Differential Calculus

avg roc: $\frac{\Delta y}{\Delta x}$

central diff approx: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$

First Principles:

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

SPs: $\frac{dy}{dx} = 0$

If $g(x) = f^{-1}(x)$, $g'(x) = \frac{1}{f'(f^{-1}(x))}$

Coord geom:

$y - y_1 = m(x - x_1)$

Parallel: $m_1 = m_2$ Perp: $m_1 \cdot m_2 = -1$

mid: $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Dist: $\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

SP Nature: sign test

$f(x) = x^3 - 11x^2 + 24x$

x	1	$\frac{4}{3}$	2
$f'(x)$	5	0	-8
shape	/	—	\

Vert Asymp
- denom $= 0$
- arg of log $= 0$
- arg of tan $= \frac{\pi}{2}$

Hor Asymp
- y approaches as $x \to \pm\infty$

NON SP, I
$\frac{d^2y}{dx^2} = 0$
- sign test

$\frac{d^3y}{dx^3} \neq 0 = $ POI
else sign test

Range: Plot a graph

Dom: $\sqrt{\geq 0}$, $\log_a(>0)$, $\frac{1}{\neq 0}$, $\frac{1}{\sqrt{>0}}$

Graphs: intercepts, SPs, asymps, endpoints, POIs, intersections

Inverses
ref in $y = x$
$f(x)$ must be 1 to 1
$f(f^{-1}(x)) = x$
ALL FEATURES: x & y flipped

$f(x) = x$ gives POIs (not all)

- comp sq sometimes + rejection

Functions

$\underbrace{\text{one/many}}_{x}$ to $\underbrace{\text{one/many}}_{y}$

- Vert/Hor line test

ODD: $f(-x) = -f(x)$

Even: $f(-x) = f(x)$

(can be neither)

Function Notation

$f: \text{dom} \to \text{codom}(R),\ f(x) = x^2+2x+1$

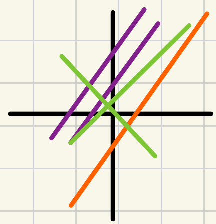
$(f \pm g)(x)$ and $(fg)(x)$ only defined over $\text{dom}(f \cap g)$

Composite functions

- $(f \circ g)(x)$ only defined if ran $g \subseteq$ dom f
- if defined, $\text{dom}(f \circ g)(x) = \text{dom } g$ (inner)

Simultaneous Linear equations

1. 1 sol: $m \neq$, c irrelevant
2. 0 sol: $m =$, $c \neq$
3. ∞ sol: $m =$, $c =$



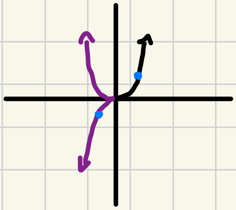
Families of Functions

Power Functions (USE surd [ ] for odd fraction powers)

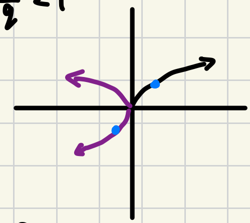
$x^{\pm\frac{p}{q}}$ $\quad \frac{\text{odd}}{\text{odd}}$: 1st & 3rd Q, $\frac{\text{even}}{\text{odd}}$: 1st & 2nd Q, $\frac{\text{odd}}{\text{even}}$: 1st Q

● = (1,1) or (-1,-1)
● = all
● = odd only

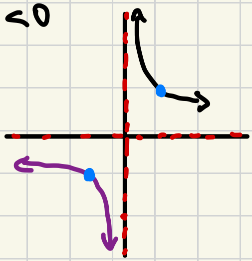
if $\frac{p}{q} > 1$



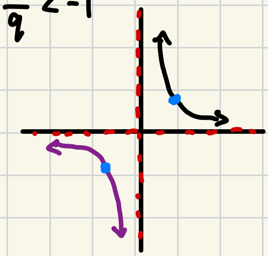
if $0 < \frac{p}{q} < 1$



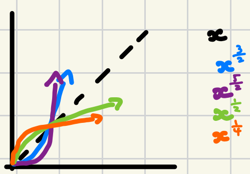
if $-1 < \frac{p}{q} < 0$



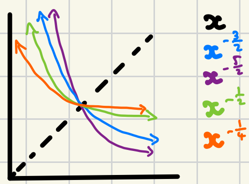
if $\frac{p}{q} < -1$



Note: all even



Note: all even



Transformations

- Dilation by a factor of _ from the vert/hori axis.
- Reflection in the vert/hori axis.
- Translation _ units in the neg/pos vert/hori direction.

