Waveguide-Loss Evaluation in Circular Hollow Waveguides and Its Ray-Optical Treatment

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***Abstract*—A ray-optical representation of the attenuation constants in circular hollow waveguides characterized by a surface impedance and admittance has been derived from the characteristic equation based on the boundary-value problems. It is shown that in the first order of approximation, the attenuation constants of all modes including hybrid modes can be derived as if the corresponding rays were meridional. The effect of the Goos–Hänchen shift is also considered, which is significantly different from that in an intuitive formula given in circular dielectric waveguides.**

I. Introduction

The ray-optics method is a simple concept used to study phenomena in wave propagation. As the method itself is an approximate one, predictions by the ray-optics method should be checked carefully by the wave-optics method based on Maxwell's equations. Among some different features in the ray- and wave-optics methods are light-acceptance property [1] and cutoff condition of optical fibers [2]–[3]. These appear when the guiding structures are cylindrical ones, different from a slab geometry.

The Goos–Hänchen shift is a lateral shift of a reflected wave when the wave is incident on some plane or curved boundary. Many works have been published concerning this, however, only a few papers have dealt with the Goos–Hänchen shift in the connection with the ray-optical representation of the dispersion characteristics.

In connection with the waveguide-loss evaluation, it has been analytically proven that in planar hollow waveguides the ray-optical expression of the attenuation constants of the modes coincides with that predicted by the wave-optics method when the Goos–Hänchen shift is taken into account [4]. However, in circular waveguides the ray-optics method is often used intuitively [5] without any analytical proof. As far as we know, only Snyder and Mitchell [6] proved analytically that the complex solution of the eigenvalue equation gives the correct ray attenuation for refracted rays in circular dielectric fibers by neglecting the Goos–Hänchen shift.

In this paper, we have directly derived the ray-optical expression of the attenuation constants of modes in general circular hollow waveguides characterized by a surface impedance and admittance [7]–[8] from the eigenvalue equation. The expression is completely different from that of heuristic one on circular dielectric waveguides [5] when the Goos–Hänchen shift is taken into account. It is also shown that in the first order of approximation, the attenuation constants of the modes can be derived as if all rays corresponding to the modes were meridional when the reflection coefficient is properly introduced.

Throughout the paper, it is assumed that the core of the waveguides is free from absorptions.

II. Derivation of Ray-Optical Representation of Attenuation Constants

Consider a circular waveguide with the core radius T and a refractive index n_0 in $r \leq T$. We assume that generalized guided or leaky modes propagate along the z axis in the cylindrical coordinate system (r, θ, z) with the complex axial phase constant β and that electromagnetic properties of media in $r > T$ can be specified by a normalized surface impedance z_{TE} and admittance y_{TM} at $r = T$ as follows [7]:

$$\left.\frac{E_\theta}{H_z}\right)_{r=T} = \frac{\omega\mu_0}{n_0 k_0} z_{TE}$$
$$\left.\frac{H_\theta}{E_z}\right)_{r=T} = -\frac{n_0 k_0}{\omega\mu_0} y_{TM}. \tag{1}$$

Introducing the transverse (radial) phase constant u/T defined by

$$\frac{u}{T} = (n_0^2 k_0^2 - \beta^2)^{1/2} \tag{2}$$

one can obtain the characteristic equation for the modes as follows [8]:

$$\left[\frac{J_n'(u)}{uJ_n(u)} + j\frac{z_{TE}}{n_0 k_0 T}\right]\left[\frac{J_n'(u)}{uJ_n(u)} + j\frac{y_{TM}}{n_0 k_0 T}\right] = \frac{n^2}{u^4} \tag{3}$$

where $n_0 k_0 T \gg 1$ and the azimuthal dependence cos $(n\theta + \theta_0)$ for the E_z or sin $(n\theta + \theta_0)$ for the H_z component is assumed.

