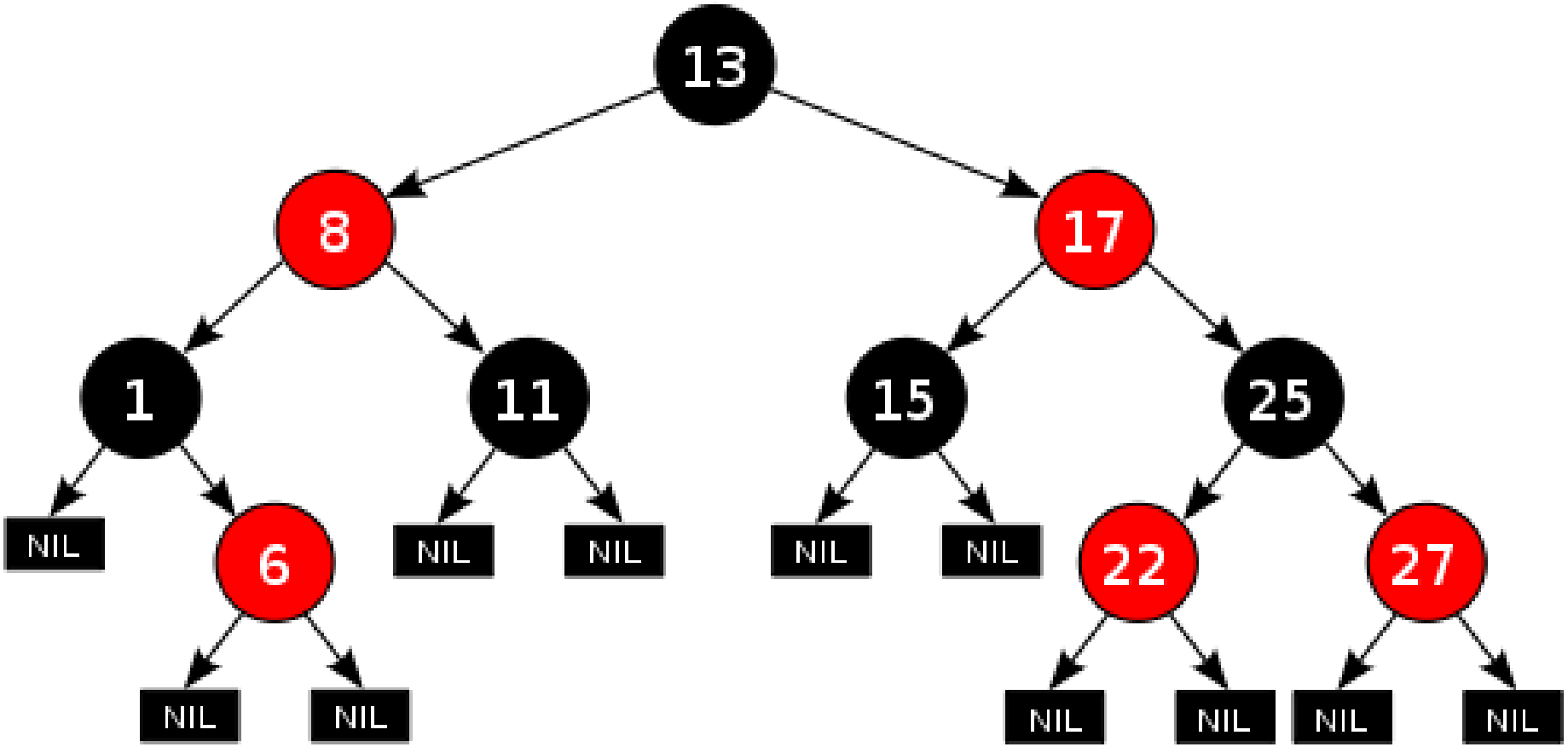
Red-Black Trees



Source: wikipedia.org

Red-Black Tree Properties (Definition of RB Trees)

A red-black tree is a BST with following properties:

1. Every node is either red or black.
2. The root is black.
3. Every leaf is NIL and black.
4. Both children of each red node are black.
5. All root-to-leaf paths contain the same number of black nodes.

More Properties

1. No root-to-leaf path contains two consecutive red nodes.
2. For each node x, all paths from x to descendant leaves contain the same number of black nodes. This number, not counting x, is the black height of x, denoted $bh(x)$.
3. No root-to-leaf path is more than twice as long as any other.

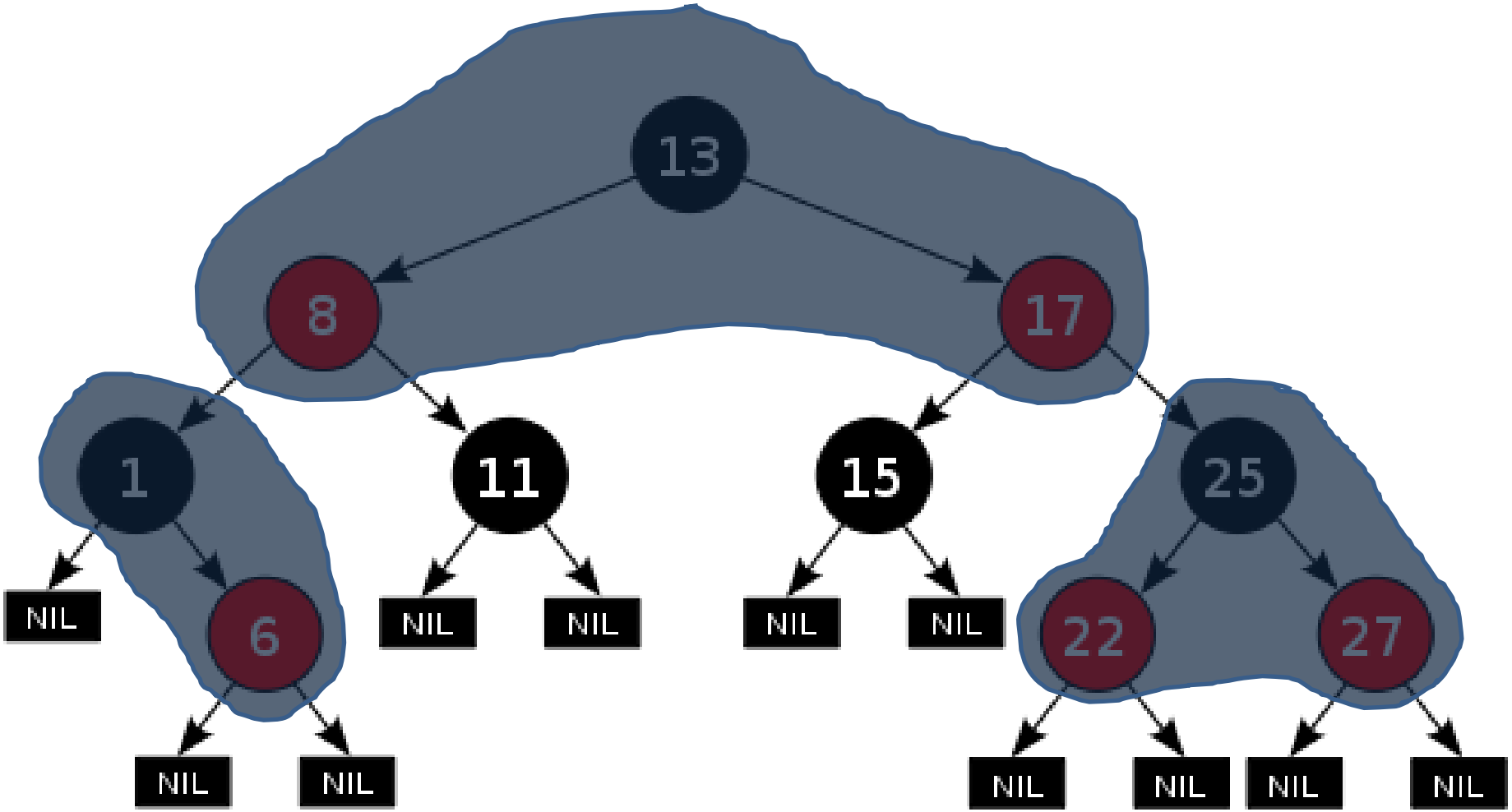
Theorem. A red-black tree with n internal nodes has height $\leq 2 \log(n + 1)$.

