

S04: Control de Error y RK-F

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Recordando

En los métodos anteriores que estudiamos, utilizamos una aproximación a una EDO a un paso y dada por

$$w_{i+1} = w_i + h_i \phi(t_i, w_i, h_i)$$

sin embargo en estos métodos descuidamos el error el cual puede crecer a medida que nos alejamos de las condiciones iniciales.

Un método **ideal** para aproximar estas soluciones tendría que considerar lo anterior

$$w_{i+1} = w_i + h_i \phi(t_i, w_i, h_i)$$

y además convinar una tolerancia

$$|y(t_i) - w_i| \leq \varepsilon$$

(una cota global del error) no importando si eso reduce el número de muestras que aproximemos, siempre y cuando tengamos la tolerancia.

Desarrollemos el algoritmo

Algunas suposiciones

Supongamos que tenemos aproximaciones w_i y \tilde{w}_i que corresponden a métodos de orden 4 y 5, respectivamente, es decir,

$$w_{i+1} = w_i + h\phi(t_i, w_i, h) + O(h^4)$$

$$\tilde{w}_{i+1} = \tilde{w}_i + h\psi(t_i, \tilde{w}_i, h) + O(h^5).$$

Denotaremos por $\tau_i(h)$ el error en el primer método y $\tilde{\tau}_i(h)$ en el segundo método. Esto es lo mismo que

$$\tau_i(h) = \frac{y(t_i) - w_i}{h}$$

$$\tilde{\tau}_i(h) = \frac{y(t_i) - \tilde{w}_i}{h}.$$

Pero (con un poco de algebra) esto es equivalente a

$$\tau_i(h) = \tilde{\tau}_i(h) + \frac{\tilde{w}_i - w_i}{h}.$$

Por lo que el error local de truncamiento es (aproximadamente)

$$\tau_{i+1} = \frac{\tilde{w}_{i+1} - w_{i+1}}{h}.$$

Más aún no queremos restringirnos a estimar el error, si no adaptar también el tamaño de paso para que este error tenga la tolerancia global.

Si supongo que mi error se comporta como $\tau_{i+1}(h) \approx Kh^n$, entonces $\tau_{i+1}(qh) \approx \frac{q^n}{h} \tau_{i+1}(h)$, por lo que mi tolerancia debe escogerse basada en:

$$\frac{q^n}{h} |\tilde{w}_{i+1} - w_{i+1}| \leq \varepsilon$$

por lo que el factor de aumento de paso debe cumplir

$$q \leq \left(\frac{\varepsilon h}{|\tilde{w}_{i+1} - w_{i+1}|} \right)^{1/n}.$$

El método de Runge-Kutta-Fehlberg.

El método utiliza Runge-Kutta con error de truncamiento de orden 5, por lo las dos aproximaciones que utilizaremos estarán dadas por:

$$\tilde{w}_{i+1} = w_i + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

$$w_{i+1} = w_i + \frac{2}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5.$$

donde los parámetros están dados por:

$$\begin{aligned} k_1 &= hf(t_i, w_i) \\ k_2 &= hf\left(t_i + \frac{h}{4}, w_i + \frac{1}{4}k_1\right) \\ k_3 &= hf\left(t_i + \frac{3h}{8}, w_i + \frac{3}{32}k_1 + \frac{9}{32}k_2\right) \\ k_4 &= hf\left(t_i + \frac{12h}{13}, w_i + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \\ k_5 &= hf\left(t_i + h, w_i + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \\ k_6 &= hf\left(t_i + \frac{h}{2}, w_i - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right) \end{aligned}$$

¿Cómo determinar el tamaño del paso y qué debe satisfacer computacionalmente?

Inicialmente comenzaremos con un tamaño de paso h para calcular w_{i+1} y \tilde{w}_{i+1} , lo cual nos determinará el q en ese paso. El incremento q debe satisfacer: * Cuando $q < 1$: rechazar el valor inicial de h en el paso i y repetir los calculos. * Cuando $q \leq 1$: aceptar el valor calculado en el paso i usando h , y cambiar el tamaño de paso a qh para el paso $i + 1$.

El pseudo-codigo

Objetivo: aproximar la solución al problema de valores iniciales

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

Entrada: * a, b puntos finales, * α condición inicial, * Tol la tolerancia en nuestro método, * $hmax$ tamaño de paso máximo, * $hmin$ tamaño de paso mínimo.

