Contents

Chapter 1

Real Numbers

1.1 Axiomatic Formulation of Real Numbers

The building axioms of real numbers is devided into three groups based on the properties they are describing.

1. Field axioms.

2. Order axioms.

3. Completeness axiom.

1.1.1 Field Axioms

A field is a non-empty set $\mathbb{F}$ with two binary operations *addition*, $+$, and *multiplication*, $\cdot$. For all $x, y, z \in \mathbb{F}$:

Axiom 1. Addition and multiplication are commutitive.

$$x + y = y + x, \quad x \cdot y = y \cdot x$$

Axiom 2. Addition and multiplication are associative.

$$x + (y + z) = (x + y) + z, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Axiom 3. Multiplication distributes over addition.

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Axiom 4. There exists a number 0 such that for every number x:

$$x + 0 = 0 + x = x$$

Axiom 5. There exists a number 1 such that for every number x:

$$x \cdot 1 = 1 \cdot x = x$$

Axiom 6. For every number x, there exists a number y such that:

$$x + y = 0$$

y is called the negative of x and is denoted by $-x$.

Axiom 7. For every number $x \neq 0$, there exists a number y such that:

$$x \cdot y = 1$$

y is called the reciprocal of x and is denoted by x^{-1} or $\dfrac{1}{x}$.

1.1.2 Order Axioms

The order axioms establishes an ordering on the numbers of $\mathbb{F}$ to determine which element is larger or smaller. To achieve an ordering, we define the set of positive real numbers $\mathbb{F}^+ \subset \mathbb{F}$.

Axiom 8. The $\mathbb{F}^+$ is closed under addition and multiplication.

$$\forall x, y \in \mathbb{F}^+, \quad (x + y) \in \mathbb{F}^+ \text{ and } (x \cdot y) \in \mathbb{F}^+$$

Axiom 9. $0 \notin \mathbb{F}^+$.

Axiom 10. For every number $x \neq 0$, either $x \in \mathbb{F}^+$ or $-x \in \mathbb{F}^+$.

We then define the binary operator $>$ such that $x > y$ whenever $(x - y) \in \mathbb{F}^+$.

1.1.3 Completeness Axiom

Given that $(\mathbb{F}, +, \cdot, >)$ is an ordered field, we define the followings:

Definition (Upper bound): A set $S \in \mathbb{F}$ has an upper bound if for some $a \in \mathbb{F}$ is greater or equal to all element of S. That is, $\forall x \in S \; a \geq x$. We say that S is bounded from above.

Definition (The least upper bound): $a \in \mathbb{F}$ is the least upper bound of a set $S \subset \mathbb{F}$ if it is smaller than every upper bound of S. We say a is the supremum of S, denoted by $a = \sup S$.

Note that, if the least upper bound exists, it must be unique.

Axiom 11. If S is a non-empty set that bounded from above that it has supremum.

Theorem 1.1. *There exists a unique set that satisifies all the axioms above. It is denoted by $\mathbb{R}$, the set of real numbers.*

Proof. The existence of $\mathbb{R}$ is proved in many ways. One way to construct real numbers uses *Dedekind Cuts*. Let the pair of rational sets (A, B) be a partition of $\mathbb{Q}$ such that:

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

2. $\forall x, y \in \mathbb{Q}$ s.t. $x < y$, $y \in A \implies x \in A$.

3. $\nexists x \in A$ s.t. $\forall y \in A, x \geq y$.

For convenience we let A represent the pair (A, B) as A completely determines B. We define $+$, $\cdot$, and $>$ as follows:

$$\begin{aligned} A + B &= \{a + b : a \in A, b \in B\} \\ \nvdash &= \{a : a < 0\} \\ -A &= \{a' : \forall a \in A, a' < -a\} \end{aligned}$$

For $\cdot$, we first take two set A and B that have some positive elements.

$$A \cdot B = \{a \cdot b : a \in A \wedge a \leq 0, b \in B \wedge b \leq 0\} \cup \nvdash$$

If A or B did not have any positive elements, we first take the negative of the set, and then multiply the two sets and take the negative of the product. Similarly, we define the reciprocal of A if A has a positive element.

$$\begin{aligned} \nVdash &= \{a : a < 1\} \\ A^{-1} &= \{a' : \forall a \in A, a > 0, a' < \frac{1}{a}\} \end{aligned}$$

Lastly:

$$A > B \text{ if } A \supset B$$

Also, if a non-empty set S of real numbers is bounded from above, then it has a supremum in $\mathbb{R}$ equal to $\bigcup S$. It is left to the reader that the $(\mathbb{R}, +, \cdot, >)$ satisifies the axioms above.

The set of real numbers is unique in sense that if $(\mathbb{R}, +, \cdot, >)$ and $(\mathbb{R}', +', \cdot', >')$ both satisify the axioms, then there exists bijective mapping $\alpha : \mathbb{R} \to \mathbb{R}'$ such that:

$$\begin{aligned} \alpha(x + y) &= \alpha(x) +' \alpha(y) \\ \alpha(0) &= 0' \\ \alpha(x \cdot y) &= \alpha(x) \cdot' \alpha(y) \\ \alpha(1) &= 1' \\ x < y &\iff \alpha(x) <' \alpha(y) \end{aligned}$$

Lastly, if S is a non-empty set in $\mathbb{R}$ and $\alpha(S) = \{\alpha(x) : x \in S\}$, then S is has an upper bound if and only if $\alpha(S)$ has an upper bound. Furthermore $\alpha(\sup S) = \sup \alpha(S)$. ■

Results 1.

1. The set of natural numbers $\mathbb{N}$ in $\mathbb{R}$ is not bounded from above.

2. Let $x \in \mathbb{R}$ be such that for all $n \in \mathbb{N}$
$$0 \leq x \leq \frac{1}{n}$$
then $x = 0$.

3. (Archimedean Property) For all $a, b > 0$ there exists $n \in \mathbb{N}$:
$$na > b$$

4. Consider $I_n = [a_n, b_n]$ $\forall n \in \mathbb{N}$ such that $I_1 \supset I_2 \supset \ldots$. Then $\cap I_n$ is not empty. Moreover, if for each $e > 0$ there exists n such that $b_n - a_n < e$, then $\cap I_n$ is a single point.

5. $\sqrt{2} \in \mathbb{R}$. In addition, for all $p > 0$, there is a positive real number q such that $q^2 = p$

Exercises

1. Prove that the addition and multiplication identity elements are unique.

2. Show that the Trichotomy law holds for $(>)$. That is, exactly one of the following three is true.
$$x > y \qquad x = y \qquad y > x$$

3. Show that $1 \in \mathbb{F}^+$.

4. Show that if $x > -1$ and $n \in \mathbb{N}$:
$$(1+x)^n \geq 1 + nx$$
and equality only holds when $n = 1$.

5. Let $F_p = \{0, 1, \ldots, p-1\}$ where p is a prime number. Define $+$ and $\cdot$ to be the modular addition and product modulus p, respectively. Investigate whether if F_p can be ordered.

6. Consider the set of all rational polynomials $\mathbb{Q}[x]$:
$$\mathbb{Q}[x] = \left\{ \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0} : a_i, b_j \in \mathbb{Q}, b_n \neq 0 \right\}$$
Show that $\mathbb{Q}[x]$ under the normal addition and multiplication is a field. Furthermore, show that $\mathbb{Q}^+[x] = \{q \in \mathbb{Q}[x] : a_m \cdot b_n > 0\}$ constitutes an ordering on $\mathbb{Q}[x]$.

Chapter 2

Metric Space

2.1 Introduction

Let X be a non-empty set and $x, y \in X$ then if there exists a non-negative real number $d(x, y)$ with following three properties:

1. $d(x, y) = 0$ if and only if $x = y$ (Positive definiteness).
2. $d(x, y) = d(y, x)$ (Symmetry).
3. $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

the combination (X, d) is called a **metric space** and $d(x, y)$ is called the **metric**, or also **distance** function.

Example 2.1. The Euclidean space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ with $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ makes a metric space. To prove this we must show the above properties work:

1. if $d(x, y) = 0$ then:
$$\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = 0$$
Therefore each of the terms must be zero:
$$(x_i - y_i)^2 = 0 \quad \forall i \leq n$$
$$x_i - y_i = 0 \implies x_i = y_i$$
Thus $x = y$
2. It is obvious that $(x_i - y_i)^2 = (y_i - x_i)^2$ and therefore $d(x, y) = d(y, x)$
3. The triangle inequality immediately follows from the Cauchy-Schwartz inequality.

