2022
Higher School Certificate
Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations
- Write your student name and/or number at the top of every page

Total marks – 100

Section I – 10 marks (pages 3 - 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 - 11)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

This paper MUST NOT be removed from the examination room.

STUDENT NAME/NUMBER...

STUDENT NAME/NUMBER.......................................

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Select the alternative A, B, C, D that best answers the question and indicate your choice with a cross (X) in the appropriate space on the grid below.

	A	B	C	D
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

STUDENT NAME/NUMBER.......................................

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1. m and n are two positive integers such that $c = m^2 + n^2$, $a = m^2 - n^2$, $b = 2mn$ and a, b, c have no common factors. Which one of the following statements is false?

   (A) $c^2 = a^2 + b^2$

   (B) $a > c - b$

   (C) a and c are both odd

   (D) b is not divisible by 4

2. Given that $|z-2| = 2$ and $\arg(z-2) = \frac{2\pi}{3}$, which of the following is an expression for z ?

   (A) $z = 1 + i\sqrt{3}$

   (B) $z = \left(2+\sqrt{3}\right) + i$

   (C) $z = \left(2-\sqrt{3}\right) + i$

   (D) $z = 3 + i\sqrt{3}$

3. $OABC$ is a rectangle with $\overrightarrow{OA} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\overrightarrow{OC} = 6\underset{\sim}{i} + 4\underset{\sim}{j} + a\underset{\sim}{k}$ for some constant a. What is the value of a ?

   (A) -2

   (B) -3

   (C) -5

   (D) -6

4. Which of the following is an expression for $\displaystyle\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

   (A) $\sin^{-1}\frac{1}{2}(x-1) + c$

   (B) $\sin^{-1}\frac{1}{2}(x+1) + c$

   (C) $\sin^{-1}\frac{1}{\sqrt{2}}(x+1) + c$

   (D) $\sin^{-1}\frac{1}{\sqrt{2}}(x-1) + c$

STUDENT NAME/NUMBER..

5. A particle is moving in a line with Simple Harmonic Motion. At any time t seconds it has displacement x metres from a fixed point on the line and velocity v ms^{-1} given by $v^2 = 9 - 4(x-1)^2$. What is the amplitude of the motion?

(A) $1 \cdot 5$ metres

(B) 2 metres

(C) $2 \cdot 5$ metres

(D) 3 metres

6. Consider the statement

For any function f(x), f(x) is not continuous at $x = c \Rightarrow$ *f(x) is not differentiable at* $x = c$.

Which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is false and the converse statement is true.

(C) The contrapositive statement is true and the converse statement is false.

(D) The contrapositive statement is true and the converse statement is true.

7. Which of the following is an expression for $e^{i3\theta} + e^{i\theta}$?

(A) $2\sin\theta\, e^{i2\theta}$

(B) $2\cos\theta\, e^{i2\theta}$

(C) $2\sin 2\theta\, e^{i\theta}$

(D) $2\cos 2\theta\, e^{i\theta}$

8. The points A, B and C are collinear where $\overrightarrow{OA} = \underset{\sim}{i} - \underset{\sim}{j}$, $\overrightarrow{OB} = -3\underset{\sim}{j} - \underset{\sim}{k}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} + a\underset{\sim}{j} + b\underset{\sim}{k}$ for some constants a and b. What are the values of a and b?

(A) $a = -1$ and $b = -1$

(B) $a = -1$ and $b = 1$

(C) $a = 1$ and $b = -1$

(D) $a = 1$ and $b = 1$

STUDENT NAME/NUMBER..

9. If $\int_1^4 f(x)\,dx = k$ for some constant k, what is the value of $\int_1^4 f(5-x)\,dx$?

(A) $-k$

(B) $5-k$

(C) $k+5$

(D) k

10. A stone is projected from a point O with speed $V\sqrt{2}$ ms^{-1} at an angle 45° above the horizontal. The stone moves in a vertical plane under gravity where the acceleration due to gravity is g ms^{-2}. At time t seconds the position vector of the stone relative to O is $\underset{\sim}{r}(t) = Vt\,\underset{\sim}{i} + \left(Vt - \frac{1}{2}gt^2\right)\underset{\sim}{j}$.

Which one of the following statements about the trajectory of the stone is correct?

(A) Its horizontal range is 2 times its maximum height.

(B) Its horizontal range is 3 times its maximum height.

(C) Its horizontal range is 4 times its maximum height.

(D) Its horizontal range is 5 times its maximum height.

STUDENT NAME/NUMBER..

Marks

Section II

90 Marks
Attempt Questions 11-16.
Allow about 2 hours and 45 minutes for this section.

Answer the questions in writing booklets provided. Use a separate writing booklet for each question.
In Questions 11-16 your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) **Use a separate writing booklet.**

(a) The complex numbers z and w are represented in the Argand diagram by the points Z and W respectively. $z = 2 + 2\sqrt{3}\,i$ and the point W is obtained by rotating the point Z in a clockwise direction about the origin through an angle of 90°.

(i) Find z and w in modulus/argument form. **2**

(ii) Find zw and $\frac{z}{w}$ in modulus/argument form. **2**

(b)(i) Use the substitution $u = \frac{1}{x}$ to show that $\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}\,dx = 0$ for any constant $a > 1$. **2**

(ii) Hence find in simplest exact form the value of $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1} x}{x}\,dx$. **3**

(c) Consider the equation $f(z) = 0$ where $f(z) = z^3 - 11z^2 + 55z - 125$.

(i) Find the three roots of the equation in the form $a + i\,b$, where a and b are real. **3**

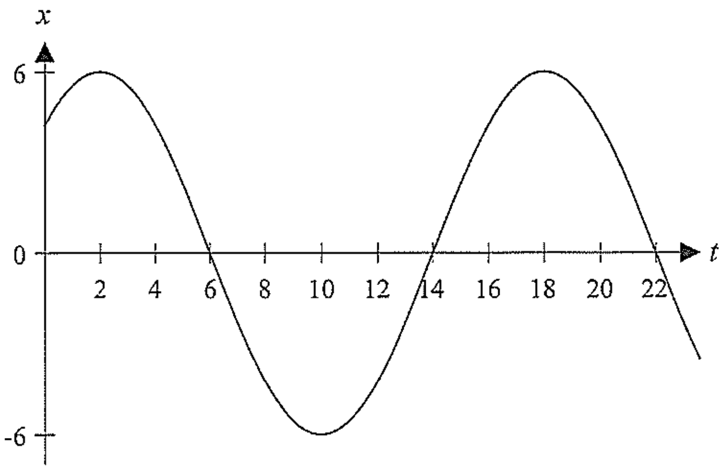
(ii) Show that the points A , B and C in the Argand diagram representing these roots lie on a circle of the form $|\, z \,| = k$ for some constant k, and find the area of ΔABC . **3**

STUDENT NAME/NUMBER......................................

Marks

Question 12 (15 marks) **Use a separate writing booklet.**

(a)



The graph shows the displacement x cm from the centre of motion at time t seconds for a particle performing Simple Harmonic Motion in a straight line.

