Contents

Chapter 1

Frequency Domain Analysis

1.1 Fourier Series

For a periodic signal $x(t)$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{2\pi j \frac{n}{T_0} t} \qquad\qquad x_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-2\pi j \frac{n}{T_0} t} \,\mathrm{d}t$$

and

$$x_{\pm}(t) = \begin{cases} x(t) & x \ \text{ is continuous at } \ t \\ \dfrac{x(t^+) + x(t^-)}{2} & x \ \text{ is discontinuous at } \ t \end{cases}$$

for angular frequency $\omega_0 = 2\pi f_0$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} \qquad\qquad x_n = \frac{\omega_0}{2\pi} \int_{T_0} x(t) e^{-jn\omega_0 t} \,\mathrm{d}t$$

$f_0 = \frac{1}{T_0}$ is called the **fundamental frequency** and its n_{th} is called the n_{th} **harmonic**.

1.2 Fourier Transform

For non-periodic signals $x(t)$:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \,\mathrm{d}t & x_{\pm}(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \,\mathrm{d}f \\ X(f) &= \mathcal{F}\{x(t)\} & x_{\pm}(t) &= \mathcal{F}^{-1}\{X(f)\} \\ X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \,\mathrm{d}t & x_{\pm}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j\omega t} \,\mathrm{d}\omega \end{aligned}$$

$X(f)$ is called the **spectrum** of $x(t)$, or the **voltage spectrum**. From the relationship between the inverse Fourier transform of Fourier transform of a signal we define

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi j f t} \,\mathrm{d}f = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \,\mathrm{d}f$$

That is, all frequencies in $\delta(t)$ are with unit magnitude and zero phase.

$$\delta(t) = \mathcal{F}^{-1}\{1\} \qquad\qquad \delta(f) = \mathcal{F}\{1\}$$

1.3 Properties of Fourier transform

Linearity. For two signals $x(t)$ and $y(t)$ and complex constants a and b

$$\begin{aligned}\mathcal{F}\{ax(t)+by(t)\} &= \int_{-\infty}^{\infty}(ax(t)+by(t))e^{-2\pi jft}\,\mathrm{d}t \\ &= a\int_{-\infty}^{\infty}x(t)e^{-2\pi jft}\,\mathrm{d}t + b\int_{-\infty}^{\infty}y(t)e^{-2\pi jft}\,\mathrm{d}t \\ &= a\mathcal{F}\{x(t)\}+b\mathcal{F}\{y(t)\}\end{aligned}$$

Duality. For any signal $x(t)$

$$x(f)=\mathcal{F}\{\mathcal{F}\{x(t)\}(-\omega)\}$$

since

$$\begin{aligned}\mathcal{F}\{\mathcal{F}\{x(t)\}(-\omega)\} &= \int_{-\infty}^{\infty}\mathcal{F}\{x(t)\}(-\omega)e^{2\pi jf\omega}\,\mathrm{d}\omega \\ &= \mathcal{F}^{-1}\{\mathcal{F}\{x(t)\}\}(f) \\ &= x(f)\end{aligned}$$

Time shift. A shift of t_0 in time domain causes a phase shift in the frequency domain.

$$\begin{aligned}\mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty}x(t-t_0)e^{-2\pi jft}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}x(t)e^{-2\pi jf(t+t_0)}\,\mathrm{d}t \\ &= e^{-2\pi jft_0}\mathcal{F}\{x(t)\}\end{aligned}$$

Scaling. Suppose $a \neq 0$ is real

$$\begin{aligned}\mathcal{F}\{x(at)\} &= \int_{-\infty}^{\infty}x(at)e^{-2\pi jft}\,\mathrm{d}t \\ &= \frac{1}{a}\operatorname{sign}(a)\int_{-\infty}^{\infty}x(t)e^{-2\pi jf\frac{t}{a}}\,\mathrm{d}t \\ &= \frac{1}{|a|}\mathcal{F}\left\{\frac{f}{a}\right\}\end{aligned}$$

Convolution. For two signals $x(t)$ and $y(t)$

$$\begin{aligned}\mathcal{F}\{x(t)*y(t)\} &= \int_{-\infty}^{\infty}(x(t)*y(t))e^{-2\pi jft}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}y(t-\tau)e^{-2\pi jf(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}y(t-\tau)e^{-2\pi jf(t-\tau)}\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}\mathcal{F}\{y(t)\}\,\mathrm{d}\tau \\ &= \mathcal{F}\{x(t)\}\mathcal{F}\{y(t)\}\end{aligned}$$