We can expand the Euclidean norm by defining Minkowski *p-norm* also called L^p*-norm* for $1 \leq p \leq \infty$ as follows:

$$d_p(x, y) = \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}}$$

and by taking the limit, $p \to \infty$ we find out that:

$$d_\infty(x, y) = \max_i\{|x_i - y_i|\}$$

Example 2.2. We can define **discrete distance** as follows:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

and it is pretty straightforward to show that the three properties hold.

Definition: The **open ball** $B_r(a)$ with radius r centered at a is the set of all points:

$$B_r(a) = \{x \in X : d(x,a) < r\}$$

and the **closed ball** $\overline{B}_r(a)$ with radius r centered at a is the set of all points:

$$\overline{B}_r(a) = \{x \in X : d(x,a) \leq r\}$$

The **sphere** $S_r(a)$ with radius r centered at a is the set of all the points:

$$S_r(a) = \{x \in X : d(x,a) = r\}$$

Definition (Open Set): Let (X,d) be a metric space. A subset $U \subset X$ is an open set if for all $a \in U$ there exists $\rho > 0$ such that:

$$B_\rho(a) \subset U$$

Definition (Internal Point): A point $a \in X$ is called an internal point of U if $\exists \rho > 0$ that the ball $B_\rho(a)$ contained in U.

Definition (Interior): The interior of a set U denoted by U° or $\mathrm{int}(U)$ is the set of all its interior points.

Definition (Adherent Point): A point $a \in X$ is called an adherent point of U if $\forall \rho > 0$ the ball $B_\rho(a)$ contains a point in U.

Definition (Limit Point): A point $a \in X$ is called a limit point of U if $\forall \rho > 0$ the set $B_\rho(a) - \{a\}$ contains a point in U. The set of all limit points is denoted by S' or $\lim S$.

Note: For any limit point $a \in U$ every open ball $B_r(a)$ contains infinitely many points in U.

Definition (Closed Set): Let (X,d) be a matric space. A subset $C \subset X$ is closed set if it contains all of its adherent point.

Definition (Closure): The closure of a set U denoted by $\overline{U}$ or $\mathrm{cl}\, U$ is set of all its adherent points.

Note: The closure of a set is a closed set.

Theorem 2.1. *Subset $C \subset X$ is closed if and only if $X - C$ is open.*

Proof. Firstly we prove the necessity condition that is C is closed if $X - C$. We employ proof by contradiction. Let C be a closed subset of X such that its complement is not open. That is, for some $a \in (X - C)$ there is no $\rho > 0$ exists such that $B_\rho(a) \subset (X - C)$. In other words, for all ρ, $\exists p \in B_\rho(a)$ s.t $p_\rho \in C$. Which implies that a is an adherent point of C but since C is closed then $a \in C$ which is a contradiction. Similarly, one can show the sufficiency condition. ■

Corollary 2.2. *X and $\emptyset$ are both closed and open.*

Remark 1. (Equivalent Definitions)

1. An open set is a union of open balls. Conversely, a union of open balls is an open set.

   Proof. For every $a \in U$ there is a ball $B_\rho(a) \subset U$ thus $\bigcup_{a\in U} B_\rho(a) \subset U$ and since $a \in B_\rho(a)$ we must have $\bigcup_{a\in U} B_\rho(a) \supset U$ hence $U = \bigcup_{a\in U} B_\rho(a)$.

   Now let $U = \bigcup B_\rho(a)$ we need to show that U is open. Let $b \in U$ then b must be a point in at least one of those balls. Let $b \in B_r(c)$ and $\rho = r - d(b,c)$. We will show that $B_\rho(b) \subset B_r(c) \subset U$, for any $x \in B_\rho(b)$ by triangle inequality we have $d(x,c) \le d(x,b) + d(b,c) < \rho + d(b,c) = r$ which means $x \in B_r(c)$. □

2. A set is open if and only if all of its members are interior points. Therefor, $U = \operatorname{int} U$.

3. Let $I = \{S \subset U : S \text{ is open}\}$ then $\operatorname{int} U = \displaystyle\bigcup_{S\in I} S$.

4. Let $I = \{S \subset U : S \text{ is closed}\}$ then $\operatorname{cl} U = \displaystyle\bigcup_{S\in I} S$.

Let (X,d) be a metric space and $Y \subset X$ then Y may inherit its metric from X and (Y,d) would also be a metric space and is called a **metric subspace** of X. We will investigate the nature of open and closed sets in subspaces. Let $B^Y_\rho(y) = \{p \in Y : d(y,p) < \rho\}$ Then, it is easy to see that:

$$B^Y_\rho(y) = B_\rho(y) \bigcap Y$$

Corollary 2.3. *Let (X,d) be a metric space and $Y \subset X$ is a metric subspace of X then $U \subset Y$ is an open subset of Y if and only if there is a open set $V \subset X$ such that $U = V \cap Y$. Similarly, for any closed set $C \subset Y$ there is a closed set $D \subset X$ such that $C = D \cap Y$.*

Proof. Ofcourse if $U \subset Y$ is open in Y then by definition it can be represent as a union of open ball $B^Y_r(a)$. Each of these balls is the intersection of a $B^X_r(a) \cap Y$. Therefore

$$U = \bigcup B^Y_r(a) = \bigcup \left(B^X_r(a) \cap Y\right) = \left(\bigcup B^X_r(a)\right) \cap Y = V \cap Y$$

Furthermore, if $a \in V \cap Y$ then there exists a ball $B^X_r(a) \subset V$. Therefore

$$B^Y_r(a) = B^X_r(a) \cap Y \subset V \cap Y = U$$

The case for closed subsets can be proved using the complements. ■

Exercises

1. Show that:

   (a) Every union of open sets is open.

   (b) Every finite union of closed sets is closed.

   (c) Every intersection of closed sets is closed.

   (d) Every intersection of open sets is open.

2. Show that $\operatorname{cl} S = S \cup \lim S$

2.2 Convergence

Let (X, d) be a metric space. A **sequence** is a function in form of $a : \{k, k+1, k+2, \dots\} \to X$ where $k \in \mathbb{Z}$. Conventionaly, instead of $a(n)$ a_n is used. The sequence $\{a_n\}$ is **convergent** to $a \in X$ if for all $\epsilon > 0$ there exists N such that:

$$n \geq N : d(a, a_n) < \epsilon$$

and it is denoted by $a_n \to a$ or $a = \lim_{n\to\infty} a_n$ In that case, the set $\{a_k, a_{k+1}, \dots\}$ is bounded in X, that is, there exist $K > 0$ and a point $b \in X$ such that $\forall n,\ a_n \in B_K(b)$.

The problem with defintion of convergence is its dependence on a convergence point so naturally the following question comes up. Is there a way to show the convergence of sequence based on itself? For that we need to define **Cauchy sequence**. A sequence $\{a_n\}$ is a Cauchy sequence if:

$$\forall \epsilon > 0,\ \exists N \quad \text{s.t.} \quad n, m \geq N \implies d(a_n, a_m) < \epsilon$$

Theorem 2.4. *Every convergent sequence is a Cauchy sequence.*

Proof. For a given $\epsilon > 0$ we know there exist N such that:

$$n \geq N \implies d(a_n, a) < \frac{\epsilon}{2}$$

and equivalently:

$$m \geq N \implies d(a_m, a) < \frac{\epsilon}{2}$$

and since by triangle inequality we have:

$$d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

■

Definition (Subsequence): We call $\{b_n\}$ a **subsequence** of $\{a_n\}$ if there is a sequence of positive integers $n_1 < n_2 < n_3 < \dots$ such that for each k, $b_k = a_{n_k}$.

Exercises

1. Show that if a sequence $\{a_n\}$ is convergent, then its limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.

2. Prove that every subsequence of a convergent sequence converges and it converges to the same limit.

2.3 Completeness

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Proposition 2.5. *$\mathbb{R}$ with the normal Euclidean norm is a complete metric space.*

To prove it, we need the following lemmas.

