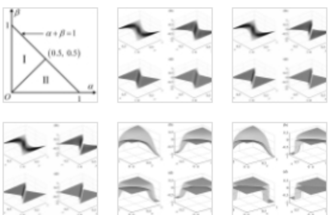



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# An analytical solution for two and three dimensional nonlinear Burgers' equation

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## Highlights

- An analytical solution for 2D and 3D Burgers' equation is presented.
- The solution can describe shock wave phenomenon for larger Reynolds number.
- The solution provides a benchmark test for numerical algorithms.

## Abstract

This paper derives analytical solutions for the two dimensional and the three dimensional Burgers' equation. The two-dimensional and three-dimensional Burgers' equation are defined in a square and a cubic space domain, respectively, and a particular set of boundary and initial conditions is considered. The analytical solution for the two dimensional Burgers' equation is given by the quotient of two infinite series which involve Bessel, exponential, and trigonometric functions. The analytical solution for the three dimensional Burgers' equation is given by the quotient of two infinite series which involve hypergeometric, exponential, trigonometric and power functions. For both cases, the solutions can describe shock wave phenomena for large Reynolds numbers ($R_e \geq 100$), which is useful for testing numerical methods.

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## Keywords

Analytical solution; Burgers' equation; Hopf–Cole transformation; Shock wave

## 1. Introduction

Burgers' equation is an important and basic partial differential equation from fluid mechanics. This equation combines the characteristics of the first order wave equation and heat conduction equation, and is used as a first order wave equation and heat conduction equation, and is used as a tool to describe the interaction between convection and diffusion.

Burgers' equation was first introduced by Bateman [1] in 1915. Then, Burgers [2], [3], introduced this equation to fluid mechanics and used it to describe the turbulent flow in a channel caused by the interaction of the opposite effects of convection and diffusion, and hence the equation was referred to as "Burgers' equation". Burgers' equations are the same as the incompressible Navier–Stokes' equations with the pressure gradient terms removed. Therefore, this equation can be considered as a simplified form of Navier–Stokes' equations. Burgers' equation is also of great use in some other fields, such as jet flows [4], growth of molecular interfaces [5], traffic flows [6], shock waves [7], gas dynamics [8], longitudinal elastic waves in an isotropic solid [9], and so forth. Due to its wide applicability, many researchers have analytically or numerically developed solutions for Burgers' equation.

In 1951, Hopf [10] and Cole [11] have independently shown that, for any initial condition, the Burgers' equation can be transformed into a linear homogeneous heat transfer equation that can be solved exactly, thus the exact solution to the Burgers' equation can be expressed in the form of Fourier series. Meanwhile, Cole [11] studied the general properties of Burgers' equation and found that the solution of Burgers' equation has the typical features of a shock wave, i.e. the nonlinear term of Burgers' equation tends to steepen the wave fronts, while the viscous term of Burgers' equation prevents the formation of the actual discontinuities. In 1972, based on the previous results, Benton and Platzman [12] surveyed 35 analytical solutions of the one-dimensional Burgers' equation. Recently, other analytical solutions for one-dimensional Burgers' equation have been derived by Wood [13], Schiffner et al. [14] and Hesameddini and Gholampour [15].

To the best of our knowledge, up to now, there are only four analytical solutions for the two dimensional Burgers' equation, which were given by Fletcher [16], Cao [17], Liao [18] and Biazar and Aminikhah [19], respectively. Among them, only the solution given by Fletcher [16] can present the shock wave phenomenon. Moreover, there are no analytical solution available for the three dimensional Burgers' equation.

The Burgers' equation with a special initial condition, which is a combination of a sine function and a cosine function, has been commonly used to describe the shock wave phenomenon for large Reynolds numbers numerically [20], [21], [22], [23]. However, there is no analytical solution available in the literature for Burgers' equation with this specific initial conditions. In this paper, an analytical solution for the 2D and the 3D Burgers' equation with this specific initial condition is proposed, which can be used to describe the shock wave phenomenon accurately and provides a benchmark test for numerical algorithms.

## 2. The 2D Burgers' equation and Hopf–Cole transformation

In this section, the analytical solution for the two dimensional Burgers' equation with a special set of initial conditions and boundary conditions is derived. The two-dimensional Burgers' equation can be written as:

$$\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) &= 0, \\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) &= 0, \end{aligned} \tag{1}$$

in which $R_e$ is the Reynolds number, the space domain is $0 < x < 1$ and $0 < y < 1$, and the time domain is $t > 0$. The initial conditions and boundary conditions [20], [21], [22], [23] are:

$$\begin{aligned} u(x,y,0) &= u_0(x,y) = \sin(\pi x)\cos(\pi y), \\ v(x,y,0) &= v_0(x,y) = \cos(\pi x)\sin(\pi y), \end{aligned} \tag{2}$$

$$u(0,y,t) = u(1,y,t) = v(x,0,t) = v(x,1,t) = 0. \tag{3}$$

The shock wave phenomenon for the solution of Eqs. (1)–(3) has been shown numerically in many references [20], [21], [22], [23].