Salida: * t tiempo muestro, * w aproximación solución, * h tamaño de paso usado.

Paso 1: Tomemos $t = a$, $w = \alpha$, $h = hmax$ y $Flag = 1$.

Paso 2: While(Flag==1) haga pasos 3-11: *Paso 3:* Calcule los parametros k_1, \dots, k_6 . *Paso 4:* Tome

$$error = \frac{1}{h} \left| \frac{1}{360}k_1 - \frac{128}{4275}k_3 - \frac{2197}{75240}k_4 + \frac{1}{50}k_5 + \frac{2}{55}k_6 \right|$$

Paso 5: If($error \leq Tol$) haga paso 6 y 7: Paso 6: Tome $t = t + h$. Tome

$$w = w + \frac{2}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

Paso 7: guarde (t, w, h) Paso 8: Tomemos $q = 0.84(Tol/error)^{1/4}$. Paso 9: Si $q \leq 0.1$ tome $h = 0.1h$, si $q \geq 4$ entonces $h = 4h$, de lo contrario $h = qh$. Paso 10: si $h > hmax$, entonces $h = hmax$. Paso 11: Si $t \leq b$, entonces $Flag = 0$, de lo contrario si $t + h > b$, entonces $h = b - t$, si $h < hmin$ $Flag = 0$ y Output("h minimo excedido"). Paso 12: Terminar proceso.

```
RKF <- function(dy,dx,y,hmax=0.1,hmin=0.0001,start=0.0,end=2.0,Tol=0.0000001,y0=0.5){
  t<- start
  w <- y0
  h <- hmax
  Flag <- 1
  sols <- numeric(length=(end-start)/hmin)
  real <- numeric(length=(end-start)/hmin)
  error <- numeric(length=(end-start)/hmin)
  sols[1]<-w
  error[1]<-w
  times <- numeric(length=(end-start)/hmin)
  times[1] <- t
  i<-1
  while (Flag==1) {
    k1<-h*dy.dx(t,w)
    k2<-h*dy.dx(t+h/4,w+k1/4)
    k3<- h*dy.dx(t+3*h/8, w+3*k1/32+9*k2/32)
    k4<- h*dy.dx(t+12*h/13,w+1932*k1/2197-7200*k2/2197+7296*k3/2197)
    k5<-h*dy.dx(t+h, w+439*k1/216-8*k2+3680*k3/513-845*k4/4140)
    k6<- h*dy.dx(t+h/2,w-8*k1/27+2*k2-3544*k3/2565+1859*k4/4104-11*k5/40)
    Error <- abs(k1/360-128*k3/4275-2197*k4/75240+k5/50+2*k6/55)/h
    if(Error<=Tol){
      t=t+h
      times[1+i]<-t
      w=w+25*k1/216+1408*k3/2565+2197*k4/4104-k5/5
      sols[1+i]<- w
      real[1+i]<- y(times[1+i])
      error[i+1]<- abs(w-real[1+i])
      i<-i+1
    }
  }
  d<- 0.84*((Tol/Error)^0.25)
  if(d<=0.1){
    h <- 0.1*h}
  else{
    if(d>=4){
      h <- 4*h }
    else{
      h <- d*h
    }
  }
  if(h>hmax){
    h<-hmax
  }
  if(t>=end){
    Flag <- 0}
  else{
```

```

    if(t+h>end) {
      h <- end-t
    } else{
      if(h<hmin){
        Flag <- 0
        return('tamaño de paso menor al minimo. Proceso terminado insatisfactoriamente')
      }
    }
  }
}

df <- data.frame(Tn=times, Yn=sols, Yo=real, ErrorRKF=error)
return(df)
}

