Contents

Chapter 1

Real Numbers

1.1 Axiomatic Formulation of Real Numbers

The building axioms of real numbers is devided into three groups based on the properties they are describing.

1. Field axioms.
2. Order axioms.
3. Completeness axiom.

1.1.1 Field Axioms

A field is a non-empty set $\mathbb{F}$ with two binary operations *addition*, $+$, and *multiplication*, $\cdot$. For all $x, y, z \in \mathbb{F}$:

Axiom 1. Addition and multiplication are commutitive.

$$x + y = y + x, \quad x \cdot y = y \cdot x$$

Axiom 2. Addition and multiplication are associative.

$$x + (y + z) = (x + y) + z, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Axiom 3. Multiplication distributes over addition.

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

Axiom 4. There exists a number 0 such that for every number x:

$$x + 0 = 0 + x = x$$

Axiom 5. There exists a number 1 such that for every number x:

$$x \cdot 1 = 1 \cdot x = x$$

Axiom 6. For every number x, there exists a number y such that:

$$x + y = 0$$

y is called the negative of x and is denoted by $-x$.

Axiom 7. For every number $x \neq 0$, there exists a number y such that:

$$x \cdot y = 1$$

y is called the reciprocal of x and is denoted by x^{-1} or $\dfrac{1}{x}$.

1.1.2 Order Axioms

The order axioms establishes an ordering on the numbers of $\mathbb{F}$ to determine which element is larger or smaller. To achieve an ordering, we define the set of positive real numbers $\mathbb{F}^+ \subset \mathbb{F}$.

Axiom 8. The $\mathbb{F}^+$ is closed under addition and multiplication.

$$\forall x, y \in \mathbb{F}^+, \quad (x + y) \in \mathbb{F}^+ \text{ and } (x \cdot y) \in \mathbb{F}^+$$

Axiom 9. $0 \notin \mathbb{F}^+$.

Axiom 10. For every number $x \neq 0$, either $x \in \mathbb{F}^+$ or $-x \in \mathbb{F}^+$.

We then define the binary operator $>$ such that $x > y$ whenever $(x - y) \in \mathbb{F}^+$.

1.1.3 Completeness Axiom

Given that $(\mathbb{F}, +, \cdot, >)$ is an ordered field, we define the followings:

Definition (Upper bound): A set $S \in \mathbb{F}$ has an upper bound if for some $a \in \mathbb{F}$ is greater or equal to all element of S. That is, $\forall x \in S\ a \geq x$. We say that S is bounded from above.

Definition (The least upper bound): $a \in \mathbb{F}$ is the least upper bound of a set $S \subset \mathbb{F}$ if it is smaller than every upper bound of S. We say a is the supremum of S, denoted by $a = \sup S$.

Note that, if the least upper bound exists, it must be unique.

Axiom 11. If S is a non-empty set that bounded from above that it has supremum.

Theorem 1.1. *There exists a unique set that satisifies all the axioms above. It is denoted by* $\mathbb{R}$*, the set of real numbers.*

Proof. The existence of $\mathbb{R}$ is proved in many ways. One way to construct real numbers uses *Dedekind Cuts*. Let the pair of rational sets (A, B) be a partition of $\mathbb{Q}$ such that:

1. $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
2. $\forall x, y \in \mathbb{Q}$ s.t. $x < y,\ y \in A \implies x \in A$.
3. $\nexists x \in A$ s.t. $\forall y \in A, x \geq y$.

For convenience we let A represent the pair (A, B) as A completely determines B. We define $+$, $\cdot$, and $>$ as follows:

$$\begin{aligned} A + B &= \{a + b \mid a \in A, b \in B\} \\ \mathit{0} &= \{a \mid a < 0\} \\ -A &= \{a' \mid \forall a \in A, a' < -a\} \end{aligned}$$

For $\cdot$, we first take two set A and B that have some positive elements.

$$A \cdot B = \{a \cdot b \mid a \in A \wedge a \leq 0, b \in B \wedge b \leq 0\} \;\cup\; \mathit{0}$$

If A or B did not have any positive elements, we first take the negative of the set, and then multiply the two sets and take the negative of the product. Similarly, we define the reciprocal of A if A has a positive element.

$$\begin{aligned} \mathit{1} &= \{a \mid a < 1\} \\ A^{-1} &= \left\{ a' \,\middle|\, \forall a \in A, a > 0, a' < \frac{1}{a} \right\} \end{aligned}$$

Lastly:

$$A > B \text{ if } A \supset B$$

Also, if a non-empty set S of real numbers is bounded from above, then it has a supremum in $\mathbb{R}$ equal to $\bigcup S$. It is left to the reader that the $(\mathbb{R}, +, \cdot, >)$ satisifies the axioms above.

The set of real numbers is unique in sense that if $(\mathbb{R}, +, \cdot, >)$ and $(\mathbb{R}', +', \cdot', >')$ both satisify the axioms, then there exists bijective mapping $\alpha : \mathbb{R} \to \mathbb{R}'$ such that:

$$\begin{aligned} \alpha(x + y) &= \alpha(x) +' \alpha(y) \\ \alpha(0) &= 0' \\ \alpha(x \cdot y) &= \alpha(x) \cdot' \alpha(y) \\ \alpha(1) &= 1' \\ x < y &\iff \alpha(x) <' \alpha(y) \end{aligned}$$

Lastly, if S is a non-empty set in $\mathbb{R}$ and $\alpha(S) = \{\alpha(x) \mid x \in S\}$, then S is has an upper bound if and only if $\alpha(S)$ has an upper bound. Furthermore $\alpha(\sup S) = \sup \alpha(S)$. ■

Results 1.

