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Part I

Digital Communication

Chapter 1

Frequency Domain Analysis

1.1 Fourier Series

For a periodic signal $x(t)$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{2\pi j \frac{n}{T_0} t} \qquad\qquad x_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-2\pi j \frac{n}{T_0} t} \,\mathrm{d}t$$

and

$$x_{\pm}(t) = \begin{cases} x(t) & x \ \text{ is continuous at } \ t \\ \dfrac{x(t^+) + x(t^-)}{2} & x \ \text{ is discontinuous at } \ t \end{cases}$$

for angular frequency $\omega_0 = 2\pi f_0$:

$$x_{\pm}(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t} \qquad\qquad x_n = \frac{\omega_0}{2\pi} \int_{T_0} x(t) e^{-jn\omega_0 t} \,\mathrm{d}t$$

$f_0 = \frac{1}{T_0}$ is called the **fundamental frequency** and its n_{th} is called the n_{th} **harmonic**.

1.2 Fourier Transform

For non-periodic signals $x(t)$:

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} \,\mathrm{d}t & x_{\pm}(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} \,\mathrm{d}f \\ X(f) &= \mathcal{F}\{x(t)\} & x_{\pm}(t) &= \mathcal{F}^{-1}\{X(f)\} \\ X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \,\mathrm{d}t & x_{\pm}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(f) e^{j\omega t} \,\mathrm{d}\omega \end{aligned}$$

$X(f)$ is called the **spectrum** of $x(t)$, or the **voltage spectrum**. From the relationship between the inverse Fourier transform of Fourier transform of a signal we define

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi j f t} \,\mathrm{d}f = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \,\mathrm{d}f$$

That is, all frequencies in $\delta(t)$ are with unit magnitude and zero phase.

$$\delta(t) = \mathcal{F}^{-1}\{1\} \qquad\qquad \delta(f) = \mathcal{F}\{1\}$$

1.3 Properties of Fourier transform

Linearity. For two signals $x(t)$ and $y(t)$ and complex constants a and b

$$\begin{aligned}\mathcal{F}\{ax(t)+by(t)\} &= \int_{-\infty}^{\infty}(ax(t)+by(t))e^{-2\pi jft}\,\mathrm{d}t \\ &= a\int_{-\infty}^{\infty}x(t)e^{-2\pi jft}\,\mathrm{d}t + b\int_{-\infty}^{\infty}y(t)e^{-2\pi jft}\,\mathrm{d}t \\ &= a\mathcal{F}\{x(t)\}+b\mathcal{F}\{y(t)\}\end{aligned}$$

Duality. For any signal $x(t)$

$$x(f)=\mathcal{F}\{\mathcal{F}\{x(t)\}(-\omega)\}$$

since

$$\begin{aligned}\mathcal{F}\{\mathcal{F}\{x(t)\}(-\omega)\} &= \int_{-\infty}^{\infty}\mathcal{F}\{x(t)\}(-\omega)e^{2\pi jf\omega}\,\mathrm{d}\omega \\ &= \mathcal{F}^{-1}\{\mathcal{F}\{x(t)\}\}(f) \\ &= x(f)\end{aligned}$$

Time shift. A shift of t_0 in time domain causes a phase shift in the frequency domain.

$$\begin{aligned}\mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty}x(t-t_0)e^{-2\pi jft}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}x(t)e^{-2\pi jf(t+t_0)}\,\mathrm{d}t \\ &= e^{-2\pi jft_0}\mathcal{F}\{x(t)\}\end{aligned}$$

Scaling. Suppose $a\neq 0$ is real

$$\begin{aligned}\mathcal{F}\{x(at)\} &= \int_{-\infty}^{\infty}x(at)e^{-2\pi jft}\,\mathrm{d}t \\ &= \frac{1}{a}\operatorname{sign}(a)\int_{-\infty}^{\infty}x(t)e^{-2\pi jf\frac{t}{a}}\,\mathrm{d}t \\ &= \frac{1}{|a|}\mathcal{F}\left\{\frac{f}{a}\right\}\end{aligned}$$

Convolution. For two signals $x(t)$ and $y(t)$

$$\begin{aligned}\mathcal{F}\{x(t)*y(t)\} &= \int_{-\infty}^{\infty}(x(t)*y(t))e^{-2\pi jft}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}y(t-\tau)e^{-2\pi jf(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}y(t-\tau)e^{-2\pi jf(t-\tau)}\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty}x(\tau)e^{-2\pi jf\tau}\mathcal{F}\{y(t)\}\,\mathrm{d}\tau \\ &= \mathcal{F}\{x(t)\}\mathcal{F}\{y(t)\}\end{aligned}$$

Parseval's property. For two signals $x(t)$ and $y(t)$ with Fourier transform $X(f)$ and $Y(f)$

$$\int_{-\infty}^{\infty} x(t)\overline{y(t)}\,\mathrm{d}t = \int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,\mathrm{d}f$$

since

$$\begin{aligned}
\int_{-\infty}^{\infty} X(f)\overline{Y(f)}\,\mathrm{d}f &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\,\mathrm{d}t\right)\overline{\left(\int_{-\infty}^{\infty} y(t)e^{-2\pi jft}\,\mathrm{d}t\right)}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-2\pi jft}\overline{y(\tau)}e^{2\pi jf\tau}\,\mathrm{d}\tau\,\mathrm{d}t\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\overline{y(\tau)}e^{2\pi jf(\tau-t)}\,\mathrm{d}f\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\overline{y(\tau)}\delta(\tau-t)\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} x(t)\overline{y(t)}\,\mathrm{d}t
\end{aligned}$$

Rayleigh's propery. For any signal $x(t)$ with Fourier transform $X(f)$

$$\int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$$

Autocorrelation. The time autocorrelation of the signal $x(t)$ is defined by

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}\,\mathrm{d}t = x(t) * \overline{x(-t)}$$

Then,

$$\mathcal{F}\{R_x(\tau)\} = |X(f)|^2$$

Differentiation.

$$\mathcal{F}\left\{\frac{\mathrm{d}}{\mathrm{d}t}x(t)\right\} = 2\pi j\mathcal{F}\{x(t)\}$$

Integration.

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau\right\} = \frac{X(f)}{2\pi jf} + \frac{1}{2}X(0)\delta(f)$$

Moments

$$\int_{-\infty}^{\infty} t^n x(t)\,\mathrm{d}t = \left(\frac{j}{2\pi}\right)^n \left[\frac{\mathrm{d}^n}{\mathrm{d}f^n}X(f)\right]_{f=0}$$

1.4 Power and Energy

Define

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t \qquad\qquad \mathcal{P}_x = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2\,\mathrm{d}t$$

A signal is **energy-type** if $\mathcal{E}_x < +\infty$ and it is **power-type** if $0 < \mathcal{P}_x < +\infty$. A signal can not be both, but it can be neither.

Remark 1. Average power is expressed in units of dBm or dBw as

$$(S)_{\text{dBw}} = 10\log_{10}(S)_{\text{watts}}$$
$$(S)_{\text{dBm}} = 10\log_{10}(S)_{\text{milliwatts}}$$

1.4.1 Energy-type

Let $x(t)$ be a energy-type signal. The **autocorrelation** of $x(t)$ is

$$\begin{aligned} R_x(\tau) &= x(\tau) * \overline{x(-\tau)} \\ &= \int_{-\infty}^{\infty} x(t)\overline{x(t-\tau)}\,\mathrm{d}t \\ \implies \mathcal{E}_x &= R_x(0) \end{aligned}$$

By Rayleigh's property

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |X(f)|^2\,\mathrm{d}f$$

The Fourier transform exists for The **energy spectral density** $\mathcal{G}(f) = \mathcal{F}\{R_x(\tau)\} = |X(f)|^2$, represent energy per hertz of bandwidth.

1.4.2 Power-type

Let $x(t)$ be a power type signal. The **time average autocorrelation** function

$$\begin{aligned} R_x(\tau) &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\overline{x(t-\tau)}\,\mathrm{d}t \\ \implies \mathcal{P}_x &= R_x(0) \end{aligned}$$

The **power spectral density** $\mathcal{S}(f) = \mathcal{F}\{R_x(\tau)\}$ and

$$\mathcal{P}_x = \int_{-\infty}^{\infty} \mathcal{S}(f)\,\mathrm{d}f$$

Remark 2. The power spectral density does not uniquely determine the signal. As it only retains the magnitude information and all phase information is lost.

Suppose $x(t)$ is a power-type signal passing through a filter with impluse response $h(t)$:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ R_y(\tau) &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} y(t)\overline{y(t-\tau)}\,\mathrm{d}t \\ &= \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} \left(\int_{-\infty}^{\infty} h(u)x(t-u)\,\mathrm{d}u\right)\left(\int_{\infty}^{\infty} \overline{h(v)x(t-\tau-v)}\,\mathrm{d}v\right)\mathrm{d}t \\ &= \lim_{T\to\infty}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} h(u)\overline{h(v)}x(t-u)\overline{x(t-\tau-v)}\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)} \lim_{T\to\infty}\int_{-\frac{T}{2}+u}^{\frac{T}{2}+u} x(w)\overline{x(w+u-\tau-v)}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{\infty}^{\infty} h(u)\overline{h(v)}R_x(v+\tau-u)\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{-\infty}^{\infty} (R_x(v+\tau) * h(v+\tau))\overline{h(v)}\,\mathrm{d}u\,\mathrm{d}v \\
&= R_x(\tau) * h(\tau) * \overline{h(-\tau)}
\end{aligned}
$$

Which implies that

$$\mathcal{S}_y(f) = \mathcal{S}_x(f)H(f)\overline{H(f)} = \mathcal{S}_x(f)|H(f)|^2$$

1.5 Sampling of bandlimited signals

$f_s = 2W$ is the **Nyquist rate** and $f_s - 2W$ is **guard band**.

1.6 Bandpass signal

A **bandpass signal** has non-zero frequencies around a small neighborhood of some high frequency f_0. That is, $X(f) = 0$ for $|f - f_0| \geq W$ where $W < f_0$. A **bandpass system** passes frequencies around some f_0 or equivalently, the impluse response is a bandpass signal. f_0 is called the **central frequency** even tho it might not be the center of signal's bandwidth.