```

1. Vamos a trabajar con

$y' = y/t - (y/t)^2$ con $1 \leq t \leq 4$, $y(1) = 1$ y solución real: $y(t) = \frac{t}{1+\ln(t)}$.

```

library(tidyverse)

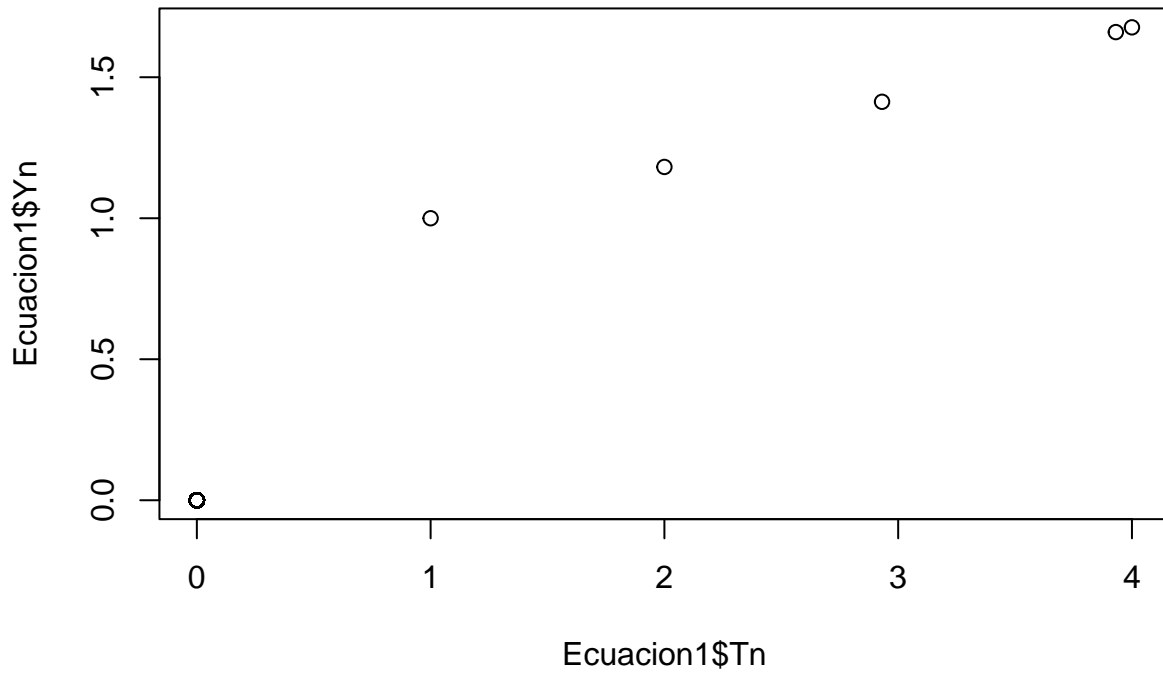
## -- Attaching packages ----- tidyverse 1.3.2 --
## v ggplot2 3.3.6      v purrr 0.3.5
## v tibble 3.1.8       v dplyr 1.0.10
## v tidyr 1.2.1        v stringr 1.4.1
## v readr 2.1.3        v forcats 0.5.2
## -- Conflicts ----- tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag() masks stats::lag()

library(latex2exp)
library(gridExtra)

##
## Attaching package: 'gridExtra'
##
## The following object is masked from 'package:dplyr':
##
## combine

dy.dt<-function(t,y){y/t-(y/t)^2}
y<-function(t){t/(1+log(t))}
Ecuacion1<-RKF(dy.dt,y,hmax=1,hmin=.0001,start=1.0,end=4.0,Tol=10E-4,y0=1.0)
plot(Ecuacion1$Tn,Ecuacion1$Yn)

```



```
dy.dt<-function(t,y){1+y/t+(y/t)^2}
y<-function(t){t*tan(log(t))}
RKF(dy.dt,y,hmax=0.01,hmin=0.000001,start=1.0,end=3.0,Tol=0.0001,y0=0.0)
```

##	Tn	Yn	Yo	ErrorRKF
## 1	1.00	0.00000000	0.00000000	0.000000e+00
## 2	1.01	0.01005013	0.01005017	3.649517e-08
## 3	1.02	0.02020125	0.02020132	7.407309e-08
## 4	1.03	0.03045432	0.03045444	1.127488e-07
## 5	1.04	0.04081032	0.04081047	1.525375e-07
## 6	1.05	0.05127017	0.05127036	1.934550e-07
## 7	1.06	0.06183481	0.06183504	2.355173e-07
## 8	1.07	0.07250514	0.07250542	2.787407e-07
## 9	1.08	0.08328209	0.08328242	3.231419e-07
## 10	1.09	0.09416655	0.09416692	3.687380e-07
## 11	1.10	0.10515940	0.10515982	4.155463e-07
## 12	1.11	0.11626153	0.11626199	4.635845e-07
## 13	1.12	0.12747382	0.12747433	5.128707e-07
## 14	1.13	0.13879713	0.13879769	5.634233e-07
## 15	1.14	0.15023234	0.15023296	6.152611e-07
## 16	1.15	0.16178032	0.16178099	6.684032e-07
## 17	1.16	0.17344192	0.17344264	7.228691e-07
## 18	1.17	0.18521801	0.18521878	7.786786e-07
## 19	1.18	0.19710944	0.19711028	8.358520e-07
## 20	1.19	0.20911709	0.20911799	8.944098e-07
## 21	1.20	0.22124182	0.22124277	9.543730e-07