Lemma 2.6. *If $\{a_n\}$ is a Cauchy sequence in a matric space (X, d) then the set $S = \{a_k, a_{k+1}, \dots\}$ is bounded.*

Proof. For a fixed $\epsilon > 0$ we know there exists N such that:

$$m, n \geq N \implies d(a_n, a_m) < \epsilon$$

especially:

$$n \geq N \implies d(a_n, a_N) < \epsilon$$

Since there is only finitely many indices less than N then we can determine the largest $d(a_N, a_m)$for all m less than N lets denote it by A. Finally, let $K = \max\{\epsilon, A\}$ then $B_K(a_N)$ contains all the elements of sequence. □

Lemma 2.7. *If one of the subsequences of Cauchy sequence is convergent then the Cauchy sequence is convergent to the same element.*

Proof. Let $a_{n_k} \to a$ when $k \to \infty$ That is, for a given $\epsilon > 0$, $\exists N_1$ such that:

$$k \geq N_1 \implies d(a_{n_k}, a) < \frac{\epsilon}{2}$$

and since $\{a_n\}$ is a Cauchy sequence then we also know that there exists N_2 such that:

$$q, m \geq N_2 \implies d(a_m, a_q) < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ and $n_q \geq N$ consequently:

$$n_q, m \geq N \implies d(a_m, a_{n_q}) < \frac{\epsilon}{2}$$

and by the triangle inequality we have:

$$d(a_m, a) \leq d(a_m, a_{n_q}) + d(a_{n_q}, a) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which proves the convergence of $a_n \to a$. □

Proof. Let $\{a_n\}$ be a Cauchy sequence. Then by Lemma 2.6, the sequence is bounded and there is a closed interval $I_0 = [a, b]$ in which all a_n lie. Consider the closed intervals $\left[a, \frac{a+b}{2}\right]$ and $\left[\frac{a+b}{2}, b\right]$. Since the sequence has infinitely many terms then there are infinitely many terms in at least one of the two intervals. Let that interval be I_1 and choose $x_1 \in I_1$ where $x_1 = a_{n_1}$ for some n_1. Repeat the process for I_1 to get I_2 and $x_2 = a_{n_2}$ where $n_2 > n_1$. Since there are infinitely many terms in I_2 we can find such n_2. By continuing this process we have a subsequence $\{x_k\}$ and a sequence of nested closed sets $\{I_k = [a_k, b_k]\}$. Since for all $\epsilon > 0$ there exists K such that $b_K - a_K < \epsilon$ then the intersection of $\{I_k\}$ is a point, say y. We claim that that $x_k \to y$, that is:

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \implies |x_n - y| < \epsilon$$

Since $y = \bigcap I_k$ then $y \in I_k$ for all k, especially $y \in I_n$. Therefore, $|x_n - y|$ is smaller than or equal to the length of I_n which is $\dfrac{b-a}{2^n} \leq \dfrac{b-a}{2^N}$. By setting $N > \log_2 \dfrac{b-a}{\epsilon}$ we have:

$$|x_n - y| \leq \frac{b-a}{2^n} \leq \frac{b-a}{2^N} < \epsilon$$

Therefore $\mathbb{R}$ is a complete metric under Euclidean norm. ■

Let $(X, d)$and (X', d')be two metric spaces. Define the following norms on the descartian product $X \times X'$:

1. $D_1((x, x'), (y, y')) = d(x, y) + d'(x, y)$
2. $D_2((x, x'), (y, y')) = \sqrt{d(x, y)^2 + d'(x', y')^2}$
3. $D_3((x, x'), (y, y')) = \max\{d(x, y)\,, d'(x', y')\}$

by letting $p_1 = (x, x')$ and $p_2 = (y, y')$:

$$D_3(p_1, p_2) \leq D_2(p_1, p_2) \leq D_1(p_1, p_2) \leq 2D_3(p_1, p_2)$$

Then it is easy to see that if a sequence $\{a_n\}$ is convergent under one of these norms, it is convergent to the same value under the other two. The same is true if the sequence is a Cauchy sequence.

By induction we can generalize it to $X_1 \times X_2 \times \cdots \times X_n$. For example, $\mathbb{R}^n$ is complete metric under all the three norms introduced above. That is, every Cauchy sequence in $\mathbb{R}^n$ is convergent. To show this assume the sequence $\{x_i\}$ is a Cauchy sequence under, WLOG, D_1:

$$\forall \epsilon > 0,\ \exists N \quad \text{s.t.} \quad i, j \geq N \implies D_1(x_i, x_j) < \epsilon$$

Then for the k-th coordinate:

$$|x_{i_k} - x_{j_k}| < D_1(x_i, x_j) < \epsilon$$

Therefore, for every coordinate, the image of the sequence on that coordinate is a Cauchy sequence. Since $\mathbb{R}$ is complete then $\{x_{i_k}\}_i$ is convergent to some x_k for all k. We claim that $x_i \to x = (x_1, \ldots, x_n)$ as $i \to \infty$:

$$D_1(x, x_i) = |x_{i_1} - x_1| + |x_{i_2} - x_2| + \cdots + |x_{i_n} - x_n|$$

We have shown that $\{x_{i_k}\}_i$ is convergent to x_k then there must be $N_1, N_2, \ldots N_n$ such that for all k:

$$\forall \epsilon, \quad i \geq N_k \implies |x_{i_k} - x_k| < \frac{\epsilon}{n}$$

Setting $N = \max\limits_{1 \leq k \leq n} N_k$:

$$D_1(x, x_i) < n \cdot \frac{\epsilon}{n} = \epsilon$$

Theorem 2.8. *Let (X, d) be a complete metric space and $Y \subset X$ is a complete metric space if and only Y is a closed subset of X.*

Proof. It is clear that Y being closed is necessary for Y being a complete metric subspace. To show that is also sufficient, we need ot show that if Y is a complete metric subspace then it is closed. Assume the contrary, that is there exists an adherent point of Y, $a \notin Y$. Since a is an adherent point of Y then for all $\rho > 0$ there exists a point $x \in B_\rho(a)$ such that $x \in Y$. For each n let $\rho = \frac{1}{n}$ and choose a point $x_n \in Y$ It is clear that $\{x_n\}$ is convergent to a. From Theorem 2.4 $\{x_n\}$ is a Cauchy sequence. Since Y is complete then a must be in Y which is a contradiction. ∎

Exercises

1. Show that if a sequence $\{a_n\}$ is convergent, then its limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.

2. Prove that every subsequence of a convergent sequence converges and it converges to the same limit.

2.4 Continuity

Definition (Continuity): Let (X, d) and (X', d') be two metric spaces and $f : X \to X'$ is a function. We say f is continuous if for every open subset V of X' the pre-image of it is an open in X:

$$f^{\text{pre}}(V) = f^{-1}(V) = \{x \in X : f(x) \in V\}$$

Furthermore, f is continuous at a point $x \in X$ when for all subset W of X' that $f(x)$ is a internal point of W, then there is an open set U containing x such that $\{f(y) : y \in U\} \subset W$. In other words x is an internal point of $f^{\text{pre}}(W)$.

Proposition 2.9. *f is continuous if and only if f is continuous at every point $x \in X$.*

Proof. Firstly, if f is continuous we show that f is continuous at every point $x \in X$. Let V be an open set around $f(x)$ then $x \in f^{\text{pre}}(V)$ must be an internal point since $f^{\text{pre}}(V)$ is open. Secondly, if f is continuous at every point $x \in X$ then f is continuous. Let $V = \{f(x) : x \in U\}$ be an open set in X'. For some $x \in U$, $f(x)$ is an internal point of V and since f is continuous at x, x is an internal point of U which means every point $x \in U$ is an internal point of U and thus $U = f^{\text{pre}}(V)$ is open. ■

Theorem 2.10 ($\epsilon - \delta$ condition). *Continuity at a point x is equivalent to the existence a $\delta > 0$ for all $\epsilon > 0$ such that:*

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$$

Proof. Let $V = \{f(y) : d'(f(x), f(y)) < \epsilon\}$ then V is open and hence $f(x)$ is an internal point of V. By continuity at point x, x must be an internal point of $f^{\text{pre}}(V)$. In other words, there exists a $\delta > 0$ such that $U = \{y : d(x, y) < \delta\} \subset f^{\text{pre}}(V)$. Take an open set $U \subset X'$, then assuming the $\epsilon - \delta$ condition, we will show that $f^{\text{pre}}(U)$ is open. Let $y \in U$ then there is $x \in f^{\text{pre}}(U)$ such that $f(x) = y$. From openness of U, there is a $\epsilon > 0$ such that $B_\epsilon(y) \subset U$, also by continuity condition, there exists a $\delta > 0$ such that:

$$d(x, z) < \delta \implies d'(f(x), f(z)) < \epsilon$$

The openness of $f^{\text{pre}}(U)$ is equivalent to $B_\delta(x) \subset f^{\text{pre}}(U)$, which clearly holds, since for any $z \in B_\delta(x) \implies f(z) \in B_\epsilon(y) \subset U$. ■

Example 2.3. Let (X, d) be a metric space with $d(x, y)$ being the discrete metric, $f : X \to X'$ where (X', d') is an arbitary metric space. Then f is always continuous. Since for every point a the open ball $B_{\frac{1}{2}}(a) = \{a\}$, and union of open sets is an open set itself, then every subset of X is open.

