

COMP2022|2922 Models of Computation

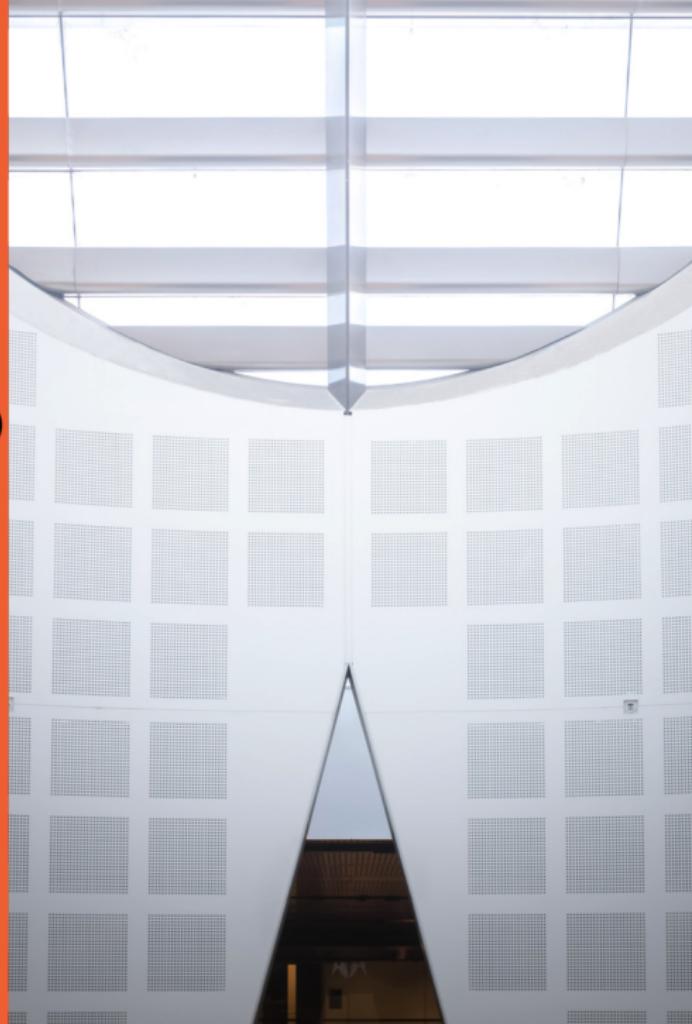
Deterministic Finite Automata (DFA)

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Questions we will study (see week 1)

Q. How can we specify languages?

- ✓ 1. In English.
- ✓ 2. In mathematics/set-theoretic notation, and recursive definitions.
- ✓ 3. By regular expressions R
- ✓ 4. By (context-free) grammars G
- ✓ 5. By automata M , and more generally by machine models of computation.

Automata in a nutshell

An automaton is like a symbol-processing program that:

- Takes a string as input.
- Can only use variables with finite domains: state taking finitely many values.
1, 2, 3 or 4
- Can read the input string character by character: `get_char()`
- Can test for end of input: `end_of_input()`
- Must decide to "Accept" or "Reject" the input string.

M
1, 2, 3 or 4

Why study such a simple model of computation?

1. It is part of computing culture.
 - first appeared in McCulloch and Pitt's model of a neural network (1943)
 - then formalised by Kleene (American mathematician) as a model of stimulus and response
2. It has numerous practical applications.
 - scanner (aka lexical analyser)
 - pattern matching
 - communication protocols with bounded memory
 - circuits with feedback
 - finite-state reactive systems
 - finite-state controllers
 - non-player characters in computer games
 - ...
3. It is simple to implement/code.

Program representation of automata

What strings over alphabet $\Sigma = \{a, b\}$ does the following code accept?

```
1 state = 1
2 while not end_of_input():
3     x = get_char()
4     if state==1 and x=="b" then state=2
5     else
6         if state==2 and x=="a" then state=3
7 if state==2 return "Accept"
8 else return "Reject"
```

Vote now! (on mentimeter)

- 1. All strings that match the RE $(a \mid b)^*$.
- 2. All strings that do contain a b but not an a . ✓
- 3. All strings that match the RE a^*bb^* .

Graphical representation of automata

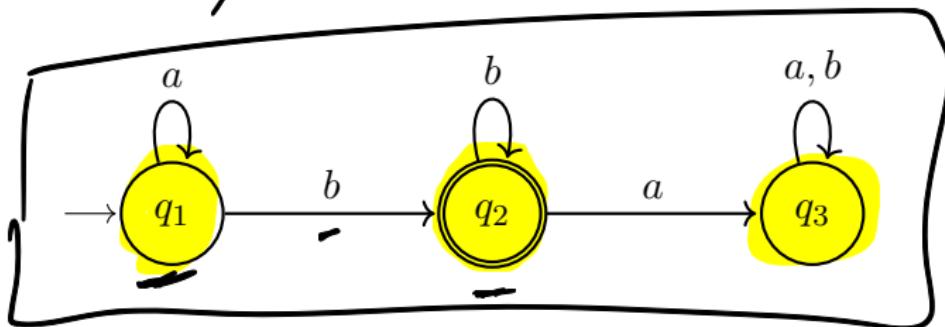
Such programs have a graphical representation as a directed edge-labeled graph:

- Vertices represent states.
- Labeled-edges $q \xrightarrow{a} q'$ represent transitions between states.
- The initial state is marked with an incoming arrow.
- Accepting states are marked with an extra circle.

