Contents

Chapter 1

Probability

1.1 Convergence

1. Convergence in probability.

$$T_n \xrightarrow{\mathbb{P}} T \iff \mathbb{P}(|T_n - T| \geq \epsilon) \xrightarrow[n\to\infty]{} 0 \quad \forall \epsilon > 0$$

2. Almost surely convergence.

$$T_n \xrightarrow{\text{a.s.}} T \iff \mathbb{P}(\{\omega \,|\, T_n(\omega) \to T(\omega)\}) = 1$$

3. Convergence in distribution.

$$T_n \xrightarrow{(d)} T \iff \mathbb{P}(T_n \leq x) \xrightarrow[n\to\infty]{} \mathbb{P}(T \leq x)$$

   for all x at which F_T is continuous.

4. $\mathcal{L}^p$ Convergence.

$$T_n \xrightarrow{:\surd} T \iff \mathbb{E}[|T_n - T|^p] \xrightarrow[n\to\infty]{} 0 \tag{1.1}$$

   For $p \geq 1$.

Theorem 1.1 (Weak/Strong law of large numbers). *Let $X_1, \ldots, X_n$ be i.i.d. with $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}[X]$ both finite. Then*

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow[n\to\infty]{\mathbb{P}/\,\text{a.s.}} \mu$$

Theorem 1.2 (Central limit theorem). *Let $X_1, \ldots, X_n$ be i.i.d with $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{Var}[X]$ both finite. Then*

$$\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow[n\to\infty]{(d)} \mathrm{N}(0,1)$$

Theorem 1.3 (Hoeffding inequality). *Let $X_1, \ldots, X_n$ be i.i.d with $\mu = \mathbb{E}[X]$, $X \in [a,b]$ then for all $\epsilon > 0$*

$$\mathbb{P}\big(\big|\bar{X}_n - \mu\big| \geq \epsilon\big) \leq 2e^{-\frac{2n\epsilon^2}{(b-a)^2}}$$

Proposition 1.4. *The followings are equivalent*

1. $T_n \xrightarrow{(d)} T$.
2. $\mathbb{E}[f(T_n)] \xrightarrow[n\to\infty]{} \mathbb{E}[f(T)]$ *for all continuous and bounded* f.
3. $\mathbb{E}\big[e^{ixT_n}\big] \xrightarrow[n\to\infty]{} \mathbb{E}\big[e^{ixT}\big]$ *for all* x.

Proposition 1.5. *We have the following relationships*

$$T_n \xrightarrow{\text{a.s.}} T \implies T_n \xrightarrow{\mathbb{P}} T \implies T_n \xrightarrow{(d)} T$$

and

$$T_n \xrightarrow{L^1} T \implies T_n \xrightarrow{\mathbb{P}} T$$

and

$$T_n \xrightarrow{L^p} T \implies T_n \xrightarrow{L^q} T, \quad \forall q \le p$$

Proposition 1.6. *Let* f *be a continuous function then*

$$T_n \xrightarrow{\text{a.s.}/\mathbb{P}/(d)} T \implies f(T_n) \xrightarrow{\text{a.s.}/\mathbb{P}/(d)} f(T)$$

Proposition 1.7. *If* $U_n \xrightarrow{\text{a.s.}/\mathbb{P}} U$ *and* $V_n \xrightarrow{\text{a.s.}/\mathbb{P}} V$ *then*

1. $$U_n + V_n \xrightarrow{\text{a.s.}/\mathbb{P}} U + V$$

2. $$U_n V_n \xrightarrow{\text{a.s.}/\mathbb{P}} UV$$

3. $$\frac{U_n}{V_n} \xrightarrow{\text{a.s.}/\mathbb{P}} \frac{U}{V}, \quad V \neq 0\,\text{a.s.}$$

These propositions hold for convergence in distribution if the pair $(U_n, V_n) \xrightarrow{(d)} (U, V)$.

Theorem 1.8 (Slutsky's theorem). *Let* X_n, Y_n *be a sequence of random variable such that*

$$X_n \xrightarrow{(d)} X \qquad Y_n \xrightarrow{\mathbb{P}} c$$

where c *is a constant then* $(X_n, Y_n) \xrightarrow{(d)} (X, c)$. *In particular*

$$X_n + Y_n \xrightarrow{(d)} X + c, \quad X_n Y_n \xrightarrow{(d)} cX$$

Theorem 1.9 (Delta method). *Let* X_n *be a sequence of random variable such that*

$$\sqrt{n}(X_n - \mu) \xrightarrow{(d)} \mathrm{N}\big(0, \sigma^2\big)$$

where μ *and* σ *are finite valued constants. Then for any function* g *that* $g'(\mu)$ *exists and is non-zero*

$$\sqrt{n}(g(X_n) - g(\mu)) \xrightarrow{(d)} \mathrm{N}\big(0, \sigma^2\big(g'(\mu)^2\big)\big)$$

Proof. By Taylor's polynomial, there exists μ^* between X_n and μ such that

$$g(X_n) - g(\mu) = (X_n - \mu)g'(\mu^*)$$

then since $g'(\mu^*)$ convergers in probability – because of weak law of large number, continuity of g', and continuous mapping theorem – to $g'(\mu)$, by Slutsky's theorem

$$\sqrt{n}(X_n - \mu)g'(\mu^*) \xrightarrow{(d)} \mathrm{N}\Big(0, \sigma^2 (g'(\mu))^2\Big)$$

which proves the Delta method. ■

Chapter 2

Parametric Models