1.6.1 Analysis of monochromatic signals

Monochromatic signals are bandpass with $W = 0$.

$$x(t) = A\cos(2\pi f_0 t + \theta)$$

The **phasor** is defined as $\hat{X} = Ae^{j\theta}$. Consider an LTI system with impluse response $H(f)$. Then, the phasor of the output of signal $x(t)$ is $\hat{Y} = AH(f_0)e^{j\theta}$ and the frequency of the output signal is the same, namely f_0. To obtain the phasor of the input consider the signal

$$
\begin{aligned}
z(t) &= Ae^{2\pi j f_0 t + j\theta} \\
&= A\cos(2\pi f_0 t + \theta) + jA\sin(2\pi f_0 t + \theta) \\
&= x(t) + jx_q(t) = x(t) + jx\Big(t - \frac{\pi}{2}\Big)
\end{aligned}
$$

where $x_q(t)$ is a 90° phase shift version of the original signal– q stands for *quadrature*. Then,

$$\hat{X} = z(t)e^{-2\pi j f_0 t}$$

Note that, $Z(f)$ can be obtained from $X(f)$ by deleting the negative frequencies and multiplying the positive frequencies by a factor of two.

1.6.2 Analysis of a general bandpass signal

For a general bandpass signal, let $Z(f)$ be the signal obtained from deleting the negative frequencies of $X(f)$ and multiplying the positive frequencies by a factor of two. That is,

$$Z(f) = 2U_{-1}(f)X(f)$$

where $U_{-1}(f)$ is the Heaviside step function. $z(t)$ is called the **analytic signal corresponding to** $x(t)$ or the **pre-envelope of** $x(t)$. The inverse Fourier of $U_{-1}(f)$ is calculated as follows

$$\begin{aligned}
\mathcal{F}^{-1}\{U_{-1}(f)\} &= \mathcal{F}\{U_{-1}(-\tau)\}(t) \\
&= \mathcal{F}\{1 - U_{-1}(\tau)\}(t) \\
&= \delta(t) - \left(\frac{1}{2\pi jt} + \frac{1}{2}\delta(t)\right) \\
&= \frac{1}{2}\delta(t) - \frac{1}{2\pi jt} \\
&= \frac{1}{2}\delta(t) + \frac{j}{2\pi t}
\end{aligned}$$

Therefore,

$$\begin{aligned}
z(t) &= x(t) * \left(\delta(t) + \frac{j}{\pi t}\right) \\
&= x(t) + jx(t) * \frac{1}{\pi t} \\
&= x(t) + jx'(t)
\end{aligned}$$

$x'(t)$ is called the **Hilbert transform of** $x(t)$. Hilbert transform, as derived below, is equivalent to a $-\frac{\pi}{2}$ shift for positive frequencies and a $\frac{\pi}{2}$ shift for negative frequencies.

$$\mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\,\mathrm{sign}(f) = e^{-j\frac{\pi}{2}\,\mathrm{sign}(f)}$$

$H(f) = -j\,\mathrm{sign}(f)$ is called the **quadrature filter**. Then, consider the signal $x_l(t) = z(t)e^{-2\pi jf_0t}$ or equivalently $X_l(f) = Z(f + f_0)$ wher f_0 is the centeral frequency of $x(t)$. $x_l(t)$ is called the **lowpass representation of bandpass signal** $x(t)$. In general $x(t)$ is a complex-valued signal, hence we can decompose it into real and imaginary parts

$$x_l(t) = x_c(t) + jx_s(t)$$

$x_c(t)$ is called the **in-phase** and $x_s(t)$ is called the the **quadrature** components of $x(t)$. Then,

$$\begin{aligned}
z(t) &= x_l(t)e^{2\pi jf_0t} \\
&= (x_c(t)\cos(2\pi f_0t) - x_s(t)\sin(2\pi f_0t)) + j(x_c(t)\sin(2\pi f_0t) + x_s(t)\cos(2\pi f_0t))
\end{aligned}$$

hence

$$\begin{aligned}
x(t) &= x_c(t)\cos(2\pi f_0t) - x_s(t)\sin(2\pi f_0t) \\
x'(t) &= x_c(t)\sin(2\pi f_0t) + x_s(t)\cos(2\pi f_0t)
\end{aligned}$$

these two equations are called the **bandpass to lowpass transformations**.

Define the **envelope** of $x(t)$, $V(t)$, as

$$V(t) = \sqrt{(x_c(t))^2 + (x_s(t))^2}$$

and the **phase** of $x(t)$, $\Theta(t)$, as

$$\Theta(t) = \arctan \frac{x_s(t)}{x_c(t)}$$

Then,

$$\begin{aligned} x_l(t) &= V(t)e^{j\Theta(t)} \\ z(t) &= V(t)e^{2\pi j f_0 t + j\Theta(t)} \\ x(t) &= V(t)\cos(2\pi f_0 t + \Theta(t)) \\ x'(t) &= V(t)\sin(2\pi f_0 t + \Theta(t)) \end{aligned}$$

Chapter 2

Random Signal Theory

2.1 Introduction

Marcum Q function

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp\left(-\frac{z^2}{2}\right)$$

and if $X \sim \mathrm{N}(\mu, \sigma^2)$ then

$$\mathbb{P}(X > a) = Q\left(\frac{a-\mu}{\sigma}\right)$$

We have the following upper bounds

$$Q(x) \leq \frac{1}{2} e^{-x^2/2}$$

and

$$Q(x) < \frac{1}{\sqrt{2\pi} x} e^{-x^2/2}$$

for all $x \geq 0$. For lower bound

$$Q(x) > \frac{1}{\sqrt{2\pi} x}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2}$$

$\bar{X}$ has multivariate Gaussian distribution with mean μ and covariance matrix Σ

$$f_{\bar{X}}(\bar{x}) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(\bar{x}-\mu)^T \Sigma^{-1} (\bar{x}-\mu)\right)$$

2.2 Random Process

A set of indexed random variables $\{X_t\}_{t \in T}$ is a random process. We denote a random process as $X(t, \omega)$ where $\omega \in \Omega$ and $t \in T = \mathbb{R}$. For a specific ω_0, $X(t, \omega_0) = x_0(t)$ is a time function called **member function**, **sample function**, or a **realization function**. The totality of all sample functions is called an **ensemble**. For a specific t_0, $X(t_0, \omega)$ is a random variable.

Definition: A process $X(t)$ is described by its M_{th} **order statistics** if for all $m \leq M$ and all $(t_1, \ldots, t_m) \in \mathbb{R}^m$ the joint PDF of $(X(t_1), \ldots, X(t_n))$ is given.

2.2.1 Statistical averages

The mean of an stochastic process

$$\mu_X(t) = \mathbb{E}[X(t)]$$

The autocorrelation of an stochastic process

$$R_{XX}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$$

Statistical averages are also called ensemble averages.

2.2.2 Stationarity in wide sense

A stochastic process with constants mean and time invariant autocorrelation is called **stationary in wide sense**.

$$\mu_X(t) = K, \qquad R_{XX}(t_1 + t, t_2 + t) = R_{XX}(t_1, t_2)$$

for all t_1, t_2, t.

Definition: A random process $X(t)$ with mean $\mu_X(t)$ and autocorrelation $R_{XX}(t + \tau, t)$ is called **cyclostationary** if both the mean and autocorrelation are periodic in t with some period T_0, that is

$$\mu_X(t + T_0) = m_X(t)$$

and

$$R_{XX}(t + \tau + T_0, t + T_0) = R_{XX}(t + \tau, t)$$

for all t and τ.

2.2.3 Time averages

The time average mean and autocorrelation of a random process is defined as

$$\langle \mu_X \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)\,\mathrm{d}t$$
$$\langle R_{XX}(\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t)X(t+\tau)\,\mathrm{d}t$$

Both time averages are random variables and depend on ω. Ensemble averages and time averages are equal in mean squared sense. A random variable X is equal to a constant b in MS sense if $\mathbb{E}[X] = b$ and $\mathbb{E}[(X - b)^2] = 0$.

2.2.4 Ergodicity

A wide-sense stationary process is **ergodic in mean** if $\langle \mu_X \rangle$ converges to μ_X in mean squared as $T \to \infty$. A wide-sense stationary process is **ergodic in autocorrelation** if $\langle R_{XX}(\tau) \rangle$ converges to $R_{XX}(\tau)$ in mean squared as $T \to \infty$.

2.2.5 Power Spectral density of stationary random process

We can define the random variables for energy $\mathscr{E}_X$ and power $\mathscr{P}_X$

$$\mathscr{E}_X = \int_{-\infty}^{\infty} X^2(t)\,\mathrm{d}t$$
$$\mathscr{P}_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)\,\mathrm{d}t$$

Then, the power content $\mathcal{P}_X$ and energy content $\mathcal{E}_X$ of a stochastic process $X(t)$ are defined as

$$\mathcal{E}_X = \mathbb{E}[\mathscr{E}_X] = \int_{-\infty}^{\infty} R_{XX}(t,t)\,\mathrm{d}t$$
$$\mathcal{P}_X = \mathbb{E}[\mathscr{P}_X] = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_{XX}(t,t)\,\mathrm{d}t$$

Foe stationary processes if $\mathcal{E}_X < \infty$, then $R_X X(0) = 0$ and hence $X(t)$ is zero almost everywhere.