Since the vorticity of the initial condition is zero, i.e.:

$$\frac{\partial u(x,y,0)}{\partial y} = \frac{\partial v(x,y,0)}{\partial x}. \tag{4}$$

The Hopf–Cole transformation can be applied to Eq. (1), which gives the solution of Eq. (1) in the following form [10], [11]:

$$u = -2\mu \frac{\partial \phi}{\partial x}/\phi,\ v = -2\mu \frac{\partial \phi}{\partial y}/\phi, \tag{5}$$

where $\mu = R_e^{-1}$. It is easy to verify that $\phi(x, y, t)$ satisfies the following equation:

$$\frac{\partial \phi}{\partial t} = \mu\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + c\left(t\right). \tag{6}$$

The solution of Eq. (6) can be written in the following form:

$$\phi\left(x, y, t\right) = w\left(x, y, t\right) \exp\left(\int_0^t c\left(t\right) \mathrm{d}t\right). \tag{7}$$

According to Eqs. (5) and (7), the choice of $c(t)$ does not affect the solution of $u$ and $v$, so we can set $c(t) = 0$ and then Eq. (6) can be simplified to the following heat transfer equation:

$$\frac{\partial \phi}{\partial t} = \mu\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right). \tag{8}$$

The initial condition for Eq. (8) is given by:

$$\phi\left(x, y, 0\right) = \exp\left[-\frac{1}{2\mu} H\left(x, y\right)\right] = \exp\left[\frac{\cos(\pi x)\cos(\pi y) - 1}{2\mu\pi}\right], \tag{9}$$

where

$$H\left(x, y\right) = \frac{1}{2}\left[\int_0^x \left[u_0\left(s, y\right) + u_0\left(s, 0\right)\right] \mathrm{d}s + \int_0^y \left[v_0\left(x, s\right) + v_0\left(0, s\right)\right] \mathrm{d}s\right], \tag{10}$$

and the boundary conditions are:

$$\frac{\partial \phi}{\partial x}(0, y, t) = \frac{\partial \phi}{\partial x}(1, y, t) = \frac{\partial \phi}{\partial y}(x, 0, t) = \frac{\partial \phi}{\partial y}(x, 1, t) = 0. \tag{11}$$

### 2.1. The analytical solution for the 2D Burgers' equation

Applying the separation of variables method, the solution for the heat transfer Eq. (8) with initial condition (9) and boundary condition (11) can be given by:

$$\phi\left(x, y, t\right) = \exp\left(-2\lambda\right) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} C_{nm} E_{mn}\left(t\right) \cos\left(n\pi x\right) \cos\left(m\pi y\right), \tag{12}$$

in which:

$$C_{mn} = \int_0^1 \int_0^1 \exp\left[2\lambda \cos\left(\pi x\right) \cos\left(\pi y\right)\right] \cos\left(n\pi x\right) \cos\left(m\pi y\right) \mathrm{d}x\mathrm{d}y \tag{13}$$

$$E_{mn}\left(t\right) = \exp\left[-\left(n^2 + m^2\right) \pi^2 \mu t\right],$$

$$A_{mn} = \begin{cases} 1, & \text{if } n = 0 \text{ and } m = 0 \\ 2, & \text{if } n = 0 \text{ and } m \neq 0 \\ 2, & \text{if } n \neq 0 \text{ and } m = 0 \\ 4, & \text{if } n \neq 0 \text{ and } m \neq 0 \end{cases}, \tag{14}$$

$$\lambda = \frac{1}{4\mu\pi}. \tag{15}$$

Then, according to Hopf–Cole transformation, the solution of the Burgers' equation can be written as:

$$\begin{aligned} u\left(x, y, t\right) &= 2\pi\mu \frac{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n A_{mn} C_{nm} E_{mn}\left(t\right) \sin\left(n\pi x\right) \cos\left(m\pi y\right)}{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} C_{nm} E_{mn}\left(t\right) \cos\left(n\pi x\right) \cos\left(m\pi y\right)}, \\ v\left(x, y, t\right) &= 2\pi\mu \frac{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} m A_{mn} C_{nm} E_{mn}\left(t\right) \cos\left(n\pi x\right) \sin\left(m\pi y\right)}{\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_{mn} C_{nm} E_{mn}\left(t\right) \cos\left(n\pi x\right) \cos\left(m\pi y\right)}. \end{aligned} \tag{16}$$

Eq. (16) was obtained by using conventional methods such as the one described above. However, the key of computing solution (16) lies in the calculation of the coefficients $C_{mn}$, which is difficult to be computed due to the double integral consisting of exponential and trigonometric functions.

The main contribution of this paper is to show that the coefficients $C_{nm}$ can be transformed into the following form:

$$C_{nm} = \begin{cases} 0, & \text{if } n+m \text{ is odd} \\ I_{(n+m)/2}(\lambda)\, I_{(n-m)/2}(\lambda), & \text{if } n+m \text{ is even} \end{cases}, \tag{17}$$

where $I_n(\lambda)$ is the modified Bessel function of the first kind and of order $n$. With Eqs. (16) and (17), the solution of the Burgers' equation can be obtained accurately and efficiently. In the following section, we will prove Eq. (17).

