CHAPTER 3: WHO PLAYS VIDEO GAMES

Math 189 : Data Analysis and Inference : Spring 2020

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Introduction

The data

Background

Investigations

Theory

* Every year, $3,000 - 4,000$ students enroll in statistics courses at UC Berkeley.
* Half of these students take introductory statistics courses to satisfy quantitative reasoning requirement.
* To aid the instruction of these students, a committee of faculty and students have designed a series of computer labs.

LABS

* The labs are meant to extend the traditional syllabus for a course by providing an interactive learning environment that offers students an alternative method for learning the concepts of statistics and probability.
* Some have linked labs to video games.
* To help committee design the labs a survey of undergraduate students who were enrolled in a lower-division statistics course was conducted.
* The survey's aim was to determine the extent to which the students play video games and which aspects of video games they find most and least fun.

Survey

* Students who were enrolled in advanced statistics course conducted the study.
* They developed the questionnaire, selected the students to be sampled and collected the data.
* In this study you will have the opportunity to analyze the results from the sample survey to offer advice to the design committee.

What makes a survey a survey?

- Scientific methodology
- Data collection from an individual
- Usually samples from a large population
- Conducted for the purpose of . . .
  - Description
  - Exploration
  - Explanation

Characteristics of Good Survey Research

- Quantitative
- Self-monitoring
- Contemporary
- Replicable
- Systematic
- Impartial
- Representative
- Theory-based

General Sampling Issues

- Basic rule–all individuals must have equal chance of being selected
- May be more accurate data than a census
- If all members of a population were identical, sampling would not be necessary
- Aim for a sample that is generalizable to total population of interest

Description

* Out of 314 students in Statistics 2, Section 1, during Fall 1994, 95 were selected at random to participate in the survey.
* Complete surveys were obtained from 91 out of 95 students.
* The data available here are the students responses to the questionnaire.

The Survey asks students to identify how often they play video games and what they like and dislike about the games.

* The answers to these questions were coded numerically as described here.
  ** Time : # of hours played in the week prior to survey
  ** Like to play : 1=never played, 2=very much, 3=somewhat, 4=not really, 5=not at all.
  ** Where play : 1=arcade, 2=home system, 3=home computer, 4=arcade and either home computer of system, 5= home computer and system, 6=all three.
  ** How often : 1=daily, 2=weekly, 3=monthly, 4=semesterly.
  ** Play if busy : 1=yes, 0=no.
  ** Playing educational : 1=yes, 0=no.
  ** Sex : 1=male, 0=female.
  ** Age : Students age in years.
  ** Computer at home : 1=yes, 0=no.
  ** Hate math: 1=yes, 0=no.
  ** Work: # of hours worked the week prior to the survey.
  ** Own PC : 1=yes, 0=no.
  ** PS has CD-Rom: 1=yes, 0=no.
  ** Have email : 1=yes, 0=no.
  ** Grade expected : 4=A,3=B,2=C,1=D,0=F.

time	like	where	freq	busy	educ	sex	age	home	math	work	own	cdrom	email	grade
2	3	3	2	0	1	0	19	1	0	10	1	0	1	4
0	3	3	3	0	0	0	18	1	1	0	1	1	1	2
0	3	1	3	0	0	1	19	1	0	0	1	0	1	3
0.5	3	3	3	0	1	0	19	1	0	0	1	0	1	3
0	3	3	4	0	1	0	19	1	1	0	0	0	1	3
0	3	2	4	0	0	1	19	0	0	12	0	0	0	3
0	4	3	4	0	0	1	20	1	1	10	1	0	1	3
0	3	3	4	0	0	0	19	1	0	13	0	0	1	3
2	3	2	1	1	1	1	19	0	0	0	0	0	0	4
0	3	3	4	0	1	1	19	1	1	0	1	0	1	4
0	3	1	4	0	0	0	20	1	0	0	1	0	0	3
0	3	2	4	0	0	0	19	1	0	0	1	0	1	4
0	2	4	1	0	1	0	19	1	1	0	0	0	1	4
3	3	3	2	1	0	0	18	0	0	0	0	0	1	3
1	3	5	2	0	1	0	18	1	1	14	1	0	1	3

Snapshot of the data

Missing data

If a question was not answered or improperly answered, then it was coded as 99.

Those respondents who had never played a video game or who did not at all like playing video games were asked to skip many of the questions.

