1

Question 1

Compute the limits of the following sequences.

1a

(a) $x_n = \sqrt{4n^2 + 3n + 3} - 2n$

(5 marks)

$$x_n = \sqrt{4n^2+3n+3} - 2n \cdot \frac{\sqrt{4n^2+3n+3}+2n}{\sqrt{4n^2+3n+3}+2n}$$

$$= \frac{4n^2+3n+3-(2n)^2}{\sqrt{4n^2+3n+3}+2n}$$

$$= \frac{4n^2+3n+3-4n^2}{\sqrt{4n^2+3n+3}+2n}$$

$$= \frac{3n+3}{\sqrt{4n^2+3n+3}+2n}$$

$$= \frac{3+\frac{3}{n}}{\sqrt{\frac{4n^2+3n+3}{n^2}}+2}$$

$$= \frac{3+\frac{3}{n}}{\sqrt{4+\frac{3}{n}+\frac{3}{n^2}}+2}$$

$$\lim(x_n) = \frac{3}{\sqrt{4}+2} = \frac{3}{4}$$

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(b) $y_n = \sum_{i=1}^{n} \frac{i}{3n^2+i}$ (6 marks)

$$y_n = \sum_{i=1}^{n} \frac{i}{3n^2+i}$$

$$\frac{i}{3n^2+n} \leq \frac{i}{3n^2+i} \leq \frac{i}{3n^2}$$

$$\sum_{i=1}^{n} \frac{i}{3n^2} = \frac{1}{3n^2} \sum_{i=1}^{n} i$$

$$= \frac{1}{3n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{n+1}{6n}$$

$$\lim\left(\frac{n+1}{6n}\right) = \lim\left(\frac{1+\frac{1}{n}}{6}\right)$$

$$= \frac{1}{6}$$

$$\sum_{i=1}^{n} \frac{i}{3n^2+n} = \frac{1}{3n^2+n} \sum_{i=1}^{n} i$$

$$= \frac{1}{3n^2+n} \cdot \frac{n(n+1)}{2}$$

$$= \frac{1}{n(3n+1)} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n+1}{6n+2}$$

$$= \frac{1+\frac{1}{n}}{6+\frac{2}{n}}$$

$$\lim\left(\frac{1+\frac{1}{n}}{6+\frac{2}{n}}\right) = \frac{1}{6}$$

By Squeeze Theorem, $\lim(y_n) = \frac{1}{6}$.

(c) $z_n = \frac{2^n}{n!}$. Note that $n! = 1 \times 2 \times \ldots \times n$ is the factorial of n!

(6 marks)

c

$$\frac{z_{n+1}}{z_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$= \frac{2}{n+1}$$

$$\lim\left(\frac{z_{n+1}}{z_n}\right) = 0 < 1$$

By the ratio test, $\lim(z_n) = 0$.

Question 2

2

Give an example of two sequences $(x_n, n \in \mathbb{N})$ and $(y_n, n \in \mathbb{N})$ satisfying the following two conditions:

(a) $\lim_{n\to\infty} \frac{x_n}{y_n} = 1$ and

(b) $\lim_{n\to\infty} (x_n - y_n)$ does not exist.

(6 marks)

$x_n = n + (-1)^n$ $\qquad$ $y_n = n$

$$\frac{x_n}{y_n} = \frac{n + (-1)^n}{n}$$

$$= 1 + \frac{(-1)^n}{n}$$

$$\lim\left(\frac{x_n}{y_n}\right) = 1$$

$$x_n - y_n = (-1)^n$$

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Question 3

Assume $(x_n, n \in \mathbb{N})$ is a decreasing sequence of real numbers such that $x_n > 0$ for all $n \in \mathbb{N}$. Let $s_n = \sum_{i=1}^{n} x_i$ be the associated series. If $\lim_{n\to\infty} s_n$ exists, prove that $\lim_{n\to\infty} nx_n = 0$.

(12 marks)

Since (x_n) is a decreasing sequence and $x_n > 0$ for all $n \in \mathbb{N}$, then $0 < x_n \leq x_1$. Thus, (x_n) is bounded. Let $|x_n| < M$ for all $n \in \mathbb{N}$. By the Monotone Convergence Theorem, $\lim(x_n)$ exists.