1. The set of natural numbers $\mathbb{N}$ in $\mathbb{R}$ is not bounded from above.
2. Let $x \in \mathbb{R}$ be such that for all $n \in \mathbb{N}$
$$0 \leq x \leq \frac{1}{n}$$
then $x = 0$.
3. (Archimedean Property) For all $a, b > 0$ there exists $n \in \mathbb{N}$:
$$na > b$$
4. Consider $I_n = [a_n, b_n]$ $\forall n \in \mathbb{N}$ such that $I_1 \supset I_2 \supset \dots$. Then $\cap I_n$ is not empty. Moreover, if for each $e > 0$ there exists n such that $b_n - a_n < e$, then $\cap I_n$ is a single point.
5. $\sqrt{2} \in \mathbb{R}$. In addition, for all $p > 0$, there is a positive real number q such that $q^2 = p$

Exercises

1. Prove that the addition and multiplication identity elements are unique.

2. Show that the Trichotomy law holds for $(>)$. That is, exactly one of the following three is true.
$$x > y \qquad x = y \qquad y > x$$

3. Show that $1 \in \mathbb{F}^+$.

4. Show that if $x > -1$ and $n \in \mathbb{N}$:
$$(1+x)^n \geq 1 + nx$$
and equality only holds when $n = 1$.

5. Let $F_p = \{0, 1, \ldots, p-1\}$ where p is a prime number. Define $+$ and $\cdot$ to be the modular addition and product modulus p, respectively. Investigate whether if F_p can be ordered.

6. Consider the set of all rational polynomials $\mathbb{Q}[x]$:
$$\mathbb{Q}[x] = \left\{ \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0} \,\middle|\, a_i, b_j \in \mathbb{Q}, b_n \neq 0 \right\}$$
Show that $\mathbb{Q}[x]$ under the normal addition and multiplication is a field. Furthermore, show that $\mathbb{Q}^+[x] = \{q \in \mathbb{Q}[x] \mid a_m \cdot b_n > 0\}$ constitutes an ordering on $\mathbb{Q}[x]$.

Chapter 2

Topology and Metric Spaces

2.1 Topology

Let X be a set. A **topology** on X is a collection $\mathscr{T}$ of subsets of X called **open set** having the following properties

1. If $U_\alpha \in \mathscr{T}$ where $\alpha \in A$ for any set A then, $\cup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

2. If $U_\alpha \in \mathscr{T}$ where $\alpha \in A$ for any finite set A then, $\cap_{\alpha \in A} U_\alpha \in \mathscr{T}$.

3. $X, \emptyset \in \mathscr{T}$.

A topological space is a pair $(X, \mathscr{T})$, where $\mathscr{T}$ is a topology on X.

Example 2.1. On any set X we can define two topologies. The *trivial topology* on X consists of $\{\emptyset, X\}$. The *discrete topology* on X is $\mathscr{P}(X)$.

If $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies on X, we say that $\mathscr{T}_1$ is *weaker/coarser* than $\mathscr{T}_2$, or that $\mathscr{T}_2$ is *stronger/finer* than $\mathscr{T}_1$, if $\mathscr{T}_1 \subset \mathscr{T}_2$.

Definition: Let X be a topological space and $x \in X$, N is called a **neighbourhood** of x if there exists an open set G such that $x \in G \subset N$. We say that x is an interior point of N if N is neighbourhood of x.

Proposition 2.1. *A subset U is open if and only if U is a neighbourhood for all $x \in U$.*

Proof. Let U be a neighbourhood for all of its points. That is, for every $x \in U$ there is an open set G_x such that $x \in G_x \subset U$. Then, $\cup_x G_x \subset U$ however, $U \subset \cup_x G_x$. Therefore, $U = \cup_x G_x$ and by the first axiom U is an open set. If U is an open set, then U is a neighbourhood for each point $x \in U$. ∎

Definition: A subset F of X is called **closed** if F^c is open. **Closure** of a set E is the intersection of all closed sets that include E and it is denoted by $\operatorname{cl} E$ or $\bar{E}$.

Proposition 2.2. *Let X be a topological space and E a subset of X. Then, $\operatorname{cl} E$ is closed.*

Proof. We know that $\operatorname{cl} E = \cap_{E \subset F} F$ where F are closed set. Then, $(\operatorname{cl} E)^c = \cup_{E \subset F} F^c$ which is an open set and hence $\operatorname{cl} E$ is closed. ∎

Proposition 2.3. *If E is subset of a topological space X and $xinX$, then $x \in \operatorname{cl} E$ if and only if $U \cap E = \emptyset$ for every open neighbourhood of U of x.*

Proof. Suppose there exists an open neighbourhood of x, U such that, $U \cap E = \emptyset$. Then, $F = \operatorname{cl} E \cap U^c$ is a closed that constains E hence, $\operatorname{cl} E \subset F$ and $x \notin \operatorname{cl} E$. If $x \notin \operatorname{cl} E$, then $(\operatorname{cl} E)^c$ is an open neighbourhood of x which does not meet E. ■

Definition: A **limit point** of E is a point $x \in X$ such that $E \cap U \backslash \{x\} \neq \emptyset$ for every neighbourhood U of x. The set of all limit points of E is denoted by E' or $\lim E$. If $x \in E$ but $x \notin \operatorname{cl} E$, then x is called an **isolated point** of E.

Definition: A subset E of a topological space X is **perfect** if every point of E is a limit point.

Definition: Let E be subset of a topological space X then, the **interior** of E is the union of all open sets that are contained in E, denoted by E° or $\operatorname{int} E$.

Proposition 2.4. *For any set E in topological space X, interior of E is the set of all interior points of E and $\operatorname{cl} E = (\operatorname{int} E)^c$.*

Definition: The **boundary** of E is defined as $\operatorname{cl} E \backslash \operatorname{int} E$ and it is denoted by $\operatorname{bdry} E$ or ∂E.

Proposition 2.5. *Let E be subset of a topological space X. E is closed if and only if $E = \operatorname{cl} E$ and E is open if and only if $E = \operatorname{int}$. Furthermore, $\operatorname{bdry} E = \emptyset$ if and only if E is both open and closed.*

Proof. Note that $E \subset \operatorname{cl} E$ and when E is closed, $\operatorname{cl} E \subset E$ therefore, $E = \operatorname{cl} E$. ■

Definition: A set D in a topological space X is called **dense** when, $\operatorname{cl} D = X$. More generally, D is dense in subset E, if $E \subset \operatorname{cl} D$. A topological space in which there exists a countable dense set is called **separable**.

Proposition 2.6. *D is dense in E if and only if for all $x \in E$, $D \cap U \neq \emptyset$ for any open neighbourhood U of x.*

Proof. If there exists $x \in E$ such that an open neighbourhood U of x does not intersect D, then $x \notin \operatorname{cl} D$ and hence $D \not\supset E$. If for each $x \in E$ every neighbourhood U of x intersects D, then $x \in \operatorname{cl} D$ hence $E \subset \operatorname{cl} D$. ■

Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. Then, $(Y, \mathscr{T}_y)$ is a **topological subspace** where $\mathscr{T}_y = \{U \cap Y \mid U \in \mathscr{T}\}$ is the **relative topology** of Y.