The was a second part of the survey that covers whether the student likes or dislikes playing games and why.

These questions are different from the others in the more than one response may be given.

Type	Percent
Action	50%
Adventure	28%
Simulation	17%
Sports	39%
Strategy	63%

Table: What types of games do you play? (at most three answers)

Table 1 summarizes the types of games played.

-The student is asked to check all types that he or she plays. For example, 50% of the students responding to this question said that they play action games.

- Not all students responded to this question, in part because this who said thatchy have never played a video game or do not at all like to play video games were instructed to skip this question.

Why?	Percent
Graphics/Realism	26%
Relaxation	66%
Eye/hand coordination	5%
Mental Challenge	24%
Felling of mastery	28%
Bored	27%

Table: Why do you play the games you checked above? (at most three answers)

Table 2 summarizes reasons for playing the game.

- Students who did answer this question were also asked to provide reasons why they play the games they do. They were asked to select up to three such reasons. Their responses are presented in Table 2

Dislikes	Percent
Too much time	48%
Frustrating	26%
Lonely	6%
Too many rules	19%
Costs too much	40%
Boring	17%
Friend’s don’t play	17%
It is pointless	33%

Table: What don’t you like about video game playing? (at most three answers)

Table 3 summarizes what students didn’t like about the games.

- Finally, table 3, contains summary of what the students do not like about video games. All students were asked to answer this question, and again they were asked to select up to three reasons for not liking video games.

- Third part of the survey collect general information about the student: age, sex, etc.

All of the population studied were undergraduates enrolled in Introductory Probability and Statistics, Section 1, during Fall 1994.

- The class is a lows-division prerequisite for students intending to major in business

- During the Fall the class met on MWF from 1-2pm in a large lecture hall that seats four hundred.

- In addition to three hours of lecture, students attended a small, one-hour discussion section that met on T/Th.

- There were ten discussion sections for the class, each with approximately 30 students.

The survey methodology(cont.)

The list of all students who had taken the second exam of the semester was used to select the students to be surveyed.

- The exam was given a week prior to the survey.

- To choose 95 students for the study, each student was assigned a number from 1 to 314.

- A pseudo random number generator selected 95 numbers between 1 to 314.

- To encourage honest responses, the students anonymity was preserved.

To limit the number of nonrespondents, a three stage system of data collection was employed.

- Data collectors visited both the Tu and Th meetings of the discussion sections i noted week the survey was conducted.

- The students had taken an exam the week before the survey, and the graded exam papers were returned to them during the discussion section in the week of the survey.

- On Friday, those students who had not been reach during the discussion section were located during the lecture.

- A total of 91 students completed the survey.

-To encourage accuracy in reporting, the data collectors were asked to briefly inform the student of the purpose of the survey and of the guarantee of anonymity.

Video Games can be classified according to the device on which they are played and according to the kind of skills needed to play the game.

	Eye/hand	Puzzle	Plot	Strategy	Rules
Action	×				
Adventure		×	×		
Simulation				×	×
Strategy				×	×
Role-play		×	×		×

Table: Classification of five main types of video games

Table 4 summarizes the attributes typically found in each category.

Device: arcade, console, PC

Arcade games: fast and emphasize eye/hand coordination

Console games: action, adventure or strategy games

PC games: simulation and role-play exclusively and other types as well

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* [Scenario 4:] Next consider the "attitude" questions. In general, do you think the students enjoy playing video games? If you had to make a short list of the most important reasons why students like/dislike video games, what would you put on the list? Don't forget that those students who say that they have never played video games or do not at all like video games are asked to skip over some of these questions. So, there may be many nonrespondents to the questions as to whether they think video games are educational, where they play video games, etc.

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* [Scenario 5:] Look for the differences between those who like to play video games and those who don't. To do this, use the questions in the last part of the survey, and make comparisons between male and female students, those who work for pay and those who don't, those who own a computer and those who don't. Graphical display and cross-tabulations are particularly helpful in making these kinds of comparisons. Also, you may want to collapse the range of responses to a question down to two or three possibilities before making these comparisons.

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* [Scenario 6:] (optional) Just for fun, further investigate the grade that students expect in the course. How will does it match the target distribution used in grade assignment of 20% A's, 30%B's,40% c's and 10%D's or lower? If the nonrespondents were failing students who no longer bothered to come to the discussion section, would this change the picture ?

