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Chapter 1

Multivariable Calculus

1.1 Linear Algebra

1.1.1 Vector Spaces

Definition (Normed vector space): Let V be a vector space. A **norm** is a real valued function $\| \cdot \| : V \rightarrow \mathbb{R}$ which has the following properties

1. $\forall x \in V, \|x\| > 0$.
2. $\|x\| = 0 \implies x = 0$.
3. $\forall x \in V \forall \alpha \in \mathbb{F}, \|\alpha x\| = |\alpha| \|x\|$.
4. $\forall x, y \in V \quad \|x + y\| \leq \|x\| + \|y\|$.

Each normed vector space induces a metric space (V, d) where $d(x, y) = \|y - x\|$.

Theorem 1.1. *In every normed space $(V, \| \cdot \|)$ we have*

$$|||v| - |w|| \leq \|v - w\|$$

Hence the norm is Lipschitz continuous.

Definition: Assume V is a vector space and let $\| \cdot \|_1, \| \cdot \|_2$ be two norms for V . They are said to be equivalent when

$$\exists c_1, c_2 > 0 \forall x : \quad c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$$

To check if the above definition is indeed an equivalence relation, we must show that following:

Reflexive $\| \cdot \|_1 \sim \| \cdot \|_1$.

Symmetric $\| \cdot \|_1 \sim \| \cdot \|_2 \implies \| \cdot \|_2 \sim \| \cdot \|_1$.

Transitive $\| \cdot \|_1 \sim \| \cdot \|_2, \| \cdot \|_2 \sim \| \cdot \|_3 \implies \| \cdot \|_1 \sim \| \cdot \|_3$.

Remark 1. Equivalent norms induce equivalent metrics, hence they induce the same topology.

Theorem 1.2. *All norms defined on a finite dimensional vector space V are equivalent.*

Proof. Let $\|\cdot\|$ be an arbitrary norm on V and $\{e_1, e_2, \dots, e_n\}$ be a basis of V . Let $\|\cdot\|_2$ be L_2 -norm (Euclidean norm). It will suffice to show $\|\cdot\| \sim \|\cdot\|_2$. Let

$$M = \max(\|e_1\|, \dots, \|e_n\|)$$

Take $x \in V$, writing $x = \sum_{i=1}^n \xi_i e_i$ we have:

$$\|x\| = \left\| \sum_{i=1}^n \xi_i e_i \right\| \leq \sum_{i=1}^n |\xi_i| \|e_i\| \leq M \sqrt{n} \|x\|_2$$

Taking $c_2 = M\sqrt{n}$ proves the right inequality. For the left inequality we need the following lemma

Lemma 1.3. *If V is a normed vector space with $\|\cdot\|_2$, as defined above, is viewed as metric space $(V, \|\cdot\|_2)$ then $\|\cdot\| : V \rightarrow \mathbb{R}$ is continuous.*

Proof. Let $x_0 \in V$ and M be defined as above. For any $\epsilon > 0$ consider $\delta = \frac{\epsilon}{M\sqrt{n}}$ then if $\|x - x_0\|_2 < \delta$

$$|\|x\| - \|\cdot\| x_0| \leq \|x - x_0\| \leq M\sqrt{n} \|x - x_0\|_2 \leq \epsilon$$

Now consider the sphere of radius $r = 1$ centered at 0, $S_1(0) = S_1 = \{x \in V : \|x\|_2 = 1\}$. One can show that S is compact. Therefore, $\|x\|$ assumes its minimum on S . Let $a = \|x_0\|$ be the minimum. Since $0 \notin S$ then $a > 0$. By letting $y = x/\|x\|_2$, we have $y \in S$ and thus $a \leq \|y\|$ which is

$$a \|x\|_2 \leq \|x\|$$

Taking $c_1 = a$ proves the theorem. ■

Theorem 1.4. *Let $(V, \|\cdot\|)$ be a normed space. The following are equivalent*

1. V is finite dimensional.
2. every bounded closed set in V is compact.
3. the closed unit ball in V is compact.

Proof.