A. Attenuation Constants of the TE_{0q} and TM_{0q} Modes

We first discuss the case $n = 0$, i.e., the TE_{0q} and TM_{0q} modes. The characteristic equation (3) becomes

$$\frac{J_1(u)}{uJ_0(u)} = j\frac{1}{n_0 k_0 T} \times \begin{cases} z_{TE}, & TE_{0q} \text{ modes} \\ y_{TM}, & TM_{0q} \text{ modes} \end{cases} \tag{4}$$

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which leads to

$$\frac{J_0(u) + jJ_1(u)}{J_0(u) - jJ_1(u)} = -r_m(u) \tag{5}$$

where the reflection coefficient $r_m(u)$ at the boundary $r = T$ is defined by

$$r_m(u) = \begin{cases} \dfrac{z_{TE} - z_0}{z_{TE} + z_0}, & TE_{0q} \text{ modes} \\ \dfrac{y_{TM} + y_0}{y_{TM} - y_0}, & TM_{0q} \text{ modes.} \end{cases} \tag{6}$$

A normalized impedance z_0 and admittance y_0 in the core ($r < T$) are defined by

$$z_0 = y_0 = n_0k_0T/u. \tag{7}$$

Equation (6) is exactly the reflection coefficient for the meriodional ray which corresponds to the TE_{0q} and TM_{0q} modes.

By dividing u into real and imaginary parts

$$u = u_0 + ju_i \tag{8}$$

and assuming that u_i is much smaller than u_0, one can find from (5)

$$|r_m(u)|^2 = 1 - 4u_i\left[1 - \frac{1}{u_0}\frac{J_0(u_0)\,J_1(u_0)}{J_0^2(u_0) + J_1^2(u_0)}\right]. \tag{9}$$

On the other hand, we simply obtain an equation

$$\begin{aligned} |r_m(u)|^2 &= |r_m(u_0) + jr'_m(u_0)\,u_i|^2 \\ &= |r_m(u_0)|^2\left\{1 + 2u_i \operatorname{Re}\left[j\frac{r'_m(u_0)}{r_m(u_0)}\right]\right\} \end{aligned} \tag{10}$$

when u_i is small. One should note that (10) is independent of (9). Furthermore, one cannot put $|r_m(u_0)|^2 = 1$, because $r_m(u_0)$ is a general complex function of u_0 when z_{TE} or y_{TM} is complex, as seen from (6).

By use of (9) and (10), u_i is expressed as follows:

$$\begin{aligned} u_i &= \frac{1 - R_m}{4}\left\{1 + \frac{1}{2}\operatorname{Re}\left[j\frac{r'_m(u_0)}{r_m(u_0)}\right] - \frac{1}{u_0}\frac{J_0(u_0)\,J_1(u_0)}{J_0^2(u_0) + J_1^2(u_0)}\right\} \\ &= \frac{1 - R_m}{4}\left\{1 + \operatorname{Re}\left[j\frac{r'_m(u_0)}{r_m(u_0)}\right]\right\} \end{aligned} \tag{11}$$

where it is assumed that z_{TE} and y_{TM} are independent of β or u (see Appendix A) and $|r_m(u_0)|^2 \simeq 1$ for low-loss waveguides. $R_m = |r_m(u_0)|^2$ represents the power reflection coefficient of the plane wave at $r = T$ impinging at the angle θ_z

$$\theta_z = \sin^{-1}(u_0/n_0k_0T) \tag{12}$$

relative to the z axis as shown in Fig. 1.

Combining (2) and (11), one can express the modal-power attenuation constant 2α as

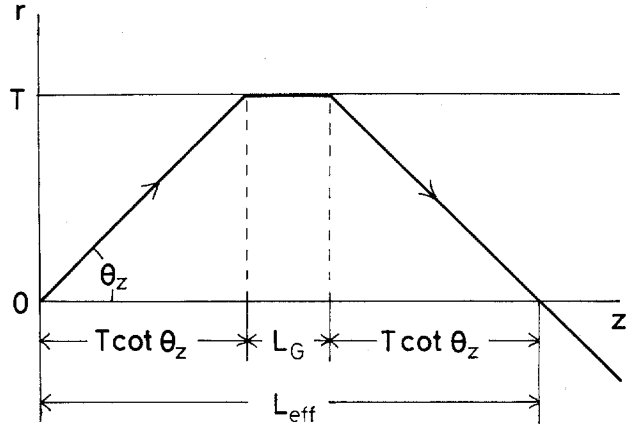

Fig. 1. Ray picture of reflection at $r = T$ showing a lateral shift of reflected ray, where θ_z is defined by $\sin^{-1}(u_0/n_0k_0T)$ or $\tan^{-1}(u_0/\beta_0T)$.