Note that, it's easy to prove that the solution of the Burgers' equation given by (16) and (17) has the following properties:

$$\begin{aligned} u(x,y,t) &= -u(1-x,1-y,t), \\ u(x,y,t) &= -u\left(\frac{x-y}{2},\frac{x-y}{2},t\right), x>y \text{ and } x+y=1, \\ u(x,y,t) &= v(y,x,t), \\ \frac{\partial u(x,y,t)}{\partial y} &= \frac{\partial v(x,y,t)}{\partial x}. \end{aligned} \tag{18}$$

### 2.2. The proof for Eq. (17)

Firstly, we will prove the first part of Eq. (17), i.e. $C_{nm} = 0$ when $n + m$ is odd. The integration in Eq. (13) can be divided into two parts, i.e.:

$$\begin{aligned} C_{mn} = & \int_0^1 \int_0^{1-y} \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos(n\pi x)\cos(m\pi y)\,\mathrm{d}x\mathrm{d}y \\ & + \int_0^1 \int_{1-y}^1 \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos(n\pi x)\cos(m\pi y)\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{19}$$

Applying the change of variables $x = 1 - \xi$ and $y = 1 - \eta$ to the second term on the right hand-side in Eq. (19), yields:

$$\begin{aligned} C_{mn} = & \int_0^1 \int_0^{1-y} \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos(n\pi x)\cos(m\pi y)\,\mathrm{d}x\mathrm{d}y \\ & + \int_1^0 \int_{1-\eta}^0 \left\{ \begin{array}{l} \exp\left\{2\lambda\cos\left[\pi(1-\xi)\right]\cos\left[\pi(1-\eta)\right]\right\} \times \\ \cos\left[n\pi(1-\xi)\right]\cos\left[m\pi(1-\eta)\right] \end{array} \right\} \mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{20}$$

By using the following identity:

$$\cos\left[n\pi(1-\xi)\right]\cos\left[m\pi(1-\eta)\right] = -\cos(n\pi\xi)\cos(m\pi\eta) \quad \text{, if } n+m \text{ is odd.} \tag{21}$$

Eq. (20) can be simplified into the following equation:

$$C_{mn} = 0, \text{ if } n+m \text{ is odd.} \tag{22}$$

Therefore, the first part of Eq. (17) is proved.

Secondly, we will prove the second part of Eq. (17). For notational convenience, let:

$$F_{mn} = I_{(n+m)/2}(\lambda)\, I_{(n-m)/2}(\lambda). \tag{23}$$

Then, according to the integral representation of Bessel function [24], we have:

$$\begin{aligned} I_{(n+m)/2}(\lambda) &= \int_0^1 \exp\left[\lambda\cos(\pi x)\right]\cos\left[\left(\frac{n+m}{2}\right)\pi x\right]\mathrm{d}x, \\ I_{(n-m)/2}(\lambda) &= \int_0^1 \exp\left[\lambda\cos(\pi y)\right]\cos\left[\left(\frac{n-m}{2}\right)\pi y\right]\mathrm{d}y. \end{aligned} \tag{24}$$

Then, Eq. (23) can be rewritten as:

$$F_{mn} = \int_0^1 \int_0^1 \exp\left[\lambda\cos(\pi x) + \lambda\cos(\pi y)\right]\cos\left[\left(\tfrac{n+m}{2}\right)\pi x\right]\cos\left[\left(\tfrac{n-m}{2}\right)\pi y\right]\mathrm{d}x\mathrm{d}y. \tag{25}$$

Applying the sum to product identities of the triangular function to the first term of the integrand on the right hand-side in Eq. (25) and applying product-to-sum identities of the triangular function to the second and third terms of the integrand on the right hand-side in Eq. (25), one obtains:

$$\begin{aligned} F_{mn} &= \frac{1}{2}\int_0^1\int_0^1 \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}x\mathrm{d}y \\ &\quad -\frac{1}{2}\int_0^1\int_0^1 \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\alpha\right)\sin\left(m\pi\beta\right)\mathrm{d}x\mathrm{d}y \\ &\quad +\frac{1}{2}\int_0^1\int_0^1 \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\beta\right)\cos\left(m\pi\alpha\right)\mathrm{d}x\mathrm{d}y \\ &\quad -\frac{1}{2}\int_0^1\int_0^1 \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\beta\right)\sin\left(m\pi\alpha\right)\mathrm{d}x\mathrm{d}y, \end{aligned} \tag{26}$$

where

$$\alpha = \tfrac{x+y}{2},\quad \beta = \tfrac{x-y}{2}. \tag{27}$$

Then, changing the variables $x$ and $y$ to α and β gives:

$$\begin{aligned} F_{mn} &= \int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad +\int_{-1/2}^{0}\int_{-\beta}^{1+\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad -\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\alpha\right)\sin\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad -\int_{-1/2}^{0}\int_{-\beta}^{1+\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\alpha\right)\sin\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad +\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\beta\right)\cos\left(m\pi\alpha\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad +\int_{-1/2}^{0}\int_{-\beta}^{1+\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\beta\right)\cos\left(m\pi\alpha\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad -\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\beta\right)\sin\left(m\pi\alpha\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad -\int_{-1/2}^{0}\int_{-\beta}^{1+\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\sin\left(n\pi\beta\right)\sin\left(m\pi\alpha\right)\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned} \tag{28}$$

According to the properties of the triangular function, it's easy to prove that, on the right hand-side in Eq. (28), the first term is equal to the second term, the third term is opposite to the fourth term, the fifth term is equal to the sixth term, and the seventh term is opposite to the eighth term. Therefore, Eq. (28) can be simplified into the following equation:

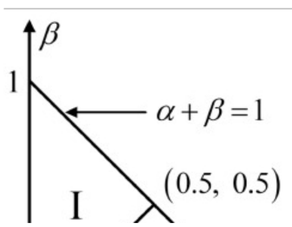
$$\begin{aligned} F_{mn} &= 2\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad +2\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(m\pi\alpha\right)\cos\left(n\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta. \end{aligned} \tag{29}$$