(i) Write an expression for x as a function of t. **2**

(ii) Find the distance travelled by the particle in the first minute of its motion after observation began at time $t = 0$. **2**

(b)(i) Show that for $k \geq 2$, $\tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) = \tan^{-1}\left(\frac{2}{k^2}\right)$. **2**

(ii) Hence find $\lim_{n \to \infty} \sum_{k=1}^{n} \tan^{-1}\left(\frac{2}{k^2}\right)$. **3**

(c)(i) If $I_n = \int_1^e (1 - \ln x)^n \, dx$ for $n = 0,\ 1,\ 2, \ldots$, show that $I_n = -1 + nI_{n-1}$ for $n = 1, 2, 3, \ldots$. **2**

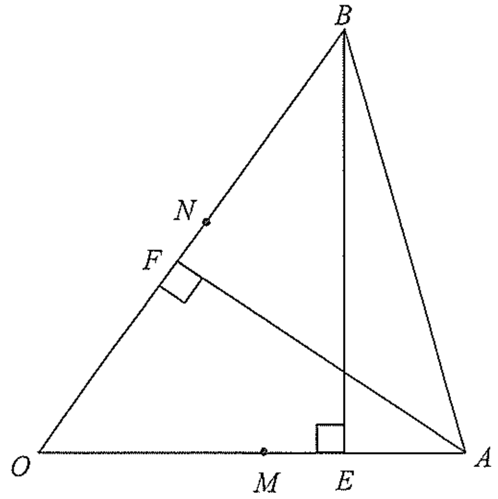
(ii) Use Mathematical Induction to show that $I_n = n!\,e - 1 - \sum_{r=1}^{n} {}^nP_r$ for all positive integers $n \geq 1$. **4**

STUDENT NAME/NUMBER.......................................

Marks

Question 13 (15 marks) **Use a separate writing booklet.**

(a)



Not to scale

In ΔOAB, BE is the altitude from B to OA and AF is the altitude from A to OB. **3**

M, N are the midpoints of OA, OB respectively. $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.

Use vector methods to show that $|OM||OE| = |ON||OF|$.

(b) A body of mass m kg is travelling in a horizontal straight line so that the resultant force on the body is a resistance force of magnitude $\frac{1}{10}m\sqrt{1+v}$ when its speed is v ms^{-1}. Initially the speed of the body is 15 ms^{-1}.

(i) Find the time taken for the body to come to rest. **2**

(ii) Find the distance travelled by the body in coming to rest. **3**

(c) Consider the lines L_1, L_2 determined by the vector equations

$$L_1 : \underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad L_2 : \underset{\sim}{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(i) Show that L_1 and L_2 intersect and are perpendicular, stating the coordinates of the point of intersection. **3**

(ii) Deduce that the plane containing the lines L_1 and L_2 has an equation determined by **2**

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + a \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$ for parameters a and b, and hence that this plane has equation $y + z = 1$.

(iii) Find the perpendicular distance from the origin to this plane. **2**

STUDENT NAME/NUMBER..

Marks

Question 14 (15 marks) **Use a separate writing booklet.**

(a)(i) Show that $\frac{n}{n+1}\,{}^{2n}C_n = {}^{2n}C_{n-1}$ for $n \geq 1$. **1**

(ii) Show that $\frac{1}{n+1}\,{}^{2n}C_n$ is an integer for $n \geq 1$. **2**

(b) Find the perpendicular distance between the parallel lines L_1 and L_2 with vector equations $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\underset{\sim}{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ respectively. **3**

(c) Prove that there exist no positive integers m and n such that $4m^2 - n^2 = 25$. **3**

(d)(i) Use de Moivre's theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **2**

(ii) Solve the equation $\cos 5\theta = -1$ for $0 \leq \theta \leq 2\pi$. **4**

Hence show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$ and find the value of $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}$.

STUDENT NAME/NUMBER.......................................

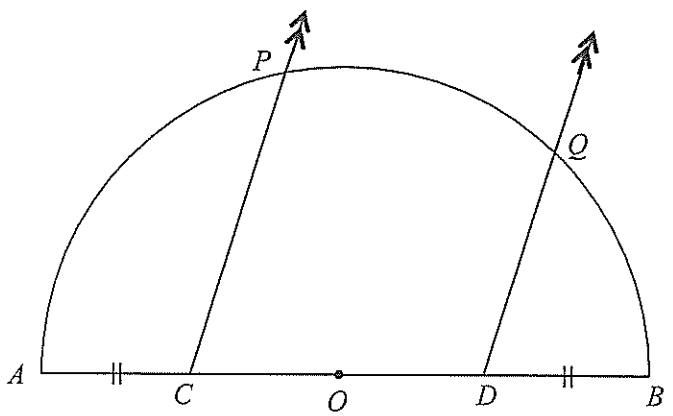
Marks

Question 15 (15 marks) **Use a separate writing booklet.**

(a) Use the substitution $t=\tan\frac{x}{2}$ to find in simplest exact form the value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\sin x}\,dx\ .$$

(b)



A semi-circle is drawn on diameter AB. O is the midpoint of AB and points C and D lie on AB such that $AC=BD$. Parallel lines are drawn through C and D intersecting the semi-circle at P and Q respectively. $\overrightarrow{OC}=\underset{\sim}{c}$ and $\overrightarrow{CP}=\underset{\sim}{p}$.

(i) Explain why $\overrightarrow{DQ}=\lambda\underset{\sim}{p}$ for some scalar $\lambda>0$ then show that $(1-\lambda)\underset{\sim}{p}.\underset{\sim}{p}+2\,\underset{\sim}{c}.\underset{\sim}{p}=0$. **3**

(ii) Hence show that $\angle CPQ=90°$. **1**

(c) Consider the functions $f_n(x)=e^x-\left(1+\sum_{r=1}^{n}\frac{x^r}{r!}\right)$, $n=1,2,3,\ldots$

(i) Show that $f_n(0)=0$ and $f_{n+1}{}'(x)=f_n(x)$ for $n=1,2,3,\ldots$. **2**

(ii) Show that $f_1(x)>0$ for all $x>0$ and hence $1+x<e^x$ for all $x>0$. **2**

(iii) Use Mathematical induction to show that for all positive integers $n\geq1$, **3**

$$1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}<e^x \quad \text{for all } x>0.$$

STUDENT NAME/NUMBER......................................

Marks

Question 16 (15 marks) **Use a separate writing booklet.**

(a) Prove $\forall\, p \in \mathbb{Z}^{+},\ p \textit{ divides } \left((p-1)!+1\right) \Rightarrow p \textit{ is prime.}$ **3**

(b) A srone is projected horizontally with speed $V\,\mathrm{ms}^{-1}$ from a point O at the top edge of a vertical cliff H metres above horizontal ground. The stone moves in a vertical plane under gravity and subject to air resistance. The acceleration due to gravity is $g\ \mathrm{ms}^{2}$. At time t seconds the stone has position vector $x\,\underset{\sim}{i} + y\,\underset{\sim}{j}$ relative to O, velocity

vector $v_x\,\underset{\sim}{i} + v_y\,\underset{\sim}{j}$, and acceleration vector $-\dfrac{v_x}{g}\underset{\sim}{i} - \left(\dfrac{v_y}{g}+g\right)\underset{\sim}{j}$.

The stone hits the ground after T seconds at a distance R metres from the foot of the cliff.