Parseval's property. For two signals $x(t)$ and $y(t)$ with Fourier transform $X(f)$ and $Y(f)$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,\mathrm{d}t = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,\mathrm{d}f$$

since

$$\begin{aligned}
\int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,\mathrm{d}f &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\,\mathrm{d}t\right)\overline{\left(\int_{-\infty}^{\infty} y(t)e^{-2\pi jft}\,\mathrm{d}t\right)}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\overline{y(\tau)}e^{2\pi jf\tau}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\overline{y(\tau)}e^{2\pi jf(\tau-t)}\,\mathrm{d}f\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\overline{y(\tau)}\delta(\tau-t)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t)\overline{y(t)}\,\mathrm{d}t
\end{aligned}$$

Rayleigh's propery. For any signal $x(t)$ with Fourier transform $X(f)$

$$\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$$

Autocorrelation. The time autocorrelation of the signal $x(t)$ is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}\,\mathrm{d}t = x(t) * \overline{x(-t)}$$

Then,

$$\mathcal{F}\{R_x(\tau)\} = |X(f)|^2$$

Differentiation.

$$\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right\} = 2\pi j\mathcal{F}\{x(t)\}$$

Integration.

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau\right\} = \frac{X(f)}{2\pi jf} + \frac{1}{2}X(0)\delta(f)$$

Moments

$$\int_{-\infty}^{\infty} t^n x(t)\,\mathrm{d}t = \left(\frac{j}{2\pi}\right)^n\left[\frac{\mathrm{d}^n}{\mathrm{d}f^n}X(f)\right]_{f=0}$$

1.4 Power and Energy

Define

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t \qquad\qquad \mathcal{P}_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,\mathrm{d}t$$

A signal is **energy-type** if $\mathcal{E}_x < +\infty$ and it is **power-type** if $0 < \mathcal{P}_x < +\infty$. A signal can not be both, but it can be neither.

Remark 1. Average power is expressed in units of dBm or dBw as

$$\begin{aligned}(S)_{\text{dBw}} &= 10\log_{10}(S)_{\text{watts}}\\ (S)_{\text{dBm}} &= 10\log_{10}(S)_{\text{milliwatts}}\end{aligned}$$

1.4.1 Energy-type

Let $x(t)$ be a energy-type signal. The **autocorrelation** of $x(t)$ is

$$\begin{aligned}R_x(\tau) &= x(\tau) * \overline{x(-\tau)}\\ &= \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}\,\mathrm{d}t\\ \implies \mathcal{E}_x &= R_x(0)\end{aligned}$$

By Rayleigh's property

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$$

The Fourier transform exists for The **energy spectral density** $\mathcal{G}(f) = \mathcal{F}\{R_x(\tau)\} = |X(f)|^2$, represent energy per hertz of bandwidth.

1.4.2 Power-type

Let $x(t)$ be a power type signal. The **time average autocorrelation** function

$$\begin{aligned}R_x(\tau) &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}\,\mathrm{d}t\\ \implies \mathcal{P}_x &= R_x(0)\end{aligned}$$

The **power spectral density** $\mathcal{S}(f) = \mathcal{F}\{R_x(\tau)\}$ and

$$\mathcal{P}_x = \int_{-\infty}^{\infty} \mathcal{S}(f)\,\mathrm{d}f$$

Remark 2. The power spectral density does not uniquely determine the signal. As it only retains the magnitude information and all phase information is lost.

Suppose $x(t)$ is a power-type signal passing through a filter with impluse response $h(t)$:

$$\begin{aligned}y(t) &= x(t) * h(t)\\ R_y(\tau) &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\overline{y(t-\tau)}\,\mathrm{d}t\\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\int_{-\infty}^{\infty} h(u)x(t-u)\,\mathrm{d}u\right)\left(\int_{\infty}^{\infty}\overline{h(v)x(t-\tau-v)}\,\mathrm{d}v\right)\mathrm{d}t\\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(u)\overline{h(v)}x(t-u)\overline{x(t-\tau-v)}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v\\ &= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)}\lim_{T\to\infty}\int_{-\frac{T}{2}+u}^{\frac{T}{2}+u} x(w)\overline{x(w+u-\tau-v)}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)}R_x(v+\tau-u)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\infty}^{\infty}(R_x(v+\tau) * h(v+\tau))\overline{h(v)}\,\mathrm{d}u\,\mathrm{d}v \\
&= R_x(\tau) * h(\tau) * \overline{h(-\tau)}
\end{aligned}$$

Which implies that

$$\mathcal{S}_y(f) = \mathcal{S}_x(f)H(f)\overline{H(f)} = \mathcal{S}_x(f)|H(f)|^2$$

1.5 Sampling of bandlimited signals

$f_s = 2W$ is the **Nyquist rate** and $f_s - 2W$ is **guard band**.