Proof of Theorem

- Consider a red black tree with height h.
- Collapse all red nodes into their (black) parent nodes to get a tree with all black nodes.
- Each internal node has 2 to 4 children.
- The height of the collapsed tree is $h' \geq h / 2$, and all external nodes are at the same level.
- Number of internal nodes in collapsed tree is

$$n \geq 1 + 2 + 2^2 + \cdots + 2^{h'-1} = 2^{h'} - 1 \geq 2^{h/2} - 1.$$

- So, $h \leq 2\log_2(n+1)$.



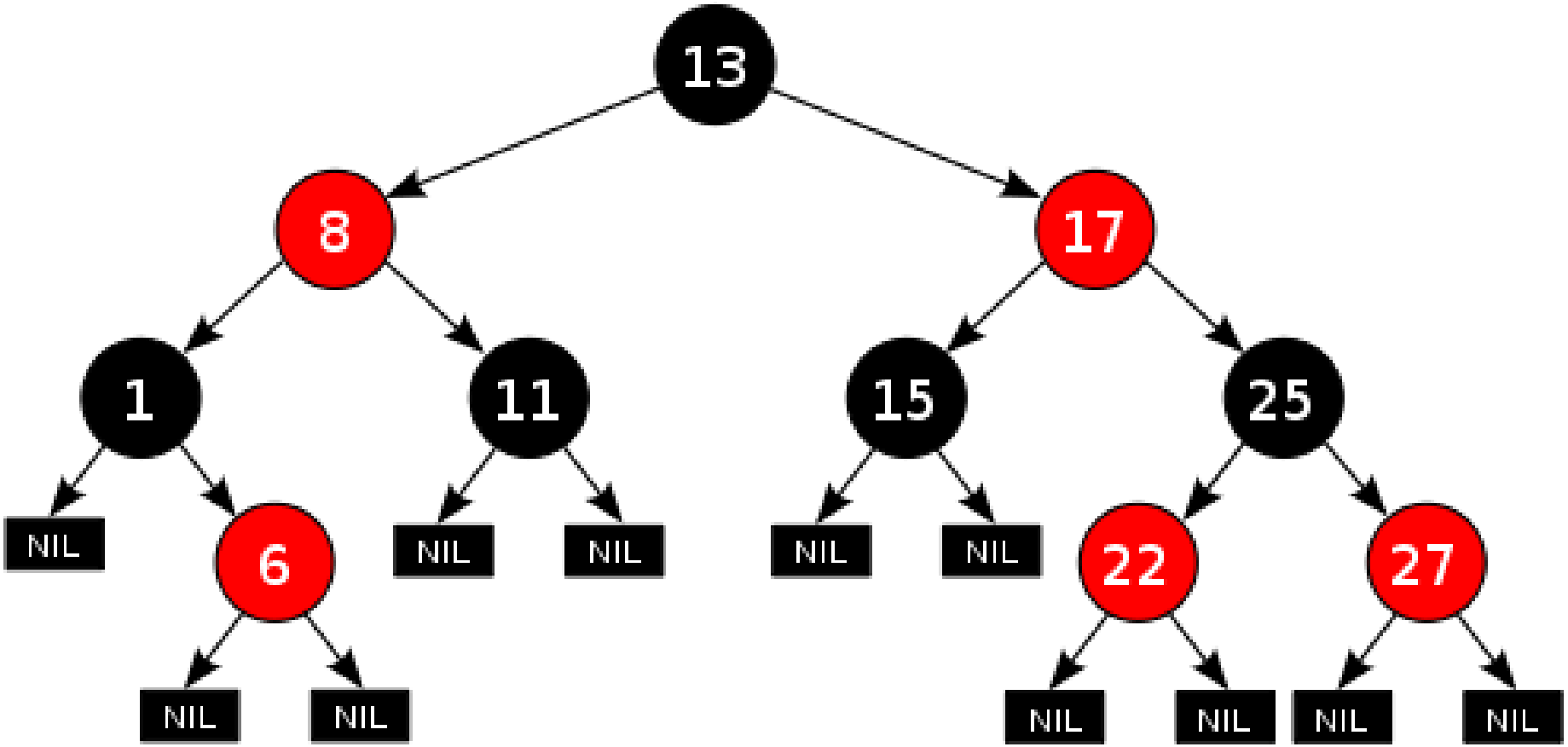
Source: wikipedia.org

Insert a node z

- Insert *z* as in a regular BST; color it red.
- If any violation to RB properties, fix it.
- Possible violations:
  - The root is red. (Case 0)

    To fix up, make it black.
  - Both z and z's parent are red.

    To fix up, consider three cases. (Actually, six cases: I, II, III, I', II', III')

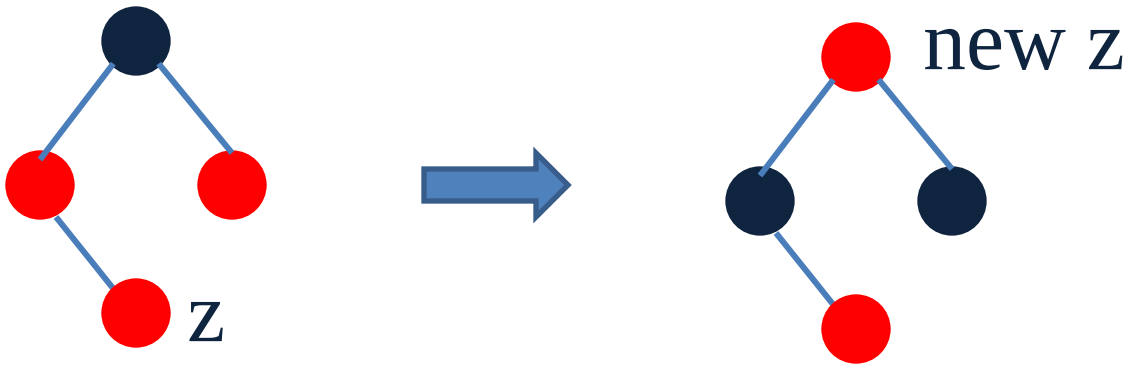


Source: wikipedia.org

Insert Fixup: Case I

The parent and “uncle” of z are both red:

- Color the parent and uncle of z black;
- Color the grandparent of z red;
- Repeat on the grandparent of z.

