Contents

Chapter 1

Multivariable Calculus

1.1 Linear Algebra

1.1.1 Vector Spaces

Definition (Normed vector space): Let V be a vector space. A **norm** is a real valued function $\| \cdot \| : V \to \mathbb{R}$ which has the following properties

1. $\forall x \in V, \ \|x\| > 0$.

2. $\|x\| = 0 \implies x = 0$.

3. $\forall x \in V \ \forall \alpha \in \mathbb{F}, \ \|\alpha x\| = |\alpha| \, \|x\|$.

4. $\forall x, y \in V \ \|x + y\| \leq \|x\| + \|y\|$.

Each normed vector space induces a metric space (V, d) where $d(x, y) = \|y - x\|$.

Theorem 1.1. *In every normed space $(V, \| \cdot \|)$ we have*

$$| \|v\| - \|w\| | \leq \|v - w\|$$

Hence the norm is Lipschitz continuous.

Definition: Assume V is a vector space and let $\| \cdot \|_1, \ \| \cdot \|_2$ be two norms for V. They are said to be equivalent when

$$\exists c_1, c_2 > 0 \ \forall x : \qquad c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|$$

To check if the above definition is indeed an equivalence relation, we must show that following:

Reflexive $\| \cdot \|_1 \sim \| \cdot \|_1$.

Symmetric $\| \cdot \|_1 \sim \| \cdot \|_2 \implies \| \cdot \|_2 \sim \| \cdot \|_1$.

Transitive $\| \cdot \|_1 \sim \| \cdot \|_2, \ \| \cdot \|_2 \sim \| \cdot \|_3 \implies \| \cdot \|_1 \sim \| \cdot \|_3$.

Remark 1. Equivalent norms induce equivalent metrics, hence they induce the same topology.

Theorem 1.2. *All norms defined on a finite dimensional vector space V are equivalent.*

Proof. Let $\| \cdot \|$ be an arbitrary norm on V and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Let $\| \cdot \|_2$ be L_2-norm (Euclidean norm). It will suffice to show $\| \cdot \| \sim \| \cdot \|_2$. Let

$$M = \max(\|e_1\|, \ldots, \|e_n\|)$$

Take $x \in V$, writing $x = \sum_{i=1}^n \xi_i e_i$ we have:

$$\|x\| = \left\| \sum_{i=i}^n \xi_i e_i \right\| \leq \sum_{i=1}^n |\xi_i| \, \|e_i\| \leq M\sqrt{n} \, \|x\|_2$$

Taking $c_2 = M\sqrt{n}$ proves the right inequality. For the left inequality we need the following lemma

Lemma 1.3. *If V is a normed vector space with $\| \cdot \|_2$, as defined above, is viewed as metric space $(V, \| \cdot \|_2)$ then $\| \cdot \| : V \to \mathbb{R}$ is continuous.*

Proof. Let $x_0 \in V$ and M be defined as above. For any $\epsilon > 0$ consider $\delta = \frac{\epsilon}{M\sqrt{n}}$ then if $\|x - x_0\|_2 < \delta$

$$|\|x\| - \|x_0\|| \leq \|x - x_0\| \leq M\sqrt{n} \, \|x - x_0\| \leq \epsilon$$

    Now consider the sphere of radius $r = 1$ centered at 0, $S_1(0) = S_1 = \{x \in V : \|x\|_2 = 1\}$. One can show that S is compact. Therefore, $\|x\|$ assumes its minimum on S. Let $a = \|x_0\|$ be the minimum. Since $0 \notin S$ then $a > 0$. By letting $y = x/\|x\|_2$, we have $y \in S$ and thus $a \leq \|y\|$ which is

$$a \|x\|_2 \leq \|x\|$$

Taking $c_1 = a$ proves the theorem.   ∎

Example 1.1. The closed unit ball in the infinite dimensional vector space $C([0,1], \mathbb{R})$ with $\|f\| = \max f(x)$ is not compact. Take $f_n(x) = x^n$. Obviously $\|f_n\| = 1$, however f_n doesn't uniformly converge and hence f_n doesn't have a limit in $C([0,1], \mathbb{R})$ with the max norm. Consider the following norm

$$\|f\|_I = \int_0^1 |f(x)| \, \mathrm{d}x$$

Note that $\| \cdot \|_I$ and $\| \cdot \|_{\max}$ are not equivalent. Let $g(x) = 0$ for all $x \in [0,1]$. Then

$$\|f_n - g\|_I = \frac{1}{n+1} \to 0 \quad \text{as} \, n \to \infty.$$

Definition (Banach space): A normed vector space V that is complete is a **Banach space**. A **Hilbert Space** is a Banach space whose norm is induced by an inner product.