1.6 Bandpass signal

A **bandpass signal** has non-zero frequencies around a small neighborhood of some high frequency f_0. That is, $X(f) = 0$ for $|f - f_0| \geq W$ where $W < f_0$. A **bandpass system** passes frequencies around some f_0 or equivalently, the impluse response is a bandpass signal. f_0 is called the **central frequency** even tho it might not be the center of signal's bandwidth.

1.6.1 Analysis of monochromatic signals

Monochromatic signals are bandpass with $W = 0$.

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

The **phasor** is defined as $\hat{X} = Ae^{j\theta}$. Consider an LTI system with impluse response $H(f)$. Then, the phasor of the output of signal $x(t)$ is $\hat{Y} = AH(f_0)e^{j\theta}$ and the frequency of the output signal is the same, namely f_0. To obtain the phasor of the input consider the signal

$$\begin{aligned}
z(t) &= Ae^{2\pi j f_0 t + j\theta} \\
&= A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) \\
&= x(t) + jx_q(t) = x(t) + jx\left(t - \frac{\pi}{2}\right)
\end{aligned}$$

where $x_q(t)$ is a 90° phase shift version of the original signal– q stands for *quadrature*. Then,

$$\hat{X} = z(t)e^{-2\pi j f_0 t}$$

Note that, $Z(f)$ can be obtained from $X(f)$ by deleting the negative frequencies and multiplying the positive frequencies by a factor of two.

1.6.2 Analysis of a general bandpass signal

For a general bandpass signal, let $Z(f)$ be the signal obtained from deleting the negative frequencies of $X(f)$ and multiplying the positive frequencies by a factor of two. That is,

$$Z(f) = 2U_{-1}(f)X(f)$$

where $U_{-1}(f)$ is the Heaviside step function. $z(t)$ is called the **analytic signal corresponding to** $x(t)$ or the **pre-envelope of** $x(t)$. The inverse Fourier of $U_{-1}(f)$ is calculated as follows

$$\begin{aligned}\mathcal{F}^{-1}\{U_{-1}(f)\} &= \mathcal{F}\{U_{-1}(-\tau)\}(t)\\ &= \mathcal{F}\{1 - U_{-1}(\tau)\}(t)\\ &= \delta(t) - \left(\frac{1}{2\pi jt} + \frac{1}{2}\delta(t)\right)\\ &= \frac{1}{2}\delta(t) - \frac{1}{2\pi jt}\\ &= \frac{1}{2}\delta(t) + \frac{j}{2\pi t}\end{aligned}$$

Therefore,

$$\begin{aligned}z(t) &= x(t) * \left(\delta(t) + \frac{j}{\pi t}\right)\\ &= x(t) + jx(t) * \frac{1}{\pi t}\\ &= x(t) + jx'(t)\end{aligned}$$

$x'(t)$ is called the **Hilbert transform of** $x(t)$. Hilbert transform, as derived below, is equivalent to a $-\frac{\pi}{2}$ shift for positive frequencies and a $\frac{\pi}{2}$ shift for negative frequencies.

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\,\mathrm{sign}(f) = e^{-j\frac{\pi}{2}\,\mathrm{sign}(f)}$$

$H(f) = -j\,\mathrm{sign}(f)$ is called the **quadrature filter**. Then, consider the signal $x_l(t) = z(t)e^{-2\pi jf_0t}$ or equivalently $X_l(f) = Z(f + f_0)$ wher f_0 is the centeral frequency of $x(t)$. $x_l(t)$ is called the **lowpass representation of bandpass signal** $x(t)$. In general $x(t)$ is a complex-valued signal, hence we can decompose it into real and imaginary parts

$$x_l(t) = x_c(t) + jx_s(t)$$

$x_c(t)$ is called the **in-phase** and $x_s(t)$ is called the the **quadrature** components of $x(t)$. Then,

$$\begin{aligned}z(t) &= x_l(t)e^{2\pi jf_0t}\\ &= (x_c(t)\cos(2\pi f_0t) - x_s(t)\sin(2\pi f_0t)) + j(x_c(t)\sin(2\pi f_0t) + x_s(t)\cos(2\pi f_0t))\end{aligned}$$

hence

$$\begin{aligned}x(t) &= x_c(t)\cos(2\pi f_0t) - x_s(t)\sin(2\pi f_0t)\\ x'(t) &= x_c(t)\sin(2\pi f_0t) + x_s(t)\cos(2\pi f_0t)\end{aligned}$$

these two equations are called the **bandpass to lowpass transformations**.