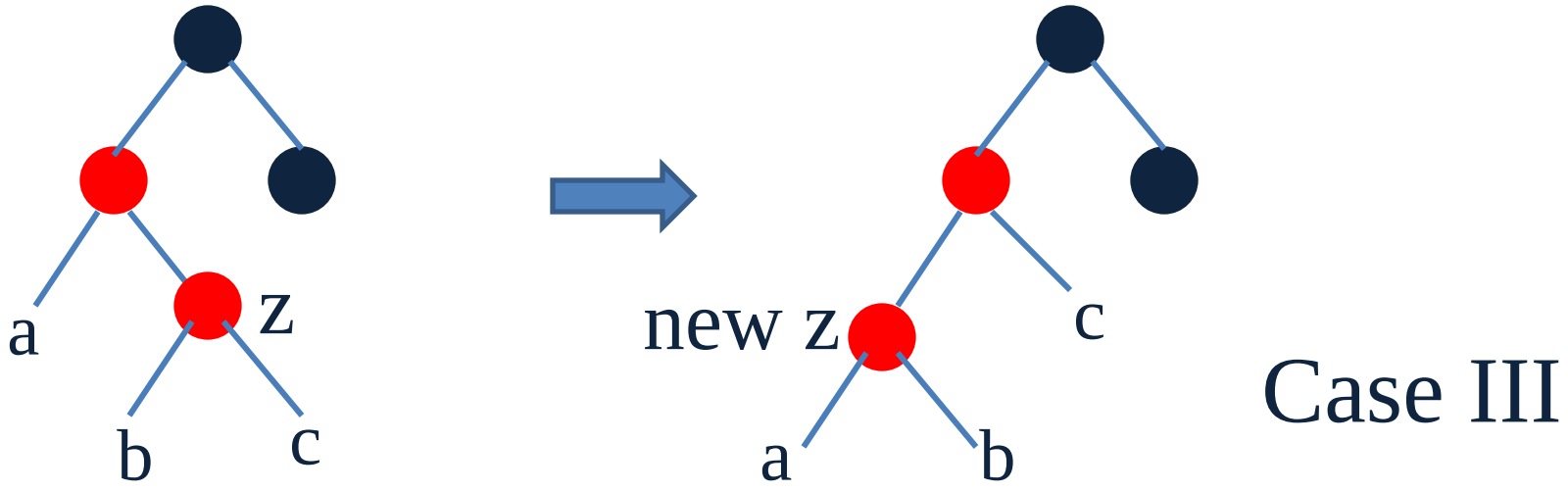


Insert Fixup: Case II

The parent of z is red, the uncle of z is black,

z's parent is a left child , z is a right child:

- Left Rotate on z's parent;
- Make z's left child the new z; it becomes Case III.

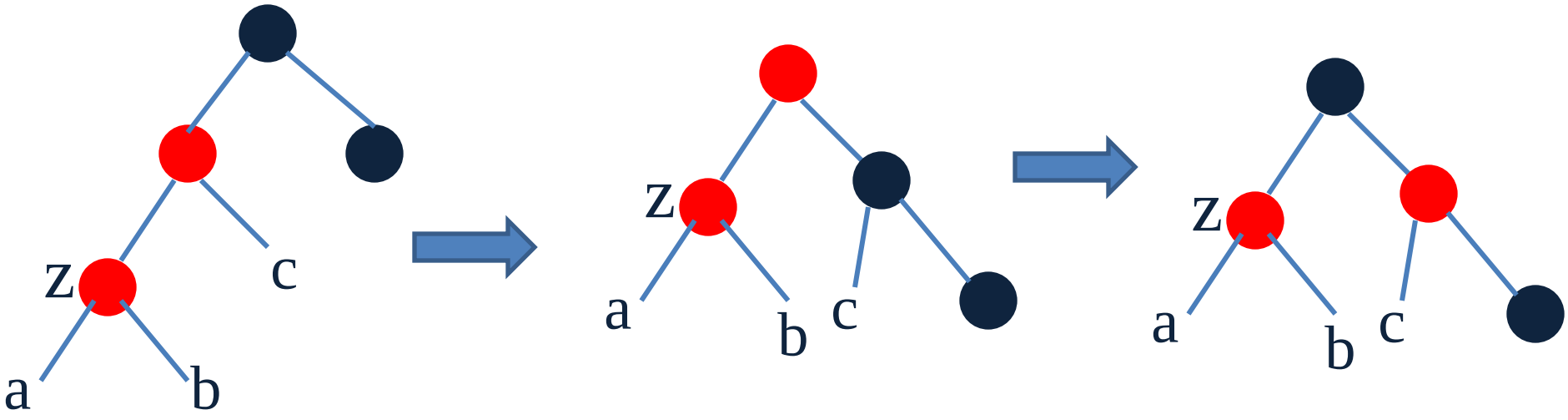


Case III

Insert Fixup: Case III

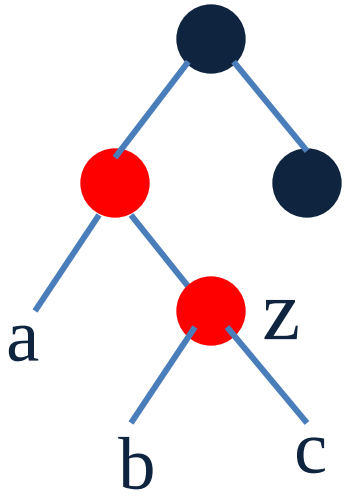
The parent of z is Red and the “uncle” is Black, z is a left child, and its parent is a left child:

- Right Rotate on the grandparent of z.
- Switch colors of z’s parent and z’s sibling.
- Done!

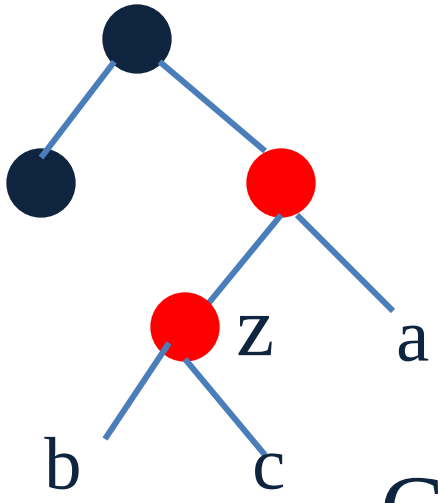


Case II’

Symmetric to Case II



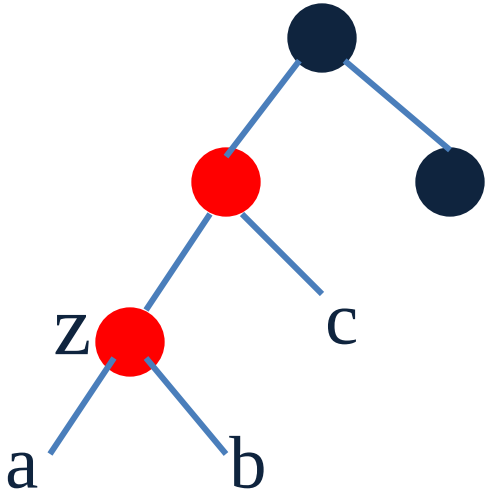
Case II



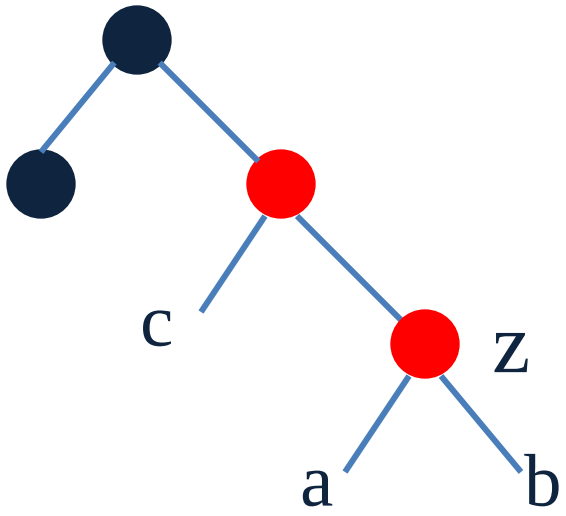
Case II’

Case III’

Symmetric to Case III



Case III



Case III’

