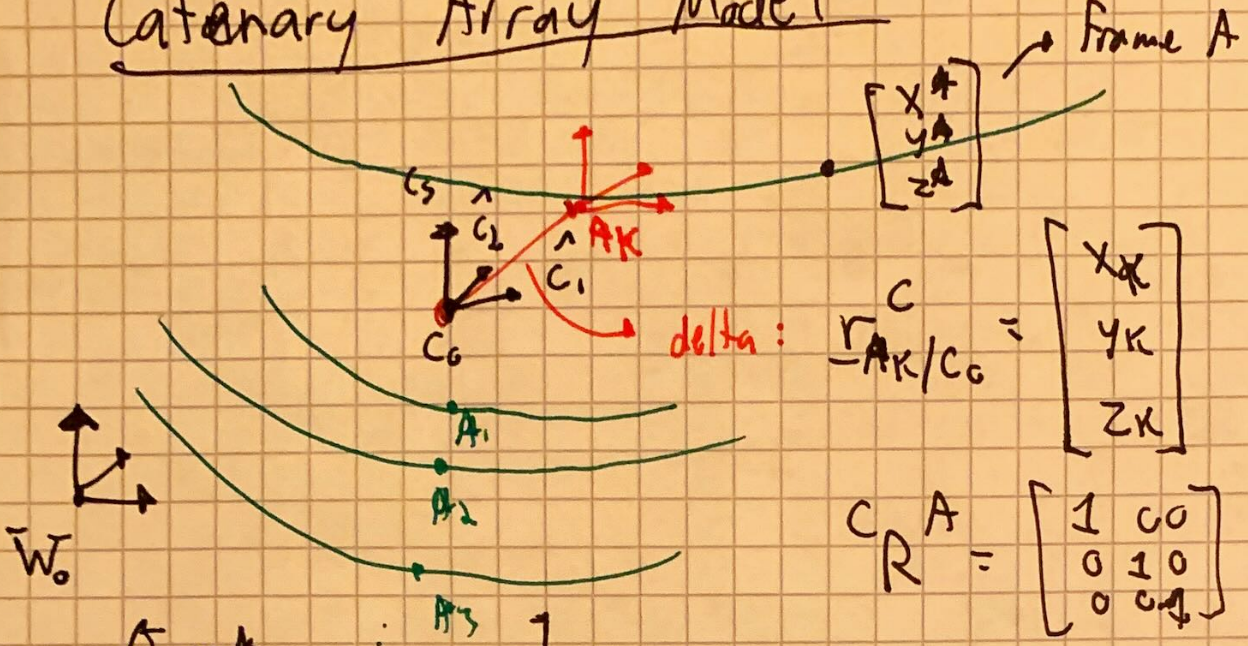
Catenary Array Model



delta : $\underline{r}^{C}_{A_K/C_0} = \begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix}$

$${}^{C}R^{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (same orientation)

$$\underline{r}^{A_K}_{j} = \begin{bmatrix} x^A_j \\ y^A = 0 \\ z^A = \mathrm{cat}(x^A_j) \end{bmatrix}$$

$$\underline{r}^{C}_{jK} = \cancel{{}^{C}R^{A}}^{\,1}\, \underline{r}^{A}_{jK} + \underline{r}^{C}_{A_K/C} = \begin{bmatrix} x_j + x_K \\ y_K \\ \mathrm{cat}(x_j) + z_K \end{bmatrix}$$

j: index of local x

K: index of cat in the array

$$\underline{r}^{W}_{jK} = {}^{W}R^{C}\, \underline{r}^{C}_{jK} + \underline{r}^{W}_{C_0/W_0} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_j + x_K \\ y_K \\ \mathrm{cat}(x_j) + z_K \end{pmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Array Model / Point Cloud Matching Cost ("Sample")

$$J(p) = \underbrace{\underline{R}^T \underline{C}}_{\text{pts match cost}} + \underbrace{\underline{p}_e^T Q \underline{p}_e}_{\text{regulation cost}} \qquad \underline{p}_e = \underline{p}_{nom} - \underline{p}$$

$$= \sum_{i=1}^{m} R_i C_i \quad \rightarrow R_i = \frac{1}{m} \rightarrow \text{average cost} \quad \forall i$$