Let $X(t)$ be a random process and $X_T(f)$ be the random process from considering the Fourier transforms of truncated sample functions. That is,

$$X_T(f,\omega) = \mathcal{F}\{x_T(t,\omega)\}$$

Then, the **power density spectrum** or **power spectral density** is defined as

$$\mathcal{G}_X(f) = \lim_{T\to\infty} \frac{\mathbb{E}\left[|X_T(f)|^2\right]}{T}$$

Furthermore,

$$\langle\mathcal{G}_X(f)\rangle = \lim_{T\to\infty} \frac{\left|\int_{-T/2}^{T/2} X(t)\exp(-2\pi j f t)\,\mathrm{d}t\right|^2}{T}$$

For an ergodic random process

$$\langle\mathcal{G}_X(f)\rangle \overset{\text{MS}}{=} \mathcal{G}_X(f)$$

Theorem 2.1 (Wiener-Khinchin). *If for all finite τ and any interval I of length $|\tau|$, the autocorrelation function R_{XX} satisfies the condition*

$$\left|\int_I R_{XX}(t+\tau,t)\,\mathrm{d}t\right| < \infty$$

Then,

$$\mathcal{G}_X(f) = \mathcal{F}\left\{\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_{XX}(t+\tau,t)\,\mathrm{d}t\right\}$$

Thus, if $X(t)$ is stationary with

$$\int_{-\infty}^{\infty} |R_{XX}(\tau)|\,\mathrm{d}\tau < \infty$$

then

$$\mathcal{G}_X(f) = \int_{-\infty}^{\infty} R_{XX}(\tau) \exp(-2\pi j f \tau) \, \mathrm{d}\tau$$

If $X(t)$ is cyclostationary with

$$\left| \int_0^{T_0} R_{XX}(t+\tau, t) \, \mathrm{d}t \right| < \infty$$

then

$$\mathcal{G}_X(f) = \mathcal{F}\big\{\bar{R}_{XX}(\tau)\big\}$$

where

$$\bar{R}_{XX}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} R_{XX}(t+\tau, t) \, \mathrm{d}t$$

Moreover, $\mathbb{E}\big[(X(t))^2\big]$ can be thought of as the average power dissipated by random process across 1 ohm resistor.

$$\mathbb{E}\big[(X(t))^2\big] = R_{XX}(0) = \int_{-\infty}^{\infty} \mathcal{G}_X(f) \, \mathrm{d}f$$

1. $\langle X(t) \rangle$ is the DC component.
2. $\big\langle (X(t))^2 \big\rangle$ is the total average power.
3. $\langle (X(t)) \rangle^2$ is the DC power.
4. $\big\langle (X(t))^2 \big\rangle - \langle (X(t)) \rangle^2$ is the AC power.
5. $\sqrt{\big\langle (X(t))^2 \big\rangle - \langle (X(t)) \rangle^2}$ is rms value.

2.2.6 PSD of a sum process

Suppose $Z(t) = X(t) + Y(t)$ is the sum of two jointy stationary process. It can be readily verified that $Z(t)$ is stationary and

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

and

$$\mathcal{G}_Z(f) = \mathcal{G}_X(f) + \mathcal{G}_Y(f) + 2\Re(\mathcal{G}_{XY} f)$$

where

$$\mathcal{G}_{XY}(f) = \mathcal{F}\{R_{XY}(\tau)\}$$

If $X(t)$ and $Y(t)$ are uncorrelated and at least one of them is zero mean, then $R_{XY}(\tau) = 0$ and

$$\mathcal{G}_Z(f) = \mathcal{G}_X(f) + \mathcal{G}_Y(f)$$

2.3 Systems and random signals

The response of a system to a random signal, is a random signal itself. Therefore, we need tools to investigate the relationships between to random processes.

Definition: Two random process $X(t)$ and $Y(t)$ are **independent** if for all t_1, t_2, the random variables $X(t_1)$ and $Y(t_2)$ are independent. Similarly, $X(t)$ and $Y(t)$ are **uncorrelated** if for all t_1, t_2, the random variables $X(t_1)$ and $Y(t_2)$ are uncorrelated.

Definition: The **cross-correlation** function between two random process $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)]$$

Two processes $X(t)$ and $Y(t)$ are **jointly wide-sense stationary**, if both are stationary and the cross-correlation function depends on $\tau = t_1 - t_2$.

2.3.1 Response of memoryless channel

$$Y(t) = g(X(t))$$

2.3.2 Response of LTI System

Suppose stationary process $X(t)$ is passed through a LTI system with impulse response $h(t)$. Then, the input and output prcesses $X(t)$ and $Y(t)$ are jointly stationary. Moreover,

$$\begin{aligned}
Y(t) &= X(t) * h(t) \\
&= \int_{-\infty}^{\infty} X(\tau)h(t-\tau)\,\mathrm{d}\tau \\
\implies \mathbb{E}[Y(t)] &= \int_{-\infty}^{\infty} \mathbb{E}[X(\tau)]h(t-\tau)\,\mathrm{d}\tau \\
\mathbb{E}[Y(t)] &= \mu_X \int_{-\infty}^{\infty} h(\tau)\,\mathrm{d}\tau = \mu_X H(0)
\end{aligned}$$

and

$$\begin{aligned}
R_{YX}(\tau) &= \mathbb{E}[Y(t)X(t-\tau)] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} X(\eta)h(t-\eta)X(t-\tau)\,\mathrm{d}\eta\right] \\
&= \int_{-\infty}^{\infty} R_{XX}(t-\tau-\eta)h(t-\eta)\,\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty} R_{XX}(\eta-\tau)h(\eta)\,\mathrm{d}\eta \\
&= \int_{-\infty}^{\infty} R_{XX}(\tau-\eta)h(\eta)\,\mathrm{d}\eta \\
&= R_{XX}(\tau) * h(\tau) \\
\implies R_{YY}(\tau) &= \mathbb{E}[Y(t+\tau)Y(t)] \\
&= \mathbb{E}\left[\int_{-\infty}^{\infty} Y(t+\tau)X(t-\eta)h(\eta)\,\mathrm{d}\eta\right]
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} R_{YX}(\tau+\eta)h(\eta)\,\mathrm{d}\eta \\
&= R_{YX}(\tau) * h(-\tau) \\
&= R_{XX}(\tau) * h(\tau) * h(-\tau) \\
\implies \mathcal{G}_X(f) &= \mathcal{F}\{R_{YY}(t)\} = \mathcal{G}_X(f)|H(f)|^2
\end{aligned}
$$

2.3.3 Special classes of random processes

Gaussian random process

$X(t)$ is Gaussian if

$$
f_X(x_1,\dots,x_n;t_1,\dots,t_n) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\left(-\frac{1}{2}(\bar{x}-\bar{\mu})^T \Sigma^{-1} (\bar{x}-\bar{\mu})\right) \qquad \forall n, \forall t_1,\dots,t_n
$$

where $\Sigma = \left[\mathbb{E}[(X(t_i)-\mathbb{E}[X(t_i)])(X(t_j)-\mathbb{E}[X(t_j)])]\right]_{i,j}$ $\bar{\mu} = \left[\mathbb{E}[X(t_i)]\right]_i$.

$X(t)$ is zero mean stationary Gaussian if $\mathbb{E}[X(t)] = 0$ and $R_{XX}(t_1+t, t_2+t) = R_{XX}(t_1,t_2)$ for all t_1, t_2, t. Then, $\Sigma = \left[\mathbb{E}[R_{XX}(t_i,t_j)]\right]_{i,j}$ and $\bar{\mu} = \bar{0}$.

Theorem 2.2. *For a Gaussian process, $\mu_X(t)$ and $R_{XX}(t_1,t_2)$ gives a complete statistical description of the process.*

Corollary 2.3. *For Gaussian processes, WSS and strictly stationarity are equivalent.*

Theorem 2.4. *The output of an LTI system on a Gaussian input is Gaussian.*

Theorem 2.5. *A sufficient condition for the ergodicity of the stationary zero-mean Gaussian process is*

$$
\int_{-\infty}^{\infty} |R_{XX}(\tau)|\,\mathrm{d}\tau < \infty
$$

Markoff Sequence

Suppose $X(t)$ is defined for countable indices and it assumes a finte set of values. That is, $X(t)$ is discrete-time and discrete-amplitude. $X(t)$ is a Markoff chain if

$$
\mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1},\dots,X_1 = a_1) = \mathbb{P}(X_n = a_n \mid X_{n-1} = a_{n-1})
$$

Let $p_i(n) = \mathbb{P}(X_n = a_i)$ and $p_{i,j}(n,m) = \mathbb{P}(X_n = i \mid X_m = j)$ for $n > m$ then

$$
p_j(n) = \sum_{i=1}^{N} p_{j,i}(n,m)p_i(m)
$$

If $p_{i,j}(n+1,n) = p_{i,j}(n,n-1)$ for all n, then $\{X_n\}$ is called **homogeneous**. Finally, let

$$
P(k) = \begin{bmatrix} p_1(k) \\ \vdots \\ p_N(k) \end{bmatrix} \qquad \varphi = \begin{bmatrix} p_{1,1}(n,n-1) & \dots & p_{1,N}(n,n-1) \\ \vdots & \ddots & \vdots \\ p_{N,1}(n,n-1) & \dots & p_{N,N}(n,n-1) \end{bmatrix}
$$

Therefore

$$
P(k) = \varphi P(k-1)
$$

A Markov chain is stationary if $P(k+1) = P(k)$ for all k.

2.4 Noise in communication systems

Determined through expriments (thermodynamics and quantum mechanic) noise voltage $V(t)$ that appears accross the terminal of a resistor of R Ohms has a Gaussian distribution with $\mu_V = 0$ and

$$\mathbb{E}\big[V^2(t)\big] = \frac{(2\pi kT)^2}{3h}R$$

where k is Boltzmann constant, h Plank constant, T is temperature in Kelvins. Then,

$$\mathcal{G}_V(f) = \frac{2Rh|f|}{\exp(h|f|/(kT)) - 1}$$

$\mathcal{G}_V(f)$ is flat over $|f| < 0.1\frac{kT}{h}$. However, for modeling

$$\mathcal{G}_V(f) = 2RkT$$

but in this case $\mathbb{E}[V^2(t)] = \infty$. Yet it is alright since $V(t)$ is subjected to filtering and hence $\mathbb{E}[V^2(t)]$ will be finite.

Definition: A noise signal having flat power spectral density over a wide range frequency is called white noise.

$$\mathcal{G}_V(f) = \frac{\eta}{2}$$

The factor $1/2$ is included to indicate that $\mathcal{G}_V(f)$ is a two-sided psd.

At room temperature, $\mathcal{G}_V(f)$ drops to 90% of its maximum at about $f \approx 2 \times 10^12$ Hertz, which is beyond the frequencies employed in the conventionaly communication systems.

