Chap 2 : Linear regreession with least square estimate

Prof M.Tognolini HEIG-VD, Feb. 2022

Exercice 2.2 Signal approximation and de-noising

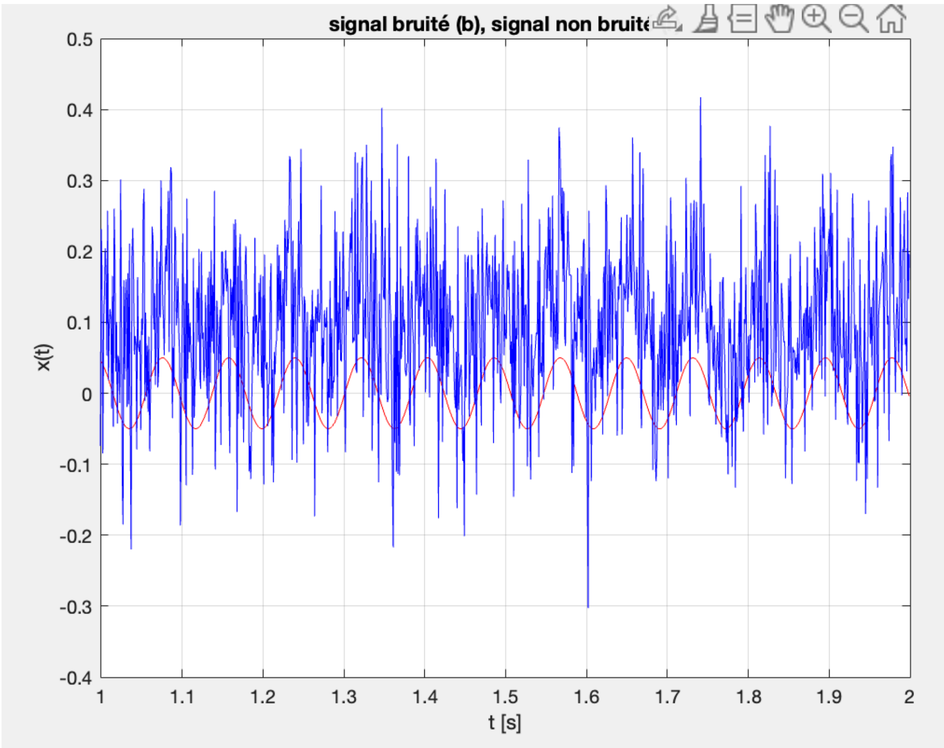


Figure 1 : Noisy signal (blue) , original and un-noisy signaal (red)

In this exemple of noisy signal see in Figure 1 , we need to recover the original signal. The original signal has a frequency between 10 and 15 Hz with an unknown amplitude and phase. The whole signal is a record of 10'000 samples with a sampling frequency of 1 kHz.

in order to use the least square approximation we write a n overdetermined eqaution system that for each sample of the signal x[n] correspond a sum of pondered basis signals $\phi_1 = [1, 1, 1, 1, ...]$ and $\phi_i = sin(2\pi f_i t)$ and $\phi_j = cos(2\pi f_j t)$:

$x[n] = k_1\phi_1[n] + k_2\phi_2[n] + \cdots + k_m\phi_m[n]$ (1)

This is a linear system of equations with the unknown $k_1 \cdots k_m$ bit it is overdetermined because the number of equation is much greather than the number on unknown.

in matrix form we could write it as :

$\Phi[n] \cdot \Theta = x^T[n]$ (2)