Corollary 1.4. *Any finite dimensional normed vector space V over a normed complete field $\mathbb{F}$ is a Banach space.*

1.1.2 Linear Maps

Let V and W be a vector spaces over $\mathbb{F}$. A map $T : V \to W$ is **linear** if

$$T(x + \lambda y) = T(x) + \lambda T(y)$$

for all $x, y \in V$ and $\lambda \in \mathbb{F}$.

Definition: Let $(V, \| \cdot \|_V)$ and $(W, \| \cdot \|_W)$ be normed spaces then, a linear transformation $T : V \to W$ is **bounded** if there exists a constant $C > 0$ such that

$$\|Tv\|_W \leq C \|v\|_V$$

for all $v \in V$. We denote the set of all linear map from $V \to W$ as $\mathcal{L}(V, W)$ and the set of all bounded linear maps as $\mathcal{B}(V, W)$. If $T \in \mathcal{L}(V, W)$ is bijective such that $T^{-1} \in \mathcal{L}(V, W)$, then T is called an **isomorphism** and V, W are **isomorphic**. An operator $T \in \mathcal{L}(V, W)$ is called **isometric** if $\|Tv\|_W = \|v\|_V$ for all $v \in V$.

Definition: If $(V, \| \cdot \|_V), (W, \| \cdot \|_W)$ are normed spaces then the **operator norm** of a linear transformation $T : V \to W$ is

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \,\middle|\, v \neq 0 \right\}$$

Proposition 1.5. *Let $T : U \to V$ and $T' : V \to W$ be two linear transformations.*

$$\|T' \circ T\| \leq \|T\| \, \|T'\|$$

Proof. for an arbitrary non-zero $x \in U$

$$\|T' \circ T(x)\|_W \leq \|T'\| \, \|Tx\|_V \leq \|T'\| \, \|T\| \, \|x\|_U$$

which implies

$$\|T' \circ T\| \leq \|T\| \, \|T'\|$$

Theorem 1.6. *Let $(V, \| \cdot \|_V)$ and $(W, \| \cdot \|_W)$ be normed spaces and $T : V \to W$ be a linear transformation. The following are equivalent*

1. *$\|T\|$ is finite.*

2. *T is bounded.*

3. *T is Lipschitz continuous.*

4. *T is continuous at a point.*

5. *$\sup_{\|v\|_V = 1} \|Tv\|_W < \infty$.*

Proof. item 1 $\Rightarrow$ item 2: Obviously

$$\frac{\|Tv\|_W}{\|v\|_V} \leq \|T\|$$
$$\implies \|Tv\|_W \leq \|T\| \, \|v\|_V$$

note that if $v = 0$ then $Tv = 0$ as well and thus the last inequality holds for all $v \in V$.

item 2 $\Rightarrow$ item 3:

$$\|Tv - Tu\|_W = \|T(u - v)\|_W \leq C \|u - v\|_V$$

item 3 $\Rightarrow$ item 4: Trivial.

item 4 $\Rightarrow$ item 5: Let T be continuous at $u \in V$. Then there is a $\delta > 0$ such that

$$\|v - u\| < \delta \implies \|Tv - Tu\|_W = \|T(v - u)\|_W < 1$$

Now for an arbitrary non-zero v we have

$$\left\| \left(\frac{\delta v}{2 \|v\|_V} + u \right) - u \right\|_V < \delta$$

Therefore

$$\left\| T\left(\frac{\delta v}{2 \|v\|_V} \right) \right\|_W < 1$$
$$\left\| T\left(\frac{v}{\|v\|_V} \right) \right\|_W < \frac{2}{\delta}$$

item 5 $\Rightarrow$ item 1: Let $v \in V$ be an arbitrary vector. Then

$$\sup \left\| T\left(\frac{v}{\|v\|_V} \right) \right\|_W < \infty$$
$$\implies \sup \frac{\|Tv\|_W}{\|v\|_W} < \infty$$

Theorem 1.7. *If V is a finite dimensional normed vector space then any linear transformation $T : V \to W$ is continuous.*

Proof. Since V is finite dimensional, according to Theorem 1.2, any two norms are equivalent. Hence, take $\| \cdot \|_2$ to be Euclidean norm over a basis $\{e_1, \ldots, e_n\}$. Let x be such that $\|x\|_2 < \delta$ for some $\delta > 0$. Therefore, $|\xi_i| < \delta^2$

$$\|Tx\|_W = \left\| \sum \xi_i T(e_i) \right\|_W \leq \sum |\xi_i| \|T(e_i)\|_W \leq \delta^2 K$$

where $K = \max \|T(e_i)\|_W$. By letting $\delta = \sqrt{\frac{\epsilon}{K}}$ we proved continuity at 0 and hence the continuity by Theorem 1.6. ∎

Theorem 1.8. *For two normed vector spaces V, W, $(\mathcal{B}(V, W), \|T\|)$ is a normed vector space. Moreover, it is a Banach space when W is a Banach space.*

Proof. Clearly $\mathcal{B}(V, W)$ is a vector space. For its norm $\|T\|$ we have

1. $\|T\| \geq 0$ by definition.

