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Chapter 1

Matrices

1.0.1 Gaussian elimination

We can define three elementary operations that do not change the solution of the system.

Definition (Elementary row operations): Consider a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, we define three elementary row operations on A:

1. multiplication of one row by a non-zero scalar c, denoted by $e_r(c)$.
2. replacement of r_{th} row of A by row r plus c times row $s \neq r$ that is $r_{\text{new}} = r + cs$ and it is denoted by $e_{rs}(c)$
3. Interchange two rows e_{rs}.

Theorem 1.1. *To each elementary row operation e there corresponds an elementary row operation e^{-1}, of the same type as e, such that $e^{-1}(e(A)) = e(e^{-1}(A)) = A$ for any A. Furthermore, each elementary operation has a $n \times n$ matrix representation.*

Proof. It is easily verified by hand that the inverses exist and are of the same kind. ∎

At each step of Gaussian elimination, one takes an element to be the **pivot** and eliminate the element from the other equations. At the end, the system has been reduced to a **upper triangle** (maybe with permuation). For **Gauss-Jordan method** the pivot needs to be one and the element must be eliminate from all equations.

1.0.2 LU decomposition

With some assumption one can transform matrix A to an upper triangular matrix U via elementary row operations. That is, there exists an elementary matrix E such that $EA = U$. It can be verified that E is lower triangular matrix and hence $L = E^{-1}$ is lower triangular and $A = LU$. The assumptions made can be simplified if we allow permuations. That is

Theorem 1.2. *There exists a permuation of a matrix A that has a LU decomposition. In other words, there exists a permuation matrix P such that*

$$PA = LU$$

Theorem 1.3. *If A is strongly invertible matrix that is, every submatrix $A(1 : i, 1 : i)$ is invertible (leading minors are non-zero) then there exists a unique lower unitriangular matrix L and a unique upper triangular matrix U such that $A = LU$.*

Theorem 1.4. *If the decomposition $A = LU$ exist where L is a lower unitriangular matrix, then the decomposition $A = LDU$ exists where D is a diagonal matrix and L and U are lower and upper unitriangular matrices respectively. Furthermore, if such LDU decomposition exists and $A = A^T$ then $A = LDL^T$*

Chapter 2

Vector Spaces

A **vector space**, also called **linear space**, consist of the followings:

1. a field $\mathbb{F}$ of scalars.
2. a set V of vectors.
3. a vector addition $+$, with the following properties:
   (a) V is closed under addition.
   (b) addition is commutative.
   (c) addition is associative.
   (d) addition has a unique identity element 0.
   (e) for each vector $\alpha \in V$, $\exists \beta \in V$ s.t. $a + b = 0$.
4. a scalar multiplication with the following properties:
   (a) V is closed under scalar multiplication.
   (b) $(c_1 c_2)\alpha = c_1(c_2\alpha)$.
   (c) $c(\alpha + \beta) = c\alpha + c\beta$.
   (d) $(c_1 + c_2)\alpha = c_1\alpha + c_2\alpha$
   (e) scalar multiplication has a unique identity element 1.

Example 2.1. $V = \mathbb{R}$ and $\mathbb{F} = \mathbb{R}$ is a vector space. Furthermore, $V = \mathbb{R}^n$ over $\mathbb{F} = \mathbb{R}$ is a vector space with the scalar multiplication $c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n)$.

Example 2.2. $V = \mathbb{R}$ and $\mathbb{F} = \mathbb{Z}$ is not a vector space as $\mathbb{Z}$ is not a field.

Definition: A vector $\beta \in V$ is said to be a **linear combination** of the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ if there exists scalars $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that:

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \sum_{i=1}^{n} c_i\alpha_i$$

2.1 Subspaces

Let V be a vector space over field $\mathbb{F}$. A susbspace of V is a subset W of V which is itself a vector space over $\mathbb{F}$ with the operations of vector addition and scalar multiplication on V.

Theorem 2.1. *A non-empty subset W of V is a subspace of V if and only if for each pair of vectors $\alpha, \beta \in W$ and each scalar $c \in \mathbb{F}$ the vector $c\alpha + \beta \in W$.*

Proof. Necessity: Suppose W is a non-empty subset of V with the above property. Since W is not empty then there exists a vector $\alpha \in W$ and therefore, $(-1)\alpha + \alpha = 0 \in W$. For each $c \in \mathbb{F}$, $c\alpha + 0 = c\alpha \in W$. Finally, if $\beta \in W$ as well then $1\alpha + \beta \in W$. Therefore, W satisfies all the conditions and is a linear subspace of V.

