# Contents

# Chapter 1

# The Fundamental Theorem of Arithmetic

induction, well-ordering principle, divisibility, gcd is commutative,associative, and distributive, relatively prime, primes, fundamental theorem of arithmetic.

## 1.1 The series of reciprocals of the primes

**Theorem 1.1.** *The infinite series* $\sum \frac{1}{p_n}$ *diverges.*

*Proof.* Suppose the sum converges instead and let $k$ be such that

$$\sum_{n=k+1}^{\infty} \frac{1}{p_n} \leq \frac{1}{2}$$

Let $Q = p_1 \dots p_k$, then for all $r \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{r} \frac{1}{1+nQ} &\leq \sum_{t=1}^{\infty} \left( \sum_{m=k+1}^{\infty} \frac{1}{p_m} \right)^t \\
&\leq \sum_{t=1}^{\infty} \left( \frac{1}{2} \right)^t \\
&= 1
\end{aligned}$$

By allowing $r \to \infty$, we get

$$\sum_{n=1}^{\infty} \frac{1}{1+nQ} \leq 1$$

However, this is a constradiction as the sum diverges as

$$\sum_{n=1}^{\infty} \frac{1}{1+nQ} \leq \sum_{n=1}^{\infty} \frac{1}{Q+nQ} \leq \frac{1}{Q} \sum_{n=2}^{\infty} \frac{1}{n}$$

Therefore, $\sum \frac{1}{p_n}$ must diverge. ■

Euclidean algorithm, division algorithm, gcd algorithm.

## Exercises

1. If $(a,b)=1$ and if $c \mid a$ and $d \mid b$, then $(c,d)=1$.

   *Solution.* Let $e=(c,d)$, since $e \mid c$, then $e \mid a$ and similarly, $e \mid b$. Therefore, $e \mid (a,b)$ which means $e=1$. ▷

2. If $(a,b)=(a,c)=1$, then $(a,bc)=1$.

   *Solution.* Let $d=(a,bc)$ and $e=(b,d)$. Then, $e \mid d$ and hence $e \mid a$, as a result $e \mid (a,b)$ which means $e=1$. Note that, $d \mid bc$ but $(b,d)=1$ thus, $d \mid c$. Since $d \mid a$, then $d \mid (a,c)$ and hence $d=1$. ▷

3. If $(a,c)=1$, then $(a,bc)=(a,b)$.

   *Solution.* Let $d=(a,bc)$ and $e=(c,d)$. Then, $e \mid d$ and hence $e \mid a$, as a result $e \mid (a,c)$ which means $e=1$. Note that, $d \mid bc$ but $(c,d)=1$ thus, $d \mid b$. Since $d \mid a$, then $d \mid (a,b)$. Moreover, $(a,b) \mid d$ since $(a,b) \mid a$ and $(a,b) \mid bc$. Therefore, $d=(a,b)$. ▷

4. If $m \neq n$ compute the $\gcd\big(a^{2^m}+1, a^{2^n}+1\big)$ in terms of $a$.

   *Solution.* WLOG assume $n<m$ and note that
$$a^{2^m}-1=a^{2^{m-n}\cdot 2^n}-1=\big(a^{2^n}-1\big)\big(a^{2^n}+1\big)\big(a^{2\cdot 2^n}+1\big)\dots\Big(a^{2^{m-n-1}\cdot 2^n}+1\Big)$$
   and hence
$$a^{2^n}+1 \mid a^{2^m}-1$$
   Therfore,
$$\big(a^{2^n}+1, a^{2^m}+1\big)=\big(2, a^{2^n}+1\big)=\begin{cases}1 & a \text{ is even}\\ 2 & a \text{ is odd}\end{cases}$$
   ▷

5. If $a>1$, then $(a^m-1, a^n-1)=a^{(m,n)}-1$.

   *Solution.* If $m=n$, then the result hold obviously. Suppose $n<m$ and note that
$$a^m-1=(a^{m-n})(a^n-1)+(a^{m-n}-1)$$
   and therefore, $(a^m-1,a^n-1)=(a^{m-n}-1,a^n)$. By applying the Euclidean algorithm we arrive at the conclusion. ▷

6. Given $n>0$, let $S$ be a set whose elements are positive integers $\leq 2n$ such that if $a$ and $b$ are in $S$ and $a \neq b$, then $a \nmid b$. What is the maximum number of integers that $S$ can contain?