Goals

In this section we will use as out primary example the problem of estimating the average amount of time students in the class spent playing video games in the week prior to the survey.

* To determine the exact amount of time for the entire class we would need to interview all of the students (over 3000 of them).
* Alternatively, a subset of them can be interviewed, and the information collected from this subset could provide an approximation to the full group.
  * In this section we discuss one rule for selecting a subset of student to be surveyed, the simple random sample.
  * The simple random sample is a probability method for selecting the students..
  * Probability methods are useful because through chance we can make useful statements about the relation between the sample and the entire group.
  * With a probability method, we know the chance of each possible sample.

Goals (cont.)

Terminology

* Population units make up the group that we want to know more about
  * In this lab, the units are the students enrolled in the 1994 Fall semester class of Introductory Probability and Statistics.
* Population size, usually denoted by N, is the total number of units in the population. For very large population, often the exact size of the population is not known. Here we have 314 students in the class.
* Unit characteristic is a particular piece of information about each member of the population.
  * The characteristic that interests us in our example is the amount of time the student played video games in the week prior to the survey.
* Population parameter is a summary of the characteristic for all units in the population, such as the average value of the characteristic.
  * The population parameter of interests to us here is the average amount of time students in the class spent playing video games in the week prior to the survey.

Terminology

* Sample units are those members of the population selected for the sample.
* Sample size usually denoted by n, is the number of units chosen for the sample. We will use 91 for our sample size, and ignore the four who did not respond.
* Sample statistic is a numerical summary of the characteristic of the units sampled. The statistic estimated the population parameter. Since the population parameter in our example is the average time spent playing video games by all students in the class in the week prior to the survey, a reasonable sample statistic is the average time spent playing video games by ll students in the sample.

The simple random sample is a very simple probability model for assigning probabilities to all samples of size n from a population of size N.

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In general with N population units and a sample of size n, there are N choose n possible samples. The probability rule that defined the simple random sample is that each one of the $\binom{N}{n}$ samples is equally likely to be selected. That is, each unique sample of n units has the same chance, $1/\binom{N}{n}$, of being selected. From this probability, we can make statements about the variations we would expect to see across repeated samples.

- Assign each unit a number from 1 to N.
- Write each number on a ticket, put all of the tickets in the box, mix them up,
- Draw n tickets one at a time from the box without replacement.

Number of different units in the sample

Unit # 1

* The chance that the unit # 1 is the first to be selected for the sample is 1/N.
* Likewise, the unit # 1 has chance 1/N of being the second unit chosen for the sample.
* All together, the unit #1 has the chance of n/N of being in the sample.

Unit # 2

* By symmetry, The chance that the unit # 2 is the first to be selected for the sample is 1/N.
* Likewise, the unit # 2 has chance 1/N of being the second unit chosen for the sample.
* All together, the unit #2 has the chance of n/N of being in the sample.

Each unit has the same chance of being in the sample.

There is dependence between selections.

* The chance that the unit # 1 is chosen first and the unit # 2 is chosen second is $\frac{1}{N(N-1)}$.
* This chance is the same for any two units in the population, hence the chance that # 1 and # 2 are both in the sample, is $\frac{n(n-1)}{N(N-1)}$.
* In our example

$$\mathbb{P}(\text{ unit \# 1 in the sample }) = 91/314,$$

$$\mathbb{P}(\text{ unit \# 1 and unit \#2 are in the sample }) = \frac{91 \times 90}{314 \times 313}.$$

Probability distribution of the units chosen in the sample

* Let $I(1), I(2), \ldots$ represent the first,second, $\ldots$ number drawn from the list $1, 2, \ldots, N$.
* Then,

$$\mathbb{P}\left(I(1) = 1\right) = 1/N$$

$$\mathbb{P}\left(I(1) = 1 \text{and } I(2) = N\right) = \frac{1}{N(N-1)}$$

and in general, for $1 \leq j_1 \neq j_2 \neq \cdots \neq j_n \leq N$

$$\mathbb{P}\left(I(1) = j_1, I(2) = j_2, \ldots, I(n) = j_n\right) = \frac{1}{N(N-1)\cdots(N-n+1)}$$

The simple random sample method puts a probability structure on the sample. Different samples have different characteristic and different sample statistics.

....

