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Question 1

Compute the limit $\lim_{n\to\infty} x_n$ of the sequence $(x_n, n \in \mathbb{N})$ if

1a (a) $x_n = \frac{\sin n + 2n}{n}$

(5 marks)

$$x_n = \frac{\sin n + 2n}{n} = \frac{\sin n}{n} + 2$$

$$\lim(x_n) = \lim\left(\frac{\sin n}{n}\right) + \lim(2)$$

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

By Squeeze Theorem,

$$\lim(x_n) = 0 + 2$$
$$= 2$$

(b) $x_n = \frac{(-1)^n + n}{n^2}$

(5 marks)

1b $$x_n = \frac{(-1)^n + n}{n^2} = \frac{(-1)^n}{n^2} + \frac{1}{n}$$

$$n^2 \geq n$$

$$-\frac{1}{n} \leq \frac{1}{n^2} \leq \frac{1}{n}$$

By Squeeze Theorem,

$$\lim(x_n) = 0 + 0 = 0$$

1c

(c) $x_n = \sqrt{n^2 + 8n} - n$

(5 marks)

$$x_n = \sqrt{n^2+8n} - n = \sqrt{n^2+8n} - n \cdot \frac{\sqrt{n^2+8n}+n}{\sqrt{n^2+8n}+n}$$

$$= \frac{n^2+8n-n^2}{\sqrt{n^2+8n}+n}$$

$$= \frac{8n}{\sqrt{n^2+8n}+n}$$

$$= \frac{8}{\sqrt{1+\frac{8}{n}}+1}$$

$$\lim(x_n) = \frac{8}{\sqrt{1}+1}$$

$$= 4$$

2a

Question 2

(a) Compute the function limit

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}.$$

(5 marks)

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)}$$

$$= \frac{1}{2}$$

$$\frac{\frac{1}{2\sqrt{x}}}{1}$$

$$= \frac{1}{2}$$

2b (b) Determine whether the function

$$f(x) = \begin{cases} \frac{x}{|x|} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$$

is continuous at $x = 0$.

(5 marks)

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$ $\quad f(0) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{-x} = -1$

$\lim_{x \to 0} f(x)$ does not exist. Therefore, f is not continuous at $x = 0$.

Question 3

Consider two bounded nonempty sets $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

(10 marks)

3 Since A and B are bounded. There exists M_A, M_B such that $x \in A$, $|x| \leq M_A$ and $x \in B$, $|x| \leq M_B$. If $x \in A \cup B$, let $M = \sup\{M_A, M_B\}$, $|x| \leq M$, so $A \cup B$ is bounded. By the Completeness property, $\sup(A \cup B)$ exists.

Let $s = \sup(A \cup B)$.

For all $x \in A \cup B$, $x \leq s$. Therefore, s is an upper bound of $A \cup B$. Hence, s is an upper bound of A and B. Thus, $s \geq \sup A$, $s \geq \sup B$. Then, $s \geq \sup\{\sup A, \sup B\}$.

Let $s = \sup\{\sup A, \sup B\}$

If $s = \sup A$, $s \geq x$ for all $x \in A$.

If $s = \sup B$, $s \geq x$ for all $x \in B$.

Therefore, $s \geq x$ for all $x \in A \cup B$. Hence, s is an upper bound of $A \cup B$. Then, $s \geq \sup(A \cup B)$.

Hence, $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

4

Question 4

Assume $f(x)$ is a continuous function on $\mathbb{R}$ and define the set $S = \{x \in \mathbb{R} | f(x) = 0\}$. If there is a sequence $(x_n, n \in \mathbb{N})$ in S such that $x_n \in S$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \overline{x}$, show that $\overline{x} \in S$.

(10 marks)

Since f is continuous on $\mathbb{R}$, f is continuous at $\bar{x}$. Then, every sequence (x_n) that converge to $\bar{x}$, we have that the sequence $(f(x_n))$ converge to $f(\bar{x})$.

Given $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $n \geq K$, $|f(x_n) - f(\bar{x})| < \varepsilon$.

$$
\begin{aligned}
|f(x_n) - f(\bar{x})| &= |0 - f(\bar{x})| \\
&= |f(\bar{x})| \\
&< \varepsilon.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, $f(\bar{x})$ must be 0. Therefore, $\bar{x} \in S$.

Question 5

If $f(x)$ is uniformly continuous on the set $S \subseteq \mathbb{R}$ and $|f(x)| \geq C > 0$ for all $x \in S$, where C is a positive constant. Apply epsilon-delta arguments to prove that the function defined as $g(x) = \frac{1}{f(x)}$ is also uniformly continuous on S.

(15 marks)

5

$$|f(x)| \geq C$$

$$\frac{1}{|f(x)|} \leq \frac{1}{C}$$

$$\frac{1}{|f(x) \cdot f(y)|} \leq \frac{1}{C^2}$$

We have f is uniformly continuous. Given $\varepsilon_0 > 0$, then exists $\delta > 0$ with $x, y \in S$ such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon_0$. Let $\varepsilon_0 = C^2 \cdot \varepsilon$.

