Contents

Chapter 1

Matrices

1.0.1 Gaussian elimination

We can define three elementary operations that do not change the solution of the system.

Definition (Elementary row operations): Consider a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, we define three elementary row operations on A:

1. multiplication of one row by a non-zero scalar c, denoted by $e_r(c)$.
2. replacement of r_{th} row of A by row r plus c times row $s \neq r$ that is $r_{\text{new}} = r + cs$ and it is denoted by $e_{rs}(c)$
3. Interchange two rows e_{rs}.

Theorem 1.1. *To each elementary row operation e there corresponds an elementary row operation e^{-1}, of the same type as e, such that $e^{-1}(e(A)) = e(e^{-1}(A)) = A$ for any A. Furthermore, each elementary operation has a $n \times n$ matrix representation.*

Proof. It is easily verified by hand that the inverses exist and are of the same kind. ■

At each step of Gaussian elimination, one takes an element to be the **pivot** and eliminate the element from the other equations. At the end, the system has been reduced to a **upper triangle** (maybe with permuation). For **Gauss-Jordan method** the pivot needs to be one and the element must be eliminate from all equations.

1.0.2 LU decomposition

With some assumption one can transform matrix A to an upper triangular matrix U via elementary row operations. That is, there exists an elementary matrix E such that $EA = U$. It can be verified that E is lower triangular matrix and hence $L = E^{-1}$ is lower triangular and $A = LU$. The assumptions made can be simplified if we allow permuations. That is

Theorem 1.2. *There exists a permuation of a matrix A that has a LU decomposition. In other words, there exists a permuation matrix P such that*

$$PA = LU$$

Theorem 1.3. *If A is strongly invertible matrix that is, every submatrix $A(1:i, 1:i)$ is invertible (leading minors are non-zero) then there exists a unique lower unitriangular matrix L and a unique upper triangular matrix U such that $A = LU$.*

Theorem 1.4. *If the decomposition $A = LU$ exist where L is a lower unitriangular matrix, then the decomposition $A = LDU$ exists where D is a diagonal matrix and L and U are lower and upper unitriangular matrices respectively. Furthermore, if such LDU decomposition exists and $A = A^T$ then $A = LDL^T$*

1.1 Matrix Subspaces

1.1.1 Column space

The **column space** of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, usually denoted by $\operatorname{colsp} A$ is the span of its columns.

$$\operatorname{colsp} A = \{Ax \mid x \in \mathbb{F}^n\}$$

It is obvious that $\dim \operatorname{colsp} A$ is the number of linearly independent columns of A.

1.1.2 Null space

The **null space** or **kernel** of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ is all the vectors $x \in \mathbb{F}^n$ such that $Ax = 0$, and it is denoted by $\ker A$.

$$\ker A = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

To find the null space of a matrix A, we need to solve the homogeneous equation $Ax = 0$. One way of doing this is employing the Guassian elimination to find a row reduced form of A.

Example 1.1. We want to find the null space of

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 6 & 1 & 7 & 1 \\ 4 & 8 & 1 & 9 & 4 \end{bmatrix}$$

we have

$$\begin{aligned} A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 6 & 1 & 7 & 1 \\ 4 & 8 & 1 & 9 & 4 \end{bmatrix} & \qquad \to \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -5 \\ 0 & 0 & -3 & -3 & -4 \end{bmatrix} \\ \to \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} & \qquad \to \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

hence we must solve the following system

$$\begin{cases} x_1 + 2x_2 + x_3 + 3x_4 + 2x_5 = 0 \\ \qquad\qquad - x_3 - \ \ x_4 - \ \ x_5 = 0 \\ \qquad\qquad\qquad\qquad\quad - 3x_5 = 0 \end{cases}$$

hence all the solutions are

$$\alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad \forall \alpha, \beta \in \mathbb{R}$$

Moreover, by turning A to its row reduced echolon form we have

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

hence we can re-write

$$\begin{aligned} Rx &= \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix}}_{x_{\text{pivot}}} + \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_2 \\ x_4 \end{bmatrix}}_{x_{\text{free}}} = 0 \end{aligned}$$

by removing the zero rows we get that

$$\begin{aligned} &I x_{\text{pivot}} + F x_{\text{free}} = 0 \\ \implies &\begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = \begin{bmatrix} -F \\ I \end{bmatrix} x_{\text{free}} \end{aligned}$$

note that, for some permuation matrix P

$$\begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = Px$$

which implies that

$$x = P^{-1} \begin{bmatrix} x_{\text{pivot}} \\ x_{\text{free}} \end{bmatrix} = \underbrace{P^T \begin{bmatrix} -F \\ I \end{bmatrix}}_{\text{null matrix}} x_{\text{free}}$$

Proposition 1.5. *For every matrix $A \in \mathcal{M}_{m \times n}(\mathbb{F})$ we have*

1. *pivot columns are linearly independent and free columns are a linear of combination of them.*

2. *non-zero rows are linearly independent.*

3. *the number of pivot columns is equal to the number of independent columns and number of independent rows.*

Definition: The **rank** of a matrix $\operatorname{rank} A$ to be number of pivot columns. A **full row rank** matrix is a matrix that all of its rows are linearly independent and a **full column rank** is a matirx that all of its columns are linearly independent. Lastly, a matrix that all of its columns and row are linearly independent is called **full rank**.

Furthermore, the number free columns is equal to the $\operatorname{null} = \dim \ker A$ and it is equal to $n - \operatorname{rank} A$.

Proposition 1.6. *For the linear system $Ax = b$ we have:*

1. *If A is full rank then only one solution.*
2. *If A is full row rank then infinitely many solution.*
3. *If A is full column rank then either no solution or one solution.*
4. *Otherwise, then the system either has no or infinitely many solutions.*

1.1.3 Row space

Similar to column space, **row space** is the space spanned by the rows of A which is the same as $\operatorname{colsp} A^T$

Proposition 1.7.

$$\ker A \cap \operatorname{colsp} A^T = \{0\}$$

Theorem 1.8.

$$\operatorname{null} A + \operatorname{rank} A = n$$