$$2\alpha = \frac{u_0}{2\beta_0T^2}\,\frac{1 - R_m}{1 + \operatorname{Re}\left[j\dfrac{r'_m(u_0)}{r_m(u_0)}\right]} \tag{13}$$

where β_0 is the real part of β defined by

$$\beta_0 = [n_0^2k_0^2 - (u_0/T)^2]^{1/2} = n_0k_0\cos\theta_z. \tag{14}$$

Equation (13) can be rewritten as

$$2\alpha = \frac{1 - R_m}{L_{eff}} \tag{15}$$

where L_{eff} is divided into two parts

$$L_{eff} = L_m + L_G. \tag{16}$$

L_m is the axial distance of the meridional-ray propagation between successive reflection, and L_G corresponds to the Goos–Hänchen shift, as shown in Fig. 1. They are expressed as follows:

$$L_m = 2T\cot\theta_z \tag{17}$$

$$L_G = 2T\cot\theta_z \operatorname{Re}\left[j\frac{r'_m(u_0)}{r_m(u_0)}\right]. \tag{18}$$

Equation (15) is exactly the ray-optical expression of the power-loss coefficients of the modes in circular waveguides. One should note that the correction of the Goos–Hänchen shift to the simple geometrical distance L_m to evaluate the attenuation constants is twice that in a slab geometry [4], although the corresponding rays to the TE_{0q} and TM_{0q} modes are meridional. This difference always exists no matter how the diameter of the hollow core becomes large and shows that cylindrical configurations can never be treated by planar ones. This fact also cannot be explained by the heuristic formula [5], used to derive a power attenuation constant in dielectric waveguides

$$2\alpha = \frac{1 - R_s}{L_s + L_G\cos\theta_z/\sin\theta_N} \tag{19}$$

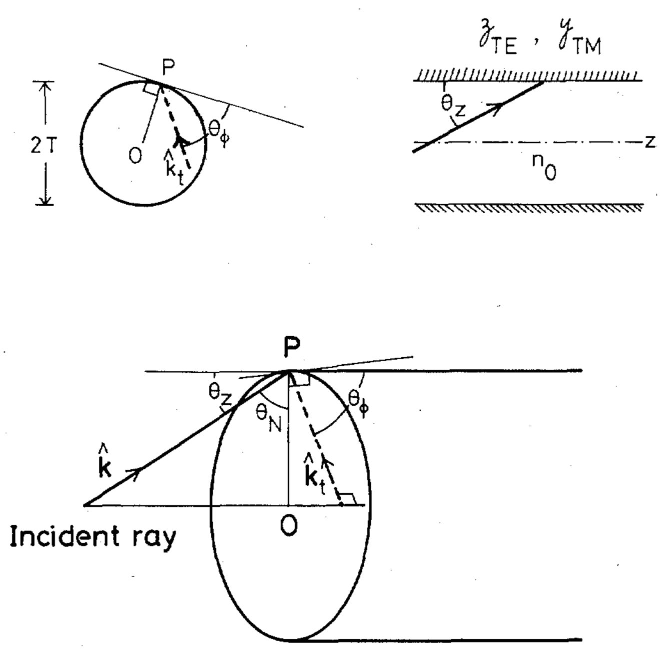

Fig. 2. Illustration of angles defined at incidence on the boundary [6]. P is the point of incidence; O is the center of the circular cross section; and θ_N is the angle between the normal to the boundary, given by line OP and the incidence-ray direction $\hat{k}$. $\hat{k}_t$ is the projection of the ray onto the cylindrical cross section.

which seems to describe the attenuation of any mode, where L_s is the axial distance of the ray propagation between successive reflections, L_G is a shift at each reflection in the plane of the incident ray and the normal at the point of reflection, and θ_N is the angle between the incident ray and normal, respectively (see Fig. 2). As (19) is expective to describe attenuation of all modes R_s, which means the power reflection coefficient of the plane wave corresponding to the skew rays, is used to distinguish it from R_m, used for meridional rays. As far as the meridional rays are concerned, R_s reduces to R_m and $\cos \theta_z = \sin \theta_N$ in (19).

In the next section, we shall analytically derive a ray-optical representation of the attenuation constant of the hybrid modes, which is different from the heuristic formula (19).