Equivalently, f is continuous at a if for all $\epsilon > 0$, a is an internal point of $f^{\text{pre}}(B_\epsilon(f(a)))$. That is there exists $\delta > 0$ such that, $B_\delta(a) \subset f^{\text{pre}}(B_\epsilon(f(a)))$.

Theorem 2.11. *Let (X, d) and (X', d') be two metric spaces and $f : X \to X'$. f is continuous at $a \in X$ if and only if for every sequence $\{a_n\}$ in X with $a_n \to a$ we have $f(a_n) \to f(a)$.*

Proof. Let f be continuous at a and $a_n \to a$. From continuity of f, for each given ϵ, there is a δ such that:

$$d(x,a) < \delta \implies d'\left(f(x), f(a)\right) < \epsilon$$

From the convergence of $\{a_n\}$, for each given δ, there is a N such that:

$$\forall n \geq N \implies d(a_n, a) < \delta$$

By merging these two equations we will get:

$$\forall n \geq N \implies d(a_n, a) < \delta \implies d'(f(a_n), f(a)) < \epsilon$$

which was what was wanted.

If f is not continuous, there must be an $\epsilon > 0$ that for all $\delta > 0$, for some $x \in B_\delta(a)$, $d'\left(f(x), f(a)\right) \geq \epsilon$. Especially, for each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ and x_n have the described property. Since $x_n \to a$ by our assumption $f(x_n) \to f(a)$, which is a contradiction and thus f is continuous. ■

Exercises

1. Let $(X, d), (X', d')$, and (X'', d'') be metric spaces and $f : X \to X'$, $g : X' \to X''$ be two functions. If f is continous at a and g is continous at $f(a)$, then $g \circ f$ is continuous at a.

2. Let (X_i, d_i), $i = 1, \dots k$ be metric spaces. Define D to be any of the three discussed metric over $X = X_1 \times X_2 \times \cdots \times X_k$. Then the projection function, $\pi_j(x) : X \to X_j$ is continuous for all j.
$$\pi_j(x_1, x_2, \dots, x_n) = x_j$$

3. Let X, D be defined as above, (X', d') be a matic space, and $f : X' \to X$. f is continous at $a' \in X'$ if and only if $\pi_j \circ f$ is continuous for all $j = 1, \dots, k$.

4. The four algebraic operations are continuous on their domain.
$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad +(x,y) = x + y$$
$$- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad -(x,y) = x - y$$
$$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \times(x,y) = x \times y$$
$$\div : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}, \quad \div(x,y) = x \div y$$
Where the metric of $\mathbb{R}$ on the right hand side is the common Euclidean metric, and on the left hand side is any of the three metric.

2.5 Compactness

A subset $K \subset X$ is **compact** if for all sequence $\{a_n\}$ in K there exists a subsequence of $\{a_n\}$ that converges to $a \in K$.

Corollary 2.12. *If K is compact then K must be closed and bounded.*

Proof. Obviously if K is not closed then there must be a limit point $a \notin K$ such that the sequence $\{a_n\}$ converges to a. We have shown every subsequence of a convergent sequence converges to the same value, therefore K is not compact. If K is unbounded then for each point $a \in K$ for all $n \in \mathbb{N}$, the ball $B_n(a)$ has a point other than a in K. Then we can select a_n to be a point. Cleary no subsequence of $\{a_n\}$ can be convergent. ■

Theorem 2.13. *If $K \subset X$ is compact and C is a closed subset of X such that $C \subset K$, then C is compact.*

Proof. Take a sequence $\{a_n\} \in C$. Since $\{a_n\} \in K$ then it has a convergent subsequence $b_k = a_{n_k}$. Let $b \in K$ be the point of convergence of $\{b_k\}$. Since $\{b_k\} \in C$ and C is closed then $b \in C$ and therefore C is compact. ■

Proposition 2.14. *A subset in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Using the same idea as Proof 5 one can show the case for $n = 1$. Assume the propsition is true for $n = k-1$ and let $K \in \mathbb{R}^k$ be a closed and bounded set and $\{a_n\} \in K$. Furthermore, let $\{b_n\}$ be the projection of $\{a_n\}$ onto $\mathbb{R}^{k-1}$ and $\{c_n\}$ be the projection of $\{a_n\}$ on to k_{th} dimension. By induction, there exists a convergent subsequence $\{b_{n_m}\}$. For $\{c_{n_m}\}$ there exists a convergent subsequence $\{c^i_{n_m}\}_i$ as well. It is easy to see that $\{a^i_{n_m}\}_i$ is a convergent subsequence of $\{a_n\}$. ■

Corollary 2.15. $[a, b]$ *is compact in $\mathbb{R}$.*

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We define:

$$\limsup a_n = \overline{\lim}\, a_n = \lim_{n\to\infty} \Big(\sup \{a_k : k \geq n\} \Big)$$

$$\liminf a_n = \underline{\lim}\, a_n = \lim_{n\to\infty} \Big(\inf \{a_k : k \geq n\} \Big)$$

Note: The limits, $\limsup a_n$ and $\liminf a_n$, always exists. Albeit they might be infinite.

Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$, and A^* is the set of all limit points of all subsequence of $\{a_n\}$. We know that A^* is not empty and since $\{a_n\}$ is bounded and then A^* must be bounded as well. Thus, by completeness axiom, A^* has infimum and supremum. Moreover, $\sup A^*, \inf(A^*) \in A^*$.

Proposition 2.16. *A bounded sequence $\{a_n\}$ is convergent if and only if $\limsup a_n = \liminf a_n$.*

Corollary 2.17. *If K is a compact subset of $\mathbb{R}$ then K has minimum and maximum. That is, there are $M, m \in K$ such that $\forall x \in K,\ m \leq x \leq M$.*

Proof. Since K is bounded then it has supremum and infimum in $\mathbb{R}$. Obviously, there are convergent sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to m = \inf K$ and $b_n \to M = \sup K$. By compactness of K, $M, m \in K$. ■

Theorem 2.18. *(X, d) and (X', d') are metric spaces and $K \subset X$ is compact. If $f : X \to X'$ is continuous, then $f(K)$ is a compact subset of X'.*

Proof. Let $\{y_n\} \in f(K)$ and $\{x_n\} \in K$ are such that $f(x_n) = y_n$. Since K is compact there is a convergent subsequence $\{x_{n_k}\}$ and since f is continous $\{y_{n_k} = f(x_{n_k})\}$ is also convergent. Hence $f(K)$ is compact. ■

Corollary 2.19. *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ is continuous. If K is a compact subset of X. Then f attains maximum and minimun in $\mathbb{R}$.*

Note: For a continuous function $f : X \to X'$ it is not necessary that the image of an open/closed set to be open/closed.

Definition (Uniform continuity): Let (X, d) and (X', d') be metric spaces. $f : X \to X'$ is uniformly continuous if:

$$\forall \epsilon > 0 \; \exists \delta > 0, \; x, y \in X, \; d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$$

Proposition 2.20. *$f : X \to X'$ is uniformly continous if and only if for every pair sequence (x_n, y_n) in X satisfying $d(x_n, y_n) \to 0$ we have $d'(f(x_n), f(y_n)) \to 0$.*

Proof. Necessity: We have

$$\begin{aligned}
&\forall \epsilon \; \exists \delta \text{ s.t. } \forall x, y \in X, d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon \\
&\forall \delta \; \exists N \text{ s.t. } n \geq N \implies d(x, y) < \delta
\end{aligned}$$

combining the two brings us at the conclusion. Sufficiency: Suppose for the sake of contradtion that:

$$\exists \epsilon \; \forall \delta \; \exists x, y \in X \text{ s.t. } d(x, y) < \delta \wedge d'(f(x), f(y)) \geq \epsilon$$

then let $\delta = \frac{1}{n}$ and make the sequence pair (x_n, y_n). Clearly, $d(x_n, y_n) \to 0$ therefore, $d'(f(x), f(y)) \to 0$. Which is a contradition since $d'(f(x), f(y)) \geq \epsilon$. ■

Proposition 2.21. *(X, d) and (X', d') are matric spaces and X is compact. If $f : X \to X'$ is continuous then it is uniformly continous.*

Proof. Similarly, for the sake of contradicition suppose

$$\exists \epsilon \; \forall \delta \; \exists x, y \in X \text{ s.t. } d(x, y) < \delta \wedge d'(f(x), f(y)) \geq \epsilon$$

and let $\delta = \frac{1}{n}$ and make the sequence pair (x_n, y_n). By compactness of X, there are two convergent subsequence $\{x_{n_k}\}$ and $\{y_{n_k}\}$. Since $d(x_n, y_n) \to 0$ then if $x_{n_k} \to x$, $y_{n_k} \to x$ as well. By continuity of f, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$ and thus $d'(f(x_{n_k}), f(y_{n_k})) \to 0$. Which is a contradicition as for sufficiently large K, $k \geq K \implies d'(f(x), f(y)) \geq \epsilon$ ■

Define the **diamter** of a set S to be:

$$\operatorname{diam} S = \sup \{d(s, s') : s, s' \in S\}$$

the cleary for bounded sets we have:

$$\operatorname{diam} S < +\infty$$

Proposition 2.22. *Let (X, d) be a metric space and $\{K_n\}$ is a sequence of compact subset of X with $K_1 \supset K_2 \supset \dots$.*

1. *$\bigcap K_n$ is not empty.*
2. *If $\operatorname{diam} K_n \to 0$ then $\bigcap K_n$ is a singular point.*

Proof.