Sufficiency: If W is a subspace of V and $\alpha, \beta \in W$ with $c \in \mathbb{F}$ then $c\alpha + \beta \in W$. ■

Corollary 2.2. *Let V be a vector space over $\mathbb{F}$. The intersection of any collection subspaces of V is a subspace of V.*

Theorem 2.3. *Let V be a vector space and W_1 and W_2 be two subspaces of V such that $W_1 \cap W_2$ is a subspace of V. Then $W_1 \subset W_2$ or $W_2 \subset W_1$.*

Proof. For the sake of contradiction assume neither $W_1 \subset W_2$ nor $W_2 \subset W_1$. Then, there are vectors α_1 and α_2 such that $\alpha_1 \in W_1$ but $\alpha_2 \notin W_2$ and similarly $\alpha_2 \in W_2$ but $\alpha_2 \notin W_1$. Since $\alpha_1 + \alpha_2 \in W_1 \cup W_2$, then $\alpha_1 + \alpha_2 \in W_1$ or $\alpha_1 + \alpha_2 \in W_2$ which is a contradiction. ■

Theorem 2.4. *Let V be a vector space over the inifinite field $\mathbb{F}$. If $W_1, W_2, \ldots, W_n$ are subspaces of V and $V \subset \cup W_i$, then there exists k such that $V = W_k$.*

Proof. too long, do it urself bitch. ■

2.2 Span

Definition: Let S be a set of vector in a vector space V. The subspace spanned by S is defined to be the intersection of all subsapces of V which contains S and is denoted by $\operatorname{span} S$. That is, $\operatorname{span} S = \bigcap_{S \subset W} W$. By Corollary 2.2, $\operatorname{span} S$ is a linear subspace. Obviously, $\operatorname{span} S$ is the smallest subspace containing S because if there were $S \subset K \subset \operatorname{span} S$ since by definition $\operatorname{span} S = \cap W \subset K$, then $K = \operatorname{span} S$.

Example 2.3. Let $S = \{0\}$ then $\operatorname{span} S = \bigcap_{\{0\} \subset W} W = \{0\}$. Moreover:

$$\operatorname{span} \emptyset = \{0\} \qquad \operatorname{span} V = V$$

Theorem 2.5. *Let V is a vector space and $S \neq \emptyset$*

$$\operatorname{span} S = \{c_1\alpha_1 + \cdots + c_n\alpha_n \mid \alpha_i \in S,\ c_i \in \mathbb{F},\ n \in \natural\}$$

Proof. Let L be the set describe above. Clearly, $S \subset L$ and L is a subsapce of V hence $\operatorname{span} S \subset L$. Since $S \subset \operatorname{span} S$ and $\operatorname{span} S$ is closed under addition and scalar multiplication then $c_1\alpha_1 + \cdots + c_n\alpha_n \in \operatorname{span} S$ hence $L \subset \operatorname{span} S$. ■

Definition: If $S_1, S_2, \dots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

where $\alpha_i \in S_i$ is called the **sum** of subsets $S_1, S_2, \dots, S_k$ and is denoted by

$$S_1 + \cdots + S_k = \{\alpha_1 + \alpha_2 + \cdots + \alpha_k \mid \alpha_i \in S_i\}$$

Theorem 2.6. *Let $W_1, W_2, \dots, W_k$ be subspaces of vector space V. Then*

$$W = W_1 + W_2 + \cdots + W_k$$

is a subspaces of V. Moerover, $W = \operatorname{span} \cup W_i$.