Suppose $\lim(S_n)$ exists, then S_n is a Cauchy sequence. Given $\varepsilon > 0$, there exists $H \in \mathbb{N}$ such that for all $m, n \geq H$, then $|S_m - S_n| < \varepsilon$. We also have that $x_{n+1} = S_{n+1} - S_n$. Since the k-tail sequence (S_{n+k}) must also converge to the same limit, $\lim(x_n) = 0$.

Since (x_n) is a decreasing sequence,

$m = 2^k \quad n = 2^{k-1}$

$$
\begin{aligned}
S_{2^k} - S_{2^{k-1}} &= \sum_{i=1}^{2^k} x_i - \sum_{i=1}^{2^{k-1}} x_i \\
&= \sum_{i=2^{k-1}+1}^{2^k} x_i \\
&\geq (2^k - 2^{k-1})\, x_{2^k} \\
&= 2^{k-1}(2-1)\, x_{2^k} \\
&= 2^{k-1} x_{2^k}
\end{aligned}
$$

$$2(S_{2^k} - S_{2^{k-1}}) \geq 2^k x_{2^k} \Rightarrow \qquad 0 < 2^k x_{2^k} \leq 2(S_{2^k} - S_{2^{k-1}})$$

By Squeeze Theorem, $\lim(nx_n) = 0$.

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Question 4

Let $(x_n, n \in \mathbb{N})$ be a sequence of real numbers and define $X_n = \sum_{i=1}^{n} |x_{i+1} - x_i|$. Assume $X_n \leq M$ for any $n \in \mathbb{N}$, where $M > 0$ is a positive constant.

Prove:

(a) The sequence $(X_n, n \in \mathbb{N})$ is convergent.

(5 marks)

4a (X_n) is the partial sums of nonnegative terms, so (X_n) is an increasing sequence. We have $X_i \leq X_n$ for all $n \in \mathbb{N}$. Also $X_n \leq M$, where $M > 0$. Thus, X_n is bounded. By the Monotone Convergence Theorem, X_n is convergent.

(b) Use the Cauchy Criterion to show that the sequence $(x_n, n \in \mathbb{N})$ is also convergent.

(8 marks)

4b
$$|X_m - X_n| = \sum_{i=1}^{m} |x_{i+1} - x_i| - \sum_{i=1}^{n} |x_{i+1} - x_i|$$
$$= \sum_{i=n+1}^{m} |x_{i+1} - x_i|$$
$$= |x_{n+2} - x_{n+1}| + |x_{n+3} - x_{n+2}| + \dots + |x_{m+1} - x_m|$$
$$\geq |x_{n+2} - x_{n+1} + x_{n+3} - x_{n+2} + \dots + x_{m+1} - x_m|$$
$$= |x_{m+1} - x_{n+1}|$$

Since (X_n) is convergent, (X_n) is a Cauchy sequence. Given $\varepsilon > 0$, there exists H such that for all $m, n \geq H$, $|X_m - X_n| < \varepsilon$.

We have $|X_{m+1} - X_{n+1}| < \varepsilon$.

Also, $|x_m - x_n| < |X_{m+1} - X_{n+1}| < \varepsilon$.

Thus, $|x_m - x_n| < \varepsilon$.

Hence, (x_n) is a Cauchy sequence, thus convergent.

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Question 5

Assume $f(x)$ is a function on $\mathbb{R}$ and $\lim_{x\to 0} f(x) = L$, where L is a real number. Suppose $a > 0$ is a constant and we define $g(x) = f(ax)$. Apply epsilon-delta arguments to prove that $\lim_{x\to 0} g(x) = L$.

(10 marks)

Since $\lim_{y\to 0} f(y) = L$, given $\varepsilon > 0$, there exists $\delta_0 > 0$ such that if $0 < |y - 0| < \delta_0$, then $|f(y) - L| < \varepsilon$.

Let $y = ax$, $\delta = \frac{\delta_0}{a}$

We have if $0 < |ax - 0| < \delta_0$, then $|f(ax) - L| < \varepsilon$.