1. Consider the sequence $\{a_n\}$ such that $a_n \in K_n$. Since $a_n \in K_1$ for all n, then there is a convergent subsequence $\{a_{n_k}\}$ with $a_{n_k} \to a$. $a \in K_1$, however, $a \in K_2$ and so on, as well. Therefore $a \in \bigcap K_n$.
2. Let $a, b \in \bigcap K_n$. Then, $a, b \in K_n$ for all n and we must have that $d(a, b) \leq \operatorname{diam} K_n$. Therefore, $a = b$. ■

Exercises

1. Prove that $\sqrt{|\ x\ |} : \mathbb{R} \to \mathbb{R}$ is uniformly continous.

2.6 Connectedness

Definition: (X, d) a metric space. X is disconnected if the open sets A, B are found such that:

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A \bigcap B = \emptyset, \quad A \bigcup B = X$$

X is said to be connected if it is not disconnected. $S \subset X$ is connected if it is connected as a subspace of X.

Example 2.4. The following subsets of $\mathbb{R}$ are disconnected:

1. $S = [-1, 0[\cup]0, 1]$
2. $\mathbb{Q}$
3. $S = [1, 0] \cup [1, 2]$

Definition: $S \subset \mathbb{R}$ is an interval if when $a, c \in S$ and $a < b < c$ then $b \in S$.

Example 2.5. $\mathbb{R}$ and its intervals are connected. In fact the only connected subsets of $\mathbb{R}$ are its intervals.

Theorem 2.23. *(X, d) and (X', d') are metric spaces. $f : X \to X'$ is continuous and S is a connected subset of X. Then, $f(S)$ is connected in X'.*

Corollary 2.24 (Mean value theorem). *If $f : [a, b] \to \mathbb{R}$ is a continous function and $f(a) = A, f(b) = B$ then for every C between A and B there exists a $c \in [a, b]$ such that $f(c) = C$.*

Proposition 2.25. *If $S \subset X$ is a connected set then every $S \subset T \subset \bar{S}$ is connected.*

Definition: $G_f : M \to M \times N$ to be the graph of f, that is $G_f = \{(x, f(x)) | x \in M\}$.

Theorem 2.26. *The graph of a continous function over a connected set is connected.*

Example 2.6. topological curve is connected and also its closure is connected.

Proposition 2.27. *Let (X, d) be a metric space and (S_α) is a collection of connected sets in X. If $x \in S_\alpha \forall \alpha$ then the union of S_α is connected.*

Definition (Path connected): S is path connected if for every pair of points $p, q \in S$ there exists a continous function $\gamma : [a, b] \to S$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Theorem 2.28. *if a set S is path connected then it is connected but the inverse is not true.*

Example 2.7. infinite broom is path connected but toplogical sine curve is not.

Proposition 2.29. *If f is continuous on a path connected set then the image of f is path connected.*

Proposition 2.30. *Every open set of $\mathbb{R}$ is the union of countably many disjoint open intervals.*

Exercises

1.

2.7 Covering

Definition (Covering): Let (X, d) be a metric space. A covering for such space is a collection of U_α of open subsets of X such that $\bigcup U_\alpha = X$. Similarly, for $S \subset X$, a covering is a collection of U_α of open subsets of X such that $S \subset \bigcup U_\alpha$.

Definition (Sub covering): A finite subcovering of $\bigcup U_\alpha$ is a collection of finitely many U_α such that their union covers the same space. That is, there exists a U_{α_n} for $n \leq k$ such that:

$$U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots U_{\alpha_k} = X$$

Example 2.8. $\mathbb{R}$ and covering $U_x =]x-1, x+1[$, no finite subcovering but countably many.

Theorem 2.31. *Compactness is equivalent to the existence a finite subcovering for every covering.*

Proof. To prove the theorem, let us define:

Definition: For a metric space (X, d) is **covering compact** if every covering reduces to a finite subcovering.

We will show for metric spaces compactness is equivalent to covering compact. lebegue number ■

Example 2.9. from definition show that $[a, b]$ is covering compact.

Exercises

1. Show that $\mathbb{Q} \cap [0, 1]$ is not covering compact, directly from the definition.

2.8 Cantor Set

Definition: define cantro set

Definition (Perfect space): define perfect set

Proposition 2.32. *Cantor set is a perfect space.*

Definition: Totally disconnected

Proposition 2.33. *Cantor set is totally disconnected.*

Theorem 2.34. *Let K be a complete, totally disconnected, and compact metric space. Then K is homeomorphic to cantor set, in that, there is a continuous function $h : K \to C$ such that h^{-1} is continuous as well.*

Exercises

1. Show that $\mathbb{Q} \cap [0, 1]$ is not covering compact, directly from the definition.

Chapter 3

Differentiation

Definition: Let I be an interval in $\mathbb{R}$. If a is an interior point of I, then we say that $f : I \to \mathbb{R}$ is differentiable at a when the following limit exists:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

The limit, if exists, is denoted by $f'(a)$. If a is an end point and the length of the interval is greater than zero, then the limit only exists from one direction.

Equivalently, there exists a line l, not parallel to y-axis, in form of $l : A(x) = mx + b$, that is tangent to f at $x = a$. In this case:

$$\lim_{x\to a} \frac{f(x) - [mx + b]}{x - a} = 0 \qquad A(a) = f(a)$$

In a general case, two functions f, g are tangent to each other at $x = a$ if:

$$f(a) = g(a) \qquad \lim_{x\to a} \frac{f(x) - g(x)}{x - a} = 0 \tag{3.1}$$

Corollary 3.1.

1. *f is differentiable at a if it is continuous at a.*

2. *If $f'(a) > 0$, there exists $\delta > 0$ such that for $x \in \,]a - \delta, a[\, \cap I \implies f(x) < f(a)$ and for $x \in \,]a, a + \delta[\, \cap I \implies f(x) < f(a)$. And if $f'(a) < 0$ the inequality sign are reversed. Therefore, if f has a local extremum at a, then in case $f'(a)$ exists, $f'(a) = 0$.*

Example 3.1. a function that its derivate is not continuous (with $\sin \frac{1}{x}$).

Theorem 3.2 (Rolle's theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable on the interval. If $f(a) = 0, f(b) = 0$, then there exists $c \in \,]a, b[$ such that:*

$$f'(c) = 0$$

Proof. If $f \equiv 0$ on $[a, b]$ then its derivative $f'(x) \equiv 0$ on $[a, b]$. If $f(x) \neq 0$ for some $x \in]a, b[$ then it must have a non-zero maximum or minimum at some $c \in]a, b[$. Since $[a, b]$ is compact then by continuity of f, $f([a, b])$ is also compact in $\mathbb{R}$ and therefore f attains its maximum or minimum. We know that at least one of its extremities must lie in $]a, b[$, say point c, hence by Item 2 $f'(c) = 0$. ∎

Theorem 3.3 (Mean value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable on the interval,then there exists $c \in \,]a, b[$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define $g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$. Then it is clear that $g(a) = g(b) = 0$ and g is continous and differentiable on the interval. Then by Theorem 3.2 there exists $c \in]a, b[$ such that $g'(c) = 0$. Equivalently:

$$\begin{aligned} g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ \implies f'(c) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

which concludes the proof. ∎

Corollary 3.4 (Growth Estimate). *If $|f'(x)| \leq M$ in $]a, b[$ then f satisfies the global lipschitz condition for all $x, y \in [a, b]$ $|f(x) - f(y)| \leq M|x - y|$.*

Corollary 3.5. *Let $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x) < 0$ (or $f'(x) > 0$) for all $x \in]a, b[$ then f is strictly increasing (or decreasing) on $[a, b]$.*

Theorem 3.6. *$f : [a, b] \to \mathbb{R}$ is continuous and differentiable on $]a, b[$ then for $f'(]a, b[)$ the intermediate value theorem holds and thus it is an interval.*

Proof. Let $x_1, x_2 \in]a, b[$. WLOG assume $f'(x_1) < f'(x_2)$, we wish to prove that for all $y^* \in]f'(x_1), f'(x_2)[$ there is a $x^* \in]x_1, x_2[$ such that $f(x^*) = y^*$. Put $g(x) = f(x) - y^*x$. By differentiability of f on $[a, b]$, g is differentiable on $[a, b]$. Then, $g'(x_1) = f'(x_1) - y^* < 0$ and $g'(x_2) = f'(x_2) - y^* > 0$, therefore there are $t_1, t_2 \in]x_1, x_2[$ such that $g(t_1) < g(x_1)$ and $g(t_2) < g(x_2)$. Since g is continuous on $[x_1, x_2]$ then it must attains its minimum at some $x^* \in [x_1, x_2]$. However x^* can't be x_1 or x_2 and hence $x^* \in]x_1, x_2[$. It is then easy to see that $f'(x^*) = y^*$. ∎

Definition (Darboux continous): A function f is Darboux continous if it posseses the intermediate value property.

