

Conditionnal coefficient for extremal graphical models

CAPEL Alexandre

2024-12-04

Contents

1	Extremal/Asymptotical independence	1
1.1	Case $d = 2$	1
1.2	Case $d > 2$	2
2	The conditional extremal coefficient χ_{ij}^A	2
2.1	The trivariate coefficient χ_{ijk}	2
2.2	Case where $A = \{k\}$	3
2.3	Definition for any A	3
2.4	Simulation study for the Hüsler-Reiss model	3
3	Proofs	5
3.1	Alternative writing for the trivariate coefficient	5
3.2	Inequalities of section 2	6

Let $V = \{1, \dots, d\}$ the set of nodes of a graph $\mathcal{G} = (V, E)$.

We want to define a new coefficient used for characterize the conditionnal dependance to apply for a pairwise Markov property in the extremal graphical model setting (Engelke and Hitz 2020).

1 Extremal/Asymptotical independence

Let X a random d -dimensional vector where the marginal distribution are standard Fréchet.

1.1 Case $d = 2$

One famous quantity in extreme value theory is the bivariate extremal coefficient which is defined as follow :

$$\chi_{1,2} = \lim_{q \rightarrow 1} \mathbb{P}(F(X_1) > q \mid F(X_2) > q) = \lim_{q \rightarrow 1} \chi_{1,2}(q)$$

if the right-hand term exists, where F is the distribution function of a standard Fréchet.

We say that X_1 and X_2 are asymptotically independent if $\chi_{1,2} = 0$ and asymptotically dependent otherwise.

If X is in the attraction domain of generalized Pareto distribution Y of exponent measure Λ , there is an equivalence between :

$$\chi_{1,2} = 0 \iff \left(\forall x = (x_1, x_2) > 0, \Lambda(x) = \Lambda_1(x_1) + \Lambda_2(x_2) \right)$$

And we know in (Strokorob 2020) that is equivalent to the extended extremal independance $Y_1 \perp\!\!\!\perp_e Y_2$.

1.2 Case $d > 2$

For some dimension d , we have similar results. Let A and B a partition of V . With the extended extremal independance of (Strokorob 2020), one can show a similar equivalence as above where :

$$Y_A \perp\!\!\!\perp_e Y_B \iff \left(\forall x = (x_A, x_B) > 0, \Lambda(x) = \Lambda_A(x_A) + \Lambda_B(x_B) \right)$$

We can also define the Sum of Extremal COefficient of the partition $\{A, B\}$ by :

$$SECO_{A,B} = \Lambda_A(\mathbb{1}_A) + \Lambda_B(\mathbb{1}_B) - \Lambda(\mathbb{1})$$

where Λ_C is the exponent measure of the extreme limit of the sub-vector X_C .

With (Ferreira 2011), we can show that $SECO_{A,B} = 0$ if and only if we have $Y_A \perp\!\!\!\perp_e Y_B$.

Motivation

For both situation, we can stress that there is a list of equivalent assertion which are following the same structure : extremal (or asymptotical) independance, exponent measure decomposition, and coefficient equality.

So we can wonder, in the context of extremal conditional independence, if we can define a conditional exponent measure Λ^C to have the relation :

$$Y_A \perp\!\!\!\perp_e Y_B \mid Y_C \iff \left(\forall x = (x_A, x_B) > 0, \Lambda^C(x) = \Lambda_A^C(x_A) + \Lambda_B^C(x_B) \right)$$

where A, B, C is a partition of V .

Furthermore, can we find some coefficient θ_{AB}^C (computable from data) and a constant $c \in \mathbb{R}$ such that :

$$\theta_{AB}^C = c \iff Y_A \perp\!\!\!\perp_e Y_B \mid Y_C$$

2 The conditional extremal coefficient χ_{ij}^A

The aim of the section is to present a new coefficient for extreme graphical models and try to link the general case to the univariate one, which is simpler to compute. As above, let X a random d -dimensional vector where the marginal distribution are standard Frechet.

2.1 The trivariate coefficient χ_{ijk}

We define the trivariate extremal coefficient as :

$$\chi_{ijk} = \lim_{q \rightarrow 1} \mathbb{P}(F(X_i) > q, F(X_j) > q \mid F(X_k) > q) = \lim_{q \rightarrow 1} \chi_{ijk}(q)$$

if the right-hand term exists.

