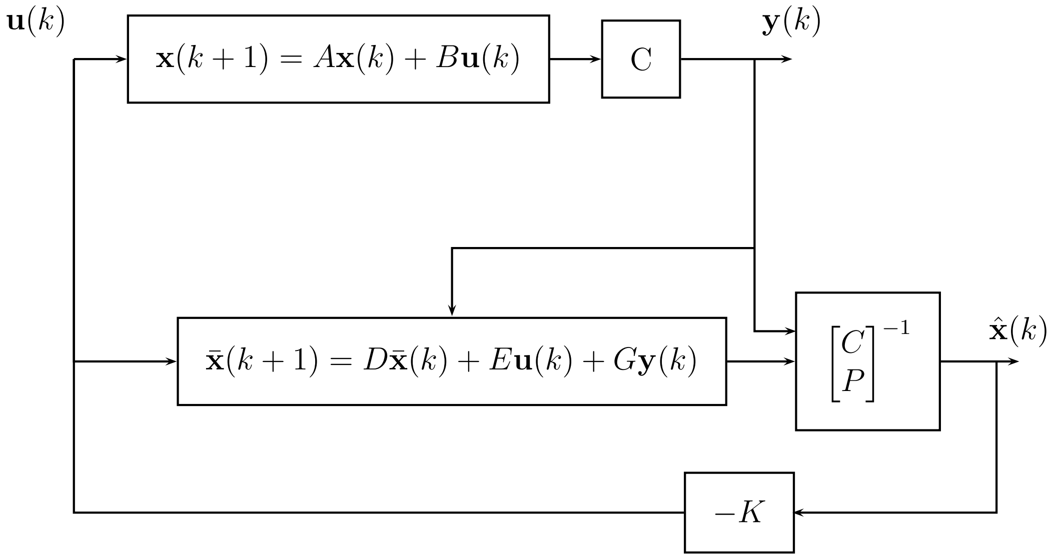


Figure 2: Controller with reduced order observer

# 2 Deadbeat Control by State Feedback and Deadbeat Observers

Consider the system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + Bu(k) \\ y(k) &= C\mathbf{x}(k) \end{aligned}$$

where $A \in R^{n\times n}$, $B \in R^{n\times 1}$ and $C \in R^{1\times n}$. With the state feedback control $u(k) = -K\mathbf{x}(k)$ the closed loop system becomes

$$\mathbf{x}(k+1) = (A - BK)\mathbf{x}(k)$$

Desired characteristic equation:

$$z^n + \alpha_1 z^{n-1} + \cdots + \alpha_{n-1}z + \alpha_n = 0$$

We pick a $K$ such that coefficients of $|zI - (A - BK)|$ match with those of the desired characteristic equation.

Let us consider a special case when $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. The desired characteristic equation in this case becomes

$$z^n = 0$$

By Cayley-Hamilton theorem:

$$(A - BK)^n = 0$$

Thus

$$\mathbf{x}(k) = (A - BK)^k \mathbf{x}(0) = 0, \quad \text{for } k \geq n$$

In other words, any initial state $\mathbf{x}(0)$ is driven to the equilibrium state $\mathbf{x} = 0$ in at most $n$ steps.

Thus the control law that assigns all the poles to origin can be viewed as a deadbeat control.

Similarly when all observer poles are at zero, we refer to that a deadbeat observer.

In deadbeat response, settling time depends on the sampling period. For a very small $T$, settling time is also very small, but the control signal becomes very high. Designer has to make a trade off between the two.

# Module 10: Output Feedback Design
## Lecture Note 1

We have discussed earlier that due to unavailability of system states, which are necessary for state feedback control, one need to estimate the unmeasurable states.

In the same context we will discuss another important topic in controller design which uses partial state feedback or output feedback for economical reasons.

## 1 Incomplete State Feedback

Consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

where $\mathbf{x}(k) \in R^{n\times 1}$, $\mathbf{u}(k) \in R^{m\times 1}$, $\mathbf{y}(k) \in R^{p\times 1}$. Let us consider an input

$$\mathbf{u}(k) = -G\mathbf{x}(k)$$

Let us assume that the state $x_i(k)$ is unavailable for feedback, where $i \in [1, n)$. The corresponding columns of $G$ thus become zero. Let

$$G = WG^*$$

where $W \in R^{m\times 1}$ and $G^* \in R^{1\times n}$.

Objective is to choose $W$ such that $(A,\ BW)$ is controllable. With partial state feedback, the columns of $G^*$ that correspond to the zero columns of $G$ must equal zero.

$G^*$ for single input case is related to desired closed loop poles, system parameters and $W$. It can be shown that

$$G^* = -[\Delta_{o1}\ \ \Delta_{o2}\ \ \cdots\ \ \Delta_{on}]K^{-1}$$

$\Delta_{oi} = \Delta_o(\lambda_i)$, where, $\Delta_o(.)$ represents the open loop characteristic equation and $\lambda_i$ represents the $i^{th}$ desired closed loop eigenvalue.

$$K = [k_1 \;\; k_2 \;\; \cdots \;\; k_n]$$

where $k_i = k(\lambda_i)$ and $k(z) = adj(zI - A)BW$.

When one or more columns of $G^*$ are forced to be zero, we need to put constraints on the desired closed loop eigenvalues. The following example will illustrate the design procedure more clearly.

**Example 1:** Consider the system

$$\mathbf{x}(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

The state feedback control is defined as

$$u(k) = -G\mathbf{x}(k)$$

where

$$G = [g_1 \;\; g_2]$$

Let us assume that $x_2$ is unavailable for feedback. Thus $g_2 = 0$. With $g_2 = 0$, the characteristic equation of the closed loop system becomes

$$|zI - A + BG| = z^2 + 2z + (1 + g_1) = 0$$

Since we have only one parameter to design, two eigenvalues cannot be arbitrarily chosen simultaneously. Dividing both sides of the above equation by $z^2 + 2z + 1$, we can write

$$1 + \frac{g_1}{z^2 + 2z + 1} = 0$$

The variations in the closed loop poles with respect to the parameter $g_1$ can be seen from the root locus plot.

We can draw both positive and negative root locus for $g_1 \geq 0$ and $g_1 < 0$ respectively. The root locus is shown in Figure 1 where the blue circle is the unit circle, red line represents root locus for $g_1 > 0$ and green line represents root locus for $g_1 < 0$.

