Contents

Part I

Quantum Light

Chapter 1

Coherent Quasi-Classical States of Harmonic Oscilator

As the energy increases the behaviour of a quantum system should resemble a classical one. We may ask whether there are quantum states that give classical predications. Yes, there are; they are called the *quasi-classical states* or *coherent* states.

1.1 Classical states

In classical mechanic the harmonic oscilator is described by

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t) &= \dfrac{1}{m}p(t) \\ \\ \dfrac{\mathrm{d}}{\mathrm{d}t}p(t) &= -m\omega^2 x(t) \end{cases}$$

Let $\hat{x}(t) = \beta x(t)$ and $\hat{p}(t) = \frac{1}{\beta\hbar}p(t)$ where $\beta = \sqrt{\frac{m\omega}{\hbar}}$. Then,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) &= \omega\hat{p}(t) \\ \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{p}(t) &= -\omega\hat{x}(t) \end{cases}$$

Let $\alpha(t) = \frac{1}{\sqrt{2}}(\hat{x}(t) + i\hat{p}(t))$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t) = -i\omega\alpha(t)$$

which gives $\alpha(t) = \alpha_0 e^{-i\omega t}$ with $\alpha_0 = \alpha(0) \in \mathbb{C}$. Everything is determied by α_0.

$$\begin{cases} \hat{x}(t) &= \dfrac{1}{\sqrt{2}}(\alpha_0 e^{-i\omega t} + \bar{\alpha_0} e^{i\omega t}) \\ \\ \hat{p}(t) &= -\dfrac{i}{\sqrt{2}}(\alpha_0 e^{-i\omega t} - \bar{\alpha_0} e^{i\omega t}) \end{cases}$$

Moreover, the total energy of the system is given by

$$\begin{aligned}
\mathcal{H}(t) &= \frac{1}{2m}(p(t))^2 + \frac{1}{2}m\omega^2(x(t))^2 \\
&= \frac{\hbar\omega}{2}(\hat{p}(t))^2 + \frac{\hbar\omega}{2}(\hat{x}(t))^2 \\
&= \hbar\omega|\alpha(t)|^2 \\
&= \hbar\omega|\alpha_0|^2
\end{aligned}$$

For classical system $\mathcal{H}$ is must greater then $\hbar\omega$, hence $|\alpha_0| \gg 1$.

1.2 Defining quasi-classical states

We want quantum states such that $\langle X\rangle, \langle P\rangle$, and $\langle H\rangle$ at any given instant are equal to the classical $x, p, \mathcal{H}$. We have

$$\begin{aligned}
\hat{X} &= \beta X = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \\
\hat{P} &= \frac{1}{\hbar\beta}P = -\frac{i}{\sqrt{2}}\left(a - a^\dagger\right) \\
\hat{H} &= \frac{1}{\hbar\omega}H = a^\dagger a + \frac{1}{2}
\end{aligned}$$

The time evolution of $\langle a\rangle$ is given by

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle a\rangle = \langle [a, H]\rangle = \hbar\omega\langle a\rangle \implies \frac{\mathrm{d}}{\mathrm{d}t}\langle a\rangle = -i\omega\langle a\rangle$$

Thus, $\langle a\rangle = \langle a\rangle(0)e^{-i\omega t}$. As a result, we get similar equations to the classical case if we set $\langle a\rangle(0) = \alpha_0$ and from $\langle H\rangle$ we get the condition

$$\hbar\omega\left\langle a^\dagger a\right\rangle + \frac{\hbar\omega}{2} \approx \hbar\omega\left\langle a^\dagger a\right\rangle = \hbar\omega|\alpha_0|^2$$

Therefore, the conditions are $\langle a\rangle(0) = \alpha_0$ and $\left\langle a^\dagger a\right\rangle(0) = |\alpha_0|^2$. These are sufficient to determine $|\psi(0)\rangle$.

Let $b(\alpha) = a - \alpha$, then

$$b^\dagger(\alpha_0)b(\alpha_0) = a^\dagger a - \alpha_0 a^\dagger - \overline{\alpha_0}a + |\alpha_0|^2$$

and we have

$$\begin{aligned}
\|b(\alpha_0)|\psi(0)\rangle\| &= \langle\psi(0)|b^\dagger(\alpha_0)b(\alpha_0)|\psi(0)\rangle \\
&= \langle\psi(0)|a^\dagger a - \alpha_0 a^\dagger - \overline{\alpha_0}a + |\alpha_0|^2|\psi(0)\rangle \\
&= \left\langle a^\dagger a\right\rangle(0) - \alpha_0\left\langle a^\dagger\right\rangle(0) - \overline{\alpha_0}\langle a\rangle(0) + |\alpha_0|^2 \\
&= |\alpha_0|^2 - \alpha_0\overline{\alpha_0} - \overline{\alpha_0}\alpha_0 - |\alpha_0|^2 = 0
\end{aligned}$$

Therefore, $a|\psi(0)\rangle = \alpha_0|\psi(0)\rangle$. Moreover, the converse is true – i.e. eigenvectors of a satisfy the quasi-classical conditions.