   *Solution.* Note that $S$ can not have more than $n$ elements. To see this, consider the sets $\big\{m2^k \,\big|\, k \geq 0,\ m2^k \leq 2n\big\}$ for $m=1,3,\dots,2n-1$. There are $n-1$ such sets and they partition the set $\{1,2,\dots,2n\}$. No two elements of $S$ can come from the same set, and as a result $|S| \leq n-1$ by pigeonhole principle. However, note that $S=\{n+1,n+2,\dots,2n\}$ satisfies the conditions and has exactly $n-1$ elements. Therefore, the maximum of $n-1$ elements is attainable for all $n>0$. ▷

7. If $n > 1$ prove that the sum $\sum_{k=1}^{n} \frac{1}{k}$ is not an integer. Also show that for any signing of the sum $\sum_{k=1}^{n} (-1)^{a_k} \frac{1}{k}$ is not an integer.

   *Solution.* Let $p$ be the largest prime less than or equal to $n$. Let $r, s \in \mathbb{Z}$ be such that $s \neq 0$ and $(r, s) = 1$.

$$\frac{r}{s} = \sum_{\substack{k=1 \\ k\neq p}}^{n} (-1)^{a_k} \frac{1}{k}$$

   We claims that $p \nmid s$. For the sake of contradiction suppose there is an integer $q$ such that $s = pq$. Then,

$$\begin{aligned} r &= s\left(\sum_{\substack{k=1 \\ k\neq p}}^{n} (-1)^{a_k} \frac{1}{k}\right) \\ &= \sum_{\substack{k=1 \\ k\neq p}}^{n} (-1)^{a_k} \frac{pq}{k} \end{aligned}$$

   Since $(p, k) = 1$ for all $k \leq n$ and $k \neq p$, then it must be the case that the sum

$$\sum_{\substack{k=1 \\ k\neq p}}^{n} (-1)^{a_k} \frac{q}{k}$$

   is an integer. Therefore, we have shown that there is integer $t$ such that $r = pt$, which contradicts our assumption that $(r, s) = 1$. Thus, $p$ does not divide $s$. To conclude, consider the sum

$$\frac{r}{s} + \frac{(-1)^{a_p}}{p} = \frac{pr + (-1)^{a_p} s}{ps}$$

   which can not be integer as $p \nmid s$. ▷

# Chapter 2

# Arithmetical Functions and Dirichlet Multiplication

**Definition:** A function $f : \mathbb{N} \to \mathbb{C}$ is an arithmetical function.

## 2.1 Mobius function

The Mobius function $\mu$, is defined as $\mu(1) = 1$ and for $n > 1$ if $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$

$$\mu(n) = \begin{cases} (-1)^k & \alpha_1 = \cdots = \alpha_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 2.1.** *If* $n \geq 1$,

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

## 2.2 The Euler totient function

The Euler totient function $\phi$ is defined as

$$\phi(n) = {\sum_{k=1}^{n}}' 1 = \left| \left\{ 1 \leq k \leq n \,\middle|\, (k, n) = 1 \right\} \right|$$

**Theorem 2.2.** *If* $n \geq 1$,

$$\sum_{d|n} \phi(d) = n$$

**Theorem 2.3.** *If* $n \geq 1$,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

### 2.2.1 The product formular for $\phi(n)$

**Theorem 2.4.** *For any* $n \geq 1$,

$$\phi(n) = n \prod_{p|n} \left( 1 - \frac{1}{p} \right)$$

**Corollary 2.5.**

1. *$\phi(p^\alpha) = (p-1)p^{\alpha-1}$.*
2. *$\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)}$ where $d = (m,n)$.*
3. *If $a \mid b$, then $\phi(a) \mid \phi(b)$.*
4. *$\phi(n)$ is even for $n \geq 3$. Moreover, if $n$ has $r$ distinct odd prime factos, then $2^r \mid \phi(n)$.*

## 2.3 The Dirichlet product

**Definition:** Let $f$ and $g$ be two arithmetical functions, their **Dirichlet product** is defined as

$$(f * g)(n) = \sum_{d \mid n} f(d) g\left(\frac{n}{d}\right)$$

Then, we can write $\phi = \mu * N$ where $N(n) = n$.