(i) Show that $x = Vg\left(1-e^{-\frac{1}{g}t}\right)$ and $y = g^{3}\left(1-e^{-\frac{1}{g}t}\right)-g^{2}t$. **4**

(ii) Hence show that $T = \dfrac{g^{2}R+VH}{Vg^{2}}$. **2**

(c)(i) Show that $e^{ik\pi} = (-1)^{k}$ for all integers $k \geq 0$. **1**

(ii) Show that $\displaystyle\sum_{k=0}^{n-1}(-1)^{k}\cos^{n}\left(\frac{k\pi}{n}\right) = \frac{1}{2^{n}}\sum_{k=0}^{n-1}\left(1+e^{i\left(\frac{2k\pi}{n}\right)}\right)^{n}$. **2**

(iii) Hence show that $\displaystyle\sum_{k=0}^{n-1}(-1)^{k}\cos^{n}\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$. **3**

Section 1 Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1	D	$c^2-a^2=\left(m^2+n^2\right)^2-\left(m^2-n^2\right)^2=4m^2n^2=b^2$. A is true. $a^2=c^2-b^2=(c-b)(c+b)>(c-b)^2$ since $b>0$. $\therefore a>c-b$ since $a>0,\ c-b>0$. B is true. b is even and $c^2=a^2+b^2$. Hence if either a or c is even, all three of a, b, c are even, but have no common factor. Hence a , c are both odd. C is true. If m and n are both odd, a and c are even and a, b, c have a common factor. Hence one of m, n is even, and hence b is divisible by 4. D is false.	MEX12-2
2	A	$z-2=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=-1+i\sqrt{3}\qquad \therefore z=1+i\sqrt{3}$	MEX12-4
3	C	$\angle AOC=90^\circ\quad \therefore \left(3\underset{\sim}{i}-2\underset{\sim}{j}+2\underset{\sim}{k}\right).\left(6\underset{\sim}{i}+4\underset{\sim}{j}+a\underset{\sim}{k}\right)=0\quad \therefore a=-5$	MEX12-3
4	B	$\int \frac{1}{\sqrt{3-2x-x^2}}\,dx=\int \frac{1}{\sqrt{4-(x+1)^2}}\,dx=\sin^{-1}\left(\frac{x+1}{2}\right)+c$	MEX12-5
5	A	$v^2=9-4(x-1)^2$. $\therefore v=0$ for $x-1=\pm\frac{3}{2},\ x=-\frac{1}{2}$ or $x=\frac{5}{2}$. Particle moves 3 m between its extreme positions. Amplitude is 1.5 m.	MEX12-6
6	C	Contrapositive: *f(x) is differentiable at* $x=c \Rightarrow$ *f(x) is continuous at* $x=c$ True Converse: *f(x) is not differentiable at* $x=c \Rightarrow$ *f(x) is not continuous at* $x=c$ False	MEX12-2
7	B	$e^{i3\theta}+e^{i\theta}=\left(1+e^{i2\theta}\right)e^{i\theta}$ $\left(1+e^{i2\theta}\right)=1+\cos 2\theta+i\sin 2\theta=2\cos\theta\left(\cos\theta+i\sin\theta\right)=2\cos\theta\, e^{i\theta}$ $\therefore e^{i3\theta}+e^{i\theta}=2\cos\theta\, e^{i2\theta}$	MEX12-4
8	D	A, B, C collinear $\Leftrightarrow\ \overline{BA}=\lambda\overline{AC}$ for some scalar λ. $\overline{BA}=\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}\qquad \overline{AC}=\underset{\sim}{i}+(a+1)\underset{\sim}{j}+b\underset{\sim}{k}\qquad \therefore a=1,\ b=1$	MEX12-3
9	D	Substituting $u=5-x$, $\int_1^4 f(5-x)\,dx=\int_4^1 f(u)(-1)\,du=\int_1^4 f(u)\,du=k$	MEX12-5
10	C	$\dot{y}=0\ \Rightarrow\ V-gt=0\quad \therefore t=\frac{V}{g}$ and $y=\frac{V^2}{g}\left(1-\frac{1}{2}\right)=\frac{V^2}{2g}$ at maximum height. $y=0\Rightarrow Vt-\frac{1}{2}gt^2=0\quad \therefore t=\frac{2V}{g}$ and $x=\frac{2V^2}{g}$ at horizontal range. $\frac{2V^2}{g}=4\times\frac{V^2}{2g}$. Hence horizontal range is 4 times maximum height.	MEX12-6

Section II

Question 11

a.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Writes both z and w in modulus/argument form	2
Substantial progress eg writes z in modulus/argument form	1

Answer

$$z=4\left(\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)=4\left(\cos\tfrac{\pi}{3}+i\sin\tfrac{\pi}{3}\right) \qquad w=4\left(\cos(\tfrac{\pi}{3}-\tfrac{\pi}{2})+i\sin(\tfrac{\pi}{3}-\tfrac{\pi}{2})\right)=4\left(\cos(-\tfrac{\pi}{6})+i\sin(-\tfrac{\pi}{6})\right)$$

a.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Writes expressions for both the product and quotient in modulus/argument form	2
Substantial progress eg writes such an expression for one of the product or quotient	1

Answer

$$zw=4\times4\left\{\cos\left(\tfrac{\pi}{3}+(-\tfrac{\pi}{6})\right)+i\sin\left(\tfrac{\pi}{3}+(-\tfrac{\pi}{6})\right)\right\}=16\left(\cos\tfrac{\pi}{6}+i\sin\tfrac{\pi}{6}\right)$$

$$\frac{z}{w}=\frac{4}{4}\left\{\cos\left(\tfrac{\pi}{3}-(-\tfrac{\pi}{6})\right)+i\sin\left(\tfrac{\pi}{3}-(-\tfrac{\pi}{6})\right)\right\}=\cos\tfrac{\pi}{2}+i\sin\tfrac{\pi}{2}$$

b.i Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Makes the required substitution to deduce result	2
Substantial progress eg makes the required substitution	1

Answer

$$u=\frac{1}{x}$$

$$du=-\frac{1}{x^2}dx$$

$$-\frac{1}{u^2}du=dx$$

$$x=\tfrac{1}{a}\Rightarrow u=a$$

$$x=a\Rightarrow u=\tfrac{1}{a}$$

$$\frac{\ln x}{1+x^2}=\frac{-\ln u}{1+\frac{1}{u^2}}=\frac{-u^2\ln u}{u^2+1}$$

$$\therefore\int_{\frac{1}{a}}^{a}\frac{\ln x}{1+x^2}\,dx=\int_{a}^{\frac{1}{a}}\frac{-u^2\ln u}{1+u^2}\left(\frac{-1}{u^2}\right)du$$

$$=-\int_{\frac{1}{a}}^{a}\frac{\ln u}{1+u^2}\,dx$$

$$=-\int_{\frac{1}{a}}^{a}\frac{\ln x}{1+x^2}\,dx$$

$$\therefore\int_{\frac{1}{a}}^{a}\frac{\ln x}{1+x^2}\,dx=0$$

Q11 (Cont.)

b.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Applies integration by parts, then evaluates in simplest exact form, using result from b.i.	3
Substantial progress eg evaluates using integration by parts and b.i. but simplification incomplete	2
Some progress eg applies integration by parts	1

Answer

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1} x}{x}\, dx = \left[\left(\tan^{-1} x\right)\ln x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} - \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\ln x}{1+x^2}\, dx$$

$$= \tfrac{\pi}{3}\ln\sqrt{3} - \tfrac{\pi}{6}\ln\tfrac{1}{\sqrt{3}} - 0$$

$$= \tfrac{\pi}{6}\ln 3 + \tfrac{\pi}{12}\ln 3$$

$$= \tfrac{\pi}{4}\ln 3$$

c.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Finds all three roots of the equation	3
Substantial progress eg finds the real root and a quadratic equation for the non-real roots	2
Some progress eg finds the real root	1

Answer

$f(z) = z^3 - 11z^2 + 55z - 125 \qquad f(5) = 0 \Rightarrow z-5$ is a factor.