For example f' of differentiable function is Darboux continuous.

Theorem 3.7 (Cauchy's mean value theorem). *$f, g : [a, b] \to \mathbb{R}$ are continuous then there exists a $c \in \,]a, b[$, such that:*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

Proof. Define $h(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$, then clearly $h(a) = 0, h(b) = 0$ and $h(x)$ is continous and differentiable on $[a, b]$. Hence by applying the theorem 3.2 for some $c \in]a, b[$ we have:

$$\begin{aligned} & h'(c) = 0 \\ & \implies f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) = 0 \\ & \implies f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \end{aligned}$$

Theorem 3.8 (L'Hopital's rule). *Suppose that* $\lim_{x\to a^+} f(x) = 0, \lim_{x\to a^+} g(x) = 0$ *where* f, g *are differentiable on a open interval* $I =]a, b[$ *for some* b *such that* $g'(x) \neq 0$ *in* I *except maybe at* $x = a$ *and the limit*

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

exists, then:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Proof. For a fixed $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|x - a| < \delta \implies \left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$$

then since $f(t), g(t) \to 0$ as $t \to a$ from right side then there must be a $t \in]a, x[$ such that

$$\left|\frac{f(x) - f(t)}{g(x) - g(t)} - \frac{f(x)}{g(x)}\right| < \frac{\epsilon}{2}$$

then simply:

$$\left|\frac{f(x)}{g(x)} - L\right| \leq \left|\frac{f(x)}{g(x)} - \frac{f(x) - f(t)}{g(x) - g(t)}\right| + \left|\frac{f(x) - f(t)}{g(x) - g(t)} - L\right| \tag{3.2}$$

$$< \frac{\epsilon}{2} + \left|\frac{f'(\theta)}{g'(\theta)} - L\right| \tag{3.3}$$

$$< \epsilon \tag{3.4}$$

Note that $\theta \in]t, x[$ and thus $|\theta - a| < \delta$ ■

Definition (Higher order derivatives): f is said to be r_{th}-differentiable at x if it is differentiable r times. The r_{th} derivative of f is denoted as $f^{(r)}$. If $f^{(r)}$ exists for all r and x then f is said to be infinitely differentiable or smooth.

Definition (Smoothness classes): The set of all f is continuosly r_{th}-differentiable is called class $\mathcal{C}^r$.

Definition (Taylor polynomial): The r_{th}-order Taylor polynomial of an r_{th}-order differentiable function f at x is

$$P_r(x, h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots + \frac{f^{(r)}(x)}{r!}h^r = \sum_{n=0}^{r} \frac{f^{(n)}(x)}{n!}h^n$$

Theorem 3.9 (Taylor approximation theorem). *Let* f *be a* r*-differentiable function at* x *then:*

1. $$\frac{f(x + h) - P_r(x, h)}{h^r} \to 0 \text{ as } h \to 0$$

2. *and* P_r *is the only* r_{th} *degree polynomial that has such property.*

3. *Furthermore, if f is r-differentiable on an interval I for every $x, y \in I$, there exists ξ between x, y such that:*

$$f(y) - P_{r-1}(x, y-x) = \frac{f^{(r)}(\xi)}{(r)!}(y-x)^r$$

Proof.

1. For the base case $r = 1$

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = f'(x) - f'(x) = 0$$

and by induction we prove the case $r = n \geq 2$

$$\lim_{h\to 0} \frac{f(x+h) - P_n(x,h)}{h^n} = 0$$
$$\iff \forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \ |h| < \delta \implies |f(x+h) - P_n(x,h)| < \epsilon |h^n|$$

Let $g(h) = f(x+h) - P_n(x,h)$ then since both $f(x+h)$ and $P_n(x,h)$ are differentiable then we apply Theorem 3.3

$$\begin{aligned} g(h) - g(0) = g(h) &= h(g'(c)) \\ &= h(f'(x+c) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{(k-1)!} c^{k-1}) \\ &= h(f'(x+c) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!} c^{k}) \\ &= h(f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^{k}) \end{aligned}$$

for some $c \in]0, h[$. Note that f' is $(n-1)$-differentiable at x thus by induction for any $\epsilon > 0$ there exists a δ such that if $c < \delta$ then:

$$|f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^k| < \epsilon |c^{n-1}|$$

which means

$$\begin{aligned} |f(x+h) - P_n(x,h)| = |g(h)| &= |h||f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^k| \\ &< |h|\epsilon|c^{n-1}| < \epsilon|h^n| \end{aligned}$$

Therefore for any ϵ if $h < \delta$ then $c < \delta$ and the result holds.

2. Let $Q_r(x,h)$ be another r_{th} degree polynomial such that

$$\lim_{h\to 0} \frac{f(x+h) - Q_r(x,h)}{h^r} = 0$$

then

$$\lim_{h\to 0} \frac{P_r(x,h) - Q_r(x,h)}{h^r} = 0$$

however this can only happen if $Q_r(x,h) = P_r(x,h)$.

3. Again for the base case $r = 1$

$$f(y) - f(x) = f'(\xi)(y - x)$$

which is the Theorem 3.3. for $r = n$ we have that

$$g(h) = f(x + h) - P_{n-1}(x, h) + Ch^n \implies g(0) = g'(0) = \cdots = g^{(n-1)} = 0$$

Set C such that $g(y - x) = 0$. Then by applying Theorem 3.2 $(n - 1)$ times

$$\begin{aligned}
g(0) = g(y-x) = 0 &\implies g'(c_1) = 0 \quad c_1 \in]0, y-x[\\
g'(0) = g'(c_1) = 0 &\implies g'(c_2) = 0 \quad c_2 \in]0, c_0[\\
&\vdots \\
g^{(n-2)}(0) = g^{(n-2)}(c_{n-2}) = 0 &\implies g^{(n-1)}(c_{n-1}) = 0 \quad c_{n-1} \in]0, c_{n-2}[\\
g^{(n-1)}(0) = g^{(n-1)}(c_{n-1}) = 0 &\implies g^{(n)}(\xi - x) = 0 \quad \xi - x \in]0, c_{n-1}[\subset]0, y-x[\\
\implies g^{(n)}(\xi - x) = f^{(n)}(\xi) + Cn! = f^{(n)}(\xi) - \frac{n!}{(y-x)^n}(f(y) - P_{n-1}(x, y-x)) = 0 & \\
\implies f(y) - P_{n-1}(x, y-x) = \frac{f^{(n)}(\xi)}{n!}(y-x)^n \qquad \xi \in]x, y[&
\end{aligned}$$

which completes the proof. ■

Theorem 3.10 (Inverse function). *Let I be an open set and $f : I \to \mathbb{R}$ is continuous and differentiable such that its derivate is non-zero. Thus, f is either monotonic. Furthermore, it is one to one then it has a differentiable inverse f^{-1}:*

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. limit algebra ■

Chapter 4

Integration

Definition (Partition): $I = [a, b] \in \mathbb{R}$. A partition of I is a finite ordered sequence of points in I.

$$P = \{x_0, x_1, \ldots, x_n | a = x_0 \leq x_1 \leq \cdots \leq x_n = b\}$$

A partition pair (P, T) is set

$$(P, T) = \{x_0, t_1, x_1, t_2, x_2 \ldots x_{n-1}, t_n, x_n | a = x_0 \leq t_1 \leq x_1 \leq \cdots \leq t_n \leq x_n = b\}$$

Moreover, define $\|P\| = \max{(x_i - x_{i-1})}$.