Remember:
- translation often opposite to sign for hori axis
- $\frac{1}{n}$ for dilation from vert axis
- sometimes you need to go back to base function (not to)
- when translating x or dilating y, replace x with new e.g. $x \to (x-1)$

Finding the rule

hyperbola: $y = \frac{a}{x-b} + c$
truncus: $y = \frac{a}{(x-b)^2} + c$

$\Big\}$ asymps: $x = b$, $y = c$

Polynomial Functions

$$q(x)\overline{)p(x)}\ \text{with quotient } h(x)$$

$\vdots$

remainder (lower degree than $q(x)$)

$\Rightarrow \dfrac{p(x)}{q(x)} = h(x) + \dfrac{\text{remainder}}{q(x)}$

D/S Of Cubes

$A^3 - B^3 = (A-B)(A^2+AB+B^2)$

$A^3 + B^3 = (A+B)(A^2-AB+B^2)$

Remainder theorem

If $\frac{p(x)}{ax+b}$ remainder $= p(-\frac{b}{a})$

$\therefore$ if $P(-\frac{b}{a}) = 0$, $ax+b$ is a factor

Rational Root Theorem

$3x^3+8x^2+2x-5$

$3 \rightarrow B = \pm 1, \pm 3$

$-5 \rightarrow a = \pm 1, \pm 5$

$Bx+a \rightarrow P(\frac{-a}{B}) = 0$ (must be relatively prime)

Try $B=3, a=1 : P(-\frac{1}{3}) \neq 0$

Try $\underline{B=3, a=5} : P(-\frac{5}{3}) = 0$ $\therefore 3x+5$ is a factor of $P(x)$

Quadratic functions

quad $=0$, linear factors $=0$

e.g. $(x-3)(x+1) = 0$

$x=3, x=-1$

Finding TP

1. comp sq

e.g. x^2+6x+2

$=(x+3)^2-9+2$

$=(x+3)^2-7$

$\therefore$ TP at $x=-3$

Note: must factor out coeff of x^2 first

2. $\dfrac{-b}{2a}$

e.g. x^2+6x+2

$\dfrac{-6}{2(1)} = -3$

3. $\dfrac{x\text{-int} + x\text{-int}}{2}$

$(x-1)(x+7)$ } (Diff from 1&2)

$\dfrac{1+(-7)}{2} = -3$

4. $\dfrac{dy}{dx} = 0$

$y = x^2+6x+2$

$\dfrac{dy}{dx} = 0 = 2x+6$

$x = \dfrac{-6}{2}$

$= -3$

Sketching

- Sketch for non-linear inequalities
- intercepts
- Shape (reflected or not)
- TP

Discriminant and Quad formula

$\Delta = b^2-4ac$

$x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\left.\begin{array}{l}\Delta > 0: 2 \text{ sols} \\ \Delta = 0: 1 \text{ sol} \\ \Delta < 0: \text{no sols}\end{array}\right\}$ (real)

Finding rule

x-ints: $a(x-b)(x-c)$

TP: $a(x-h)^2+k$

rand points: ax^2+bx+c

Cubic functions

Factorising:

① long division:

$$\begin{array}{r} x^2-3x-10 \\ x-1 \overline{)\; x^3-4x^2-7x+10} \\ x^3-x^2 \\ \hline -3x^2-7x \\ -3x^2+3x \\ \hline -10x+10 \\ -10x+10 \\ \hline 0 \end{array}$$