The automaton **accepts** the strings that label a path from the initial state to a final state.

Graphical representation of an automaton

Program \rightsquigarrow automaton



$\underbrace{aaa}_{\pi_0} \underline{b} \underbrace{bbb}_{\pi_0}$

$L(Q^* L L^*)$

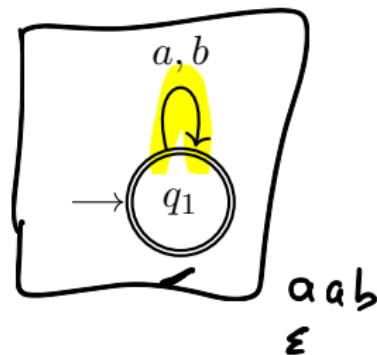
Example

is the set of

What strings ~~does~~ this automaton accept?

^

$$\Sigma = \{a,b\}$$



Vote now! (on mentimeter)

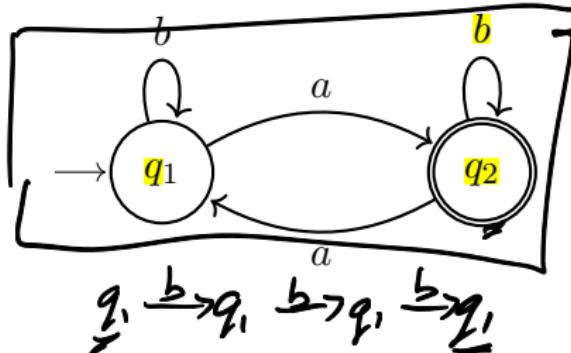


2. All strings that end in an a .
3. All strings with an odd number of a 's.
4. All strings that do not contain an a .

Example

What strings does this automaton accept?

State stores the parity of the #a's seen so far.



q_1 represents strings with even # a's

$-q_2 \dots$ odd ...

bbb

$q_1 \xrightarrow{b} q_1 \xrightarrow{b} q_1 \xrightarrow{b} q_1$

Vote now! (on mentimeter)

- ? 1. All strings X
- ? 2. All strings that end in an a . X
- ✓ 3. All strings with an odd number of a 's.
- ? 4. All strings that do not contain an a . X

$q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_1$

Example

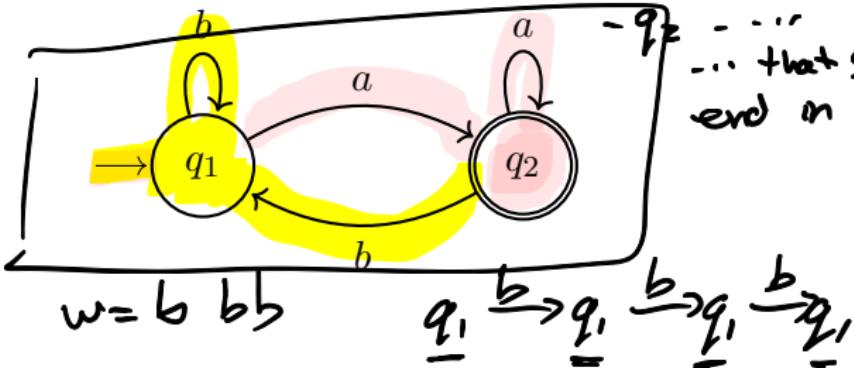
What strings does this automaton accept?

- q_1 represents all strings that do not end in "a".

- q_2 ... that do not end in "a".

bbb ✗

aa /
a ✓



Vote now! (on mentimeter)

- ? 1. All strings ✗
- ✓ 2. All strings that end in an a.
- ? 3. All strings with an odd number of a's. ✗
- ? 4. All strings that do not contain an a. ✗

DFA

FSM finite state machines

Definition

A **deterministic finite automaton (DFA)** M consists of 5 items

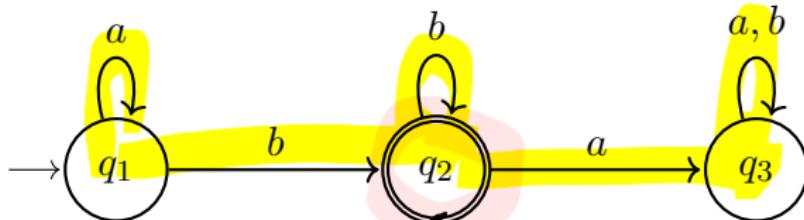
$(Q, \Sigma, \delta, q_0, F)$ where

1. Q is a finite set of **states**,
2. Σ is the **(input) alphabet**,
3. $\delta: Q \times \Sigma \rightarrow Q$ is the **transition function**,
 - If $\delta(q, a) = q'$ we write $q \xrightarrow{a} q'$, called a **transition**.
4. $q_0 \in Q$ is the **start state**, and
5. $F \subseteq Q$ is the set of **accept states** (aka **final states**).

$$Q = \{q_1, q_2\}$$



Example



$$M = (Q, \Sigma, \delta, q_1, F)$$

$$\delta: Q \times \Sigma \rightarrow Q$$

$$Q = \{q_1, q_2, q_3\}$$

$$\Sigma = \{a, b\}$$

$$F = \{q_2\}$$

δ	a	b
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_3	q_1

The language recognised by M

Definition

DFA $M = (Q, \Sigma, \delta, q_0, F)$, string $w = w_1 w_2 \cdots w_n$ over Σ .