Sample statistic has a probability distribution related to the sampling procedure.

Computing expected value of the sample statistic.....

* Let x_1 be the value of the characteristic for unit # 1. x_2 for unit #2, $\cdots$
* In our example x_i is the time spent playing video games by the student # i: $i = 1 \ldots, 314$.
* Population average is

$$\mu = \frac{1}{N} \sum_{i=1}^{N} x_i$$

as our population parameter.
* Let $x_{I(j)}$ represent the value of the characteristic for the j-th unit sampled.

Note: $x_{I(1)}$ is a random variable

* In our example $x_{I(j)}$ represent the value of the time spent playing video games by the j-th unit sampled: $j = 1, \cdots, 91$.

$$\mathbb{E}\left(x_{I(j)}\right) = \sum_{i=1}^{N} x_i \mathbb{P}\left(I(j) = i\right) = \sum_{i=1}^{N} x_i \frac{1}{N} = \mu$$

* The sample average is the sample statistic that estimates the population parameter μ,

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n} x_{I(j)}.$$

Note: $\bar{x}$ is a random variable

* $\mathbb{E}(\bar{x}) = \mu$

Note: We have shown that the sample average is an unbiased estimator of the population parameter

* Next we find the standard deviation of $\bar{x}$. To do this, we first find the variance of $x_{I(j)}$

$$\mathrm{Var}(X_{I(j)}) = \mathbb{E}\left(x_{I(j)} - \mu\right)^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2 = \sigma^2$$

where we used σ^2 to denote population variance.

Computing variance of the sample statistic.....

* Then we compute the variance of the sample average $\bar{x}$ as follows

$$\begin{aligned}
\mathrm{Var}(\bar{x}) &= \frac{1}{n^2}\mathrm{Var}\left(\sum_{j=1}^{n} x_{I(j)}\right) && (1)\\
&= \frac{1}{n^2}\sum_{j=1}^{n}\mathrm{Var}\left(x_{I(j)}\right) + \frac{1}{n^2}\sum_{j=1, j\neq k}^{n}\mathrm{Cov}\left(x_{I(j)}, x_{I(k)}\right) && (2)\\
&= \frac{1}{n}\sigma^2 + \frac{n-1}{n}\mathrm{Cov}\left(x_{I(1)}, x_{I(2)}\right) && (3)
\end{aligned}$$

The last equality follows from noting that all pairs $\left(x_{I(j)}, x_{I(k)}\right)$ are identically distributed. The covariance between any two sampled units $x_{I(j)}$ and $x_{I(k)}$ is not 0 because the sampling procedure makes them dependent.

$$\mathrm{Cov}\left(x_{I(1)}, x_{I(2)}\right) = -\frac{\sigma^2}{N-1}$$

$$\mathrm{Var}(\bar{x}) = \frac{1}{n}\sigma^2\frac{N-n}{N-1}, \qquad \mathrm{SD}(\bar{x}) = \frac{1}{\sqrt{n}}\sigma\sqrt{\frac{N-n}{N-1}}$$

The factor

$$\frac{N-n}{N-1} = 1 - \frac{n-1}{N-1} \sim 1 - \frac{n}{N}$$

in the variance and SD is called the finite population correction factor. The ratio n/N is called the sampling fraction. It is very small when the sample size is small relative to the population size.

This is frequently the case in sampling, and when this happens

$$\mathrm{Var}(\bar{x}) \sim \sigma^2/n$$

and the finite population correction factor is often ignored.

In our example, it cannot be ignored as

$$\frac{\sqrt{314-91}}{\sqrt{314-1}} = 0.84$$

Variance....

Notice that without the correction factor, the variance is the same as if we made the draws with replacement (or sampling from an infinite population)

With a simple random sample, the standard deviation for the estimator can be computed in advance, dependent on the population variance σ^2. If σ^2 is known approximately, then the sample size can be chosen to give an acceptable level of accuracy for the estimator.

Often a pilot study, results from a related study, or a worst-case estimate of σ^2 is used in planning the sample size for the survey.

Estimators for Standard Errors

Standard deviations for estimators are typically called standard errors (SEs)

. They indicate the size of the deviation of the estimator from its expectation.

