Chapter 1

Series

Theorem 1.1. *A series s_n is convergent if and only if for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that:*

$$m, n \geq N \implies \left| \sum_{i=n} m a_m \right| \leq \epsilon$$

Proof. Obviously any convergent sequence is Cauchy. Furthermore, due to completeness of $\mathbb{R}$ every Cauchy sequence is convergent. ■

Corollary 1.2. *The series s_n is convergent if $a_n \to 0$.*

Theorem 1.3.

1. *If $|a_n| < b_n$ for all $n > N$ for a sufficiently large N then convergence of $\sum b_n$ implies the convergence $\sum a_n$.*

2. *If $0 < b_n < a_n$ for all $n > N$ for a sufficiently large N then divergence of $\sum b_n$ implies the divergence $\sum a_n$.*

Corollary 1.4. *Absolute convergence implies convergence.*

Theorem 1.5 (Integral Test). *Consider the improper integral $\int_0^\infty f$ and the series $\sum_{i=1}^\infty a_k$*

1. *$0 \leq a_k \leq f(x)$ for sufficiently large k and each $x \in \,]k-1, k]$, then the convergence of integral implies the convergence of the series.*

2. *Similarly if $0 \leq f(x) \leq a_k$ for sufficiently large k and each $x \in \,[k, k+1[$, then the divergence of integral implies the divergence of the series.*

Definition: The exponential growth rate of the series $\sum a_n$

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{a_k}$$

Theorem 1.6 (Root test). *If $\alpha < 1$ the series is convergent and if $\alpha > 1$ it is divergent. If $\alpha = 1$ the test is inconclusive.*

Theorem 1.7 (Ratio test). *Let the ratio between successive terms of the series a_k be $r_k = |\frac{a_{k+1}}{a_k}|$*

$$\rho = \limsup r_k \qquad \lambda = \liminf r_k$$

If $\rho < 1$ then the series converges, if $\lambda > 1$ then the series diverges, and otherwise the ratio test is inconclusive.

Theorem 1.8. *Let $a_1 \geq a_2 \geq \cdots \geq 0$ be a decreasing non-negative sequence then the alternating series*

$$\sum_{n=1}^{\infty} a_n (-1)^n \tag{1.1}$$

is convergent.

Theorem 1.9. *Suppose $\sum c_k x^k$ is a power series. Its radius of convergence R is unique and is such that for $|x| < R$ the power series converges and for $|x| > R$ diverges.*

$$R = \frac{1}{\limsup \sqrt[k]{c_k}}$$