One should note that for positive values of $g_1$, both roots are always outside the unit circle. Again, for negative values of $g_1$, one of the roots is inside the unit circle while the other is

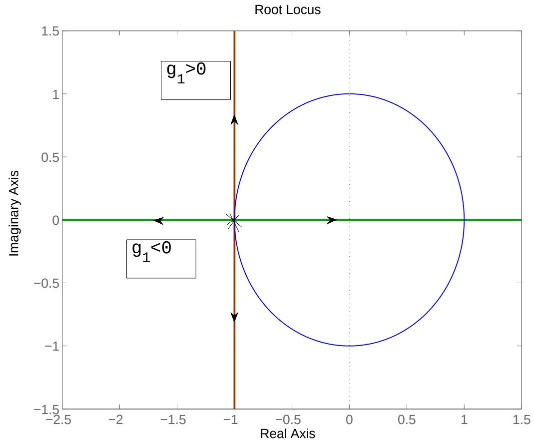


Figure 1: Root locus for Example 1 with $g_1$ as a variable parameter

always outside the unit circle.

One can thus conclude that for the given example when only $x_1$ is available for feedback, the system cannot be stabilized for any value of $g_1$.

Now, if we consider instead of $x_1$, only $x_2$ is available for feedback. Then $G = [0 \quad g_2]$. The characteristic equation of the closed loop system becomes

$$|zI - A + BG| = z^2 + (2 + g_2)z + 1 = 0$$

Dividing both sides of the above equation by $z^2 + 2z + 1$, we can write

$$1 + \frac{g_2 z}{z^2 + 2z + 1}$$

We can draw both positive and negative root locus for $g_2 \geq 0$ and $g_2 < 0$ respectively. The root locus is shown in Figure 2 where the red line represents root locus for $g_2 > 0$ and the green line represents root locus for $g_2 < 0$.

In this case when $g_2 > 0$, one of the roots is inside the unit circle while the other is always outside the unit circle. When $g_2 < 0$, for some range roots are on the unit circle, otherwise one of the roots is always outside the unit circle.

Thus we see that for the given system, not only can the eigenvalues be placed at desired locations, but the system is also not stabilizable.

## 2 Output feedback design

Output feedback control uses the system outputs since outputs of a system are always measurable and available for feed back.

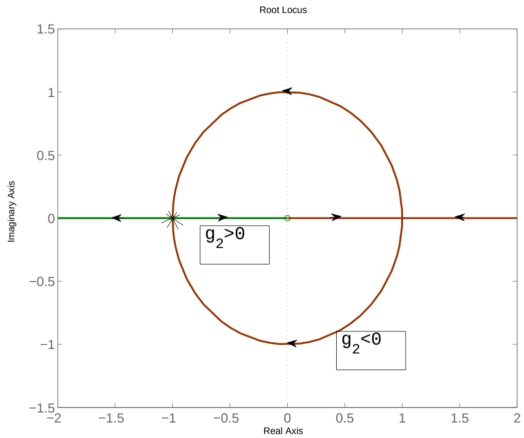


Figure 2: Root locus for Example 1 with $g_2$ as a variable parameter

Let us consider the following system

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k)$$
$$\mathbf{y}(k) = C\mathbf{x}(k)$$

where $\mathbf{x}(k) \in R^{n\times 1},\ \mathbf{u}(k) \in R^{m\times 1},\ \mathbf{y}(k) \in R^{p\times 1}$. Let us consider an input

$$\mathbf{u}(k) = -G\mathbf{y}(k) \qquad \text{where, } G \in R^{m\times p}$$

Objective is to choose $G$ such that eigenvalues of the closed loop system are at desired locations. However, since $p \le m \le n$, all eigenvalues cannot be arbitrarily assigned.

We will see that the number of eigenvalues that can be assigned arbitrarily eventually depends on the ranks of $C$ and $B$.

Let us first consider the single input case. Putting the expression of $u(k)$ into the system equation,

$$\mathbf{x}(k+1) = (A - BGC)\mathbf{x}(k)$$

$GC$ can be designed using pole placement technique if the pair $(A,\ B)$ is controllable. Since $C$ is not a square matrix $G$ cannot be directly solved from $GC = K$.

For $m = 1$, $G \in R^{1\times p}$, $C \in R^{p\times n}$, $B \in R^{n\times 1}$ and $GC \in R^{1\times n}$.

There are $p$ gain elements in $G$ but only $r$ of them are free as independent parameters where $r$ is the rank of $C$ and $r \leq p$. For example, if

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$G = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

then

$$GC = \begin{bmatrix} g_1 & g_2 & 2g_2 \end{bmatrix}$$

Only two elements of $GC$ are independent. This implies that only 2 of the total 3 eigenvalues can be placed arbitrarily.

For single input cases if the rank of $C$ is equal to the order of the system, i.e., $n$, the output feedback is equivalent to complete state feedback.

Say the order is 3. The state feedback gain $K$ is designed as $\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$
Let us assume $n$ to be 3 and

$$C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank of $C$ is also 3. Now,

$$G = \begin{bmatrix} g_1 & g_2 & g_3 \end{bmatrix}$$

Thus

$$GC = \begin{bmatrix} g_1 & g_2 & g_1 + g_2 + g_3 \end{bmatrix}$$

$g_1$, $g_2$, $g_3$ can be solved as

$$g_1 = k_1, \quad g_2 = k_2, \quad g_3 = k_3 - g_1 - g_2$$

For multi input systems $B \in R^{n \times m}$. Let $B^* = BW$ where $W \in R^{m \times 1}$ thus $B^* \in R^{n \times 1}$.

Similarly, let, $G = WG^*$ where $G^* = [g_1{}^* \ g_2{}^* \ \ g_3{}^* \ \ \ldots\ldots \ \ g_p{}^* \in R^{1 \times p}]$. Then,

$$BGC = BWG^*C = B^*G^*C$$

The characteristic equation of the closed loop system becomes

$$|zI - (A - BGC)| = |zI - (A - B^*G^*C)| = 0$$

$G^*C$ can be determined using pole placement technique.

Unlike the single input case, the solution of the feed back gain here depends on the ranks of $C$ as well as $B$.

If the rank of $C$ is greater than or equal to the rank of $B$, the elements of $W$ can be arbitrarily chosen if the pair (A, B) are controllable.