$f(z) = (z-5)(z^2 - 6z + 25)$

$= (z-5)\left\{(z-3)^2 + 16\right\} \qquad$ Roots of $f(z) = 0$ are $5,\ 3+4i,\ 3-4i$

c.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Shows the points A, B, C lie on the circle $\|z\|=5$ and finds the area of the triangle	3
Substantial progress eg shows points lie on $\|z\|=5$ and states the coordinates of the points	2
Some progress eg finds the coordinates of the points or deduces each root satisfies $\|z\|=5$	1

Answer

All three roots have a modulus of 5, hence A, B, C lie in the Argand diagram on a circle of radius 5 and centre $(0,0)$ with equation $|z| = 5$.

Without loss of generality, let A, B, C have coordinates $(5,0)$, $(3,4)$, $(3,-4)$ respectively.

Then $BC = 8$ and the perpendicular from A to BC has length $5-3=2$.

Area of $\Delta ABC = \frac{1}{2}\times 8 \times 2 = 8$ sq. units.

Question 12

a.i Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Uses the amplitude, period and position of the graph to deduce its equation	2
Substantial progress eg finds the amplitude and period of the graph	1

Answer

Amplitude is 6 cm. Period is 16 s. $\therefore \frac{2\pi}{n} = 16 \quad \therefore n = \frac{\pi}{8}$.

Graph is a translation 2 units to the right of $x = 6\cos\frac{\pi}{8}t$. $\quad \therefore x = 6\cos\frac{\pi}{8}(t-2)$

a.ii Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Finds the distance travelled in the first minute	2
Substantial progress eg finds the initial position, or the distance travelled in one oscillation	1

Answer

$60 = 16\times 3 + 8 + 4$ Hence 60 seconds is the time for 3 oscillations, plus one half an oscillation, plus twice the time taken to travel from its initial position to its position at time $t = 2$.

$t = 0 \Rightarrow x = 6\cos\left(-\frac{\pi}{4}\right) = 3\sqrt{2}$. Hence particle travels $\left(6 - 3\sqrt{2}\right)$ cm in first 2 seconds.

The distance travelled in $3\frac{1}{2}$ oscillations is $3\frac{1}{2}\times 24\text{ cm} = 84\text{ cm}$.

Hence distance travelled in the first minute is $84 + 2\left(6-3\sqrt{2}\right)\text{ cm} = \left(96 - 6\sqrt{2}\right)\text{ cm}$

b.i Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Verifies the LHS angle satisfies $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ands finds its tan ratio to deduce result	2
Substantial progress eg writes an expression for tan of the LHS	1

Answer

$$\tan\left\{\tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right)\right\} = \frac{\frac{1}{k-1} - \frac{1}{k+1}}{1 + \left(\frac{1}{k-1}\right)\left(\frac{1}{k+1}\right)}$$
$$= \frac{(k+1)-(k-1)}{(k-1)(k+1)+1}$$
$$= \frac{2}{k^2}$$

For $k \ge 2$, $0 < \frac{1}{k+1} < \frac{1}{k-1} \le 1$

$$\therefore 0 < \tan^{-1}\left(\frac{1}{k+1}\right) < \tan^{-1}\left(\frac{1}{k-1}\right) \le \frac{\pi}{4}$$
$$\therefore 0 < \tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) \le \frac{\pi}{4}$$
$$\therefore -\frac{\pi}{2} < \tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) < \frac{\pi}{2}$$

$$\therefore \tan^{-1}\left(\frac{1}{k-1}\right) - \tan^{-1}\left(\frac{1}{k+1}\right) = \tan^{-1}\left(\frac{2}{k^2}\right)$$

Q12 (Cont.)

b.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses the result from b.i. to simplify the sum and obtain the required limit	3
Substantial progress eg correct process but makes one error or omission in simplification	2
Some progress eg uses result from b.i. and realises some terms cancel out	1

Answer

$$\begin{aligned}
\sum_{k=1}^{n}\tan^{-1}\left(\frac{2}{k^2}\right) &= \tan^{-1}2+\sum_{k=2}^{n}\tan^{-1}\left(\frac{2}{k^2}\right)\\
&= \tan^{-1}2+\sum_{k=2}^{n}\left\{\tan^{-1}\left(\frac{1}{k-1}\right)-\tan^{-1}\left(\frac{1}{k+1}\right)\right\}\\
&= \tan^{-1}2+\sum_{k=2}^{n}\tan^{-1}\left(\frac{1}{k-1}\right)-\sum_{k=2}^{n}\tan^{-1}\left(\frac{1}{k+1}\right)\\
&= \tan^{-1}2+\sum_{k=1}^{n-1}\tan^{-1}\left(\frac{1}{k}\right)-\sum_{k=3}^{n+1}\tan^{-1}\left(\frac{1}{k}\right)\\
&= \tan^{-1}2+\tan^{-1}1+\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}\\
&= \left(\tan^{-1}2+\tan^{-1}\frac{1}{2}\right)+\frac{\pi}{4}-\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}\\
&= \frac{\pi}{2}+\frac{\pi}{4}-\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1}
\end{aligned}$$

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n}\tan^{-1}\left(\frac{2}{k^2}\right)=\frac{3\pi}{4}-\lim_{n\to\infty}\left\{\tan^{-1}\frac{1}{n}+\tan^{-1}\left(\frac{1}{n+1}\right)\right\}=\frac{3\pi}{4}$$

c.i. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Applies integration by parts and simplifies to obtain required result	2
Substantial progress eg applies integration by parts with one error in simplification	1

Answer

$$\begin{aligned}
I_n &= \int_1^e (1-\ln x)^n\,dx\\
&= \left[x(1-\ln x)^n\right]_1^e-\int_1^e nx(1-\ln x)^{n-1}\left(-\frac{1}{x}\right)dx\ , \quad n=1,2,3,\ldots\\
&= -1+n\int_1^e(1-\ln x)^{n-1}\,dx\\
&= -1+nI_{n-1}
\end{aligned}$$

Q12 (Cont.)

c.ii. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Carries out the process of Mathematical induction to prove the required result	4
Substantial progress eg correct process with one minor error or omission	3
Moderate progress eg verifies truth of P_1 ; uses recurrence relation substituting for I_k if P_k true	2
Some progress eg verifies truth of P_1	1

Answer

Let P_n, $n=1,2,3,\ldots$, be the sequence of propositions defined by $P_n: \quad I_n = n!\,e-1-\sum_{r=1}^{n} {}^nP_r$.

Consider P_1 : $\quad I_1=-1+I_0=-1+\int_1^e 1\,dx=-1+(e-1)=e-2 \quad$ and $\quad 1!\,e-1-{}^1P_1=e-1-1=e-2$

Hence P_1 is true.

If P_k is true : $\quad I_k = k!\,e-1-\sum_{r=1}^{k} {}^kP_r \quad$ *

Consider P_{k+1} : $\quad I_{k+1}=-1+(k+1)I_k$

$$=-1+(k+1)\left(k!\,e-1-\sum_{r=1}^{k}{}^kP_r\right) \quad \text{if } P_k \text{ is true using *}$$

$$=-1+(k+1)k!\,e-(k+1)-\sum_{r=1}^{k}(k+1)\frac{k!}{(k-r)!}$$

$$=-1+(k+1)!\,e-(k+1)-\sum_{r=1}^{k}\frac{(k+1)!}{\{(k+1)-(r+1)\}!}$$

$$=-1+(k+1)!\,e-(k+1)-\sum_{r=1}^{k}{}^{k+1}P_{r+1}$$

$$=-1+(k+1)!\,e-(k+1)-\sum_{r=2}^{k+1}{}^{k+1}P_r$$

$$=-1+(k+1)!\,e-(k+1)-\left(\sum_{r=1}^{k+1}{}^{k+1}P_r-{}^{k+1}P_1\right)$$

$$=-1+(k+1)!\,e-(k+1)-\left(\sum_{r=1}^{k+1}{}^{k+1}P_r-(k+1)\right)$$

$$=(k+1)!\,e-1-\sum_{r=1}^{k+1}{}^{k+1}P_r$$

Hence if P_k is true, then P_{k+1} is true. But P_1 is true. Hence by Mathematical Induction, P_n is true for all positive integers $n\geq 1$.