**Theorem 2.6.**

1. *$f * g = g * f$.*
2. *$(f * g) * k = f * (g * k)$.*

**Definition:** The identity function, $I(n) = \left\lfloor \frac{1}{n} \right\rfloor$.

**Theorem 2.7.** *For any arithmetical function $f$, $I * f = f * I = f$.*

**Theorem 2.8.** *If $f$ is an arithmetical function with $f(1) \neq 0$, there is a unique arithmetical function $f^{-1}$, called the Dirichlet inverse of $f$ such that*

$$f * f^{-1} = f^{-1} * f = I$$

*Moreover, $f^{-1}$ is given by $f^{-1}(1) = \frac{1}{f(1)}$ and for $n > 1$*

$$f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d \mid n \\ d < n}} f\left(\frac{n}{d}\right) f^{-1}(d)$$

**Remark 1.** The set of all arithmetical functions $f$ with $f(1) \neq 0$ is an Abelian group under Dirichlet multiplication.

**Proposition 2.9.** $(f * g)^{-1} = f^{-1} * g^{-1}$.

**Definition:** The unit function $u(n) = 1$ for all $n$. Since $\sum_{d \mid n} \mu(d) = I(n)$, then $\mu * u = I$ and thus by uniqueness of inverse $\mu^{-1} = u$.

**Theorem 2.10 (Mobius inversion formula).** *If*

$$f(n) = \sum_{d \mid n} g(n)$$

*then,*

$$g(n) = \sum_{d \mid n} f(d) \mu\left(\frac{n}{d}\right) \tag{2.1}$$

*Proof.* Since $f = g * u$, then $g = f * u^{-1} = f * \mu$. ∎

## 2.4 The Mangoldt function $\Lambda$

**Definition:** For every integer $n \geq 1$, we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and } m \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 2.11.** *For $n \geq 1$,*

$$\log(n) = \sum_{d|n} \Lambda(n)$$

*and*

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) = -\sum_{d|n} \mu(d) \log(d)$$

## 2.5 Multiplicative functions

**Definition:** An arithmetical function $f$ is **multiplicative** if $f \not\equiv 0$ and

$$f(mn) = f(m)f(n)$$

whenver $(m,n) = 1$. The function $f$ is said to be **completely multiplicative** if for all $m, n$

$$f(mn) = f(m)f(n)$$

**Remark 2.** Multiplicative functions for a subgroup under $*$.

**Proposition 2.12.** *If $f$ is multiplicative, then $f(1) = 1$.*

**Theorem 2.13.** *Given an arithmetical function $f$ with $f(1) = 1$*

1. *$f$ is multiplicative if and only if $f(\prod p_i^{\alpha_i}) = \prod f(p_i^{\alpha_i})$*
2. *If $f$ is multiplicative, then $f$ is completely multiplicative if $f(p^\alpha) = (f(p))^\alpha$.*

**Theorem 2.14.** *If $f$ and $g$ are both multiplicative, then $f * g$ is multiplicative. If $g$ and $f * g$ are both multiplicative, then $f$ is multiplicative.*