Definition: A **base** for a topolgy $\mathscr{T}$ is a collection of open set $\mathscr{B} \subset \mathscr{T}$ such that each $U \in \mathscr{T}$ is a union of open sets in $\mathscr{B}$. That is, $U = \bigcup_{G \in \mathscr{B}} G$.

2.2 Metric spaces

Let X be a non-empty set and $x, y \in X$ then if there exists a non-negative real number $d(x,y)$ with following three properties:

1. $d(x,y) = 0$ if and only if $x = y$ (Positive definiteness).
2. $d(x,y) = d(y,x)$ (Symmetry).
3. $d(x,y) \leq d(x,z) + d(z,y)$ (Triangle inequality).

the combination (X, d) is called a **metric space** and $d(x,y)$ is called the **metric**, or also **distance** function.

Example 2.2. The Euclidean space $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ with $d(x,y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ makes a metric space. To prove this we must show the above properties work:

1. if $d(x,y) = 0$ then:
$$\sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2} = 0$$
Therefore each of the terms must be zero:
$$(x_i - y_i)^2 = 0 \quad \forall i \leq n$$
$$x_i - y_i = 0 \implies x_i = y_i$$
Thus $x = y$
2. It is obvious that $(x_i - y_i)^2 = (y_i - x_i)^2$ and therefore $d(x,y) = d(y,x)$
3. The triangle inequality immediately follows from the Cauchy-Schwartz inequality.

We can expand the Euclidean norm by defining Minkowski *p-norm* also called L^p*-norm* for $1 \leq p \leq \infty$ as follows:

$$d_p(x,y) = \left(\sum_i |x_i - y_i|^p\right)^{\frac{1}{p}}$$

and by taking the limit, $p \to \infty$ we find out that:

$$d_\infty(x,y) = \max_i\{|x_i - y_i|\}$$

Example 2.3. We can define **discrete distance** as follows:

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

and it is pretty straightforward to show that the three properties hold.

Definition: The **open ball** $B_r(a)$ with radius r centered at a is the set of all points:

$$B_r(a) = \{x \in X : d(x,a) < r\}$$

and the **closed ball** $\overline{B}_r(a)$ with radius r centered at a is the set of all points:

$$\overline{B}_r(a) = \{x \in X : d(x,a) \leq r\}$$

The **sphere** $S_r(a)$ with radius r centered at a is the set of all the points:

$$S_r(a) = \{x \in X : d(x,a) = r\}$$

Definition: Let (X, d) be a metric space. A subset $U \subset X$ is an open set if for all $a \in U$ there exists $\rho > 0$ such that $B_\rho(a) \subset U$.

Given this defintion of open sets, we can define a topolgy on metric space X.

1. Firstly, we need to show that every union of open sets is open itself. Let U_α be some open sets indexed by A and let $x \in \cup_\alpha U_\alpha$. Then, there exists a $\alpha \in A$ such that $x \in U_\alpha$. Since, U_α is open, there exists a ball $B_r(x)$ which is contained in U_α. Clearly, $B_r(x) \in \cup_\alpha U_\alpha$ and hence $\cup_\alpha U_\alpha$ is open.

2. Secondly, we show that intersection of finite collection of open sets in open. Let U_α be open sets indexed by a finite set A and let $x \in \cap_\alpha U_\alpha$. For each $\alpha \in A$, there exists a ball $B_{r_\alpha}(x)$ such that $B_{r_\alpha}(x) \subset U_\alpha$. Let $r = \min_\alpha r_\alpha$ and note that $B_r(x) \subset B_{r_\alpha}(x) \subset U_\alpha$ for all $\alpha \in A$. Thus, $B_r(x) \subset \cap_\alpha U_\alpha$ hence, $\cap_\alpha U_\alpha$ is open.

3. Thirdly, we show that X and $\emptyset$ are open. $\emptyset$ is trivially open as it has no element. And $B_r(x) \subset X$ by defintion for all r hence, X is open as well.

Consider the following re-defintions of concepts introduced in the previous section.

Definition (Internal Point): A point $a \in X$ is called an internal point of U if $\exists \rho > 0$ that the ball $B_\rho(a)$ contained in U.

Definition (Interior): The interior of a set U denoted by U° or $\text{int}(U)$ is the set of all its interior points.

Definition (Adherent Point): A point $a \in X$ is called an adherent point of U if $\forall \rho > 0$ the ball $B_\rho(a)$ contains a point in U.

Definition (Limit Point): A point $a \in X$ is called a limit point of U if $\forall \rho > 0$ the set $B_\rho(a) - \{a\}$ contains a point in U. The set of all limit points is denoted by S' or $\lim S$.

Note: For any limit point $a \in U$ every open ball $B_r(a)$ contains infinitely many points in U.

Definition (Closed Set): A subset $C \subset X$ is closed set if it contains all of its adherent point.

Definition (Closure): The closure of a set U denoted by $\overline{U}$ or $\text{cl}\, U$ is set of all its adherent points.

Note: The closure of a set is a closed set.

We then, show that these re-definitions are equivalent to the topological defintions.

Theorem 2.7. *Subset $C \subset X$ is closed if and only if $X - C$ is open.*

Proof. Firstly we prove the necessity condition that is C is closed if $X - C$ is open . We employ proof by contradiction. Let C be a closed subset of X such that its complement is not open. That is, for some $a \in (X - C)$ there is no $\rho > 0$ exists such that $B_\rho(a) \subset (X - C)$. In other words, for all ρ, $\exists p \in B_\rho(a)$ s.t $p_\rho \in C$. Which implies that a is an adherent point of C but since C is closed then $a \in C$ which is a contradiction. Similarly, one can show the sufficiency condition. ■

Corollary 2.8. *X and $\emptyset$ are both closed and open.*

Remark 1. (Equivalent Definitions)

1. An open set is a union of open balls. Conversely, a union of open balls is an open set.

   Proof. For every $a \in U$ there is a ball $B_\rho(a) \subset U$ thus $\bigcup_{a\in U} B_\rho(a) \subset U$ and since $a \in B_\rho(a)$ we must have $\bigcup_{a\in U} B_\rho(a) \supset U$ hence $U = \bigcup_{a\in U} B_\rho(a)$.