Interchanging variables α and β for the second term on the right hand-side in Eq. (29) gives:

$$\begin{aligned} F_{mn} &= 2\int_0^{1/2}\int_\beta^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta \\ &\quad +2\int_0^{1/2}\int_\alpha^{1-\alpha} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\beta\mathrm{d}\alpha. \end{aligned} \tag{30}$$

The two integrands in Eq. (30) are the same and the regions of integration are shown in Fig. 1. According to Fig. 1, Eq. (30) can be written as:

$$F_{mn} = 2\int_0^1\int_0^{1-\beta} \exp\left[2\lambda\cos\left(\pi\alpha\right)\cos\left(\pi\beta\right)\right]\cos\left(n\pi\alpha\right)\cos\left(m\pi\beta\right)\mathrm{d}\alpha\mathrm{d}\beta. \tag{31}$$



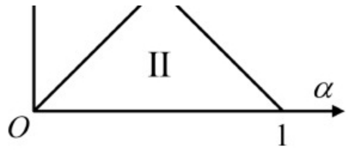





Fig. 1. The regions of integration.

Eq. (31) can be rewritten in the following form:

$$F_{mn} = \int_0^1 \int_0^{1-\beta} \exp\left[2\lambda\cos(\pi\alpha)\cos(\pi\beta)\right]\cos(n\pi\alpha)\cos(m\pi\beta)\,\mathrm{d}\alpha\mathrm{d}\beta \\ + \int_0^1 \int_0^{1-\beta} \exp\left[2\lambda\cos(\pi\alpha)\cos(\pi\beta)\right]\cos(n\pi\alpha)\cos(m\pi\beta)\,\mathrm{d}\alpha\mathrm{d}\beta. \tag{32}$$

By making the change of variables $x = \alpha$ and $y = \beta$ to the first term on the right hand-side in Eq. (32) and making the change of variables $x = 1 - \alpha$ and $y = 1 - \beta$ to the second term on the right hand-side in Eq. (32), we have:

$$F_{mn} = \int_0^1 \int_0^{1-y} \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos(n\pi x)\cos(m\pi y)\,\mathrm{d}x\mathrm{d}y \\ + \int_1^0 \int_1^{1-y} \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos\left[n\pi(1-x)\right]\cos\left[m\pi(1-y)\right]\mathrm{d}x\mathrm{d}y. \tag{33}$$

By using the following equation:

$$\cos\left[n\pi(1-x)\right]\cos\left[m\pi(1-y)\right] = \cos(n\pi x)\cos(m\pi y), \quad \text{if } n+m \text{ is even.} \tag{34}$$

Eq. (33) can be simplified into the following equation:

$$F_{mn} = \int_0^1 \int_0^1 \exp\left[2\lambda\cos(\pi x)\cos(\pi y)\right]\cos(n\pi x)\cos(m\pi y)\,\mathrm{d}x\mathrm{d}y, \quad \text{if } n+m \text{ is even.} \tag{35}$$

Comparing Eq. (35) with Eq. (13) gives:

$$F_{mn} = C_{mn}, \quad \text{if } n+m \text{ is even.} \tag{36}$$

Therefore, the second part of Eq. (17) is proved.

## 3. The 3D Burgers' equation and Hopf–Cole transformation

In this section, the analytical solution for the three dimensional Burgers' equation with a special set of initial conditions and boundary conditions is derived. The three-dimensional Burgers' equation can be written as:

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \frac{1}{R_e}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) &= 0, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \frac{1}{R_e}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) &= 0, \\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - \frac{1}{R_e}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) &= 0, \end{aligned} \tag{37}$$

in which $R_e$ is the Reynolds number, the space domain is $0 < x < 1$, $0 < y < 1$ and $0 < z < 1$, and the time domain is $t > 0$. The initial conditions and boundary conditions are:

$$\begin{aligned} u(x,y,z,0) &= u_0(x,y,z) = \sin(\pi x)\cos(\pi y)\cos(\pi z), \\ v(x,y,z,0) &= v_0(x,y,z) = \sin(\pi y)\cos(\pi x)\cos(\pi z), \\ w(x,y,z,0) &= w_0(x,y,z) = \sin(\pi z)\cos(\pi x)\cos(\pi y), \end{aligned} \tag{38}$$

$$\begin{aligned} u(0,y,z,t) &= u(1,y,z,t) = 0, \\ v(x,0,z,t) &= v(x,1,z,t) = 0, \\ w(x,y,0,t) &= w(x,y,1,t) = 0. \end{aligned} \tag{39}$$

Since the vorticity of the initial condition is zero, i.e.:

$$\begin{aligned}\frac{\partial u(x,y,z,0)}{\partial y} &= \frac{\partial v(x,y,z,0)}{\partial x},\\ \frac{\partial w(x,y,z,0)}{\partial y} &= \frac{\partial v(x,y,z,0)}{\partial z},\\ \frac{\partial w(x,y,z,0)}{\partial x} &= \frac{\partial u(x,y,z,0)}{\partial z}.\end{aligned} \tag{40}$$

The Hopf–Cole transformation can be applied to Eq. (37), which gives the solution of Eq. (37) in the following form [10], [11]:

$$u = -2\mu\frac{\partial\phi}{\partial x}/\phi, \quad v = -2\mu\frac{\partial\phi}{\partial y}/\phi, \quad w = -2\mu\frac{\partial\phi}{\partial z}/\phi. \tag{41}$$