Available power is the maximum power that can be delivered to a load from a source having a fixed but non-zero resistence.

$$P_{\max} = \frac{\mathbb{E}[V^2(t)]}{4R}$$

Available power psd is $\mathcal{G}_V(f) = kT/2$.

We will assume the thermal noist is stationary, ergodic, zero-mean, white Gaussian noise whose power spectrum is $N_0/2$ where $N_0 = kT$.

Chapter 3

Information Theory

Remark 3. For a more complete and through treatment refer to the notes on the subject.

3.1 Measure of information

Information content of a message is inversely proportional to the likelihood of that message. Let $m_1, m_2, \ldots, m_q$ be q messages with probability $p_1, p_2, \ldots, p_q$ respectively, such that $p_1 + \cdots + p_q = 1$. Then, information content of m_k, $I(m_k)$ must satisfy the followings

1. $I(m_k) > I(m_j)$ if $m_k < m_j$.
2. $I(m_k) \to 0$ as $p_k \to 0$.
3. $I(m_k) \geq 0$ when $0 \leq p_k \leq 1$.

Furthermore, for two independent messages m_1 and m_2

$$I(m_1, m_2) = I(m_1) + I(m_2)$$

One continuous function that satisfies these requirements is $I(m_k) = -\log p_k$ where the base of the logarithm determines the unit of information, e.g. base e is nats, 2 is bit, 10 is Hartley/decit.

3.1.1 Average information content

For a statistically independent source that emits N symbols from a M-symbol alphabet i.i.d.

$$I_{tot} = -N \sum_{i=1}^{M} p_i \log(p_i)$$

$$H = \frac{I_{tot}}{N} = -\sum_{i=1}^{M} p_i \log(p_i)$$

Proposition 3.1. *For a source with an M-symbol alphabet, the maximum entropy is attained when the symbols are equiprobabilistic and $H_{max} = \log M$.*

Suppose r_s is the symbol rate of the source, measured in . Then, average information rate R is

$$R = r_s H$$

3.1.2 Statistically dependent source

- emits a symbol once every T_s seconds.
- A stochastic process.
- stationary Markoff.
- There are n states with a transition matrix φ, (M symbols with a residual influence lasting q symbols can be represented by $n \leq M^q$ states).
- $\mathbb{P}(X_k = s_q \mid X_1, \ldots, X_{k-1}) = \mathbb{P}(X_k = s_q \mid S_k)$ where S_k is a discrete random variable.
- At the begninning it is in one of the n states with probability $p_i(1)$.
- $p_j(k+1) = \sum_{i=1}^{n} p_i(k) p_{ij}$ for all j. $P(k+1) = \varphi P(k)$.

3.1.3 Entropy for Markov source

- Assume, discrete finite state ergodic hence stationary.
- Entropy of state i is
$$H_i = -\sum_{j=1}^{M} p_{ij} \lg p_{ij}$$
- Entropy of source
$$H = \sum_{i=1}^{n} p_i H_i$$
- therefore, $R = r_s H$.

Theorem 3.2. *Let $G_N = -\frac{1}{N} \sum_i p(m_i) \lg p(m_i)$ over all messages of length N. Then, G_N is monotonice decreasing function of N and*

$$\lim_{N \to \infty} G_N = H$$

3.2 Source encoding

Definition: The ratio of sourse information and the average encoded output bit rate is called *coding efficiency.*

3.2.1 Shannon Algorithm

Let $m_1, \ldots, m_q$ be arranged in decreasing order of probability $p_1 \leq \cdots \leq p_q$. Let $F_i = \sum_{k=1}^{i-1} p_k$ with $F_1 = 0$. Let $n_i = \lceil -\lg p_i \rceil$. Then, the code

$$c_i = (F_i)_2 \qquad \text{binary fraction of } F_i \text{ up to } n_i \text{ bits.}$$

has the following properties

1. $l(c_i) > l(c_j) \implies p_i < p_j$.

2. Codewords are all differnt. In fact, it is an instantaneous code.
3. $G_N \leq \hat{H_N} < G_N + \frac{1}{N}$.
4. The efficiency rate is $e = \frac{H}{\hat{H_n}}$.

Important parameters in design od encoder/decoder

- rate efficiency
- complexity of design
- effects of error

3.3 Communication channel

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3.4 Discrete communication channel

Consider a discrete memoryless channel. Then, the channel may be described with conditional probability $p(y|x)$. The average information rate is $D_i n = r_s H(X)$ and the average rate of information transmission is

$$D_t = (H(X) - H(X|Y))r_s = r_s I(X;Y)$$

The capacity of the channel is defined as $C = \max_{p(x)} D_t$.

Theorem 3.3. *Let C be the capacity and H be the entropy. If $r_s H \leq C$, then there exists an enconding scheme such that the output of the source can be transmitted over channel with an arbitrary small probability of error. Conversely, it is not possible to transmit information at a rate exceeding C without a positive frequency.*

Remark 4. with memory and Gilbert

3.5 Continuous channels

Remark 5. additive and multiplicative noise

- Modulator and demodulator are techniques to reduce guassian noise effect.
- Impulse noise are modeled in the discrete portion.

Theorem 3.4 (Shannon-Hartley theorem). *The capacity of a channel with bandwidth B and additive guassian band-limited white noise is*

$$C = B \lg\left(1 + \frac{S}{N}\right)$$

where S and N are the average signal power and noise power at the output channel. $N = \eta B$ if two sided spectral density of the noise is $\frac{\eta}{2}$.

Implications

1. Gives an upperlimit that can be reached
2. Exchange of S/N for bandwidth.
3. Bandwidth compression.
4. Noiseless channel has infinite capacity. For noisy channels, as bandwidth increases because the noise power increases as well, the capacity approaches a limit.

Communication at transmitting information rate of $B \lg(1 + S/N)$ is called *ideal*.

1. Most physical channels are approximately gaussian.
2. Guassian noise provides a lowerbound performace for all other types.

Remark 6. CRT

Chapter 4

Baseband Data Transmission

Remark 7. Most efficient: PAM,PDM,PPM.

–add image of page 43
r_b is bit rate, T_b is bit duration.

$$X(t) = \sum_{k=-\infty}^{\infty} a_k p_g(t - kT_b)$$

where we can assume that

$$p_g(0) = 1 \qquad\qquad a_k = \begin{cases} -a & d_k = 0 \\ a & d_k = 1 \end{cases}$$

Then,

$$Y(t) = \sum_{k=-\infty}^{\infty} A_k p_r(t - t_d - kT_b) + n_0(t)$$

where $A_k = K_c a_k$, K_c is the normalizing constant that yields $p_r(0) = 1$, and $K_c p_r(t - t_d)$ is the response of the system to $p_g(t)$.

Remark 8. Equalizing filter.

$Y(t)$ is sampled at rate of $t_m = mT_b + t_d$ and m_{th} is generated by comparing $Y(t_m)$ to some threshold.

$$Y(t) = A_m + \underbrace{\sum_{k\neq m} A_k p_r((m-k)T_b)}_{\text{Intersymbol Interference}} + \underbrace{n_0(t_m)}_{\text{channel noise}}$$

The goals are

- minimize errors introduced by noise and ISI.
- maximize r_b for a given bandwidth.
- minimize bandwidth for a given r_b.

4.1 Baseband binary PAM system

For design purposes we will assume that input data rate, overall bit error probability, characteristics of the channel are given. Channel noise is modeled by a AWGN with known spectral density $\mathcal{G}_n(f)$. Source output is assumed be equiprobable sequence of independent bits.

Remark 9. PAM: specify pulse shapes, $p_g(t), p_r(t), H_R(f), H_T(f)$.

4.1.1 Baseband pulse shaping

IN the equation for $Y(t)$, to remove ISI we must have

$$p_r(nT_b) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

Theorem 4.1. *If $P_r(f)$ satisfies Nyquist criterion.*

$$\sum_{k=-\infty}^{\infty} P_r\left(f + \frac{k}{T_b}\right) = T_b \qquad\qquad \text{for } |f| \leq \frac{1}{2T_b}$$

then

$$p_r(nT_b) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

hence, ISI can be removed if and only if bandwidth of P_r, is $|f| > \frac{r_b}{2}$. ??

In practical systems for r_b rate the available bandwidth is between $\frac{r_b}{2}$ to r_b Hz. and a class of $P_r(f)$ with a this bandwidth are **raised cosine frequence** which are commonly used. (parameter β)

1. Bandwidth $= r_b(2) + \beta$.

2. larger β implies faster decaying pules, hence less ISI due to timing errors.

3. $P_r(f)$ is real, non-negative and $\int_{-\infty}^{\infty} P_r(f)\, \mathrm{d}f = 1$.

4. $\beta = 0$ produces zero ISI at a data rate of r_b.

5. Practically impossible since time response must be zero prior to a time $t_0 > 0$. However, a delayed version $p_r(t - t_d)$ may be chose so that $p_r(t - t_d) = 0$ for $t < t_0$.

6. One may want to use the whole bandwidth to get a faster decay of $p_r(t)$.

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4.1.2 Optimum transmitting and receiving filter

A design constraint

$$P_g(f)H_T(f)H_c(f)H_R(f) = K_c P_r(f)e^{-2\pi jft_d}$$

where P_g, H_c, and P_r are assumed to be known. If P_r is chosen to have zero ISI, then P_g is a delay version of P_r. Therefore, we need to choose H_T and H_R such that thee effect of noise is minimized. Lets derive the probability of error.

$$Y(t_m) = A_m + n_0(t_m)$$

where $n_0(t_m) \sim \mathrm{N}(0, N_0)$. The threshold is assumed to be zero.

