1

Question 1

Compute the following limits of sequences $(x_n \ : \ n \geq 1)$.

1a

Question 1a

$$x_n = \frac{n^3 + 3n + 1}{3n^4 + 2n^2 + 5},$$

(5 marks)

$$\lim(x_n) = \lim\left(\frac{n^3+3n+1}{3n^4+2n^2+5}\right)$$

$$= \lim\left(\frac{\frac{1}{n} + \frac{3}{n^3} + \frac{1}{n^4}}{3 + \frac{2}{n^2} + \frac{5}{n^4}}\right)$$

$$= \frac{0}{3} = 0$$

1b

Question 1b

$$x_n = \frac{\ln(3n)}{\ln(5n^2)},$$

(5 marks)

$$\lim(x_n) = \lim\left(\frac{\ln(3n)}{\ln(5n^2)}\right)$$

$$= \lim\left(\frac{\frac{1}{3n}\cdot 3}{\frac{1}{5n^2}\cdot 10n}\right) \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim\left(\frac{\frac{1}{n}}{\frac{2}{n}}\right)$$

$$= \frac{1}{2}$$

1c

Question 1c

$$x_n = \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{4}{7}\right)^k$$

(5 marks)

$$x_n = \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{4}{7}\right)^k$$

$$= \sum_{k=1}^{n} (-1)(-1)^k \left(\frac{4}{7}\right)^k$$

$$= \sum_{k=1}^{n} (-1)\left(-\frac{4}{7}\right)^k$$

$$= \sum_{k=1}^{n} (-1)\left(-\frac{4}{7}\right)\left(-\frac{4}{7}\right)^{k-1}$$

$$= \frac{4}{7} \sum_{k=1}^{n} \left(-\frac{4}{7}\right)^{k-1}$$

$$= \frac{4}{7} \frac{1-\left(-\frac{4}{7}\right)^n}{1-\left(-\frac{4}{7}\right)}$$

$$= \frac{4}{7} \cdot \frac{1-\left(-\frac{4}{7}\right)^n}{\frac{11}{7}}$$

$$= \frac{4}{11} \cdot \left(1-\left(-\frac{4}{7}\right)^n\right)$$

$$\lim (x_n) = \frac{4}{11}$$

$$\sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$

2

Question 2

Suppose that $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function whose range is a bounded set. Define

$$f(x) = \sup\{h(x, y) \ : \ y \in \mathbb{R}\}$$

and

$$g(y) = \inf\{h(x, y) \ : \ x \in \mathbb{R}\}.$$

2a

Question 2a

Show that

$$\sup\{g(y) \ : \ y \in \mathbb{R}\} \leq \inf\{f(x) \ : \ x \in \mathbb{R}\}.$$

(6 marks)

For a given x, for all y, $h(x,y) \leq f(x)$
For a given y, for all x, $g(y) \leq h(x,y)$
For a given (x,y), $g(y) \leq h(x,y) \leq f(x)$

Since $x, y \in \mathbb{R}$ are arbitrary, for all (x,y), $g(y) \leq h(x,y) \leq f(x)$
$g(y) \leq f(x)$.

Since $g(y)$ is bounded above, by the completeness property, there exists $\sup g(y) \leq f(x)$. Also, $g(y) \leq \inf f(x)$.
We conclude that $\sup g(y) \leq \inf f(x)$.

2b

Question 2b

Give an example of a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with a bounded range such that

$$\sup\{g(y) \ : \ y \in \mathbb{R}\} < \inf\{f(x) \ : \ x \in \mathbb{R}\}.$$

(4 marks)

Let $h(x,y) = \sin x + \sin y$. Then $h(x,y) \in [-2,2]$.
$f(x) = \sup\{h(x,y) \mid y \in \mathbb{R}\}$ $g(y) = \inf\{h(x,y) \mid x \in \mathbb{R}\}$

$\inf f(x) = 0$ $\sup g(y) = 0$

$\sup g(y) < \inf f(x)$.

X

strict equality.

2b Let $h(x,y) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases}$. Then $h(x,y) \in [0,1]$.