- A run (aka **computation**) of M on w is a sequence of transitions $q_0 \xrightarrow{w_1} q_1 \xrightarrow{w_2} q_2 \xrightarrow{w_3} \cdots \xrightarrow{w_n} q_n$.
- The run is **accepting** if $q_n \in F$.
- If w has an accepting run then we say that M accepts w .

The set $L(M) = \{w \in \Sigma^* : M \text{ accepts } w\}$ is the **language recognised by M** (aka **language of M**).

In words:

- The automaton **accepts** the strings that label a path from the initial state to a final state.
- The **language recognised by M** (aka **the language of M**) written $L(M)$ is the set of strings the automaton accepts.

Regular languages

The languages of DFAs are so important, we give them a name:

Definition

A language L is called regular if $\underline{L = L(M)}$ for some DFA M .

regular expression

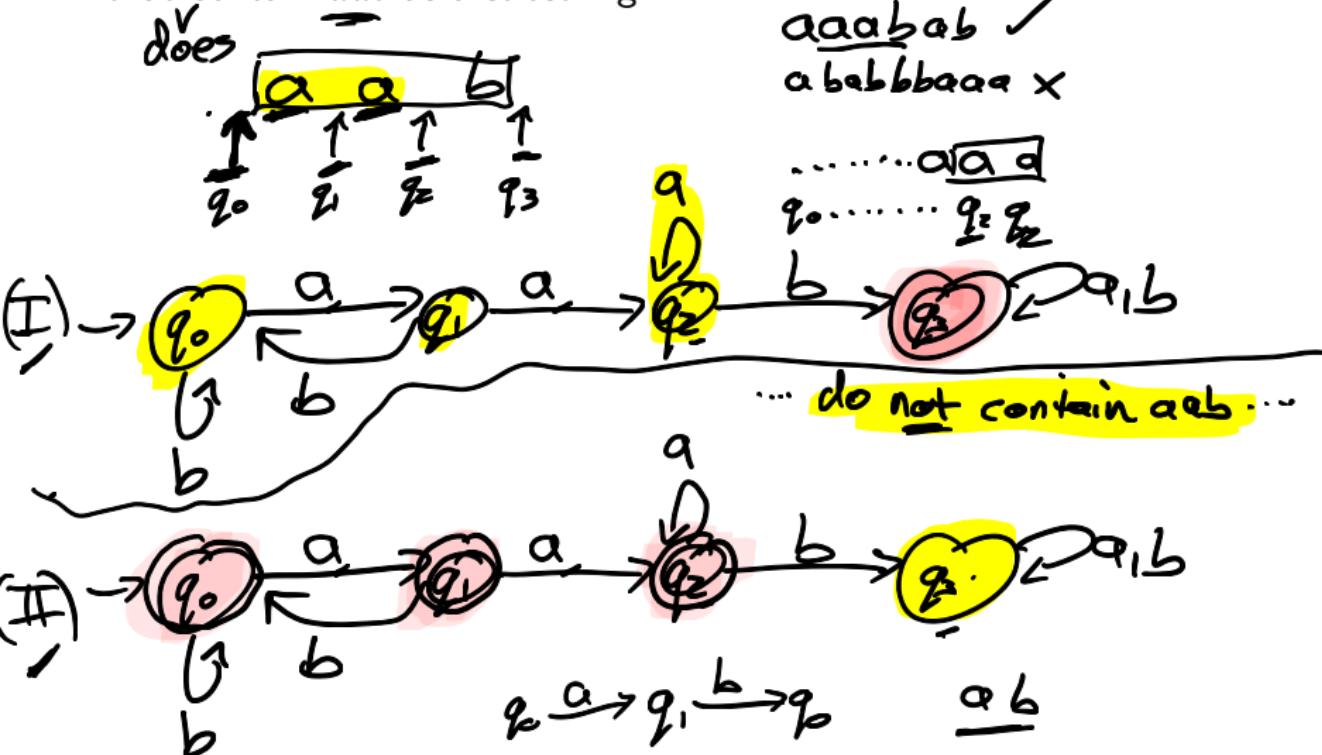
Designing automata tips (i)

Imagine you are the automaton reading the string symbol by symbol:

- what **information** about the string read so far do you need to make a decision on whether to accept or reject.
- ! ! – can you update this information if another input symbol arrives?
- • – the states will store this information.

Designing automata

Draw an automaton for the language of strings over $\Sigma = \{a, b\}$ that contain aab as a substring.



Designing automata tips (ii)

Build automata out of other automata using closure properties of the regular languages.