Demonstration

- http://www.ece.uc.edu/~franco/C321/html/RedBlack/redblack.html

BST Deletion Revisited

Delete z

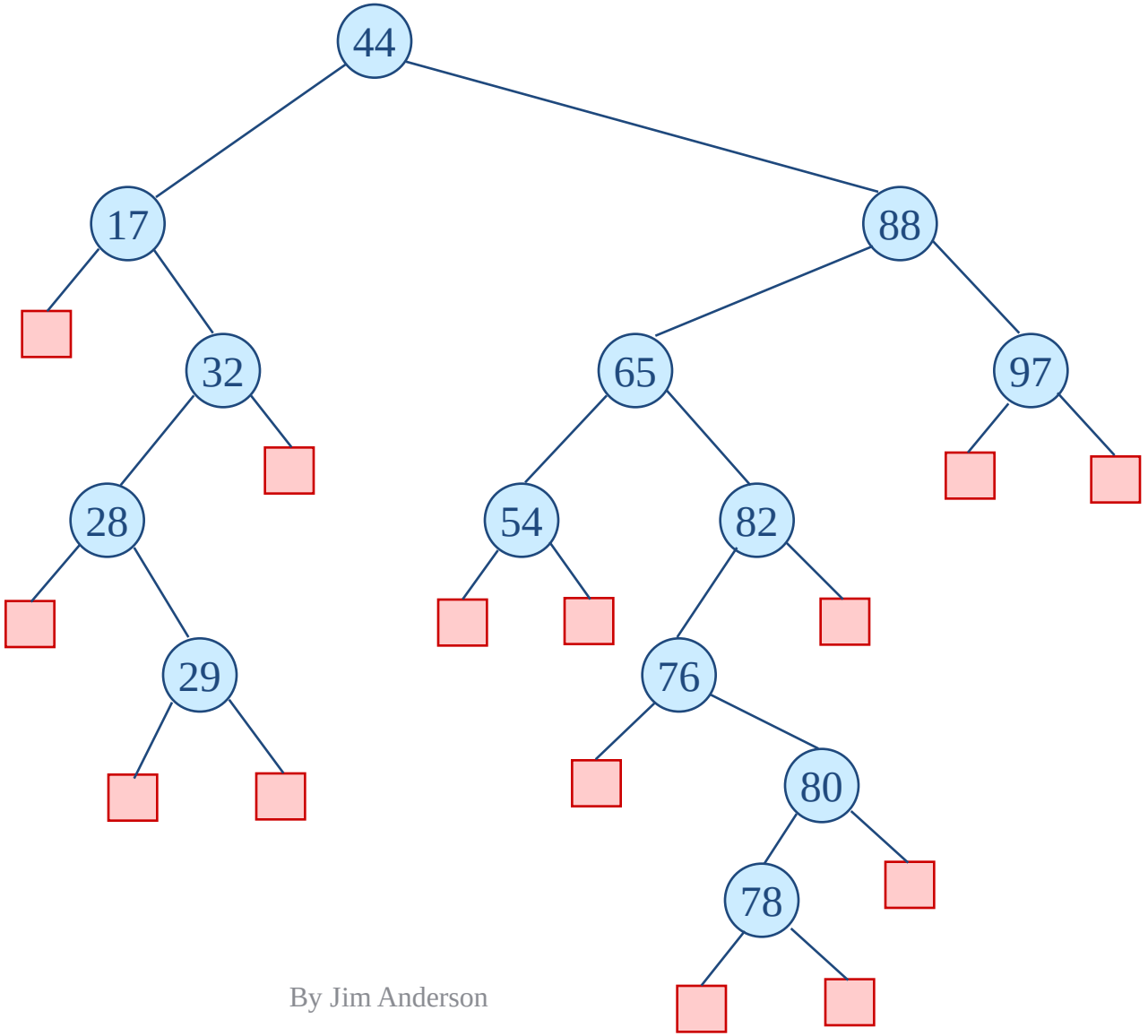
- If z has no children, we just remove it.
- If z has only one child, we splice out z.
- If z has two children, we splice out its successor y, and then replace z's key and satellite data with y's key and satellite data.

- Which physical node is deleted from the tree?

BST: Delete

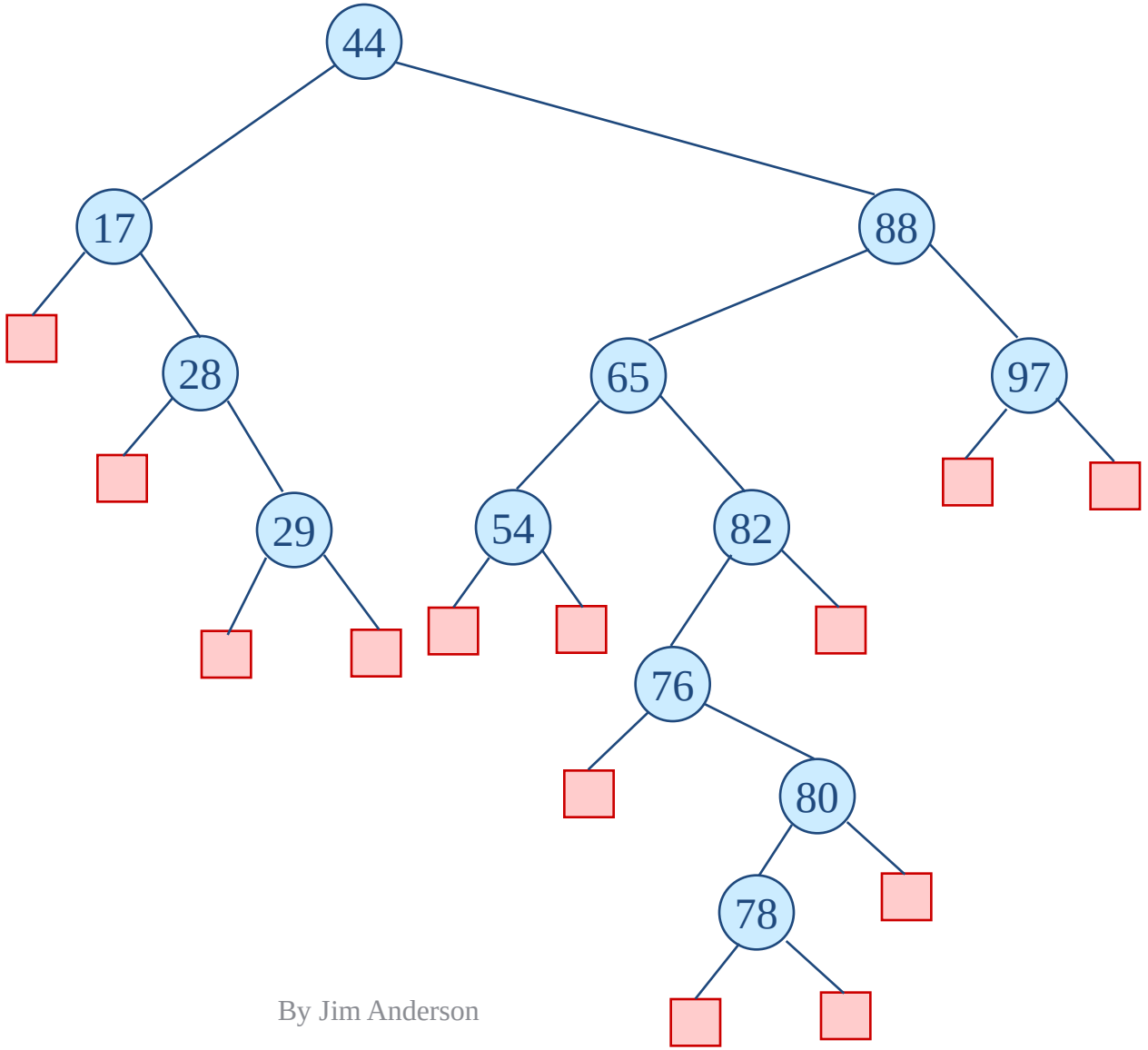
Delete(32)

Has only one child: just splice out 32.