Define the **envelope** of $x(t)$, $V(t)$, as

$$V(t) = \sqrt{(x_c(t))^2 + (x_s(t))^2}$$

and the **phase** of $x(t)$, $\Theta(t)$, as

$$\Theta(t) = \arctan\frac{x_s(t)}{x_c(t)}$$

Then,

$$\begin{aligned}
x_l(t) &= V(t)e^{j\Theta(t)} \\
z(t) &= V(t)e^{2\pi j f_0 t + j\Theta(t)} \\
x(t) &= V(t)\cos(2\pi f_0 t + \Theta(t)) \\
x'(t) &= V(t)\sin(2\pi f_0 t + \Theta(t))
\end{aligned}$$

Chapter 2

Random Signal Theory

2.1 Introduction

Marcum Q function

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp\left(-\frac{z^2}{2}\right)$$

and if $X \sim \mathrm{N}(\mu, \sigma^2)$ then

$$\mathbb{P}(X > a) = Q\left(\frac{a-\mu}{\sigma}\right)$$

We have the following upper bounds

$$Q(x) \leq \frac{1}{2} e^{-x^2/2}$$

and

$$Q(x) < \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$$

for all $x \geq 0$. For lower bound

$$Q(x) > \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2}$$

$\bar{X}$ has multivariate Gaussian distribution with mean μ and covariance matrix Σ

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(\bar{x}-\mu)^T \Sigma^{-1} (\bar{x}-\mu)\right)$$

2.2 Random Process

A set of indexed random variables $\{X_t\}_{t \in T}$ is a random process. We denote a random process as $X(t, \omega)$ where $\omega \in \Omega$ and $t \in T = \mathbb{R}$. For a specific ω_0, $X(t, \omega_0) = x_0(t)$ is a time function called **member function**, **sample function**, or a **realization function**. The totality of all sample functions is called an **ensemble**. For a specific t_0, $X(t_0, \omega)$ is a random variable.

Definition: A process $X(t)$ is described by its M_{th} **order statistics** if for all $m \leq M$ and all $(t_1, \ldots, t_m) \in \mathbb{R}^m$ the joint PDF of $(X(t_1), \ldots, X(t_n))$ is given.

2.2.1 Statistical averages

The mean of an stochastic process

$$\mu_X(t) = \mathbb{E}[X(t)]$$

The autocorrelation of an stochastic process

$$R_{XX}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$$

Statistical averages are also called ensemble averages.

2.2.2 Stationarity in wide sense

A stochastic process with constants mean and time invariant autocorrelation is called **stationary in wide sense**.

$$\mu_X(t) = K, \qquad R_{XX}(t_1 + t, t_2 + t) = R_{XX}(t_1, t_2)$$

for all t_1, t_2, t.

Definition: A random process $X(t)$ with mean $\mu_X(t)$ and autocorrelation $R_{XX}(t+\tau, t)$ is called **cyclostationary** if both the mean and autocorrelation are periodic in t with some period T_0, that is

$$\mu_X(t + T_0) = m_X(t)$$

and

$$R_{XX}(t + \tau + T_0, t + T_0) = R_{XX}(t + \tau, t)$$

for all t and τ.

2.2.3 Time averages

The time average mean and autocorrelation of a random process is defined as

$$\langle \mu_X \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\,\mathrm{d}t$$
$$\langle R_{XX}(\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau)\,\mathrm{d}t$$

Both time averages are random variables and depend on ω. Ensemble averages and time averages are equal in mean squared sense. A random variable X is equal to a constant b in MS sense if $\mathbb{E}[X] = b$ and $\mathbb{E}[(X-b)^2] = 0$.

2.2.4 Ergodicity

A wide-sense stationary process is **ergodic in mean** if $\langle \mu_X \rangle$ converges to μ_X in mean squared as $T \to \infty$. A wide-sense stationary process is **ergodic in autocorrelation** if $\langle R_{XX}(\tau) \rangle$ converges to $R_{XX}(\tau)$ in mean squared as $T \to \infty$.