If rank of $B >$ rank of $C$, we cannot arbitrarily assign all the elements of $W$ if we wish to assign maximum number of eigenvalues of closed loop systems arbitrarily.

Examples, to be discussed in the next lecture, will make the design procedure more clearly understood.

# Module 10: Output Feedback Design
## Lecture Note 2

# 1 Output feedback design examples

In the last lecture, we have discussed about the incomplete state feedback design and output feedback design. In this lecture we would solve some examples to make the procedure properly understood.

**Example 1:** Let us consider the following system

$$\begin{aligned}\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k)\\ \mathbf{y}(k) &= C\mathbf{x}(k)\end{aligned}$$

for which the $A$, $B$, $C$ matrices are as follows

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

We know, for output feedback,

$$\mathbf{u}(k) = -G\mathbf{y}(k)$$

where the matrix $G$ has to be designed. Since $C$ has rank 2 and the rank of $B$ is also 2, minimum 2 eigenvalues can be placed at desired locations. Let these two be 0.1 and 0.2. The characteristic equation of $A$ is

$$|zI - A| = z^3 + 1$$

$B^*$ is written as

$$B^* = BW = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_2 \\ w_1 \\ 0 \end{bmatrix}$$

which has two independent parameters in terms of $w_1$ and $w_2$. Controllability matrix for the pair $(A,\ B^*)$ is

$$U_c^* = \begin{bmatrix} B^* & AB^* & A^2B^* \end{bmatrix} = \begin{bmatrix} w_2 & w_1 & 0 \\ w_1 & 0 & -w_2 \\ 0 & -w_2 & -w_1 \end{bmatrix}$$

It will be non singular if ${w_1}^3 - {w_2}^3 \neq 0$. Let

$$G^* = \begin{bmatrix} {g_1}^* & {g_2}^* \end{bmatrix} C$$

Then

$$G^*C = \begin{bmatrix} {g_1}^* + {g_2}^* & {g_2}^* & 0 \end{bmatrix}$$

Since $C$ has a rank of 2, $G^*C$ has two independent parameters in terms of $g_1^*$ and $g_2^*$. The closed loop characteristic equation is

$$\phi(z) = z^3 + \alpha_3 z^2 + \alpha_2 z + \alpha_1 = 0$$

Thus

$$G^*C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} {U_c^*}^{-1} \phi(A)$$

or,

$$\begin{bmatrix} {g_1}^* + {g_2}^* \\ {g_2}^* \\ 0 \end{bmatrix} = \frac{1}{w_1^3 - w_2^3} \begin{bmatrix} -\alpha_3 w_2^2 + \alpha_2 w_1^2 - (\alpha_1 - 1) w_1 w_2 \\ \alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 \\ -\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 \end{bmatrix}$$

The last row in the above equation corresponds to the following constraint equation.

$$-\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 = 0 \tag{1}$$

Since 2 of the three eigenvalues can be arbitrarily placed, $w_1$ and $w_2$ can be arbitrary provided the condition $w_1^3 \neq w_2^3$ is satisfied. But they should be selected such that the third eigenvalue is stable. This puts an additional constraint on $w_1$ and $w_2$.

For example, the necessary condition for the closed loop system to be stable is $|\alpha| < 1$. To satisfy this condition, $w_2$ cannot be equal to zero.

For $z = 0.1$ and $0.2$ to be the roots of the characteristic equation

$$z^3 + \alpha_3 z^2 + \alpha_2 z + \alpha_1 = 0$$

the following equations must be satisfied

$$\begin{aligned} \alpha_1 + 0.001 + \alpha_3 0.01 + 0.1\alpha_2 = 0 \\ \alpha_1 + 0.008 + \alpha_3 0.04 + 0.2\alpha_2 = 0 \end{aligned}$$

Simplifying the above equations,

$$\alpha_2 + 0.3\alpha_3 + 0.07 = 0 \tag{2}$$

$$\alpha_1 - 0.02\alpha_3 - 0.006 = 0 \tag{3}$$

Solving equations (1), (2) and (3) together

$$\begin{aligned} \alpha_1 &= \frac{0.02w_1^2 + 0.0004w_2^2 + 0.006w_1w_2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2} \\ \alpha_2 &= \frac{-0.2996w_1^2 - 0.07w_1w_2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2} \\ \alpha_3 &= \frac{0.994w_1^2 - 0.07w_2^2}{0.3w_2^2 + w_1w_2 + 0.02w_1^2} \end{aligned}$$

If we set $w_1 = 0$ and $w_2 = 1$, we get $\alpha_1 = 0.00133$, $\alpha_2 = 0$ and $\alpha_3 = -0.23333$.

With the above coefficients we find the roots to be $z_1 = 0.1$, $z_2 = 0.2$ and $z_3 = -0.0667$. Thus the third pole is placed within the unit circle and the closed loop system is stable.

There also exist some other combinations of $w_1$ and $w_2$ for which $z_1 = 0.1$, $z_2 = 0.2$ and the closed loop system is stable.

Putting the values of $w_1$ and $w_2$ and corresponding $\alpha_1$, $\alpha_2$ and $\alpha_3$ in the expression of $G^*C$, we get

$$\begin{bmatrix} {g_1}^* + {g_2}^* & {g_2}^* & 0 \end{bmatrix} = \begin{bmatrix} \alpha_3 \\ -\alpha_1 + 1 \\ \alpha_2 \end{bmatrix}^T = \begin{bmatrix} -0.23333 \\ 0.99867 \\ 0 \end{bmatrix}^T$$

Thus the feedback matrix can be calculated as

$$G^* = \begin{bmatrix} -1.232 & 0.99867 \end{bmatrix}$$

Hence,

$$G = WG^* = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1.232 & 0.99867 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -1.232 & 0.99867 \end{bmatrix}$$

**Example 2:** Consider the same system as in the previous example except for the fact that now

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

which has rank 1. This implies that

$$G^*C = \begin{bmatrix} {g_1}^* + {g_2}^* & 0 & 0 \end{bmatrix}$$

which has only one independent parameter in terms of $g_1^*$ and $g_2^*$. Thus

$$\begin{bmatrix} {g_1}^* + {g_2}^* \\ 0 \\ 0 \end{bmatrix} = \frac{1}{w_1^3 - w_2^3} \begin{bmatrix} -\alpha_3 w_2^2 + \alpha_2 w_1^2 - (\alpha_1 - 1) w_1 w_2 \\ \alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 \\ -\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 \end{bmatrix}$$

Last two rows of the above equation are constrained to be zero. Thus we can only assign $w_1$ or $w_2$ arbitrarily, not both. The constraint equations are as follows.

$$\alpha_3 w_1^2 - \alpha_2 w_1 w_2 + (\alpha_1 - 1) w_2^2 = 0$$

$$-\alpha_3 w_1 w_2 + \alpha_2 w_2^2 - (\alpha_1 - 1) w_1^2 = 0$$

If we want two closed loop eigenvalues to be placed at $z = 0.1$ and $z = 0.2$, we will altogether have four equations with five unknowns. Only one of these five unknowns can be assigned arbitrarily.