### 2.5.1 Inverse of completely multiplicative functions

**Theorem 2.15.** *Let $f$ be a multiplicative function. Then, $f$ is completely multiplicative if and only if*

$$f^{-1}(n) = \mu(n)f(n)$$

**Remark 3.** Note that $N = \phi * u$ and $\phi = N * \mu$ therefore, $\phi^{-1} = \mu^{-1} * N^{-1} = u * N^{-1}$. Since $N$ is completely multiplicative, $\phi^{-1} = u * \mu N$. That is,

$$\phi^{-1}(n) = \sum_{d|n} d\mu(d)$$

**Theorem 2.16.** *If $f$ is multiplicative,*

$$\sum_{d|n} \mu(d)f(d) = \prod_{p|n} (1 - f(p))$$

## 2.6 Liouville's function $\lambda$

**Definition:** The Liouville function $\lambda$ is defined as $\lambda(1) = 1$ and if $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, then

$$\lambda(n) = (-1)^{\alpha_1 + \cdots + \alpha_k}$$

and also $\lambda^{-1}(n) = |\mu(n)|$.

## 2.7 The divisor function $\sigma_\alpha$

**Definition:** For all $\alpha \in \mathbb{C}$, $\sigma_\alpha(n) = \sum_{d|n} d^\alpha = u \times N^\alpha$

**Proposition 2.17.** *The divisor function $\sigma_\alpha$ is multiplicative. Therefore,*

$$\sigma_\alpha\left(p^k\right) = 1 + p^\alpha + \cdots + p^{k\alpha} = \begin{cases} \dfrac{p^{(k+1)\alpha} - 1}{p^\alpha - 1} & \alpha \neq 0 \\ k + 1 & \alpha = 0 \end{cases}$$

**Theorem 2.18.** *For $n \geq 1$, we have*

$$\sigma_\alpha^{-1}(n) = \sum_{d|n} d^\alpha \mu(d) \mu\left(\frac{n}{d}\right)$$

## 2.8 Generalized convolution

Let $F : ]0, \infty[ \to \mathbb{C}$ such that $F(x) = 0$ for $0 < x < 1$. Let $f$ be an arithmetical function

$$f \circ F(x) = \sum_{n \leq x} f(n) F\left(\frac{x}{n}\right)$$

is a function such that $f \circ F(x) = 0$ for $0 < x < 1$ and defined on $]0, \infty[$.

**Remark 4.** In general, $\circ$ is not commutative nor associative.

**Theorem 2.19.** *Let $f$ and $g$ be two arithmetical functions*

$$f \circ (g \circ F) = (f * g) \circ F$$

**Theorem 2.20 (Inverse formula).** *Let $f$ have inverse $f^{-1}$, then the equation*

$$G(x) = \sum_{n \leq x} f(x) F\left(\frac{x}{n}\right)$$

*implies*

$$F(x) = \sum_{n \leq x} f^{-1}(x) G\left(\frac{x}{n}\right)$$

**Theorem 2.21 (Generalized Mobius inversion).** *Let $f$ be a completely multiplicative function*

$$G(x) = \sum_{n \leq x} f(n) F\left(\frac{x}{n}\right) \iff F(x) = \sum_{n \leq x} \mu(n) f(n) G\left(\frac{x}{n}\right)$$

## 2.9 Formal power series

Definiton of formal power series as usual with equality, sum, and multiplication. Therefore, formal power series form a ring with 0 and 1. If the leading coefficient is non-zero, then the formal power series is invertible.

**Definition:** Let $f$ be an arithmetical function and $p$ be a prime

$$f_p(x) = \sum_{n=0}^{\infty} f(p^n)x^n$$

is the **Bell series of $f$ modulo** $p$.

**Theorem 2.22.** *If $f$ and $g$ are multiplicative, then $f = g$ if and only if $f_p = g_p$ for all $p$.*

**Example 2.1.**

$$\mu_p(x) = 1 - x \qquad I_p(x) = 1 \qquad \lambda_p(x) = \frac{1}{1+x}$$

$$\phi_p(x) = \frac{1-x}{1-px} \qquad u_p(x) = \frac{1}{1-x} \qquad N_p^\alpha(x) = \frac{1}{1-p^\alpha x}$$