Similar to the derivation of Eq. (8), it is easy to verify that $\phi(x, y, z, t)$ satisfies the following heat conduction equation:

$$\frac{\partial\phi}{\partial t} = \mu\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right), \tag{42}$$

with the initial condition:

$$\phi(x,y,z,0) = \exp\left[-\frac{1}{2\mu}H(x,y,z)\right] = \exp\left[\frac{\cos(\pi x)\cos(\pi y)\cos(\pi z)-1}{2\mu\pi}\right], \tag{43}$$

where

$$H(x,y,z) = \frac{1}{3}\begin{bmatrix}\int_0^x [u_0(s,y,z) + u_0(s,0,z) + u_0(s,0,0)]\,\mathrm{d}s \\ +\int_0^y [v_0(x,s,z) + v_0(x,s,0) + v_0(0,s,0)]\,\mathrm{d}s \\ +\int_0^z [w_0(x,y,s) + w_0(0,y,s) + w_0(0,0,s)]\,\mathrm{d}s\end{bmatrix}, \tag{44}$$

and the boundary conditions:

$$\begin{aligned}\frac{\partial\phi}{\partial x}(0,y,z,t) &= \frac{\partial\phi}{\partial x}(1,y,z,t) = 0,\\ \frac{\partial\phi}{\partial y}(x,0,z,t) &= \frac{\partial\phi}{\partial y}(x,1,z,t) = 0,\\ \frac{\partial\phi}{\partial z}(x,y,0,t) &= \frac{\partial\phi}{\partial z}(x,y,1,t) = 0.\end{aligned} \tag{45}$$

### 3.1.. The analytical solution for the 3D Burgers' equation

Applying the separation of variables method, the solution for the heat conduction Eq. (42) with initial condition (43) and boundary condition (45) can be given by:

$$\phi(x,y,z,t) = \exp(-2\lambda)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} A_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\cos(n\pi y)\cos(l\pi z), \tag{46}$$

in which:

$$\begin{aligned}C_{mnl} &= \int_0^1\int_0^1\int_0^1 \left\{\begin{matrix}\exp[2\lambda\cos(\pi x)\cos(\pi y)\cos(\pi z)]\times \\ \cos(m\pi x)\cos(n\pi y)\cos(l\pi z)\end{matrix}\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ E_{mnl}(t) &= \exp\left[-\left(m^2+n^2+l^2\right)\pi^2\mu t\right],\end{aligned} \tag{47}$$

$$A_{mnl} = \begin{cases}1, & \text{if } n=m=l=0\\ 2, & \text{if } m\neq 0 \text{ and } n=0 \text{ and } l=0\\ 2, & \text{if } m=0 \text{ and } n\neq 0 \text{ and } l=0\\ 2, & \text{if } m=0 \text{ and } n=0 \text{ and } l\neq 0\\ 4, & \text{if } m\neq 0 \text{ and } n\neq 0 \text{ and } l=0\\ 4, & \text{if } m=0 \text{ and } n\neq 0 \text{ and } l\neq 0\\ 4, & \text{if } m\neq 0 \text{ and } n=0 \text{ and } l\neq 0\\ 8, & \text{if } m\neq 0 \text{ and } n\neq 0 \text{ and } l\neq 0\end{cases}, \tag{48}$$

$$\lambda = \frac{1}{4\mu\pi}. \tag{49}$$

Then, according to Hopf–Cole transformation, the solution of the Burgers' equation can be written as:

$$\begin{aligned}u &= 2\pi\mu\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} mA_{mnl}C_{mnl}E_{mnl}(t)\sin(m\pi x)\cos(n\pi y)\cos(l\pi z)}{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} A_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\cos(n\pi y)\cos(l\pi z)},\\ v &= 2\pi\mu\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} nA_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\,sin(n\pi y)\cos(l\pi z)}{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} A_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\cos(n\pi y)\cos(l\pi z)},\\ w &= 2\pi\mu\frac{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} lA_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\cos(n\pi y)\,sin(l\pi z)}{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} A_{mnl}C_{mnl}E_{mnl}(t)\cos(m\pi x)\cos(n\pi y)\cos(l\pi z)}.\end{aligned} \tag{50}$$

Eqs. (47) and (48) show that, the values of $C_{mnl}$ and $A_{mnl}$ do not change when changing the order of $m$, $n$ and $l$. Therefore, without loss of generality, we assume that $m \geq n \geq l \geq 0$.

The key for computing solution (50) is to compute the coefficient $C_{mnl}$. However, Eq. (47) shows the difficulty of computing $C_{mnl}$ due to the triple integral involved, consisting of exponential and trigonometric functions.