But these four equations would be nonlinear in $w_1$ and $w_2$, hence difficult to solve. The simpler way would be to use the following equation

$$|zI - A + BGC| = z^3 + (g_{21} + g_{22}) z^2 + (g_{11} + g_{12}) z + 1 = 0$$

Only two coefficients can be arbitrarily assigned. Since the constant term is equal to 1, the system cannot be stabilized with output feedback.

# Module 11: Introduction to Optimal Control
## Lecture Note 1

# 1 Introduction to optimal control

In the past lectures, although we have designed controllers based on some criteria, but we have never considered optimality of the controller with respect to some index. In this context, Linear Quadratic Regular is a very popular design technique.

The optimal control theory relies on design techniques that maximize or minimize a given performance index which is a measure of the effectiveness of the controller.

Euler-Lagrange equation is a very popular equation in the context of minimization or maximization of a functional.

A functional is a mapping or transformation that depends on one or more functions and the values of the functionals are numbers. Examples of functionals are performance indices which will be introduced later.

In the following section we would discuss the Euler-Lagrange equation for discrete time systems.

## 1.1 Discrete Euler-Lagrange Equation

A large class of optimal digital controller design aims to minimize or maximize a performance index of the following form.

$$J = \sum_{k=0}^{N-1} F(k, \boldsymbol{x}(k), \boldsymbol{x}(k+1), \boldsymbol{u}(k))$$

where $F(k, \boldsymbol{x}(k), \boldsymbol{x}(k+1), \boldsymbol{u}(k))$ is a differentiable scalar function and $\boldsymbol{x}(k) \in R^n$, $\boldsymbol{u}(k) \in R^m$.

The minimization or maximization of $J$ is subject to the following constraint.

$$\boldsymbol{x}(k+1) = f(k, \boldsymbol{x}(k), \boldsymbol{u}(k))$$

The above can be the state equation of the system, as well as other equality or inequality constraints.

Design techniques for optimal control theory mostly rely on the calculus of variation, according to which, the problem of minimizing one function while it is subject to equality constraints is solved by adjoining the constraint to the function to be minimized.

Let $\boldsymbol{\lambda}(k+1) \in R^{n\times 1}$ be defined as the Lagrange multiplier. Adjoining $J$ with the constraint equation,

$$J_a = \sum_{k=0}^{N-1} F(k, \boldsymbol{x}(k), \boldsymbol{x}(k+1), \boldsymbol{u}(k)) + < \boldsymbol{\lambda}(k+1), [\boldsymbol{x}(k+1) - f(k, \boldsymbol{x}(k), \boldsymbol{u}(k))] >$$

where $< . >$ denotes the inner product.

Calculus of variation says that the minimization of $J$ with constraint is equivalent to the minimization of $J_a$ without any constraint.

Let $\boldsymbol{x}^*(k)$, $\boldsymbol{x}^*(k+1)$, $\boldsymbol{u}^*(k))$ and $\boldsymbol{\lambda}^*(k+1)$ represent the vectors corresponding to optimal trajectories. Thus one can write

$$\begin{aligned}
\boldsymbol{x}(k) &= \boldsymbol{x}^*(k) + \epsilon\boldsymbol{\eta}(k) \\
\boldsymbol{x}(k+1) &= \boldsymbol{x}^*(k+1) + \epsilon\boldsymbol{\eta}(k+1) \\
\boldsymbol{u}(k) &= \boldsymbol{u}^*(k) + \delta\boldsymbol{\mu}(k) \\
\boldsymbol{\lambda}(k+1) &= \boldsymbol{\lambda}^*(k+1) + \gamma\boldsymbol{\nu}(k+1)
\end{aligned}$$

where $\boldsymbol{\eta}(k)$, $\boldsymbol{\mu}(k)$, $\boldsymbol{\nu}(k)$ are arbitrary vectors and $\epsilon$, $\delta$, $\gamma$ are small constants.

Substituting the above four equations in the expression of $J_a$,

$$\begin{aligned}
J_a &= \sum_{k=0}^{N-1} F(k, \boldsymbol{x}^*(k) + \epsilon\boldsymbol{\eta}(k), \boldsymbol{x}^*(k+1) + \epsilon\boldsymbol{\eta}(k+1), \boldsymbol{u}^*(k) + \delta\boldsymbol{\mu}(k)) + \\
&\quad < \boldsymbol{\lambda}^*(k+1) + \gamma\boldsymbol{\nu}(k+1), [\boldsymbol{x}^*(k+1) + \epsilon\boldsymbol{\eta}(k+1) - f(k, \boldsymbol{x}^*(k) + \epsilon\boldsymbol{\eta}(k), \boldsymbol{u}^*(k) + \delta\boldsymbol{\mu}(k))] >
\end{aligned}$$

To simplify the notation, let us denote $J_a$ as

$$J_a = \sum_{k=0}^{N-1} F_a(k, \boldsymbol{x}(k), \boldsymbol{x}(k+1), \boldsymbol{u}(k)), \boldsymbol{\lambda}(k+1))$$