$f(x) = \sup\{h(x,y) \mid y \in \mathbb{R}\}$ $\qquad$ $g(y) = \inf\{h(x,y), \mid x \in \mathbb{R}\}$.
$= \{1\}$ $\qquad$ $= \{0\}$.

$\sup g(y) = 0 < 1 = \inf f(x)$.

Question 3a

3a Suppose that $X = (x_n : n \geq 1)$ is an unbounded sequence. Show that X contains a sub-sequence $(x_{n_k} : k \geq 1)$ such that

$$\lim_{k \to \infty} \frac{1}{x_{n_k}} = 0$$

(7 marks)

Since (x_n) is unbounded, for every $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $|x_n| > m$.

Let the n_k term of (x_{n_k}) be such that $|x_{n_k}| > K$.
Then, we have $\left|\frac{1}{x_{n_k}}\right| < \frac{1}{K}$.

Given $\varepsilon > 0$, by the Archimedean Property, there exists $H \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < H$

$\frac{1}{H} < \varepsilon$

$\left|\frac{1}{x_{n_k}} - 0\right| < \frac{1}{H} < \varepsilon$

Therefore, $\lim\left(\frac{1}{x_{n_k}}\right) = 0$ by definition.

3b

Question 3b

Give an example of a sequence $(x_n : n \geq 1)$ such that the set of its elements $\{x_n : n \in \mathbb{N}\}$ is a bounded set that does not contain its supremum and infimum.

(3 marks)

$$x_n = \begin{cases} 1 - \frac{1}{n} & \text{for } n = 2k,\ k \in N \\ \frac{1}{n} - 1 & \text{for } n = 2k-1,\ k \in N \end{cases}$$

$\sup(x_n) = 1$ $\inf(x_n) = -1$

4

Question 4

Compute the following function limits, explaining your steps in detail.

4a

Question 4a

$$\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 - x - 2}$$

(4 marks)

$$\lim_{x \to -1} \frac{x^2+3x+2}{x^2-x-2} = \lim_{x \to -1} \frac{(x+2)\cancel{(x+1)}}{(x-2)\cancel{(x+1)}}$$

$$= \lim_{x \to -1} \frac{x+2}{x-2}$$

$$= -\frac{1}{3}$$

4b

Question 4b

$$\lim_{x \to 2} \frac{x-2}{\sqrt{|x-2|}}$$

(6 marks)

$$\lim_{x \to 2^+} \frac{x-2}{\sqrt{|x-2|}} = \lim_{x \to 2^+} \frac{x-2}{\sqrt{x-2}}$$

$$= \lim_{x \to 2^+} \frac{x-2}{\sqrt{x-2}} \cdot \frac{\sqrt{x-2}}{\sqrt{x-2}}$$

$$= \lim_{x \to 2^+} \frac{x-2}{x-2} \cdot \sqrt{x-2}$$

$$= \lim_{x \to 2^+} \sqrt{x-2}$$

$$= 0$$

$$\lim_{x \to 2^-} \frac{x-2}{\sqrt{|x-2|}} = \lim_{x \to 2^-} \frac{x-2}{\sqrt{2-x}} = \lim_{x \to 2^-} \frac{x-2}{\sqrt{2-x}} \cdot \frac{\sqrt{2-x}}{\sqrt{2-x}}$$

$$= \lim_{x \to 2^-} \frac{-(2-x)}{2-x} \sqrt{2-x}$$

$$= \lim_{x \to 2^-} -\sqrt{2-x}$$

$$= 0$$

$$\lim_{x \to 2} \frac{x-2}{\sqrt{|x-2|}} = 0$$

5a

Question 5a

Let a, b be real numbers with $a < b$. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous on $[a, \infty)$ and uniformly continuous on $(-\infty, b]$. Apply epsilon-delta arguments to show that f is uniformly continuous on $\mathbb{R}$.

(7 marks)

We have f is uniformly continuous on $[a,\infty]$. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $x, y \in [a, \infty)$ with $|x-y| < \delta_1$ then $|f(x) - f(y)| < \varepsilon$. For the given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that if $x, y \in (-\infty, b]$, with $|x-y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$. Let $\delta = \inf\{\delta_1, \delta_2\}$. We have that if $|x-y| < \delta$, $x, y \in (-\infty, b] \cup [a, \infty)$, then $|f(x) - f(y)| < \varepsilon$. Thus, by definition, f is uniformly continuous on $\mathbb{R}$.