2.2.5 Power Spectral density of stationary random process

We can define the random variables for energy $\mathscr{E}_X$ and power $\mathscr{P}_X$

$$\mathscr{E}_X = \int_{-\infty}^{\infty} X^2(t)\,\mathrm{d}t$$
$$\mathscr{P}_X = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t)\,\mathrm{d}t$$

Then, the power content $\mathcal{P}_X$ and energy content $\mathcal{E}_X$ of a stochastic process $X(t)$ are defined as

$$\mathcal{E}_X = \mathbb{E}[\mathscr{E}_X] = \int_{-\infty}^{\infty} R_{XX}(t,t)\,\mathrm{d}t$$
$$\mathcal{P}_X = \mathbb{E}[\mathscr{P}_X] = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{XX}(t,t)\,\mathrm{d}t$$

Foe stationary processes if $\mathcal{E}_X < \infty$, then $R_X X(0) = 0$ and hence $X(t)$ is zero almost everywhere.

Let $X(t)$ be a random process and $X_T(f)$ be the random process from considering the Fourier transforms of truncated sample functions. That is,

$$X_T(f,\omega) = \mathcal{F}\{x_T(t,\omega)\}$$

Then, the **power density spectrum** or **power spectral density** is defined as

$$\mathcal{G}_X(f) = \lim_{T\to\infty} \frac{\mathbb{E}\left[|X_T(f)|^2\right]}{T}$$

Furthermore,

$$\langle \mathcal{G}_X(f)\rangle = \lim_{T\to\infty} \frac{\left|\int_{-T/2}^{T/2} X(t)\exp(-2\pi j f t)\,\mathrm{d}t\right|^2}{T}$$

For an ergodic random process

$$\langle \mathcal{G}_X(f)\rangle \overset{\text{MS}}{=} \mathcal{G}_X(f)$$

Theorem 2.1 (Wiener-Khinchin). *If for all finite τ and any interval I of length $|\tau|$, the autocorrelation function R_{XX} satisfies the condition*

$$\left|\int_I R_{XX}(t+\tau,t)\,\mathrm{d}t\right| < \infty$$

Then,

$$\mathcal{G}_X(f) = \mathcal{F}\left\{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_{XX}(t+\tau,t)\,\mathrm{d}t\right\}$$

Thus, if $X(t)$ is stationary with

$$\int_{-\infty}^{\infty} |R_{XX}(\tau)|\,\mathrm{d}\tau < \infty$$

then

$$\mathcal{G}_X(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-2\pi j f\tau)\,\mathrm{d}\tau$$

If $X(t)$ is cyclostationary with

$$\left|\int_0^{T_0} R_{XX}(t+\tau,t)\,\mathrm{d}t\right| < \infty$$

then

$$\mathcal{G}_X(f) = \mathcal{F}\big\{\bar{R}_{XX}(\tau)\big\}$$

where

$$\bar{R}_{XX}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_{XX}(t+\tau,t)\,\mathrm{d}t$$

Moreover, $\mathbb{E}\big[(X(t))^2\big]$ can be thought of as the average power dissipated by random process across 1 ohm resistor.

$$\mathbb{E}\big[(X(t))^2\big] = R_{XX}(0) = \int_{-\infty}^{\infty} \mathcal{G}_X(f)\,\mathrm{d}f$$

1. $\langle X(t)\rangle$ is the DC component.
2. $\big\langle (X(t))^2\big\rangle$ is the total average power.
3. $\langle (X(t))\rangle^2$ is the DC power.
4. $\big\langle (X(t))^2\big\rangle - \langle (X(t))\rangle^2$ is the AC power.
5. $\sqrt{\big\langle (X(t))^2\big\rangle - \langle (X(t))\rangle^2}$ is rms value.

2.2.6 PSD of a sum process

Suppose $Z(t) = X(t) + Y(t)$ is the sum of two jointy stationary process. It can be readily verified that $Z(t)$ is stationary and

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

and

$$\mathcal{G}_Z(f) = \mathcal{G}_X(f) + \mathcal{G}_Y(f) + 2\Re(\mathcal{G}_{XY} f)$$

where

$$\mathcal{G}_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\}$$

If $X(t)$ and $Y(t)$ are uncorrelated and at least one of them is zero mean, then $R_{XY}(\tau) = 0$ and

$$\mathcal{G}_Z(f) = \mathcal{G}_X(f) + \mathcal{G}_Y(f)$$

2.3 Systems and random signals

The response of a system to a random signal, is a random signal itself. Therefore, we need tools to investigate the relationships between to random processes.

