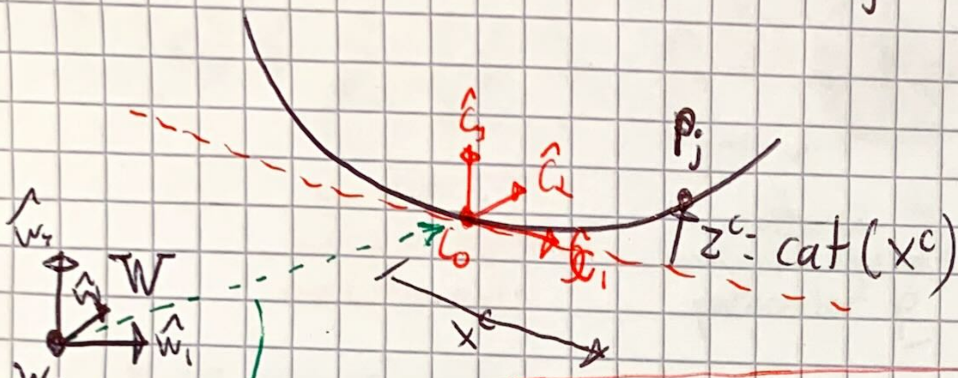
Catenary toolbox

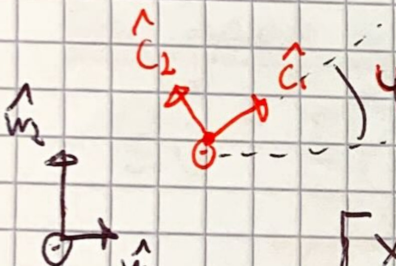
$$\underline{r}_j^c = \underline{r}^c_{p_j/c_o} = \begin{bmatrix} x^c \\ 0 \\ cat(x^c) \end{bmatrix}$$



$$\boxed{z = cat(x) = a\left[\cosh\left(\frac{x}{a}\right) - 1\right]}$$

sag parameter

$$\underline{r}_{c_o/w_o} = \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}$$ translation



ψ : rotation around $\hat{z}$

$$\underbrace{\begin{bmatrix} x_j^w \\ y_j^w \\ z_j^w \end{bmatrix}}_{} = \underbrace{\begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^{w}R^{c}} \underbrace{\begin{bmatrix} x^c \\ 0 \\ cat(x^c) \end{bmatrix}}_{\underline{r}_j^c} + \underbrace{\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}}_{\underline{r}_{c_o/w_o}}$$

$$\underline{r}_j^w = \underline{r}^w_{p_j/w_o}$$

$$\underline{r}_j^w = f(x^c, p)$$

$$p \rightarrow \begin{bmatrix} x_o \\ y_o \\ z_o \\ \psi \\ a \end{bmatrix}$$ parameter vector

3D world pts

local x in catenary frame

Catenary / point cloud matching cost

$$J(p) = \underbrace{\sum_{i=1}^{m} \frac{c_i}{m}}_{\text{average cost of measurements}} + \underbrace{p_e^T \, Q \, p_e}_{\text{distance to nominal "expected" parameter } p}$$

m → # of mesurments

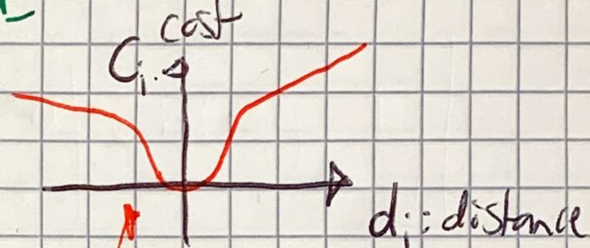
Q : regulation weigh matrix

$$p_e = p_{nom} - p$$

Cost per measurmnt pts

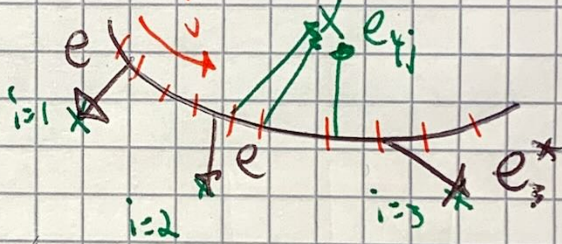
$$c_i = \log\left(1 + \left(\frac{b\,d_i}{l}\right)^{power}\right)$$

parameters of shaping curve



Errors/distance to pts

$$\underline{e}_{ij}^w = \underbrace{\underline{r}_i^w}_{\text{measured pts}} - \underbrace{\underline{r}_j^w}_{\text{catenary model pts}}$$



$$d_{ij} = \| \underline{r}_i^w - \underline{r}_j^w \| \qquad d_i^* = \min_j \| \underline{r}_i^w - \underline{r}_j^w \|$$

$$\underline{e}_i^* = \underline{r}_i^w - \underbrace{\underline{r}_{j^*}^w}_{\text{closest model pts}} \qquad j_i^* = \arg\min_j \| \underline{r}_i^w - \underline{r}_j^w \|$$