If you have DFA for L_1, L_2 then you can get DFA for

- ✓ 1. $\Sigma^* \setminus L_1$
- ✓ 2. $L_1 \cup L_2$ (and thus, by DeMorgan!, also $L_1 \cap L_2$) *tutorial*
- { 3. $L_1 L_2$
- 4. $(L_1)^*$

Let's see how to do complement (**union is in the tutorial**, **concatenation and star** are trickier and we will do it in a future lecture).

Regular languages closed under complementation

Theorem

If L is regular, then $\Sigma^* \setminus L$ is regular.

Idea

Swap accept and reject states in a DFA for L



Regular languages closed under complementation

Theorem

If L is regular, then $\Sigma^* \setminus L$ is regular.

Construction: "swap accept and reject states"

- Given DFA $M = (Q, \Sigma, \delta, q_0, F)$ recognising L
- We build a DFA M' recognising $\Sigma^* \setminus L$ as follows:
Define $M' = (Q, \Sigma, \delta, q_0, F')$ where $F' = Q \setminus F$.

Regular languages closed under complementation

Theorem

If L is regular, then $\Sigma^* \setminus L$ is regular.

Example

Give DFA for the language of strings over $\{a, b\}$ that do **not** contain aab as a substring.

Important questions about DFA

- Which languages can be described by DFAs? All languages?
 - No! (we will come back to this in a future lecture)
- There are natural decision problems associated with DFAs. Are there programs that solve them?

1. Membership problem

Input: DFA M , string w .

Output: decide if $w \in L(M)$.

2. Non-emptiness problem (tutorial)

Input: DFA M .

Output: decide if $L(M) \neq \emptyset$.

3. Equivalence problem (tutorial)

Input: DFAs M_1, M_2 .

Output: decide if $L(M_1) = L(M_2)$.

Membership problem

Input: DFA M , string w .

Output: decide if $w \in L(M)$.

```
1 def membership(M, w):
2     state = q_0
3     while not end_of_input(w):
4         x = get_char(w)
5         state = δ(state, x)
6         if state in F:
7             return "Accept"    "Yes, w ∈ L(M)"
8         else:
9             return "Reject"   "No".
```

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Nondeterministic finite automata (NFA)

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$$\delta(q, a)$$

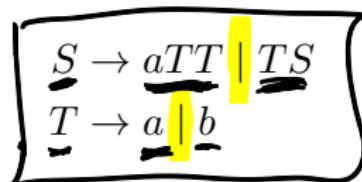
~~random~~

Agenda

1. Introduction to **nondeterminism**.
 - It refers to situations that in which the next state of the computation is not uniquely determined by the current state.
2. Nondeterministic finite automata

Motivating nondeterminism

Let's try to derive $aaabb$ by searching through the space of derivations:

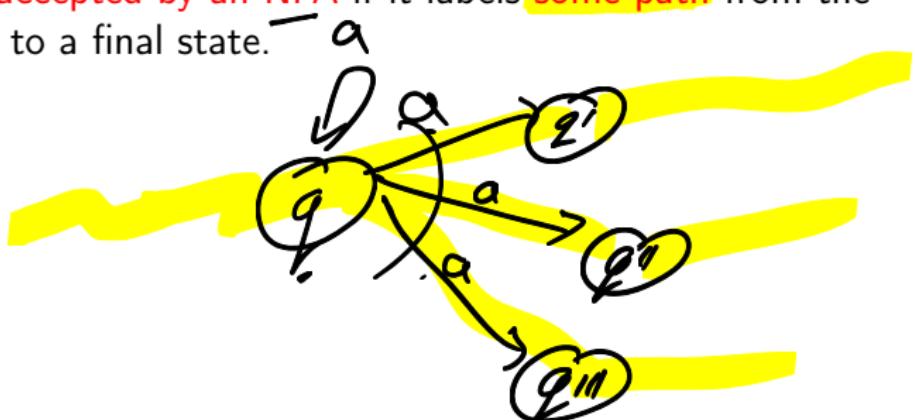


- $S \Rightarrow aTT \Rightarrow X$
- $S \Rightarrow TS \Rightarrow aS \Rightarrow aaTT \Rightarrow X$
- $S \Rightarrow TS \Rightarrow aS \Rightarrow aTS \Rightarrow aaS \Rightarrow aaaTT \Rightarrow aaaaT \Rightarrow X$
- $S \Rightarrow TS \Rightarrow aS \Rightarrow aTS \Rightarrow aaS \Rightarrow aaaTT \Rightarrow aaabT \Rightarrow aaabb$

Takeaway: a string can be derived if there is some sequence of correct guesses (even if some sequences of guesses are wrong).

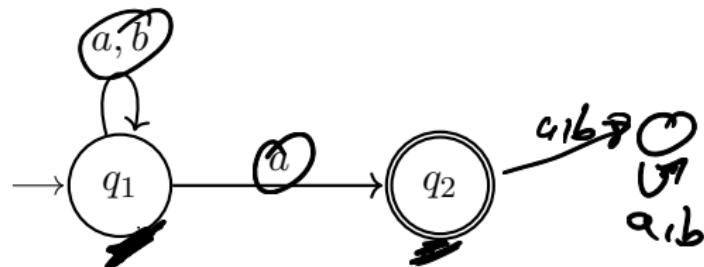
Let's introduce nondeterminism into automata.