## 22	1.21	0.23348448	0.23348550	1.015763e-06
## 23	1.22	0.24584594	0.24584702	1.078602e-06
## 24	1.23	0.25832708	0.25832822	1.142911e-06
## 25	1.24	0.27092876	0.27092996	1.208714e-06
## 26	1.25	0.28365185	0.28365313	1.276032e-06
## 27	1.26	0.29649724	0.29649858	1.344891e-06
## 28	1.27	0.30946581	0.30946722	1.415313e-06
## 29	1.28	0.32255844	0.32255993	1.487322e-06
## 30	1.29	0.33577604	0.33577760	1.560945e-06
## 31	1.30	0.34911950	0.34912113	1.636204e-06
## 32	1.31	0.36258972	0.36259143	1.713127e-06
## 33	1.32	0.37618762	0.37618941	1.791739e-06
## 34	1.33	0.38991411	0.38991599	1.872067e-06
## 35	1.34	0.40377014	0.40377209	1.954138e-06
## 36	1.35	0.41775662	0.41775865	2.037979e-06
## 37	1.36	0.43187450	0.43187662	2.123618e-06
## 38	1.37	0.44612473	0.44612694	2.211084e-06
## 39	1.38	0.46050827	0.46051057	2.300406e-06
## 40	1.39	0.47502610	0.47502849	2.391614e-06
## 41	1.40	0.48967918	0.48968166	2.484737e-06
## 42	1.41	0.50446851	0.50447109	2.579806e-06
## 43	1.42	0.51939508	0.51939776	2.676853e-06
## 44	1.43	0.53445991	0.53446269	2.775910e-06
## 45	1.44	0.54966401	0.54966689	2.877008e-06
## 46	1.45	0.56500841	0.56501139	2.980180e-06
## 47	1.46	0.58049416	0.58049724	3.085461e-06
## 48	1.47	0.59612230	0.59612549	3.192884e-06
## 49	1.48	0.61189390	0.61189720	3.302485e-06
## 50	1.49	0.62781004	0.62781345	3.414298e-06
## 51	1.50	0.64387180	0.64387533	3.528360e-06
## 52	1.51	0.66008030	0.66008394	3.644708e-06
## 53	1.52	0.67643663	0.67644040	3.763379e-06
## 54	1.53	0.69294194	0.69294582	3.884411e-06
## 55	1.54	0.70959736	0.70960137	4.007844e-06
## 56	1.55	0.72640405	0.72640818	4.133716e-06
## 57	1.56	0.74336317	0.74336743	4.262068e-06
## 58	1.57	0.76047592	0.76048031	4.392942e-06
## 59	1.58	0.77774348	0.77774801	4.526379e-06
## 60	1.59	0.79516708	0.79517174	4.662421e-06
## 61	1.60	0.81274794	0.81275274	4.801113e-06
## 62	1.61	0.83048731	0.83049225	4.942498e-06
## 63	1.62	0.84838644	0.84839153	5.086622e-06
## 64	1.63	0.86644661	0.86645185	5.233530e-06
## 65	1.64	0.88466913	0.88467451	5.383270e-06
## 66	1.65	0.90305528	0.90306082	5.535889e-06
## 67	1.66	0.92160641	0.92161210	5.691435e-06
## 68	1.67	0.94032385	0.94032970	5.849959e-06
## 69	1.68	0.95920897	0.95921499	6.011511e-06
## 70	1.69	0.97826315	0.97826933	6.176141e-06
## 71	1.70	0.99748779	0.99749413	6.343904e-06
## 72	1.71	1.01688429	1.01689080	6.514852e-06
## 73	1.72	1.03645410	1.03646079	6.689039e-06
## 74	1.73	1.05619867	1.05620554	6.866522e-06
## 75	1.74	1.07611948	1.07612653	7.047356e-06