Expanding $F_a$ using Taylor series around $\boldsymbol{x}^*(k),\ \boldsymbol{x}^*(k+1),\ \boldsymbol{u}^*(k))$ and $\boldsymbol{\lambda}^*(k+1)$, we get

$$\begin{aligned} F_a(k, \boldsymbol{x}(k), \boldsymbol{x}(k+1), \boldsymbol{u}(k), \boldsymbol{\lambda}(k+1)) = F_a(k, \boldsymbol{x}^*(k), \boldsymbol{x}^*(k+1), \boldsymbol{u}^*(k), \boldsymbol{\lambda}^*(k+1)) + \\ \left\langle \epsilon\boldsymbol{\eta}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k)} \right\rangle + \left\langle \epsilon\boldsymbol{\eta}(k+1), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k+1)} \right\rangle + \left\langle \delta\boldsymbol{\mu}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{u}^*(k)} \right\rangle + \\ \left\langle \gamma\boldsymbol{\nu}(k+1), \frac{\partial F_a^*(k)}{\partial \boldsymbol{\lambda}^*(k+1)} \right\rangle + \text{ higher order terms} \end{aligned}$$

where

$$F_a^*(k) = F_a(k, \boldsymbol{x}^*(k), \boldsymbol{x}^*(k+1), \boldsymbol{u}^*(k), \boldsymbol{\lambda}^*(k+1))$$

The necessary condition for $J_a$ to be minimum is

$$\begin{aligned} \left.\frac{\partial J_a}{\partial \epsilon}\right|_{\epsilon=\delta=\gamma=0} &= 0 \\ \left.\frac{\partial J_a}{\partial \delta}\right|_{\epsilon=\delta=\gamma=0} &= 0 \\ \left.\frac{\partial J_a}{\partial \gamma}\right|_{\epsilon=\delta=\gamma=0} &= 0 \end{aligned}$$

Substituting $F_a$ into the expression of $J_a$ and applying the necessary conditions,

$$\sum_{k=0}^{N-1} \left[ \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k)} \right\rangle + \left\langle \boldsymbol{\eta}(k+1), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k+1)} \right\rangle \right] = 0 \tag{1}$$

$$\sum_{k=0}^{N-1} \left\langle \boldsymbol{\mu}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{u}^*(k)} \right\rangle = 0 \tag{2}$$

$$\sum_{k=0}^{N-1} \left\langle \boldsymbol{\nu}(k+1), \frac{\partial F_a^*(k)}{\partial \boldsymbol{\lambda}^*(k+1)} \right\rangle = 0 \tag{3}$$

Equation (1) can be rewritten as

$$\begin{aligned} \sum_{k=0}^{N-1} \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k)} \right\rangle &= -\sum_{k=1}^{N} \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \\ &= -\sum_{k=0}^{N-1} \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle + \\ &\quad \left. \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \right|_{k=0} - \left. \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \right|_{k=N} \end{aligned}$$

where

$$F_a^*(k-1) = F_a(k-1, \boldsymbol{x}^*(k-1), \boldsymbol{x}^*(k), \boldsymbol{u}^*(k-1), \boldsymbol{\lambda}^*(k))$$

Rearranging terms in the last equation, we get

$$\sum_{k=0}^{N-1} \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k)} + \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle + \left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \Bigg|_{k=0}^{k=N} = 0 \tag{4}$$

According to the fundamental lemma of calculus of variation, equation (4) is satisfied for any $\boldsymbol{\eta}(k)$ only when the two components of the equation are individually zero. Thus,

$$\frac{\partial F_a^*(k)}{\partial \boldsymbol{x}^*(k)} + \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} = 0 \tag{5}$$

$$\left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \Bigg|_{k=0}^{k=N} = 0 \tag{6}$$

Equation (5) is known as the discrete Euler-Lagrange equation and equation (6) is called the transversality condition which is nothing but the boundary condition needed to solve equation (5).

Discrete Euler-Lagrange equation is the necessary condition that must be satisfied for $J_a$ to be an extremal.

With reference to the additional conditions (2) and (3), for arbitrary $\boldsymbol{\mu}(k)$ and $\boldsymbol{\nu}(k+1)$,

$$\frac{\partial F_a^*(k)}{\partial u_j^*(k)} = 0, \quad j = 1, 2, \cdots m \tag{7}$$

$$\frac{\partial F_a^*(k)}{\partial \lambda_i^*(k+1)} = 0, \quad i = 1, 2, \cdots n \tag{8}$$

Equation (8) leads to

$$\boldsymbol{x}^*(k+1) = f(k, \boldsymbol{x}^*(k), \boldsymbol{u}^*(k))$$

which means that the state equation should satisfy the optimal trajectory. Equation (7) gives the optimal control $\boldsymbol{u}^*(k)$ in terms of $\boldsymbol{\lambda}^*(k+1)$

In a variety of the design problems, the initial state $\boldsymbol{x}(0)$ is given. Thus $\boldsymbol{\eta}(0) = 0$ since $\boldsymbol{x}(0)$ is fixed. Hence the tranversality condition reduces to

$$\left\langle \boldsymbol{\eta}(k), \frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)} \right\rangle \Bigg|_{k=N} = 0$$

Again, a number of optimal control problems are classified according to the final conditions.

If $\boldsymbol{x}(N)$ is given and fixed, the problem is known as fixed-endpoint design. On the other hand, if $\boldsymbol{x}(N)$ is free, the problem is called a free endpoint design.

For fixed endpoint ($\boldsymbol{x}(N)$ = fixed, $\boldsymbol{\eta}(N) = 0$) problems, no transversality condition is required to solve.

For free endpoint the transversality condition is given as follows.

$$\left.\frac{\partial F_a^*(k-1)}{\partial \boldsymbol{x}^*(k)}\right|_{k=N} = 0$$

For more details, one can consult **Digital Control Systems** by B. C. Kuo.

# Module 11: Introduction to Optimal Control
## Lecture Note 2

## 1 Performance Indices

Whenever we use the term optimal to describe the effectiveness of a given control strategy, we do so with respect to some performance measure or index.

We generally assume that the value of the performance index decreases with the quality of the control law.

Constructing a performance index can be considered as a part of the system modeling. We would now discuss some typical performance indices which are popularly used.

Let us first consider the following system

$$\begin{aligned}\boldsymbol{x}(k+1) &= A\boldsymbol{x}(k) + B\boldsymbol{u}(k), \quad \boldsymbol{x}(k_0) = \boldsymbol{x}_0 \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k)\end{aligned}$$

Suppose that the objective is to control the system such that over a fixed interval $[N_0, N_f]$, the components of the state vector are as small as possible. A suitable performance to be minimized is

$$J_1 = \sum_{k=N_0}^{N_f} \boldsymbol{x}^T(k)\boldsymbol{x}(k)$$

When $J_1$ is very small, $\|\boldsymbol{x}(k)\|$ is also very small.