* When σ^2 is unknown, a common estimator for it is

$$s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_{I(j)} - \bar{x})^2$$

* To estimate $\mathrm{Var}(\bar{x})$, we can then use

$$\frac{s^2}{n}\frac{N-n}{N-1}$$

* The reason for using s^2 is that the sample, when chosen by the simple random sample method, should look roughly like a small -scaled version of the population, so we plug in s^2 for σ^2 in the variance of $\bar{x}$.

Estimators for Standard Errors (cont.)

* In fact we can make a slightly better estimate for $\mathrm{Var}(\bar{x})$
* To estimate $\mathrm{Var}(\bar{x})$, we can then use

$$\mathbb{E}s^2 = \frac{N}{N-1}\sigma^2$$

* Hence, an unbiased estimator of σ^2 is then

$$s^2 \frac{N-1}{N}$$

* Hence an unbiased estimator of $\mathrm{Var}(\bar{x})$ is

$$\frac{s^2}{n}\frac{N-n}{N}$$

Note: There is essentially no difference between these two estimators of $\mathrm{Var}(\bar{x})$ for any reasonably sized population.

Population Totals and Percentages

Sometimes the population parameter is a proportion or percentage, such as the proportion of students who played a video game in the week prior to the survey or the percentage of students who own PCs.

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When the parameter is a proportion, it makes sense for the characteristic value x_i to be 0 or 1 to denote the absence or presence of the characteristic, respectively. For example, for $i = 1, \ldots, 314$

$$x_i = \begin{cases} 1 & \text{if the ith student in the population owns a PC} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Then $\tau = \sum x_i$ counts all of the students who own PCs in the population, and

$$\pi = \frac{1}{N} \sum x_i$$

is the proportion of students in the population who owns PCs.

In this case $\bar{x}$ remains an unbiased estimate of π, the population average, and $N\bar{x}$ estimates τ.

A simpler form for the population variance and the unbiased estimator of $\mathrm{Var}(\bar{x})$ can be obtained because

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \pi)^2 = \pi(1 - \pi).$$

Then the estimator for the standard error is

$$\hat{SE}(\bar{x}) = \frac{\sqrt{\bar{x}(1 - \bar{x})}}{\sqrt{n - 1}} \frac{\sqrt{N - n}}{\sqrt{N}}$$

Often the symbols $\hat{\mu}$, $\hat{\pi}$ and $\hat{\tau}$ are used in place of $\bar{x}$, $N\bar{x}$ to denote sample estimated of the parameters μ, π and τ.. The following table contains the expectations and standard errors for estimators of a population average, proportion and total.

	Average	Proportion	Total
Parameter	μ	π	τ
Estimator	$\bar{x}$	$\bar{x}$	$N\bar{x}$
Expectation	μ	π	τ
Standard Error	$\frac{\sigma}{\sqrt{n}} \frac{\sqrt{N-n}}{\sqrt{N-1}}$	$\frac{\sqrt{\pi(1-\pi)}}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}$	$N \frac{\sigma}{\sqrt{n}} \frac{\sqrt{N-n}}{\sqrt{N-1}}$
Estimator of SE	$\frac{s}{\sqrt{n}} \frac{\sqrt{N-n}}{\sqrt{N}}$	$\frac{\sqrt{\bar{x}(1-\bar{x})}}{\sqrt{n-1}} \sqrt{\frac{N-n}{N}}$	$N \frac{s}{\sqrt{n}} \frac{\sqrt{N-n}}{\sqrt{N}}$

Table: Properties of sample statistics

If the sample size is large, then the probability distribution of the sample average is often well approximated by the normal curve.

Central Limit Theorem

If $X_1, \ldots, X_n$ are independent, identically distributed with mean μ and variance σ^2 then, for n large, the probability distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

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In simple random sampling, the $x_{I(j)}$ are identically distributed but nonindependent. However, the normal approximation can still hold if, in addition to the sample size bing large, it is not too large relative to the population size.

Is it Normal?

* If the sampling proportion n/N is small, then the $x_{I(j)}$ are nearly independent.
* There are no hard rules for how large n must be or how small n/N must be before we can use the normal approximation.
* You can use simulation to check it.
* The central limit theorem is very powerful result. It implies that for any population distribution, under simple random sampling (for appropriate n and n/N), the sample average has an approximate normal distribution.