Proof. Let $\alpha \in \operatorname{span} \cup W_i$ and $\beta \in W$. By Theorem 2.5, $\alpha = \alpha_1 + \cdots + \alpha_n$ where $\alpha_i \in \cup W_i$. Define $\alpha_1' = \sum_{\alpha_i \in W_1} \alpha_i$, $\alpha_2' = \sum_{\alpha_i \in W_2 \setminus W_1} \alpha_i$ and so on. Clearly, $\alpha = \alpha_1' + \alpha_2' + \cdots + \alpha_k'$ thus $\alpha \in W$ and $\operatorname{span} \cup W_i \subset W$. By definition, $\beta = \beta_1 + \cdots + \beta_k$ where $\beta_i \in W_i$ and therefore, $\beta_i \in \cap W_i \subset \operatorname{span} \cup W_i$. Since $\operatorname{span} \cup W_i$ is a subspace of V then $\beta \in \operatorname{span} \cup W_i$ which implies $W = \operatorname{span} \cup W_i$. ∎

2.3 Basis and Dimension

Definition (Linearly dependent): A set S of vector space V is said to be **linearly dependent**if there are $c_1, \dots, c_n \in \mathbb{F}$ where at least one of the c_i is non-zero and $\alpha_1, \dots, \alpha_n$ such that

$$c_1\alpha_1 + \cdots + c_n\alpha_n = 0$$

Furthermore, S is **linearly independent** if it is not linearly dependent. That is, if $c_1\alpha_1 + \cdots + c_n\alpha_n = 0$ then $c_i = 0$ for all $i = 1, \dots, n$.

Corollary 2.7. *If S is linearly dependent and $S \subset S'$ then S' is linearly dependent as well. If S is linearly independent and $S' \subset S$ then S' is linearly independent.*

Definition (Basis): Let V be a vector space over field $\mathbb{F}$. $S \subset V$ is a **basis** of V if it is linearly indepenent and spans V. Additionally, V is finite dimensional if V has a finite basis.

Theorem 2.8. *Let V be a vector space which is spanned by a $\beta_1, \dots, \beta_m$. Then any independent subset of V contains at most m elements.*

Proof. needs matrices maybe. ∎

Corollary 2.9. *If V is a finite dimensional vector space then any two bases of V have the same finite number of elements.*

Proof. Let α, β be two bases of V. By Theorem 2.8, $|\alpha| \leq |\beta|$ and $|\beta| \leq |\alpha|$ therefore they must have the same number of elements. Since, there exists a finite basis γ then α and β must be finite as well. ∎

Definition (Dimension): **Dimension** of a finite dimensional vector space V, denoted by $\dim V$ is the number of elements of one of its basis.

Example 2.4. $\dim\{0\} = 0$ since $\operatorname{span}\emptyset = \{0\}$ and therefore $\dim\{0\} = |\emptyset| = 0$.

Corollary 2.10. *Let V be a finite dimensional vector space and let $n = \dim V$. Then*

1. *any subset of V whichi has more then n elements is linearly dependent.*

2. *no subset of V which contains fewer than n vectors can span V.*

Proof. 1. It is an immediate consequence of Theorem 2.8.

2. If there was a set $\beta = \{\beta_1, \ldots, \beta_m\}$ with $m < n$ such that $V = \operatorname{span}\beta$ then there must be a basis β' such that $|\beta'| \leq m$. Which contradicts Corollary 2.9 ∎

Lemma 2.11. *Let S be a linearly independent subset of vector space V. Suppose β is a vector in V not spanned by S. Then the set $S \cap \{\beta\}$ is linearly independent.*

Proof. Suppose $\alpha_1, \ldots, \alpha_m$ are distinct vectors in S and that

$$c_1\alpha_1 + \cdots + c_m\alpha_m + b\beta = 0 \tag{2.1}$$

then if $b \neq 0$

$$\beta = -\frac{1}{b}(c_1\alpha_1 + \cdots + c_m\alpha_m)$$

and thus $\beta \in \operatorname{span} S$ which is a contradiction. ∎

Theorem 2.12. *If W is a subspace of a finite dimensional vector space V then every linearly independent subset of W is finite and part of a basis of V.*

Proof. Clearly, since $W \subset V$ then every linearly independent subset S of W must be finite and $|S| \leq \dim V$. Let $S = \{\alpha_1, \alpha_m\} \subset W$ be linearly independent. If $\operatorname{span} S = V$ we're done otherwise, there is $\alpha_{m+1} \in V$ such that $\alpha_{m+1} \notin \operatorname{span} S$. By Lemma 2.11, $S_1 = S \cup \{\alpha_{m+1}\}$ is linearly independent. Now if $\operatorname{span} S_1 = V$ we are done. If not we continue in similar fashion. Since V is finite and any independent set can not have more than $\dim V$ elements, then in $\dim V - m$ steps the set $S \cap \{\alpha_{m+1}, \ldots, \alpha_n\}$ is a basis for V. ∎