In that case, if the vector X is in the attraction domain of Y , the corresponding generalized Pareto distribution, then we also get :

$$\chi_{ijk} = \mathbb{P}(Y_i > 1, Y_j > 1 \mid Y_k > 1) \tag{1}$$

We would like to use this coefficient in order to characterize the conditionnal independance (or dependance) between the component in a similar sense of the extremal conditionnal independance (see (Engelke and Hitz 2020)).

2.2 Case where $A = \{k\}$

By analogy on the extremal coefficient χ_{ij} , we define the conditionnal extremal coefficient by :

$$\chi_{ij}^k = \lim_{q \rightarrow 1} \mathbb{P}(F(X_i) > q \mid F(X_j) > q, F(X_k) > q) = \frac{\chi_{ijk}}{\chi_{jk}}$$

in the same setting as above.

Of course, the equation in the last section gives us another way to compute this quantity using the corresponding generalised Pareto distribution Y :

$$\chi_{ij}^k = \mathbb{P}(Y_i > 1 \mid Y_j > 1, Y_k > 1)$$

2.3 Definition for any A

Now, let's define for some $A \subset V$.

Let $i, j \in V$, then the conditionnal extremal coefficient is defined as :

$$\chi_{ij}^A = \lim_{q \rightarrow 1} \mathbb{P}(F(X_i) > q \mid F(X_j) > q, \|F(X_A)\|_\infty > q) = \lim_{q \rightarrow 1} \chi_{ij}^A(q)$$

where $F(X_A) = (F(X_k))_{k \in A}$, if the right-hand limit exists.

We could use different extension for a general subset A of V as for example $\lim_{q \rightarrow 1} \mathbb{P}(F(X_i) > q \mid F(X_j) > q, F(X_A) > q)$, but let's focus on the first one before.

First, we can write another expression for this coefficient, similar to the previous section, which is :

$$\chi_{ij}^A = \mathbb{P}(Y_i > 1 \mid Y_j > 1, \|Y_A\|_\infty > 1)$$

(the demonstration is the same as the case where $A = \{k\}$)

More, we have some inequalities between the conditional coefficient χ_{ij}^A and all sub-coefficient χ_{ij}^k for $k \in A$.

Proposition. Let $A \subset V$ and $i, j \notin A$. Then we have :

- $\chi_{ij}^k \leq |A| \chi_{ij}^A$ for all $k \in A$.
- $\chi_{ij}^A \leq \sum_{k \in A} \chi_{ij}^k$.

In particular, we have thanks to these inequalities, the equivalence :

$$\left(\forall k \in A, \chi_{ij}^k = 0 \right) \iff \chi_{ij}^A = 0$$

An appreciated properties will be that a conditional independence between two nodes occurs if and only if, $\chi_{ij}^A = 0$ with $A = V \setminus \{i, j\}$ and so $\chi_{ij}^k = 0$ for all $k \neq i, j$.

2.4 Simulation study for the Hüsler-Reiss model

We would like to test the last assumption in the case of a Hüsler-Reiss model, where the conditional independence can be verified easily. For $d = 3$ for example, let's define the variogram matrix of our Hüsler-Reiss model :

$$\Gamma = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}$$

Then, thanks to the extended precision matrix Θ defined in (Hentschel, Engelke, and Segers 2023) we have that :

- $Y_1 \perp\!\!\!\perp Y_2 | Y_3 \iff a = b + c$
- $Y_1 \perp\!\!\!\perp Y_3 | Y_2 \iff b = a + c$
- $Y_2 \perp\!\!\!\perp Y_3 | Y_1 \iff c = a + b$

More, in the Hüsler-Reiss model, the extremal coefficient (bivariate an trivariate) can be computed fastly using :

- $\chi_{ij} = 2 - 2\Phi(\sqrt{\Gamma_{ij}}/2)$
- $\chi_{ijk} = 3 - 2\Phi(\sqrt{\Gamma_{ij}}/2) - 2\Phi(\sqrt{\Gamma_{jk}}/2) - 2\Phi(\sqrt{\Gamma_{ik}}/2) + \sum_{k=1}^3 \Phi_2((\sqrt{\Gamma_{ik}}/2)_{i \neq k}; A_k \Sigma^{(k)} A_k)$ with $A_k = \text{diag}((\sqrt{\Gamma_{ik}}^{-1})_{i \neq k})$ and where Φ is the distribution function of a standard Gaussian variable and $\Phi_2(\cdot; \Sigma)$ the distribution function of a gaussian vector of covariance matrix Σ .