② brackets method

$x^3-4x^2-7x+10$

② $x^3-x^2-3x^2+3x-10x+10$

① $x^2(x-1)-3x(x-1)-10(x-1)$

③ $(x-1)(x^2-3x-10)$

Equations

ax^3+bx^2+cx+d

$a(x-b)(x-c)(x-d)$ } x-ints

$a(x-b)(x-c)^2$ ← TP + x-int

$a(x-h)^3+k$ (h,k) → SPI

Sketching

- Sketch for non-linear Inequalities
- intercepts
- shape (reflected or not)
- TPS/SPI/PI

Quartic

$ax^4+bx^3+cx^2+dx+e$

Polynomials of form $a(x-h)^n+k$ have an SPI/TP at (h,k)

Exponentials and Logarithms

Index Laws

- $a^m a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $a^0 = 1$
- $a^{-m} = \frac{1}{a^m}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Graphing

$y = a^x + c$

$y = c$: hori asymp

$y = \log_a(x-b)$

$x = b$: vert asymp

Log laws

- $\log_a(b) = c \iff a^c = b$, $a > 0, a \neq 1, b > 0$
- $\log_a(1) = 0$
- $\log_a(m) + \log_a(n) = \log_a(mn)$
- $\log_a(m) - \log_a(n) = \log_a\left(\frac{m}{n}\right)$
- $\log_a\left(\frac{1}{m}\right) = -\log_a(m)$
- $\log_a(m^p) = p\log_a(m)$, $m > 0$

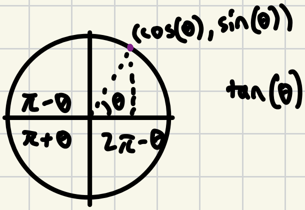
Base change:

- $\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$
- $a^c = b^{kc}$, $k = \log_b(a)$

Made with Goodnotes

Trigonometric Functions

Unit circle



$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Complementary angles

$\sin(\frac{\pi}{2} - \theta) = \cos(\theta)$

$\cos(\frac{\pi}{2} - \theta) = \sin(\theta)$

$\sin(\frac{\pi}{2} + \theta) = \cos(\theta)$

$\cos(\frac{\pi}{2} + \theta) = -\sin(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1$

Careful of quadrants

e.g. find $\tan(\theta)$ if $\cos(\theta) = -\frac{2}{3}$, $\frac{\pi}{2} < \theta < \pi$

$\sin^2(\theta) + \frac{4}{9} = 1$

$\sin(\theta) = \frac{\pm\sqrt{5}}{3}$ 2nd quad, sin pos

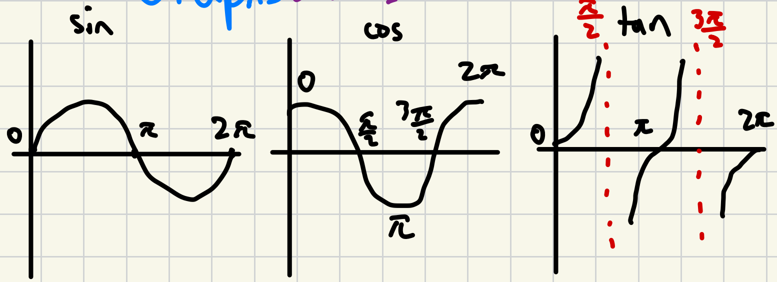
$\therefore \sin(\theta) = \frac{\sqrt{5}}{3}$

$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{-\frac{2}{3}}{\frac{\sqrt{5}}{3}}$

$= -\frac{2}{3} \times \frac{3}{\sqrt{5}}$

$= -\frac{2}{\sqrt{5}}$

Graphs [0, 2π]



amp: |a| (x-dil)
period (T): $\frac{2\pi}{n}$
median: b
for $a\,{}^{\sin}_{\cos}(nx - c) + b$

no amp
period (T): $\frac{\pi}{n}$
vert asymp at inside $= \frac{\pi}{2}$

- use graphs for non-linear inequalities

Differential Calculus (Application)

tangent: same m at point (x, y)

normal: $-\frac{1}{m}$ at point (x, y)

local min and max diff to global min and max

$\lim_{x \to a^{\circ}} f(x)$

$\circ = -$: from below →

$\circ = +$: from above ←

Strictly in/decreasing

In: $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

De: $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

If $\frac{dy}{dx} < 0$, always strictly decreasing

- If strictly increasing, $f(x) = f^{-1}(x)$ same as $f(x) = x = f^{-1}(x)$
- Else, may or may not