The main contribution of this paper is to show that the coefficient $C_{mnl}$ can be transformed into the following form:

$$C_{nml} = \frac{1}{[(m+n)/2]!\,[(m-n)/2]!}\left(\frac{\lambda}{2}\right)^{m} \times \begin{cases} \sum_{j=1}^{(i+1)/2} \gamma_j \frac{(m+2j-2)!!}{(m+2j-1)!!}\, {}_3F_4\left(\frac{m+1}{2}, \frac{m}{2}+1, \frac{m}{2}+j; \frac{m+n}{2}+1, \frac{m-n}{2}+1, m+1, \frac{m-1}{2}+j+1; \lambda^2\right), \\ \text{if } n, m, l \text{ are all odd} \\ \sum_{j=0}^{l/2} \gamma_j \frac{(m+2j-1)!!}{(m+2j)!!}\, {}_3F_4\left(\frac{m+1}{2}, \frac{m}{2}+1, \frac{m+1}{2}+j; \frac{m+n}{2}+1, \frac{m-n}{2}+1, m+1, \frac{m}{2}+j+1; \lambda^2\right), \\ \text{if } n, m, l \text{ are all even} \\ 0, \text{otherwise} \end{cases}, \tag{51}$$

where $(n)!$ is the factorial of integer $n$; the double factorial $(n)!!$ is defined as:

$$(2n)!! = (2n) \times (2n-2) \times \cdots \times 2, (2n-1)!! = (2n-1) \times (2n-3) \times \cdots \times 1, \tag{52}$$

and ${}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z)$ is a generalized hypergeometric series, which is defined in Section 9.14 of Ref. [24] as:

$${}_pF_q\left(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k (\beta_2)_k \cdots (\beta_q)_k} \frac{z^k}{k!}, \tag{53}$$

in which $(a)_k$ is the Pochhammer symbol and is defined as:

$$\begin{aligned} &\alpha_0 = 1 \\ &(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1), k \geq 1. \end{aligned} \tag{54}$$

The coefficient $\gamma_j$ in Eq. (51) is defined by the following equation:

$$\cos(lz) = \begin{cases} \sum_{j=0}^{l/2} \gamma_j \cos^{2j}(z), \text{if } l \text{ is even} \\ \sum_{j=1}^{(l+1)/2} \gamma_j \cos^{2j-1}(z), \text{if } l \text{ is odd} \end{cases}, \tag{55}$$

and can be computed using the following equations:

$$\begin{aligned} \cos(2z) &= 2\cos^2(z) - 1 \\ \cos(nz) &= 2\cos(z)\cos[(n-1)z] - \cos[(n-2)z]. \end{aligned} \tag{56}$$

### 3.2. The proof of Eq. (51)

In Section 2.2, we have proved the following relationship:

$$\begin{aligned} &\int_0^1 \int_0^1 \exp[2\lambda \cos(\pi x)\cos(\pi y)] \cos(m\pi x)\cos(n\pi y)\, dx dy \\ &= \begin{cases} 0, & \text{if } n+m \text{ is odd} \\ I_{(n+m)/2}(\lambda)\, I_{(n-m)/2}(\lambda), & \text{if } n+m \text{ is even} \end{cases}. \end{aligned} \tag{57}$$

Therefore, Eq. (47) can be written as:

$$C_{mnl} = \begin{cases} 0, & \text{if } n+m \text{ is odd} \\ \int_0^1 \cos(l\pi z)\, I_{(m+n)/2}[\lambda\cos(\pi z)]\, I_{(m-n)/2}[\lambda\cos(\pi z)]\, dz, & \text{if } n+m \text{ is even} \end{cases}. \tag{58}$$

Applying the change of variable $\xi = \cos(\pi z)$ to Eq. (58) and using Eq. (55), we have:

$$C_{mnl} = \begin{cases} \sum_{j=1}^{(l+1)/2} \gamma_j D_{m,n,2j-1}, & \text{if } n+m \text{ is even and } l \text{ is odd} \\ \sum_{j=0}^{l/2} \gamma_j D_{m,n,2j}, & \text{if } n+m \text{ is even and } l \text{ is even} \\ 0, & \text{if } n+m \text{ is odd} \end{cases}, \tag{59}$$

where

$$D_{m,n,j} = \frac{1}{\pi}\int_{-1}^{1} \frac{I_{(m+n)/2}(\lambda\xi) I_{(m-n)/2}(\lambda\xi)\xi^j}{\sqrt{1-\xi^2}} \mathrm{d}\xi. \tag{60}$$

According to Eq. (60), if $m$ and $n$ are both odd, one of $(m+n)/2$ and $(m-n)/2$ is odd and another is even, then $I_{(m+n)/2}(\lambda\xi) I_{(m-n)/2}(\lambda\xi) / \sqrt{1-\xi^2}$ is an odd function with respect to ξ, so $D_{m,n,j} = 0$ if $j$ is even. In a similar way, if $m$ and $n$ are both even, $(m+n)/2$ and $(m-n)/2$ are either both even, or both odd, then $I_{(m+n)/2}(\lambda\xi) I_{(m-n)/2}(\lambda\xi) / \sqrt{1-\xi^2}$ is an even function with respect to ξ, so $D_{m,n,j} = 0$ if $j$ is odd. Therefore, we have:

$$\begin{cases} D_{m,n,j} \neq 0, & \text{possibly if } n, m, j \text{ are all odd, or } n, m, j \text{ are all even} \\ D_{m,n,j} = 0, & \text{otherwise} \end{cases}. \tag{61}$$

Using Eq. (59) and (61), we have:

$$\begin{cases} C_{nml} \neq 0, & \text{possibly if } n, m, l \text{ are all odd, or } n, m, l \text{ are all even} \\ C_{nml} = 0, & \text{otherwise} \end{cases}. \tag{62}$$

Up to now, Eq. (62) proves the third part of Eq. (51), and Eq. (59) shows that computing $D_{m,n,j}$ is the key to simplify $C_{nml}$. Therefore, in the following of this section, the analytical formula for $D_{m,n,j}$ defined by Eq. (60) is given, which will prove the first and second parts of Eq. (51).