If we want to minimize the output over a fixed interval $[N_0, N_f]$, a suitable performance would be

$$\begin{aligned}J_2 &= \sum_{k=N_0}^{N_f} \boldsymbol{y}^T(k)\boldsymbol{y}(k) \\ &= \sum_{k=N_0}^{N_f} \boldsymbol{x}^T(k)C^TC\boldsymbol{x}(k)\end{aligned}$$

If $C^T C = Q$, which is a symmetric matrix,

$$J_2 = \sum_{k=N_0}^{N_f} \boldsymbol{x}^T(k) Q \boldsymbol{x}(k)$$

When the objective is to control the system in such a way that the control input is not too large, the corresponding performance index is

$$J_3 = \sum_{k=N_0}^{N_f} \boldsymbol{u}^T(k) \boldsymbol{u}(k)$$

Or,

$$J_4 = \sum_{k=N_0}^{N_f} \boldsymbol{u}^T(k) R \boldsymbol{u}(k)$$

where the weight matrix $R$ is symmetric positive definite.

We cannot simultaneously minimize the performance indices $J_1$ and $J_3$ because minimization of $J_1$ requires large control input whereas minimization of $J_3$ demands a small control. A compromise between the two conflicting objects is

$$\begin{aligned} J_5 &= \lambda J_1 + (1-\lambda) J_3 \\ &= \sum_{k=N_0}^{N_f} \left[\lambda \boldsymbol{x}^T(k)\boldsymbol{x}(k) + (1-\lambda)\boldsymbol{u}^T(k)\boldsymbol{u}(k)\right] \end{aligned}$$

A generalization of the above performance index is

$$J_6 = \sum_{k=N_0}^{N_f} \left[\boldsymbol{x}^T(k) Q \boldsymbol{x}(k) + \boldsymbol{u}^T(k) R \boldsymbol{u}(k)\right]$$

which is the most commonly used quadratic performance index.

In certain applications, we may wish the final state to be close to 0. Then a suitable performance index is

$$J_7 = \boldsymbol{x}^T(N_f) F \boldsymbol{x}(N_f)$$

When the control objective is to keep the state small, the control input not too large and the final state as close to 0 as possible, we can combine $J_6$ and $J_7$, to get the most general performance index

$$J_8 = \frac{1}{2}\boldsymbol{x}^T(N_f) F \boldsymbol{x}(N_f) + \frac{1}{2}\sum_{k=N_0}^{N_f} \left[\boldsymbol{x}^T(k) Q \boldsymbol{x}(k) + \boldsymbol{u}^T(k) R \boldsymbol{u}(k)\right]$$

1/2 is introduced to simplify the manipulation.

Sometimes we want the system state to track a desired trajectory throughout the interval $[N_0, N_f]$. In that case the performance index $J_8$ can be modified as

$$\begin{aligned} J_9 &= \frac{1}{2}\left[(\boldsymbol{x}_d(N_f) - \boldsymbol{x}(N_f))^T F(\boldsymbol{x}_d(N_f) - \boldsymbol{x}(N_f))\right] \\ &\quad + \frac{1}{2}\sum_{k=N_0}^{N_f}\left[(\boldsymbol{x}_d(k) - \boldsymbol{x}(k))^T Q(\boldsymbol{x}_d(k) - \boldsymbol{x}(k)) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)\right] \end{aligned}$$

For infinite time problem, the performance index is

$$J = \sum_{k=N_0}^{\infty}\left[\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)\right]$$

In most cases, $N_0$ is considered to be 0.

**Example:** Consider the dynamical system

$$\begin{aligned} \boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(k), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ y(k) &= [2 \quad 0]\boldsymbol{x}(k) \end{aligned}$$

Suppose that we want to minimize the output as well as the input with equal weightage along the convergence trajectory. Construct the associated performance index.

Since the initial condition of the system is $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and we have to minimize the performance index over the whole convergence trajectory, we need to take summation from 0 to $\infty$.

Again, since the output and input are to be minimized with equal weightage, we can write the cost function or performance index as

$$\begin{aligned} J &= \sum_{k=0}^{\infty}(y^2(k) + u^2(k)) \\ &= \sum_{k=0}^{\infty}(\boldsymbol{x}^T(k)[2 \quad 0]^T[2 \quad 0]\boldsymbol{x}(k) + u^2(k)) \\ &= \sum_{k=0}^{\infty}(\boldsymbol{x}^T(k)\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}\boldsymbol{x}(k) + u^2(k)) \end{aligned}$$

Comparing with the standard cost function, we can say that here $Q = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 1$.

In the next lecture we will discuss design of Linear Quadratic Regulator (LQR) by solving Algebraic Riccati Equation (ARE). To derive ARE, we need the following theorem.

Consider the system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k)$$

where $\boldsymbol{x}(k) \in R^n$, $\boldsymbol{u}(k) \in R^m$ and $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

**Theorem 1:** *If the state feedback controller $\boldsymbol{u}^*(k) = -K\boldsymbol{x}(k)$ is such that*

$$\min_{\boldsymbol{u}}(\Delta V(\boldsymbol{x}(k)) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)) = 0 \tag{1}$$

*for some Lyapunov function $V(k) = \boldsymbol{x}^T(k)P\boldsymbol{x}(k)$, then $\boldsymbol{u}^*(k)$ is optimal. Here the cost function is*

$$J(u) = \sum_{k=0}^{\infty}(\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k))$$

*and we assume that the closed loop system is asymptotically stable.*

**Proof:** Equation (1) can also be represented as

$$\Delta V(\boldsymbol{x}(k))|_{\boldsymbol{u}=\boldsymbol{u}^*} + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k)$$

Hence, we can write

$$\Delta V(\boldsymbol{x}(k))|_{\boldsymbol{u}=\boldsymbol{u}^*} = (V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)))|_{\boldsymbol{u}=\boldsymbol{u}^*} = -\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) - \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k)$$

We can sum both sides of the above equation from 0 to $\infty$ and get

$$V(\boldsymbol{x}(\infty)) - V(\boldsymbol{x}(0)) = -\sum_{k=0}^{\infty}(\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k))$$

Since the closed loop system is stable by assumption, $\boldsymbol{x}(\infty) = 0$ and hence $V(\boldsymbol{x}(\infty)) = 0$. Thus

$$V(\boldsymbol{x}(0)) = \sum_{k=0}^{\infty}(\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k))$$

Now, $V(\boldsymbol{x}(0)) = \boldsymbol{x}_o^T P\boldsymbol{x}_0$.

