26^{th} TJIMO

Alexandria, Virginia

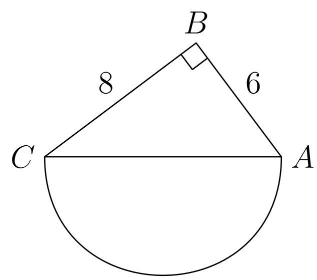
Round: **Individual**

1. The $\star$ operator is defined as the following:

$$a \star b = a^2 + ab - b.$$

Find $(3 \star 4) \star 7$.

2. Suppose a semicircle lies on the hypotenuse of right triangle ABC with legs $AB = 6$ and $BC = 8$. What is the area of the entire region? Express your answer as a common fraction in terms of π.



3. If $x^2 + y^2 = 6$ and $x + y = 6$, find xy.

4. The side length of the base of a square pyramid is 4. If its height is 9, what is its volume?

5. Alec has n lollipops, and he can divide them evenly among his 7 friends and himself at his birthday party. If Saroja crashes his party, then he can still divide them evenly among everyone. What is the least possible value of n?

6. Allen has 6 apricots: 2 blue, 2 red, 1 green, and 1 black. What is the minimum number of apricots he can pick to guarantee picking a red apricot?

7. A gardener wants to enclose a rectangular garden with 28 meters of fencing. What is the maximum area the gardener can enclose?

8. Helena flies toward Washington DC at a speed of 400 mi/hr from Barcelona, and Tara simultaneously flies from DC to Barcelona at a speed of 320 mi/hr. The distance between DC and Barcelona is 4000 miles. At what distance from Barcelona will the two planes pass each other? Express your answer as a common fraction.

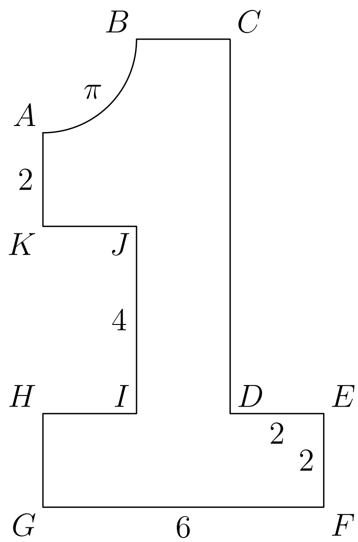
9. Longan, lychees, and genips are members of the soapbery family. In a system of trading, four longans are equivalent to five lychees, and nine lychees can be traded for twelve genips. If a unit is equivalent to seven gemips, how many units do you have if you own 24 of each soapberry? Express your answer as a common fraction.

10. Ashley and Kevin are playing a game. If Ashley gets a number, she adds 2 and gives it to Kevin. If Kevin gets a number, he adds 3 and sends it to Ashley. The first person to receive a number over 1000 wins. If Kevin starts by giving Ashley 0, then who wins?

Time limit: 1 hour.

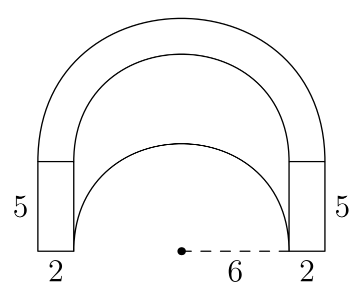
11. How many even positive divisors does 7000000 have?

12. Find the area of the "1" shape, given that the upper curve AB is a quarter-circle arc with length π, $AK = 2$, $IJ = 4$, $FG = 6$, $EF = 2$, and $DE = 2$. Express your answer in terms of π.



13. Saroja, Tim, Daniel, Sara, Ildoo, and Tara are waiting in line to buy tickets to prom. Because Tara rejected Ildoo's proposal to the dance, they refuse to stand next to each other in line. How many ways are there for the six kids to line up such that Tara and Ildoo do not stand next to each other?

14. Bill, Antonio and Hitesh are constructing a metal tube for their HUM project. Hitesh first buys a solid metal cylinder of radius 8 and height 5. Next, Antonio drills a large hole through the middle of the cylinder so that the remaining tube has a thickness of 2 units. Suddenly, Bill swings his sword and slices their tube in half! The result is a semicircular tube, which is shown below. Find the semicircular tube's surface area. Express your answer in terms of π.