Let $|\alpha\rangle$ denote the eigenvector of a with eigenvalue α. Let $|\alpha\rangle = \sum c_n(\alpha)|n\rangle$. Then,

$$\begin{aligned} a|\alpha\rangle &= a\Big(\sum c_n(\alpha)|n\rangle\Big) \\ &= \sum \sqrt{n} c_n(\alpha)|n-1\rangle \\ &= \sum \sqrt{n+1} c_{n+1}(\alpha)|n\rangle \\ \alpha|\alpha\rangle &= \sum \alpha c_n(\alpha)|n\rangle \\ \implies c_{n+1}(\alpha) &= \frac{\alpha}{\sqrt{n+1}} c_n(\alpha) \\ \implies c_n(\alpha) &= \frac{\alpha^n}{\sqrt{n!}} c_0(\alpha) \end{aligned}$$

Since $|\alpha\rangle$ is normalized

$$\sum_{n=0}^{\infty} \left| \frac{\alpha^n}{\sqrt{n!}} c_0(\alpha) \right|^2 = |c_0(\alpha)|^2 \sum_{n=0}^{\infty} \frac{|\alpha^2|^n}{n!} = |c_0(\alpha)|^2 e^{|\alpha|^2} = 1 \implies c_0(\alpha) = e^{-\frac{|\alpha|^2}{2}}$$

Therefore, probability distribution of the states of $|\alpha\rangle$ is Poisson.

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Furthermore, $\mathbb{P}(|n\rangle) = \frac{\alpha^2}{n} \mathbb{P}(|n-1\rangle)$ hence the maximum value of $\mathbb{P}(|m\rangle)$ is achieved when $m = \lfloor |\alpha|^2 \rfloor$.

$$\begin{aligned} \langle H \rangle &= \sum_n \mathbb{P}(|n\rangle)\left(n + \frac{1}{2}\right)\hbar\omega = \left(|\alpha|^2 + \frac{1}{2}\right)\hbar\omega \approx E_m \\ \langle H^2 \rangle &= \sum_n \mathbb{P}(|n\rangle)\left(n + \frac{1}{2}\right)^2 \hbar^2\omega^2 = \left(|\alpha|^4 + 2|\alpha|^2 + \frac{1}{4}\right)\hbar^2\omega^2 \\ \implies \Delta H &= \hbar\omega|\alpha| \\ \implies \frac{\Delta H}{\langle H \rangle} &\approx \frac{1}{|\alpha|} \ll 1 \end{aligned}$$

when $|\alpha| \gg 1$. And for $\langle X\rangle, \langle P\rangle$ we have

$$\begin{aligned} \langle X \rangle &= \sqrt{\frac{2\hbar}{m\omega}} \Re\alpha & \langle P \rangle &= \sqrt{2m\hbar\omega} \Im\alpha \\ \langle X^2 \rangle &= \frac{\hbar}{2m\omega}\big((\alpha + \overline{\alpha})^2 + 1\big) & \langle P \rangle &= \frac{m\hbar\omega}{2}\big(1 - (\alpha - \overline{\alpha})^2\big) \\ \implies \Delta X &= \sqrt{\frac{\hbar}{2m\omega}} & \Delta P &= \sqrt{\frac{m\hbar\omega}{2m}} \end{aligned}$$

which implies that $\Delta X \Delta P = \hbar/2$. Lastly, note that

$$\langle N \rangle_\alpha = |\alpha|^2 \qquad\qquad \Delta N_\alpha = |\alpha|$$

Thus, to obtain a coherent state, close to classical state, we must linearly superpose a very large number of states since $\Delta N_\alpha \gg 1$. However, the relative value of the dispersion over N is very small.

$$\frac{\langle N \rangle_\alpha}{\Delta N_\alpha} = \frac{1}{|\alpha|} \ll 1$$

1.3 Displacement Operator

Let $D(\alpha) = e^{\alpha a^\dagger - \overline{\alpha} a}$ be the displacement operator. Note that $[\alpha a^\dagger, \overline{\alpha} a] = |\alpha|^2$ and hence

$$D(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\overline{\alpha} a}$$

Proposition 1.1. *The displacement operator $D(a)$ is a unitary operator that transform $|0\rangle$ to $|\alpha\rangle$. That is,*

$$|\alpha\rangle = D(\alpha)|0\rangle$$

Lemma 1.2. $\langle x|e^{\lambda X} = e^{\lambda x}\langle x|$ *and* $\langle x|e^{-i\lambda/\hbar P} = \langle x - \lambda|$.