- A **nondeterministic finite automaton (NFA)** is like a DFA except that states can have more than one outgoing transition on the same input symbol.
- A string is **accepted by an NFA** if it labels some path from the initial state to a final state.



NFA

What strings over alphabet $\Sigma = \{a, b\}$ are accepted by this NFA?



Vote now! (on mentimeter)

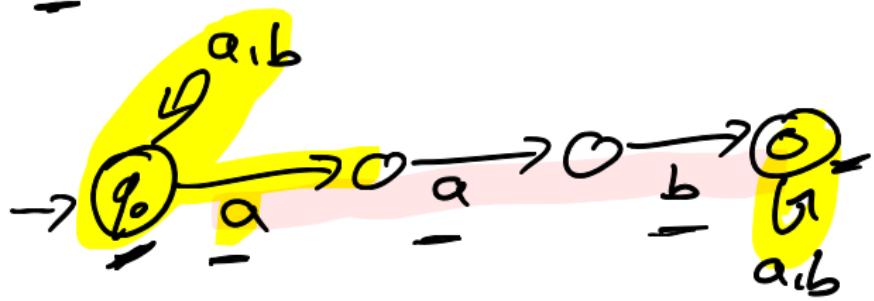
The set of

1. all strings. ? x
2. strings that have at least one a . ? x
3. strings that end in an a .
4. strings that start with an a .



Pattern matching made easy!

- NFAs are good for specifying languages of the form "the string has x as a substring".
- E.g., $x = \underline{aab}$



aab

Definition of NFA

Definition

A nondeterministic finite automaton (NFA) $M = (Q, \Sigma, \delta, q_0, F)$ is the same as a DFA except that

$$\begin{aligned}\delta: Q \times \Sigma &\rightarrow P(Q) \\ \delta: Q \times \Sigma_\epsilon &\rightarrow P(Q),\end{aligned}$$

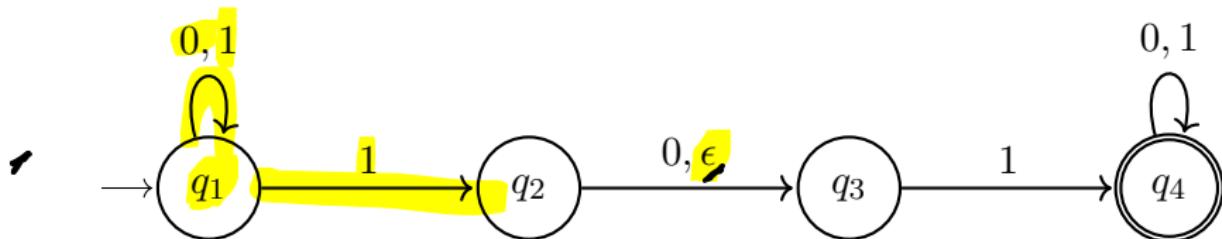
$P(Q)$ is the set of all subsets of Q .

called the transition relation.

- If $q' \in \delta(q, a)$ we write $q \xrightarrow{a} q'$, called a transition.
- Here $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$.
- So, we also allow epsilon-transitions
 - This amounts to transitions that do not consume the next input symbol.



NFA with Epsilon transitions



$$\underline{M} = (Q, \Sigma, \delta, q_0, F)$$

$$Q = \{q_1, q_2, q_3, q_4\}$$

$$\Sigma = \{0, 1\}$$

q_0 is q_1

$$F = \{q_4\}$$

δ	0	1	ϵ	
q_1	$\{q_1\}$	\emptyset	$\{q_1, q_2\}$	\emptyset
q_2	$\{q_3\}$	\emptyset	\emptyset	$\{q_3\}$
q_3	\emptyset	$\{q_4\}$	$\{q_4\}$	\emptyset
q_4	$\{q_4\}$	$\{q_4\}$	\emptyset	\emptyset

Definition of NFA

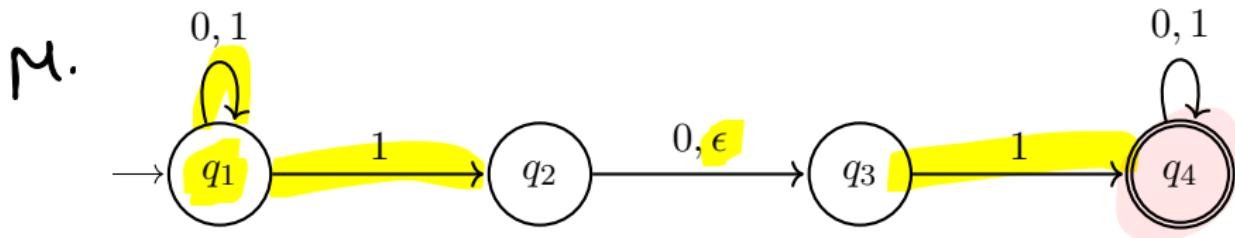
- A **run** (aka **computation**) of an NFA M on string w is a sequence of transitions $q_0 \xrightarrow{y_1} q_1 \xrightarrow{y_2} q_2 \dots \xrightarrow{y_m} q_m$ such that each $y_i \in \Sigma_\epsilon$ and $w = y_1 y_2 \dots y_m$.¹
- The run is **accepting** if $q_m \in F$.
- If w has at least one accepting run, then we say that w is **accepted** by M .
- The language **recognised** by M is $L(M) = \{w \in \Sigma^* : w \text{ is accepted by } M\}$.