## 76	1.75	1.09621802	1.09622525	7.231600e-06
## 77	1.76	1.11649581	1.11650323	7.419313e-06
## 78	1.77	1.13695438	1.13696199	7.610554e-06
## 79	1.78	1.15759528	1.15760309	7.805384e-06
## 80	1.79	1.17842010	1.17842810	8.003866e-06
## 81	1.80	1.19943043	1.19943864	8.206064e-06
## 82	1.81	1.22062791	1.22063632	8.412042e-06
## 83	1.82	1.24201416	1.24202278	8.621866e-06
## 84	1.83	1.26359086	1.26359970	8.835604e-06
## 85	1.84	1.28535970	1.28536876	9.053325e-06
## 86	1.85	1.30732240	1.30733167	9.275098e-06
## 87	1.86	1.32948068	1.32949018	9.500996e-06
## 88	1.87	1.35183633	1.35184606	9.731091e-06
## 89	1.88	1.37439111	1.37440108	9.965457e-06
## 90	1.89	1.39714685	1.39715706	1.020417e-05
## 91	1.90	1.42010539	1.42011584	1.044731e-05
## 92	1.91	1.44326859	1.44327928	1.069495e-05
## 93	1.92	1.46663834	1.46664929	1.094718e-05
## 94	1.93	1.49021657	1.49022777	1.120407e-05
## 95	1.94	1.51400522	1.51401668	1.146572e-05
## 96	1.95	1.53800626	1.53801799	1.173220e-05
## 97	1.96	1.56222171	1.56223371	1.200360e-05
## 98	1.97	1.58665359	1.58666587	1.228002e-05
## 99	1.98	1.61130396	1.61131653	1.256154e-05
## 100	1.99	1.63617493	1.63618778	1.284825e-05
## 101	2.00	1.66126862	1.66128176	1.314026e-05
## 102	2.01	1.68658717	1.68660061	1.343765e-05
## 103	2.02	1.71213278	1.71214652	1.374054e-05
## 104	2.03	1.73790767	1.73792172	1.404900e-05
## 105	2.04	1.76391409	1.76392846	1.436316e-05
## 106	2.05	1.79015434	1.79016902	1.468312e-05
## 107	2.06	1.81663071	1.81664572	1.500898e-05
## 108	2.07	1.84334559	1.84336093	1.534085e-05
## 109	2.08	1.87030135	1.87031703	1.567884e-05
## 110	2.09	1.89750042	1.89751645	1.602307e-05
## 111	2.10	1.92494528	1.92496165	1.637366e-05
## 112	2.11	1.95263841	1.95265514	1.673072e-05
## 113	2.12	1.98058237	1.98059946	1.709438e-05
## 114	2.13	2.00877972	2.00879718	1.746476e-05
## 115	2.14	2.03723309	2.03725093	1.784198e-05
## 116	2.15	2.06594514	2.06596337	1.822619e-05
## 117	2.16	2.09491857	2.09493719	1.861750e-05
## 118	2.17	2.12415612	2.12417514	1.901605e-05
## 119	2.18	2.15366057	2.15368000	1.942199e-05
## 120	2.19	2.18343476	2.18345460	1.983546e-05
## 121	2.20	2.21348156	2.21350181	2.025659e-05
## 122	2.21	2.24380388	2.24382456	2.068554e-05
## 123	2.22	2.27440469	2.27442581	2.112246e-05
## 124	2.23	2.30528700	2.30530856	2.156750e-05
## 125	2.24	2.33645386	2.33647588	2.202082e-05
## 126	2.25	2.36790840	2.36793088	2.248258e-05
## 127	2.26	2.39965376	2.39967671	2.295295e-05
## 128	2.27	2.43169315	2.43171659	2.343209e-05
## 129	2.28	2.46402984	2.46405376	2.392018e-05