BST: Delete

Delete(32)

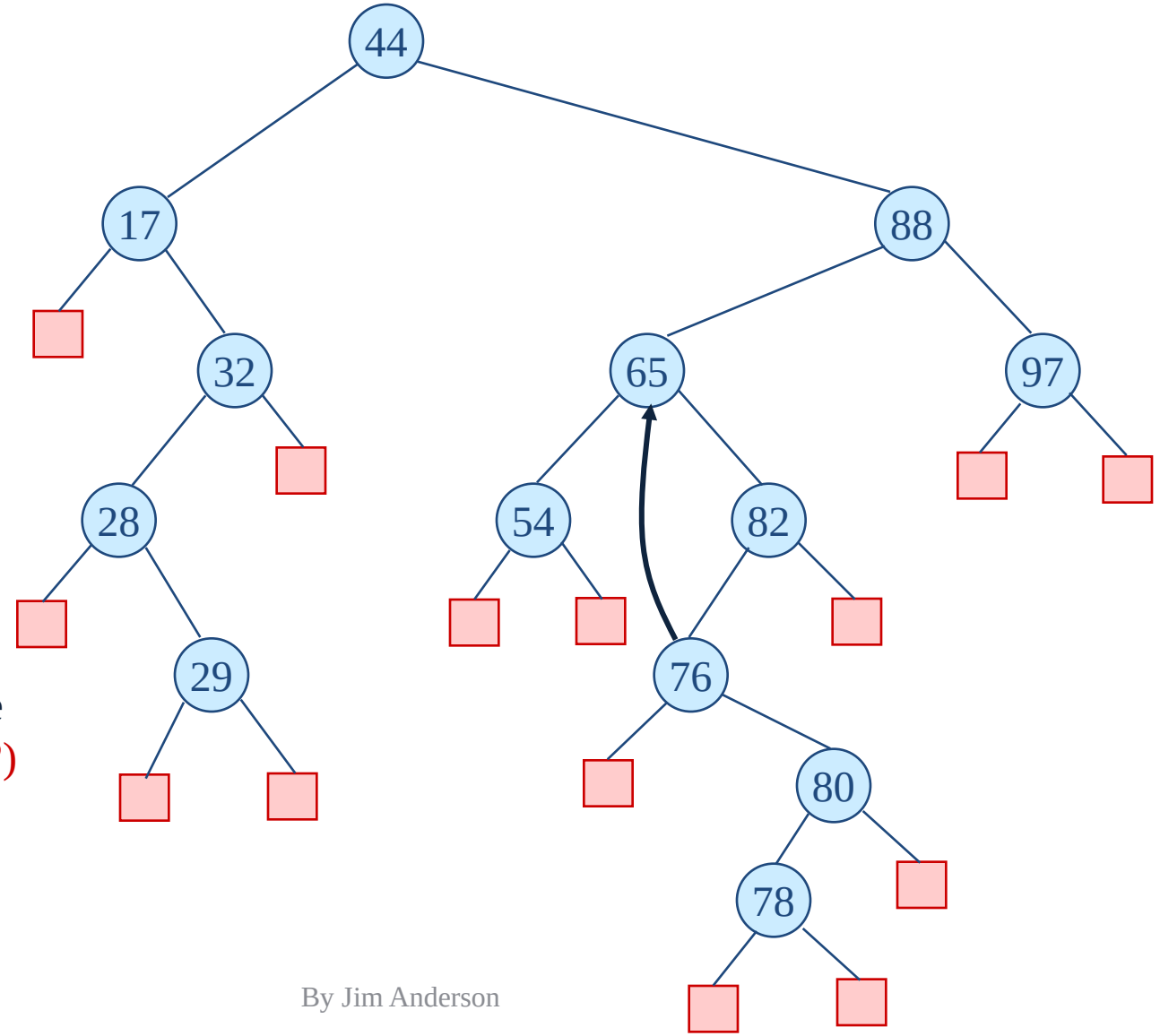


BST: Delete

Delete(65)

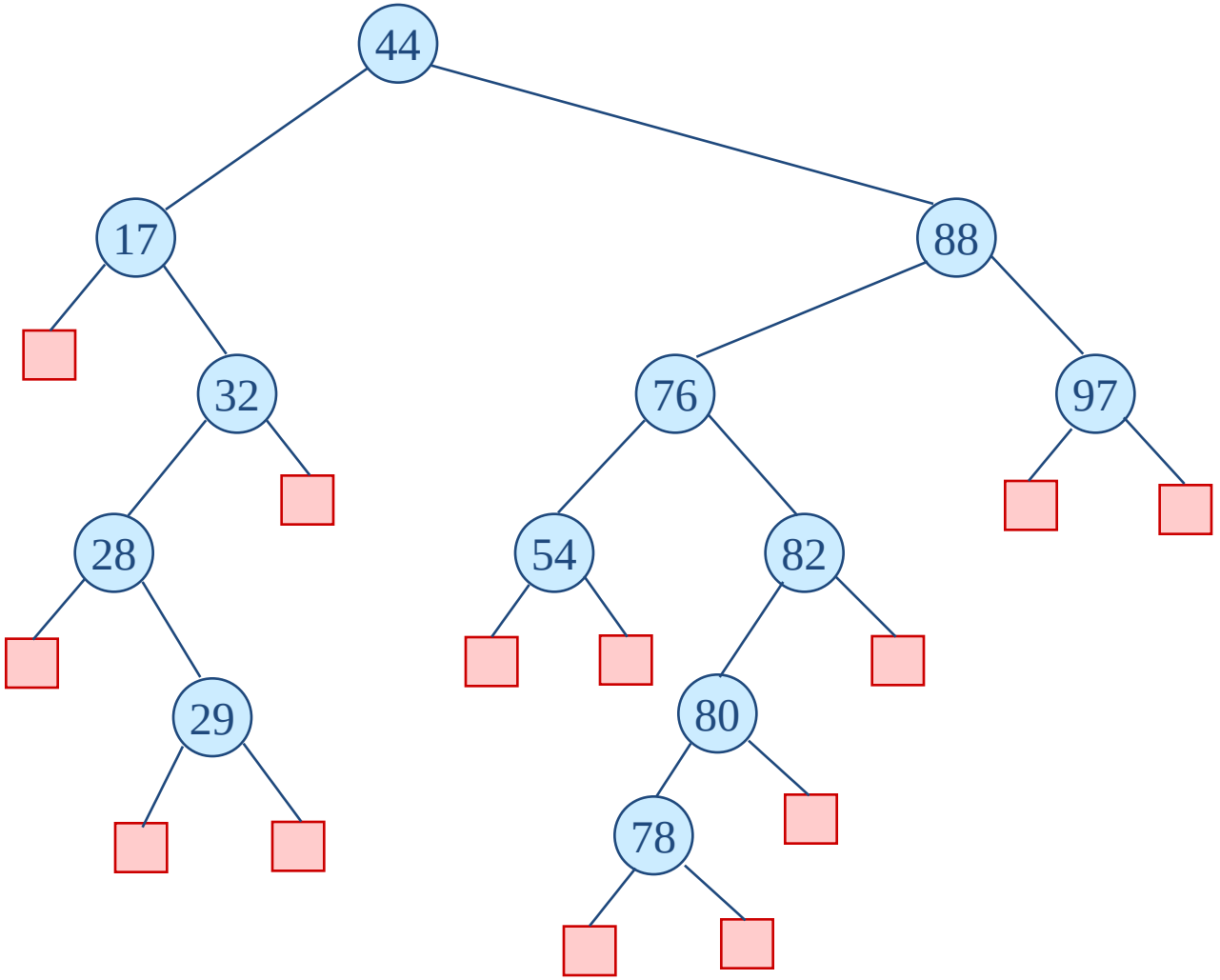
Has two children:
Replace 65 by successor, 76, and splice out successor.