Cost per measurement

$$C_i = \log_{base}\left(1 + \frac{[b\, d_i]^{power}}{l}\right)$$

error/distance to pts

$$d_i = \min_{jk} \left\| \underline{r}_i^w - \underline{r}_{jk}^w \right\| = \underbrace{\min_{k} \left[\overbrace{\min_{j} \left\| \underline{r}_i^w - \underline{r}_{jk}^w \right\|}^{\text{closest distance to each } k \text{ cable}} \right]}_{\text{closest of all cable}}$$

Model points

$$\underline{r}_{jk}^w = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_j + x_k \\ y_k \\ cat(x_j) + z_k \end{pmatrix} + \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Cat

$$z_j = cat(x_j) = a\left[\cosh\left(\frac{x_j}{a}\right) - 1\right]$$

Array Model / Point Cloud Matching Cost

$$J(p) = \underline{R}^T \underline{C} + \underline{p_e}^T Q\, \underline{p_e} \qquad \underline{p_e} = \underline{p}_{nom} - \underline{p}$$

$$\boxed{\frac{\partial J}{\partial \underline{p}}} = \underline{R}^T \frac{\partial \underline{C}}{\partial \underline{p}} + 2\, \underline{p_e}^T Q \frac{\cancel{\partial \underline{p_e}}^{\,-1}}{\partial \underline{p}}$$

$$= \sum_{1}^{m} R_i \left(\frac{\partial C_i}{\partial \underline{p}}\right) - 2\left[\underline{p}_{nom} - \underline{p}\right]^T Q$$

closest point j^* on closest cable k^*

$$\boxed{\frac{\partial C_i}{\partial \underline{p}}} = \frac{\partial C_i}{\partial d_i}\, \frac{\partial d_i}{\partial \underline{e}_i}\, \frac{\partial \underline{e}_i}{\partial \underline{r}_{j^*k^*}}\, \frac{\partial \underline{r}_{j^*k^*}}{\partial \underline{p}}$$

$$C_i = \log_{base}\left(1 + \frac{[b\, d_i]^{power}}{\ell}\right) \quad \Big| \quad \frac{\partial C_i}{\partial d_i} = \frac{1}{\ln(base)} \left(\frac{\cancel{\ell}}{\cancel{\ell} + [b d_i]^{power}}\right) \frac{b}{\cancel{\ell}}\, power\, [b d_i]^{(power-1)}$$

$$= \boxed{\frac{b\, power\, [b\, d_i]^{power-1}}{\ln(base)\left[\ell + [b d_i]^{power}\right]}}$$

$$d_i = \sqrt{\underline{e}_i^T \underline{e}_i} = \left(\underline{e}_i^T \underline{e}_i\right)^{1/2} \quad \Big| \quad \frac{\partial d_i}{\partial \underline{e}_i} = \frac{2\, \underline{e}_i^T}{2\left(\underline{e}_i^T \underline{e}_i\right)^{1/2}} = \boxed{\frac{\underline{e}_i^T}{d_i}}$$

$$\underline{e}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}^w - \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_j + x_k \\ y_k \\ cat(x_j) + z_k \end{pmatrix}^c - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}^w$$

$$\frac{\partial \underline{e}_i}{\partial \underline{p}} = \begin{array}{c} \begin{bmatrix} -1 & 0 & 0 & [x_j + x_k] s\psi + y_k c\psi & 0 & -c\psi & +s\psi & 0 \\ 0 & -1 & 0 & -[x_j + x_k] c\psi + y_k s\psi & 0 & -s\psi & -c\psi & 0 \\ 0 & 0 & -1 & 0 & 1 + \frac{x_j}{a} \sinh\left(\frac{x_j}{a}\right) - \cosh\left(\frac{x}{a}\right) & 0 & 0 & -1 \end{bmatrix} \\ \begin{array}{cccccccc} x_0 & y_0 & z_0 & \psi & a & x_k & y_0 & z_0 \end{array} \end{array}$$

