1. Let $x, y \in \mathbb{R},\ x < y$ and $n \in \mathbb{N}$. Define $\delta := \frac{y-x}{n}$:

$$
\begin{aligned}
|f(x) - f(y)| &\le \sum_{i=1}^{n} |f(x+i\delta) - f(x+(i-1)\delta)| && (1)\\
&\le \sum_{i=1}^{n} |x+i\delta - (x+(i-1)\delta)|^2 && (2)\\
&= n\delta^2 = \frac{(y-x)^2}{n} && (3)\\
\implies |f(x) - f(y)| &= 0, \qquad \forall x, y \in \mathbb{R} && (4)\\
\implies f(x) &\equiv c && (5)
\end{aligned}
$$

2.

   (a) $f(x)$ is uniformly continuous by letting $\delta = \left(\frac{\epsilon}{H+1}\right)^{\frac{1}{\alpha}}$. Since $f(x)$ is continuous then for any sequence $\{x_n\}$ such that $x_n \to a$, $\{f(x_n)\}$ is convergent. Let its limit to be L_a. Define L_b similarly for b. Then by the uniqueness of the limits

$$
g(x) = \begin{cases} f(x) & a < x < b \\ L_a & x = a \\ L_b & x = b \end{cases} \tag{6}
$$

   is unique and continuous. WLOG we prove that:

$$
\forall x \in [a, b] \ \ |g(x) - g(a)| \le 2H\,|x-a|^{\alpha} \tag{7}
$$

   let $x_n \to a$ and fix $\epsilon \le H\,|x-a|^{\alpha}$. Then for some $n >= N_g$ we have:

$$
\begin{cases} |g(x_n) - g(a)| < \epsilon \\ x_n < x \implies |x_n - a| < |x - a| \end{cases} \tag{8}
$$

   thus:

$$
\begin{aligned}
|g(a) - g(x)| &\le |g(a) - g(x_n)| + |g(x_n) - g(x)| && (9)\\
&\le \epsilon + H\,|x_n - x|^{\alpha} && (10)\\
&\le H\,|x-a|^{\alpha} + H\,|a-x|^{\alpha} = 2H\,|x-a|^{\alpha} && (11)
\end{aligned}
$$

   (b) it means ur mama gay.

   (c) same as item 1.

3. (a) use mean value theorem

   (b) increasing (not strict).

4. conjugate / l'Hopital

5. true by MVT.

$$
\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} f'(c_x) = L \tag{12}
$$

Old book 6. because of the endpoint can be the minimum and thus they only need to smaller from the values to one side, not both.

6. l'Hopital proof

7. l'Hopital proof

8. $f(x) = \sqrt{1-x^2}$ - it seems to follow MVT

9. define $g(x) = f(x) - x$ bound it by a linear function. Prove existence and then use monoticity to prove uniqueness.

10. no clue

11. By Taylor's theorem:

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h - \frac{f''(x)}{2}h^2}{h^2} = 0 \tag{13}$$

$$\lim_{h\to 0} \frac{f(x-h) - f(x) + f'(x)h - \frac{f''(x)}{2}h^2}{h^2} = 0 \tag{14}$$

$$\implies \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h) - f''(x)h^2}{h^2} = 0 \tag{15}$$

$f(x) = x^2 \sin \frac{1}{x}$

12. for (i) and (ii) simply use the conditions to bound the following, also use the fact that limit exists ofcourse.

$$\left|\frac{1}{\beta_n - \alpha_n}\right| |f(\beta_n) - f(\alpha_n) - (\beta_n - \alpha_n)f'(0)| \tag{16}$$

$$\leq \left|\frac{1}{\beta_n - \alpha_n}\right| |f(\beta_n) - f(0) - \beta_n f'(0)| + \left|\frac{1}{\beta_n - \alpha_n}\right| |f(\alpha_n) - f(0) - \alpha_n f'(0)| \tag{17}$$

$$= \left|\frac{\beta_n}{\beta_n - \alpha_n}\right| \left|\frac{f(\beta_n) - f(0)}{\beta_n} - f'(0)\right| + \left|\frac{\alpha_n}{\beta_n - \alpha_n}\right| \left|\frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0)\right| \tag{18}$$

for (iii) use MVT and by continuity of derivative arrive at the desired result. for (b) do the work, no biggie :))

13. hint induction

14. $f(x) = x^{r+1} \sin \frac{1}{x} \in \mathcal{C}^r$ but it not $\mathcal{C}^{r+1}$

15.

    (a) Clearly on the derivative at $x = 0$ may fail to exist. We claim that $\frac{d^n}{dx^n}\mathcal{E}(0) = 0$

    (b) yes it is analytic.

    (c) the product of two smooth function is smooth. the rest is obvious

16. L is closed then $U = L^c$ is open in $\mathbb{R}$. Therefore there are countable disjoint open intervals I_n such that $U = \cup I_n$. Let $I_i =]a_i, b_i[$. Define:

$$\beta_L(x) = \sum_{i=1}^{\infty} e^2 \mathcal{E}(x - a_i)\mathcal{E}(b_i - x) \tag{19}$$