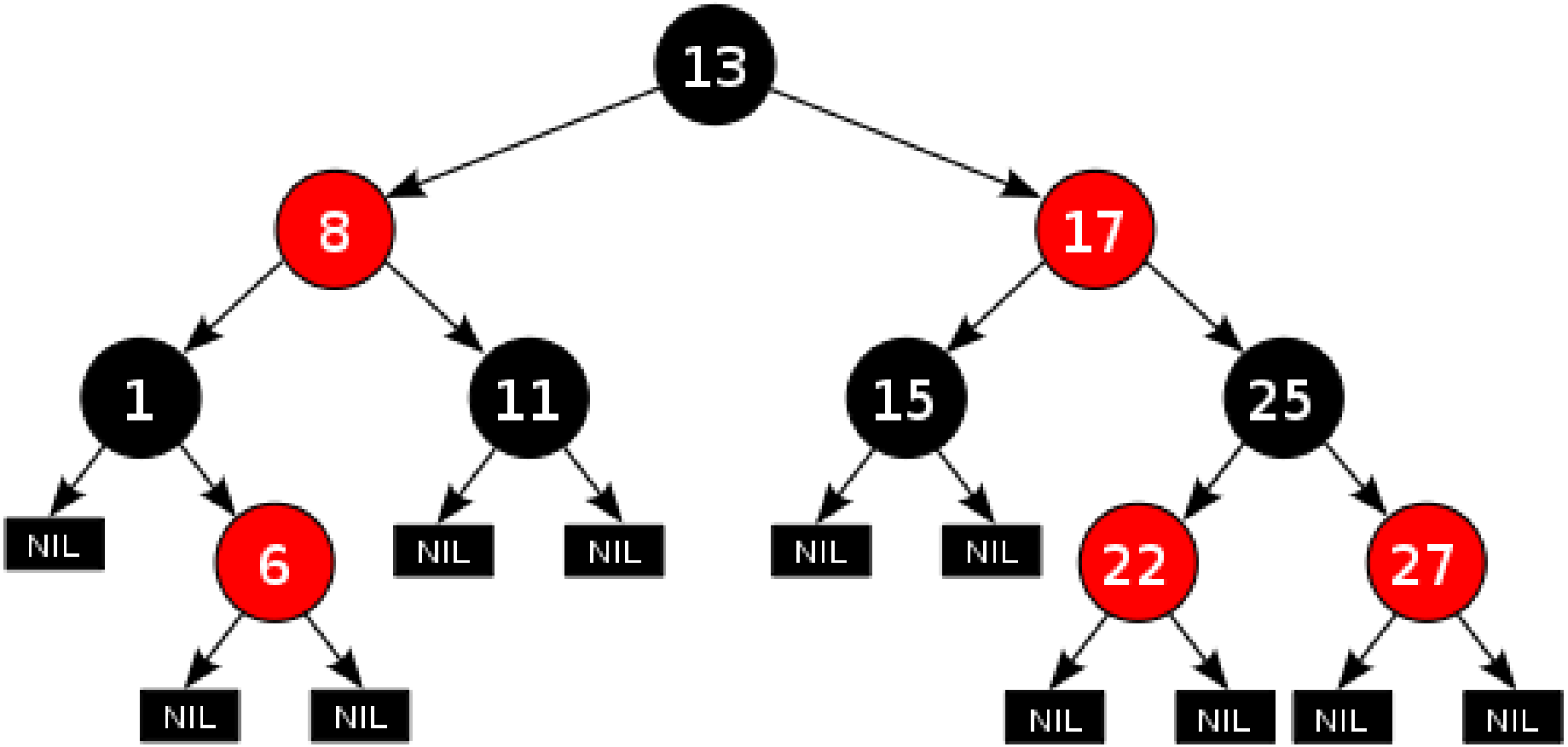
Note: Successor can have at most one child. (Why?)



BST: Delete

Delete(65)





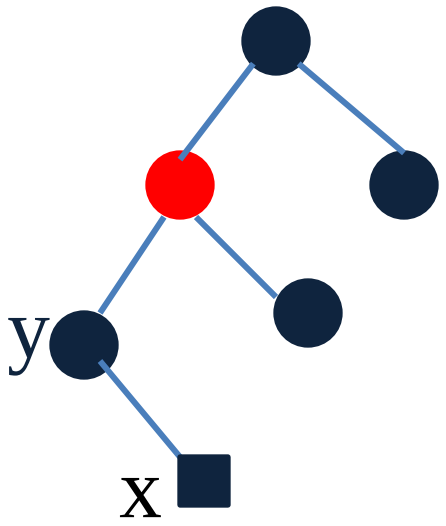
Source: wikipedia.org

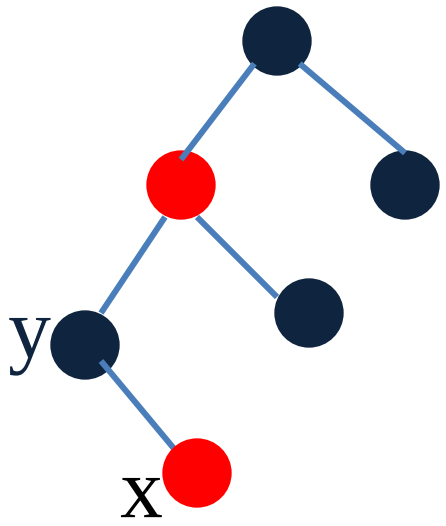
Delete *z*

- Delete *z* as in a regular BST.
- If z had two (non-nil) children, when copying y's key and satellite data to z, do not copy the color, (i.e., keep z's color).
- Let y be the node being removed or spliced out.

  (Note: either y = z or y = successor(z) .)
- If y is red, no violation to the red-black properties.
- If y is black, then one or more violations may arise and we need to restore the red-black properties.

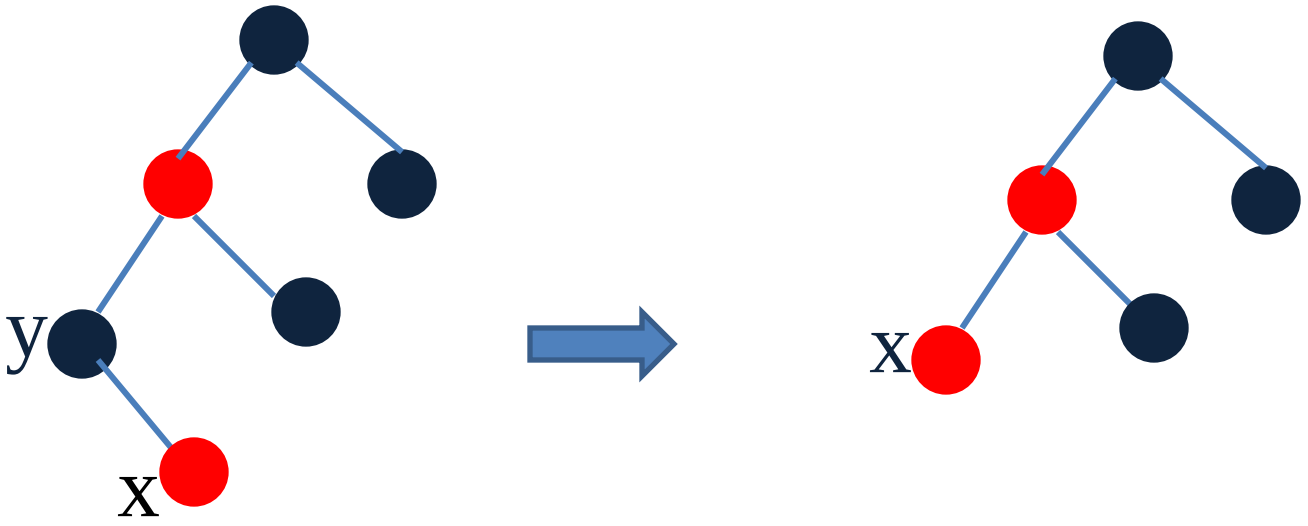
- Let x denote the child of y before it was spliced out.
- x is either nil (leaf) or was the only non-nil child of y.