Remark 10. To minimize error threshold should be set $\frac{N_0}{2A}\ln\left(\frac{\mathbb{P}(d_m=0)}{\mathbb{P}(d_m=1)}\right)$.

$$\begin{aligned} P_e = \mathbb{P}\left(\hat{d} \neq d\right) &= \mathbb{P}(Y(t_m) > 0 \mid d_m = 0)\mathbb{P}(d_m = 0) + \mathbb{P}(Y(t_m) < 0 \mid d_m = 1)\mathbb{P}(d_m = 1) \\ &= \frac{1}{2}[\mathbb{P}(n_0(t_m) < -A) + \mathbb{P}(n_0(t_m) > A)] \\ &= \frac{1}{2}\mathbb{P}(|n_0(t_m)| > A) \end{aligned}$$

Since $n_0(t_m)$ is assumed to be zero mean gaussian at the input to $H_R(f)$, then

$$N_0 = \int_{-\infty}^{\infty} \mathcal{G}_n(f)|H_R(f)|^2 \,\mathrm{d}f$$

Hence

$$\begin{aligned} P_e &= \frac{1}{2}\int_{|x|>A} \frac{1}{\sqrt{2\pi N_0}} \exp\left(-\frac{x^2}{2N_0}\right) \mathrm{d}x \\ &= \Phi\left(\frac{A}{\sqrt{N_0}}\right) = 1 - Q\left(\frac{A}{\sqrt{N_0}}\right) \end{aligned}$$

Thus to minimize P_e we should maximize $\frac{A}{\sqrt{N_0}}$. To do this, we express $\frac{A^2}{N_0}$ in terms of H_T and H_R. Recall

$$X(t) = \int_{k=-\infty}^{\infty} a_k p_g(t - kT_b)$$

equiprobable and independent imples $X(t)$ is a random waveform with psd

$$\begin{aligned} \mathcal{G}_X(f) &= \frac{|P_g(f)|^2}{T_b}\mathbb{E}\left[a_k^2\right] \\ &= a^2\frac{|P_g(f)|^2}{T_b} \end{aligned}$$

psd of the transmitted signal is

$$\mathcal{G}_Z(f) = |H_T(f)|^2\mathcal{G}_X(f)$$

The average power

$$\mathcal{S}_T = \frac{a^2}{T_b}\int_{-\infty}^{\infty} |P_g(f)|Y2|H_T(f)|^2 \,\mathrm{d}f$$

setting $A_k = k_c a_k$ and $A = K_c a$

$$\mathcal{S}_T = \frac{A^2}{K_c^2 T_b} \int_{-\infty}^{\infty} |P_g(f)|Y2|H_T(f)|^2 \, \mathrm{d}f$$

Let $I = \int_{-\infty}^{\infty} |P_g(f)|Y2|H_T(f)|^2 \, \mathrm{d}f$ and $N_0 == \int_{-\infty}^{\infty} \mathcal{G}_n(f)|H_R(f)|^2 \, \mathrm{d}f$

$$A^2 = K_c^2 T_b S_T I^{-1}$$

thus

$$\frac{A^2}{N_0} = \frac{K_c^2 T_b S_T}{IJ}$$

hence we should minimize $\gamma^2 = IJ$

Part II

Signals and Systems

Part III

Basics

Chapter 5

Introduction

Signals are functions if independent variables that carry information. It can be continuous or discrete and be multi-dimensional. A system responds to applies input signals, and its response is described in terms of one or more output signals. A continuous time system receives and gives continuous signals and discrete time system receives and gives discrete signals. Systems can be connected together in series, parallel, or feedback loop.

5.1 Signal Properties

Definition: The energy of a continuous signal over interval $[t_1, t_2]$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 \, \mathrm{d}t$$

and similarly for a discrete signal over interval $n_1 \leq n \leq n_2$

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

The power of signal is the time averaged energy of that signal. The total energy of signal is defined to be

$$E_\infty = \lim_{T\to\infty} \int_{-T} T|x(t)|^2 \, \mathrm{d}t$$

for a continuous time signal and

$$E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2$$

Lastly, the total power of a signal is

$$P_\infty = \lim_{T\to\infty} \frac{1}{T} \int_{-T} T|x(t)|^2 \, \mathrm{d}t$$

and

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

5.2 System Properties

Memoryless A system is memoryless if it depends only on the present input. On the other hand, a system has memory if it depends on present and past values. For example, *accumulator* and *delay* are two such systems.

$$y[n] = \sum_{k=-\infty}^{n} x[n], \qquad y[n] = x[n-1]$$

Invertible A system is invertible if there exists a system $y(t) \to w(t)$ such that $w(t) = x(t)$, for all t.

Causal A system is casual or *nonancitipative* if it depends on past and present value. Mathematicall, a system $x(t) \to y(t)$ is causal if

$$x_1(t) = x_2(t), \quad \forall t \leq t_0 \implies y_1(t) = y_2(t), \quad \forall t \leq t_0$$

Stability Informally means that small change in the input does not converge. That is, bounded input results in bounded output.

Time invariant Time shift input results in an identical time shift in the output signal.

$$x(t) \to y(t) \implies x(t-t_0) \to y(t-t_0)$$

Linearity A system is linear if

$$ax_1(t) + x_2(t) \to ay_1(t) + y_2(t), \quad \forall a \in \mathbb{C}$$

Proposition 5.1. *A linear system is causal if and only if it satisfies the condition of initial rest*

$$x(t) = 0, \quad \forall t \leq t_0 \implies y(t) = 0, \quad \forall t \leq t_0$$

Proof. Consider a linear system such that the responses to $x_1(t)$ and $x_2(t)$ are $y_1(t)$ and $y_2(t)$, respectively. Suppos this system is causal. By linearity,$x_2(t) = 0 \implies y_2(t) = 0$ and hence

$$x_1(t) = x_2(t) = 0, \quad \forall t \leq t_0 \implies y_1(t) = y_2(t) = 0, \quad \forall t \leq t_0$$

Suppose the system has the initial rest condition. By linearity,

$$x_1(t) - x_2(t) \to y_1(t) - y_2(t)$$

Then, if $x_1(t) = x_2(t), \quad \forall t \leq t_0$ then

$$x_1(t) = x_2(t) \implies x_1(t) - x_2(t) = 0 \implies y_1(t) - y_2(t) = 0 \implies y_1(t) = y_2(t), \quad \forall t \leq t_0 \quad \blacksquare$$

Chapter 6

Linear Time Invariant Signals

6.1 Discrete signals

Let $x[n]$ be a discrete signal then it can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

which is called *sifting property*. Consider a discrete LTI system $x[n] \to y[n]$, by sifting property

$$y[n] = x[n] * h[n]$$

where $h[n]$ is the response to $\delta[n]$. Hence a LTI system can be completely characterized by its response to $\delta[n]$.

6.2 Continuous signal

Similarly we can write the sifting propery as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau = x(t) * \delta(t)$$

and a continuous LTI system $x(t) \to y(t)$ can be written in terms of its response $h(t)$ to unit impluse $\delta(t)$

$$y(t) = x(t) * h(t)$$

6.3 Properties of convolution and LTI

For simplicity we only bring the continuous, however, the equivalent discrete form also holds.

Commutative $x(t) * y(t) = y(t) * x(t)$.

Distributive $x(t) * (y(t) + z(t)) = x(t) * y(t) + x(t) * z(t)$.

Associative $x(t) * (y(t) * z(t)) = (x(t) * y(t)) * z(t)$.

- LTI is memoryless if $h(t) = 0$ for $t \neq 0 \implies h(t) = K\delta(t) \implies y(t) = Kx(t)$.

- LTI system is invertible if there exists $g(t)$ such that $h(t) * g(t) = \delta(t)$.
- LTI system is causal if $h(t) = 0$ for $t < 0$.
- LTI is stable iff
$$\int_{-\infty}^{\infty} |h(\tau)| \,\mathrm{d}\tau < \infty$$

6.4 Singularity functions

We can view the unit impluse as a short signal that has integral of 1 and hence signals with such property are similar. Consider the LTI system

$$y(t) = \frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}$$

Then the unit impluse response is $u_n(t)$ such that

$$y(t) = x(t) * u_n(t) = (x(t) * u_{n-1}(t)) * u_1(t)$$

and therefore

$$u_n(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{n}$$

$u_1(t)$ is called the unit doublet and it is defined

$$u_1(t) = \frac{\mathrm{d}\delta(t)}{\mathrm{d}t}$$

Similarly for integral

$$u_{-1}(t) = u(t) = \int_{-\infty}^{t} \delta(\tau) \,\mathrm{d}\tau \implies x(t) * u(t) = \int_{-\infty}^{t} x(\tau) \,\mathrm{d}\tau$$

for the double integral

$$u_{-2}(t) = u_{-1}(t) u_{-1}(t) = \int_{-\infty}^{t} u_{-1}(\tau) \,\mathrm{d}\tau$$

similarly for higher order integrals

$$u_{-n}(t) = \underbrace{u_{-1}(t) * u_{-1}(t) * \cdots * u_{-1}(t)}_{n}$$

Lastly we can denote $\delta(t) = u_0(t)$ to then arrive at

$$u_r(t) * u_s(t) = u_{r+s}(t)$$

for all $u, s \in \mathbb{Z}$.

Part IV

Fourier

Chapter 7

Fourier Series Representation

7.1 Eigenfunctions

For discrete and continuous LTI systems we have

$$x(t) = e^{st} \to y(t) = H(s)e^{st},\ s \in \mathbb{C}$$
$$x[n] = z^n \to y[n] = H(x)z^n,\ z \in \mathbb{C}$$

A signal for which the output signal of a system is a constant multiple of itself is called the **eigenfunction** of the system and the constant is called the **eigenvalue**. To show the above equations (we assume the convergence of the integral and the sum at the last steps.)