Question 13

a. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Uses vector methods, including the properties of dot products, to deduce the required result	3
Substantial progress eg writes the vectors involved as scalar multiples of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, then either finds expressions for $\lvert OM\rvert\lvert OE\rvert$ and $\lvert ON\rvert\lvert OF\rvert$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, or uses the right angles to express $\underset{\sim}{a}\cdot\underset{\sim}{a}$ and $\underset{\sim}{b}\cdot\underset{\sim}{b}$ in terms of $\underset{\sim}{a}\cdot\underset{\sim}{b}$	2
Some progress eg writes the vectors involved as scalar multiples of $\underset{\sim}{a}$ and $\underset{\sim}{b}$	1

Answer

$\overrightarrow{OM}=\frac{1}{2}\underset{\sim}{a}$ and $\overrightarrow{OE}=\lambda\underset{\sim}{a}$ for some scalar λ. Then $\lvert OM\rvert\lvert OE\rvert=\frac{1}{2}\lambda\,\underset{\sim}{a}\cdot\underset{\sim}{a}$

$\overrightarrow{ON}=\frac{1}{2}\underset{\sim}{b}$ and $\overrightarrow{OF}=\mu\underset{\sim}{b}$ for some scalar μ. Then $\lvert ON\rvert\lvert OF\rvert=\frac{1}{2}\mu\,\underset{\sim}{b}\cdot\underset{\sim}{b}$

$\overrightarrow{AF}\perp\overrightarrow{OB}$ $\quad\therefore(\mu\underset{\sim}{b}-\underset{\sim}{a})\cdot\underset{\sim}{b}=0$ $\quad\therefore\mu\,\underset{\sim}{b}\cdot\underset{\sim}{b}=\underset{\sim}{a}\cdot\underset{\sim}{b}$

Similarly $\overrightarrow{BE}\perp\overrightarrow{OA}$ gives $\lambda\,\underset{\sim}{a}\cdot\underset{\sim}{a}=\underset{\sim}{b}\cdot\underset{\sim}{a}$. But $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{a}$. Hence $\lvert OM\rvert\lvert OE\rvert=\lvert ON\rvert\lvert OF\rvert=\frac{1}{2}\underset{\sim}{a}\cdot\underset{\sim}{b}$

b.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes an appropriate expression for the acceleration then integrates to find the required time	2
Substantial progress eg integrates to find a relationship between v and t	1

Answer

Let the body come to rest in T seconds

$$\ddot{x}=-\tfrac{1}{10}\sqrt{1+v}$$
$$\frac{dv}{dt}=-\tfrac{1}{10}\sqrt{1+v}$$
$$\int_{15}^{0}\frac{1}{\sqrt{1+v}}\,dv=-\tfrac{1}{10}\int_{0}^{T}dt$$
$$2\left[\sqrt{1+v}\right]_{15}^{0}=-\tfrac{1}{10}T$$
$$2(1-4)=-\tfrac{1}{10}T$$
$$T=60$$

Hence body comes to rest in 1 minute.

Q13 (Cont.)

b.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Writes an appropriate expression for the acceleration then integrates to find the required distance	3
Substantial progress eg correct process with one error or omission in evaluation	2
Some progress eg obtains integrals that relate v and x using *variables separable*	1

Answer

Let the body come to rest after travelling X metres.

$$v\frac{dv}{dx} = -\tfrac{1}{10}\sqrt{1+v}$$

$$\int_{15}^{0} \frac{v}{\sqrt{1+v}}\,dv = -\tfrac{1}{10}\int_{0}^{X} dx$$

$$\int_{15}^{0}\left(\sqrt{1+v} - \frac{1}{\sqrt{1+v}}\right)dv = -\tfrac{1}{10}X$$

$$\left[\tfrac{2}{3}(1+v)^{\frac{3}{2}} - 2\sqrt{1+v}\right]_{15}^{0} = -\tfrac{1}{10}X$$

$$\tfrac{2}{3}(1-64) - 2(1-4) = -\tfrac{1}{10}X$$

$$X = 360$$

Hence body travels 360 m in coming to rest.

c.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Shows the lines intersect at right angles, stating the point of intersection	3
Substantial progress eg shows the lines intersect stating the point of intersection	2
Some progress eg shows the direction vectors perpendicular, or finds λ, μ at intersection	1

Answer

At any intersection point
$$\begin{aligned} 3+2\lambda &= -1+\mu \quad (1) \\ 2-\lambda &= 1+\mu \quad (2) \\ -1+\lambda &= -\mu \quad (3) \end{aligned}$$
is a set of 3 consistent simultaneous equations.

Considering (1) and (2) : $(1)-(2) \Rightarrow 1+3\lambda = -2$ and hence $\lambda = -1,\ \mu = 2$.
Substituting in (3) : LHS $= -2 =$ RHS

Hence the set of 3 equations is consistent with solution $\lambda = -1,\ \mu = 2$.

At intersection $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$. Also $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2-1-1 = 0$.

Hence the lines intersect at right angles at $(1, 3, -2)$

Q13 (Cont.)

c.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Deduces the parametric form of the plane through L_1 and L_2 and finds its Cartesian form	2
Substantial progress eg explains why the form given includes both lines, or converts this parametric equation to Cartesian form	1

Answer

Unit vectors in the directions of the perpendicular vectors $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ form the basis of a plane in 3D space so that linear combinations of such vectors determine a plane through the origin. Addition of the vector $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$ translates this plane to pass through the point of intersection of the lines L_1 and L_2.

Hence the unique plane containing lines L_1 and L_2 has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + a\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{for parameters } a, b.$$

Note that $b = 0$ gives a vector equation for L_1 and $a = 0$ gives a vector equation for L_2 since $(1, 3, -2)$ lies on both lines.

For all points in this plane,
$$\begin{aligned} x &= 1 + 2a + b \quad (1) \\ y &= 3 - a + b \quad (2) \\ z &= -2 + a - b \quad (3) \end{aligned}$$
(2)+(3) gives $y + z = 1$

Hence the equation of the plane is $y + z = 1$.

Q13 (cont.)

c.iii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the perpendicular distance from the origin to the plane	2
Substantial progress eg finds a vector from the origin that is perpendicular to the plane, or writes an expression for the distance (or its square) from a general point in the plane to the origin	1

Answer

Let $P(x, y, 1-y)$ be a point in the plane $y + z = 1$. The perpendicular distance from the origin to this plane is the square root of the minimum value of $|OP|^2 = x^2 + y^2 + (1-y)^2 = x^2 + 2\left(y - \frac{1}{2}\right)^2 + \frac{1}{2}$.

This expression has a minimum value of $\frac{1}{2}$ when $x = 0$ and $y = \frac{1}{2}$.

Hence the perpendicular distance from the origin to the plane $y + z = 1$ is $\frac{1}{\sqrt{2}}$ units.

Q13 (cont.)
Alternative solution to 13c.iii.

Consider the vector $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ perpendicular to both $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

$$\begin{aligned} 2u - v + w &= 0 \quad (1) \\ u + v - w &= 0 \quad (2) \end{aligned} \qquad (1)+(2) \Rightarrow 3u = 0 \qquad \therefore u = 0,\ v = w\ .$$

Hence $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a vector through the origin which is perpendicular to both $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$,

and hence to the plane defined in (ii), and meets this plane at the point where $x = 0,\ y = z$ and $y + z = 1$, that is at the point $\left(0, \frac{1}{2}, \frac{1}{2}\right)$. Hence the perpendicular distance from the origin to the plane in (ii) is

$\sqrt{0 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \dfrac{1}{\sqrt{2}}$ units.

