Question 1a

Compute the limit $\lim_{n\to\infty} x_n$ of the sequence $(x_n, n \in \mathbb{N})$, where

$$x_n = \frac{2n^3}{5n^3 + 17n^2 + 2}.$$

(5 marks)

1a

$$x_n = \frac{2n^3}{5n^3 + 17n^2 + 2}$$

$$= \frac{2}{5 + \frac{17}{n} + \frac{2}{n^3}}$$

$$\lim(x_n) = \frac{2}{5}$$

Question 1b

Compute the limit $\lim_{n\to\infty} x_n$ of the sequence $(x_n, n \in \mathbb{N})$, where

$$x_n = \sum_{i=1}^{n} \frac{1}{n^2 + 2n + 3}.$$

(5 marks)

1b

$$x_n = \sum_{i=1}^{n} \frac{1}{n^2+2n+3}$$

$$= \frac{n}{n^2+2n+3}$$

$$= \frac{\frac{1}{n}}{1 + \frac{2}{n} + \frac{3}{n^2}}$$

$$\lim(x_n) = \frac{0}{1} = 0$$

$n \quad 3$

$n \quad 1$

Question 1c

Compute the upper limit , i.e., $\limsup_{n\to\infty} x_n$, of the sequence $(x_n, n \in \mathbb{N})$,

$$x_n = (-1)^n + \frac{2n}{n+2}.$$

(5 marks)

1c $\lim\left(\frac{2n}{n+2}\right) = \lim\left(\frac{2}{1+\frac{2}{n}}\right)$

$= 2$

Let $n=2k$. $\lim(x_{2k}) = 1+2$

$=3$.

Let $n=2k-1$ $\lim(x_{2k-1}) = -1+2$

$=1$

$\lim\sup(x_n) = \sup\{1,3\}$

$= 3$.

Question 1d

Compute the function limit

$$\lim_{x\to 0} x\left(\cos\frac{1}{x} + x\right).$$

(5 marks)

1d $\lim_{x\to 0} x(\cos\frac{1}{x} + x) = \lim_{x\to 0} x\cos\frac{1}{x} + \lim_{x\to 0} x^2$

$1 \leq \cos\frac{1}{x} \leq -1$

$x \leq x\cos\frac{1}{x} \leq -x$

By Squeeze Theorem,

$\lim_{x\to 0} -x = \lim_{x\to 0} x = \lim_{x\to 0} x\cos\frac{1}{x} = 0$

$\lim_{x\to 0} x(\cos\frac{1}{x} + x) = 0$

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Question 2

Give an example of sequence $(x_n, n \in \mathbb{N})$ such that it is bounded but not Cauchy. Show the reason.

(10 marks)

$x_n = (-1)^n$

For any $n \in \mathbb{N}$, $|x_{n+1} - x_n| = |(-1)^{n+1} - (-1)^n|$
$= 2$

If $\varepsilon = 2$, then for every $H \in \mathbb{N}$, if $n \geq H$,

$|x_{n+1} - x_n| \geq 2 = \varepsilon$. Thus x_n is not convergent. By
Theorem 2.3, x_n is not cauchy.

Question 3

Prove that the sequence $(x_n = \frac{1}{n} + \sin n,\ n \in \mathbb{N})$ has a convergent subsequence.

(10 marks)

3 $x_n = \frac{1}{n} + \sin n$

We have $\frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$. We also have $-1 \leq \sin n \leq 1$.
Therefore x_n is bounded above. Also $\frac{1}{n} > 0$, this means that x_n is bounded below. Thus, x_n is bounded. By the Bolzano-Weierstrass Theorem, x_n has a convergent subsequence.

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Question 4

Consider a sequence $(x_n, n \in \mathbb{N})$. Assume that for any $n \in \mathbb{N}$ there is $|x_{n+1} - x_n| < c^n$, where $0 < c < 1$ is a constant. Show that $(x_n, n \in \mathbb{N})$ is a Cauchy sequence.

(10 marks)

We need to show that for every $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that for every $n, m \geq H$, $|x_m - x_n| < \varepsilon$.

Since $0 < c < 1$, then if $m > n$, $c^n > c^m$.

$$|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + x_{m-2} + \dots + - x_{n+1} + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq c^{m-1} + c^{m-2} + \dots + c^n$$

$$= c^n (1 + c + \dots + c^{m-1-n})$$

$$= c^n \left(\frac{1 - c^{m-n}}{1 - c} \right)$$

$$< c^n \frac{1}{1-c}$$

$$\frac{c^N}{1-c} < \varepsilon$$

$$c^N < \varepsilon (1-c)$$

$$N \ln c < \ln(\varepsilon(1-c))$$

$$N > \frac{\ln(\varepsilon(1-c))}{\ln c} \qquad (\ln c < 0)$$

Given $\varepsilon > 0$. Choose N such that for all $m, n \in \mathbb{N}$,

$N > \frac{\ln(\varepsilon(1-c))}{\ln c} \Rightarrow |x_m - x_n| < \varepsilon$.