The Riemann sum of a function f on the interval $[a, b]$ with respect to the pair partition (P, T) is:

$$R(f, P, T) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

Definition (Riemann Integrability): If there exist a number S that for all $\epsilon > 0$ there exist a $\delta > 0$ such that for all partition (P, T) that if $\|P\| < \delta$ implies $|S - R(f, P, T)| < \epsilon$, then f is Riemann integrable and S is the integral of f on I denoted by

$$\int_a^b f$$

. Furthermore, if S exists then it is unique. Denote the set of all Riemann integrable function on an interval I as $\mathcal{R}_I$.

Theorem 4.1. *Suppose $f \in \mathcal{R}_I$ then f is bounded.*

Proof. By Riemann integrability of f, for a $\epsilon > 0$ there is $\delta > 0$ such that for any partition pair (P, T) that $\|P\| < \delta$ then $|S - R(f, P, T)| < \epsilon$. Consider twp partition pair (P, T) and (P, T') on I with $\|P\| < \delta$ and $t_i = t'_i$ for all i except j. Then by triangle inequality:

$$\begin{aligned} &|R(f, P, T) - R(f, P, T')| \leq |S - R(f, P, T)| + |S - R(f, P, T')| \leq 2\epsilon \\ \implies &|R(f, P, T) - R(f, P, T')| = (x_j - x_{j-1})|f(t_j) - f(t'_j)| \leq 2\epsilon \end{aligned}$$

Corollary 4.2.

1. *If $f, g \in \mathcal{R}_I$ and $c \in \mathbb{R}$ then $f + cg \in \mathcal{R}_I$ and*

$$\int_a^b f + cg = \int_a^b f + c \int_a^b g$$

2. *For a constant function* $f(x) = c$ *its integral is* $c(b-a)$

3. *If* $f(x) \geq 0$ *then* $\int_a^b f \geq 0$

Definition (Commen refinement): Let P_1, P_2 be two partitions on an interval I. Their common refinement $P^* = P_1 \vee P_2$ is defined as

$$P^* = \{z_0 \leq z_1 \leq \cdots \leq z_m | z_i \in P_1 \cup P_2\}$$

Definition (Darboux Integral): Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function. Define the upper Darboux and lower Darboux sums with respect to a partition P as follow

$$\mathrm{U}(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \qquad M_i = \sup_{x \in [x_{i-1}, x_i]} \{f(x)\}$$
$$\mathrm{L}(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} \{f(x)\}$$

Consider P' a refinement of P, then the following inequalities hold:

$$\mathrm{L}(f,P) \leq \mathrm{L}(f,P') \leq \mathrm{U}(f,P') \leq \mathrm{U}(f,P)$$

Therefore as the partition gets refined the upper sum decrease and the lower sum increase. Since both of these sums are bounded then by the completeness axiom the upper and lower integral

$$\overline{\int_a^b} f = \inf \{\mathrm{U}(f,P)\}$$
$$\underline{\int_a^b} f = \sup \{\mathrm{L}(f,P)\}$$

exist. In case they are equal, f is said to be Darboux integrable.

Theorem 4.3. *Darboux integrability is equivalent to Riemann integrability and the value of integrals are equal.*

Proof. Firstly, assume f is bounded and Darboux integrable. Equivalently, for any $\epsilon_1 > 0$ there exists a partition P such that

$$\mathrm{U}(f,P) - \mathrm{L}(f,P) < \epsilon_1$$

Let $\|P\| = \delta_1$ and $0 < \delta < \delta_1$ such that if a partition Q has $\|Q\| < \delta$ then for all partition pairing $|I - R(f,Q,T)| < \epsilon$. Consider $P^* = Q \vee P$. It is clear that

$$\mathrm{U}(f,P^*) - \mathrm{L}(f,P^*) < \epsilon_1 \quad \text{and} \quad \|P^*\| < \delta$$

To estimate $\mathrm{U}(f,Q)$ and $\mathrm{L}(f,Q)$ consider their difference with $\mathrm{U}(f,P^*)$ and $\mathrm{L}(f,P^*)$, respectively.

$$\mathrm{U}(f,Q) - \mathrm{U}(f,P^*) = \sum_{i=1}^{n} M_i^Q(x_i^Q - x_{i-1}^Q) - \sum_{i=1}^{n^*} M_i^*(x_i^* - x_{i-1}^*)$$

The sums are different only in $x_i^* \in P$. Therefore, their difference is in the intervals that are have an endpoint in P and for each of these interval the difference is $(M_j^Q - M_i^*)(x_i^* - x_{i-1}^*)$, note that j is dependent on i, hence

$$\mathrm{U}(f,Q) - \mathrm{U}(f,P^*) = \sum_{i=1}^{n^P} ((M_j^Q - M_i^*)(x_i^* - x_{i-1}^*) < 2Mn^P\delta$$

where M is the bound of f. Similary for the lower bounds we get:

$$\mathrm{L}(f,P^*) - \mathrm{L}(f,Q) = \sum_{i=1}^{n^P} ((m_i^* - m_j^Q)(x_i^* - x_{i-1}^*) < 2Mn^P\delta$$

As a result if set δ such that $\mathrm{U}(f,Q) - \mathrm{L}(f,Q) < \epsilon$ we will be done, since for any partition T $R(f,Q,T), I \in [\mathrm{L}(f,Q), \mathrm{U}(f,Q)]$ hence $|I - R(f,Q,T) < \epsilon$. To do so notice

$$\mathrm{U}(f,Q) - \mathrm{L}(f,Q) = \mathrm{U}(f,Q) - \mathrm{U}(f,P^*) + \mathrm{U}(f,P^*) - \mathrm{L}(f,P^*) + \mathrm{L}(f,P^*) - \mathrm{L}(f,Q) < 4Mn^P\delta + \epsilon_1$$

which will be less ϵ if

$$\delta = \min\left(\frac{\epsilon}{6Mn^P}, \delta_1\right), \qquad \epsilon_1 = \frac{\epsilon}{3}$$

Secondly, assum f is Riemann integrable. Then for any fixed $\epsilon > 0$ then for any two pair partition $(P,T), (P,T')$ such that $\|P\| < \delta$ then

$$R(f,P,T) - R(f,P,T') < \frac{\epsilon}{3}$$

Then choose T such that

$$\mathrm{U}(f,P) - R(f,P,T) < \frac{\epsilon}{3}$$

that is, choose t_i such that

$$M_i - f(t_i) < \frac{\epsilon}{3(b-a)}$$

$$\implies \sum_{i=1}^{n} (M_i - f(t_i))(x_i - x_{i-1}) < \frac{\epsilon}{3(b-a)} \sum_{i=1}^{n} x_i - x_{i-1} < \frac{\epsilon}{3}$$

Similarly one can choose T' so that

$$R(f,P,T') - \mathrm{L}(f,P) < \frac{\epsilon}{3}$$

Therefore:

$$\mathrm{U}(f,P) - \mathrm{L}(f,P) = \mathrm{U}(f,P) - R(f,P,T) + R(f,P,T) - R(f,P,T') + R(f,P,T') - \mathrm{L}(f,P) < \epsilon$$

example: f and g differ in only one point.

Definition (Zero set): A set $A \subset \mathbb{R}$ is a zero set if for each $\epsilon > 0$ there is a countable covering of A of open intervals $]a_i, b_i[$ such that:

$$\sum_{i=1}^{\infty} b_i - a_i \leq \epsilon \tag{4.1}$$

If a property holds for all points except those in a zero set then one says that the property holds almost everywhere.