Definition: Two random process $X(t)$ and $Y(t)$ are **independent** if for all t_1, t_2, the random variables $X(t_1)$ and $Y(t_2)$ are independent. Similarly, $X(t)$ and $Y(t)$ are **uncorrelated** if for all t_1, t_2, the random variables $X(t_1)$ and $Y(t_2)$ are uncorrelated.

Definition: The **cross-correlation** function between two random process $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$$

Two processes $X(t)$ and $Y(t)$ are **jointly wide-sense stationary**, if both are stationary and the cross-correlation function depends on $\tau = t_1 - t_2$.

2.3.1 Response of memoryless channel

$$Y(t) = g(X(t))$$

2.3.2 Response of LTI System

Suppose stationary process $X(t)$ is passed through a LTI system with impulse response $h(t)$. Then, the input and output prcesses $X(t)$ and $Y(t)$ are jointly stationary. Moreover,

$$\begin{aligned}
Y(t) &= X(t) * h(t) \\
&= \int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,\mathrm{d}\tau \\
\implies \mathbb{E}[Y(t)] &= \int_{-\infty}^{\infty} \mathbb{E}[X(\tau)]h(t-\tau)\,\mathrm{d}\tau \\
\mathbb{E}[Y(t)] &= \mu_X \int_{-\infty}^{\infty} h(\tau)\,\mathrm{d}\tau = \mu_X H(0)
\end{aligned}$$

and

$$\begin{aligned}
R_{YX}(\tau) &= \mathbb{E}[Y(t)X(t-\tau)] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} X(\eta)h(t-\eta)X(t-\tau)\,\mathrm{d}\eta\right] \\
&= \int_{-\infty}^{\infty} R_{XX}(t-\tau-\eta)h(t-\eta)\,\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty} R_{XX}(\eta-\tau)h(\eta)\,\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau-\eta)h(\eta)\,\mathrm{d}\eta \\
&= R_{XX}(\tau) * h(\tau) \\
\implies R_{YY}(\tau) &= \mathbb{E}[Y(t+\tau)Y(t)] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} Y(t+\tau)X(t-\eta)h(\eta)\,\mathrm{d}\eta\right]
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} R_{YX}(\tau+\eta)h(\eta)\,\mathrm{d}\eta \\
&= R_{YX}(\tau) * h(-\tau) \\
&= R_{XX}(\tau) * h(\tau) * h(-\tau) \\
\implies \mathcal{G}_X(f) &= \mathcal{F}\{R_{YY}(t)\} = \mathcal{G}_X(f)|H(f)|^2
\end{aligned}$$

2.3.3 Special classes of random processes

Gaussian random process

$X(t)$ is Gaussian if

$$f_X(x_1,\dots,x_n;t_1,\dots,t_n) = \frac{1}{\sqrt{(2\pi)^n \det\Sigma}} \exp\left(-\frac{1}{2}(\bar{x}-\bar{\mu})^T\Sigma^{-1}(\bar{x}-\bar{\mu})\right) \qquad \forall n, \forall t_1,\dots,t_n$$

where $\Sigma = \big[\mathbb{E}[(X(t_i)-\mathbb{E}[X(t_i)])(X(t_j)-\mathbb{E}[X(t_j)])]\big]_{i,j}$ $\bar{\mu} = \big[\mathbb{E}[X(t_i)]\big]_i$.

$X(t)$ is zero mean stationary Gaussian if $\mathbb{E}[X(t)] = 0$ and $R_{XX}(t_1+t, t_2+t) = R_{XX}(t_1,t_2)$ for all t_1, t_2, t. Then, $\Sigma = \big[\mathbb{E}[R_{XX}(t_i,t_j)]\big]_{i,j}$ and $\bar{\mu} = \bar{0}$.