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \left| \frac{f(y) - f(x)}{f(x) \cdot f(y)} \right|$$

$$= \frac{|f(x) - f(y)|}{|f(x) \cdot f(y)|}$$

$$< \frac{1}{C^2} \cdot C^2 \cdot \varepsilon$$

$$= \varepsilon$$

Therefore, $g(x)$ is also uniformly continuous on S.

Question 6

Give an example of functions $f(x)$ and $g(x)$ such that both $f(x)$ and $g(x)$ is discontinuous at a point $\overline{x}$ but $f(x) + g(x)$ is continuous at $\overline{x}$. Justify your answer.

(15 marks)

6

$$f(x) = \begin{cases} 0 & x = \bar{x} \\ 1 & x \neq \bar{x} \end{cases}$$

$$g(x) = \begin{cases} 1 & x = \bar{x} \\ 0 & x \neq \bar{x} \end{cases}$$

$\lim_{x \to \bar{x}} f(x) = 1$ $\quad f(\bar{x}) = 0$

Let $\varepsilon_0 = 1$. Given $\delta > 0$. We have $|f(x) - f(\bar{x})| = 1 \geq \varepsilon_0$ whenever $x \neq \bar{x} \Rightarrow 0 < |x - \bar{x}| < \delta$. Thus, $f(x)$ is not continuous at $\bar{x}$.

Let $\varepsilon_0 = 1$. Given $\delta > 0$. We have $|g(x) - g(\bar{x})| = 1 \geq \varepsilon_0$ whenever $x \neq \bar{x} \Rightarrow 0 < |x - \bar{x}| < \delta$. Thus, $g(x)$ is not continuous at $\bar{x}$.

$f(x) + g(x) = 1$

Given $\varepsilon > 0$. Let $\delta = \varepsilon$. If $|x - \bar{x}| < \delta$, then
$$|f(x) + g(x) - f(\bar{x}) - g(\bar{x})| = 0 < \varepsilon.$$

Thus, $f+g$ is continuous at $\bar{x}$.

Question 7

Use the Monotone Convergence Theorem to show that the sequence $(x_n, n \in \mathbb{N})$ is convergent, where the sequence is given by $x_n = \sum_{i=1}^{n}(\frac{1}{n^2} + \frac{1}{n^3})$.

(10 marks)

7 $x_n = \sum_{i=1}^{n}\left(\frac{1}{n^2} + \frac{1}{n^3}\right)$

$= n\left(\frac{1}{n^2} + \frac{1}{n^3}\right)$

$= \frac{1}{n} + \frac{1}{n^2}$

$0 < \frac{1}{n} \leq 1 \qquad 0 < \frac{1}{n^2} \leq 1$

$0 < \frac{1}{n} + \frac{1}{n^2} \leq 2$

Thus, x_n is bounded.

$x_1 = 1 + 1 = 2$

$x_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ Thus, $x_2 \leq x_1$.

Suppose $x_k \leq x_{k-1}$.

$\frac{1}{k} + \frac{1}{k^2} \leq \frac{1}{k-1} + \frac{1}{k-1}$

$\frac{1}{k+1} + \frac{1}{(k+1)^2} \leq \frac{1}{k} + \frac{1}{k^2}$ $\qquad k+1 > k$, $(k+1)^2 > k^2$

Thus, by induction, (x_n) is a decreasing sequence.
Therefore, by Monotone Convergence Theorem, (x_n) is convergent.

Question 8

Let $(x_n, n \in \mathbb{N})$ and $(y_n, n \in \mathbb{N})$ be two bounded sequence. Prove that $\liminf_{n\to+\infty}(x_n + y_n) \geq \liminf_{n\to+\infty} x_n + \liminf_{n\to+\infty} y_n$.

(15 marks)

8 Let $x = \liminf (x_n)$ $y = \liminf (y_n)$.
Since x is the limit inferior, given $\frac{\varepsilon}{2} > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x - \frac{\varepsilon}{2} > x_n$ but an infinite number of $n \in \mathbb{N}$ such that $x + \frac{\varepsilon}{2} > x_n$.

Since y is the limit inferior, given $\frac{\varepsilon}{2} > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $y - \frac{\varepsilon}{2} > y_n$ but an infinite number of $n \in \mathbb{N}$ such that $y + \frac{\varepsilon}{2} > y_n$.

Therefore, there are a finite number of $n \in \mathbb{N}$ such that
$x + y - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} > x_n + y_n$

$$x + y - \varepsilon > x_n + y_n$$

but an infinite number of $n \in \mathbb{N}$ such that
$x + y + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} > x_n + y_n$
$x + y + \varepsilon > x_n + y_n$
This implies $\liminf (x_n + y_n) = x + y$.
Therefore, $\liminf (x_n + y_n) \geq \liminf (x_n) + \liminf (y_n)$