$-{}^wR^c$

Gradient Cost "Sample"

$$\underline{e}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}^W - \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}_\psi \begin{pmatrix} x_j + x_k \\ y_k \\ \text{cat}(x_j) + z_k \end{pmatrix}^C - \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix}^W$$

$$\underline{p}^T = [\, x_o \,;\, y_o \,;\, z_o \,;\, \psi \,;\, a \,;\, \overbrace{d_1 \,;\, d \ldots}^{\text{internal offsets}} \,]$$

$$\frac{\partial \underline{e}_i}{\partial \underline{p}} = \left[\begin{array}{ccc:c:c:c} -1 & 0 & 0 & [x_j + x_k]s\psi + [y_k]c\psi & 0 & \\ 0 & -1 & 0 & -[x_j + x_k]c\psi + [y_k]s\psi & 0 & \\ 0 & 0 & -1 & 0 & 1 + \frac{x_j}{a}\sinh\left(\frac{x_j}{a}\right) - \cosh\left(\frac{x_j}{a}\right) & \end{array}\right] \ldots$$

offsets

$$\underline{r}_k = \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} = f(\overbrace{d_1, \ldots, d_l}^{\text{offsets}})$$

$$\frac{\partial \underline{e}_i}{\partial d_l} = \frac{\partial \underline{e}_i}{\partial \underline{r}_k} \cdot \frac{\partial \underline{r}_k}{\partial d_l} = \overbrace{\underbrace{\begin{bmatrix} -c\psi & +s\psi & 0 \\ -s\psi & -c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{-{}^W R(\psi)^C} \begin{bmatrix} \frac{\partial \underline{r}_k}{\partial d_l} \end{bmatrix}}$$

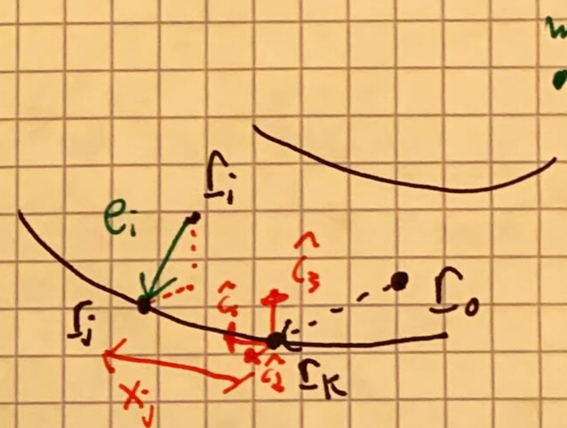
supplied by model

ex:

$$\underline{r}_1 = \begin{bmatrix} 0 \\ d_1 \\ 0 \end{bmatrix} \qquad \frac{\partial \underline{r}_1}{\partial d_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Array Model / Point Cloud Matching cost ("x")

$$J(p) = \underbrace{\underline{R}^T \underline{c}}_{\text{pts match cost}} + \underbrace{\underline{p}_e^T Q\, \underline{p}_e}_{\text{regulation term}} \longrightarrow \underline{p}_e = \underline{p}_{nom} - \underline{p}$$



Cost per measurements

$$c_i = \log_{base}\left(1 + \left[\frac{b\, d_i}{l}\right]^{power}\right)$$

distance to closest catenary

$$d_i = \sqrt{\underline{e}_i^T \underline{e}_i} = \|\underline{e}_i\|$$

error in local cat frame

measurement i

closest point j on curve K

$$d_i = \min_K \|\underline{e}_{iK}\| = \min_K \|\underline{r}_i^{c*} - \underline{r}_{jK}^{c*}\|$$

Closest point on catenary curve K

$$\boxed{x_j^c + x_K^c = x_i^c}$$

$$\underline{r}_{jK}^c = \begin{bmatrix} x_j \\ 0 \\ cat(x_j) \end{bmatrix} + \begin{bmatrix} x_K \\ y_K \\ z_K \end{bmatrix} = \begin{bmatrix} x_i^c \\ y_K \\ cat(x_i^c - x_K^c) + z_K \end{bmatrix}$$