5b

Question 5b

Prove that $f(x) = \sin(x^2)$ is not uniformly continuous on $[0, \infty)$.

(8 marks)

There exists an ε_0 and two sequences (x_n) and (y_n) in $[0, \infty)$ such that $\lim(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.

Let $\varepsilon_0 = 1$, $x_n = \sqrt{2\pi n}$, $y_n = \sqrt{2\pi n + \frac{\pi}{2}}$

$$x_n - y_n = \sqrt{2\pi n} - \sqrt{2\pi n + \frac{\pi}{2}} \cdot \frac{\sqrt{2\pi n} + \sqrt{2\pi n + \frac{\pi}{2}}}{\sqrt{2\pi n} + \sqrt{2\pi n + \frac{\pi}{2}}}$$

$$= \frac{2\pi n - 2\pi n - \frac{\pi}{2}}{\sqrt{2\pi n} + \sqrt{2\pi n + \frac{\pi}{2}}}$$

$$= \frac{-\frac{\pi}{2}}{\sqrt{2\pi n} + \sqrt{2\pi n + \frac{\pi}{2}}}$$

We have $\lim(x_n - y_n) = 0$. $|f(x_n) - f(y_n)| = |\sin(2\pi n) - \sin(2\pi n + \frac{\pi}{2})|$

$= 1$

$\geq \varepsilon_0$

Therefore, $f(x)$ is not uniformly continuous.

6

Question 6

Define a function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 2x^3 + ax + b\,, & x < 2 \\ a(2^x) + bx^2\,, & 2 \le x < 4 \\ x^2 + bx + a\,, & x \ge 4 \end{cases}.$$

Find the values of a, b such that $f(x)$ is continuous at $x = 2$ and $x = 4$.

(10 marks)

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} a(2^x) + bx^2$

$= 4a + 4b$

$f(2) = a(2^2) + b(2)^2$
$= 4a + 4b$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x^3 + ax + b$

$= 16 + 2a + b$

$16 + 2a + b = 4a + 4b \quad — (1)$

$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x^2 + bx + a$

$= 16 + 4b + a$

$f(4) = 4^2 + 4b + a$
$= 16 + 4b + a$

$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} a(2^x) + bx^2$

$= 16a + 16b$

$16 + 4b + a = 16a + 16b \quad — (2)$

$16 + 4b + a - 16a - 16b = 0$

$16 - 15a - 12b = 0$

$16 - 15\left(\frac{16-3b}{2}\right) - 12b = 0$

$16 + 2a + b = 4a + 4b$

$16 - 3b = 2a$

$a = \frac{16-3b}{2}$

$16 - 120 + \frac{45}{2}b - 12b = 0$

$22\frac{1}{2}b - 12b = \cancel{120}\ 104$

$10\frac{1}{2}b = 104$

$21b = 208$

$$b = \frac{208}{21}$$

$$a = \frac{16 - 3\left(\frac{208}{21}\right)}{2} = 8 - \frac{208}{14}$$

$$= -\frac{48}{7}$$

Question 7a

7a

Show that the function $f(x) = x^3 + 2x - 2$ has a root in the interval $(0, 1)$.

(5 marks)

$f(0) = -2$ $\quad f(1) = 1 + 2 - 2 = 1$

Since $f(x)$ is a polynomial, it is continuous everywhere. We have $f(0) \cdot f(1) < 0$. Then, by the Intermediate Value Theorem, there exists $x_0 \in (0,1)$ such that $f(x_0) = 0$.

Question 7b

7b

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(0) > 0$ and $f(2) < 4$. Prove that there is a number $c \in (0, 2)$ such that $f(c) = c^2$.

(6 marks)

Let $g(x) = f(x) - x^2$ $\quad g(0) = f(0) - 0^2 > 0$ $\quad g(2) = f(2) - 2^2 = f(2) - 4 < 0$

We have $f(x)$ continuous everywhere, $g(x)$ being the sum of two continuous functions, so $g(x)$ is continuous everywhere. We have $g(0) \cdot g(2) < 0$. Then, by IVM there exists $c \in (0,2)$ such that $f(c) - c^2 = 0 \Rightarrow f(c) = c^2$.