   Now let $U = \bigcup B_\rho(a)$ we need to show that U is open. Let $b \in U$ then b must be a point in at least one of those balls. Let $b \in B_r(c)$ and $\rho = r - d(b,c)$. We will show that $B_\rho(b) \subset B_r(c) \subset U$, for any $x \in B_\rho(b)$ by triangle inequality we have $d(x,c) \le d(x,b) + d(b,c) < \rho + d(b,c) = r$ which means $x \in B_r(c)$. □

2. A set is open if and only if all of its members are interior points. Therefore, $U = \operatorname{int} U$.

3. Let $I = \{S \subset U : S \text{ is open}\}$ then $\operatorname{int} U = \bigcup\limits_{S\in I} S$.

4. Let $I = \{U \subset S : S \text{ is closed}\}$ then $\operatorname{cl} U = \bigcap\limits_{S\in I} S$.

Let (X,d) be a metric space and $Y \subset X$ then Y may inherit its metric from X and (Y,d) would also be a metric space and is called a **metric subspace** of X. We will investigate the nature of open and closed sets in subspaces. Let $B^Y_\rho(y) = \{p \in Y : d(y,p) < \rho\}$ Then, it is easy to see that:

$$B^Y_\rho(y) = B_\rho(y) \bigcap Y$$

Corollary 2.9. *Let (X,d) be a metric space and $Y \subset X$ is a metric subspace of X then $U \subset Y$ is an open subset of Y if and only if there is a open set $V \subset X$ such that $U = V \cap Y$. Similarly, for any closed set $C \subset Y$ there is a closed set $D \subset X$ such that $C = D \cap Y$.*

Proof. Ofcourse, if $U \subset Y$ is open in Y then by definition it can be represent as a union of open ball $B^Y_r(a)$. Each of these balls is the intersection of a $B^X_r(a) \cap Y$. Therefore

$$U = \bigcup B^Y_r(a) = \bigcup \left(B^X_r(a) \cap Y\right) = \left(\bigcup B^X_r(a)\right) \cap Y = V \cap Y$$

Furthermore, if $a \in V \cap Y$ then there exists a ball $B^X_r(a) \subset V$. Therefore

$$B^Y_r(a) = B^X_r(a) \cap Y \subset V \cap Y = U$$

The case for closed subsets can be proved using the complements. ■

Exercises

1. Show that $\operatorname{cl} S = S \cup \lim S$

2.3 Convergence

Let X be a topological space. A **sequence** is a function in form of $a : \{k, k+1, k+2, \ldots\} \to X$ where $k \in \mathbb{Z}$. Conventionally, instead of $a(n)$, a_n is used. The sequence $\{a_n\}$ is **convergent** to $a \in X$ if for all neighbourhood U of a there exists N such that:

$$n \geq N \implies a_n \in U$$

and it is denoted by $a_n \to a$ or $a = \lim_{n\to\infty} a_n$. Given a topolgy, convergence is not necessarily well-behaved. For example, in the trivial topolgy, if $a_n \to a$, then $a_n \to b$ for any $b \in X$. To do away with this we consider **Hausdorff spaces**.

Definition: Let X be a topological space. X is a Hausdorff space if for any two $x, y \in X$ where $x \neq y$, there exists disjoint open sets U and V such that $x \in U$ and $y \in V$.

Proposition 2.10. *Let X be a Hausdorff space and x_n is a sequence in X. If $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.*

Proof. If $x \neq y$, then there are disjoint open set U and V with $x \in U$ and $y \in V$. If $x_n \to x$, then there exists N such that for $n \geq N$, $x_n \in U$. But this implies that for $n \geq N$, $x_n \notin V$. Meaning, there exists no N' that for $n \geq N'$, $x_n \in V$ hence, x_n does not converge to y. ∎

In the case of a metric space, the set $\{a_k, a_{k+1}, \ldots\}$ is bounded in X, that is, there exist $K > 0$ and a point $b \in X$ such that $\forall n,\ a_n \in B_K(b)$.

Another problem with the defintion of convergence is its dependence on a convergence point. So naturally the following question comes up. Is there a way to show the convergence of sequence based on itself? For that, we need to define **Cauchy sequence**. A sequence $\{a_n\}$ in a metric space X is a Cauchy sequence if:

$$\forall \epsilon > 0,\ \exists N \quad \text{s.t.} \quad n, m \geq N \implies d(a_n, a_m) < \epsilon$$

Lemma 2.11. *In a metric space X, convergence of a_n to a is equivalent to*

$$\forall \epsilon > 0,\ \exists N \quad \text{s.t.} \quad n \geq N \implies d(a_n, a) < \epsilon$$

Proof. If $a_n \to a$, then for $B_\epsilon(a)$ there exists N such that $n \geq N \implies a_n \in B_\epsilon(a)$. On the other hand, for any neighbourhood U we can find $\epsilon > 0$ such that $B_\epsilon(a) \subset U$. Hence, if the metric convergence condition holds, then topological convergence holds, as well. ∎

Theorem 2.12. *Every convergent sequence is a Cauchy sequence.*

Proof. For a given $\epsilon > 0$ we know there exist N such that:

$$n \geq N \implies d(a_n, a) < \frac{\epsilon}{2}$$

and equivalently:

$$m \geq N \implies d(a_m, a) < \frac{\epsilon}{2}$$

and since by the triangle inequality we have:

$$d(a_m, a_n) \leq d(a_m, a) + d(a_n, a) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

∎

Definition (Subsequence): We call $\{b_n\}$ a **subsequence** of $\{a_n\}$ if there is a sequence of positive integers $n_1 < n_2 < n_3 < \dots$ such that for each k, $b_k = a_{n_k}$.

Exercises

1. Show that if a sequence $\{a_n\}$ is convergent, then its limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.

2. Prove that every subsequence of a convergent sequence converges and it converges to the same limit.

2.4 Completeness

A metric space (X, d) is **complete** if every Cauchy sequence converges.

Proposition 2.13. *$\mathbb{R}$ with the normal Euclidean norm is a complete metric space.*

To prove it, we need the following lemmas.