**Theorem 2.23.** *Let $f$ and $g$ be two arithmetical functions and $h = f * g$, then $h_p = f_p g_p$ for all $p$.*

As a result,

$$(\sigma_\alpha)_p(x) = N_p^\alpha(x)u_p(x) = \frac{1}{1-p^\alpha x}\frac{1}{1-x} = \frac{1}{1-(p^\alpha+1)x+p^\alpha x^2} = \frac{1}{1-\sigma_\alpha(p)+p^\alpha x^2}$$

**Definition:** The derivative arithmetical function $f$ is defined by

$$f'(n) = f(n)\log(n)$$

**Theorem 2.24.**

1. $(f+g)' = f' + g'$.
2. $(f*g)' = f' * g + f * g'$.
3. $(f^{-1})' = -f' * (f*f)^{-1}$ *provided that* $f(1) \neq 0$.

## 2.10 The Selberg theorem

**Theorem 2.25.** *For $n \geq 1$,*

$$\Lambda(n)\log(n) + \sum_{d|n} \Lambda(d)\Lambda\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d)\log^2\left(\frac{n}{d}\right)$$

# Chapter 3

# Averages of Arithmetical Functions

Arithmetical functions fluctuate a lot, by taking averages we can determine their behaviour

$$\tilde{f}(n) = \frac{1}{n}\sum_{k=1}^{n} f(k)$$

## 3.1 Asymptotic equality of function

$f(x) \in O(g(x))$ if there exists $M > 0$ and $a$ such that for all $x \geq a$, $|f(x)| \leq M|g(x)|$. Usually, $g$ is taken to be positive.

**Definition:** If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, then $f$ is asymptotic to $g$ as $x \to \infty$ and we write $f(x) \sim g(x)$ as $x \to \infty$.

## 3.2 Euler's summation formula

**Theorem 3.1.** *If $f$ has a continuous derivative $f'$ on the interval $[y, x]$, where $0 < y < x$, then*

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)\,\mathrm{d}t + \int_y^x (t - \lfloor t \rfloor) f'(t)\,\mathrm{d}t \\ + f(x)(\lfloor x \rfloor - x) - f(y)(\lfloor y \rfloor - y)$$

## 3.3 Some elementary asymptotic formula

**Definition:** The Euler-Mascheroni constant is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right)$$

**Definition:** The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

where $s \in \mathbb{C}$ is a complex variable.

**Theorem 3.2.** *If $x \geq 1$ we have*

$$\sum_{n \leq x} \frac{1}{n} = \log n + \gamma + O\left(\frac{1}{x}\right) \tag{3.1}$$

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O\big(x^{-s}\big) \qquad s > 0 \wedge s \neq 1 \tag{3.2}$$

$$\sum_{n > x} \frac{1}{n^s} = O\big(x^{1-s}\big) \qquad s > 1 \tag{3.3}$$

$$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha}) \qquad \alpha \geq 0 \tag{3.4}$$

## 3.4 The average order of $d(n)$

**Theorem 3.3.** *For all $x \geq 1$,*

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O\big(\sqrt{x}\big)$$

*The error term can be improved to $O\big(x^{12/37+\epsilon}\big)$ for all $\epsilon > 0$.*

## 3.5 The average order of $\sigma_\alpha(n)$

**Theorem 3.4.** *For all $x \geq 1$*

$$\sum_{n \leq x} \sigma_1(x) = \frac{1}{2}\zeta(2)x^2 + O(x \log x)$$
$$\sum_{n \leq x} \sigma_{-1}(x) = \zeta(2)x + O(\log x)$$

*If $\alpha > 0$ and $\alpha \neq 1$, then*

$$\sum_{n \leq x} \sigma_\alpha(x) = \frac{1}{\alpha+1}\zeta(\alpha+1)x^{\alpha+1} + O\big(x^{\beta}\big)$$
$$\sum_{n \leq x} \sigma_{-\alpha}(x) = \zeta(\alpha+1)x + O\big(x^{\delta}\big)$$

*where $\beta = \max\{1, \alpha\}$ and $\delta = \max\{0, 1-\alpha\}$.*

## 3.6 The average order $\phi(n)$

**Theorem 3.5.** *For $x > 1$ we have*

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2}x^2 + O(x \log x)$$

## 3.7 An application

**Definition:** Two lattice point $P$ and $Q$ are mutually visible if the line segment connecting them contains no other lattice point.