In words:

An NFA M describes the language $L(M)$ of all strings that label paths from the start state to an accepting state.

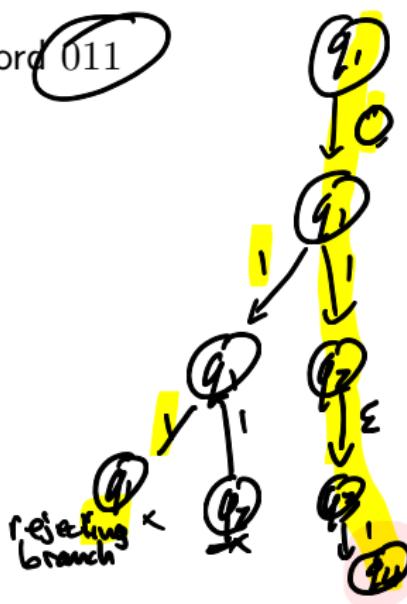
¹Recall that $\Sigma_\epsilon = \Sigma \cup \{\epsilon\}$, and $\epsilon x = x\epsilon = x$ for all strings x .

NFA with Epsilon transitions



input word 011

run tree



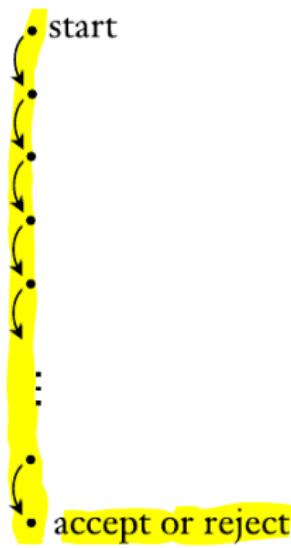
So 011 is accepted by
M.

accepting run/branch.

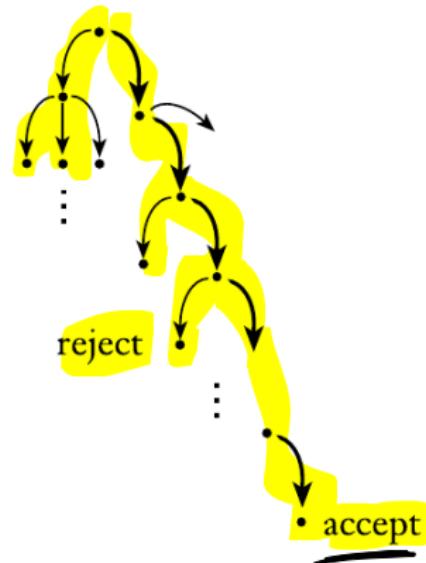
rejecting branch

Comparing NFAs and DFAs

Deterministic computation



Nondeterministic computation



Comparing NFAs and DFAs

1. In a DFA, every input has exactly one run.² The input string is accepted if this run is accepting.
2. In an NFA, an input may have zero, one, or more runs. The input string is accepted if at least one of its runs is accepting.
3. For every DFA M there is an NFA N such that $L(M) = L(N)$.
 - Idea: $q' = \delta(q, a)$ in M becomes $\{q'\} = \delta(q, a)$ in N .
 - You can think of a DFA as an NFA in which there is no nondeterminism.

$$\delta': Q \times \Sigma \rightarrow Q$$



$$\delta: Q \times \Sigma_{\epsilon} \rightarrow P(Q)$$

²With our convention of not drawing rejecting sinks, an input of a DFA may have no runs too.

Where are we going?

- We are going to show that DFA, NFA and regular expressions specify the same set of languages!
- We will do this with a series of transformations:

1. From Regular Expressions to NFAs
2. From NFAs to NFAs without ϵ -transitions
3. From NFAs without ϵ -transitions to DFAs *subset construction*
4. From DFAs to Regular Expressions

From Regular Expressions to NFAs

" R is equivalent to N "

Theorem

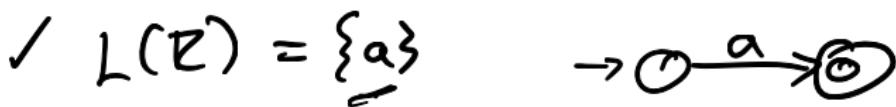
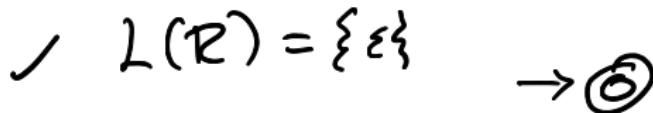
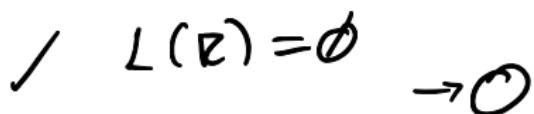
For every $\text{RE } R$ there is an NFA N such that $L(R) = L(N)$.

The construction is by recursion on the structure of R .