Normal distribution can be used to provide confidence intervals for the population parameter. One interval estimate of μ, called 68% confidence interval is

$$\left(\bar{x} - \frac{\sigma}{\sqrt{n}}, \bar{x} + \frac{\sigma}{\sqrt{n}}\right)$$

A 95% confidence interval is

$$\left(\bar{x} - 2\frac{\sigma}{\sqrt{n}}, \bar{x} + 2\frac{\sigma}{\sqrt{n}}\right)$$

Interpretation

By Central Limit Theorem the chance that $\bar{x}$ is within one (or two) standard error(s) of μ is approximately 68% (or 95%).

Note: Sample statistic $\bar{x}$ is random, so we can think of confidence intervals as random intervals. Different samples lead to different confidence intervals. If we take many simple random samples where for each sample we compute CI, then we expect 95% of CI's to contain μ.

That is, ignoring the finite sample population correction factor,

$$\mathbb{P}\left(\bar{x} - 2\frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + 2\frac{\sigma}{\sqrt{n}}\right) = \mathbb{P}\left(\mu - 2\frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu + 2\frac{\sigma}{\sqrt{n}}\right) \tag{5}$$

$$= \mathbb{P}\left(2 \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq 2\right) \sim 0.95 \tag{6}$$

In practice, we often don't know σ, and we substitute s in place of σ in order to make the confidence interval. With this substitution we often call the interval, approximate confidence interval.

To summarize the ideas introduced in this section, consider the problem of estimating the proportion of females in the class.

In this case we know that there are 131 females in the class. Therefore, we have all of the population information

$$\pi = 131/314, \qquad \sigma^2 = 0.2431, \qquad N = 314$$

Because we have simple random sample, the probabilty distribution for $x_{I(j)}$ matches the population distribution, i.e.

$$\mathbb{P}\left(x_{I(j)} = 1\right) = 131/314 = 0.4172, \qquad \mathbb{P}\left(x_{I(j)} = 0\right) = 183/314 = 0.5818.$$

This means that

$$\mathbb{E}(x_{I(j)}) = 0.4172, \qquad \mathrm{Var}(x_{I(j)}) = 0.2431, \qquad \mathbb{E}(\bar{x}) = 0.4172$$

$$\mathrm{SE}(\bar{x}) = \sqrt{\frac{0.2431}{91} \times \frac{223}{313}} = 0.044$$

The exact distribution of $\bar{x}$ can be found:

$$\mathbb{P}(\bar{x} = m/91) = \mathbb{P}(\text{the sample has } m \text{ females}) = \frac{\binom{131}{m}\binom{183}{91-m}}{\binom{314}{91}}$$

This is known as hypergeometric distribution.

In a real sampling problem, the exact distribution of $\bar{x}$ is not known. However, it is known

- the distribution of $x_{I(j)}$ matches the propagation distribution
- $\bar{x}$ is the unbiased estimator of the population parameter
- provided n is large and n/N small, the distribution of $\bar{x}$ is roughly normal

This is enough to construct confidence intervals.

In this example n is large but $n/N = 91/314$ is not small. We will use bootstrap to check if we can use normal approximation.

In our example, 38 out of 91 students were females.

- Our estimate for the population parameter is $\bar{x} = 38/91 = 0.42$
- Our estimate of the standard error is

$$\frac{\sqrt{\frac{38}{91}(1-\frac{38}{91})}}{\sqrt{91-1}} \times \frac{\sqrt{314-91}}{\sqrt{314}} = 0.044$$

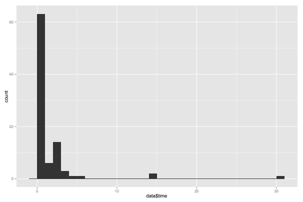
- Hence, our CI is $(0.33, 0.51)$

Finally, in this example. the actual proportion of women in the sample was very close to the expected proportion of women, $\mathbb{E}(\bar{x})$.

- We may want to calculate the probability that we would get as close or closer to the expected proportion.
- Sometimes, samples seem to be too close to what we expect: if the chance of getting data as close or closer to the expected value is small, say 1/100, then we might suspect the sampling procedure.
  - Expected number of women in the sample is $91 \times \pi = 37.97$
  -
$$\mathbb{P}(\text{exactly 38 of 91 students were women}) = \frac{\binom{131}{38}\binom{183}{91-38}}{\binom{314}{91}} = 0.10$$

From the histogram of the time spent playing video games by the students in the sample, we see that the sample distribution is extremely skewed.