Theorem 2.2. *For a Gaussian process, $\mu_X(t)$ and $R_{XX}(t_1,t_2)$ gives a complete statistical description of the process.*

Corollary 2.3. *For Gaussian processes, WSS and strictly stationarity are equivalent.*

Theorem 2.4. *The output of an LTI system on a Gaussian input is Gaussian.*

Theorem 2.5. *A sufficient condition for the ergodicity of the stationary zero-mean Gaussian process is*

$$\int_{-\infty}^{\infty} |R_{XX}(\tau)|\,\mathrm{d}\tau < \infty$$

Markoff Sequence

Suppose $X(t)$ is defined for countable indices and it assumes a finte set of values. That is, $X(t)$ is discrete-time and discrete-amplitude. $X(t)$ is a Markoff chain if

$$\mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1},\dots,X_1 = a_1) = \mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1})$$

Let $p_i(n) = \mathbb{P}(X_n = a_i)$ and $p_{i,j}(n,m) = \mathbb{P}(X_n = i \mid X_m = j)$ for $n > m$ then

$$p_j(n) = \sum_{i=1}^{N} p_{j,i}(n,m)p_i(m)$$

If $p_{i,j}(n+1,n) = p_{i,j}(n,n-1)$ for all n, then $\{X_n\}$ is called **homogeneous**. Finally, let

$$P(k) = \begin{bmatrix} p_1(k) \\ \vdots \\ p_N(k) \end{bmatrix} \qquad \varphi = \begin{bmatrix} p_{1,1}(n,n-1) & \dots & p_{1,N}(n,n-1) \\ \vdots & \ddots & \vdots \\ p_{N,1}(n,n-1) & \dots & p_{N,N}(n,n-1) \end{bmatrix}$$

Therefore

$$P(k) = \varphi P(k-1)$$

A Markov chain is stationary if $P(k+1) = P(k)$ for all k.

2.4 Noise in communication systems

Determined through expriments (thermodynamics and quantum mechanic) noise voltage $V(t)$ that appears accross the terminal of a resistor of R Ohms has a Gaussian distribution with $\mu_V = 0$ and

$$\mathbb{E}\big[V^2(t)\big] = \frac{(2\pi kT)^2}{3h}R$$

where k is Boltzmann constant, h Plank constant, T is temperature in Kelvins. Then,

$$\mathcal{G}_V(f) = \frac{2Rh|f|}{\exp(h|f|/(kT)) - 1}$$

$\mathcal{G}_V(f)$ is flat over $|f| < 0.1\frac{kT}{h}$. However, for modeling

$$\mathcal{G}_V(f) = 2RkT$$

but in this case $\mathbb{E}[V^2(t)] = \infty$. Yet it is alright since $V(t)$ is subjected to filtering and hence $\mathbb{E}[V^2(t)]$ will be finite.

Definition: A noise signal having flat power spectral density over a wide range frequency is called white noise.

$$\mathcal{G}_V(f) = \frac{\eta}{2}$$

The factor 1/2 is included to indicate that $\mathcal{G}_V(f)$ is a two-sided psd.

At room temperature, $\mathcal{G}_V(f)$ drops to 90% of its maximum at about $f \approx 2 \times 10^12$ Hertz, which is beyond the frequencies employed in the conventionaly communication systems.

Available power is the maximum power that can be delivered to a load from a source having a fixed but non-zero resistence.

$$P_{\max} = \frac{\mathbb{E}[V^2(t)]}{4R}$$

Available power psd is $\mathcal{G}_V(f) = kT/2$.

We will assume the thermal noist is stationary, ergodic, zero-mean, white Gaussian noise whose power spectrum is $N_0/2$ where $N_0 = kT$.

Chapter 3

Information Theory

Remark 3. For a more complete and through treatment refer to the notes on the subject.

3.1 Measure of information

Information content of a message is inversely proportional to the likelihood of that message. Let $m_1, m_2, \ldots, m_q$ be q messages with probability $p_1, p_2, \ldots, p_q$ respectively, such that $p_1 + \cdots + p_q = 1$. Then, information content of m_k, $I(m_k)$ must satisfy the followings

1. $I(m_k) > I(m_j)$ if $m_k < m_j$.
2. $I(m_k) \to 0$ as $p_k \to 0$.
3. $I(m_k) \geq 0$ when $0 \leq p_k \leq 1$.

Furthermore, for two independent messages m_1 and m_2

$$I(m_1, m_2) = I(m_1) + I(m_2)$$

One continuous function that satisfies these requirements is $I(m_k) = -\log p_k$ where the base of the logarithm determines the unit of information, e.g. base e is nats, 2 is bit, 10 is Hartley/decit.

