Contents

Chapter 1

Growth of Functions

Definition:

Θ-notation asymptotically tight bound

$$\Theta(g(n)) = \{f(n) \mid \exists c_1, c_2, n_0 \text{ s.t. } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \;\; \forall n \geq n_0\}$$

O-notation asymptotic upper bound

$$O(g(n)) = \{f(n) \mid \exists c, n_0 \text{ s.t. } 0 \leq f(n) \leq cg(n) \;\; \forall n \geq n_0\}$$

Ω-notation asymptotic lower bound

$$\Omega(g(n)) = \{f(n) \mid \exists c, n_0 \text{ s.t. } 0 \leq cg(n) \leq f(n) \;\; \forall n \geq n_0\}$$

o-notation asymptotically smaller

$$o(g(n)) = \{f(n) \mid \forall c > 0, \, \exists n_0 \text{ s.t. } 0 \leq f(n) < cg(n) \;\; \forall n \geq n_0\}$$

ω-notation asymptotically larger

$$\omega(g(n)) = \{f(n) \mid \forall c > 0, \, \exists n_0 \text{ s.t. } 0 \leq cg(n) < f(n) \;\; \forall n \geq n_0\}$$

Proposition 1.1.

1. *For any two function $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.*
2. *$f(n) = \omega(g(n))$ if and only if $g(n) = o(f(n))$ and $f(n) = \Omega(g(n))$ if and only if $g(n) = O(f(n))$*

A function $f(n)$ is **polylogarithmically bounded** if $f(n) = O\big(\lg^k n\big)$. Any exponential function with a base strictly greater than 1 grows faster than any polynomial function and any polynomial function grows faster than any polylogarithmic function.

Remark 1 (Stirling's approximation).

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right) \tag{1.1}$$

Theorem 1.2. *Let $a \geq 1$ and $b > 1$ be constants. The recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(b)$$

has the bounds

1. *If $f(n) = O\big(n^{\log_b a - \epsilon}\big)$ for some $\epsilon > 0$ then $T(n) = \Theta\big(n^{\log_b a}\big)$.*
2. *If $f(n) = \Theta\big(n^{\log_b a}\big)$ then $T(n) = \Theta\big(n^{\log_b a} \lg n\big)$.*
3. *If $f(n) = \Omega\big(n^{\log_b a + \epsilon}\big)$ for some $\epsilon > 0$ and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n then $T(n) = \Theta(f(n))$.*