B. Attenuation Constants of the Hybrid Modes

We consider the hybrid modes, i.e, $n \neq 0$. The corresponding rays are not meridional but skew. For simplicity, we assume

$$|z_{TE}|/n_0 k_0 T << 1$$

and

$$|y_{TM}|/n_0 k_0 T << 1. \tag{20}$$

In this case, the characteristic equation (3) is simplified as follows:

$$\frac{J_{n \mp 1}(u)}{J_n(u)} = \mp j \frac{Z}{Z_0} \tag{21}$$

where upper and lower lines correspond to the HE_{nq} and EH_{nq} nodes, respectively, and Z and Z_0 are defined by

$$Z = \frac{1}{2}(z_{TE} + y_{TM}) \tag{22}$$

$$Z_0 = \frac{1}{2}(z_0 + y_0). \tag{23}$$

After some algebra, we can rewrite (21) as

$$\frac{J_n(u) \mp jJ_{n \mp 1}(u)}{J_n(u) \pm jJ_{n \mp 1}(u)} = -r_m(u) \tag{24}$$

where $r_m(u)$ is defined by

$$r_m(u) = \frac{Z - Z_0}{Z + Z_0} \tag{25}$$

which is similar to (6) for the TE_{0q} and TM_{0q} modes. The reason why the suffix m corresponding to the meridional rays is used for the skew rays will be made clear later.

Following the similar procedures in Section II-A, one can express u_i as follows:

$$u_i = \frac{1}{4} \frac{1 - R_m}{1 + \frac{1}{2}\operatorname{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right] - (2n \mp 1)\,Q} \tag{26}$$

where we put $|r_m(u_0)|^2 = R_m$ and Q is defined by

$$Q = \frac{1}{u_0} \frac{J_n(u_0)\, J_{n \mp 1}(u_0)}{J_n^2(u_0) + J_{n \mp 1}^2(u_0)}. \tag{27}$$

By using the result in Appendix B, one finds

$$u_i = \frac{1 - R_m}{4} \frac{1}{1 + (1 \mp n)\operatorname{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right]}. \tag{28}$$

Therefore the power-loss coefficient 2α of the hybrid modes is expressed as follows:

$$2\alpha = \frac{u_0}{2\beta_0 T^2} \frac{1 - R_m}{1 + (1 \mp n)\operatorname{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right]} \tag{29}$$

or

$$2\alpha = \frac{1 - R_m}{L_m + L_G} \tag{30}$$

where L_m and L_G are defined by

$$L_m = 2T \cot \theta_z \tag{31}$$

$$L_G = 2T(1 \mp n) \cot \theta_z \operatorname{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right]. \tag{32}$$

One should note that (30) is exactly the ray-optical expression of the power-loss coefficients of the hybrid modes when the Goos–Hänchen shift is taken into account. From

(30)–(32), one can deduce the following significant facts:

1) correction terms of the Goos–Hänchen shifts of the HE_{nq} and EH_{nq} modes have different signs for each other to the geometrical length L_m for $n \geq 2$, and
2) power-loss coefficients can be derived as if rays corresponding to the hybrid modes were meridional with the angle θ_z in reference to the z axis when the reflection coefficient is defined by (25).

The fact 1) cannot be explained by (19) at all. As far as 2) is concerned, one may wonder why the effect of the skewness of the rays is not involved in the expression of (30). In the next section, we answer the question clearly.

When $|z_{TE}|$ and $|y_{TM}|$ are much smaller than n_0k_0T/u_0 and the effect of the Goos–Hänchen shift is ignored, we can deduce α from (13) and (29) as

$$\alpha = n_0k_0 \frac{u_0^2}{(n_0k_0T)^3} \times \begin{cases} \mathrm{Re}\,(z_{TE}), & TE_{0q} \text{ modes} \\ \mathrm{Re}\,(y_{TM}), & TM_{0q} \text{ modes} \\ \frac{1}{2}\,\mathrm{Re}\,(z_{TE} + y_{TM}), & HE_{nq} \text{ and } EH_{nq} \text{ modes.} \end{cases} \quad (33)$$

Therefore one should note that there are no degeneracies between the meridional (TE_{0q} and TM_{0q}) modes and any skew (hybrid) mode.

C. Relation Between Attenuation Constants by Meridional- and Skew-Ray Formulations

For simplicity, we neglect the effect of the Goos–Hänchen shift in (30), i.e.,

$$2\alpha = \frac{1 - R_m}{L_m}. \quad (34)$$

To evaluate the attenuation constants predicted by skew-ray formulation, we define the angles θ_ϕ as shown in Fig. 2. The angles θ_N, θ_z, and θ_ϕ are expressed as follows [6]:

$$\cos\theta_\phi = n/u_0 \quad (35)$$

$$\cos\theta_N = (u_0^2 - n^2)^{1/2}/n_0k_0T = \sin\theta_z \sin\theta_\phi. \quad (36)$$

From the analogy of the reflection problem of the plane wave at the boundary between two homogeneous media [9], one may reasonably define the reflection coefficient r_s for skew rays at $r = T$ as

$$r_s = \frac{Z - 1/\cos\theta_N}{Z + 1/\cos\theta_N} = \frac{Z - Z_0u_0/(u_0^2 - n^2)^{1/2}}{Z + Z_0u_0/(u_0^2 - n^2)^{1/2}} \quad (37)$$

which shows that r_s is different from r_m in (25) when $n \neq 0$.