Continuity:

lim from left and right approach same value and the value is = to the value of the function at that point

Differentiability

Differentiable at $x = a$ if $f'(a)$ exists

(must be continuous at $x = a$)

- endpoints nor differentiable

↗ $f(x)$ diff at $x = a \Rightarrow f(x)$ cont at $x = a$

check on notes given

On $f(x)$	On $f'(x)$
SP at $x = a$ (sign or 2nd derivative for nature)	x-int at $x = a$
Non-S PI at $x = a$	TP at $x = a$
$f(x)$ strictly increasing $(a, b$	$f'(x) > 0$, (a, b)
... strictly decreasing	$f'(x) < 0$, (a, b)
vert asymp at $x = a$	vert asymp at $x = a$

Notes from book

Integral Calculus

antiderivative: $\int f(x)\,dx = F(x) + C$

Note: don't mess up + & − with differentiation of cos & sin

integration:

$\int \sin \to -\cos$

$\int \cos \to \sin$

Properties:

- $\int af(x)\,dx = a\int f(x)\,dx$
- $\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int \alpha f(x) + \beta g(x)\,dx = \alpha\int f(x)\,dx + \beta\int g(x)\,dx$

Standard:

- $\int x^r\,dx = \frac{x^{r+1}}{r+1} + C$, $r \in \mathbb{Q}$ (rational) $\setminus \{-1\}$
- $\int (ax+b)^r\,dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + C$, $r \in \mathbb{Q} \setminus \{-1\}$
- $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + C$, $ax+b > 0 \Rightarrow x > -\frac{b}{a}$

Note: may need to rearrange/polynomial division/index laws into standard form. e.g. $\sqrt{ax+b} \to (ax+b)^{\frac{1}{2}}$

- $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$, where $k \neq 0$
  - ↳ $a^{kx} = e^{(k\log_e(a))x}$
- $\int \sin(kx+a)\,dx = -\frac{1}{k}\cos(kx+a) + C$
- $\int \cos(kx+a)\,dx = \frac{1}{k}\sin(kx+a) + C$

Integration BY RECOGNITION

$$\frac{d}{dx}[F(x)] = f(x) \Rightarrow F(x) = \int f(x)\,dx + C$$

Often Q of 2 parts:

a) Find derivative of ____

b) Hence, find $\int$ ____ dx

e.g. part a)

$\frac{dy}{dx} = e^{-x} - xe^{-x}$, $y = xe^{-x}$

part b) find $\int xe^{-x}\,dx$

$y = \int e^{-x} - xe^{-x}\,dx$

↳ xe^{-x}

$\therefore xe^{-x} = -e^{-x} + C - \int xe^{-x}\,dx$

$\therefore \int xe^{-x}\,dx = -e^{-x} - xe^{-x} + C$

DEFINITE Integrals

high: b

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

low: a

Note $a < b$

Note area is always positive

Definite Integrals can be divided into sub intervals

e.g. $\int_a^b f(x)\,dx = \int_a^{\beta} f(x)\,dx + \int_{\beta}^{b} f(x)\,dx$, where $a < \beta < b$

Area under a curve:

$\int_a^b f(x)\,dx$ gives signed area between $f(x)$ & x axis between $x = a$ & $x = b$

For pos area: $\int_a^b |f(x)|\,dx$ (shortcut)

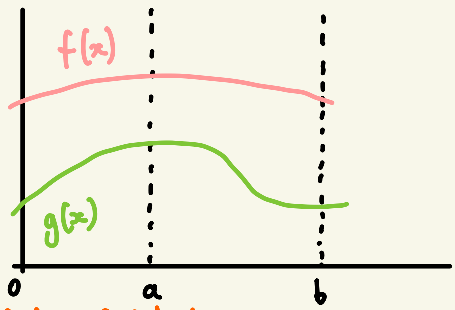
Area between 2 curves

f & g continuous over $[a, b]$, $f(x) \geq g(x)$ over $[a,b]$

then area bounded by the 2 curves & $x=a$ & $x=b$ is given by $\int_a^b f(x)-g(x)\,dx$

that is:

$$\int_a^b (\text{Upper curve} - \text{Lower curve})\,dx$$



Shortcut: Area $= \int_a^b |f(x)-g(x)|\,dx$ → typically also works if top func changes, but check