## 130	2.29	2.49666713	2.49669155	2.441740e-05
## 131	2.30	2.52960840	2.52963332	2.492391e-05
## 132	2.31	2.56285706	2.56288250	2.543992e-05
## 133	2.32	2.59641659	2.59644255	2.596560e-05
## 134	2.33	2.63029053	2.63031703	2.650115e-05
## 135	2.34	2.66448246	2.66450951	2.704677e-05
## 136	2.35	2.69899605	2.69902365	2.760266e-05
## 137	2.36	2.73383500	2.73386317	2.816902e-05
## 138	2.37	2.76900308	2.76903183	2.874607e-05
## 139	2.38	2.80450415	2.80453348	2.933402e-05
## 140	2.39	2.84034209	2.84037202	2.993309e-05
## 141	2.40	2.87652088	2.87655142	3.054352e-05
## 142	2.41	2.91304454	2.91307571	3.116553e-05
## 143	2.42	2.94991720	2.94994900	3.179936e-05
## 144	2.43	2.98714301	2.98717545	3.244525e-05
## 145	2.44	3.02472622	3.02475932	3.310346e-05
## 146	2.45	3.06267115	3.06270493	3.377423e-05
## 147	2.46	3.10098220	3.10101665	3.445783e-05
## 148	2.47	3.13966382	3.13969898	3.515452e-05
## 149	2.48	3.17872057	3.17875644	3.586459e-05
## 150	2.49	3.21815708	3.21819367	3.658830e-05
## 151	2.50	3.25797804	3.25801536	3.732595e-05
## 152	2.51	3.29818825	3.29822632	3.807782e-05
## 153	2.52	3.33879257	3.33883142	3.884423e-05
## 154	2.53	3.37979598	3.37983561	3.962548e-05
## 155	2.54	3.42120352	3.42124394	4.042188e-05
## 156	2.55	3.46302033	3.46306156	4.123377e-05
## 157	2.56	3.50525163	3.50529369	4.206147e-05
## 158	2.57	3.54790276	3.54794566	4.290532e-05
## 159	2.58	3.59097913	3.59102290	4.376567e-05
## 160	2.59	3.63448627	3.63453091	4.464289e-05
## 161	2.60	3.67842979	3.67847533	4.553733e-05
## 162	2.61	3.72281543	3.72286188	4.644938e-05
## 163	2.62	3.76764901	3.76769639	4.737942e-05
## 164	2.63	3.81293646	3.81298479	4.832784e-05
## 165	2.64	3.85868385	3.85873314	4.929506e-05
## 166	2.65	3.90489732	3.90494760	5.028148e-05
## 167	2.66	3.95158315	3.95163444	5.128755e-05
## 168	2.67	3.99874775	3.99880006	5.231368e-05
## 169	2.68	4.04639762	4.04645098	5.336034e-05
## 170	2.69	4.09453940	4.09459383	5.442799e-05
## 171	2.70	4.14317986	4.14323538	5.551710e-05
## 172	2.71	4.19232590	4.19238252	5.662816e-05
## 173	2.72	4.24198454	4.24204230	5.776166e-05
## 174	2.73	4.29216295	4.29222187	5.891813e-05
## 175	2.74	4.34286844	4.34292853	6.009808e-05
## 176	2.75	4.39410844	4.39416974	6.130206e-05
## 177	2.76	4.44589056	4.44595309	6.253062e-05
## 178	2.77	4.49822254	4.49828632	6.378434e-05
## 179	2.78	4.55111227	4.55117733	6.506379e-05
## 180	2.79	4.60456780	4.60463417	6.636959e-05
## 181	2.80	4.65859736	4.65866506	6.770235e-05
## 182	2.81	4.71320931	4.71327837	6.906271e-05
## 183	2.82	4.76841222	4.76848267	7.045132e-05