Lemma 2.14. *If $\{a_n\}$ is a Cauchy sequence in a metric space (X, d) then the set $S = \{a_k, a_{k+1}, \ldots\}$ is bounded.*

Proof. For a fixed $\epsilon > 0$ we know there exists N such that:

$$m, n \geq N \implies d(a_n, a_m) < \epsilon$$

especially:

$$n \geq N \implies d(a_n, a_N) < \epsilon$$

Since there is only finitely many indices less than N then we can determine the largest $d(a_N, a_m)$for all m less than N lets denote it by A. Finally, let $K = \max\{\epsilon, A\}$ then, $B_K(a_N)$ contains all the elements of sequence. □

Lemma 2.15. *If one of the subsequences of Cauchy sequence is convergent, then the Cauchy sequence is convergent to the same element.*

Proof. Let $a_{n_k} \to a$ when $k \to \infty$ That is, for a given $\epsilon > 0$, $\exists N_1$ such that:

$$k \geq N_1 \implies d(a_{n_k}, a) < \frac{\epsilon}{2}$$

and since $\{a_n\}$ is a Cauchy sequence then we also know that there exists N_2 such that:

$$q, m \geq N_2 \implies d(a_m, a_q) < \frac{\epsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ and $n_q \geq N$ consequently:

$$n_q, m \geq N \implies d\big(a_m, a_{n_q}\big) < \frac{\epsilon}{2}$$

and by the triangle inequality we have:

$$d(a_m, a) \leq d\big(a_m, a_{n_q}\big) + d\big(a_{n_q}, a\big) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which proves the convergence of $a_n \to a$. □

We present a proof for the completeness of $\mathbb{R}$ under the Euclidean norm.

Proof. Let $\{a_n\}$ be a Cauchy sequence. Then by Lemma 2.14, the sequence is bounded and there is a closed interval $I_0 = [a, b]$ in which all a_n lie. Consider the closed intervals $\left[a, \dfrac{a+b}{2}\right]$ and $\left[\dfrac{a+b}{2}, b\right]$. Since the sequence has infinitely many terms then there are infinitely many terms in at least one of the two intervals. Let that interval be I_1 and choose $x_1 \in I_1$ where $x_1 = a_{n_1}$ for some n_1. Repeat the process for I_1 to get I_2 and $x_2 = a_{n_2}$ where $n_2 > n_1$. Since there are infinitely many terms in I_2 we can find such n_2. By continuing this process we have a subsequence $\{x_k\}$ and a sequence of nested closed sets $\{I_k = [a_k, b_k]\}$. Since for all $\epsilon > 0$ there exists K such that $b_K - a_K < \epsilon$ then the intersection of $\{I_k\}$ is a point, say y. We claim that that $x_k \to y$, that is:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \implies |x_n - y| < \epsilon$$

Since $y = \bigcap I_k$ then $y \in I_k$ for all k, especially $y \in I_n$. Therefore, $|x_n - y|$ is smaller than or equal to the length of I_n which is $\dfrac{b-a}{2^n} \leq \dfrac{b-a}{2^N}$. By setting $N > \log_2 \dfrac{b-a}{\epsilon}$ we have:

$$|x_n - y| \leq \frac{b-a}{2^n} \leq \frac{b-a}{2^N} < \epsilon$$

Therefore $\mathbb{R}$ is a complete metric under Euclidean norm. ■

Let (X, d) and (X', d')be two metric spaces. Define the following norms on the Cartesian product $X \times X'$:

1. $D_1((x, x'), (y, y')) = d(x, y) + d'(x, y)$
2. $D_2((x, x'), (y, y')) = \sqrt{d(x, y)^2 + d'(x', y')^2}$
3. $D_3((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}$

Let $p_1 = (x, x')$ and $p_2 = (y, y')$:

$$D_3(p_1, p_2) \leq D_2(p_1, p_2) \leq D_1(p_1, p_2) \leq 2D_3(p_1, p_2)$$

Then, it is easy to see that if a sequence $\{a_n\}$ is convergent under one of these norms, it is convergent to the same value under the other two. The same is true if the sequence is a Cauchy sequence.

By induction we can generalize it to $X_1 \times X_2 \times \ldots \times X_n$. For example, $\mathbb{R}^n$ is complete metric under all the three norms introduced above. That is, every Cauchy sequence in $\mathbb{R}^n$ is convergent. To show this assume the sequence $\{x_i\}$ is a Cauchy sequence under, WLOG, D_1:

$$\forall \epsilon > 0, \, \exists N \quad \text{s.t.} \quad i, j \geq N \implies D_1(x_i, x_j) < \epsilon$$

Then for the k-th coordinate:

$$|x_{i_k} - x_{j_k}| < D_1(x_i, x_j) < \epsilon$$

Therefore, for every coordinate, the image of the sequence on that coordinate is a Cauchy sequence. Since $\mathbb{R}$ is complete then $\{x_{i_k}\}_i$ is convergent to some x_k for all k. We claim that $x_i \to x = (x_1, \ldots, x_n)$ as $i \to \infty$:

$$D_1(x, x_i) = |x_{i_1} - x_1| + |x_{i_2} - x_2| + \cdots + |x_{i_n} - x_n|$$

We have shown that $\{x_{i_k}\}_i$ is convergent to x_k then there must be $N_1, N_2, \ldots N_n$ such that for all k:

$$\forall \epsilon, \quad i \geq N_k \implies |x_{i_k} - x_k| < \frac{\epsilon}{n}$$

Setting $N = \underset{1 \leq k \leq n}{max} N_k$:

$$D_1(x, x_i) < n \cdot \frac{\epsilon}{n} = \epsilon$$

Theorem 2.16. *Let (X, d) be a complete metric space and $Y \subset X$ is a complete metric space if and only Y is a closed subset of X.*

Proof. It is clear that Y being closed is necessary for Y being a complete metric subspace. To show that is also sufficient, we need ot show that if Y is a complete metric subspace then it is closed. Assume the contrary, that is there exists an adherent point of Y, $a \notin Y$. Since a is an adherent point of Y then for all $\rho > 0$ there exists a point $x \in B_\rho(a)$ such that $x \in Y$. For each n let $\rho = \frac{1}{n}$ and choose a point $x_n \in Y$ It is clear that $\{x_n\}$ is convergent to a. From Theorem 2.12 $\{x_n\}$ is a Cauchy sequence. Since Y is complete then a must be in Y which is a contradiction. ■

Theorem 2.17 (Baire Category Theorem). *Let X be a complete metric space and G_n be a sequence of dense open sets in X. Then, $\cap_{n=1}^{\infty} G_n$ is dense in X.*

Theorem 2.18. *Let (X, d) be a complete metric space, and suppose that $f : X \to X$ has the property that there exists $\alpha < 1$ such that*

$$d(f(x), f(y)) < \alpha d(x, y)$$

for all $x, y \in X$. Then, there exists a unique point $x \in X$ such that $f(x)x$. Moreover, if $x_0 \in X$, and $x_{n+1} = f(x_n)$, then $x = \lim_{n\to\infty} x_n$.