The relationship between the modified Bessel functions of the first kind $I_k(\xi)$ and the Bessel functions of the first kind $J_k(\xi)$ gives (Eq. (3) in Section 8.406 of Ref. [24]):

$$I_k(\xi) = i^{-k} J_k(i\xi), k \text{ is integer}, \tag{63}$$

where $i^2 = -1$. Using Eq. (63), we have:

$$I_{(m+n)/2}(\lambda\xi) I_{(m-n)/2}(\lambda\xi) = i^{-m} J_{(m+n)/2}(i\lambda\xi) J_{(m-n)/2}(i\lambda\xi). \tag{64}$$

According to Eq. (1) in Section 8.442 of Ref. [24], we have:

$$J_p(\xi) J_q(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k (p+q+k+1)_k}{k!(p+k)!(q+k)!} \left(\frac{\xi}{2}\right)^{p+q+2k}, p \text{ and } q \text{ are both integers}, \tag{65}$$

in which $(n)_m$ is the Pochhammer symbol given in Eq. (54). The combination of Eqs. (64) and (65) gives:

$$I_{(m+n)/2}(\lambda\xi) I_{(m-n)/2}(\lambda\xi) = \sum_{k=0}^{\infty} \frac{(m+k+1)_k}{k![(m+n)/2+k]![(m-n)/2+k]!} \left(\frac{\lambda\xi}{2}\right)^{m+2k}. \tag{66}$$

Substituting Eq. (66) into Eq. (60) gives:

$$D_{m,n,j} = \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{(m+k+1)_k}{k![(m+n)/2+k]![(m-n)/2+k]!} \left(\frac{\lambda}{2}\right)^{m+2k} \int_{-1}^{1} \frac{\xi^{m+j+2k}}{\sqrt{1-\xi^2}} \mathrm{d}\xi. \tag{67}$$

Eq. (61) shows that $D_{m,n,j}$ is possibly nonzero when $m$,$n$ and $j$ are all even, or $m$, $n$ and $j$ are all odd. For these two cases, $m + j + 2k$ is even, so we have (Eq. (3) in Section 3.248 of Ref. [24]):

$$\int_0^1 \frac{\xi^{2n}}{\sqrt{1-\xi^2}} \mathrm{d}\xi = \frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}. \tag{68}$$

Therefore, when $m$, $n$ and $j$ are all even, or $m$, $n$ and $j$ are all odd, Eq. (67) can be written as:

$$D_{m,n,j} = \sum_{k=0}^{\infty} \frac{(m+k+1)_k}{k![(m+n)/2+k]![(m-n)/2+k]!} \left(\frac{\lambda}{2}\right)^{m+2k} \frac{(m+j+2k-1)!!}{(m+j+2k)!!}. \tag{69}$$

A simplification of Eq. (69) gives:

$$\begin{aligned} D_{m,n,j} &= \left(\frac{\lambda}{2}\right)^m \frac{(m+j-1)!!}{[(m+n)/2]!\,[(m-n)/2]!\,(m+j)!!} \\ &\times \sum_{k=0}^{\infty} \left[ \begin{array}{l} \frac{\lambda^{2k}}{k!} \frac{1}{2^{2k}} \frac{[(m+n)/2]!}{[(m+n)/2+k]!} \times \frac{[(m-n)/2]!}{[(m-n)/2+k]!} \times \\ \frac{(m+j)!!}{(m+j+2k)!!} \times \frac{(m+j-1+2k)!!}{(m+j-1)!!} \times (m+k+1)_k \end{array} \right]. \end{aligned} \tag{70}$$

According to the definition of the factorial, the double factorial (see Eq. (52)) and the Pochhammer symbol (see Eq. (54)), we have the following equations:

$$\begin{aligned}\frac{[(m+n)/2]!}{[(m+n)/2+k]!} &= \frac{1}{[(m+n)/2+1]\,[(m+n)/2+2]\cdots[(m+n)/2+k]}\\ &= \frac{1}{[(m+n)/2+1]_k},\end{aligned} \tag{71}$$

$$\begin{aligned}\frac{[(m-n)/2]!}{[(m-n)/2+k]!} &= \frac{1}{[(m-n)/2+1]\,[(m-n)/2+2]\cdots[(m-n)/2+k]}\\ &= \frac{1}{[(m-n)/2+1]_k},\end{aligned} \tag{72}$$

$$\begin{aligned}\frac{(m+j)!!}{2^{-k}(m+j+2k)!!} &= \frac{1}{2^{-k}(m+j+2)(m+j+4)\cdots(m+j+2k)}\\ &= \frac{1}{\left(\frac{m+j}{2}+1\right)\left(\frac{m+j}{2}+2\right)\cdots\left(\frac{m+j}{2}+k\right)}\\ &= \frac{1}{\left(\frac{m+j}{2}+1\right)_k},\end{aligned} \tag{73}$$

$$\begin{aligned}\frac{2^{-k}(m+j-1+2k)!!}{(m+j-1)!!} &= 2^{-k}(m+j+1)(m+j+3)\cdots(m+j-1+2k)\\ &= \left(\frac{m+j+1}{2}\right)\left(\frac{m+j+1}{2}+1\right)\cdots\left(\frac{m+j+1}{2}+k-1\right)\\ &= \left(\frac{m+j+1}{2}\right)_k.\end{aligned} \tag{74}$$