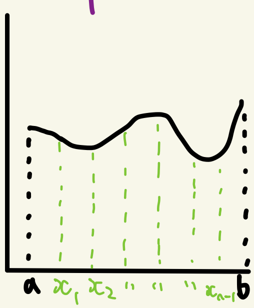
e.g. 

Steps

1. Find POI(s) of $f(x)$ & $g(x)$
2. Sketch $f(x)$ & $g(x)$ → show POIs
3. Shade required area

"By hand" methods: by recognition & using inverse of integrand

Trapezium Approximation



$$= \int_a^b f(x)\,dx \approx h\left(\tfrac{1}{2}f(a) + f_1 + f_2 + \ldots + f_{n-1} + \tfrac{1}{2}f(b)\right)$$

$$= \frac{h}{2}\left(f(a) + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f(b)\right)$$

- h - width of each trapezium
- n - number of trapezia

OR: $x_0 = a$, $x_n = b$: $\dfrac{x_n - x_0}{2n} = \dfrac{h}{2}$ sub in alt form

$$A_{\text{trapezium}} = \frac{1}{2}(\text{sum of parallel side lengths})(\text{width})$$

Applications

The average value of $f(x)$ over $a \leq x \leq b$ equals:

$\frac{1}{b-a}\int_a^b f(x)\,dx$ (Not rate of change / value not slope)

Uses

- Finding area
- Avg value of a function
- Probability: Continuous random variables. Calculating probability. Expected value and variance

Techniques

- "Standard" forms
- Integration by recognition
- Integration using the inverse function

Integration By Substitution (I think)

$$I = \int_0^{\log_e(2)} e^{-3x+1}\,dx$$

let $u = -3x+1$

$$\Rightarrow \frac{du}{dx} = -3 \Rightarrow dx = \frac{du}{-3}$$

→ sub og into u equation

$$\therefore I = \int_1^{-3\log_e(2)+1} e^u\left(\frac{du}{-3}\right)$$

$$= -\frac{1}{3}\left[e^u\right]_1^{-3\log_e(2)+1}$$

$$= -\frac{1}{3}\left(\frac{e}{8} - e\right)$$

$$= \frac{7e}{24}$$

Tricks to throw you off:

$$\int_{\frac{\pi}{2}}^{\pi} \rightarrow \int_{\pi}^{\frac{\pi}{2}} \Rightarrow -\text{area}$$

Probability

Normal Probability

$\cap$ and
$\cup$ or
$'$ complement (not)
ε set
$\emptyset$ null set

$0 \leq Pr(A) \leq 1$

$Pr(A') = 1 - Pr(A)$

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$Pr(A|B) = \dfrac{Pr(A \cap B)}{Pr(B)}$

Independent

$Pr(A \cap B) = Pr(A) \times Pr(B)$

$Pr(A|B) = Pr(A)$

Mutually exclusive

$Pr(A \cap B) = 0$

- Use Venn diagrams, Karnaugh table, tree diagrams

Probability & Number Different

Discrete Random Variables

Rules

$0 \leq Pr(x) \leq 1$ for all x

$\sum_x p(x) = 1$

$Pr(a \leq X \leq b) = \sum_{x=a}^{x=b} p(x)$

Mean (expected value): $E(X) = \mu_x = \sum_x x \cdot Pr(X=x)$

$E(aX+b) = aE(X)+b$

$E(X+Y) = E(X)+E(Y)$

Variance: $Var(X) = \sigma_x^2 = E\left((X-\mu_x)^2 \cdot Pr(X=x)\right)$

OR $= \sigma_x^2 = E(X^2) - (E(X))^2$

$Var(aX+b) = a^2 Var(X)$

Standard deviation: $sd(X) = \sigma_x = \sqrt{Var(X)}$

$sd(aX+b) = |a| sd(X)$

E.g. combinatorics Q

4L, 3B, 7G, 6E (select group of 5)