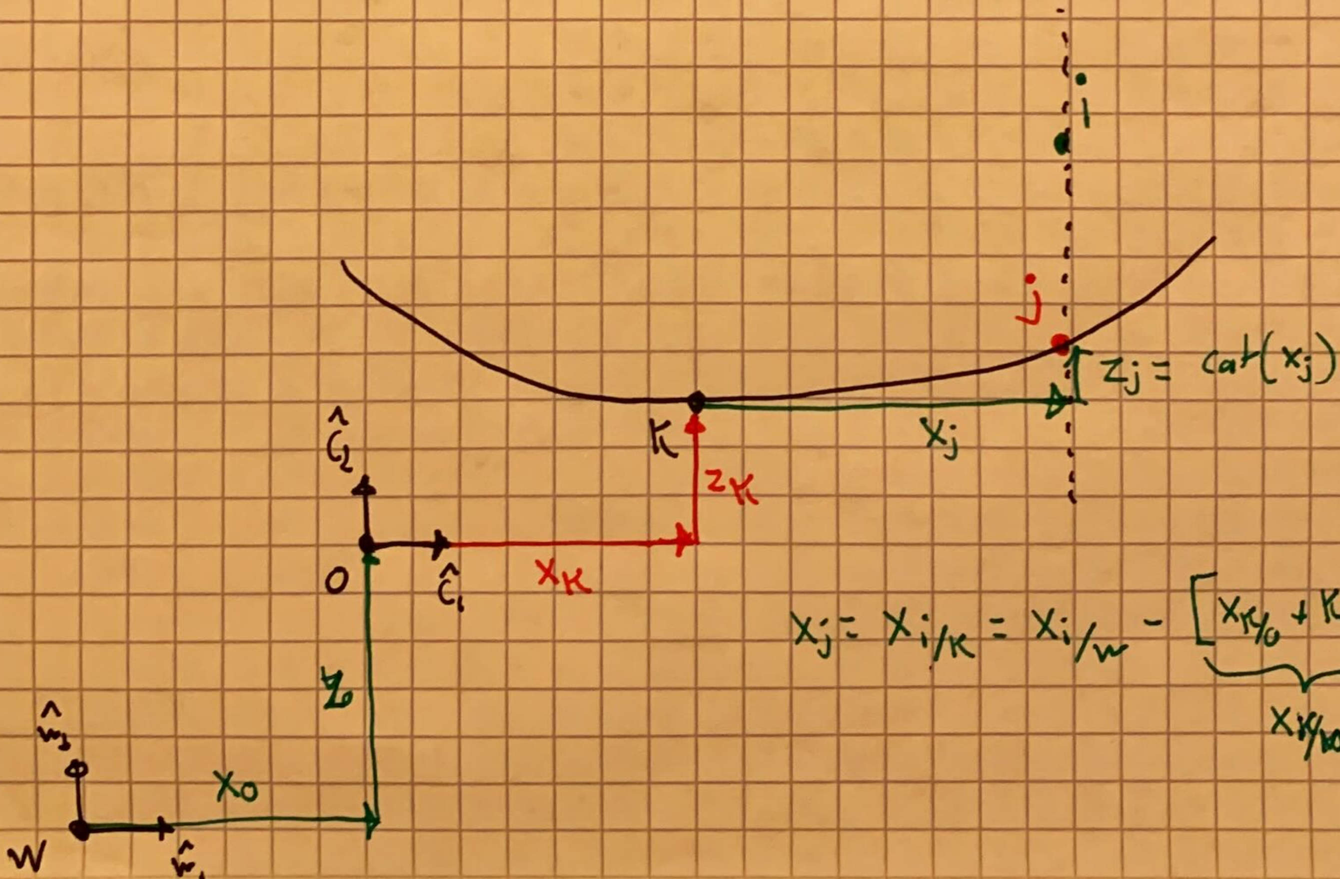
$$\underline{r}_i^c = \begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\underbrace{\begin{bmatrix} x_i^w \\ y_i^w \\ z_i^w \end{bmatrix}}_{\underline{r}_i^w} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right)$$

Array/point cloud matching cost (part 2) ("x")

$$\underline{e}_{ik}^{c} = \underbrace{\begin{bmatrix} c\psi & s\psi & 0 \\ -s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} x_i^w \\ y_i^w \\ z_i^w \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \right)}_{\underline{r}_{i/o}^{c}} - \underbrace{\begin{bmatrix} x_i^c \\ y_k \\ \underbrace{\mathrm{cat}(x_i^c - x_k^c)}_{z_{j/k}} + z_k \end{bmatrix}}_{\underline{r}_{j/o}^{c}}.$$

$$\underline{e}_{ik}^{c} = \begin{bmatrix} 0 \\ -s\psi\,[x_i^w - x_0] + c\psi\,[y_i^w - y_0] - y_k \\ z_i^w - z_k - z_0 - \mathrm{cat}\left(c\psi\,[x_i^w - x_0] + s\psi\,[y_i^w - y_0] - x_k\right) \end{bmatrix}$$

$$\mathrm{cat}(x) = a\left[\cosh\left(\frac{x}{a}\right) - 1\right]$$



$$x_j = x_{i/k} = x_{i/w} - \underbrace{\left[x_{k/o} + x_{o/w}\right]}_{x_{k/w}}$$