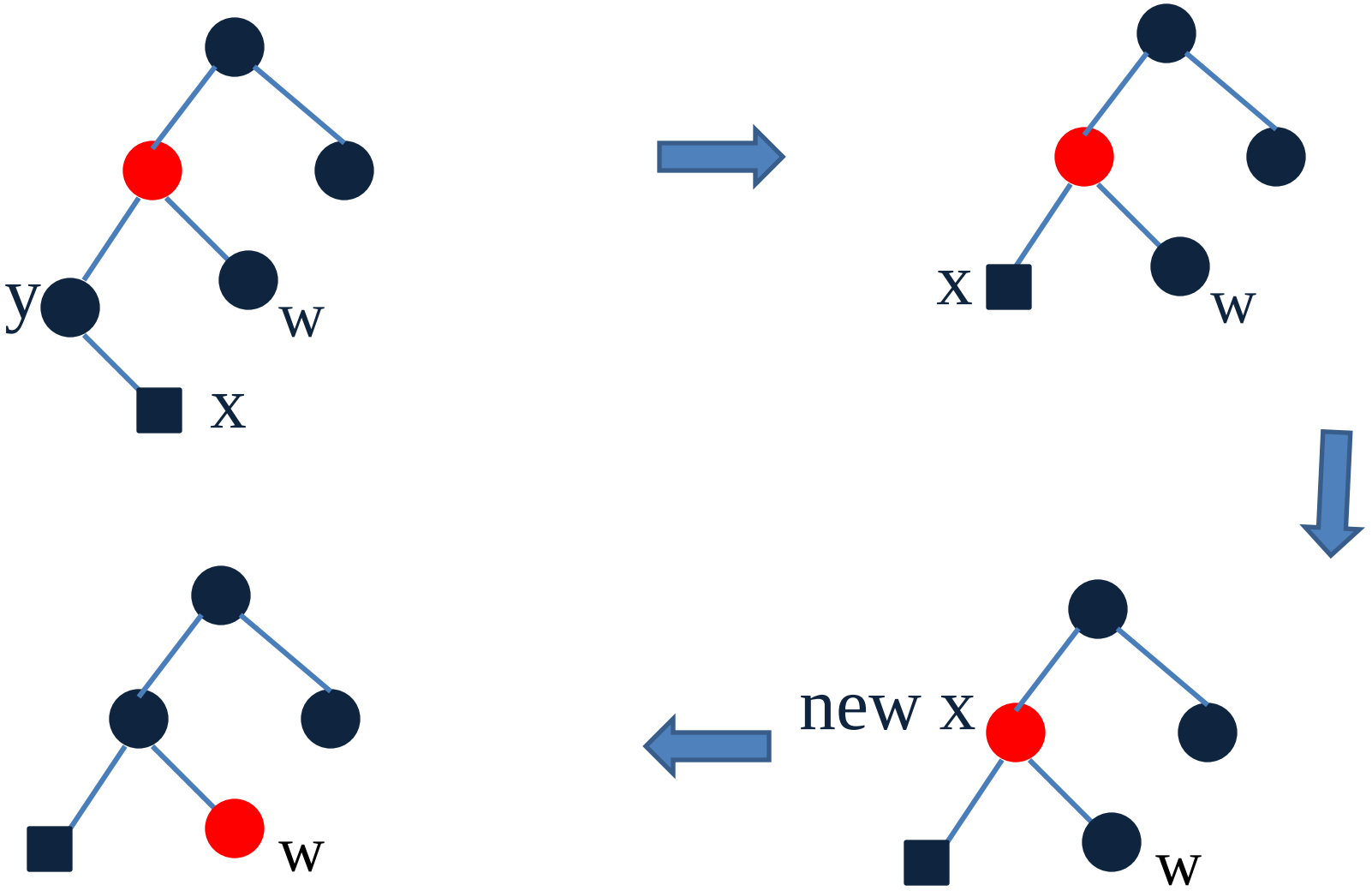


Restoring RB Properties

- Easy: If x is red, just change it to black.
- More involved: If x is black.



Example: x is black



Restoring RB Properties

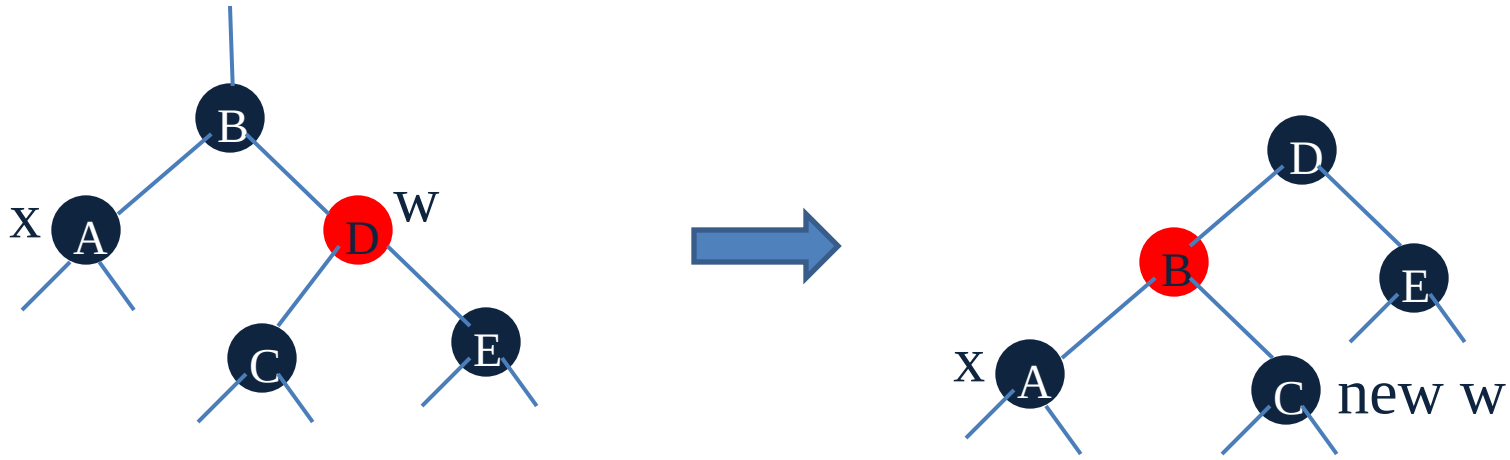
- Assume: x is black and is a left child.
- The case where x is black and a right child is similar (symmetric).
- Four cases:
  1. x' sibling w is red.
  2. x's sibling w is black; both children of w are black.
  3. x's sibling w is black; left child of w is red, right child black.
  4. x's sibling w is black; right child of w is red.

Main idea

- Regard the pointer x itself as black.
- Counting x, the tree satisfies RB properties.
- Transform the tree and move x up until:
  - x points to a red node, or
  - x is the root, or
  - RB properties are restored.
- At any time, maintain RB properties, with x counted as black.

