Contents

Chapter 1

Frequency Domain Analysis

1.1 Fourier Series

For a periodic signal $x(t)$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{2\pi j \frac{n}{T_0} t} \qquad x_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-2\pi j \frac{n}{T_0} t} \, \mathrm{d}t$$

for angular frequency $\omega_0 = 2\pi f_0$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} \qquad x_n = \frac{\omega_0}{2\pi} \int_{T_0} x(t) e^{-jn\omega_0 t} \, \mathrm{d}t$$

1.2 Fourier Transform

For non-periodic signals $x(t)$:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \, \mathrm{d}t \qquad x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \, \mathrm{d}f$$

$$X(f) = \mathcal{F}\{x(t)\} \qquad x(t) = \mathcal{F}^{-1}\{X(f)\}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, \mathrm{d}t \qquad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j\omega t} \, \mathrm{d}\omega$$

$X(f)$ is called the **spectrum** of $x(t)$, or the **voltage spectrum**. From the relationship between the inverse Fourier transform of Fourier transform of a signal we define

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi j f t} \, \mathrm{d}f = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \, \mathrm{d}f$$

That is, all frequencies in $\delta(t)$ are with unit magnitude and zero phase.

$$\delta(t) = \mathcal{F}^{-1}\{1\} \qquad \delta(f) = \mathcal{F}\{1\}$$

1.3 Power and Energy

Define

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t \qquad \mathcal{P}_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 \, \mathrm{d}t$$

A signal is **energy-type** if $E_x < +\infty$ and it is **power-type** if $0 < P_x < +\infty$. A signal can not be both, but it can be neither.

1.3.1 Energy-type

Let $x(t)$ be a energy-type signal. The **autocorrelation** of $x(t)$ is

$$\begin{aligned} R_x(\tau) &= x(\tau) * \overline{x(-\tau)} \\ &= \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}\,\mathrm{d}t \\ \implies \mathcal{E}_x &= R_x(0) \end{aligned}$$

By Rayleigh's property

$$\mathcal{E}_x = \int_{-\infty}^{\infty}$$

The **energy spectral density** $\mathcal{G}(f) = \mathcal{F}\{R_x(\tau)\} = |X(f)|^2$, represent energy per hertz of bandwidth.

1.3.2 Power-type

Let $x(t)$ be a power type signal. The **time average autocorrelation** function

$$\begin{aligned} R_x(\tau) &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}\,\mathrm{d}t \\ \implies \mathcal{P}_x &= R_x(0) \end{aligned}$$

The **power spectral density** $\mathcal{S}(f) = \mathcal{F}\{R_x(\tau)\}$ and

$$\mathcal{P}_x = \int_{-\infty}^{\infty} \mathcal{S}(f)\,\mathrm{d}f$$

Suppose $x(t)$ is a power-type signal passing through a filter with impluse response $h(t)$:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ R_y(\tau) &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\overline{y(t-\tau)}\,\mathrm{d}t \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\int_{-\infty}^{\infty} h(u)x(t-u)\,\mathrm{d}u\right)\left(\int_{\infty}^{\infty} \overline{h(v)x(t-\tau-v)}\,\mathrm{d}v\right)\mathrm{d}t \\ &= \lim_{T\to\infty} \int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(u)\overline{h(v)}x(t-u)\overline{x(t-\tau-v)}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)} \lim_{T\to\infty}\int_{-\frac{T}{2}+u}^{\frac{T}{2}+u} x(w)\overline{x(w+u-\tau-v)}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)}R_x(v+\tau-u)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty} (R_x(v+\tau) * h(v+\tau))\overline{h(v)}\,\mathrm{d}u\,\mathrm{d}v \\ &= R_x(\tau) * h(\tau) * \overline{h(-\tau)} \end{aligned}$$

Which implies that

$$\mathcal{S}_y(f) = \mathcal{S}_x(f)H(f)\overline{H(f)} = \mathcal{S}_x(f)|H(f)|^2$$

1.4 Sampling of bandlimited signals

$f_s = 2W$ is the **Nyquist rate** and $f_s - 2W$ is **guard band**.

1.5 Bandpass signal

Chapter 2

Information Theory

Remark 1. For a more complete and through treatment refer to the notes on the subject.