15. Martians have two temperature scales, Cahrenheit and Felsius, which are linearly related. Lava boils at 314 Cahrenheit and 100 Felsius, while lava freezes at 272 Cahrenheit and 30 Felsius. Find the temperature such that the value on the Cahrenheit scale is equal to the value on the Felsius scale.

Time limit: 1 hour.

16. Joie has just registered her TJVMT account, and she has aptly made her password "piejoie". How many distinct ways are there to rearrange the letters in her password such that the vowels 'e', 'i', 'e', 'i', and 'o' are in that order from left to right?

17. Find the last digit of $(3^{1000} + 1^{1000} + 4^{1000})^{1000}$.

18. Jessica has five different colors of lipstick, each with its own matching colored purse. She happens to meet Gerard on the street, and drops all of her five purses in excitement. He picks them up for her, and puts the lipsticks back into the purses, but he does all this while staring into Jessica's eyes, paying no attention to matching the colors of lipstick and purse. What is the probability that exactly one of the lipsticks is in the right colored purse? Express your answer as a common fraction.

19. A, B, C, D, E, F, G, and H represent distinct digits from 0-9, where $A \neq 0$ and $E \neq 0$. They satisfy the relation below:

$$\begin{array}{ccccc} & A & B & C & D \\ + & E & F & G & B \\ \hline A & H & C & A & A \end{array}$$

Find $B + C + D + G$.

20. An archery target is divided into concentric rings such that the area between any two consecutive rings is $12\sqrt{17}$. Given that the area of the center ring (or bullseye) is also $12\sqrt{17}$, find the ratio of the radius of the 100th ring to that of the 4th ring.

21. Joe has a six digit telephone number, such that each digit is distinct. His first two digits are 2 and 5. Each digit afterwards is determined such that every 3 consecutive digits form a multiple of 11. Find Joe's entire 6-digit telephone number.

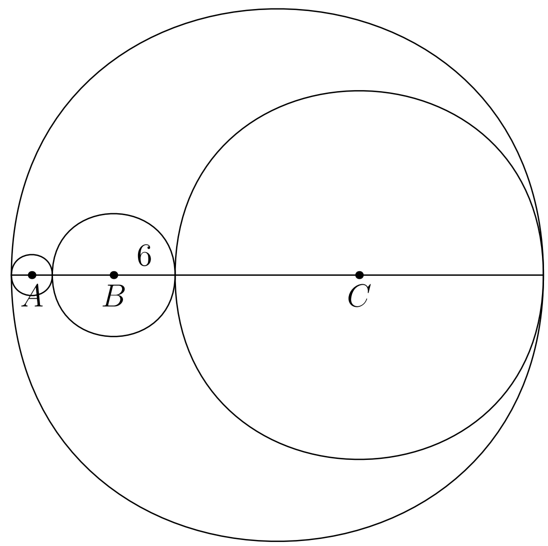
22. Sam starts out on the point $(0, 0)$ on a coordinate grid. If Sam can only move right or up on the coordinate grid and cannot pass through the point $(3, 3)$, how many ways can Sam get to $(7, 7)$?

23. Given that x and y are positive integers such that $17x + 19y = 401$, find xy.

24. Solve for x: $(2^x)^3 - 3(2^x)^2 + 3(2^x) - 1 = 0$.

Time limit: 1 hour.

25. A big circle O has diameter 52. There are three circles A, B, and C inside circle O such that circles A and C are internally tangent to O; circle B is externally tangent to circles A and C, and is between those two circles; and the radii of circles A, B, C, and O are collinear. (See picture below.) Given that the radii a, b, c of circles A, B, C form a geometric sequence, and that the diameter of circle B is 12, find the area inside circle O, but outside circles A, B, and C. Express your answer in terms of π.



26. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that do not contain three consecutive elements.

27. An ant is walking on a triangle. Each second, it will move to an adjacent vertex. What is the probability that it will return to the vertex where it started in five seconds? Express your answer as a common fraction.

28. In triangle ABC, we have $AB = 20$, $BC = 21$, and $AC = 29$. M is the midpoint of AC, and N lies on BC such that $BN = 18$. What is the value of MN? Express your answer as a common fraction.

29. Find the number of ordered triples (a, b, c) of positive integers such that the least common multiple of a, b, and c is 2014^2.

30. Compute $\sum_{k=0}^{19} 5k^4 + 10k^3 + 10k^2 + 5k + 1$.

Time limit: 1 hour.