This observation raises a question of whether the probability distribution of the sample average follows normal curve.



* Without knowledge of the population, we cannot answer this question completely.
* Bootstrap can help.

Bootstrap algorithms are simple and general tools for

(a) assessing estimators' accuracy via variance estimation, and
(b) producing confidence intervals and p-values.

Bootstrap applies to finite samples and provides numerical solutions for non-standard situations so that it is particularly appealing when dealing with finite populations and complex sampling designs

According to the simple random sample probability model, the distribution of the sample should look roughly similar to that of the population.

We could create a new population of 314 based on the sample and use this population, which we call the the bootstrap population, to find the probability distribution of the sample average.

The following table helps:

Time	Count	Bootstrap Population
0	57	197
0.1	1	3
0.5	5	17
1	5	17
1.5	1	4
2	14	48
3	3	11
4	1	3
5	1	4
14	2	7
30	1	3

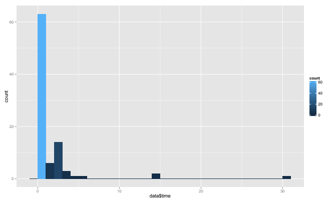
For every unit in the sample, we make $314/91 = 3.45$ units in the bootstrap population with the same time value and round off to the nearest integer.

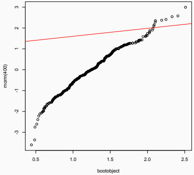
Next, to determine the probability distribution of the sample average when the sample is taken from the bootstrap population, we use a computer.

* Select a simple random sample of 91 from the bootstrap population, called a bootstrap sample, and take it advantage.
* Then we take another sample of 91 and take its average.
* A histogram of bootstrap sample averages, each constructed from a simple random damsel of 91 from the bootstrap population, appears in the Figure bellow.

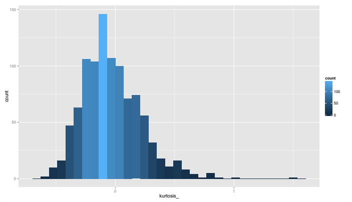
We took 400 bootstrap samples from the bootstrap population, in order to make a reasonable simulation of a probability distribution of the bootstrap average.

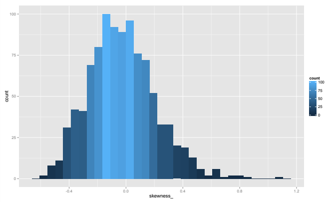
From the qq plot we see it to be close to normal distribution





* To further validate this claim we can compare the kurtosis - 2.67 and skewness - 0.19 for the 400 bootstrap sample averages to a simulated distribution of skewness and kurtosis for samples of size 400 from a normal distribution
* See figure bellow.





The bootstrap (cont.)

The method described here is one version of the bootstrap.

The bootstrap technique derives its name from the expression " to pull yourself up by your own bootstraps",

In the sampling context, we study the relation between bootstrap samples and the bootstrap population, where both the samples and the population are known, in order to learn about the relationship between out actual sample and the population, where the latter is unknown.

Bootstrap (cont.)-R code for the plots above

```
bootobject= NULL
for ( i in 1:400)
{
bootobject[i]=mean(sample(as.vector(data$time),size=91,replace=TRUE))
}
 m=qplot(bootobject, geom="histogram")
 m + geom_histogram(aes(fill = ..count..))
 require(e1071)
 kurtosis_=NULL
 for (i in 1:1000)
 {
kurtosis_[i]=kurtosis(rnorm(400))
 }
 m=qplot(kurtosis_, geom="histogram")
 m + geom_histogram(aes(fill = ..count..))
skewness_=NULL
for (i in 1:1000)
 {
skewness_[i]=kurtosis(rnorm(400))
 }
 m=qplot(skewness_, geom="histogram")
 m + geom_histogram(aes(fill = ..count..))
```

Jackknife

The single-elimination jackknife re-sampling technique

* Somewhat similar to the bootstrap re-sampling technique
* Do experiment once (here: 1000 measurements)
* Make 1000 samples of 999 measurements each ? with the i-th measurement eliminated in the i-th sample
* Compute the mean (or the wanted quantity) of each sample

Contrasts:

* The bootstrap method handles skewed distributions better
* The jackknife method is suitable for smaller original data samples