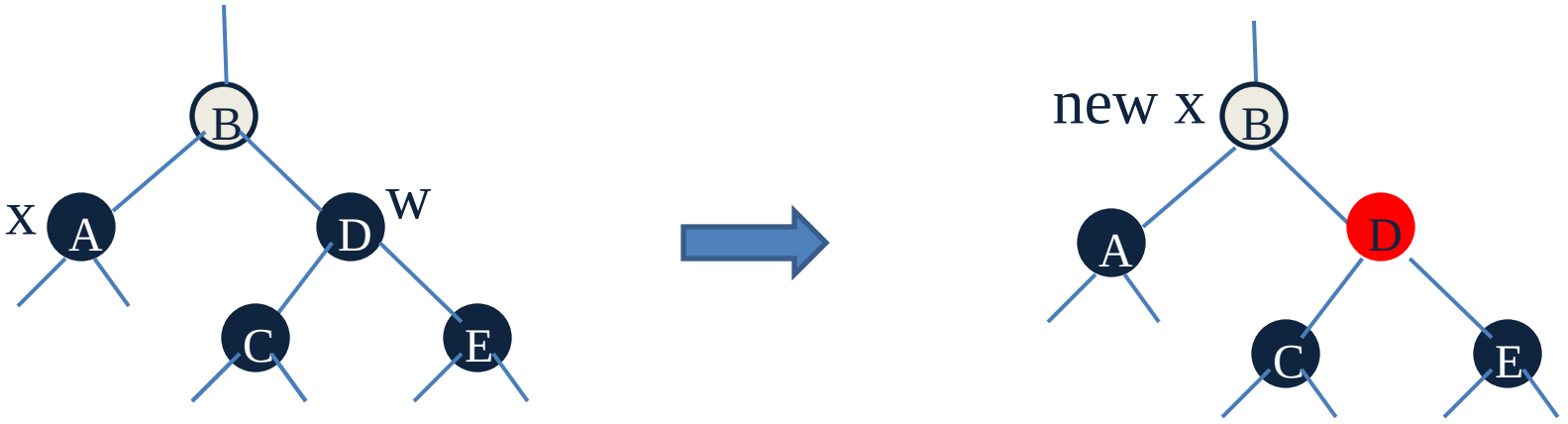
x is a black left child: Case 1

- x' sibling w is red.
- Left rotate on B; change colors of B and D.
- Transform to Case 2, 3, or 4 (where w is black).



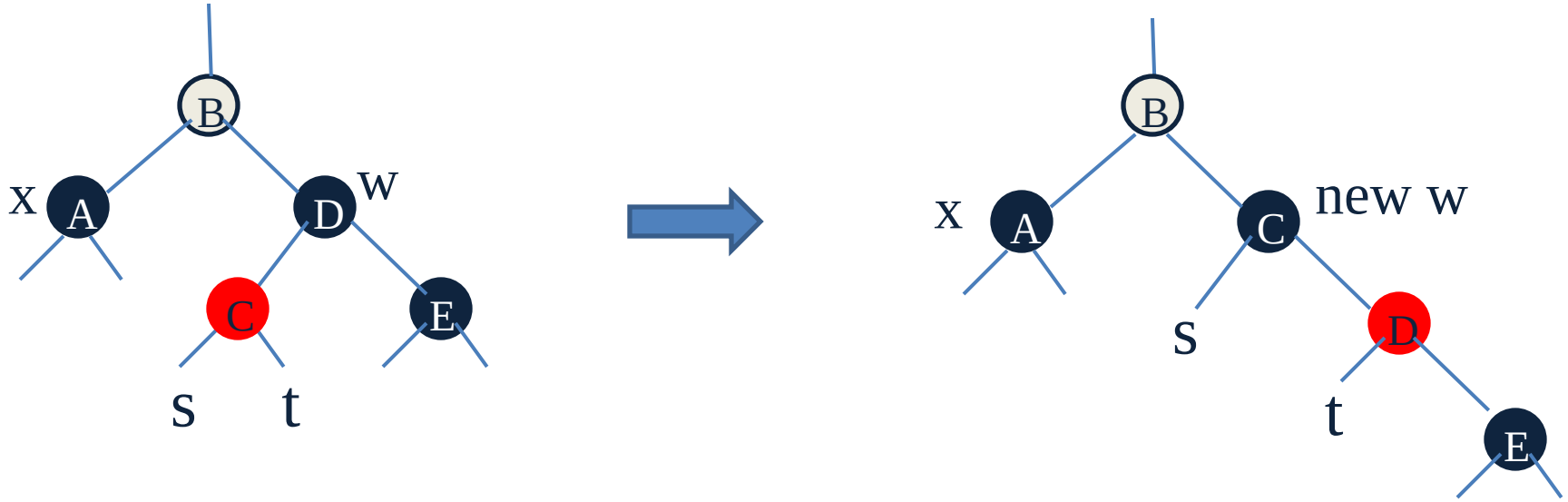
x is a black left child : Case 2

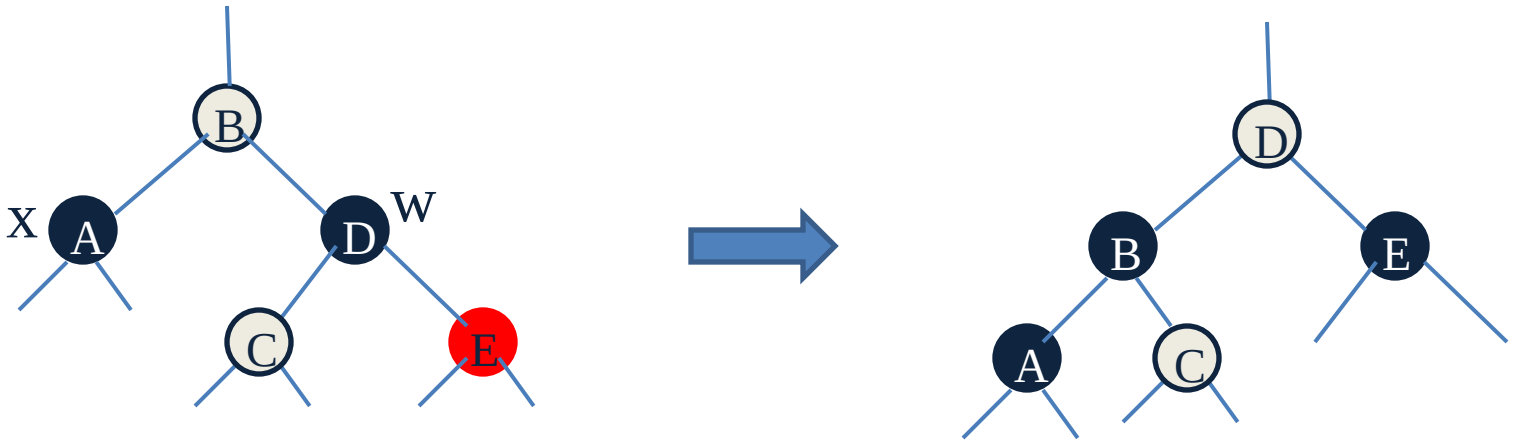
- x's sibling w is black; both children of w are black.
- Move x up, and change w's color to red.
- If new x is red, change it to black; else, repeat.



x is a black left child : Case 3

- x's sibling w is black; w's left child is red, right child black.
- Right rotate on w (D); switch colors of C and D.
- C becomes the new w.
- Transform to Case 4.



X is a black left child : Case 4

- x's sibling w is black; w's right child is red.
- Left rotate on B; switch colors of B and D; change E to black.
- Done!