1. The base cases are $R = \emptyset, R = \epsilon, R = a$ for $a \in \Sigma$
 - We must show that each of these languages is recognised by some NFA.
2. The recursive cases are $R = (R_1 | R_2)$, $R = (R_1 R_2)$, and $R = R_1^*$
 - We must show that if N_1, N_2 are NFAs, then there are NFAs recognising $L(N_1 \cup N_2)$, $L(N_1)L(N_2)$, and $L(N_1)^*$.

From Regular Expressions to NFAs

The base cases are $R = \emptyset$, $R = \epsilon$, $R = a$ for $a \in \Sigma$.



NFAs are closed under union

Lemma

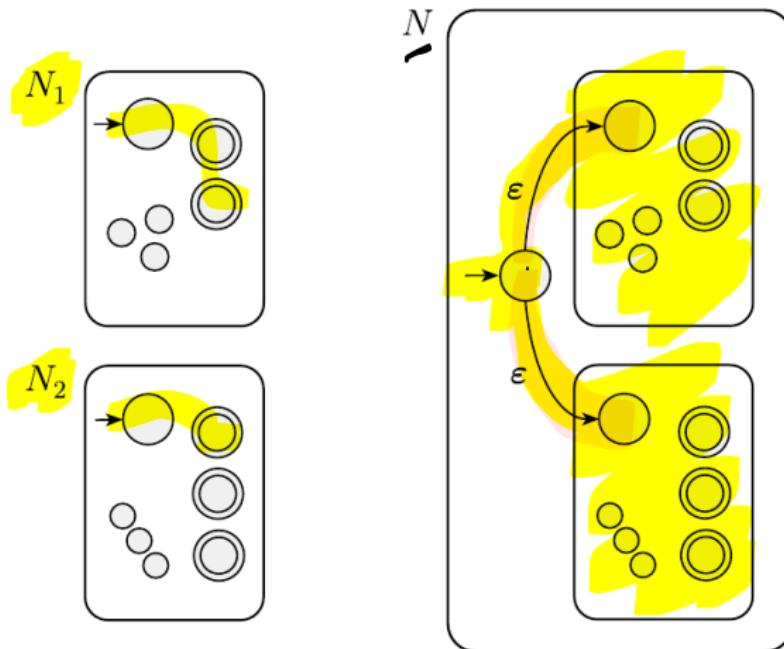
If N_1, N_2 are NFAs, there is an NFA N recognising $L(N_1) \cup L(N_2)$

Idea: "Simulate N_1 or N_2 "

- Given NFAs $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ construct NFA N that guesses which of N_1 or N_2 to simulate. How?
- N has states $Q_1 \cup Q_2 \cup \{q_0\}$ so that it can simulate N_1, N_2 .
- N guesses from q_0 whether to go to the start state of N_1 or N_2 .

NFAs are closed under union

$$L(N) = L(N_1) \cup L(N_2)$$



NFAs are closed under concatenation

Lemma

If N_1, N_2 are NFAs, there is an NFA N recognising $L(N_1)L(N_2)$

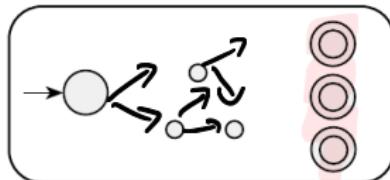
Idea: "Simulate N_1 followed by N_2 "

- Given NFAs $N_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ construct NFA N that **guesses** how to break the input into two pieces, the first accepted by N_1 , the second by N_2 . How?
- N has states $Q_1 \cup Q_2$ so that it can simulate N_1, N_2 .
- At some point when N_1 is in a final state, **guess** that it is time to move to the initial state of N_2 .