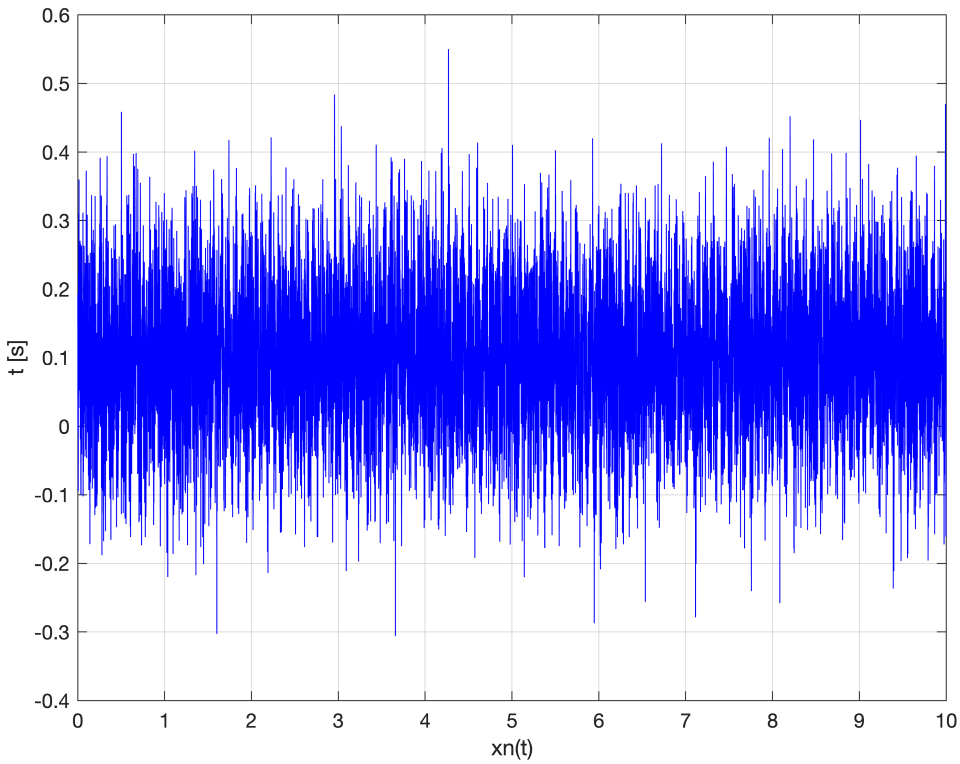
where

$\Theta = [\phi_1^T[n], \phi_2^T[n], \cdots, \phi_m^T[n]]$ (3)

So we could find the solution by solving by pseudo inverse of

Import the data and plot

First import the data in the file NoisySignal.mat and plot it.



In this signal we hide a harmonic signal with a unknown frequency between 10 and 15 Hz with unknown amplitude and phase.

The goal is to retrive the signal from the noise and determine his parameters.

Regressors vectors build

First we need to build the regression vecrors $\Phi_i[n]$ containing each one a base function with N samples as the length of the signal x.

as regressor vertors we need to generate à constant =1 vector because we could se the x signal has an ofset, and sin vectors having an amplitude of 1 and a frequency f between 10 and 15 Hz with a step of 0.1 Hz (because the resolution of frequency we could resolve could be > 1/Duration). The same of the regressors with cos function at the same frequency:

First build a vector containing all the freq values from 10 to 15 Hz with a step of 0.1 Hz let's call it f.

Second generate the regressors : $\phi_1 = [1, 1, 1, 1, ...]$ and $\phi_i = sin(2\pi f_i t)$ and $\phi_j = cos(2\pi f_j t)$

and third generate the regressors matrix Φ containing all the M ragressors as column vectors :

You couls notice the regressors are orthogonals (test it), what is the energy of each regressors ?

Normalize the regressors to have an energy equal 1

Projection of the signal x into the new vectorspace composed by regressors

Solving the overdetermined equation system equation (2) with pseudo inverse matrix

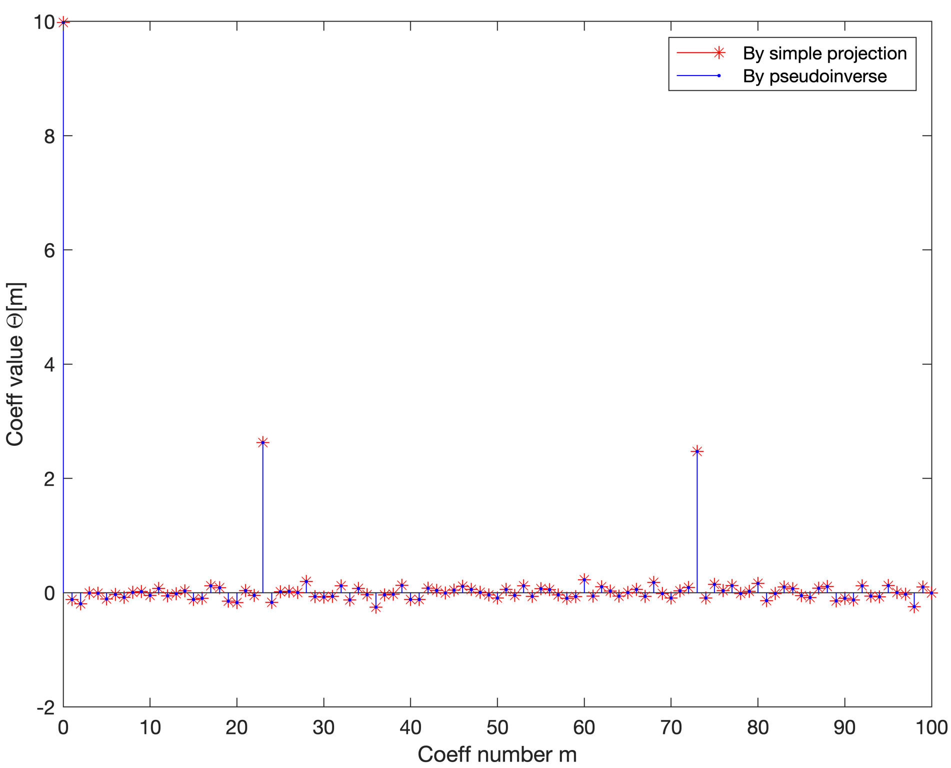
$P = (\Phi[n]^T \Phi[n])^{-1} \Phi[n]^T \quad \Theta = P \cdot x^T[n]$ (4)

In our case the regressors are orthonormal and the product $(\Phi[n]^T \Phi[n]) = I$ is an identity matrix with size MxM.