We produce a random matrix restricted by the relation $a = b + c$ (to get a graph with no node linking 1 and 2). The coefficients b and c are simulated with by an uniform distribution on $[0, 5]$.

```
tar_load(Gamma_cond_indep_12)      # load our simulated variogram
print(Gamma_cond_indep_12)
```

```
##      [,1]      [,2]      [,3]
## [1,] 0.00000 5.75331 4.73967
## [2,] 5.75331 0.00000 1.01364
## [3,] 4.73967 1.01364 0.00000
```

Of course, we have to check the conditional negative definiteness of the matrix (to know if the matrix is a possible Hüsler-Reiss variogram). In order to do that we use the `graphicalExtremes`.

```
checkGamma(Gamma_cond_indep_12, returnBoolean = T)      # function checking the validity of the matrix

## [1] TRUE
```

Now, we are able to compute the trivariate coefficient, and consequently check if this is zero or not.

```
tar_load(chi_trivariate_example)      # load the value of the trivariate coef for our matrix
print(chi_trivariate_example)
```

```
## [1] 0.2001413
```

So finally it is not 0...

This is not very surprising because the condition that we wanted is about a coefficient which is totally symmetric between i, j and k , so the conditional independence (in the $d = 3$ case at least) of two implies the conditional independence for all and then the graph is empty.

But, let's check another thing. We are going to simulate several matrix and see what is the trend for the value of the conditional extremal coefficient.

TODO

```
tar_load(chi_trivariate_simulation_results)

chi_trivariate_simulation_results |>
  ggplot() + aes(x = num_sim, y = tri_chi) +
  geom_line(col = "darkorange") +
  coord_cartesian(ylim = c(0, 1))+
  xlab("Simulation number") +
  ylab("Trivariate coefficient")+
  ggtitle("Trivariate coefficient variation") +
```

```

theme_ipsum()

chi_trivariate_simulation_results |>
  ggplot() + aes(x = tri_chi) +
  geom_boxplot(col = "#474F58", fill = "darkorange") +
  xlab("Trivariate coefficient")+
  ggtitle("Trivariate coefficient boxplot simulation") +
  theme_ipsum()

```

3 Proofs

3.1 Alternative writing for the trivariate coefficient

We will show that we can compute the trivariate coefficient with the formula :

$$3 - \Lambda_{ij}(1, 1) - \Lambda_{ik}(1, 1) - \Lambda_{jk}(1, 1) + \Lambda_{ijk}(1, 1, 1)$$

and the two writing can be expressed with the latter.

First, let's take a look to the limit side :

$$\begin{aligned}
\chi_{ijk}(q) &= \frac{1}{1-q} [1 - 3q + \mathbb{P}(F(X_i) < q, F(X_j) < q) + \mathbb{P}(F(X_i) < q, F(X_k) < q) \\
&\quad + \mathbb{P}(F(X_j) < q, F(X_k) < q) - \mathbb{P}(F(X_i) < q, F(X_j) < q, F(X_k) < q)] \\
&= 3 - \frac{1 - \mathbb{P}(F(X_i) < q, F(X_j) < q)}{1-q} - \frac{1 - \mathbb{P}(F(X_i) < q, F(X_k) < q)}{1-q} \\
&\quad - \frac{1 - \mathbb{P}(F(X_j) < q, F(X_k) < q)}{1-q} + \frac{1 - \mathbb{P}(F(X_i) < q, F(X_j) < q, F(X_k) < q)}{1-q} \\
&= 3 - \frac{1}{F^{-1}(q)} \left[\frac{F^{-1}(q)(1 - \mathbb{P}(X_{ij} < F^{-1}(q)))}{1-q} - \frac{F^{-1}(q)(1 - \mathbb{P}(X_{ik} < F^{-1}(q)))}{1-q} \right. \\
&\quad \left. - \frac{F^{-1}(q)(1 - \mathbb{P}(X_{jk} < F^{-1}(q)))}{1-q} + \frac{F^{-1}(q)(1 - \mathbb{P}(X_{ijk} < F^{-1}(q)))}{1-q} \right]
\end{aligned}$$

But, as X is a random vector with standard Frechet marginals, X is a regular variation random vector, that means that :

$$\lim_{u \rightarrow \infty} u(1 - \mathbb{P}(X_C < u)) = \Lambda_C(\mathbb{I}_C)$$

where Λ_C is the exponent measure of the marginal extreme limit of X_C .