each type has 2 leaders. Pr(2E leaders | all 4 types are in group)

$$= \frac{{}^2C_2 \times {}^4C_1 \times {}^3C_1 \times {}^7C_1}{4032}$$

$\hookrightarrow$ sum
$\begin{cases} 2L: {}^4C_2 \times 3 \times 7 \times 6 \\ 2B: {}^3C_2 \times 7 \times 6 \times 4 \\ 2G: {}^7C_2 \times 6 \times 4 \times 3 \\ 2E: {}^6C_2 \times 4 \times 3 \times 7 \end{cases}$

Binomial Distribution

$X \sim \text{Binomial}(n=?, p=?)$

n: num of trials; p: Pr(success) in one trial

Requirements
- Success or failure
- Pr(success) same for every trial
- Trials independent

$Pr(X=x) = {}^nC_x\, p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots n$

sub in n & p to get probability mass function

$E(X) = np$

$Var(X) = np(1-p)$

Normal approximation to Binomial

$\mu = np, \quad \sigma = \sqrt{np(1-p)}$

Continuous Random Variables

function is a pdf only if:

1. $f(x) \geq 0$ for all x
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

For a continuous random variable X with pdf $f(x)$:

1. $Pr(a < X < b) = \int_a^b f(x)\,dx$
2. $Pr(X = a) = 0$

$\therefore < \; = \; \leq$

Inverse probability problem

$Pr(X < k) = \frac{p}{100}: \int_{-\infty}^{k} f(x)\,dx = \frac{p}{100}$

$Pr(X > k) = \frac{p}{100}: \int_{k}^{\infty} f(x)\,dx = \frac{p}{100}$

special case ($p = 50$):

$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2}$ OR $\int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$ OR $\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx$

Must graph even when pdf $= 0$

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$

$Var(X) = E(X^2) - \mu^2, \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$sd(X) = \sqrt{Var(X)}$

Integrals to infinity:

WRITE: $\int_0^\infty x e^{-x}\,dx$

$= \lim_{t \to \infty} \int_0^t x e^{-x}\,dx$

$= \lim_{t \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^t$

$= \lim_{t \to \infty} \dots$

$= \lim \dots (1) - \lim \dots (te^{-t}) - \lim \dots (e^{-t})$

$e^{-t} \downarrow > t \uparrow$

$= 1 - 0 - 0$

$= 1$

Made with Goodnotes

Normal Distribution

Standard Normal

standard normal pdf: $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

$Z \sim \text{Normal}(\mu = 0, \sigma = 1)$

- symmetric about $\mu = 0$

$\Pr(Z \geq a) = \Pr(Z \leq -a)$, $\Pr(Z \leq a) = 1 - \Pr(Z \geq a)$

General Normal

pdf: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

- symmetric about μ

mean = median = mode ?

Relationship

$Z = \frac{X - \mu}{\sigma}$, $X \sim \text{Normal}(\mu, \sigma)$ &

$Z \sim \text{Normal}(\mu = 0, \sigma = 1)$

$\Pr(X \leq a) = \Pr\left(Z \leq \frac{a - \mu}{\sigma}\right)$

68-95-99.7 rule

$\Pr(\mu - \sigma \leq k \leq \mu + \sigma) \approx 0.68$

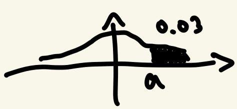
$\Pr(\mu - 2\sigma \leq k \leq \mu + 2\sigma) \approx 0.95$

$\Pr(\mu - 3\sigma \leq k \leq \mu + 3\sigma) \approx 0.997$

When transforming normal distribution rule, check $\int_{-\infty}^{\infty}$ after and dilate by factor to make it $= 1$

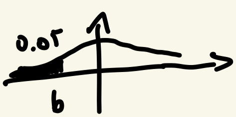
Special Q (unknown mean & sd) (TF)

$X \sim \text{Normal}(\mu = ?, \sigma = ?)$, $\Pr(X > 60.4) = 0.03$, $\Pr(X < 59.7) = 0.05$



$\Pr(Z > a) = 0.03$

$a = 1.8808 \Rightarrow a = \frac{60.4 - \mu}{\sigma}$



$\Pr(Z < b) = 0.05$

$b = -1.6449 \Rightarrow b = \frac{59.7 - \mu}{\sigma}$

} simult solve for μ, σ

Calculating Probabilities

① define the r.v.