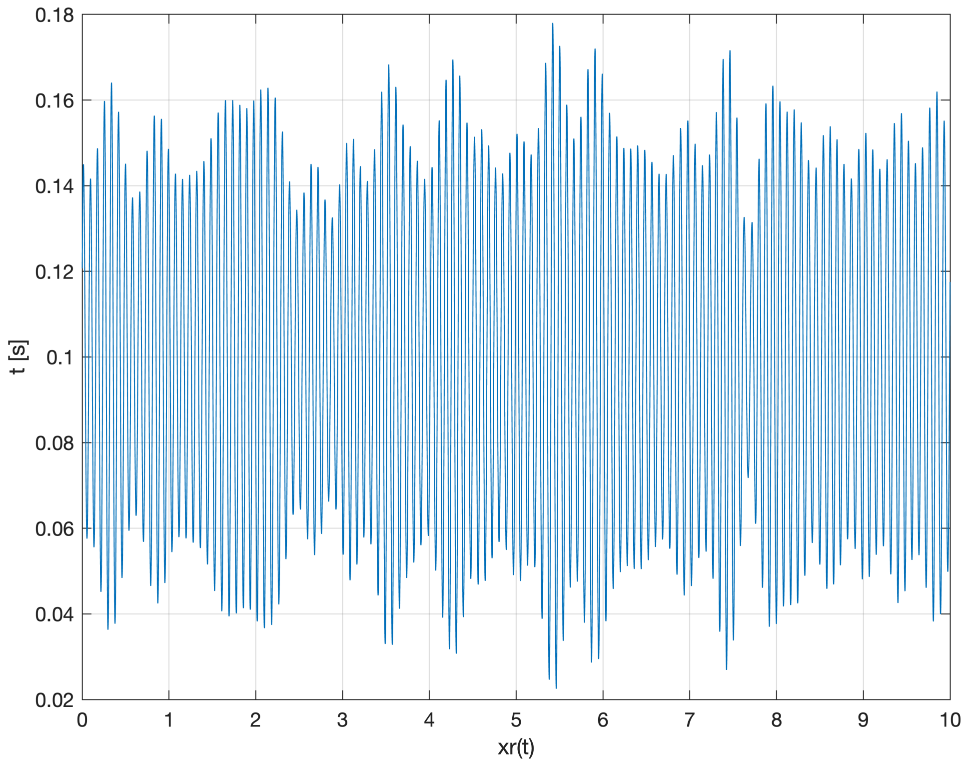
The consequence of this fact is the simplicity of the coefficients computation that are given by :

$\Theta = (\Phi[n]^T \Phi[n])^{-1} \Phi[n]^T x[n]^T = \Phi[n]^T x[n]^T$ (5)

Compute the coefficients Θ using the pseudoinverse P as in (4), and the simple projection givent in (5). Plot the coefficients.



We could compute the approximation of the signal using all the coefficients , and this is the best approximation of x called $\hat{x}[n] = \Phi_n \cdot \Theta$ (6)



The error of the approximation $e[n] = x[n] - \hat{x}[n]$ could be computerd as his energy We $W_e = ||e||^2/N^2$

```
We = 9.8771e-07

Wxr = 1.1358e-06

Wxn = 2.1235e-06
```

Signal denoising

If the goal is to retrive a specific signal that we know some details as the shape (sinus, ...) we just take the specific coefficients (those with the bigger magnitude), and only with this coefficients we compute the retrived signal. In this case we take only 2 coefficients corresponding the sinus and cosinus regressors at the same frequency.

The frequency fmax of the signal corresponds to the maximum coefficient value index (icos):

```
MaxCos = 2.6281
icos = 23

MaxSin = 2.4742
isin = 23

fmax = 12.2000
```

The signal recovery is then computed with 2 coefficients MaxCos and MaxSin , the amplitude Axr RMS cold be computed and also tha phase Phxr.

```
Axr = 3.6096e-04

Phxr = 0.7552
```

Plot the retrived signal xrsin(t) and the original signal x(t) we give for verification in the dataset file NoisySignal.mat.

the error of retriving signal has an energy   has to be < 3e-10:

```
Wexrsin = 1.7086e-10
```