## 184	2.83	4.82421479	4.82428666	7.186885e-05
## 185	2.84	4.88062595	4.88069926	7.331599e-05
## 186	2.85	4.93765475	4.93772954	7.479346e-05
## 187	2.86	4.99531048	4.99538678	7.630199e-05
## 188	2.87	5.05360260	5.05368044	7.784232e-05
## 189	2.88	5.11254077	5.11262018	7.941523e-05
## 190	2.89	5.17213484	5.17221586	8.102151e-05
## 191	2.90	5.23239489	5.23247755	8.266199e-05
## 192	2.91	5.29333120	5.29341553	8.433750e-05
## 193	2.92	5.35495426	5.35504031	8.604891e-05
## 194	2.93	5.41727481	5.41736260	8.779711e-05
## 195	2.94	5.48030379	5.48039337	8.958302e-05
## 196	2.95	5.54405239	5.54414380	9.140759e-05
## 197	2.96	5.60853206	5.60862533	9.327178e-05
## 198	2.97	5.67375446	5.67384964	9.517661e-05
## 199	2.98	5.73973155	5.73982867	9.712309e-05
## 200	2.99	5.80647551	5.80657463	9.911230e-05
## 201	3.00	5.87399883	5.87409998	1.011453e-04
## 202	3.00	5.87399883	5.87409998	1.011453e-04
## 203	0.00	0.00000000	0.00000000	0.000000e+00
## 204	0.00	0.00000000	0.00000000	0.000000e+00
## 205	0.00	0.00000000	0.00000000	0.000000e+00
## 206	0.00	0.00000000	0.00000000	0.000000e+00
## 207	0.00	0.00000000	0.00000000	0.000000e+00
## 208	0.00	0.00000000	0.00000000	0.000000e+00
## 209	0.00	0.00000000	0.00000000	0.000000e+00
## 210	0.00	0.00000000	0.00000000	0.000000e+00
## 211	0.00	0.00000000	0.00000000	0.000000e+00
## 212	0.00	0.00000000	0.00000000	0.000000e+00
## 213	0.00	0.00000000	0.00000000	0.000000e+00
## 214	0.00	0.00000000	0.00000000	0.000000e+00
## 215	0.00	0.00000000	0.00000000	0.000000e+00
## 216	0.00	0.00000000	0.00000000	0.000000e+00
## 217	0.00	0.00000000	0.00000000	0.000000e+00
## 218	0.00	0.00000000	0.00000000	0.000000e+00
## 219	0.00	0.00000000	0.00000000	0.000000e+00
## 220	0.00	0.00000000	0.00000000	0.000000e+00
## 221	0.00	0.00000000	0.00000000	0.000000e+00
## 222	0.00	0.00000000	0.00000000	0.000000e+00
## 223	0.00	0.00000000	0.00000000	0.000000e+00
## 224	0.00	0.00000000	0.00000000	0.000000e+00
## 225	0.00	0.00000000	0.00000000	0.000000e+00
## 226	0.00	0.00000000	0.00000000	0.000000e+00
## 227	0.00	0.00000000	0.00000000	0.000000e+00
## 228	0.00	0.00000000	0.00000000	0.000000e+00
## 229	0.00	0.00000000	0.00000000	0.000000e+00
## 230	0.00	0.00000000	0.00000000	0.000000e+00
## 231	0.00	0.00000000	0.00000000	0.000000e+00
## 232	0.00	0.00000000	0.00000000	0.000000e+00
## 233	0.00	0.00000000	0.00000000	0.000000e+00
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```
## 24970 0.00 0.00000000 0.00000000 0.000000e+00
## 24971 0.00 0.00000000 0.00000000 0.000000e+00
## 24972 0.00 0.00000000 0.00000000 0.000000e+00
## 24973 0.00 0.00000000 0.00000000 0.000000e+00
## 24974 0.00 0.00000000 0.00000000 0.000000e+00
## 24975 0.00 0.00000000 0.00000000 0.000000e+00
## 24976 0.00 0.00000000 0.00000000 0.000000e+00
## 24977 0.00 0.00000000 0.00000000 0.000000e+00
## 24978 0.00 0.00000000 0.00000000 0.000000e+00
## 24979 0.00 0.00000000 0.00000000 0.000000e+00
## 24980 0.00 0.00000000 0.00000000 0.000000e+00
## 24981 0.00 0.00000000 0.00000000 0.000000e+00
## 24982 0.00 0.00000000 0.00000000 0.000000e+00
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## 24984 0.00 0.00000000 0.00000000 0.000000e+00
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## 24986 0.00 0.00000000 0.00000000 0.000000e+00
## 24987 0.00 0.00000000 0.00000000 0.000000e+00
## 24988 0.00 0.00000000 0.00000000 0.000000e+00
## 24989 0.00 0.00000000 0.00000000 0.000000e+00
## 24990 0.00 0.00000000 0.00000000 0.000000e+00
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## 24994 0.00 0.00000000 0.00000000 0.000000e+00
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## 24998 0.00 0.00000000 0.00000000 0.000000e+00
## 24999 0.00 0.00000000 0.00000000 0.000000e+00
## [ reached 'max' / getOption("max.print") -- omitted 1975001 rows ]
```