Contents

Chapter 1

Multivariable Calculus

1.1 Linear Algebra

1.1.1 Vector Spaces

Definition (Normed vector space): Let V be a vector space. A **norm** is a real valued function $\|\cdot\| : V \to \mathbb{R}$ which has the following properties

1. $\forall x \in V,\ \|x\| > 0.$
2. $\|x\| = 0 \implies x = 0.$
3. $\forall x \in V\ \forall \alpha \in \mathbb{F},\ \|\alpha x\| = |\alpha|\,\|x\|.$
4. $\forall x, y \in V\ \ \|x + y\| \leq \|x\| + \|y\|.$

Each normed vector space induces a metric space (V, d) where $d(x, y) = \|y - x\|$.

Theorem 1.1. *In every normed space* $(V, \|\cdot\|)$ *we have*

$$|\|v\| - \|w\|| \leq \|v - w\|$$

Hence the norm is Lipschitz continuous.

Definition: Assume V is a vector space and let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms for V. They are said to be equivalent when

$$\exists c_1, c_2 > 0\ \forall x : \qquad c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|$$

To check if the above definition is indeed an equivalence relation, we must show that following:

Reflexive $\|\cdot\|_1 \sim \|\cdot\|_1$.

Symmetric $\|\cdot\|_1 \sim \|\cdot\|_2 \implies \|\cdot\|_2 \sim \|\cdot\|_1$.

Transitive $\|\cdot\|_1 \sim \|\cdot\|_2$, $\|\cdot\|_2 \sim \|\cdot\|_3 \implies \|\cdot\|_1 \sim \|\cdot\|_3$.

Remark 1. Equivalent norms induce equivalent metrics, hence they induce the same topology.

Theorem 1.2. *All norms defined on a finite dimensional vector space V are equivalent.*

Proof. Let $\|\cdot\|$ be an arbitrary norm on V and $\{e_1, e_2, \dots, e_n\}$ be a basis of V. Let $\|\cdot\|_2$ be L_2-norm (Euclidean norm). It will suffice to show $\|\cdot\| \sim \|\cdot\|_2$. Let

$$M = \max(\|e_1\|, \dots, \|e_n\|)$$

Take $x \in V$, writing $x = \sum_{i=1}^{n} \xi_i e_i$ we have:

$$\|x\| = \left\| \sum_{i=i}^{n} \xi_i e_i \right\| \leq \sum_{i=1}^{n} |\xi_i| \, \|e_i\| \leq M\sqrt{n} \, \|x\|_2$$

Taking $c_2 = M\sqrt{n}$ proves the right inequality. For the left inequality we need the following lemma

Lemma 1.3. *If V is a normed vector space with $\|\cdot\|_2$, as defined above, is viewed as metric space $(V, \|\cdot\|_2)$ then $\|\cdot\| : V \to \mathbb{R}$ is continuous.*

Proof. Let $x_0 \in V$ and M be defined as above. For any $\epsilon > 0$ consider $\delta = \frac{\epsilon}{M\sqrt{n}}$ then if $\|x - x_0\|_2 < \delta$

$$|\|x\| - \|x_0\|| \leq \|x - x_0\| \leq M\sqrt{n} \, \|x - x_0\| \leq \epsilon$$

Now consider the sphere of radius $r = 1$ centered at 0, $S_1(0) = S_1 = \{x \in V : \|x\|_2 = 1\}$. One can show that S is compact. Therefore, $\|x\|$ assumes its minimum on S. Let $a = \|x_0\|$ be the minimum. Since $0 \notin S$ then $a > 0$. By letting $y = x/\|x\|_2$, we have $y \in S$ and thus $a \leq \|y\|$ which is

$$a \, \|x\|_2 \leq \|x\|$$

Taking $c_1 = a$ proves the theorem. ■

Example 1.1. The closed unit ball in the infinite dimensional vector space $C([0,1], \mathbb{R})$ with $\|f\| = \max f(x)$ is not compact. Take $f_n(x) = x^n$. Obviously $\|f_n\| = 1$, however f_n doesn't uniformly converge and hence f_n doesn't have a limit in $C([0,1], \mathbb{R})$ with the max norm. Consider the following norm

$$\|f\|_I = \int_0^1 |f(x)| \, \mathrm{d}\, x$$

Note that $\|\cdot\|_I$ and $\|\cdot\|_{\max}$ are not equivalent. Let $g(x) = 0$ for all $x \in [0,1]$. Then

$$\|f_n - g\|_I = \frac{1}{n+1} \to 0 \quad \text{as} n \to \infty.$$

Definition (Banach space): A normed vector space V that is complete is a **Banach space**. A **Hilbert Space** is a Banach space whose norm is induced by an inner product.

Corollary 1.4. *Any finite dimensional normed vector space V over a normed complete field $\mathbb{F}$ is a Banach space.*

1.1.2 Linear Maps

Let V and W be a vector spaces over $\mathbb{F}$. A map $T : V \to W$ is **linear** if

$$T(x + \lambda y) = T(x) + \lambda T(y)$$

for all $x, y \in V$ and $\lambda \in \mathbb{F}$.

Definition: Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces then, a linear transformation $T : V \to W$ is **bounded** if there exists a constant $C > 0$ such that

$$\|Tv\|_W \le C \|v\|_V$$

for all $v \in V$. We denote the set of all linear map from $V \to W$ as $\mathcal{L}(V, W)$. If $T \in \mathcal{L}(V, W)$ is bijective such that $T^{-1} \in \mathcal{L}(V, W)$, then T is called an **isomorphism** and V, W are **isomorphic**. An operator $T \in \mathcal{L}(V, W)$ is called **isometric** if $\|Tv\|_W = \|v\|_V$ for all $v \in V$.

Definition: If $(V, \|\cdot\|_V), (W, \|\cdot\|_W)$ are normed spaces then the **operator norm** of a linear transformation $T : V \to W$ is

$$\|T\| = \sup \left\{ \frac{\|Tv\|_W}{\|v\|_V} \middle| v \neq 0 \right\}$$

Proposition 1.5. *Let $T : U \to V$ and $T' : V \to W$ be two linear transformations.*

$$\|T' \circ T\| \le \|T\| \|T'\|$$

Proof. for an arbitrary non-zero $x \in U$

$$\|T' \circ T(x)\|_W \le \|T'\| \|Tx\|_V \le \|T'\| \|T\| \|x\|_U$$

which implies

$$\|T' \circ T\| \le \|T\| \|T'\|$$

Theorem 1.6. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces and $T : V \to W$ be a linear transformation. The following are equivalent*

1. *$\|T\|$ is finite.*
2. *T is bounded.*
3. *T is Lipschitz continuous.*
4. *T is continuous at a point.*
5. $\sup_{\|v\|_V = 1} \|Tv\|_W < \infty$.

Proof. item 1 $\Rightarrow$ item 2: Obviously

$$\frac{\|Tv\|_W}{\|v\|_V} \le \|T\|$$
$$\implies \|Tv\|_W \le \|T\| \|v\|_V$$

note that if $v = 0$ then $Tv = 0$ as well and thus the last inequality holds for all $v \in V$.

item 2 $\Rightarrow$ item 3:

$$\|Tv - Tu\|_W = \|T(u-v)\|_W \leq C\,\|u-v\|_V$$

item 3 $\Rightarrow$ item 4: Trivial.
item 4 $\Rightarrow$ item 5: Let T be continuous at $u \in V$. Then there is a $\delta > 0$ such that

$$\|v-u\| < \delta \implies \|Tv - Tu\|_W = \|T(v-u)\|_W < 1$$

Now for an arbitrary non-zero v we have

$$\left\| \left(\frac{\delta v}{2\,\|v\|_V} + u \right) - u \right\|_V < \delta$$

Therefore

$$\left\| T\left(\frac{\delta v}{2\,\|v\|_V} \right) \right\|_W < 1$$
$$\left\| T\left(\frac{v}{\|v\|_V} \right) \right\|_W < \frac{2}{\delta}$$

item 5 $\Rightarrow$ item 1: Let $v \in V$ be an arbitrary vector. Then

$$\sup \left\| T\left(\frac{v}{\|v\|_V} \right) \right\|_W < \infty$$
$$\implies \sup \|Tv\|_W < \infty$$

Theorem 1.7. *If V is a finite dimensional normed vector space then any linear transformation $T: V \to W$ is continuous.*

Proof. Since V is finite dimensional, according to Theorem 1.2, any two norms are equivalent. Hence, take $\|\cdot\|_2$ to be Euclidean norm over a basis $\{e_1, \ldots, e_n\}$. Let x be such that $\|x\|_2 < \delta$ for some $\delta > 0$. Therefore, $|\xi_i| < \delta^2$

$$\|Tx\|_W = \left\| \sum \xi_i T(e_i) \right\|_W \leq \sum |\xi_i|\,\|T(e_i)\|_W \leq \delta^2 K$$

where $K = \max \|T(e_i)\|_W$. By letting $\delta = \sqrt{\frac{\epsilon}{K}}$ we proved continuity at 0 and hence the continuity by Theorem 1.6. ■

Theorem 1.8. *For two normed vector spaces V, W, $(\mathcal{L}(V,W), \|T\|)$ is a normed vector space. Moreover, it is a Banach space when W is a Banach space.*

Proof. Clearly $\mathcal{L}(V,W)$ is a vector space. For its norm $\|T\|$ we have

1. $\|T\| \geq 0$ by definition.

2. if $\alpha \in \mathbb{F}_W$ then

$$\|\alpha T\| = \sup\left\{ \frac{\|(\alpha T)v\|_W}{\|v\|_V} \middle| v \neq 0 \right\} = |\alpha| \sup\left\{ \frac{\|Tv\|_W}{\|v\|_V} \middle| v \neq 0 \right\} = |\alpha|\,\|T\|$$

3. for the triangle inequality

$$\begin{aligned}
\|T_1 + T_2\| &= \sup\left\{\frac{\|(T_1+T_2)v\|_W}{\|v\|_V}\right\} \\
&\le \sup\left\{\frac{\|T_1v\|_W + \|T_2v\|_W}{\|v\|_V}\right\} \\
&= \sup\left\{\frac{\|T_1v\|_W}{\|v\|_V}\right\} + \sup\left\{\frac{\|T_2v\|_W}{\|v\|_V}\right\} \\
&= \|T_1\| + \|T_2\|
\end{aligned}$$

Theorem 1.9. *Let $(V, \|\cdot\|)$ be a normed space. Then any linear transformation $T : \mathbb{R}^n \to V$ is continuous. Furthermore, if T is a bijection, it is a heomeomorphis.*

Proof.

Definition: Let $V_1, V_2, \ldots, V_n$ be finite dimensional normed vector spaces. Then the function $\phi : V_1 \times \cdots \times V_n$ is n**-linear** if by fixing any $n-1$ component, ϕ is linear relative to the remaining components.

Theorem 1.10. *$f : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation is invertible if and only if there exists a c such that:*

$$c\,\|x\| \le \|f(x)\|$$

Proof. A linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one if and only if it is surjective because $\dim \operatorname{Im} f + \dim \ker f = n$. Hence, we only need to show that f is one-to-one. ■

Exercises

1. Show that for a linear transformation T, $\|T\| = \sup_{\|v\|_V \le 1} \|Tv\|_W$.

1.2 Derivative

Let V, W be finite dimensional vector spaces and $f : U \subset V \to W$ where U is open. Then f is differentiable at x_0 when a linear transformation $T : V \to W$ such that

$$\lim_{\|h\|\to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

Or equivalently there exists a sublinear function $R(h)$ such that

$$f(x_0 + h) - f(x_0) - Th = R(h) \qquad \frac{R(h)}{\|h\|} \to 0$$

T if it exists is unique, represented by $f'(x_0)$, $\mathrm{D}\, f$, or $\mathrm{d}\, f(x)$ and called the **total derivative** or **Fréchet derivative**.

Example 1.2. Every linear function $f : V \to W$ with $f(v) = Tv + b$ where $b \in W$ is differentiable and $\mathrm{D}\, f(v) = T$. Since

$$\|h\|_V \, \delta \implies \|f(v + h) - f(v) - \mathrm{D}(h)\|_W = \|T(v + h) - Tv - Th\|_W = 0 < \epsilon \, \|h\|_V$$

Hence, the derivative of any linear function is constant. Consider $S : V \times V \to V$ with $S(v, u) = v + u$. S is differentiable because S is linear (why?).

Example 1.3. Let $\mu : \mathbb{R} \times V \to V$ with $\mu(r, x) = rx$. Then μ is differentiable and $(\mathrm{D}\, \mu(r, x))(t, h) = rh + ta$.

Proposition 1.11. *Differentiability of f at x implies continuity at x.*

Proposition 1.12. *Assume $f : U \subset V \to W$ is differentiable at x_0 and let $u \in V$ be a non-zero vector then*

$$\lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t} = f'(x_0)\,(u)$$

Proof. Let $h = tu$ then

$$\begin{aligned} R(tu) &= f(x_0 + tu) - f(x_0) - T(tu) \\ &= f(x_0 + tu) - f(x_0) - tT(u) \\ \implies \frac{R(tu)}{t} &= \frac{f(x_0 + tu) - f(x_0)}{t} - T(u) \\ \implies \lim_{t\to 0} \frac{f(x_0 + tu) - f(x_0)}{t} &= T(u) \end{aligned}$$

Definition (Directional derivative): If we let $\|u\| = 1$ then the limit in Propostion 1.12 becomes the **directional derivative** of f in the direction of u and is denoted by $\mathrm{D}_u\, f$.

Remark 2. The existence of all directional derivatives of f doesnt imply its differentiability or even its continuity.

Theorem 1.13. $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ *has all of its partial derivative and they're continuous then f is differentiable.*

Proposition 1.14. *Let $f, g : V \to W$ be differentiable at x and $h : W \to U$ be differentiable at $y = f(x)$. Furthermore, let c be an scalar then*

1. $\mathrm{D}(f + cg) = \mathrm{D}(f) + c\,\mathrm{D}(g)$.
2. $h \circ f$ *is differentiable at x and*

$$\mathrm{D}(h \circ f) = (\mathrm{D}(h) \circ f) \circ \mathrm{D}(f)$$

Theorem 1.15 (Leibnitz rule). *Let $V_1, V_2, \ldots, V_n$ be finite dimensional vector spaces and $f : V_1 \times \cdots \times V_n \to W$ is a n-linear function. f is differentiable at $a = (a_1, \ldots, a_n)$ and*

$$(\mathrm{D}\, f(a))(h_1, \ldots h_n) = f(h_1, a_2, \ldots, a_n) + f(a_1, h_2, \ldots, a_n) + \cdots + f(a_1, a_2, \ldots, h_n)$$

Proposition 1.16. *$f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0 if and only if every component is differentiable at x_0.*