Gradient of "x" array cost

→ Same as "sample" up to

$$\frac{\partial c_i}{\partial \underline{p}} = \underbrace{\frac{\partial c_i}{\partial d_i} \frac{\partial d_i}{\partial \underline{e}_i}}_{\text{same}} \underbrace{\frac{\partial \underline{e}_i}{\partial \underline{p}}}_{\text{new}}$$

$z_j = cat(x_j)$

$$\underline{e}_i = \begin{bmatrix} 0 \\ -s\psi[x_i^w - x_0] + c\psi[y_i^w - y_0] - y_k \\ z_i^w - z_k - z_0 - cat(\underbrace{c\psi[x_i^w - x_0] + s\psi[y_i^w - y_0] - x_k}_{x_j}) \end{bmatrix}$$

$$\underline{p}^T = [\, x_0 \;\; y_0 \;\; z_0 \;\vdots\; \psi \;\vdots\; a \;\vdots\; \underbrace{\cdots \;\; d_e \;\; \cdots}_{\text{offsets}} \,]$$

$$\frac{\partial \underline{e}_i}{\partial \underline{p}} = \begin{bmatrix} 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & \vdots \\ s\psi & -c\psi & 0 & \vdots & [x_0 - x_i^w]c\psi + [y_0 - y_i^w]s\psi & \vdots & 0 & \vdots \\ -\frac{\partial z_j}{\partial x_0} & -\frac{\partial z_j}{\partial y_0} & -1 & \vdots & -\frac{\partial z_j}{\partial \psi} & \vdots & -\frac{\partial z_j}{\partial a} & \vdots \end{bmatrix}$$

$$\left[\cdots \quad \frac{\partial \underline{e}_i}{\partial d_e} = \frac{\partial \underline{e}_i}{\partial \underline{r}_k} \underbrace{\frac{\partial \underline{r}_k}{\partial d_e}}_{\text{supplied by model}} \quad \cdots \right]$$

$$\frac{\partial \underline{e}_i}{\partial \underline{r}_k} \rightarrow \begin{bmatrix} \frac{\partial \underline{e}_i}{\partial x_k} & \vdots & \frac{\partial \underline{e}_i}{\partial y_k} & \vdots & \frac{\partial \underline{e}_i}{\partial z_k} \end{bmatrix}$$

$$\frac{\partial \underline{e}_i}{\partial x_k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -\frac{\partial z_j}{\partial x_k} & 0 & -1 \end{bmatrix}$$

$$z_j = \mathrm{cat}(x_j) = a\left[\cosh\left(\frac{x_j}{a}\right) - 1\right]$$

$$x_j = c\psi\left[x_i^w - x_0\right] + s\psi\left[y_i^w - y_0\right] - x_k$$

$$\frac{\partial z_j}{\partial a} = \frac{\partial\left[a\cosh\left(\frac{x_j}{a}\right) - a\right]}{\partial a} = \cosh\left(\frac{x_j}{a}\right) + a\frac{\partial\left[\cosh\left(\frac{x_j}{a}\right)\right]}{\partial a} - 1 = \cosh\left(\frac{x_j}{a}\right) - \frac{x_j}{a}\sinh\left(\frac{x_j}{a}\right) - 1$$

$$\frac{\partial z_j}{\partial x_j} = \frac{\partial\left[a\cosh\left(\frac{x_j}{a}\right) - a\right]}{\partial x_j} = \sinh\left(\frac{x_j}{a}\right)$$

$$\frac{\partial x_j}{\partial x_0} = -c\psi$$

$$\frac{\partial x_j}{\partial y_0} = -s\psi$$

$$\frac{\partial x_j}{\partial \psi} = s\psi\left[x_0 - x_i^w\right] + c\psi\left[y_i^w - y_0\right] = -s\psi\left[x_i^w - x_0\right] + c\psi\left[y_i^w - y_0\right]$$

$$\frac{\partial x_j}{\partial x_k} = -1$$