2. if $\alpha \in \mathbb{F}_W$ then

$$\|\alpha T\| = \sup \left\{ \frac{\|(\alpha T)v\|_W}{\|v\|_V} \middle| v \neq 0 \right\} = |\alpha| \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \middle| v \neq 0 \right\} = |\alpha| \|T\|$$

3. for the triangle inequality

$$\|T_1 + T_2\| = \sup \left\{ \frac{\|(T_1 + T_2)v\|_W}{\|v\|_V} \right\}$$

$$\leq \sup \left\{ \frac{\|T_1 v\|_W + \|T_2 v\|_W}{\|v\|_V} \right\}$$

$$= \sup \left\{ \frac{\|T_1 v\|_W}{\|v\|_V} \right\} + \sup \left\{ \frac{\|T_2 v\|_W}{\|v\|_V} \right\}$$

$$= \|T_1\| + \|T_2\|$$

Suppose W is a Banach space and $\{T_i\} \in \mathcal{B}(V, W)$ is a Cauchy sequence. Then for all $v \in V$

$$\forall \epsilon \, \exists N \text{ s.t. } m, n > N \implies \|T_m v - T_n v\|_W \leq \|T_m - T_n\| < \epsilon$$

$\{T_i v\}$ is a Cauchy sequence. Since W is complete then $T_i v \to Tv$ for some function T. We claim that T is a bounded linear map and is the limit of $T_i \to T$.

$$T(v + cu) = \lim_{i \to \infty} T_i(v + cu) = \lim_{i \to \infty} T_i v + c T_i u$$

$$= Tv + cTu$$

Note that $|\|T_m\| - \|T_n\|| \leq \|T_m - T_n\|$ ans hence $\|T_i\|$ is a Cauchy in sequence in $\mathbb{R}$ that has a limit t. There exists a N such that $|\|T_n\| - t| < 1$ for all $n \geq N$.

$$\frac{\|Tv\|_W}{\|v\|_V} = \lim_{i \to \infty} \frac{\|T_i v\|_W}{\|v\|_V} < t + 1$$

hence T is bounded and $T \in B(V, W)$. Finally, we show that $T_i \to T$. For an arbitrary $v \neq 0$ and $\epsilon > 0$ there exist N such that

$$n \geq N \implies \|T_i v - Tv\|_W < \epsilon \|v\|_V$$

which means that

$$\|T_i - T\| = \sup \frac{\|T_i v - Tv\|_W}{\|v\|_V} < \epsilon$$

Therefore $T_i \to T$ as desired. $\blacksquare$

Theorem 1.9. *Let $(V, \|\cdot\|)$ be a normed space. Then any linear transformation $T : \mathbb{R}^n \to V$ is continuous. Furthermore, if T is a bijection, it is a homeomorphis.*

Proof. Since $\mathbb{R}^n$ is finite then by Theorem 1.7, T is continuous. Assuming T is bijective, we must show that its inverse T^{-1} is continuous as well. Since T is a bijection then T is a linear isomorphism and $\dim V = \dim \mathbb{R}^n = n$ hence T^{-1} is a continuous map. $\blacksquare$

Theorem 1.10. $T : \mathbb{R}^n \to \mathbb{R}^n$ *linear transformation is invertible if and only if there exists a c such that:*

$$c \, \|x\| \leq \|Tx\|$$

Proof. If T is invertible then $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is bounded and thus

$$\left\| T^{-1} x \right\| \leq c \left\| x \right\|$$

and since T is bijective then there exists y such that $x = Ty$ which implies

$$\left\| y \right\| \leq c \left\| Ty \right\|$$

If there exists such c then $\left\| Tx \right\| > 0$ for all non-zero x and hence $\ker T = 0$ which implies that T is a bijection and hence T is invertible. ∎

Definition: Let $V_1, V_2, \ldots, V_n$ be normed vector spaces. Then the function $\phi : V_1 \times \cdots \times V_n \to W$ is n-**linear** if by fixing any $n - 1$ component, ϕ is linear relative to the remaining components.