Exercises

1. Show that if a sequence $\{a_n\}$ is convergent, then its limit is unique. That is, if $a_n \to a$ and $a_n \to b$ as $n \to \infty$ then $a = b$.

2. Prove that every subsequence of a convergent sequence converges and it converges to the same limit.

2.5 Continuity

Definition (Continuity): Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces and $f : X \to Y$ be a function. We say f is continuous if for every open subset V of Y the pre-image of it is an open set in X.

$$f^{\text{pre}}(V) = f^{-1}(V) = \{x \in X : f(x) \in V\}$$

Furthermore, f is continuous at a point $x \in X$ when for all subset W of Y that $f(x)$ is an internal point of, there is an open set U containing x such that $\{f(y) : y \in U\} \subset W$. In other words x is an internal point of $f^{\text{pre}}(W)$.

Proposition 2.19. *f is continuous if and only if f is continuous at every point $x \in X$.*

Proof. Firstly, if f is continuous we show that f is continuous at every point $x \in X$. Let V be an open set around $f(x)$ then $x \in f^{\text{pre}}(V)$ must be an internal point since $f^{\text{pre}}(V)$ is open. Secondly, if f is continuous at every point $x \in X$ then f is continuous. Let $V = \{f(x) : x \in U\}$ be an open set in Y. For any $x \in U$, $f(x)$ is an internal point of V and since f is continuous at x, x is an internal point of U which means every point $x \in U$ is an internal point of U and thus $U = f^{\text{pre}}(V)$ is open. ■

Theorem 2.20 ($\epsilon - \delta$ condition). *Let (X, d) and (Y, d') be two metric space. Continuity at a point x is equivalent to the existence a $\delta > 0$ for all $\epsilon > 0$ such that:*

$$d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$$

Proof. Let $V = \{f(y) : d'(f(x), f(y)) < \epsilon\}$ then V is open and hence $f(x)$ is an internal point of V. By continuity at point x, x must be an internal point of $f^{\text{pre}}(V)$. In other words, there exists a $\delta > 0$ such that $U = \{y : d(x, y) < \delta\} \subset f^{\text{pre}}(V)$. Take an open set $U \subset Y$, then assuming the $\epsilon - \delta$ condition, we will show that $f^{\text{pre}}(U)$ is open. Let $y \in U$ then there is $x \in f^{\text{pre}}(U)$ such that $f(x) = y$. From openness of U, there is a $\epsilon > 0$ such that $B_\epsilon(y) \subset U$, also by continuity condition, there exists a $\delta > 0$ such that:

$$d(x, z) < \delta \implies d'(f(x), f(z)) < \epsilon$$

The openness of $f^{\text{pre}}(U)$ is equivalent to $B_\delta(x) \subset f^{\text{pre}}(U)$, which clearly holds, since for any $z \in B_\delta(x) \implies f(z) \in B_\epsilon(y) \subset U$. ■

Example 2.4. Let (X, d) be a metric space with $d(x, y)$ being the discrete metric, $f : X \to X'$ where (X', d') is an arbitary metric space. Then f is always continuous. Since for every point a the open ball $B_{\frac{1}{2}}(a) = \{a\}$, and union of open sets is an open set itself, then every subset of X is open.

Equivalently, f is continuous at a if for all $\epsilon > 0$, a is an internal point of $f^{\text{pre}}(B_\epsilon(f(a)))$. That is there exists $\delta > 0$ such that, $B_\delta(a) \subset f^{\text{pre}}(B_\epsilon(f(a)))$. More generally, if X has the discrete topolgy or X' has the trivial topolgy, then $f : X \to X'$ is always continous.

Proposition 2.21. *Let $f : (X, \mathscr{T}_1) \to (X, \mathscr{T}_2)$ be the identity function. f is continous if and only if $\mathscr{T}_1$ is stronger that $\mathscr{T}_2$.*

Proof. If f is continous, then every open set $V \in \mathscr{T}_2$ is an open set in $\mathscr{T}_1$. Hence, $\mathscr{T}_2 \subset \mathscr{T}_1$. If $\mathscr{T}_2 \subset \mathscr{T}_1$ and V is an open set in $\mathscr{T}_2$, then $V = f^{\text{pre}}(V) \in \mathscr{T}_1$ is open and thus f is continous. ■

Theorem 2.22. *Let (X, d) and (X', d') be two metric spaces and $f : X \to X'$. f is continuous at $a \in X$ if and only if for every sequence $\{a_n\}$ in X with $a_n \to a$ we have $f(a_n) \to f(a)$.*

Proof. Let f be continuous at a and $a_n \to a$. From continuity of f, for each given ϵ, there is a δ such that:

$$d(x, a) < \delta \implies d'(f(x), f(a)) < \epsilon$$

From the convergence of $\{a_n\}$, for each given δ, there is a N such that:

$$\forall n \geq N \implies d(a_n, a) < \delta$$

By merging these two equations we will get:

$$\forall n \geq N \implies d(a_n, a) < \delta \implies d'(f(a_n), f(a)) < \epsilon$$

which was what was wanted.

If f is not continuous, there must be an $\epsilon > 0$ that for all $\delta > 0$, for some $x \in B_\delta(a)$, $d'(f(x), f(a)) \geq \epsilon$. Especially, for each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ and x_n have the described property. Since $x_n \to a$ by our assumption $f(x_n) \to f(a)$, which is a contradiction and thus f is continuous. ■

Definition: Let X and Y be two topological spaces and $f : X \to Y$. f is a **homeomorphism** of X to Y if f is bijective and, f and f^{-1} are continous. Furthermore, X and Y are **homeomorphic** if there exists a homeomorphism between them.

