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Chapter 1

The Fundamental Theorem of Arithmetic

induction, well-ordering principle, divisibility, gcd is commutative, associative, and distributive, relatively prime, primes, fundamental theorem of arithmetic.

1.1 The series of reciprocals of the primes

Theorem 1.1. *The infinite series $\sum \frac{1}{p_n}$ diverges.*

Proof. Suppose it does not, let $Q = p_1 \dots p_k$. Then, for all $r \geq 1$,

$$\sum_{n=1}^r \frac{1}{1+nQ} \leq \sum_{t=1}^{\infty} \left(\sum_{m=k+1}^{\infty} \frac{1}{p_m} \right)^t \quad \blacksquare$$

Euclidean algorithm, division algorithm, gcd algorithm. **Exercises**

1. $1, 2, (a, c) = 1 \implies (a, bc) = (a, b), 18, 28, 29, 30$

Chapter 2

Arithmetical Functions and Dirichlet Multiplication

Definition: A function $f : \mathbb{N} \rightarrow \mathbb{C}$ is an arithmetical function.

2.1 Mobius function

The Mobius function μ , is defined as $\mu(1) = 1$ and for $n > 1$ if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

$$\mu(n) = \begin{cases} (-1)^k & \alpha_1 = \cdots = \alpha_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.1. If $n \geq 1$,

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

2.2 The Euler totient function

The Euler totient function ϕ is defined as

$$\phi(n) = \sum_{k=1}^n ' 1 = \left| \left\{ 1 \leq k \leq n \mid (k, n) = 1 \right\} \right|$$

Theorem 2.2. If $n \geq 1$,

$$\sum_{d|n} \phi(d) = n$$

Theorem 2.3. If $n \geq 1$,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

2.2.1 The product formular for $\phi(n)$

Theorem 2.4. For any $n \geq 1$,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p} \right)$$

Corollary 2.5.

1. $\phi(p^\alpha) = (p-1)p^{\alpha-1}$.
2. $\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)}$ where $d = (m, n)$.
3. If $a \mid b$, then $\phi(a) \mid \phi(b)$.
4. $\phi(n)$ is even for $n \geq 3$. Moreover, if n has r distinct odd prime factors, then $2^r \mid \phi(n)$.

2.3 The Dirichlet product

Definition: Let f and g be two arithmetical functions, their **Dirichlet product** is defined as

$$(f * g)(n) = \sum_{d \mid n} f(d)g\left(\frac{n}{d}\right)$$

Then, we can write $\phi = \mu * N$ where $N(n) = n$.

Theorem 2.6.

1. $f * g = g * f$.
2. $(f * g) * k = f * (g * k)$.

Definition: The identity function, $I(n) = \lfloor \frac{1}{n} \rfloor$.

Theorem 2.7. For any arithmetical function f , $I * f = f * I = f$.

Theorem 2.8. If f is an arithmetical function with $f(1) \neq 0$, there is a unique arithmetical function f^{-1} , called the *Dirichlet inverse* of f such that

$$f * f^{-1} = f^{-1} * f = I$$

Moreover, f^{-1} is given by $f^{-1}(1) = \frac{1}{f(1)}$ and for $n > 1$

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} f\left(\frac{n}{d}\right) f^{-1}(d)$$

Remark 1. The set of all arithmetical functions f with $f(1) \neq 0$ is an Abelian group under Dirichlet multiplication.

Proposition 2.9. $(f * g)^{-1} = f^{-1} * g^{-1}$.

Definition: The unit function $u(n) = 1$ for all n . Since $\sum_{d \mid n} \mu(d) = I(n)$, then $\mu * u = I$ and thus by uniqueness of inverse $\mu^{-1} = u$.

Theorem 2.10 (Möbius inversion formula). If

$$f(n) = \sum_{d \mid n} g(d)$$

then,

$$g(n) = \sum_{d \mid n} f(d)\mu\left(\frac{n}{d}\right) \tag{2.1}$$

Proof. Since $f = g * u$, then $g = f * u^{-1} = f * \mu$. ■

2.4 The Mangoldt function Λ

Definition: For every integer $n \geq 1$, we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 2.11. For $n \geq 1$,

$$\log(n) = \sum_{d|n} \Lambda(n)$$

and

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) = - \sum_{d|n} \mu(d) \log(d)$$

2.5 Multiplicative functions

Definition: An arithmetical function f is **multiplicative** if $f \not\equiv 0$ and

$$f(mn) = f(m)f(n)$$

whenever $(m, n) = 1$. The function f is said to be **completely multiplicative** if for all m, n

$$f(mn) = f(m)f(n)$$

Remark 2. Multiplicative functions for a subgroup under $*$.

Proposition 2.12. If f is multiplicative, then $f(1) = 1$.

Theorem 2.13. Given an arithmetical function f with $f(1) = 1$

1. f is multiplicative if and only if $f(\prod p_i^{\alpha_i}) = \prod f(p_i^{\alpha_i})$
2. If f is multiplicative, then f is completely multiplicative if $f(p^\alpha) = (f(p))^\alpha$.

Theorem 2.14. If f and g are both multiplicative, then $f * g$ is multiplicative. If g and $f * g$ are both multiplicative, then f is multiplicative.

2.5.1 Inverse of completely multiplicative functions

Theorem 2.15. Let f be a multiplicative function. Then, f is completely multiplicative if and only if

$$f^{-1}(n) = \mu(n)f(n)$$

Remark 3. Note that $N = \phi * u$ and $\phi = N * \mu$ therefore, $\phi^{-1} = \mu^{-1} * N^{-1} = u * N^{-1}$. Since N is completely multiplicative, $\phi^{-1} = u * \mu N$. That is,

$$\phi^{-1}(n) = \sum_{d|n} d\mu(d)$$

Theorem 2.16. If f is multiplicative,

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$$

2.6 Liouville's function λ

Definition: The Liouville function λ is defined as $\lambda(1) = 1$ and if $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then

$$\lambda(n) = (-1)^{\alpha_1 + \dots + \alpha_k}$$

and also $\lambda^{-1}(n) = |\mu(n)|$.

2.7 The divisor function σ_α

Definition: For all $\alpha \in \mathbb{C}$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha = u \times N^\alpha$

Proposition 2.17. *The divisor function σ_α is multiplicative. Therefore,*

$$\sigma_\alpha(p^k) = 1 + p^\alpha + \dots + p^{k\alpha} = \begin{cases} \frac{p^{(k+1)\alpha} - 1}{p^\alpha - 1} & \alpha \neq 0 \\ k + 1 & \alpha = 0 \end{cases}$$

Theorem 2.18. *For $n \geq 1$, we have*

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right)$$