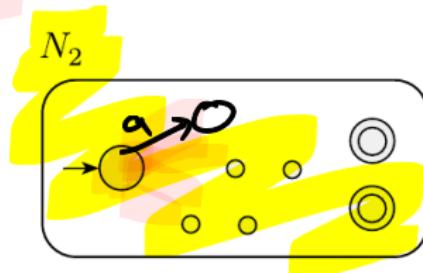
NFAs are closed under concatenation

$$L(N) = L(N_1) \cdot L(N_2)$$

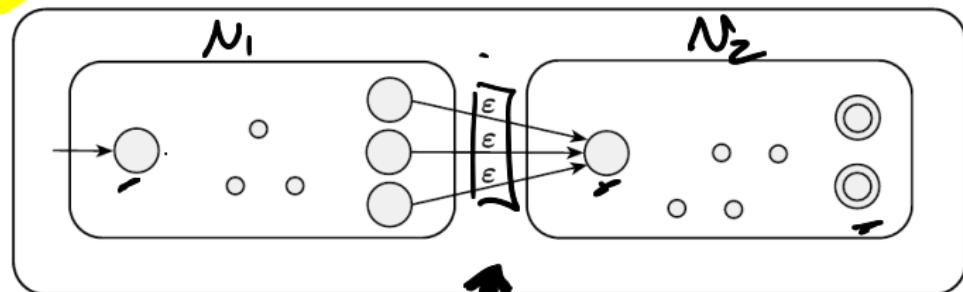
N_1



N_2



N



$w = w_1 \underline{w_2} \in L(N)$

NFAs are closed under star

Lemma

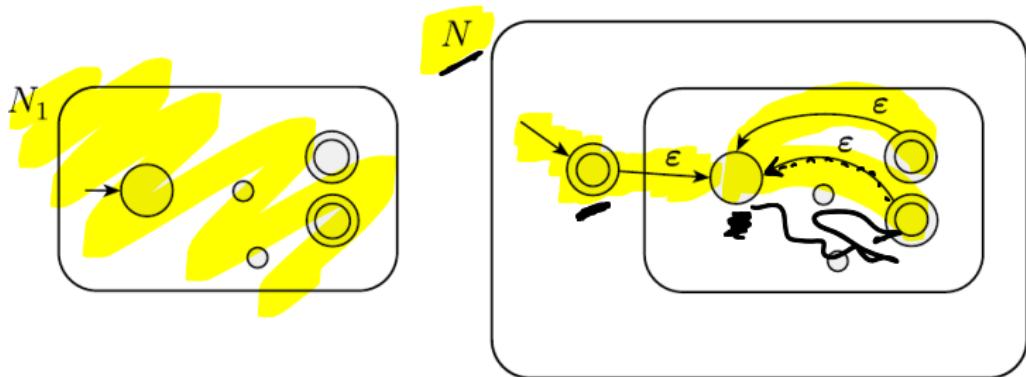
If N_1 is an NFA, there is an NFA N recognising $L(N_1)^*$

Idea: "Repeatedly simulate N_1 "

- Given NFA $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ construct NFA N that guesses how to break the input into pieces, each of which is accepted by N_1 . How?
- N has states $Q \cup \{q_0\}$, and extra transitions from final states of N_1 to the initial state of N_1 .
- The new state q_0 ensures that ϵ is accepted.

NFAs are closed under star

$$L(N) = L(N_1)^*$$

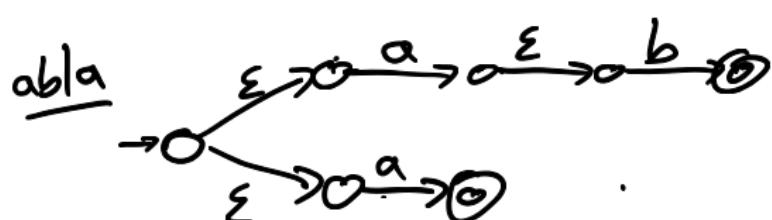
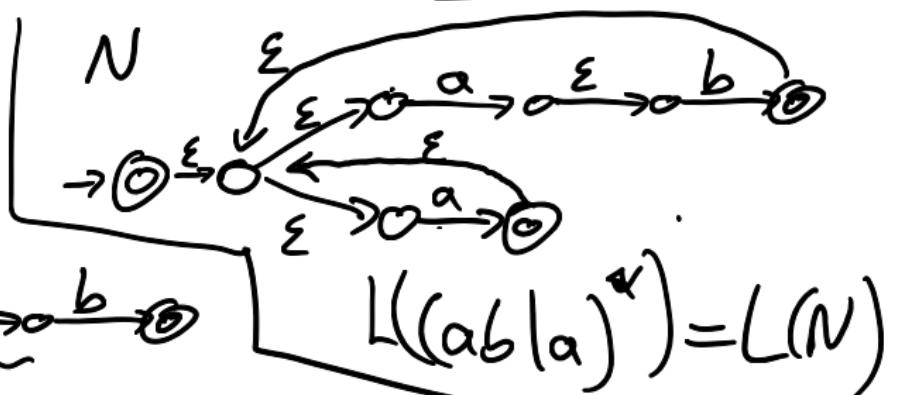


$$\underline{w_1} \in L(N_1) \quad w_2 \in L(N_1) \quad \underline{\epsilon} \in L(N_1)^* \quad \underline{w_n} \in L(N_1)$$

From RE to NFA: example

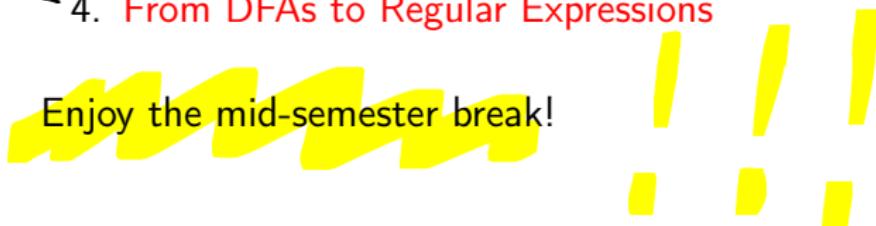
Convert the regular expression $(ab \mid a)^*$ to an NFA.

- a \rightarrow
 - b \rightarrow



Where are we going?

- We are going to show that DFA, NFA and regular expressions specify the same set of languages!
- We will do this with a series of transformations:
 1. From Regular Expressions to NFAs
 2. From NFAs to NFAs without ϵ -transitions
 3. From NFAs without ϵ -transitions to DFAs
 4. From DFAs to Regular Expressions



$$L_1 \circ L_2 \neq L_2 \circ L_1$$