② define the distribution the r.v. follows

③ define the problem in terms of a probability statement

Sampling

population proportion: $p = \frac{\text{num in population w attribute}}{\text{population size}}$ (constant)

sample proportion: $\hat{p} = \frac{\text{num in sample w attribute}}{\text{sample size}}$ (not constant)

Small populations (pop size <10n) → direct consideration/counting methods

scaffolded pr dist table

x	...
$Pr(X=x)$	...
$\hat{p} = \frac{x}{n}$	...
$Pr(\hat{P}=\hat{p}) = Pr(X=x)$	...

Large populations (pop size >10n) → Binomial distribution

$E(\hat{P}) = p$

$sd(\hat{P}) = \sqrt{\frac{p(1-p)}{n}} = \sqrt{Var(\hat{P})}$

Approx dist of sample proportion
(normal approx to binomial)

(required) $np > 5$, $n(1-p) > 5$ or even better: $np > 10$, $n(1-p) > 10$

$\hat{P} \sim Normal\left(\mu = p, \sigma = \sqrt{\frac{p(1-p)}{n}}\right)$

Confidence Intervals

- using sample proportion ($\hat{p}$) to estimate population proportion (p) = point estimate
- An interval we are reasonably sure contains the population proportion (p) is called a confidence interval for p

Approx 95% confidence interval for p when $n\hat{p} > 5$ & $n(1-\hat{p}) > 5$ (or better > 10):

$$\left(\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \; , \; \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

1.96 only for 95% confidence, otherwise k

Margin of error

$$k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Common Q1

find sample size (n)

0.95 Conf Int : $\underbrace{(0.0284, 0.0866)}_{\text{avg} = \hat{p} = 0.0550}$

sub in

$$\left(\underbrace{\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}_{=0.0284} \; , \; \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

$\hookrightarrow 0.0284 = 0.0550 - 1.96\sqrt{\frac{0.0550(1-0.0550)}{n}}$

$n \approx 200$

Common Q2

sample size to reduce width by $\frac{1}{3}$?

width = 2 M.E (margin of error)

$W_{old} = 2M.E_{old} = 2 \times 1.96\sqrt{\frac{\cdots}{200}}$

$W_{new} = \frac{2}{3} W_{old} = 2 \times 1.96\sqrt{\frac{\cdots}{n_{new}}}$

$\cancel{2} \times \cancel{1.96}\sqrt{\frac{\cdots}{n_{new}}} = \frac{2}{3} \times \cancel{2} \times \cancel{1.96}\sqrt{\frac{\cdots}{200}}$

$\frac{\cancel{\cdots}}{n_{new}} = \frac{4}{9} \cdot \frac{\cancel{\cdots}}{200}$

$n_{new} = \frac{9}{4} \times 200 = 450$

Common Q3

$\hat{p} = ?$ $n = 80$ C% conf interval: $(0.3023, 0.4377)$

$$\hat{p} = \frac{0.3023 + 0.4377}{2} = 0.37$$

$$\Big(\underbrace{\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}_{L}, \underbrace{\hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}}_{R}\Big)$$

$\frac{1}{2}(L+R) = \hat{p}$

$\frac{1}{2}(R-L) = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$$\frac{1}{2}(R-L) = \frac{1}{2}(0.4377 - 0.3023) = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.0677$$

$$k\sqrt{\frac{0.37(1-0.37)}{80}} = 0.067 \Rightarrow k = 1.25419$$

$$C\% = \frac{C}{100} = 0.7902$$

$$C = 79$$

z: