Contents

1 Linear Equation **3**
1.1 Fields 3
1.2 Matrices 4

Chapter 1

Linear Equation

1.1 Fields

The set $\mathbb{F}$ together with two operation $+$, addition, and $\cdot$, multiplication, that satisfy the follwings is called a **field**. For all $x, y, z \in \mathbb{F}$

1. Addition and multiplication are *commutative*

$$x + y = y + x \qquad\qquad x \cdot y = y \cdot x$$

2. Addition and multiplication are *associative*

$$x + (y + z) = (x + y) + z \qquad\qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

3. Multiplication distributes over addition

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

4. There exists an element 0, zero, in $\mathbb{F}$ such that $x + 0 = x$.

5. There exists an element 1 , one, in $\mathbb{F}$ such that $x \cdot 1 = x$.

6. For each element $x \in \mathbb{F}$ there corresponds a unique element $y \in \mathbb{F}$ such that $x + y = 0$. y is commonly denoted as $-x$.

7. For each non-zero element $x \in \mathbb{F}$ there corresponds a unique element $y \in \mathbb{F}$ such that $x \cdot y = 1$. y is commonly denoted as x^{-1} or $\frac{1}{x}$.

8. $\mathbb{F}$ is closed under addition and multiplication.

$$x + y \in \mathbb{F} \qquad\qquad x \cdot y \in \mathbb{F}$$

Definition (Characteristics): Let n be the least number such that

$$\underbrace{1 + 1 + \ldots 1}_{n} = 0 \tag{1.1}$$

then n is the **characteristics** of $\mathbb{F}$. If for a field there exists no such n, then its characteristics is 0.

Theorem 1.1. *If $\mathbb{F}$ is a finite field, then the number of elements of $\mathbb{F}$ must be in form of p^k where p isa prime number and $k \in \mathbb{N}$. Also fro every number in such form there exists a unique $\mathbb{F}$ with p^k elements.*

If $\mathbb{F}$ is a field then the set of all polynomials with the coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[x]$, that is

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_i \in \mathbb{F}\ \forall i,\ n \in \mathbb{N} \right\}$$

Cleary $\mathbb{F}[x]$ does not have a multiplicative inverse for some of its non-zero elements. Define $\mathbb{F}(x)$ as follow

$$\mathbb{F}(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f(x)\,, g(x) \in \mathbb{F}[x]\,,\ g(x) \neq 0 \right\}$$

which is a field. Also, note that $\mathbb{F} \subset \mathbb{F}[x] \subset \mathbb{F}(x)$.

1.2 Matrices

Let us denote the set of all metrices of size $m \times n$ with elements in $\mathbb{F}$ by $\mathcal{M}_{m\times n}(\mathbb{F})$ and if $m = n$ then it is equivalently denoted as $\mathcal{M}_n(\mathbb{F})$.

Matrix $A \in \mathcal{M}_n(\mathbb{F})$ is said to be **invertiable** or **singular** if there exists a matrix $B \in \mathcal{M}_n(\mathbb{F})$ such that $AB = BA = \mathbb{I}_n$.

Consider the following system of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \ a_{2n}x_n = y_2 \\ \qquad \vdots \\ a_{m1}x_1 + a_{n2}x_2 + a_{mn}x_n = y_m \end{cases} \tag{1.2}$$

with all $a_{ij} \in \mathbb{F}$. Then if $c_k \in F,\ k = 1, \ldots, n$:

$$(c_1a_{11} + c_2a_{21} + \ldots c_m a_{m1})x_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \ldots c_m a_{mn})x_n = c_1y_1 + \cdots + c_m y_m$$

is **linear combination** of the Equation (1.2).

Definition (Equaivalent Systems): Two systems are considered equivalent if each equation in one system is a linear combination of the other system.

Proposition 1.2. *Equivalent systems of linear equations have exactly the same solution.*

The linear system, Equation (1.2), can be represent in form of matrices $AX = Y$ where $A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \\ a_{n1} & \ldots & a_{mn} \end{bmatrix} \in \mathcal{M}_{m\times n}(\mathbb{F})$ and $X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$ then the solutions of Equation (1.2) are exactly the same as $AX = Y$.