Proposition 4.4. *The following properties hold for zero sets:*

1. *Covering of A with open intervals is equivalent to covering with closed interval.*
2. *A finit set is a zero set.*
3. *A countable union of zero set is a zero set.*

Definition (Oscillation): Suppose $f : I \to \mathbb{R}$ where I is an interval and $x \in I$ then the oscillation of f at x is

$$\begin{aligned}\mathrm{Osc}(f,x) &= \limsup_{t\to x} f(t) - \liminf_{t\to x} f(t)\\ &= \lim_{h\to 0} \operatorname{diam} f([x-h, x+h])\end{aligned}$$

Proposition 4.5. *f is continuous at x if and only if $Osc(f,x) = 0$.*

Theorem 4.6 (Riemann-Lebesgue theorem). *The function f is Riemann integrable if and only if it is bounded and the set of its discountinuities is zero set.*

Proof. First assume f is Riemann integrable. Let $\mathcal{D}$ be the set of all its discontinuities. Moreover, $\mathcal{D}_n = \{x | \mathrm{Osc}(f,x) \geq \frac{1}{n}\}$. Thus it is clear that $\mathcal{D} = \bigcup \mathcal{D}_n$. We will show that each $\mathcal{D}_n$ is a zero set. By Riemann integrability of f, for any $\epsilon > 0$ we have a partition P such that

$$\mathrm{U}(f,P) - \mathrm{L}(f,P) < \frac{\epsilon}{n}$$

We call $[x_{i-1}, x_i] \in P$ a bad interval if there exist $x \in [x_{i-1}, x_i]$ an interior point, such that $x \in \mathcal{D}_n$.

$$\begin{aligned}\sum_{\text{bad}} (M_i - m_i)(x_i - x_{i-1}) &< \mathrm{U}(f,P) - \mathrm{L}(f,P) < \frac{\epsilon}{n}\\ \frac{1}{n}\sum_{\text{bad}} (x_i - x_{i-1}) &< \sum_{\text{bad}} (M_i - m_i)(x_i - x_{i-1}) < \frac{\epsilon}{n}\\ \implies \sum_{\text{bad}} (x_i - x_{i-1}) &< \epsilon\end{aligned}$$

and since the endpoints are finite then $\mathcal{D}_n$ is a zero set and therefore, $\mathcal{D}$ is zero set. Seconde assume that $f : [a,b] \to \mathbb{R}$ is bounded and $\mathcal{D}$ is a zero set. Choose n such that:

$$\frac{1}{n} < \epsilon_1$$

for ϵ_1 that is to be determined. Since $\mathcal{D}_n \subset \mathcal{D}$ then it is a zero set as well. In other words for any ϵ_2 there is covering of $\mathcal{D}_n$, $I_1, I_2, \ldots$ such that

$$\sum \operatorname{diam} I_i < \epsilon_2$$

For any $x \notin \mathcal{D}_n$ we know that is an open interval J_x such that $M_{J_x} - m_{J_x} < \dfrac{1}{n}$. Let $I = \bigcup I_i$ and $J = \bigcup J_x$. It is clear than $I \cup J$ is a covering of $[a,b]$. Since $[a,b]$ is compact then the open covering has a Lebesgue numberλ. Let P be a partition such that $\|P\| < \lambda$ then an interval $[x_{i-1}, x_i]$ is bad if it is wholly within a I_i and it is good if it is not.

$$\begin{aligned} \mathrm{U}(f,P) - \mathrm{L}(f,P) &= \sum_{\text{good}} (M_i - m_i)(x_i - x_{i-1}) + \sum_{\text{bad}} (M_i - m_i)(x_i - x_{i-1}) \\ &< \frac{1}{n} \sum_{\text{good}} (x_i - x_{i-1}) + 2M \sum_{\text{bad}} (x_i - x_{i-1}) \\ &< \frac{b-a}{n} + 2M\epsilon_2 < (b-a)\epsilon_1 + 2M\epsilon_2 = \epsilon \end{aligned}$$

by setting $\epsilon_1 = \dfrac{\epsilon}{2(b-a)}$ and $\epsilon_2 = \dfrac{\epsilon}{4M}$. ∎

Corollary 4.7.

1. *Any continuous function $f : [a, b] \to \mathbb{R}$ is integrable.*

   Proof. Since there is no point of discontinuity then it is a zero set. □

2. *Any monotonic function $f : [a, b] \to \mathbb{R}$ is integrable.*

3. *Product of two integrable function is integrable.*

Theorem 4.8 (Fundamental theorem of calculus). *If f is an integrable function then its indefinite integral*

$$F(x) = \int_a^x f(t)dt$$

is continuous at x. Furthermore, its derivative is equal to $f(x)$ at every point x that f is continuous.

Definition: $F(x)$ is anti-derivate of $f(x) : [a, b] \to \mathbb{R}$ if

$$F'(x) = f(x)$$

for all $x \in [a, b]$.

Corollary 4.9. *Every continuous function has an anti-derivative.*

Theorem 4.10. *Anti-derivate of a Riemann integrable function if exists differs from its indefinite integral by a constant.*

Chapter 5

Series

Theorem 5.1. *A series s_n is convergent if and only if for each $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that:*

$$m, n \geq N \implies \left| \sum_{i=n} m a_m \right| \leq \epsilon$$

Proof. Obviously any convergent sequence is Cauchy. Furthermore, due to completeness of $\mathbb{R}$ every Cauchy sequence is convergent. ∎

Corollary 5.2. *The series s_n is convergent if $a_n \to 0$.*

Theorem 5.3.

1. *If $|a_n| < b_n$ for all $n > N$ for a sufficiently large N then convergence of $\sum b_n$ implies the convergence $\sum a_n$.*
2. *If $0 < b_n < a_n$ for all $n > N$ for a sufficiently large N then divergence of $\sum b_n$ implies the divergence $\sum a_n$.*

Corollary 5.4. *Absolute convergence implies convergence.*

Theorem 5.5 (Integral Test). *Consider the improper integral $\int_0^\infty f$ and the series $\sum_{i=1}^\infty a_k$*

1. *$0 \leq a_k \leq f(x)$ for sufficiently large k and each $x \in]k-1, k]$, then the convergence of integral implies the convergence of the series.*
2. *Similarly if $0 \leq f(x) \leq a_k$ for sufficiently large k and each $x \in [k, k+1[$, then the divergence of integral implies the divergence of the series.*

Definition: The exponential growth rate of the series $\sum a_n$

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{a_k}$$

Theorem 5.6 (Root test). *If $\alpha < 1$ the series is convergent and if $\alpha > 1$ it is divergent. If $\alpha = 1$ the test is inconclusive.*

Theorem 5.7 (Ratio test). *Let the ratio between successive terms of the series a_k be $r_k = |\frac{a_{k+1}}{a_k}|$*

$$\rho = \limsup r_k \qquad \lambda = \liminf r_k$$

If $\rho < 1$ then the series converges, if $\lambda > 1$ then the series diverges, and otherwise the ratio test is inconclusive.

Theorem 5.8. *Let $a_1 \geq a_2 \geq \cdots \geq 0$ be a decreasing non-negative sequence then the alternating series*

$$\sum_{n=1}^{\infty} a_n(-1)^n \tag{5.1}$$

is convergent.

Theorem 5.9. *Suppose $\sum c_k x^k$ is a power series. Its radius of convergence R is unique and is such that for $|x| < R$ the power series converges and for $|x| > R$ diverges.*

$$R = \frac{1}{\limsup \sqrt[k]{c_k}}$$

Chapter 6

Function Spaces

Definition (Point Convergence): for each point there is a ϵ

example x^n, 1/2 lines, $sqrt(x^2 + 1/n)$, rationals

Definition (Uniform Convergence): there is a ϵ for all points

Corollary 6.1. *Uniform convergence implies point convergence.*

example x/n with restriction

Theorem 6.2. *f_n are uniformly convergent and continuous then f is continuous*

Definition: $\mathcal{C}^0(X, \mathbb{R})$: all continuous function from X to $\mathbb{R}$. $d(f, g) = \sup \{|f(x) - g(x)| : x \in X\}$

Definition: $\|f\| = \sup\{|f(x)| : x \in X\}$ therefore $d(f, g) = \|f - g\|$

Theorem 6.3. *$f_n : [a, b] \to \mathbb{R}$ uniformly convergent if f_n is Riemann integrable then f is Riemann integrable*

$$\int_a^b \underbrace{\lim f_n}_{f} = \lim \int_a^b f_n$$

Lemma 6.4. *metric spaces and $f_n : X \to X'$ are uniformly convergent if f_n is bounded then f is bounded.*

Proposition 6.5. *$(\mathcal{C}_b^0, d)$ is a complete metric space.*

Definition: $\mathcal{C}_b^0(X, \mathbb{R})$ and $\mathcal{C}_b(X, \mathbb{R})$ closed subset of $\mathcal{B}(X, \mathbb{R})$

exampele: any compact metric space

Theorem 6.6. *f_n are differentiable functions*

1. *f_n' are uniformly convergent to g*
2. *f_n are point convergent*

Proposition 6.7. *$f_n : [a, b] \to \mathbb{R}$ consider $\sum f_n$*

1. *f_n are riemann integrable and the series uniformly convergent then the $\sum f_n$ is riemann integrable and*
$$\int_a^b \sum f_n = \sum \int_a^b f_n$$
2. *Similarly for derivative*

Definition (Wierstrass M test): This is super cool

power series and convergence radius of convergence of integral/derivative of power series is equal to the radius of convergence of the originalo series.

Theorem 6.8. *in the convergence cirle the power series is inifinitely integrable and differentiable, and coefficients are taylor coefficients*

analytical definition proof of analytical means analytical in interval using a pair sequence