Question 14

a.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves required result	1

Answer

$$\frac{n}{n+1}\,{}^{2n}C_n = \frac{n(2n)!}{(n+1)n!\,n!} = \frac{(2n)!}{(n+1)!(n-1)!} = {}^{2n}C_{n-1}$$

a.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Shows the given expression is an integer.	2
Substantial progress eg expresses $\frac{1}{n+1}$ in terms of $\frac{n}{n+1}$ and uses (i) but explanation incomplete	1

Answer

$$\frac{1}{n+1}\,{}^{2n}C_n = \left(1 - \frac{n}{n+1}\right){}^{2n}C_n = {}^{2n}C_n - {}^{2n}C_{n-1}$$

Since these binomial coefficients are respectively the number of ways of choosing n or $(n-1)$ items from $2n$ items, both are integers hence their difference is also an integer.

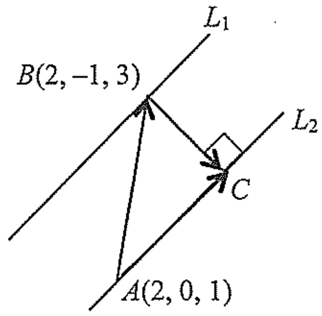
Q14 (Cont.)

b. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Finds the perpendicular distance between the parallel lines	3
Substantial progress eg correct process but makes an error in computation	2
Some progress eg draws a diagram showing points (2, −1, 3) on L_1 and (2, 0, 1) on L_2	1

Answer



Let $\underset{\sim}{p} = \overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$ and $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Then $\overrightarrow{AC} = proj_{\underset{\sim}{u}}\, \underset{\sim}{p}$ and $\overrightarrow{BC} = proj_{\underset{\sim}{u}}\, \underset{\sim}{p} - \underset{\sim}{p}$

$$proj_{\underset{\sim}{u}}\, \underset{\sim}{p} = \frac{\underset{\sim}{p} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\underset{\sim}{u} = \frac{-4}{6}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\therefore \overrightarrow{BC} = -\frac{2}{3}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3}\left\{\begin{pmatrix} -2 \\ -4 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}\right\} = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ -4 \end{pmatrix}$$

Hence the perpendicular distance between the parallel lines is $\frac{1}{3}\sqrt{2^2+1^2+4^2} = \frac{\sqrt{21}}{3}$

c. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves there are no such positive integers m and n	3
Substantial progress eg correct process but lack of clarity in explanation	2
Some progress eg factors $4m^2 - n^2$ and considers these factors as factors of 25	1

Answer

If m, n are positive integers such that $4m^2 - n^2 = 25$, then $(2m-n)(2m+n) = 25$ and hence $2m-n,\ 2m+n$ are factors of 25 with $2m-n < 2m+n$.

Then $\begin{aligned} 2m-n &= 1 \quad (1) \\ 2m+n &= 25 \quad (2) \end{aligned}$ and (1)+(2) gives $4m = 26$.

But there is no positive integer m such that $4m = 26$.

Hence by contradiction, there are no positive integers m and n such that $4m^2 - n^2 = 25$

Q14 (Cont.)

d.i. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Uses de Moivre's theorem to produce required result	2
Substantial progress eg uses de Moivre's theorem and takes real part of the binomial expansion	1

Answer

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$\begin{aligned}\therefore \cos 5\theta &= \mathrm{Re}(\cos\theta + i\sin\theta)^5 \\ &= {}^5C_0\cos^5\theta + {}^5C_2\cos^3\theta(i\sin\theta)^2 + {}^5C_4\cos\theta(i\sin\theta)^4 \\ &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\ &= \cos^5\theta(1+10+5) + \cos^3\theta(-10-10) + 5\cos\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

d.ii. Outcomes assessed: MEX12-4

Marking Guidelines

Criteria	Marks
Solves $\cos 5\theta = -1$ then forms an appropriate polynomial equation using sum, product of its roots	4
Substantial progress eg. correct process evaluating the sum as required but not the product	3
Moderate progress eg. solves $\cos 5\theta = -1$ and forms polynomial with roots $\cos\theta$ for solutions θ	2
Some progress eg solves $\cos 5\theta = -1$	1

Answer

$\cos 5\theta = -1$ for $5\theta = n\pi,\ n = 1, 3, 5, \ldots$ $\quad \therefore \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$ for $0 \le \theta \le 2\pi$.

Hence $16x^5 - 20x^3 + 5x + 1 = 0$ has roots $\cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, -1, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$.

But $\cos\frac{7\pi}{5} = \cos\left(2\pi - \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}$ and similarly $\cos\frac{9\pi}{5} = \cos\frac{\pi}{5}$

Hence using the relationships between coefficients and roots :

$$2\cos\tfrac{\pi}{5} + 2\cos\tfrac{3\pi}{5} - 1 = 0$$

$$\therefore \quad \cos\tfrac{\pi}{5} + \cos\tfrac{3\pi}{5} = \tfrac{1}{2}$$

$$-\left(\cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5}\right)^2 = -\tfrac{1}{16}$$

$$\therefore \quad \cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5} = \pm\tfrac{1}{4}$$

But $\cos\frac{\pi}{5} > 0$ and $\cos\frac{3\pi}{5} < 0$

$$\therefore \quad \cos\tfrac{\pi}{5}\cos\tfrac{3\pi}{5} = -\tfrac{1}{4}$$

Question 15

a. Outcomes assessed: MEX12-5

Marking Guidelines

Criteria	Marks
Makes substitution and evaluates the resulting definite integral	4
Substantial progress eg makes substitution and rearranges the integrand into partial fractions	3
Moderate progress eg makes substitution to obtain integrand as reciprocal of quadratic in t	2
Some progress eg writes dx in terms of dt and obtains 3+5sin$x$ (or its reciprocal) as a function of t	1

Answer

$$t=\tan\tfrac{x}{2}$$
$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$\frac{2}{1+t^2}dt=dx$$
$$x=0\Rightarrow t=0$$
$$x=\tfrac{\pi}{2}\Rightarrow t=1$$

$$3+5\sin x=\frac{3(1+t^2)+10t}{1+t^2}=\frac{(3t+1)(t+3)}{1+t^2}$$
$$\frac{1}{3+5\sin x}=\left(\frac{1+t^2}{2}\right)\left\{\frac{3}{3t+1}-\frac{1}{t+1}\right\}$$

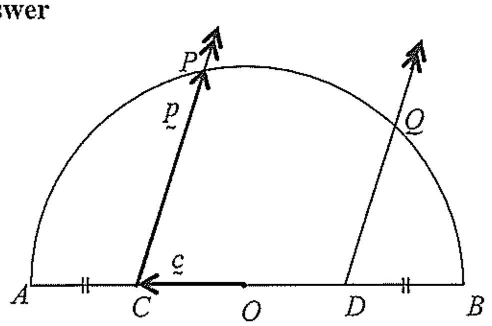
$$\int_0^{\frac{\pi}{2}}\frac{1}{3+5\sin x}\,dx=\int_0^1\left\{\frac{3}{3t+1}-\frac{1}{t+1}\right\}dt$$
$$=\left[\ln\frac{3t+1}{t+1}\right]_0^1$$
$$=\ln 2-\ln 1$$
$$=\ln 2$$

b.i. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Explains why $\overline{DQ}=\lambda \underset{\sim}{p}$ then uses $\lvert OP\rvert=\lvert OQ\rvert$ and properties of dot product to deduce result	3
Substantial progress eg correct process but error in manipulating dot products	2
Some progress eg explains why $\overline{DQ}=\lambda \underset{\sim}{p}$	1

Answer



$\overrightarrow{DQ},\ \overrightarrow{CP}$ are parallel in the same direction.

Hence $\overrightarrow{DQ}=\lambda\underset{\sim}{p}$ for some $\lambda>0$.

Then $\overrightarrow{OP}=\underset{\sim}{c}+\underset{\sim}{p}$ and $\overrightarrow{OQ}=-\underset{\sim}{c}+\lambda\underset{\sim}{p}$

But $\lvert OP\rvert=\lvert OQ\rvert$ (radii of the circle)

$$\therefore\quad(\underset{\sim}{c}+\underset{\sim}{p})\cdot(\underset{\sim}{c}+\underset{\sim}{p})=(-\underset{\sim}{c}+\lambda\underset{\sim}{p})\cdot(-\underset{\sim}{c}+\lambda\underset{\sim}{p})$$
$$\underset{\sim}{c}\cdot\underset{\sim}{c}+\underset{\sim}{p}\cdot\underset{\sim}{p}+2\underset{\sim}{c}\cdot\underset{\sim}{p}=\underset{\sim}{c}\cdot\underset{\sim}{c}+\lambda^2\underset{\sim}{p}\cdot\underset{\sim}{p}-2\lambda\,\underset{\sim}{c}\cdot\underset{\sim}{p}$$
$$(1-\lambda^2)\underset{\sim}{p}\cdot\underset{\sim}{p}+2(1+\lambda)\underset{\sim}{c}\cdot\underset{\sim}{p}=0$$
$$(1-\lambda)\underset{\sim}{p}\cdot\underset{\sim}{p}+2\underset{\sim}{c}\cdot\underset{\sim}{p}=0$$

Q15 (Cont.)

b.ii. Outcomes assessed: MEX12-3

Marking Guidelines

Criteria	Marks
Shows angle is 90° as required	1

Answer

$\overrightarrow{PQ} = -\underset{\sim}{p} - \underset{\sim}{c} - \underset{\sim}{c} + \lambda \underset{\sim}{p} = -\left\{(1-\lambda)\underset{\sim}{p} + 2\underset{\sim}{c}\right\}$

$\therefore \overrightarrow{PQ} \cdot \overrightarrow{PC} = \left\{(1-\lambda)\underset{\sim}{p} + 2\underset{\sim}{c}\right\} \cdot \underset{\sim}{p} = (1-\lambda)\underset{\sim}{p} \cdot \underset{\sim}{p} + 2\underset{\sim}{c} \cdot \underset{\sim}{p} = 0$

Hence $\angle CPQ = 90°$

c.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Shows both required results	2
Some progress eg shows $f_n(0) = 0$	1

Answer

$f_n(0) = e^0 - (1+0) = 0$

$$f_{n+1}(x) = e^x - \left(1 + \sum_{r=1}^{n+1} \frac{x^r}{r!}\right)$$

$$f_{n+1}(x) = e^x - \left(1 + x + \sum_{r=2}^{n+1} \frac{x^r}{r!}\right)$$

$$f_{n+1}'(x) = e^x - \left(1 + \sum_{r=2}^{n+1} \frac{r\,x^{r-1}}{r!}\right)$$

$$= e^x - \left(1 + \sum_{r=2}^{n+1} \frac{x^{r-1}}{(r-1)!}\right)$$

$$= e^x - \left(1 + \sum_{r=1}^{n} \frac{x^r}{r!}\right)$$

$$= f_n(x)$$

c.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves required results	2
Substantial progress eg shows $f_1(x)$ is an increasing function for $x > 0$	1

Answer

$f_1(x) = e^x - (1+x)$

$f_1'(x) = e^x - 1 > 0$ for $x > 0$

Hence $f_1(x)$ is an increasing function of x for $x > 0$ and $f_1(0) = 0$. $\therefore f_1(x) > 0$ for all $x > 0$.

Hence $1 + x < e^x$ for all $x > 0$.

Q15 (Cont.)

c.iii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses Mathematical Induction to prove the result	3
Substantial progress eg correct process but some lack of clarity or detail	2
Some progress eg defines an appropriate sequence of propositions and verifies the first is true	1

Answer

Let P_n, $n = 1, 2, 3, \ldots$ be the sequence of propositions $f_n(x) > 0$ for $x > 0$

Consider P_1 : from (ii) P_1 is true

If P_k is true : $f_k(x) > 0$ for $x > 0$

Consider P_{k+1} : $f_{k+1}{}'(x) = f_k(x) > 0$ for $x > 0$ if P_k is true

Then $f_{k+1}(x)$ is an increasing function for $x > 0$ and $f_{k+1}(0) = 0$,

giving $f_{k+1}(x) > 0$ for $x > 0$.

Hence if P_k is true, then P_{k+1} is true. But P_1 is true. Hence by Mathematical Induction, P_n is true for all positive integers $n \geq 1$. Hence $1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} < e^x$.

Question 16

a. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves the required result	3
Substantial progress eg correct process but lacks some clarity or detail	2
Some progress eg realises any positive factor of p, other than 1 or p, is one of $2, 3, 4, \ldots, p-1$	1

Answer

p is a positive integer such that p divides $((p-1)!+1)$.

If m is a factor of p, m a positive integer such that $m \neq 1$ and $m \neq p$, then m is one of the integers $2, 3, \ldots, p-1$ hence m divides $(p-1)!$. Also m divides $((p-1)!+1)$ since m is a factor of p.

Hence $\exists$ positive integers $k > j$ such that $mk = (p-1)!+1$ and $mj = (p-1)!$.

Then $m(k-j) = 1$ so that $m = 1$ and $k - j = 1$, which contradicts $m \neq 1$.

Hence there is no such factor m of p.

$\therefore \forall p \in \mathbb{Z}^+, p$ divides $((p-1)!+1) \Rightarrow p$ is prime

Q16 (Cont.)

b.i. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Integrates to obtain both x and y as functions of t	4
Substantial progress eg finds both v_x, v_y as functions of t, but only one of x, y as a function of t	3
Moderate progress eg finds only two of v_x, v_y, x, y, as functions of t	2
Some progress eg finds one of v_x, v_y, x, y, as a function of t	1

Answer

$$\frac{dv_x}{dt} = -\frac{v_x}{g}$$

$$\int \frac{1}{v_x}\,dv_x = -\frac{1}{g}\int dt$$

$$\ln A v_x = -\tfrac{1}{g}t\ , \quad A > 0 \text{ constant}$$

$$t = 0,\ v_x = V \Rightarrow AV = 1$$

$$\ln \frac{v_x}{V} = -\tfrac{1}{g}t$$

$$v_x = Ve^{-\frac{1}{g}t}$$

$$\frac{dx}{dt} = Ve^{-\frac{1}{g}t}$$

$$x = -gVe^{-\frac{1}{g}t} + c\,, \quad c \text{ constant}$$

$$t = 0, x = 0 \Rightarrow c = gV$$

$$\therefore x = gV\left(1 - e^{-\frac{1}{g}t}\right)$$

$$\frac{dv_y}{dt} = -\frac{g^2 + v_y}{g}$$

$$\int \frac{1}{g^2 + v_y}\,dv_y = -\frac{1}{g}\int dt$$

$$\ln B\left(g^2 + v_y\right) = -\tfrac{1}{g}t\ , \quad B > 0 \text{ constant}$$

$$t = 0,\ v_y = 0 \Rightarrow Bg^2 = 1$$

$$\ln \frac{g^2 + v_y}{g^2} = -\tfrac{1}{g}t$$

$$g^2 + v_y = g^2 e^{-\frac{1}{g}t}$$

$$v_y = g^2 e^{-\frac{1}{g}t} - g^2$$

$$\frac{dy}{dt} = g^2 e^{-\frac{1}{g}t} - g^2$$

$$y = -g^3 e^{-\frac{1}{g}t} - g^2 t + k, \quad k \text{ constant}$$

$$t = 0, y = 0 \Rightarrow k = g^3$$

$$\therefore\ y = g^3\left(1 - e^{-\frac{1}{g}t}\right) - g^2 t$$

b.ii. Outcomes assessed: MEX12-6

Marking Guidelines

Criteria	Marks
Proves the required relationship	2
Substantial progress eg substitutes $t = T$, $y = -H$, $x = R$ into expressions for x, y in terms of t	1

Answer

When $t = T$, $y = -H$ and $x = R$. Then $-H = \frac{g^2}{V}\left\{gV\left(1 - e^{-\frac{1}{g}T}\right)\right\} - g^2 T$

$$-H = \frac{g^2}{V}R - g^2 T$$

$$g^2 T = \frac{g^2}{V}R + H$$

$$T = \frac{g^2 R + VH}{Vg^2}$$

a.

Q16 (Cont.)

c.i. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves required result	1

Answer

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 \qquad \therefore e^{ik\pi} = \left(e^{i\pi}\right)^k = (-1)^k$$

c.ii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Proves required result	2
Substantial progress eg writes $\cos\left(\frac{k\pi}{n}\right)$ in terms of $e^{i\left(\frac{k\pi}{n}\right)}$	1

Answer

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta \qquad \therefore e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$\therefore \cos\left(\frac{k\pi}{n}\right) = \tfrac{1}{2}\left(e^{i\left(\frac{k\pi}{n}\right)} + e^{-i\left(\frac{k\pi}{n}\right)}\right) = \tfrac{1}{2}e^{-i\left(\frac{k\pi}{n}\right)}\left(e^{i\left(\frac{2k\pi}{n}\right)} + 1\right)$$

$$\sum_{k=0}^{n-1}(-1)^k\cos^n\left(\frac{k\pi}{n}\right) = \sum_{k=0}^{n-1}e^{ik\pi}\left\{\tfrac{1}{2}e^{-i\left(\frac{k\pi}{n}\right)}\left(1+e^{i\left(\frac{2k\pi}{n}\right)}\right)\right\}^n = \frac{1}{2^n}\sum_{k=0}^{n-1}\left(1+e^{i\left(\frac{2k\pi}{n}\right)}\right)^n$$

c.iii. Outcomes assessed: MEX12-2

Marking Guidelines

Criteria	Marks
Uses Binomial theorem and manipulates double summation to obtain required result	3
Substantal progress eg uses Binomial theorem then changes order of summation, identifying the sum to n terms of a geometric series.	2
Some progress eg uses the Binomial theorem to produce a double summation	1

Answer

$$\begin{aligned}
\sum_{k=0}^{n-1}(-1)^k\cos^n\left(\frac{k\pi}{n}\right) &= \frac{1}{2^n}\sum_{k=0}^{n-1}\sum_{r=0}^{n}{}^nC_r\, e^{i\left(\frac{2kr\pi}{n}\right)} \\
&= \frac{1}{2^n}\sum_{r=0}^{n}{}^nC_r\sum_{k=0}^{n-1}e^{i\left(\frac{2kr\pi}{n}\right)} \\
&= \frac{1}{2^n}\left\{{}^nC_0\sum_{k=0}^{n-1}1 + {}^nC_n\sum_{k=0}^{n-1}1 + \sum_{r=1}^{n-1}{}^nC_r\sum_{k=0}^{n-1}\left(e^{i\left(\frac{2\pi r}{n}\right)}\right)^k\right\} \\
&= \frac{1}{2^n}\left\{n + n + \sum_{r=1}^{n-1}{}^nC_r\frac{1-e^{i(2\pi r)}}{1-e^{i\left(\frac{2\pi r}{n}\right)}}\right\} \\
&= \frac{1}{2^n}\{2n+0\} \\
&= \frac{n}{2^{n-1}}
\end{aligned}$$

Independent HSC Trial Examinations 2022 Mathematics Extension 2 Mapping Grid

Question	Marks	Content	Syllabus Outcomes	Targeted Performance Bands
1	1	The nature of proof	MEX12-2	E2-E3
2	1	Introduction to complex numbers	MEX12-4	E2-E3
3	1	Further work with vectors	MEX12-3	E2-E3
4	1	Further integration	MEX12-5	E2-E3
5	1	Application of calculus to mechanics	MEX12-6	E2-E3
6	1	The nature of proof	MEX12-2	E3-E4
7	1	Introduction to complex numbers	MEX12-4	E3-E4
8	1	Further work with vectors	MEX12-3.	E3-E4
9	1	Further integration	MEX12-5	E3-E4
10	1	Application of calculus to mechanics	MEX12-6	E3-E4
11 a i	2	Introduction to complex numbers	MEX12-4	E2-E3
ii	2	Introduction to complex numbers	MEX12-4	E2-E3
b i	2	Further integration	MEX12-5	E2-E3
ii	3	Further integration	MEX12-5	E3-E4
c i	3	Introduction to complex numbers	MEX12-4	E2-E3
ii	3	Introduction to complex numbers	MEX12-4	E2-E3
12 a i	2	Application of calculus to mechanics	MEX12-6	E2-E3
ii	2	Application of calculus to mechanics	MEX12-6	E3-E4
b i	2	The nature of proof	MEX12-2	E2-E3
ii	3	The nature of proof	MEX12-2	E2-E3
c i	2	Further integration	MEX12-5	E3-E4
ii	4	Further proof by mathematical induction	MEX12-2	E3-E4
13 a	3	Further work with vectors	MEX12-3	E2-E3
b i	2	Application of calculus to mechanics	MEX12-6	E2-E3
ii	3	Application of calculus to mechanics	MEX12-6	E3-E4
c i	3	Further work with vectors	MEX12-3	E2-E3
ii	2	Further work with vectors	MEX12-3	E3-E4
iii	2	Further work with vectors	MEX12-3	E3-E4
14 a i	1	The nature of proof	MEX12-2	E2-E3
ii	2	The nature of proof	MEX12-2	E3-E4
b	3	Further work with vectors	MEX12-3	E2-E3
c	3	The nature of proof	MEX12-2	E2-E3
d i	2	Using complex numbers	MEX12-4	E3-E4
ii	4	Using complex numbers	MEX12-4	E3-E4
15 a	4	Further integration	MEX12-5	E2-E3
b i	3	Further work with vectors	MEX12-3	E2-E3
ii	1	Further work with vectors	MEX12-3	E2-E3
c i	2	The nature of proof	MEX12-2	E3-E4
ii	2	The nature of proof	MEX12-2	E3-E4
iii	3	Further proof by mathematical induction	MEX12-2	E3-E4
16 a	3	The nature of proof	MEX12-2	E3-E4
b i	4	Application of calculus to mechanics	MEX12-6	E3-E4
ii	2	Application of calculus to mechanics	MEX12-6	E3-E4
c i	1	The nature of proof	MEX12-2	E2-E3
ii	2	The nature of proof	MEX12-2	E3-E4
iii	3	The nature of proof	MEX12-2	E3-E4