But, we have $F^{-1}(q) \rightarrow \infty$ when $q \rightarrow 1$ and $-\log(q) \sim 1 - q$ so we obtain :

$$\lim_{q \rightarrow 1} \chi_{ijk}(q) = 3 - \Lambda_{ij}(1, 1) - \Lambda_{ik}(1, 1) - \Lambda_{jk}(1, 1) + \Lambda_{ijk}(1, 1, 1)$$

Now, let's develop the other term. Using the equation (16) in (Engelke and Hitz 2020), if we note λ the density of the exponent measure with respect to the Lebesgue measure, we have :

$$\mathbb{P}(Y_i > 1, Y_j > 1 \mid Y_k > 1) = \int_{[1, \infty]^3} \lambda(y) dy = \Lambda([1, \infty]^3)$$

The properties of valid density give us 3 equations that we can use to get that :

$$3 = \Lambda([0, \infty[\times [1, \infty[^2] + \Lambda([1, \infty[\times]0, \infty[\times [1, \infty[) + \Lambda([1, \infty[^2 \times]0, \infty[)$$

These quantities can be decomposed as several “volume” in the space. We can merge some volume between them to obtain this relation :

$$3 = \Lambda_{ij}(1, 1) + \Lambda_{ik}(1, 1) + \Lambda_{jk}(1, 1) - \Lambda_{ijk}(1, 1, 1) + \Lambda([1, \infty[^3)$$

and we have the right formula by isolating the $\Lambda([1, \infty[^3)$ term.

3.2 Inequalities of section 2

- $\chi_{ij}^k \leq |A| \chi_{ij}^A$:

Let $k \in A$. We have by definition that :

$$\begin{aligned} \mathbb{P}(F(X_i) > q \mid F(X_j) > q, F(X_k) > q) &= \frac{\mathbb{P}(F(X_i) > q, F(X_j) > q, F(X_k) > q)}{\mathbb{P}(F(X_j) > q, F(X_k) > q)} \\ &\leq \frac{\mathbb{P}(F(X_i) > q, F(X_j) > q, \|F(X_A)\|_\infty > q)}{\mathbb{P}(F(X_j) > q, F(X_k) > q)} \\ &\leq \chi_{ij}^A(q) \frac{\mathbb{P}(F(X_j) > q, \|F(X_A)\|_\infty > q)}{\mathbb{P}(F(X_j) > q, F(X_k) > q)} \\ &\leq \chi_{ij}^A(q) \sum_{k \in A} \frac{\mathbb{P}(F(X_j) > q, F(X_k) > q)}{\mathbb{P}(F(X_j) > q, F(X_k) > q)} \\ &\leq |A| \times \chi_{ij}^A(q) \end{aligned}$$

If we take the limit of both side, we get :

$$\chi_{ij}^k \leq |A| \times \chi_{ij}^A$$

which shows the first inequality.

- $\chi_{ij}^A \leq \sum_{k \in A} \chi_{ij}^k$:

We use the alternative writing for this proof. If we apply again the definition, we have :

$$\begin{aligned} \chi_{ij}^A &= \frac{\mathbb{P}(Y_i > 1, Y_j > 1, \|Y_A\|_\infty > 1)}{\mathbb{P}(Y_j > 1, \|Y_A\|_\infty > 1)} \\ &\leq \sum_{k \in A} \frac{\mathbb{P}(Y_i > 1, Y_j > 1, Y_k > 1)}{\mathbb{P}(Y_j > 1, \|Y_A\|_\infty > 1)} \\ &\leq \sum_{k \in A} \chi_{ij}^k \frac{\mathbb{P}(Y_j > 1, Y_k > 1)}{\mathbb{P}(Y_j > 1, \|Y_A\|_\infty > 1)} \end{aligned}$$

but as $\mathbb{P}(Y_j > 1, Y_k > 1) \leq \mathbb{P}(Y_j > 1, \|Y_A\|_\infty > 1)$ for all $k \in A$, we finally have :

$$\chi_{ij}^A \leq \sum_{k \in A} \chi_{ij}^k$$

which is what we wanted.

References

- Engelke, Sebastian, and Adrien S. Hitz. 2020. “Graphical Models for Extremes.” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 82 (4): 871–932. <https://doi.org/10.1111/rssb.12355>.
- Ferreira, H. 2011. “Dependence Between Two Multivariate Extremes.” *Statistics & Probability Letters* 81 (5): 586–91. <https://doi.org/10.1016/j.spl.2011.01.014>.
- Hentschel, Manuel, Sebastian Engelke, and Johan Segers. 2023. “Statistical Inference for Hüsler-Reiss Graphical Models Through Matrix Completions.” arXiv. <https://doi.org/10.48550/arXiv.2210.14292>.
- Strokorb, Kirstin. 2020. “Extremal Independence Old and New.” arXiv. <https://doi.org/10.48550/arXiv.2002.07808>.