Thus if a linear state feedback controller satisfies the hypothesis of the theorem the value of the resulting cost function is

$$J(\boldsymbol{u}^*) = \boldsymbol{x}_o^T P \boldsymbol{x}_0$$

To show that such a controller is indeed optimal, we will use a proof by contradiction.

Assume that the hypothesis of the theorem holds true but the controller is not optimal. Thus there exists a controller $\bar{\boldsymbol{u}}$ such that

$$J(\bar{\boldsymbol{u}}) < J(\boldsymbol{u}^*)$$

Using the theorem, we can write

$$\Delta V(\boldsymbol{x}(k))|_{\boldsymbol{u}=\bar{\boldsymbol{u}}} + \boldsymbol{x}^T(k) Q \boldsymbol{x}(k) + \bar{\boldsymbol{u}}^T(k) R \bar{\boldsymbol{u}}(k) \geq 0$$

The above can be rewritten as

$$\Delta V(\boldsymbol{x}(k))|_{\boldsymbol{u}=\bar{\boldsymbol{u}}} \geq -\boldsymbol{x}^T(k) Q \boldsymbol{x}(k) - \bar{\boldsymbol{u}}^T(k) R \bar{\boldsymbol{u}}(k)$$

Summing the above from 0 to $\infty$,

$$V(\boldsymbol{x}(0)) \leq \sum_{k=0}^{\infty} (\boldsymbol{x}^T(k) Q \boldsymbol{x}(k) + \bar{\boldsymbol{u}}^T(k) R \bar{\boldsymbol{u}}(k))$$

The above inequality implies that

$$J(\boldsymbol{u}^*) \leq J(\bar{\boldsymbol{u}})$$

which is a contradiction of our earlier assumption. Thus $\boldsymbol{u}^*$ is optimal.

For more details one may consult **Systems and Control** by Stanislaw H. Żak

# Module 11: Introduction to Optimal Control
## Lecture Note 3

## 1 Linear Quadratic Regulator

Consider a linear system modeled by

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B\boldsymbol{u}(k), \quad \boldsymbol{x}(k_0) = \boldsymbol{x}_0$$

where $\boldsymbol{x}(k) \in R^n$ and $\boldsymbol{u}(k) \in R^m$. The pair $(A,\ B)$ is controllable.

The objective is to design a stabilizing linear state feedback controller $\boldsymbol{u}(k) = -K\boldsymbol{x}(k)$ which will minimize the quadratic performance index, given by,

$$J = \sum_{k=0}^{\infty} (\boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k))$$

where, $Q = Q^T \geq 0$ and $R = R^T > 0$. Such a controller is denoted by $\boldsymbol{u}^*$.

We first assume that a linear state feedback optimal controller exists such that the closed loop system

$$\boldsymbol{x}(k+1) = (A - BK)\boldsymbol{x}(k)$$

is asymptotically stable.

This assumption implies that there exists a Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T P\boldsymbol{x}(k)$ for the closed loop system, for which the forward difference

$$\Delta V(\boldsymbol{x}(k)) = V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k))$$

is negative definite.

We will now use the theorem as discussed in the previous lecture which says if the controller $\boldsymbol{u}^*$ is optimal, then

$$\min_{\boldsymbol{u}} (\Delta V(\boldsymbol{x}(k)) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)) = 0$$

Now, finding an optimal controller implies that we have to find an appropriate Lyapunov function which is then used to construct the optimal controller.

Let us first find the $\boldsymbol{u}^*$ that minimizes the function

$$f = f(\boldsymbol{u}(k)) = \Delta V(\boldsymbol{x}(k)) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k)$$

If we substitute $\Delta V$ in the above expression, we get

$$\begin{aligned} f(\boldsymbol{u}(k)) &= \boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k) \\ &= (A\boldsymbol{x}(k) + B\boldsymbol{u}(k))^T P(A\boldsymbol{x}(k) + B\boldsymbol{u}(k)) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^T(k)R\boldsymbol{u}(k) \end{aligned}$$

Taking derivative of the above function with respect to $\boldsymbol{u}(k)$,

$$\begin{aligned} \frac{\partial f(\boldsymbol{u}(k))}{\partial \boldsymbol{u}(k)} &= 2(A\boldsymbol{x}(k) + B\boldsymbol{u}(k))^T PB + 2\boldsymbol{u}^T R \\ &= 2\boldsymbol{x}^T(k)A^T PB + 2\boldsymbol{u}^T(k)(B^T PB + R) \\ &= \mathbf{0}^T \end{aligned}$$

The matrix $B^T PB + R$ is positive definite since $R$ is positive definite, thus it is invertible. Hence,

$$\boldsymbol{u}^*(k) = -(B^T PB + R)^{-1} B^T PA\boldsymbol{x}(k) = -K\boldsymbol{x}(k)$$

where $K = (B^T PB + R)^{-1} B^T PA$. Let us denote $B^T PB + R$ by $S$. Thus

$$\boldsymbol{u}^*(k) = -S^{-1} B^T PA\boldsymbol{x}(k)$$

We will now check whether or not $\boldsymbol{u}^*$ satisfies the second order sufficient condition for minimization. Since

$$\begin{aligned} \frac{\partial^2 f(\boldsymbol{u}(k))}{\partial \boldsymbol{u}^2(k)} &= \frac{\partial}{\partial \boldsymbol{u}(k)}(2\boldsymbol{x}^T(k)A^T PB + 2\boldsymbol{u}^T(k)(B^T PB + R)) \\ &= 2(B^T PB + R) > 0 \end{aligned}$$

$\boldsymbol{u}^*$ satisfies the second order sufficient condition to minimize $f$.