Exercises

1. Let $(X, d), (X', d')$, and (X'', d'') be metric spaces and $f : X \to X'$, $g : X' \to X''$ be two functions. If f is continous at a and g is continous at $f(a)$, then $g \circ f$ is continuous at a.

2. Let (X_i, d_i), $i = 1, \ldots k$ be metric spaces. Define D to be any of the three discussed metric over $X = X_1 \times X_2 \times \ldots \times X_k$. Then, the projection function, $\pi_j(x) : X \to X_j$ is continuous for all j.
$$\pi_j(x_1, x_2, \ldots, x_n) = x_j$$

3. Let (X, D) be defined as above and let (X', d') be a matic space, and $f : X' \to X$. f is continous at $a' \in X'$ if and only if $\pi_j \circ f$ is continuous for all $j = 1, \ldots, k$.

4. The four algebraic operations are continuous on their domain.

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad +(x, y) = x + y$$
$$- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad -(x, y) = x - y$$
$$\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \times(x, y) = x \times y$$
$$\div : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R}, \quad \div(x, y) = x \div y$$

Where the metric of $\mathbb{R}$ on the right hand side is the common Euclidean metric, and on the left hand side is any of the three metric.

2.6 Covering compactness

Definition: Let X be a topological space. A **covering** for s set $E \subset X$ is a collection of U_α of open subsets of X such that $E \subset \bigcup U_\alpha$.

Definition: A **subcovering** of $\mathscr{U} = \{U_\alpha \,|\, \alpha \in A\}$ is a collection $\mathscr{V} = \{U_\alpha \,|\, \alpha \in B\}$ where $B \subset A$ and $\mathscr{V}$ covers the same set.

Definition: A subset K of X is called **compact** if every open cover of K has a finite subcover.

Example 2.5. $\mathscr{U} = \{]x-1, x+1[\,|\, x \in \mathbb{R}\}$ is covering for $\mathbb{R}$. It Obviously has a countable subcovering, however, it does not have finite subcovering.

Theorem 2.23. *Closed and bounded intervals in* $\mathbb{R}$ *are compact.*

Proposition 2.24. *If X is a compact space and $\{K_n\}$ is a sequence of non-empty closed subsets of X, with $K_{n+1} \subset K_n$ for all n, then $\bigcap_{n=1}^{\infty} K_n$ is non-empty.*

Proposition 2.25. *If E is an infinite subset of a compact set, then E has a limit point in K.*

Proposition 2.26. *A subset of a topological space is compact if and only if it is compact in iteself with the relative topology.*

Proposition 2.27. *If X is a Hausdorff space and $K \subset X$ is compact, then K is closed.*

Definition: Let X be a metric space and E a subset of X. The **diameter** of E is defined to be $\operatorname{diam} E = \sup\{d(x,y) \,|\, x, y \in E\}$. E is said to be bounded if its diameter is finite.

Definition: Let E be subset of a metric space X. E is **totally bounded** if for all $\epsilon > 0$, there exists a finite subset $\{x_1, \ldots, x_n\}$ of X such that $E \subset \bigcup_{k=1}^{n} B_\epsilon(x_k)$. A set F such that $E \subset \bigcup_{x \in F} B_\epsilon(x)$ is called an ϵ-**net** for E.

Proposition 2.28. *If E is totally bounded, then:*

1. *E is bounded.*
2. $\operatorname{cl} E$ *is totally bounded.*
3. *Any subset of E is totally bounded.*

Theorem 2.29. *A metric space X is complete if and only if X is complete and totally bounded.*

Definition: A subset E of topological space is **relatively compact** or **precompact** if $\operatorname{cl} E$ is compact.

Corollary 2.30. *A subset of a complete metric space is relatively compact if and only if it is totally bounded.*

Theorem 2.31. *A set E in $\mathbb{R}^k$ is closed and bounded if and only if it is compact.*

Note: This is generally not true for other metric spaces.

Theorem 2.32 (Weierstrass). *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Exercises

1. Show that $\mathbb{Q} \cap [0,1]$ is not covering compact, directly from the definition.

2.7 Sequential compactness

A subset $K \subset X$ is **compact** if it has the Boltzano-Weierstrass property, if for all sequences $\{a_n\}$ in K there exists a subsequence of $\{a_n\}$ that converges to a point$a \in K$.

Theorem 2.33. *Compactness is equivalent to the existence a finite subcovering for every covering.*

Proof. To prove the theorem, let us define:

We will show for metric spaces compactness is equivalent to covering compact. lebegue number ■

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. We define:

$$\limsup a_n = \overline{\lim}\, a_n = \lim_{n\to\infty}\Big(\sup\{a_k : k \geq n\}\Big)$$
$$\liminf a_n = \underline{\lim}\, a_n = \lim_{n\to\infty}\Big(\inf\{a_k : k \geq n\}\Big)$$

Note: The limits, $\limsup a_n$ and $\liminf a_n$, always exists. Albeit they might be infinite.

Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$, and A^* is the set of all limit points of all subsequence of $\{a_n\}$. We know that A^* is not empty and since $\{a_n\}$ is bounded and then A^* must be bounded as well. Thus, by completeness axiom, A^* has infimum and supremum. Moreover, $\sup A^*, \inf A^* \in A^*$.

Proposition 2.34. *A bounded sequence $\{a_n\}$ is convergent if and only if* $\limsup a_n = \liminf a_n$.

Corollary 2.35. *If K is a compact subset of $\mathbb{R}$ then K has minimum and maximum. That is, there are $M, m \in K$ such that $\forall x \in K,\ m \leq x \leq M$.*

Proof. Since K is bounded then it has supremum and infimum in $\mathbb{R}$. Obviously, there are convergent sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to m = \inf K$ and $b_n \to M = \sup K$. By compactness of K, $M, m \in K$. ■

Theorem 2.36. *(X, d) and (X', d') are metric spaces and $K \subset X$ is compact. If $f : X \to X'$ is continuous, then $f(K)$ is a compact subset of X'.*

Proof. Let $\{y_n\} \in f(K)$ and $\{x_n\} \in K$ are such that $f(x_n) = y_n$. Since K is compact there is a convergent subsequence $\{x_{n_k}\}$ and since f is continous $\{y_{n_k} = f(x_{n_k})\}$ is also convergent. Hence $f(K)$ is compact. ■

Corollary 2.37. *Let (X, d) be a metric space and $f : X \to \mathbb{R}$ is continuous. If K is a compact subset of X. Then f attains maximum and minimun in $\mathbb{R}$.*

Note: For a continuous function $f : X \to X'$ it is not necessary that the image of an open/closed set to be open/closed.