7c

Question 7c

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is discontinuous at each integer but continuous everywhere else, and $|f(x)|$ is continuous on $\mathbb{R}$.

(4 marks)

$f(x) = (-1)^{\lfloor x \rfloor}$

If $z \in \mathbb{Z}$, $\lim_{x \to z^-} f(x) = (-1)^{z-1}$ $\quad \lim_{x \to z^+} f(x) = (-1)^{z}$.

Thus, $\lim_{x \to z} f(x)$ does not exist.

If $z \notin \mathbb{Z}$, $\lim_{x \to z} f(x) = (-1)^{\lfloor z \rfloor} = f(z)$.

$|f(x)| = 1$. Since $|f(x)|$ is a constant function, it is continuous everywhere.

8a

Question 8a

Prove that

$$\lim_{x \to 1/2} \left(\frac{1}{x^3 + 1} \right) = \frac{8}{9}$$

from first principles, that is, by the using the definition of a function limit.

(9 marks)

Given $\varepsilon > 0$, if $0 < |x - \frac{1}{2}| < \delta$, then $\left| \frac{1}{x^3+1} - \frac{8}{9} \right| < \varepsilon$

$$\left| \frac{9}{9(x^3+1)} - \frac{8(x^3+1)}{9(x^3+1)} \right|$$

$$= \left| \frac{9 - 8x^3 - 8}{9(x^3+1)} \right|$$

$$= \left| \frac{1 - 8x^3}{9(x^3+1)} \right|$$

$$= \frac{1}{9} \left| \frac{1 - 8x^3}{x^3+1} \right|$$

$$= \frac{1}{9} \frac{|8x^3-1|}{|x^3+1|} = \frac{1}{9} \frac{|2x-1||4x^2+2x+1|}{|x^3+1|}$$

$$= \frac{2}{9} \frac{|x-\frac{1}{2}||4x^2+2x+1|}{|x^3+1|}$$

If $|x-\frac{1}{2}| < \frac{1}{2}$, then

$$-\frac{1}{2} < x - \frac{1}{2} < \frac{1}{2}$$

$$0 < x < 1$$

$$0 < x^3 < 1 \qquad 0 < x^2 < 1$$

$$1 < x^3+1 < 2 \qquad 0 < 4x^2 < 4$$

$$\frac{1}{2} < \frac{1}{x^3+1} < 1 \qquad 1 < 4x^2+2x+1 < 7$$

$$|4x^2+2x+1| < 7$$

$$\frac{1}{|x^3+1|} < 1$$

$$\frac{2}{9} \frac{|4x^2+2x+1|}{|x^3+1|} < \frac{14}{9}$$

Let $\delta = \inf\{\frac{1}{2}, \frac{9}{14}\varepsilon\}$. We have that if $|x-\frac{1}{2}| < \delta$, then

$$\left|\frac{1}{x^3+1} - \frac{8}{9}\right| = \frac{2}{9} \frac{|4x^2+2x+1|}{|x^3+1|} |x-\tfrac{1}{2}|$$

$$< \frac{14}{9} \cdot \frac{9}{14} \varepsilon$$

$$= \varepsilon$$

8b

Question 8b

Determine whether or not the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{x+1}{x^3+1}, & x \neq -1 \\ \frac{1}{3}, & x = -1 \end{cases}$$

is uniformly continuous on the open interval $(-2, 0)$.

(6 marks)

$f(-1) = \frac{1}{3}$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x+1}{x^3+1}$$

$$= \lim_{x \to -1} \frac{1}{3x^2}$$

$$= \frac{1}{3}$$

Let $g(x) = \frac{x+1}{x^3+1}$. g is continuous on $\mathbb{R} \setminus \{-1\}$.
We have $\lim_{x \to -1} g(x) = g(-1)$. Thus, f is continuous on $[2, 0]$.
By the Uniform Continuity Theorem, f is uniformly continuous on $[2, 0]$. Since $(2, 0) \subseteq [2, 0]$, then f is uniformly continuous on $(-2, 0)$.