2.1 Measure of information

Information content of a message is inversely proportional to the likelihood of that message. Let $m_1, m_2, \ldots, m_q$ be q messages with probability $p_1, p_2, \ldots, p_q$ respectively, such that $p_1 + \cdots + p_q = 1$. Then, information content of m_k, $I(m_k)$ must satisfy the followings

1. $I(m_k) > I(m_j)$ if $m_k < m_j$.
2. $I(m_k) \to 0$ as $p_k \to 0$.
3. $I(m_k) \geq 0$ when $0 \leq p_k \leq 1$.

Furthermore, for two independent messages m_1 and m_2

$$I(m_1, m_2) = I(m_1) + I(m_2)$$

One continuous function that satisfies these requirements is $I(m_k) = -\log p_k$ where the base of the logarithm determines the unit of information, e.g. base e is nats, 2 is bit, 10 is Hartley/decit.

2.1.1 Average information content

For a statistically independent source that emits N symbols from a M-symbol alphabet i.i.d.

$$I_{tot} = -N \sum_{i=1}^{M} p_i \log(p_i)$$

$$H = \frac{I_{tot}}{N} = -\sum_{i=1}^{M} p_i \log(p_i)$$

Proposition 2.1. *For a source with an M-symbol alphabet, the maximum entropy is attained when the symbols are equiprobabilistic and $H_{max} = \log M$.*

Suppose r_s is the symbol rate of the source, measured in . Then, average information rate R is

$$R = r_s H$$

2.1.2 Statistically dependent source

2.1.3 Entropy for Markov source

2.2 Source encoding

Definition: The ratio of sourse information and the average encoded output bit rate is called *coding efficiency.*

2.2.1 Shannon Algorithm

Let $m_1, \dots, m_q$ be arranged in decreasing order of probability $p_1 \leq \cdots \leq p_q$. Let $F_i = \sum_{k=1}^{i-1} p_k$ with $F_1 = 0$. Let $n_i = \lceil -\lg p_i \rceil$. Then, the code

$$c_i = (F_i)_2 \qquad \text{binary fraction of } F_i \text{ up to } n_i \text{ bits.}$$

has the following properties

1. $l(c_i) > l(c_j) \implies p_i < p_j$.
2. Codewords are all differnt. In fact, it is an instantaneous code.
3. $G_N \leq \hat{H}_N < G_N + \frac{1}{N}$.
4. The efficiency rate is $e = \frac{H}{\hat{H}_n}$.

Important parameters in design od encoder/decoder

- rate efficiency
- complexity of design
- effects of error

2.3 Communication channel

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2.4 Discrete communication channel

Consider a discrete memoryless channel. Then, the channel may be described with conditional probability $p(y|x)$. The average information rate is $D_i n = r_s H(X)$ and the average rate of information transmission is

$$D_t = (H(X) - H(X|Y))r_s = r_s I(X;Y)$$

The capacity of the channel is defined as $C = \max_{p(x)} D_t$.

Theorem 2.2. *Let C be the capacity and H be the entropy. If $r_s H \leq C$, then there exists an enconding scheme such that the output of the source can be transmitted over channel with an arbitrary small probability of error. Conversely, it is not possible to transmit information at a rate exceeding C without a positive frequency.*

Remark 2. with memory and Gilbert

2.5 Continuous channels

Remark 3. additive and multiplicative noise

- Modulator and demodulator are techniques to reduce guassian noise effect.
- Impulse noise are modeled in the discrete portion.

Theorem 2.3 (Shannon-Hartley theorem). *The capacity of a channel with bandwidth B and additive guassian band-limited white noise is*

$$C = B \lg\left(1 + \frac{S}{N}\right)$$

where S and N are the average signal power and noise power at the output channel. $N = \eta B$ if two sided spectral density of the noise is $\frac{\eta}{2}$.

Implications

1. Gives an upperlimit that can be reached
2. Exchange of S/N for bandwidth.
3. Bandwidth compression.
4. Noiseless channel has infinite capacity. For noisy channels, as bandwidth increases because the noise power increases as well, the capacity approaches a limit.

Communication at transmitting information rate of $B \lg(1 + S/N)$ is called *ideal*.

1. Most physical channels are approximately gaussian.
2. Guassian noise provides a lowerbound performace for all other types.

Remark 4. CRT

Chapter 3

Baseband Data Transmission