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,\mathrm{d}\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,\mathrm{d}\tau = H(s)e^{st}$$
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

7.2 Fourier series representation

7.2.1 Continuous-time

Suppose $x(t)$ is a continuous-time periodic signal with period T. The value $\omega_0 = \frac{2\pi}{T}$ then we may be able to write $x(t)$ in the form of

$$x(t) = \sum_{-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

Let $x(t)$ be a real signal then $\overline{x} = x$ hence

$$\begin{aligned} x(t) &= \sum_{-\infty}^{\infty} \overline{a_k} e^{-jk\omega_0 t} \\ &= \sum_{-\infty}^{\infty} \overline{a_{-k}} e^{-jk\omega_0 t} \implies a_k = \overline{a_{-k}} \\ &= a_0 + \sum_{k=1}^{\infty} a_k e^{jk\omega_0 t} + \overline{a_{-k}} e^{-jk\omega_0 t} \end{aligned}$$

$$
\begin{aligned}
&= a_0 + 2\sum_{k=1}^{\infty} \Re\big(a_k e^{jk\omega_0 t}\big),\ \ a_k = r_k e^{j\theta_k}, a_k = b_k + jc_k \\
&= a_0 + 2\sum_{k=1}^{\infty} r_k \cos(\theta_k + k\omega_0 t) \\
&= a_0 + 2\sum_{k=1}^{\infty} b_k \cos(k\omega_0 t) - c_k \sin(k\omega_0 t)
\end{aligned}
$$

The last two equation are the **Fourier series** representation of a real periodic signal $x(t)$ – this is called synthesis in the context of Fourier series. We also have

$$
\begin{aligned}
\int_0^T x(t) e^{-jn\omega_0 t}\,\mathrm{d}t &= \int_0^T \sum_{-\infty}^{\infty} a_k e^{j(n-k)\omega_0 t}\,\mathrm{d}t \\
&= \sum_{-\infty}^{\infty} a_k \int_0^T e^{j(n-k)\omega_0 t}\,\mathrm{d}t \\
&= a_n T
\end{aligned}
$$

therefore

$$
\begin{aligned}
a_n &= \frac{\omega_0}{2\pi}\int_0^T x(t) e^{-jn\omega_0 t}\,\mathrm{d}t = \frac{\omega_0}{2\pi}\int_c^{c+T} x(t) e^{-jn\omega_0 t}\,\mathrm{d}t \\
a_0 &= \frac{\omega_0}{2\pi}\int_0^T x(t)\,\mathrm{d}t
\end{aligned}
$$

we donote $\int_c^{c+T}$ by $\int_T$.

7.2.2 Convergence

A finite enery real signal $x(t)$

$$\int_T |x(t)|^2\,\mathrm{d}t < \infty$$

has a Fourier series representation if it satisfies the *Dirichlet conditions*

1. Over any period $x(t)$ is absolutely intergrable which implies $|a_k| < \infty$.
2. In any finite interval, $x(t)$ is of bounded variation – finite number of maximas and minimas.
3. Finite number of discontinuities in any finite interval, and the discontinuities can not be infinite.

7.2.3 Discrete-time

Let $x[n]$ be a discrete signal with period N and fundamental frequence $w_0 = \frac{2\pi}{N}$ and let

$$\phi_k(n) = e^{jk\omega_0 n} \quad k \in \mathbb{Z}$$

then since

$$\phi_k(n) = \phi_{rN+k}(n)$$

we can write $x[n]$ in form of

$$x(n) = \sum_{k=\langle N \rangle} a_k \phi_k(n)$$

where $\langle N \rangle$ is a set of N consecutive integers. Finding a_k requires solving a linear system

$$a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x[n] e^{jk\omega_0 n}$$

7.3 Properties of Fourier series

Suppose $x(t)$ is a periodic signal with period T and $\omega_0 = \frac{2\pi}{T}$ then

$$x(t) \xleftrightarrow{FS} a_k$$

where a_k is its Fourier series coefficients. Let $x(t) \xleftrightarrow{FS} a_k$ and $y(t) \xleftrightarrow{FS} b_k$, then we have the following properties

Definition:

Linearity

$$\alpha x(t) + \beta y(t) \xleftrightarrow{FS} \alpha a_k + \beta b_k$$

Time shifting

$$x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} a_k$$

Time reversal

$$\alpha x(-t) \xleftrightarrow{FS} a_{-k}$$

Time scaling

$$\alpha x(ct) \xleftrightarrow{FS} \sum_{k=-\infty}^{\infty} a_k e^{jk(c\omega_0)t}$$

It does not change the coefficients, but changes the whole thing.

Multiplication

$$x(t)y(t) \xleftrightarrow{FS} c_k = \sum_{l=-\infty}^{\infty} a_l + b_{k-l}$$

Conjugate symmetry

$$\overline{x(t)} \xleftrightarrow{FS} \overline{a_{-k}}$$

Parseval's relation

$$\int_T |x(t)|^2 \, \mathrm{d}t = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Period convolution

$$\int_T x(t)y(t-\tau) \, \mathrm{d}\tau \xleftrightarrow{FS} T\alpha a_k \beta b_k$$

Frequency shifting

$$e^{jM\omega_0 t}x(t) \overset{FS}{\longleftrightarrow} a_{k-M}$$

Differentiation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \overset{FS}{\longleftrightarrow} jk\omega_0 \alpha a_k$$

and for discrete-time signals

$$x[n] - x[n-1] \overset{FS}{\longleftrightarrow} \left(1 - e^{-jk\omega_0}\right) a_K$$

Integeration

$$\int_{\infty}^{t} x(\tau)\,\mathrm{d}\tau \overset{FS}{\longleftrightarrow} \frac{a_k}{jk\omega_0}$$

7.4 LTI systems

Consider a periodic signal $x(t)$ with

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

then for a LTI system $x(t) \to y(t)$ with $\delta(t) \to h(t)$

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} a_k \left(e^{jk\omega_0 t} * h(t)\right) \\ &= \sum_{k=-\infty}^{\infty} a_k \left(\int_{-\infty}^{\infty} e^{jk\omega_0 (t-\tau)} h(\tau)\,\mathrm{d}\tau\right) \\ &= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \left(\int_{-\infty}^{\infty} e^{-jk\omega_0 \tau} h(\tau)\,\mathrm{d}\tau\right) \\ &= \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \end{aligned}$$

For a discrete time system $\delta[n] \to h[n]$

$$\begin{aligned} y(n) &= \sum_{k=\langle N\rangle} a_K \left(e^{jk\omega_0 n} * h(n)\right) \\ &= \sum_{k=\langle N\rangle} a_K \left(\left(e^{jk\omega_0}\right)^n * h(n)\right) \\ &= \sum_{k=\langle N\rangle} a_K \left(\sum_{m=-\infty}^{\infty} e^{jk\omega_0 (n-m)} h(m)\right) \\ &= \sum_{k=\langle N\rangle} a_K e^{jk\omega_0 n} \left(\sum_{m=-\infty}^{\infty} e^{-jk\omega_0 m} h(m)\right) \\ &= \sum_{k=\langle N\rangle} a_K H\left(e^{jk\omega_0}\right) e^{jk\omega_0 n} \end{aligned}$$

7.5 Filtering

LTI systems the change frequency spectrum are called frequency-shaping filters. Systems that pass, eliminate, or attunate some frequencies are called frequency selective filters. Three types of frequency selective filters include

7.5.1 Lowpass filter

Passes low frequencies around zero and attunates or eliminates, in the ideal case, high frequencies.

$$|H|j\omega = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

–insert diagram

7.5.2 Bandpass filter

Passes frequencies around certian frequncy and attunates or eliminates, in the ideal case, other frequencies.

$$|H|j\omega = \begin{cases} 1 & \omega_{c_1} \leq |\omega| \leq \omega_{c_2} \\ 0 & \text{otherwise} \end{cases}$$

–insert diagram

7.5.3 High pass filter

Passes high frequencies around zero and attunates or eliminates, in the ideal case, low frequencies.

$$|H|j\omega = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

–insert diagram

Example 7.1. Take a simple RC circuit with input $V_s(t)$. We have

$$\begin{cases} V_R(t) + V_C(t) = V_s(t) \\ V_R(t) = Ri(t) = RC\frac{\mathrm{d}V_C(t)}{\mathrm{d}t} \end{cases}$$

Since it is an LTI system, for input $V_s(t) = e^{j\omega t}$ and output $V_C(t) = H(j\omega)$ we have

$$RCH(j\omega)j\omega e^{j\omega t} + H(j\omega)e^{j\omega} = e^{j\omega}$$
$$\implies H(j\omega) = \frac{1}{RCj\omega + 1}$$

–insert diagram which is non-ideal lowpass filter. For output $V_R(t) = H(j\omega)$ we have

$$RCH(j\omega)j\omega e^{j\omega t} + H(j\omega)e^{j\omega} = RCj\omega e^{j\omega}$$
$$\implies H(j\omega) = \frac{j\omega}{RCj\omega + 1}$$

–insert diagram which is non-ideal highpass filter.

Example 7.2 (First order recursive discrete-time filter). Consider

$$y[n] - ay[n-1] = x[n]$$

with input $x[n] = e^{j\omega n}$ the output is of form $y[n] = H(e^{j\omega})e^{j\omega n}$ then

$$\begin{aligned} H\big(e^{j\omega}\big)e^{j\omega n} + H\big(e^{j\omega}\big)e^{j\omega(n-1)} &= e^{j\omega n} \\ \implies H\big(e^{j\omega}\big)\big(1 + e^{-j\omega}\big) &= 1 \\ \implies H\big(e^{j\omega}\big) &= \frac{1}{1 + e^{-j\omega}} \end{aligned}$$

–insert diagram which is non-ideal lowpass filter for $0 < a < 1$ and a non-ideal highpass filter for $-1 < a < 0$.

Example 7.3 (Non-recursive discrete-time filter). Consider the following FIR

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k], \quad b_k \text{ are weights}$$

let $b_k = \frac{1}{M+N+1}$ then

$$\begin{aligned} H\big(e^{j\omega}\big) &= \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{j\omega(n-k)} \\ &= \frac{1}{N+M+1} \frac{e^{j\omega(N+1)} - e^{-j\omega M}}{e^{j\omega} - 1} \end{aligned}$$

which is a lowpass filter that approaches the ideal as $N + M + 1 \to \infty$.