Model pts

$$\underline{r}_j^w = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_j^l \\ 0 \\ cat(x_j^l) \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$z_j^l = cat(x_j^l) = a\left[\cosh\left(\frac{x_j^l}{a}\right) - 1\right]$$

x_j^l : Sampling $[\underbrace{x_{min}, \dots, x_{max}}_{n\_sample}]$

cost param

$[\cancel{p_n}, Q, b, l, power, x_{min}, x_{max}, n]$

Inputs

$\underline{r}_i$ & $\underline{p}_{nom}$

Catenary / point cloud cost gradient

$$J(\underline{p}) = \frac{1}{m}\sum c_i + \underline{p}_e^T Q\, \underline{p}_e \qquad \underline{p}_e = \underline{p}_{nom} - \underline{p}$$

$$\frac{\partial J}{\partial \underline{p}} = \frac{1}{m}\sum \frac{\partial c_i}{\partial \underline{p}} + 2\underline{p}_e^T Q \cancel{\frac{\partial \underline{p}_e}{\partial \underline{p}}} = -I$$

$$\boxed{\frac{\partial J}{\partial \underline{p}} = \frac{1}{m}\sum_i^m \frac{\partial c_i}{\partial \underline{p}} - 2(\underline{p}_{nom} - \underline{p})^T Q}$$

$$\frac{\partial c_i}{\partial \underline{p}} = \frac{\partial c_i}{\partial d_i^*}\,\frac{\partial d_i^*}{\partial \underline{e}_{ij}^*}\,\frac{\partial \underline{e}_{ij}^*}{\partial \underline{r}_{j^*}}\,\frac{\partial \underline{r}_{j^*}}{\partial \underline{p}}$$

$$\rightarrow c_i = \log\left(1 + \frac{(b\cdot d_i)^{power}}{l}\right) \qquad \boxed{\frac{\partial c_i}{\partial d_i} = \frac{1}{\ln(10)}\cdot\frac{\cancel{l}}{l + (b\,d_i)^{power}}\cdot\frac{power\,(b\,d_i)^{power-1}}{\cancel{l}}\cdot b}$$

$$\rightarrow d_i^* = \sqrt{\underline{e}_{ij^*}^T\,\underline{e}_{ij^*}} \qquad \boxed{\frac{\partial d_i^*}{\partial \underline{e}_{ij^*}} = \frac{1}{2\sqrt{\underline{e}^T\underline{e}}}\,\frac{\partial\, \underline{e}^T\underline{e}}{\partial \underline{e}} = \boxed{\frac{\underline{e}_{ij^*}^T}{d_i^*}}}$$

$$\rightarrow \underline{e}_{ij} = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} - \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_j^l \\ 0 \\ a\left[\cosh\left(\frac{x_j^l}{a}\right) - 1\right] \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

$$\frac{\partial \underline{e}_{ij}}{\partial \underline{p}} = \left[\begin{array}{ccc|c|c} -1 & 0 & 0 & +\sin\psi\, x_j^l & 0 \\ 0 & -1 & 0 & -\cos\psi\, x_j^l & 0 \\ 0 & 0 & -1 & 0 & -\left(\cosh\left(\frac{x_j^l}{a}\right) - \frac{x_j}{a}\sinh\left(\frac{x_j}{a}\right) - 1\right) \end{array}\right]$$