Corollary 2.13. *A proper subspace W of a finite dimensional vector space V is finite dimensional and* $\dim W < \dim V$*.*

Proof. Any basis W is linearly independent subset of V and therefore it is finite.Suppose W has basis $\{\beta_1, \ldots, \beta_m\}$ with $m \leq \dim V$. Since $\operatorname{span}\beta = W \subsetneq V$ then there is $\alpha in V$ that is not in W therefore, $\beta \cup \{\alpha\}$ is a linearly independent subset of V by Lemma 2.11. Simply

$$|\beta| = |\beta \cup \{\alpha\}| + 1 \leq n$$

and thus $\dim W < \dim V$. ∎

Theorem 2.14. *If W_1 and W_2 are finite dimensional subspaces of V (not necessarily finite dimensional), then $W_1 + W_2$ is finite dimesional and*

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Proof. $W_1 \cap W_2$ is a finite subsapce of V and thus there are $\alpha = \{\alpha_1, \dots, \alpha_a\}$ which form a basis for $W_1 \cap W_2$. Furthermore, since $W_1 \cap W_2 \subset W_1, W_2$ then there are vectors $\beta = \{\beta_1, \dots, \beta_b\}$ and $\gamma = \{\gamma_1, \dots, \gamma_c\}$ such that $\alpha \cup \beta$ is a basis for W_1 and $\alpha \cup \gamma$ is a basis for W_2. We claim that $\alpha \cup \beta \cup \gamma$ is a basis for $W_1 + W_2$. For all $\xi_1 \in W_1$ and $\xi_2 \in W_2$, $\xi_1 + \xi_2 \in W_1 + W_2$. Writing $\xi_1 = x_1\alpha_1 + \dots + x_a\alpha_a + y_1\beta_1 + \dots + y_b\beta_b$ and $\xi_2 = x'_1\alpha_1 + \dots x'_a\alpha_a + z_1\gamma_1 + \dots + z_c\gamma_c$ we get

$$\xi_1 + \xi_2 = \sum_{i=1}^{a} (x_i + x'_i)\alpha_i + \sum_{i=1}^{b} y_i\beta_i + \sum_{i=1}^{c} z_i\gamma_i \in \operatorname{span}(\alpha \cup \beta \cup \gamma)$$

and thus $W_1 + W_2 \subset \operatorname{span}(\alpha \cup \beta \cup \gamma)$. Cleary $\operatorname{span}(\alpha \cup \beta \cup \gamma) \subset W_1 + W_2$ and thus $W_1 + W_2 = \operatorname{span}(\alpha \cup \beta \cup \gamma)$. Now we need to show that $\alpha \cup \beta \cup \gamma$ is independent. Consider

$$\begin{aligned} &\sum_{i=1}^{a} x_i\alpha_i + \sum_{i=1}^{b} y_i\beta_i + \sum_{i=1}^{c} z_i\gamma_i = 0 \\ \implies &\sum_{i=1}^{a} x_i\alpha_i + \sum_{i=1}^{b} y_i\beta_i = \sum_{i=1}^{c} z_i\gamma_i \end{aligned}$$

and thus $\sum_{i=1}^{c} z_i\gamma_i \in W_1$ and as a result $\sum_{i=1}^{c} z_i\gamma_i \in W_1 \cap W_2$ therefore, writing $\sum_{i=1}^{c} z_i\gamma_i = \sum_{i=1}^{a} x'_i\alpha_i$

$$\begin{aligned} &\sum_{i=1}^{a} x_i\alpha_i + \sum_{i=1}^{b} y_i\beta_i = -\sum_{i=1}^{a} x'_i\alpha_i \\ \implies &\sum_{i=1}^{a} (x_i + x'_i)\alpha_i + \sum_{i=1}^{b} y_i\beta_i = 0 \end{aligned}$$

and since $\alpha \cup \beta$ is a basis for W_1 then $y_i = 0$. hence

$$\sum_{i=1}^{a} x_i\alpha_i + \sum_{i=1}^{c} z_i\gamma_i = 0$$

which implies $\alpha_i = 0, \gamma_i = 0$ as $\alpha \cup \gamma$ is a basis for W_2. ∎