Chapter 1

Differentiation

Definition: Let I be an interval in $\mathbb{R}$. If a is an interior point of I, then we say that $f : I \to \mathbb{R}$ is differentiable at a when the following limit exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The limit, if exists, is denoted by $f'(a)$. If a is an end point and the length of the interval is greater than zero, then the limit only exists from one direction.

Equivalently, there exists a line l, not parallel to y-axis, in form of $l : A(x) = mx + b$, that is tangent to f at $x = a$. In this case:

$$\lim_{x \to a} \frac{f(x) - [mx + b]}{x - a} = 0 \qquad A(a) = f(a)$$

In a general case, two functions f, g are tangent to each other at $x = a$ if:

$$f(a) = g(a) \qquad \lim_{x \to a} \frac{f(x) - g(x)}{x - a} = 0 \tag{1.1}$$

Corollary 1.1.

1. *f is differentiable at a if it is continuous at a.*

2. *If $f'(a) > 0$, there exists $\delta > 0$ such that for $x \in\,]a - \delta, a[\, \cap I \implies f(x) < f(a)$ and for $x \in\,]a, a + \delta[\, \cap I \implies f(x) < f(a)$. And if $f'(a) < 0$ the inequality sign are reversed. Therefore, if f has a local extremum at a, then in case $f'(a)$ exists, $f'(a) = 0$.*

Example 1.1. a function that its derivate is not continuous (with $\sin \frac{1}{x}$).

Theorem 1.2 (Rolle's theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable on the interval. If $f(a) = 0, f(b) = 0$, then there exists $c \in\,]a, b[$ such that:*

$$f'(c) = 0$$

Proof. If $f \equiv 0$ on $[a, b]$ then its derivative $f'(x) \equiv 0$ on $[a, b]$. If $f(x) \neq 0$ for some $x \in]a, b[$ then it must have a non-zero maximum or minimum at some $c \in]a, b[$. Since $[a, b]$ is compact then by continuity of f, $f([a, b])$ is also compact in $\mathbb{R}$ and therefore f attains its maximum or minimum. We know that at least one of its extremities must lie in $]a, b[$, say point c, hence by Item 2 $f'(c) = 0$. ■

Theorem 1.3 (Mean value theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous and differentiable on the interval,then there exists $c \in\,]a, b[$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof. Define $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Then it is clear that $g(a) = g(b) = 0$ and g is continous and differentiable on the interval. Then by Theorem 1.2 there exists $c \in]a, b[$ such that $g'(c) = 0$. Equivalently:

$$\begin{aligned} g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \\ \implies f'(c) = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

which concludes the proof. ■

Corollary 1.4 (Growth Estimate). *If $|f'(x)| \leq M$ in $]a, b[$ then f satisfies the global lipschitz condition for all $x, y \in [a, b]$ $|f(x) - f(y)| \leq M|x - y|$.*

Corollary 1.5. *Let $f : [a, b] \to \mathbb{R}$ is continuous and $f'(x) < 0$ (or $f'(x) > 0$) for all $x \in]a, b[$ then f is strictly increasing (or decreasing) on $[a, b]$.*

Theorem 1.6. *$f : [a, b] \to \mathbb{R}$ is continuous and differentiable on $]a, b[$ then for $f'(]a, b[)$ the intermediate value theorem holds and thus it is an interval.*

Proof. Let $x_1, x_2 \in]a, b[$. WLOG assume $f'(x_1) < f'(x_2)$, we wish to prove that for all $y^* \in]f'(x_1), f'(x_2)[$ there is a $x^* \in]x_1, x_2[$ such that $f(x^*) = y^*$. Put $g(x) = f(x) - y^* x$. By differentiability of f on $[a, b]$, g is differentiable on $[a, b]$. Then, $g'(x_1) = f'(x_1) - y^* < 0$ and $g'(x_2) = f'(x_2) - y^* > 0$, therefore there are $t_1, t_2 \in]x_1, x_2[$ such that $g(t_1) < g(x_1)$ and $g(t_2) < g(x_2)$. Since g is continuous on $[x_1, x_2]$ then it must attains its minimum at some $x^* \in [x_1, x_2]$. However x^* can't be x_1 or x_2 and hence $x^* \in]x_1, x_2[$. It is then easy to see that $f'(x^*) = y^*$. ■

Definition (Darboux continous): A function f is Darboux continous if it posseses the intermediate value property.

For example f' of differentiable function is Darboux continuous.

Theorem 1.7 (Cauchy's mean value theorem). *$f, g : [a, b] \to \mathbb{R}$ are continuous then there exists a $c \in\,]a, b[$, such that:*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

Proof. Define $h(x) = (f(x) - f(a))(g(b) - g(a)) - (g(x) - g(a))(f(b) - f(a))$, then clearly $h(a) = 0, h(b) = 0$ and $h(x)$ is continous and differentiable on $[a, b]$. Hence by applying the theorem 1.2 for some $c \in]a, b[$ we have:

$$\begin{aligned} & h'(c) = 0 \\ & \implies f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) = 0 \\ & \implies f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \end{aligned}$$

Theorem 1.8 (L'Hopital's rule). *Suppose that* $\lim_{x\to a^+} f(x) = 0, \lim_{x\to a^+} g(x) = 0$ *where* f, g *are differentiable on a open interval* $I =]a, b[$ *for some* b *such that* $g'(x) \neq 0$ *in* I *except maybe at* $x = a$ *and the limit*

$$\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L$$

exists, then:

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L$$

Proof. For a fixed $\epsilon > 0$ there exists a $\delta > 0$ such that:

$$|x - a| < \delta \implies \left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$$

then since $f(t), g(t) \to 0$ as $t \to a$ from right side then there must be a $t \in]a, x[$ such that

$$\left|\frac{f(x) - f(t)}{g(x) - g(t)} - \frac{f(x)}{g(x)}\right| < \frac{\epsilon}{2}$$

then simply:

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &\leq \left|\frac{f(x)}{g(x)} - \frac{f(x) - f(t)}{g(x) - g(t)}\right| + \left|\frac{f(x) - f(t)}{g(x) - g(t)} - L\right| && (1.2)\\
&< \frac{\epsilon}{2} + \left|\frac{f'(\theta)}{g'(\theta)} - L\right| && (1.3)\\
&< \epsilon && (1.4)
\end{aligned}$$

Note that $\theta \in]t, x[$ and thus $|\theta - a| < \delta$ ■

Definition (Higher order derivatives): f is said to be r_{th}-differentiable at x if it is differentiable r times. The r_{th} derivative of f is denoted as $f^{(r)}$. If $f^{(r)}$ exists for all r and x then f is said to be infinitely differentiable or smooth.

Definition (Smoothness classes): The set of all f is continuosly r_{th}-differentiable is called class $\mathcal{C}^r$.

Definition (Taylor polynomial): The r_{th}-order Taylor polynomial of an r_{th}-order differentiable function f at x is

$$P_r(x, h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots + \frac{f^{(r)}(x)}{r!}h^r = \sum_{n=0}^{r} \frac{f^{(n)}(x)}{n!}h^n$$

Theorem 1.9 (Taylor approximation theorem). *Let* f *be a* r*-differentiable function at* x *then:*

1. $$\frac{f(x + h) - P_r(x, h)}{h^r} \to 0 \text{ as } h \to 0$$

2. *and* P_r *is the only* r_{th} *degree polynomial that has such property.*

3. *Furthermore, if f is r-differentiable on an interval I for every $x, y \in I$, there exists ξ between x, y such that:*

$$f(y) - P_{r-1}(x, y-x) = \frac{f^{(r)}(\xi)}{(r)!}(y-x)^r$$

Proof.

1. For the base case $r = 1$

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = f'(x) - f'(x) = 0$$

and by induction we prove the case $r = n \geq 2$

$$\lim_{h\to 0} \frac{f(x+h) - P_n(x,h)}{h^n} = 0$$
$$\iff \forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \ |h| < \delta \implies |f(x+h) - P_n(x,h)| < \epsilon |h^n|$$

Let $g(h) = f(x+h) - P_n(x,h)$ then since both $f(x+h)$ and $P_n(x,h)$ are differentiable then we apply Theorem 1.3

$$\begin{aligned} g(h) - g(0) = g(h) &= h(g'(c)) \\ &= h(f'(x+c) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{(k-1)!} c^{k-1}) \\ &= h(f'(x+c) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(x)}{k!} c^{k}) \\ &= h(f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^{k}) \end{aligned}$$

for some $c \in]0, h[$. Note that f' is $(n-1)$-differentiable at x thus by induction for any $\epsilon > 0$ there exists a δ such that if $c < \delta$ then:

$$|f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^k| < \epsilon |c^{n-1}|$$

which means

$$\begin{aligned} |f(x+h) - P_n(x,h)| = |g(h)| &= |h||f'(x+c) - \sum_{k=0}^{n-1} \frac{f'^{(k)}(x)}{k!} c^k| \\ &< |h|\epsilon|c^{n-1}| < \epsilon|h^n| \end{aligned}$$

Therefore for any ϵ if $h < \delta$ then $c < \delta$ and the result holds.

2. Let $Q_r(x,h)$ be another r_{th} degree polynomial such that

$$\lim_{h\to 0} \frac{f(x+h) - Q_r(x,h)}{h^r} = 0$$

then

$$\lim_{h\to 0} \frac{P_r(x,h) - Q_r(x,h)}{h^r} = 0$$

however this can only happen if $Q_r(x,h) = P_r(x,h)$.

3. Again for the base case $r = 1$

$$f(y) - f(x) = f'(\xi)(y - x)$$

which is the Theorem 1.3. for $r = n$ we have that

$$g(h) = f(x+h) - P_{n-1}(x,h) + Ch^n \implies g(0) = g'(0) = \cdots = g^{(n-1)} = 0$$

Set C such that $g(y-x) = 0$. Then by applying Theorem 1.2 $(n-1)$ times

$$\begin{aligned}
g(0) = g(y-x) = 0 &\implies g'(c_1) = 0 \quad c_1 \in]0, y-x[\\
g'(0) = g'(c_1) = 0 &\implies g'(c_2) = 0 \quad c_2 \in]0, c_0[\\
&\vdots \\
g^{(n-2)}(0) = g^{(n-2)}(c_{n-2}) = 0 &\implies g^{(n-1)}(c_{n-1}) = 0 \quad c_{n-1} \in]0, c_{n-2}[\\
g^{(n-1)}(0) = g^{(n-1)}(c_{n-1}) = 0 &\implies g^{(n)}(\xi - x) = 0 \quad \xi - x \in]0, c_{n-1}[\subset]0, y-x[\\
\implies g^{(n)}(\xi - x) = f^{(n)}(\xi) + Cn! &= f^{(n)}(\xi) - \frac{n!}{(y-x)^n}(f(y) - P_{n-1}(x, y-x)) = 0 \\
\implies f(y) - P_{n-1}(x, y-x) &= \frac{f^{(n)}(\xi)}{n!}(y-x)^n \qquad \xi \in]x, y[
\end{aligned}$$

which completes the proof. ■

Theorem 1.10 (Inverse function). *Let I be an open set and $f : I \to \mathbb{R}$ is continuous and differentiable such that its derivate is non-zero. Thus, f is either monotonic. Furthermore, it is one to one then it has a differentiable inverse f^{-1}:*

$$f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. limit algebra ■