If $0 < |x| < \frac{\delta_0}{a} = \delta$, then $|g(x) - L| < \varepsilon$.

$|ax| < \delta_0$

$-\delta_0 < ax < \delta_0$

$-\frac{\delta_0}{a} < x < \frac{\delta_0}{a}$

$|x| < \delta$

Therefore, we conclude that $\lim_{x\to 0} g(x) = L$.

6

Question 6

Determine whether the function $f(x) = \frac{1}{x^2+1}$ is uniformly continuous on $\mathbb{R}$. Give the reasons.

(12 marks)

$$|f(x) - f(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right|$$

$$= \left|\frac{y^2+1-(x^2+1)}{(x^2+1)(y^2+1)}\right|$$

$$= \frac{|x^2-y^2|}{(x^2+1)(y^2+1)}$$

$$= \frac{|x+y|\,|x-y|}{(x^2+1)(y^2+1)} \leq \frac{|x|+|y|}{(x^2+1)(y^2+1)}|x-y|$$

Since $\frac{|x|}{x^2+1}$ is maximum at $x=1$, with $\frac{|x|}{x^2+1} = \frac{1}{2}$.

We have $\frac{|x|}{(x^2+1)(y^2+1)} < \frac{|x|}{x^2+1} \leq \frac{1}{2}$ for all $x, y \in \mathbb{R}$.

Then, $\frac{|x|+|y|}{(x^2+1)(y^2+1)} < \frac{1}{2} + \frac{1}{2} = 1$

Given $\varepsilon > 0$, let $\delta = \varepsilon$. If $|x-y| < \delta$, then

$$|f(x) - f(y)| = \frac{|x+y|\,|x-y|}{(x^2+1)(y^2+1)} \leq \frac{|x|+|y|}{(x^2+1)(y^2+1)}|x-y|$$

$$< |x-y|$$

$$< \delta$$

$$= \varepsilon.$$

Therefore, $f(x)$ is uniformly continuous by definition.

Question 7

Let $f: [0,2] \to \mathbb{R}$ be a continuous function such that $f(0) = f(2)$. Show that there exists $x_1,\ x_2 \in [0,2]$ satisfying the following two conditions:

(a) $|x_1 - x_2| = 1$ and

(b) $f(x_1) = f(x_2)$.

(15 marks)

7 Let $g(x) = f(x) - f(x+1)$, $x \in [0,1]$

$g(0) = f(0) - f(1)$ $g(1) = f(1) - f(2)$

$g(0) + g(1) = f(0) - f(1) + f(1) - f(2)$
$= f(0) - f(2) = 0$
$g(0) = -g(1)$
$g(0) \cdot g(1) = -g(1) \cdot g(1)$
< 0

$g(x)$ is continuous on $[0,1]$ since g is the sum of two continuous function. By the Intermediate Value Theorem, there exists $c \in [0,1]$ such that $g(c) = 0$

$g(c) = 0$
$f(c) - f(c+1) = 0$

Let $x_1 = c$, $x_2 = c+1$. $|x_1 - x_2| = 1$. $f(x_1) = f(x_2)$.

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Question 8

Let f and g be two continuous functions defined on $\mathbb{R}$. Define the maximum function h of f and g by $h(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$. Show that $h(x)$ is also continuous on $\mathbb{R}$.

(15 marks)

We can write $\max\{f(x), g(x)\} = \dfrac{f(x)+g(x) + |f(x)-g(x)|}{2}$.

If $f(x) \geq g(x)$, LHS $= f(x)$.

RHS $= \dfrac{f(x)+g(x)+f(x)-g(x)}{2} = f(x)$

If $g(x) > f(x)$, LHS $= g(x)$.

RHS $= \dfrac{f(x)+g(x)+g(x)-f(x)}{2} = g(x)$

Therefore, the equation is true.

We have that the sum of two continuous functions is continuous. If f is continuous, $|f| := |f(x)|$ is continuous. If f is continuous, $k \cdot f := k \cdot f(x)$ where $k \in \mathbb{R}$ is continuous. Therefore, $h(x)$ is continuous.