By using the approximations (20), one can deduce

$$1 - R_s \equiv 1 - |r_s|^2 = \frac{4(u_0^2 - n^2)^{1/2}}{n_0k_0T}\,\mathrm{Re}\,(Z). \quad (38)$$

As the axial distance L_s of the skew-ray propagation between successive reflections is expressed by

$$L_s = 2T\cot\theta_z \sin\theta_\phi \quad (39)$$

one can obtain

$$\frac{1 - R_s}{L_s} = \frac{2u_0^2}{n_0k_0\beta_0T^3}\,\mathrm{Re}\,(Z). \quad (40)$$

As the right-hand side of (40) does not depend on n explicitly, we can put $n = 0$ in the left-hand side of (40) and obtain

$$2\alpha = \frac{1 - R_s}{L_s} = \frac{1 - R_m}{L_m}. \quad (41)$$

Therefore it is concluded that (34), which apparently ignores the skewness of the rays, does include it properly. The significant point of (30) or (34) is that one can treat all rays corresponding to the hybrid modes as well as the TE_{0q} and TM_{0q} modes as if they were all meridional when the reflection coefficient is properly defined. This greatly simplifies the analyses as given in Section II-B and clearly shows the effect of the Goos–Hänchen shift from the eigenvalue equation without any knowledge of the Goos–Hänchen shift at a curved boundary [10]–[13].

III. Conclusion

Ray-optical expression of the attenuation constants of modes has been derived from the characteristic equation in circular hollow waveguides characterized by a surface impedance and admittance. It is shown that in the first order of approximation, attenuation constants of the hybrid modes can be derived as if the corresponding rays were meridional when the reflection coefficient at the boundary is properly defined. The effect of the Goos–Hänchen shift is also made clear.

Appendix A

When the normalized surface impedance z_{TE} is independent on β or u, one can easily obtain

$$j\frac{r_m'(u)}{r_m(u)} = j\frac{2}{u}\frac{z_{TE}z_0}{z_{TE}^2 - z_0^2} \quad (A1)$$

$$= -\frac{2}{u}\frac{J_0(u)\,J_1(u)}{J_0^2(u) + J_1^2(u)} \quad (A2)$$

for the TE_{0q} modes. By neglecting higher-order terms of

u_i, one obtains

$$\mathrm{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right] = -\frac{2}{u_0}\frac{J_0(u_0)\,J_1(u_0)}{J_0^2(u_0)+J_1^2(u_0)}. \tag{A3}$$

For the TM_{0q} modes, by replacing z_{TE} and z_0 in (A1) by y_{TM} and y_0, respectively, we arrive at the same equation as (A3).

APPENDIX B

For the hybrid modes, we easily obtain

$$j\frac{r_m'(u)}{r_m(u)} = j\frac{2n_0k_0TZ}{(uZ)^2-(n_0k_0T)^2} = j\frac{2}{u}\frac{ZZ_0}{Z^2-Z_0^2}. \tag{B1}$$

By substituting (21) into (B1), we obtain

$$j\frac{r_m'(u)}{r_m(u)} = \pm\frac{2}{u}\frac{J_n(u)\,J_{n\mp 1}(u)}{J_n^2(u)+J_{n\mp 1}^2(u)} \tag{B2}$$

which leads to

$$\mathrm{Re}\left[j\frac{r_m'(u_0)}{r_m(u_0)}\right] = \pm\frac{2}{u_0}\frac{J_n(u_0)\,J_{n\mp 1}(u_0)}{J_n^2(u_0)+J_{n\mp 1}^2(u_0)} \tag{B3}$$

where the higher-order terms of u_i are neglected.

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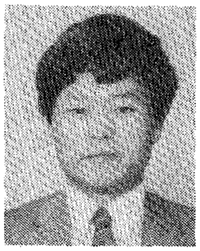

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