Chapter 8

Continuous Fourier Transform

8.1 Fourier transform for a periodic function

Suppose $x(t)$ with $x(t) = 0$ for $|t| \geq T_1$ is a finite duration aperiodic signal. Make a periodic signal $\tilde{x}(t)$ a from $x(t)$ with period $T \geq T_1$. Then $\tilde{x}$ has a Fourier series

$$\begin{aligned}
\tilde{x}(t) &= \sum_k a_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \\
a_k &= \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0}\,\mathrm{d}t \\
&= \frac{1}{T}X(jk\omega_0)
\end{aligned}$$

since for $|t| \leq T/2, x(t) = \tilde{x}(t)$ and for $|t| > T \geq T_1$, $x(t) = 0$. $X(j\omega)$ is the *Fourier integral* or *Fourier transform* of x.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t$$

Therefore,

$$\begin{aligned}
\tilde{x}(t) &= \sum_k \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t} \\
&= \frac{1}{2\pi}\sum_k X(jk\omega_0)e^{jk\omega_0 t}\omega_0
\end{aligned}$$

as $T \to \infty$, $\tilde{x} \to x$ and $\omega_0 \to 0$ hence

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\mathrm{d}\omega$$

For this to converge we need the Dirichlet condition, that is $x(t)$ must be

1. absolutely integrable.

2. bounded variation.

3. finite number of finite discontinuities.

For periodic signals the Fourier transform is a collection of impulse signals occuring at harmonic

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\mathrm{d}\omega$$

Let $X(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)$

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(2\pi\sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)\right)e^{j\omega t}\,\mathrm{d}\omega \\
&= \sum_{k=-\infty}^{\infty} a_k\int_{-\infty}^{\infty}\delta(\omega - k\omega_0)e^{j\omega t}\,\mathrm{d}\omega \\
&= \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}
\end{aligned}$$

or equivalently

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t \\
&= \sum_{k=-\infty}^{\infty} a_k\int_{-\infty}^{\infty} e^{jk\omega_0 t}e^{-j\omega t}\,\mathrm{d}t \qquad \text{(distribution integra)} \\
&= \sum_{k=-\infty}^{\infty} a_k\delta(\omega - k\omega_0)
\end{aligned}$$

8.2 Properties

Linearity

$$\mathcal{F}\{ax(t) + by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\}$$

Time shifting

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\omega t_0}\mathcal{F}\{x(t)\}$$

Conjugate symmetries

$$\mathcal{F}\left\{\overline{x(t)}\right\} = \overline{\mathcal{F}\{x(-t)\}}$$

Therefore, for a real signal x

$$\begin{aligned}
&\mathcal{F}\{x(-t)\} = \overline{\mathcal{F}\{x(t)\}} \\
\Longrightarrow\ &\mathcal{F}\{Od(x(t))\} = \frac{\mathcal{F}\{x(t)\} - \overline{\mathcal{F}\{x(t)\}}}{2} = i\Im\mathcal{F}\{x(t)\} \\
\Longrightarrow\ &\mathcal{F}\{Ev(x(t))\} = \frac{\mathcal{F}\{x(t)\} + \overline{\mathcal{F}\{x(t)\}}}{2} = \Re\mathcal{F}\{x(t)\}
\end{aligned}$$

Derivatives

$$\mathcal{F}\left\{\frac{\mathrm{d}^n}{\mathrm{d}t^n}x(t)\right\} = (j\omega)^n\mathcal{F}\{x(t)\}$$

Convolution

$$\mathcal{F}\{x(t) * y(t)\} = \mathcal{F}\{x(t)\}\mathcal{F}\{x(t)\}$$

Time scaling

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}\mathcal{F}\{x(t)\}\left(\frac{\omega}{a}\right)$$

Integral

$$\mathcal{F}\left\{\int_{\infty}^{t} x(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{j\omega}\mathcal{F}\{x(t)\} + \pi\mathcal{F}\{x(t)\}(0)\delta(\omega)$$

Inverse of derivative

$$\mathcal{F}^{-1}\left\{\frac{\mathrm{d}}{\mathrm{d}\omega}\mathcal{F}\{x(t)\}\right\} = -jtx(t)$$

Inverse of frequency shifting

$$\mathcal{F}^{-1}\{\mathcal{F}\{x(t)\}(\omega-\omega_0)\} = e^{j\omega_0 t}x(t)$$

Multiplication

$$\mathcal{F}\{x(t)y(t)\} = \mathcal{F}\{x(t)\} * \mathcal{F}\{y(t)\}$$

Parseval

$$\int_{\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{F}\{x(t)\}|^2\,\mathrm{d}\omega$$

Fourier transform of some functions

8.3 Applications

We can make frequency-selective filtering with variable center frequency (bandpass around ω_c. Solve ODEs in the following format

$$\sum_{k=0}^{K} a_k \frac{\mathrm{d}^k}{\mathrm{d}t^k}y(t) = \sum_{m=0}^{M} a_m \frac{\mathrm{d}^m}{\mathrm{d}t^m}x(t)$$

since it is an LTI system then $y(t) = x(t) * h(t)$ and

$$Y(j\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\} = H(j\omega)X(j\omega)$$

therefore

$$H(j\omega) = \frac{\sum_{m=0}^{M} b_k(j\omega)^m}{\sum_{k=0}^{K} a_k(j\omega)^k}$$

Chapter 9

Discrete Fourier Transform

Let $x[n]$ be a finite duration signal and $\tilde{x}[n]$ be the periodic form of $x[n]$, Then, as $N \to \infty$ $\tilde{x}[n] \to x[n]$.

$$\begin{aligned}
\tilde{x}[n] &= \sum_{\langle N \rangle} a_k e^{j\omega_0 kn} \qquad \omega_0 = \frac{2\pi}{n} \\
\implies a_k &= \frac{1}{N} \sum_{\langle N \rangle} \tilde{x}[n] e^{-j\omega_0 kn} \\
&= \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-j\omega_0 kn} \\
&= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega_0 kn} \\
&= \frac{1}{N} X\left(e^{j\omega_0 k}\right)
\end{aligned}$$

which implies that

$$\begin{aligned}
\tilde{x}[n] &= \sum_{\langle N \rangle} \frac{1}{N} X\left(e^{j\omega_0 k}\right) e^{j\omega_0 kn} \\
\implies x[n] &= \lim_{\omega_0 \to \infty} \frac{1}{2\pi} \sum_{0}^{\frac{2\pi}{n}} X\left(e^{j\omega_0 k}\right) e^{j\omega_0 kn} \omega_0 \\
&= \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} \, \mathrm{d}\omega
\end{aligned}$$

For convergence we need the followings to hold

1. $x[n]$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

   or has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

9.1 DFT of periodic signal

Let $x[n] = e^{j\omega_0 n}$ then

$$\begin{aligned} X\left(e^{j\omega}\right) &= \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \\ \implies x[n] &= \sum_{\langle N \rangle} a_k e^{jk\omega_0 n} \\ \implies X\left(e^{j\omega}\right) &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) \end{aligned}$$

9.2 Properties

Periodic It has a period of 2π.

$$\mathcal{F}\{x[n]\}(\omega + 2\pi) = \mathcal{F}\{x[n]\}(\omega)$$

Linear

$$\mathcal{F}\{ax[n] + by[n]\} = a\mathcal{F}\{x[n]\} + b\mathcal{F}\{y[n]\}$$

Time shifting n_0 must be an integer.

$$\mathcal{F}\{x[n - n_0]\} = e^{-j\omega n_0}\mathcal{F}\{x[n]\}$$

Frequency shifting

$$\mathcal{F}\left\{e^{j\omega_0 n}x[n]\right\} = \mathcal{F}\{x[n]\}(\omega - \omega_0)$$

Conjugate symmetry

$$\mathcal{F}\left\{\overline{x[n]}\right\} = \overline{\mathcal{F}\{x[n]\}(-\omega)}$$

and if $x[n]$ is real

$$\begin{aligned} \implies \mathcal{F}\{x[n]\} &= \overline{\mathcal{F}\{x[n]\}(-\omega)} \\ \implies \mathcal{F}\{Ev(x[n])\} &= \Re\mathcal{F}\{x[n]\} \\ \implies \mathcal{F}\{Od(x[n])\} &= j\Im\mathcal{F}\{x[n]\} \end{aligned}$$

Differencing

$$\mathcal{F}\{x[n] - x[n-1]\} = \left(1 - e^{-j\omega}\right)\mathcal{F}\{x[n]\}$$

Accumulation

$$\mathcal{F}\left\{\sum_{m=-\infty}^{n} x[m]\right\} = \frac{1}{1 - e^{-j\omega}}\mathcal{F}\{x[n]\} + \pi\mathcal{F}\{x[n]\}(0) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

Time Rerversal

$$\mathcal{F}\{x[-n]\} = \mathcal{F}\{x[n]\}(-\omega)$$

Time Expansion

$$\mathcal{F}\left\{x_{(k)}[n]\right\} = \mathcal{F}\{x[n]\}(k\omega)$$

where

$$x_{(k)}[n] = \begin{cases} x\left[\frac{n}{k}\right] & k|n \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{F}^{-1}\left\{j\frac{\mathrm{d}\mathcal{F}\{x[n]\}}{\mathrm{d}\omega}\right\} = nx[n]$$

Parsevals'

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |\mathcal{F}\{x[n]\}|^2 \,\mathrm{d}\omega$$

Convolution

$$\mathcal{F}\{x[n] * y[n]\} = \mathcal{F}\{x[n]\}\mathcal{F}\{y[n]\}$$

Multiplication

$$\mathcal{F}\{x[n]y[n]\} = \frac{1}{2\pi}\int_{2\pi} \mathcal{F}\{x[n]\}(\theta)\mathcal{F}\{y[n]\}(\omega-\theta)\,\mathrm{d}\theta$$

9.3 Linear constant coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

it is the impluse response has the following form

$$H\left(e^{j\omega}\right) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{j\omega k}}$$

Some fourier analysis would not be bad :)

Chapter 10

Time Frequency Characterization

10.1 Magnitude-phase representation of Fourier Transform

$\angle X(j\omega)$ is the relative phase and $\frac{1}{2\pi}|X(j\omega)|^2\,\mathrm{d}\omega$ is the energy of $x(t)$ in $]\omega, \omega+\mathrm{d}\omega[$. Obviously, the magnitude matters but also phase matters depending on the context.

10.2 Magnitude-phase representation of the frequency response of LTI systems

In an LTI system

$$Y(j\omega) = H(j\omega)X(j\omega) \implies \begin{cases} |Y| & = |H||X| \\ \angle Y & = \angle H + \angle X \end{cases}$$

where the $|H|$ is called the gain and $\angle H$ is the phase shift.