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Question 5

Assume a function $f(x)$ is continuous on $[0, +\infty)$ and uniformly continuous on $[1, +\infty)$. Determine whether $f(x)$ is uniformly continuous on $[0, +\infty)$.

(10 marks)

Let $A = [0,1] \subseteq [0, \infty)$. By the Uniform Continuity Theorem, since f is continuous on A, then f is uniformly continuous on A. Given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $x, y \in [0,1]$ and $|x-y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$. Also there exists $\delta_2 > 0$ such that if $x, y \in [1, \infty]$ and $|x-y| < \delta_2$, then $|f(x) - f(y)| < \varepsilon$. Let $\delta = \inf\{\delta_1, \delta_2\}$. We have that if $|x-y| < \delta$ and $x, y \in [0,1] \cup [1, \infty)$, then $|f(x) - f(y)| < \varepsilon$. Therefore, by definition, $f(x)$ is uniformly continuous on $[0, \infty)$.

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Question 6

Suppose $f(x)$ is a continuous function on the interval $[0,2]$ with $f(0)>0$ and $f(2)<4$. Show that the equation $f(x)=x^2$ has at least one solution in the interval $[0,2]$.

(10 marks)

$m, M.$ $\quad m \le f(x) \le M$ $\quad f(x)-x^2=0$

$g(x)=f(x)-x^2.$

$g(0)=f(0)-0>0$ $\quad g(2)=f(2)-4$
<0

Let $g(x)=f(x)-x^2$. We have $f(x)$ is continuous on $[0,2]$. Since x^2 is continuous everywhere, thus continuous on $[0,2]$. Then $g(x)$ is continuous on $[0,2]$. We have $g(0)=f(0)-0>0$. $g(2)=f(2)-2^2=f(2)-4<0$. By Sus Theorem 1.7, there exists $\bar{x}\in[0,2]$ such that $f(\bar{x})=0$.

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Question 7

Suppose $f(x): \mathbb{R} \to \mathbb{R}$ is a continuous function and $U \subseteq \mathbb{R}$ is an open set. Apply epsilon-delta arguments to prove that the set $f^{-1}(U) = \{x \in \mathbb{R} | f(x) \in U\}$ is also an open set.

✗ Suppose $y \in f^{-1}(U)$. Then $f(y) \in U$. Since U is an open set, then given $\varepsilon > 0$, $(f(y)-\varepsilon, f(y)+\varepsilon) \subseteq U$. By the continuity of f at y, for the given ε, there exists $\delta > 0$ such that if $|x-y| < \delta$, then $|f(x)-f(y)| < \varepsilon$. Therefore, $f(x) \in (f(y)-\varepsilon, f(y)+\varepsilon)$. Thus, $f(x) \in U$. Hence, $x \in f^{-1}(U)$.

✗ Suppose $y \in f^{-1}(U)$. Then $f(y) \in U$. Since U is an open set, given an ε-neighbourhood $V_\varepsilon(f(y))$ of $f(y)$, $V_\varepsilon(f(y)) \subseteq U$. By the continuity of f at y, for the given $V_\varepsilon(f(y))$, there exists δ-neighbourhood $V_\delta(y)$ of y such that if $x \in V_\delta(y)$, then $f(x) \in V_\varepsilon(f(y))$. Therefore, $f(x) \in U$. Hence, $x \in f^{-1}(U)$. Thus, $V_\delta(y) \subseteq f^{-1}(U)$. We conclude that $f^{-1}(U)$ is an open set.

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Question 8

Suppose $f(x)$ is an increasing function defined on an interval $X = [a, b]$. Prove that $f(x)$ is continuous at $\overline{x} = b$ if and only if $f(b) = \sup\{f(x) \mid x \in [a, b)\}$.

(15 marks)

We want to show that given $\varepsilon > 0$, there exists $f(x_\varepsilon)$ such that $f(b) - \varepsilon < f(x_\varepsilon)$ for some $x_\varepsilon \in [a, b)$.

Proof. Suppose that f is continuous at b. Then $\lim_{x \to b^-} f(x) = f(b)$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < b - x < \delta$, then $|f(x) - f(b)| < \varepsilon$. Since f is increasing,

We have $0 < b - x$.

$b > x$.

Then $f(b) \geq f(x)$. Therefore, $|f(b) - f(x)| < \varepsilon$.

$$f(b) - f(x) < \varepsilon$$
$$f(b) - \varepsilon < f(x).$$

We conclude that $f(b)$ is the supremum of $\{f(x) \mid x \in [a, b)\}$.

Suppose that $f(b) = \sup\{f(x) \mid x \in [a, b)\}$. Since $f(b)$ is the least upper bound, given $\varepsilon > 0$, there exists $f(x)$ such that $f(b) - \varepsilon < f(x)$ for some $x \in [a, b)$. Then,

$$f(b) - f(x) < \varepsilon$$
$$|f(x) - f(b)| < \varepsilon.$$

Let $\delta = b - a$. Since f is increasing, we have $f(b) > f(x)$, so $b > x$. Thus, if $0 < b - x < \delta$, then $|f(x) - f(b)| < \varepsilon$. Hence, $\lim_{x \to b^-} f(x) = f(b)$. Therefore, f is continuous at b.