Definition (Uniform continuity): Let (X, d) and (X', d') be metric spaces. $f : X \to X'$ is uniformly continuous if:

$$\forall \epsilon > 0\ \exists \delta > 0,\ x, y \in X,\ d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$$

Proposition 2.38. *$f : X \to X'$ is uniformly continous if and only if for every pair sequence $\{(x_n, y_n)\}$ in X satisfying $d(x_n, y_n) \to 0$ we have $d'(f(x_n), f(y_n)) \to 0$.*

Proof. Necessity: We have

$$\forall \epsilon \; \exists \delta \text{ s.t. } \forall x, y \in X, d(x, y) < \delta \implies d'(f(x), f(y)) < \epsilon$$
$$\forall \delta \; \exists N \text{ s.t. } n \geq N \implies d(x, y) < \delta$$

combining the two brings us at the conclusion.

Sufficiency: Suppose for the sake of contradtion that:

$$\exists \epsilon \; \forall \delta \; \exists x, y \in X \text{ s.t. } d(x, y) < \delta \wedge d'(f(x), f(y)) \geq \epsilon$$

then let $\delta = \frac{1}{n}$ and make the sequence pair $\{(x_n, y_n)\}$. Clearly, $d(x_n, y_n) \to 0$ therefore, $d'(f(x), f(y)) \to 0$. Which is a contradition since $d'(f(x), f(y)) \geq \epsilon$. ■

Proposition 2.39. *(X, d) and (X', d') are matric spaces and X is compact. If $f : X \to X'$ is continuous then it is uniformly continuous.*

Proof. Similarly, for the sake of contradicition suppose

$$\exists \epsilon \; \forall \delta \; \exists x, y \in X \text{ s.t. } d(x, y) < \delta \wedge d'(f(x), f(y)) \geq \epsilon$$

and let $\delta = \frac{1}{n}$ and make the sequence pair (x_n, y_n). By compactness of X, there are two convergent subsequence $\{x_{n_k}\}$ and $\{y_{n_k}\}$. Since $d(x_n, y_n) \to 0$ then if $x_{n_k} \to x$, $y_{n_k} \to x$ as well. By continuity of f, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$ and thus $d'(f(x_{n_k}), f(y_{n_k})) \to 0$. Which is a contradicition as for sufficiently large K, $k \geq K \implies d'(f(x), f(y)) \geq \epsilon$ ■

Exercises

1. Prove that $\sqrt{\mid x \mid} : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

2.8 Connectedness

Definition: Let X be a topological space. X is **disconnected** if there are non-empty open sets A and B are found such that

$$A \bigcap B = \emptyset, \quad A \bigcup B = X$$

X is said to be **connected** if it is not disconnected. A subset S of X is connected if it is connected when considering its relative topolgy.

Definition:

Example 2.6. The following subsets of $\mathbb{R}$ are disconnected:

1. $S = [-1, 0[\ \cup \ ]0, 1]$.
2. $\mathbb{Q}$.
3. $S = [1, 0] \cup [1, 2]$.

Definition: Two sets A and B in a topological space X are said to be **separable** if $A \cap \operatorname{cl} B = \operatorname{cl} A \cap B = \emptyset$.

Proposition 2.40. *X is connected if and only if it can not be written in form of two non-empty separable sets.*

Definition: $S \subset \mathbb{R}$ is an **intervals** if when $a, c \in S$ and $a < b < c$ then $b \in S$.

Example 2.7. $\mathbb{R}$ and its intervals are connected. In fact the only connected subsets of $\mathbb{R}$ are its intervals.

Theorem 2.41. *Let X and X' be two topological spaces. Let $f : X \to X'$ be continuous and S be a connected subset of X. Then, $f(S)$ is connected in X'.*

Corollary 2.42 (Mean value theorem). *If $f : [a, b] \to \mathbb{R}$ is a continous function and $f(a) = A, f(b) = B$ then for every C between A and B there exists a $c \in [a, b]$ such that $f(c) = C$.*

Proposition 2.43. *If $S \subset X$ is a connected set then every $S \subset T \subset \bar{S}$ is connected.*

Proposition 2.44. *Let $\{E_\alpha\}$ be collection of connected sets with $\cap_\alpha E_\alpha \neq \emptyset$. Then, $\cap_\alpha E_\alpha$ is connected.*

Definition: Let x be a point in a topological space X. The **connected component** of x, C_x, is the union of the all the connected set including x.

Proposition 2.45. *For any $x, y \in X$*

1. *C_x is connected and closed.*
2. *C_x and C_y are either disjoint or equal to each other.*

Definition: A topological space X is said to be **totally disconceted** if $C_x = \{x\}$ for all $x \in X$. A subset S of X is totally disconceted if it is totally disconceted when considering its relative topology.

Definition: The *graph* of a function $f : M \to N$, G_f, given by $G_f = \{(x, f(x)) \,|\, x \in M\}$.

Theorem 2.46. *The graph of a continous function over a connected set is connected.*

Example 2.8. Topological curve is connected and also its closure is connected.

Definition (Path connected): A set S is path connected if for every pair of points $p, q \in S$ there exists a continous function $\gamma : [a, b] \to S$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

Theorem 2.47. *If a set S is path connected, then it is connected but the inverse is not true.*

Example 2.9. Infinite broom is path connected but toplogical sine curve is not.

Proposition 2.48. *If f is continuous function and S is a path connected set, then the image of S under f is path connected.*

Proposition 2.49. *Every open set of $\mathbb{R}$ is the union of countably many disjoint open intervals.*

Exercises

1.

2.9 Cantor Set

Definition: define cantor set

Proposition 2.50. *Cantor set is a perfect set.*

Proposition 2.51. *Cantor set is totally disconnected.*

Theorem 2.52. *Let K be a complete, totally disconnected, and compact metric space. Then, K is homeomorphic to Cantor set.*

Theorem 2.53. *Let P be a non-empty perfect set in $\mathbb{R}^k$. Then, P is uncountable.*

Exercises

1. Show that $\mathbb{Q} \cap [0, 1]$ is not covering compact, directly from the definition.