Linear and non-linear phase

For unit gain all-pass $H(j\omega) = 1$ with linear phase $\angle H(j\omega) = t_0\omega$. Then,

$$y(t) = x(t - t_0)$$

Group delay

Suppose $X(j\omega)$ is zero outside a narrow band around $\omega = \omega_0$. Then, a non-linear phase can be approximated by a linear phase

$$\begin{aligned} \angle H(j\omega) &\simeq -\phi - \alpha \\ \implies Y(j\omega) &= X(j\omega)|H(j\omega)|e^{j\phi}e^{j\omega\alpha} \end{aligned}$$

α seconds delay. The group delay is defined as

$$\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}\angle H(j\omega)$$

but this might be discontinuous at $2\pi k$. So we use the un-wrapped phase

$$\tau(\omega) = -\frac{\mathrm{d}}{\mathrm{d}\omega}[\angle H(j\omega)]$$

Log-magnitude and Bode plots

$dB = 20\log_{10}|H(j\omega)|$. If ω-axis is logarithmic $\log_{10}\omega$ as well then it is called Bode plot.

10.3 Time-Domain properties of ideal frequency selective filter

a frequency selective filter

$$|H(j\omega)| = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

With linear phase (α seconds delay)

$$\angle H(j\omega) = -\alpha\omega$$

10.4 Time-domain and frequency aspects of non-ideal filters

add images in pg 175

10.5 First and second order CT system

First order continuous-time system

$$\tau\frac{\mathrm{d}y}{\mathrm{d}t} + u = x(t) \implies H(j\omega) = \frac{1}{j\omega\tau + 1}$$

which then implies that

$$\begin{aligned} h(t) &= \frac{1}{\tau}e^{-\frac{t}{\tau}}u(t) \\ s(t) &= \left(1 - e^{-\frac{t}{\tau}}\right)u(t) \end{aligned}$$

Second order continuous-time system

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\xi\omega_n\frac{\mathrm{d}y}{\mathrm{d}t} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

then

$$\begin{aligned} H(j\omega) &= \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2} \\ &= \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)} \end{aligned}$$

where $c_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$ and $c_2 = -\xi\omega_n + \omega_n\sqrt{\xi^2 - 1}$. If $\xi \neq 1$ then

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2} \qquad M = \frac{\omega_n}{2\sqrt{\xi^2 - 1}}$$

$$\Longrightarrow h(t) = M\big(e^{c_1 t} - e^{c_2 t}\big)u(t)$$

and if $\xi = 1$ then

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2} \Longrightarrow h(t) = \omega_n^2 t e^{-\omega_n t} u(t)$$

ξ is called the damping ratio and ω_n is the undamped natural frequency.

- For $0 < \xi < 1$ the response is underdamped which will overshoot and rings in step function.
- For $\xi = 1$ the response is critically damped which will have the fastest settling time.
- For $\xi > 1$ the response is overdamped which will imply it has slow settling time.

10.6 First order and second order discrete-time system

10.7 First order

$$\begin{aligned} y[n] - ay[n-1] = x[n] \Longrightarrow H\big(e^{j\omega}\big) &= \frac{1}{1 - ae^{j\omega}} \\ h[n] &= a^n u(n) \\ s[n] &= \frac{1 - a^{n+1}}{1 - a} u(n) \end{aligned}$$

10.8 Second order

$$y[n] - 2r\cos\theta y[n-1] + r^2 y[n-2] = x[n]$$

$$\begin{aligned} H\big(e^{j\omega}\big) &= \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-2j\omega}} \\ &= \frac{1}{(1 - re^{j\theta}e^{-j\omega})(1 - re^{=j\theta}e^{-j\omega})} \end{aligned}$$

If $\theta \neq 0, \pi$ then

$$H\big(e^{j\omega}\big) = \frac{A}{1 - re^{j\theta}e^{-j\omega}} + \frac{B}{1 - re^{-j\theta}e^{-j\omega}}$$

where

$$A = \frac{e^{j\theta}}{2j\sin\theta} \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

if $\theta = 0$

$$H\big(e^{j\omega}\big) = \frac{1}{(1 - re^{-j\omega})^2} \Longrightarrow h[n] = (n+1)r^n u[n]$$

and for $\theta = \pi$

$$H\big(e^{j\omega}\big) = \frac{1}{(1 + re^{-j\omega})^2} \Longrightarrow h[n] = (n+1)(-r)^n u[n]$$

Chapter 11

Sampling

11.1 Sampling theorem

Sampling with impluse train

Let T be the sampling period and $\omega_s = \frac{2\pi}{T}$ be the sampling frequency. The train impluse

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

and hence the sampled signal

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

has the Fourier transform

$$X_p(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)P(j\omega - j\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}X(j\omega) * P(j\omega)$$

with

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

and hence

$$\begin{aligned} X_p(j\omega) &= \frac{\omega_s}{2\pi}X(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s) \end{aligned}$$

Theorem 11.1. *Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then, $x(t)$ is uniquely determined by its samples $x(nT)$ if $\omega_s > 2\omega_M$, i.e. we can reconstructed $x(t)$ from its samples.*

Proof. Construct $X_p(j\omega)$ as above. Note that is has a period of ω_s. Therefore, by the fact that $X(j\omega)$ is band-limited, if $\omega_M < \omega_s - \omega_M$ or equivalently $\omega_s > 2\omega_M$ then we can reconstruct $X(j\omega)$ by applying a lowpass filter with $\omega_M < \omega_c < \omega_s - \omega_M$. ■

Definition: $2\omega_M$ is called the Nyquist rate.

Sampling with zero-order hold

it is the same as impulse train by we hold the last sample until the new sample. – add images pg 178 where

$$H_0(j\omega) = e^{-j\omega T/2}\left(\frac{2\sin\frac{\omega T}{2}}{\omega}\right)$$

and therfore to have $x(t) = r(t)$ we must have

$$H_r(j\omega) = \frac{H(j\omega)}{H_0(j\omega)}$$

where $H(j\omega)$ is the ideal lowpass needed to convert impulse train to original signal.

11.2 Reconstruction of signal from its samples using interpolation

first-order hold is the linear interpolation. Has the following filter – inset image pg 178

11.3 Effect of undersampling (Aliasing)

Do the example with a $\cos\omega_0 t$ and conclude that undersampling turns high frequencies into low frequencies. Stroboscopic effect

11.4 Discrete-time processing of continuous time signals

First we must transform continuous to discrete (C/D conversion , Analog to digital), then process the discrete signal and lastly convert it back to continuous time (D/C conversion, Digital to Analog).

C/D conversion

it can be effectively by impulse train.

$$x_p(t) = \begin{cases} x_c(nT) & t = nT \\ 0 & \text{otherwise} \end{cases} \qquad\qquad x_d[n] = x_p(nT)$$

then

$$\begin{aligned} X_d\left(e^{j\Omega}\right) &= \sum_{k=-\infty}^{\infty} x_d[k]e^{-j\Omega k} \\ &= \sum_{k=-\infty}^{\infty} x_p(nT)e^{-j\Omega k} \\ &= \sum_{k=-\infty}^{\infty} x_c(nT)e^{-j\Omega k} \end{aligned}$$

$$= X_p\left(j\frac{\Omega}{T}\right)$$

$$\implies X_d\left(e^{j\Omega}\right) = \frac{\omega_s}{2\pi}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\Omega}{T} - jk\omega_s\right)$$

D/C Conversion

we just reverse back. we turn discrete time into impulse train and then apply a lowpass filter to get the continuous signal.

The system

give a sufficiently band limited input and sampling theorem conditions hold is LTI.

$$H_c(j\omega) = \begin{cases} H_d\left(e^{j\omega T}\right) & |\omega| < \frac{\omega_s}{2} \\ 0 & \text{otherwise} \end{cases}$$

Half-sample delay

suppose we want to do

$$y_c(t) = x_c(t - \Delta)$$

Then

$$Y_c(j\omega) = e^{-j\omega\Delta}X_c(j\omega)$$

implying that

$$H_c(j\omega) = \begin{cases} e^{-j\omega\Delta} & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

and

$$H_d\left(e^{j\Omega}\right) = e^{-j\Omega\frac{\Delta}{T}}$$

given that $\frac{\Delta}{T}$ is an integer

$$y_d[n] = x_d\left[n - \frac{\Delta}{T}\right]$$

11.5 Sampling of discrete-time signal

Implust train signal

$$x_p[n] = \begin{cases} x(n) & N|n \\ 0 & \text{otherwise} \end{cases}$$

where N is the sampling period. THen,

$$\begin{aligned} x_p[n] &= x[n]p[n] \\ &= \sum_{k=-\infty}^{\infty} x[kN]\delta[n - kN] \end{aligned}$$

$$
\begin{aligned}
P\left(e^{j\omega}\right) &= \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s) \\
\implies X_p\left(e^{j\omega}\right) &= \frac{1}{2\pi}\int_{2\pi} P\left(e^{j\theta}\right)X\left(e^{j(\omega-\theta)}\right)\mathrm{d}\theta \\
&= \frac{1}{N}\int_0^{2\pi} X\left(e^{j(\omega-\theta)}\right)\sum_{k=-\infty}^{\infty}\delta(\theta-k\omega_s)\,\mathrm{d}\theta \\
&= \frac{1}{N}\sum_{k=0}^{N-1}\int_0^{2\pi} X\left(e^{j(\omega-\theta)}\right)\delta(\theta-k\omega_s)\,\mathrm{d}\theta \\
&= \frac{1}{N}\sum_{k=0}^{N-1} X\left(e^{j(\omega-k\omega_s)}\right)
\end{aligned}
$$

again no aliasing if $\omega_s > 2\omega_M$.

Discrete-time decimation

it is unnecessary to keep all the zeros.

$$x_b[n] = x_p[nN] = x[nN]$$

therefore

$$
\begin{aligned}
X_b\left(e^{j\omega}\right) &= \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x_p[k]e^{-j\omega/Nk} \\
&= X_p\left(e^{j\omega/N}\right)
\end{aligned}
$$

11.5.1 upsampling or interpolation

adding zeros $x_b[n] \to x_p[n] \to x[n]$ and then apply low-pass filter.