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Chapter 1

Multivariable Calculus

1.1 Linear Algebra

Definition (Normed vector space): Let V be a vector space. A **norm** is a real valued function $\|\cdot\| : V \to \mathbb{R}$ which has the following properties

1. $\forall x \in V,\ \|x\| > 0.$
2. $\|x\| = 0 \implies x = 0.$
3. $\forall x \in V\ \forall \alpha \in \mathbb{F},\ \|\alpha x\| = |\alpha|\,\|x\|.$
4. $\forall x, y \in V\ \ \|x + y\| \leq \|x\| + \|y\|.$

Each normed vector space induces a metric space (V, d) where $d(x, y) = \|y - x\|$.

Definition: Assume V is a vector space and let $\|\cdot\|_1,\ \|\cdot\|_2$ be two norms for V. They are said to be equivalent when

$$\exists c_1, c_2 > 0\ \forall x : \qquad c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|$$

To check if the above definition is indeed an equivalence relation, we must show that following:

Reflexive $\|\cdot\|_1 \sim \|\cdot\|_1$.

Symmetric $\|\cdot\|_1 \sim \|\cdot\|_2 \implies \|\cdot\|_2 \sim \|\cdot\|_1$.

Transitive $\|\cdot\|_1 \sim \|\cdot\|_2,\ \|\cdot\|_2 \sim \|\cdot\|_3 \implies \|\cdot\|_1 \sim \|\cdot\|_3$.

Theorem 1.1. *All norms defined on a finite dimensional vector spacev V are equivalent.*

Proof. Let $\|\cdot\|$ be an arbitrary norm on V and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Let $\|\cdot\|_2$ be L_2-norm (Euclidean norm). It will suffice to show $\|\cdot\| \sim \|\cdot\|_2$. Let

$$M = \max(\|e_1\|, \ldots, \|e_n\|)$$

Take $x \in V$, writing $x = \sum_{i=1}^{n} \xi_i e_i$ we have:

$$\|x\| = \left\| \sum_{i=i}^{n} \xi_i e_i \right\| \leq \sum_{i=1}^{n} |\cdot| \, \xi_i \, \|e_i\| \leq M\sqrt{n}\, \|x\|_2$$

Taking $c_2 = M\sqrt{n}$ proves the right inequality. For the left inequality we need the following lemma

Lemma 1.2. *If V is a normed vector space with $\|\cdot\|_2$, as defined above, is viewed as metric space $(V, \|\cdot\|_2)$ then $\|\cdot\| : V \to \mathbb{R}$ is continuous.*

Proof. Let $x_0 \in V$ and M be defined as above. For any $\epsilon > 0$ consider $\delta = \frac{\epsilon}{M\sqrt{n}}$ then if $\|x - x_0\|_2 < \delta$

$$||\|x| - \| \cdot \| \, x_0\| \leq \|x - x_0\| \leq M\sqrt{n} \, \|x - x_0\| \leq \epsilon$$

Now consider the sphere of radius $r = 1$ centered at 0, $S_1(0) = S_1 = \{x \in V : \|x\|_2 = 1\}$. One can show that S is compact. Therefore, $\|x\|$ assumes its minimum on S. Let $a = \|x_0\|$ be the minimum. Since $0 \notin S$ then $a > 0$. By letting $y = x/\|x\|_2$, we have $y \in S$ and thus $a \leq \|y\|$ which is

$$a \|x\|_2 \leq \|x\|$$

Taking $c_1 = a$ proves the theorem. ■

Theorem 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation is invertible if and only if there exists a c such that:*

$$c \|x\| \leq \|f(x)\|$$

Proof. A linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is surjective because $\dim \operatorname{Im} f + \dim \ker f = n$. Hence, we only need to show that f is one-to-one. ■