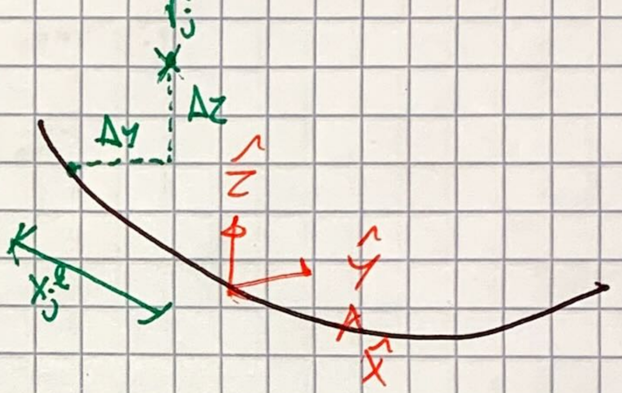
$$\underbrace{\qquad\qquad}_{\frac{\partial \underline{e}}{\partial \underline{r}_0}} \qquad \underbrace{\qquad}_{\frac{\partial \underline{e}}{\partial \psi}} \qquad \underbrace{\qquad\qquad\qquad}_{\frac{\partial \underline{e}}{\partial a}}$$

Catenary / point cloud local matching cost

$$J(p) = \sum \frac{C_i}{m} + p_e^T Q \, p_e$$

$$C_i = \log\left(1 + \frac{(b d_i)^{power}}{e}\right) \qquad d_i = \sqrt{e_i^T e_i} = \|e_i^c\|$$

error in local frame



$$e_i^l = r_i^l - \underbrace{\begin{bmatrix} x_j^l = x_i^l \\ 0 \\ cat(x_i^l) \end{bmatrix}}_{r_j^l}$$

$$r_i^l = \underbrace{\begin{bmatrix} c\psi & +s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{{}^cR^w} \left(\begin{bmatrix} x_i^w \\ y_i^w \\ z_i^w \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right)$$

$$e_i^c = \begin{bmatrix} 0 \\ -s\psi x_i^w + c\psi y_i^w + s\psi x_0 - c\psi y_0 \\ z_i^w - z_0 - cat(x_j^c) \end{bmatrix}$$

✓

$$x_j^c = c\psi x_i^w + s\psi y_i^w - c\psi x_0 - s\psi y_0$$

$$cat(x) = a\left[\cosh\left(\frac{x}{a}\right) - 1\right]$$

local matching gradient

$$\frac{\partial J}{\partial \underline{p}} = \frac{1}{m}\sum_{i}^{m} \frac{\partial C_i}{\partial \underline{p}} - 2(\underline{p}_n - \underline{p})^T Q$$

$$\frac{\partial C_i}{\partial \underline{p}} = \frac{\partial C_i}{\partial d_i} \left(\frac{\partial d_i}{\partial \underline{e}_i} \frac{\partial \underline{e}_i}{\partial \underline{p}} \right)$$

$$\frac{\partial C_i}{\partial d_i} = \frac{1}{\ln(10)} \frac{b \cdot power\, (b\, d_i)^{power-1}}{1 + (b\, d_i)^{power}}$$

$$\frac{\partial d_i}{\partial \underline{e}_i} = \frac{\underline{e}_i^T}{d_i}$$

$$\frac{\partial \underline{e}_i^c}{\partial p} = \begin{array}{c} \left[\begin{array}{ccc|c|c} 0 & 0 & 0 & 0 & 0 \\ s\psi & -c\psi & 0 & (x_o - x_i^w)c\psi + (y_o - y_i^w)s\psi & 0 \\ \frac{\partial e_z}{\partial x_o} & \frac{\partial e_z}{\partial y_o} & -1 & \frac{\partial e_z}{\partial \psi} & -\begin{pmatrix} \cosh\left(\frac{x_j^c}{a}\right) \\ -\frac{x_j^c}{a}\sinh\left(\frac{x_j^c}{a}\right) \\ -1 \end{pmatrix} \end{array}\right] \\ \begin{array}{ccc} \frac{\partial e}{\partial x_o} & \frac{\partial e}{\partial y_o} & \frac{\partial e}{\partial z_o} \end{array} \end{array}$$

$$\frac{\partial e_z}{\partial x_o} = \frac{\partial e}{\partial x_j} \frac{\partial x_j}{\partial x_o} = -\sinh\left(\frac{x_j^c}{a}\right)(-c\psi)$$

$$\frac{\partial e_z}{\partial y_o} = \frac{\partial e}{\partial x_j} \frac{\partial x_j}{\partial y_o} = -\sinh\left(\frac{x_j^c}{a}\right)(-s\psi)$$

$$\frac{\partial e_z}{\partial \psi} = \frac{\partial e}{\partial x_j} \frac{\partial x_j}{\partial \psi} = -\sinh\left(\frac{x_j^c}{a}\right)\left[-s\psi\,(x_i^w - x_o) + c\psi\,(y_i^w - y_o)\right]$$