1.1.2 Linear Maps

Let V and W be a vector spaces over \mathbb{F} . A map $T : V \rightarrow W$ is **linear** if

$$T(x + \lambda y) = T(x) + \lambda T(y)$$

for all $x, y \in V$ and $\lambda \in \mathbb{F}$.

Definition: Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces then, a linear transformation $T : V \rightarrow W$ is **bounded** if there exists a constant $C > 0$ such that

$$\|Tv\|_W \leq C \|v\|_V$$

for all $v \in V$.

Definition: If $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ are normed spaces then the **operator norm** of a linear transformation $T : V \rightarrow W$ is

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \mid v \neq 0 \right\}$$

Theorem 1.5. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces and $T : V \rightarrow W$ be a linear transformation. The following are equivalent

1. $\|T\|$ is finite.
2. T is bounded.
3. T is Lipschitz continuous.
4. T is continuous at a point.
5. $\sup_{\|v\|_V=1} \|Tv\|_W < \infty$.

Proof. item 1 \Rightarrow item 2: Obviously

$$\begin{aligned} \frac{\|Tv\|_W}{\|v\|_V} &\leq \|T\| \\ \Rightarrow \|Tv\|_W &\leq \|T\| \|v\|_V \end{aligned}$$

note that if $v = 0$ then $Tv = 0$ as well and thus the last inequality holds for all $v \in V$.

item 2 \Rightarrow item 3:

$$\|Tv - Tu\|_W = \|T(u - v)\|_W \leq C \|u - v\|_V$$

item 3 \Rightarrow item 4: Trivial.

item 4 \Rightarrow item 5: Let T be continuous at $u \in V$. Then there is a $\delta > 0$ such that

$$\|v - u\| < \delta \Rightarrow \|Tv - Tu\|_W = \|T(v - u)\|_W < 1$$

Now for an arbitrary non-zero v we have

$$\left\| \left(\frac{\delta v}{2\|v\|_V} + u \right) - u \right\|_V < \delta$$

Therefore

$$\begin{aligned} \left\| T \left(\frac{\delta v}{2\|v\|_V} + u \right) \right\|_W &< 1 \\ \left\| T \left(\frac{v}{\|v\|_V} \right) \right\|_W &< \frac{2}{\delta} \end{aligned}$$

item 5 \Rightarrow item 1: Let $v \in V$ be an arbitrary vector. Then

$$\begin{aligned} \sup \left\| T \left(\frac{v}{\|v\|_V} \right) \right\|_W &< \infty \\ \Rightarrow \sup \|Tv\|_W &< \infty \end{aligned}$$

Theorem 1.6. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ linear transformation is invertible if and only if there exists a c such that:*

$$c \|x\| \leq \|f(x)\|$$

Proof. A linear transformation $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one if and only if it is surjective because $\dim \operatorname{Im} f + \dim \ker f = n$. Hence, we only need to show that f is one-to-one. ■

Exercises

1. Show that for a linear transformation T , $\|T\| = \sup_{\|v\|_V \leq 1} \|Tv\|_W$.

1.2 Derivative

Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ where U is open. Then f is differentiable at x_0 when a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

$$\lim_{\|h\| \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

Usually, T is represented by $f'(x_0)$.

Proposition 1.7. *Assume $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at x_0 and let $u \in \mathbb{R}^n$ then*

$$\lim_{t \rightarrow 0} \frac{f(x_0 + tu) - f(x_0)}{t} = f'(x_0) \cdot u$$

Definition (Partial derivative): define

Proposition 1.8. *If f is differentiable then its partial derivatives exist.*

Proposition 1.9. *$f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at x_0 if and only if every component is differentiable at x_0 .*

Theorem 1.10. *$f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ has all of its partial derivative and they're continuous then f is differentiable.*