Proposition 1.11. *If $V_1, V_2, \ldots, V_n$ are normed vector spaces and $\phi : V_1 \times \cdots \times V_n \to W$ is a n-linear then the followings are equivalent*

1. *ϕ is continuous.*

2. *ϕ is continuous at a point.*

3. *ϕ is bounded, that is there exists a constant $C > 0$ such that*

$$\left\| \phi(v_1, \ldots, v_n) \right\|_W \leq C \left\| v_1 \right\|_{V_1} \ldots \left\| v_n \right\|_{V_n}$$

Proof. Item 1 $\implies$ Item 2: Trivial.
Item 2 $\implies$ Item 3 ∎

Exercises

1. Show that for a linear transformation T, $\left\| T \right\| = \sup_{\left\| v \right\|_V \leq 1} \left\| Tv \right\|_W$.

1.2   Derivative

Let V, W be finite dimensional vector spaces and $f : U \subset V \to W$ where U is open. Then f is differentiable at x_0 when a linear transformation $T : V \to W$ such that

$$\lim_{\|h\| \to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

Or equivalently there exists a sublinear function $R(h)$ such that

$$f(x_0 + h) - f(x_0) - Th = R(h) \qquad \frac{R(h)}{\|h\|} \to 0$$

T if it exists is unique, represented by $f'(x_0)$, $\mathrm{D}\, f$, or $\mathrm{d}\, f(x)$ and called the **total derivative** or **Fréchet derivative**.

Example 1.2. Every linear function $f : V \to W$ with $f(v) = Tv + b$ where $b \in W$ is differentiable and $\mathrm{D}\, f(v) = T$. Since

$$\|h\|_V \, \delta \implies \|f(v + h) - f(v) - \mathrm{D}(h)\|_W = \|T(v + h) - Tv - Th\|_W = 0 < \epsilon \, \|h\|_V$$

Hence, the derivative of any linear function is constant. Consider $S : V \times V \to V$ with $S(v, u) = v + u$. S is differentiable because S is linear (why?). We claim that $\mathrm{D}\, S = S$ as

$$\|S((v + h), (u + k)) - S(v, u) - S(h, k)\| = 0$$

Example 1.3. Let $\mu : \mathbb{R} \times V \to V$ with $\mu(r, x) = rx$. Then μ is differentiable and $(\mathrm{D}\, \mu(r, x))(t, h) = rh + tx$ as

$$\|\mu((r + t), (x + h)) - \mu(r, x) - (\mathrm{D}\, \mu(r, x))(t, h)\| = \|rx + rh + tx + th - rx - rh - tx\| = |h| \, \|t\|$$

Proposition 1.12. *Differentiability of f at x implies continuity at x.*

Proposition 1.13. *Assume $f : U \subset V \to W$ is differentiable at x_0 and let $u \in V$ be a non-zero vector then*

$$\lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = f'(x_0)\,(u)$$

Proof. Let $h = tu$ then

$$R(tu) = f(x_0 + tu) - f(x_0) - T(tu)$$
$$= f(x_0 + tu) - f(x_0) - tT(u)$$
$$\implies \frac{R(tu)}{t} = \frac{f(x_0 + tu) - f(x_0)}{t} - T(u)$$
$$\implies \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = T(u)$$

Definition (Directional derivative): If we let $\|u\| = 1$ then the limit in Propostion 1.13 becomes the **directional derivative** of f in the direction of u and is denoted by $\mathrm{D}_u\, f$.

Remark 2. The existence of all directional derivatives of f doesnt imply its differentiability or even its continuity.

Theorem 1.14. *$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ has all of its partial derivative and they're continuous then f is differentiable.*

Proposition 1.15. *Let $f, g : V \to W$ be differentiable at x and $h : W \to U$ be differentiable at $y = f(x)$. Furthermore, let c be an scalar then*

1. $\mathrm{D}(f + cg) = \mathrm{D}(f) + c\,\mathrm{D}(g)$.

2. $h \circ f$ is differentiable at x and

$$\mathrm{D}(h \circ f) = (\mathrm{D}(h) \circ f) \circ \mathrm{D}(f)$$

Theorem 1.16 (Leibnitz rule). *Let $V_1, V_2, \ldots, V_n$ be finite dimensional vector spaces and $f : V_1 \times \cdots \times V_n \to W$ is a n-linear function. f is differentiable at $a = (a_1, \ldots, a_n)$ and*

$$(\mathrm{D}\,f(a))(h_1, \ldots h_n) = f(h_1, a_2, \ldots, a_n) + f(a_1, h_2, \ldots, a_n) + \cdots + f(a_1, a_2, \ldots, h_n)$$

Proposition 1.17. *$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0 if and only if every component is differentiable at x_0.*