We know that $\alpha a^\dagger - \overline{\alpha} a = \lambda_x X - i\lambda_p/\hbar P$ with

$$\lambda_x = \sqrt{\frac{2m\omega}{\hbar}}\Im\alpha \qquad \lambda_p = \sqrt{\frac{2\hbar}{m\omega}}\Re\alpha$$

. Therefore, from the two statements above we have

$$\begin{aligned}
\psi_\alpha(x) = \langle x|\alpha\rangle &= \langle x|D(\alpha)|0\rangle \\
&= \langle x|e^{\lambda_x X - i\lambda_p P}|0\rangle \\
&= e^{-i\hbar\lambda_x\lambda_p/2}\langle x|e^{\lambda_x X}e^{-i\lambda_p P}|0\rangle \\
&= e^{-i\hbar\lambda_x\lambda_p/2}e^{\lambda_x x}\langle x|e^{-i\lambda_p P}|0\rangle \\
&= e^{-i\hbar\lambda_x\lambda_p/2}e^{\lambda_x x}\langle x - \lambda_p|0\rangle \\
&= e^{-i\hbar\lambda_x\lambda_p/2}e^{\lambda_x x}\phi_0(x - \lambda_p)
\end{aligned}$$

– needs correction maybe

$$\begin{aligned}
\psi_\alpha(x) &= e^{i\theta_\alpha}e^{i\langle P\rangle_\alpha x/\hbar}\phi(x - \langle X\rangle_\alpha) \\
&= e^{i\theta_\alpha}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\left(\frac{x - \langle X\rangle_\alpha}{2\Delta X_\alpha}\right)^2 + i\langle P\rangle_\alpha x/\hbar\right) \\
\implies |\psi_\alpha(x)|^2 &= \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left(-\frac{1}{2}\left(\frac{x - \langle X\rangle_\alpha}{\Delta X_\alpha}\right)\right)
\end{aligned}$$

which is a Gaussian wavepacket, which is consistent with $\Delta X_\alpha \Delta P_\alpha = \hbar/2$. Although, the quasi-classical states are not orthonormal

$$|\langle\alpha|\alpha'\rangle|^2 = e^{-|\alpha-\alpha'|^2} \neq 0$$

but they satisfy a closure relationship

$$\frac{1}{\pi}\int\int|\alpha\rangle\langle\alpha|\,\mathrm{d}\Re\alpha\Im\alpha = 1$$

–add proofs for both

1.4 Time evolution of a quasi-classical state

$$\begin{aligned}|\alpha_0(t)\rangle &= e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar}|n\rangle \\ &= e^{-|\alpha|^2/2} e^{-i\omega t/2}\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t}|n\rangle\end{aligned}$$

which means $|\alpha_0(t)\rangle = e^{-i\omega t/2}|e^{-i\omega t}\alpha_0\rangle$ and thus remains a quasi-classical state.

$$\begin{cases}\langle X\rangle_t &= \sqrt{\dfrac{2\hbar}{m\omega}}\Re(\alpha e^{-i\omega t}) \\ \langle P\rangle_t &= \sqrt{2m\hbar\omega}\Im(\alpha e^{-i\omega t}) \\ \langle H\rangle_t &= \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right)\end{cases} \qquad \begin{cases}\Delta X &= \sqrt{\dfrac{\hbar}{2m\omega}} \\ \Delta P &= \sqrt{\dfrac{m\hbar\omega}{2}} \\ \Delta H &= \hbar\omega|\alpha|\end{cases}$$

1.4.1 The motion of the Wavepacket

At t, the wave packet is still Gaussian. Following figure show the motion of the wavepacket which performs a periodic oscillation along the x-axis, without becoming distorted. It is well known that a Gaussian wavepacket, when it is free, becomes distorted as it propagates, since its width varie. However, under the effect of the parabolic potential $V(x)$, the wavepacket oscillates without becoming distorted.

Chapter 2

Field Qauntization

Chapter 3

Optical Information Processing

some algebraic definition like separability of topological space, completeness, etc. dual space.