The optimal controller can thus be constructed if an appropriate Lyapunov matrix $P$ is found. For that let us first find the closed loop system after introduction of the optimal controller.

$$\boldsymbol{x}(k+1) = (A - BS^{-1}B^T PA)\boldsymbol{x}(k)$$

Since the controller satisfies the hypothesis of the theorem, discussed in the previous lecture,

$$\boldsymbol{x}^T(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \boldsymbol{u}^{*T}(k)R\boldsymbol{u}^*(k) = 0$$

Putting the expression of $\boldsymbol{u}^*$ in the above equation,

$$\begin{aligned}
&\boldsymbol{x}^T(k)(A - BS^{-1}B^T PA)^T P(A - BS^{-1}B^T PA)\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^T PBS^{-1}RS^{-1}B^T PA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^T PA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)A^T PBS^{-1}B^T PA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)A^T PBS^{-1}B^T PA\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^T PBS^{-1}B^T PBS^{-1}B^T PA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^T PBS^{-1}RS^{-1}B^T PA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^T PA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) - 2\boldsymbol{x}^T(k)A^T PBS^{-1}B^T PA\boldsymbol{x}(k) + \\
&\quad \boldsymbol{x}^T(k)A^T PBS^{-1}(B^T PB + R)S^{-1}B^T PA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)A^T PA\boldsymbol{x}(k) - \boldsymbol{x}^T(k)P\boldsymbol{x}(k) + \boldsymbol{x}^T(k)Q\boldsymbol{x}(k) - \\
&\quad 2\boldsymbol{x}^T(k)A^T PBS^{-1}SS^{-1}B^T PA\boldsymbol{x}(k) + \boldsymbol{x}^T(k)A^T PBS^{-1}B^T PA\boldsymbol{x}(k) \\
&= \boldsymbol{x}^T(k)(A^T PA - P + Q - A^T PBS^{-1}B^T PA)\boldsymbol{x}(k) = 0
\end{aligned}$$

The above equation should hold for any value of $\boldsymbol{x}(k)$. Thus

$$A^T PA - P + Q - A^T PBS^{-1}B^T PA = 0$$

which is the well known discrete Algebraic Riccati Equation (ARE). By solving this equation we can get $P$ to form the optimal regulator to minimize a given quadratic performance index.

**Example 1:** Consider the following linear system

$$\begin{aligned}
\boldsymbol{x}(k+1) &= \begin{bmatrix} 0 & 1 \\ 0.5 & 0.8 \end{bmatrix} \boldsymbol{x}(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \quad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(k)
\end{aligned}$$

Design an optimal controller to minimize the following performance index.

$$J = \sum_{k=0}^{\infty} (x_1^2 + x_1 x_2 + x_2^2 + 0.1u^2)$$

Also, find the optimal cost.

*Solution:* The performance index $J$ can be rewritten as

$$J = \sum_{k=0}^{\infty}\left(\boldsymbol{x}^T(k)\begin{bmatrix}1 & 0.5\\ 0.5 & 1\end{bmatrix}\boldsymbol{x}(k) + 0.1u^2\right)$$

Thus, $Q = \begin{bmatrix}1 & 0.5\\ 0.5 & 1\end{bmatrix}$ and $R = 0.1$.

Let us take $P$ as

$$P = \begin{bmatrix}p_1 & p_2\\ p_2 & p_3\end{bmatrix}$$

Then,

$$A^TPA - P = \begin{bmatrix}0.25p_3 - p_1 & 0.5p_2 + 0.4p_3 - p_2\\ 0.5p_2 + 0.4p_3 - p_2 & p_1 + 1.6p_2 + 0.64p_3 - p_3\end{bmatrix}$$

$$A^TPA - P + Q = \begin{bmatrix}0.25p_3 - p_1 + 1 & 0.4p_3 - 0.5p_2 + 0.5\\ 0.4p_3 - 0.5p_2 + 0.5 & p_1 + 1.6p_2 - 0.36p_3 + 1\end{bmatrix}$$

$$A^TPB = \begin{bmatrix}0.5p_3\\ p_2 + 0.8p_3\end{bmatrix}, \quad B^TPA = \begin{bmatrix}0.5p_3 & p_2 + 0.8p_3\end{bmatrix}, \quad S = 0.1 + p_3$$

$$\begin{aligned}A^TPBS^{-1}B^TPA = &= \frac{1}{0.1+p_3}\begin{bmatrix}0.5p_3\\ p_2 + 0.8p_3\end{bmatrix}\begin{bmatrix}0.5p_3 & p_2 + 0.8p_3\end{bmatrix}\\ &= \frac{1}{0.1+p_3}\begin{bmatrix}0.25p_3^2 & 0.5p_2p_3 + 0.4p_3^2\\ 0.5p_2p_3 + 0.4p_3^2 & p_2^2 + 1.6p_2p_3 + 0.64p_3^2\end{bmatrix}\end{aligned}$$

The discrete ARE is

$$A^TPA - P + Q - A^TPBS^{-1}B^TPA \;=\; 0$$

Or,

$$\begin{bmatrix}0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1+p_3} & 0.4p_3 - 0.5p_2 + 0.5 - \dfrac{0.5p_2p_3 + 0.4p_3^2}{0.1+p_3}\\ 0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3+0.4p_3^2}{0.1+p_3} & p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2+1.6p_2p_3+0.64p_3^2}{0.1+p_3}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

We can get three equations from the discrete ARE. These are

$$0.25p_3 - p_1 + 1 - \frac{0.25p_3^2}{0.1 + p_3} = 0$$

$$0.4p_3 - 0.5p_2 + 0.5 - \frac{0.5p_2p_3 + 0.4p_3^2}{0.1 + p_3} = 0$$

$$p_1 + 1.6p_2 - 0.36p_3 + 1 - \frac{p_2^2 + 1.6p_2p_3 + 0.64p_3^2}{0.1 + p_3} = 0$$

Since the above three equations comprises three unknown parameters, these parameters can be solved uniquely, as

$$p_1 = 1.0238, \quad p_2 = 0.5513, \quad p_3 = 1.9811$$

The optimal control law can be found out as

$$\begin{aligned} u^*(k) &= -(R + B^T P B)^{-1} B^T P A \boldsymbol{x}(k) \\ &= -[0.4760 \quad 1.0265]\boldsymbol{x}(k) \\ &= -0.4760x_1(k) - 1.0265x_2(k) \end{aligned}$$

The optimal cost can be found as

$$\begin{aligned} J &= \boldsymbol{x}_0^T P \boldsymbol{x}_0 \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1.0238 & 0.5513 \\ 0.5513 & 1.9811 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= 4.1075 \end{aligned}$$