**Theorem 3.6.** *Two lattice point $(a, b)$ and $(c, d)$ are mutually visible if and only if $(a-c, b-d) = 1$.*

Consider the square $C(r) = \{(x, y) \mid |x|, |y| \leq r\}$, let $N(r) = \#C(r)$ and let $N'(r)$ be the number of visible points from the origin in $C(r)$.

**Theorem 3.7.** *The set of lattice points visible from the origin has density $\frac{6}{\pi^2}$. That is,*

$$\lim_{n\to\infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}$$

## 3.8 The average order of $\mu(n)$ and $\Lambda(n)$

**Theorem 3.8.** *We have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \mu(n) = 0$$
$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \Lambda(n) = 1$$

*Both are equivalent to prime number theorem.*

## 3.9 The partial sums of Dirichlet product

**Theorem 3.9.** *If $h = f * g$, let*

$$H(x) = \sum_{n\leq x} h(n) \qquad F(x) = \sum_{n\leq x} f(n) \qquad G(x) = \sum_{n\leq x} g(n)$$

*then we have*

$$H(x) = \sum_{n\leq x} f(n) G\left(\frac{x}{n}\right) = \sum_{n\leq x} g(n) F\left(\frac{x}{n}\right)$$

**Theorem 3.10.** *If $F(x) = \sum_{n\leq x} f(n)$ we have*

$$\sum_{n\leq x} \sum_{d\mid n} f(d) = \sum_{n\leq x} f(x) \left\lfloor \frac{x}{n} \right\rfloor = \sum_{n\leq x} F\left(\frac{x}{n}\right)$$

## 3.10 Applications to $\mu(n)$ and $\Lambda(n)$

**Theorem 3.11.** *For $x \geq 1$ we have*

$$\sum_{n\leq x} \mu(x)\Big(\frac{x}{n}\Big) = 1$$
$$\sum_{n\leq x} \Lambda(x)\Big(\frac{x}{n}\Big) = \log(\lfloor x \rfloor !)$$

**Theorem 3.12.** *For all $x \geq 1$ we have*

$$\left|\sum_{n\leq x} \frac{\mu(n)}{n}\right| \leq 1$$

*with equality hodling if $x < 2$.*

**Theorem 3.13 (Legendre's Identity).** *For all $x \geq 1$*

$$\lfloor x \rfloor ! = \prod_{p\leq x} p^{\alpha(p)}$$

*where $\alpha(p) = \sum_{m=1}^{\infty} \left\lfloor \frac{x}{p^m} \right\rfloor$.*

**Theorem 3.14.** *If $x \geq 2$*

$$\log(\lfloor x \rfloor !) = x \log x - x + O(\log x)$$

*and hence*

$$\sum_{n\leq x} \Lambda(n) \lfloor (x) n \rfloor = x \log x - x + O(\log x)$$

**Theorem 3.15.** *For $x \geq 2$*

$$\sum_{p\leq x} \lfloor (x) p \rfloor \log p = x \log x + O(x)$$

## 3.11 Another Identity for the partial sums of a Dirichlet product

**Theorem 3.16.** *If $h = f * g$, let*

$$H(x) = \sum_{n\leq x} h(n) \qquad F(x) = \sum_{n\leq x} f(n) \qquad G(x) = \sum_{n\leq x} g(n)$$

*then we have*

$$H(x) = \sum_{n\leq x} \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big) = \sum_{qd\leq x} f(d) g(q)$$

**Theorem 3.17.** *If $a, b$ are positive real numbers such that $ab = x$, then*

$$\sum_{qd\leq x} f(d)g(q) = \sum_{n\leq a} f(n) G\Big(\frac{x}{n}\Big) + \sum_{n\leq b} g(x) G\Big(\frac{x}{n}\Big) - F(a)G(b)$$