3.1.1 Average information content

For a statistically independent source that emits N symbols from a M-symbol alphabet i.i.d.

$$I_{tot} = -N \sum_{i=1}^{M} p_i \log(p_i)$$

$$H = \frac{I_{tot}}{N} = -\sum_{i=1}^{M} p_i \log(p_i)$$

Proposition 3.1. *For a source with an M-symbol alphabet, the maximum entropy is attained when the symbols are equiprobabilistic and $H_{max} = \log M$.*

Suppose r_s is the symbol rate of the source, measured in . Then, average information rate R is

$$R = r_s H$$

3.1.2 Statistically dependent source

- emits a symbol once every T_s seconds.
- A stochastic process.
- stationary Markoff.
- There are n states with a transition matrix φ, (M symbols with a residual influence lasting q symbols can be represented by $n \leq M^q$ states).
- $\mathbb{P}(X_k = s_q \mid X_1, \ldots, X_{k-1}) = \mathbb{P}(X_k = s_q \mid S_k)$ where S_k is a discrete random variable.
- At the begninning it is in one of the n states with probability $p_i(1)$.
- $p_j(k+1) = \sum_{i=1}^{n} p_i(k) p_{ij}$ for all j. $P(k+1) = \varphi P(k)$.

3.1.3 Entropy for Markov source

- Assume, discrete finite state ergodic hence stationary.
- Entropy of state i is

$$H_i = -\sum_{j=1}^{M} p_{ij} \lg p_{ij}$$

- Entropy of source

$$H = \sum_{i=1}^{n} p_i H_i$$

- therefore, $R = r_s H$.

Theorem 3.2. *Let $G_N = -\frac{1}{N} \sum_i p(m_i) \lg p(m_i)$ over all messages of length N. Then, G_N is monotonice decreasing function of N and*

$$\lim_{N \to \infty} G_N = H$$

3.2 Source encoding

Definition: The ratio of sourse information and the average encoded output bit rate is called *coding efficiency.*

3.2.1 Shannon Algorithm

Let $m_1, \ldots, m_q$ be arranged in decreasing order of probability $p_1 \leq \cdots \leq p_q$. Let $F_i = \sum_{k=1}^{i-1} p_k$ with $F_1 = 0$. Let $n_i = \lceil -\lg p_i \rceil$. Then, the code

$$c_i = (F_i)_2 \qquad \text{binary fraction of } F_i \text{ up to } n_i \text{ bits.}$$

has the following properties

1. $l(c_i) > l(c_j) \implies p_i < p_j$.

2. Codewords are all differnt. In fact, it is an instantaneous code.
3. $G_N \leq \hat{H_N} < G_N + \frac{1}{N}$.
4. The efficiency rate is $e = \frac{H}{\hat{H_n}}$.

Important parameters in design od encoder/decoder

- rate efficiency
- complexity of design
- effects of error

3.3 Communication channel

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3.4 Discrete communication channel

Consider a discrete memoryless channel. Then, the channel may be described with conditional probability $p(y|x)$. The average information rate is $D_in = r_sH(X)$ and the average rate of information transmission is

$$D_t = (H(X) - H(X|Y))r_s = r_sI(X;Y)$$

The capacity of the channel is defined as $C = \max_{p(x)} D_t$.

Theorem 3.3. *Let C be the capacity and H be the entropy. If $r_sH \leq C$, then there exists an enconding scheme such that the output of the source can be transmitted over channel with an arbitrary small probability of error. Conversely, it is not possible to transmit information at a rate exceeding C without a positive frequency.*

Remark 4. with memory and Gilbert

3.5 Continuous channels

Remark 5. additive and multiplicative noise

- Modulator and demodulator are techniques to reduce guassian noise effect.
- Impulse noise are modeled in the discrete portion.

Theorem 3.4 (Shannon-Hartley theorem). *The capacity of a channel with bandwidth B and additive guassian band-limited white noise is*

$$C = B \lg\left(1 + \frac{S}{N}\right)$$

where S and N are the average signal power and noise power at the output channel. $N = \eta B$ if two sided spectral density of the noise is $\frac{\eta}{2}$.

Implications

1. Gives an upperlimit that can be reached
2. Exchange of S/N for bandwidth.
3. Bandwidth compression.
4. Noiseless channel has infinite capacity. For noisy channels, as bandwidth increases because the noise power increases as well, the capacity approaches a limit.

Communication at transmitting information rate of $B \lg(1 + S/N)$ is called *ideal*.

1. Most physical channels are approximately gaussian.
2. Guassian noise provides a lowerbound performace for all other types.

Remark 6. CRT

Chapter 4

Baseband Data Transmission