Then, substituting Eqs. (71)–(74) into Eq. (70) gives:

$$\begin{aligned}D_{m,n,j} &= \left(\frac{\lambda}{2}\right)^m \frac{(m+j-1)!!}{[(m+n)/2]!\,[(m-n)/2]!\,(m+j)!!}\\ &\quad\times\sum_{k=0}^{\infty}\left[\frac{\lambda^{2k}}{k!}\frac{1}{2^{2k}}\frac{\left(\frac{m+j+1}{2}\right)_k\times(m+k+1)_k}{[(m+n)/2+1]_k\times[(m-n)/2+1]_k\times\left(\frac{m+j}{2}+1\right)_k}\right].\end{aligned} \tag{75}$$

Furthermore, according to the definition of the Pochhammer symbol (see Eq. (54)), we have the following equations:

$$\begin{aligned}\left(\frac{m+1}{2}\right)_k &= \left(\frac{m+1}{2}\right)\left(\frac{m+1}{2}+1\right)\left(\frac{m+1}{2}+2\right)\cdots\left(\frac{m+1}{2}+k-1\right)\\ &= 2^{-k}(m+1)(m+3)(m+5)\cdots(m+2k-1),\end{aligned} \tag{76}$$

$$\begin{aligned}\left(\frac{m}{2}+1\right)_k &= \left(\frac{m}{2}+1\right)\left(\frac{m}{2}+2\right)\left(\frac{m}{2}+3\right)\cdots\left(\frac{m}{2}+k\right)\\ &= 2^{-k}(m+2)(m+4)(m+6)\cdots(m+2k),\end{aligned} \tag{77}$$

$$(m+1+k)_k = \frac{(m+1)_{2k}}{(m+1)_k}. \tag{78}$$

The combination of Eqs. (76)–(78) gives:

$$(m+1+k)_k = \frac{2^{2k}\left(\frac{m+1}{2}\right)_k\left(\frac{m}{2}+1\right)_k}{(m+1)_k}. \tag{79}$$

Then, substituting Eq. (79) into Eq. (75) gives:

$$\begin{aligned}D_{m,n,j} &= \left(\frac{\lambda}{2}\right)^m \frac{(m+j-1)!!}{[(m+n)/2]!\,[(m-n)/2]!\,(m+j)!!}\\ &\quad\times\sum_{k=0}^{\infty}\left[\frac{\lambda^{2k}}{k!}\frac{\left(\frac{m+j+1}{2}\right)_k\times\left(\frac{m+1}{2}\right)_k\times\left(\frac{m}{2}+1\right)_k}{[(m+n)/2+1]_k\times[(m-n)/2+1]_k\times\left(\frac{m+j}{2}+1\right)_k(m+1)_k}\right].\end{aligned} \tag{80}$$

Now, according to the definition of the generalized hypergeometric series (see Eq. (53)), Eq. (80) can be simplified into the following equation:

$$D_{m,n,j} = \left(\frac{\lambda}{2}\right)^m \frac{(m+j-1)!!}{[(m+n)/2]!\,[(m-n)/2]!\,(m+j)!!} \tag{81}$$

$$\times {}_3F_4\left(\frac{\quad}{2}, \frac{\quad}{2}+1, \frac{\quad}{2}; (m+n)/2+1, (m-n)/2+1, m+1, \frac{\quad}{2}+1; \lambda^2\right).$$

Up to now, we have shown that if $m$, $n$ and $j$ are all even, or $m$, $n$ and $j$ are all odd, $D_{m,n,j}$ can be simplified to Eq. (81), otherwise $D_{m,n,j} = 0$. Then, the combination of Eqs. (59), (61) and (81) gives Eq. (51), which proves the main conclusion of this paper.

According to the analytical solution given by Eqs. (48)–(51), it is easy to prove that this solution has the following properties, i.e.:

$$\begin{aligned} w(x,y,z,t) &= v(y,z,x,t) = u(z,x,y,t) \\ u(x,y,z,t) &= u(x,1-y,1-z,t) = u(x,z,y,t) \\ u(x,y,z,t) &= -u(1-x,y,1-z,t) \\ u(x,y,z,t) &= -u(1-x,1-y,z,t). \end{aligned} \tag{82}$$

Therefore, computing the solution in the space domain of $y \geq z$ and $x + y + z \leq 1$ is sufficient.

## 4. Numerical results

### 4.1. Numerical results for the 2D Burgers' equation

For the analytical formula proposed in this paper, the main task is to compute the Bessel function accurately. However, if the double-precision floating point number is used, we cannot obtain accurate results of the Bessel function. In fact, when $R_e \geq 4509$, the value of $[I_0[R/(4\pi)]]^2$ exceeds the range of the double-precision floating point number. Even for relative small Reynolds number, calculating small differences between large numbers results in significant round-off error. Meanwhile, the truncation error should be considered as the number of the series terms increases. Therefore, in order to obtain accurate results, we have to increase the number of significant digits in floating-point number. In this paper, the Maple software is used to achieve this purpose. Maple can numerically evaluates an expression using variable precision floating point arithmetic with *Digits* decimal digit accuracy, where *Digits* is an integer and determines the accuracy of variable precision numeric computations. For example, if the *Digits* is set to 100, the number of significant digits in floating-point number is 100.

For Reynolds numbers $R_e = 100$, 1000 and 10,000, the parameter *Digits* in the Maple software is set to 72, 164 and 1050, respectively. Then, the proposed analytical formula is used to compute the solutions of the 2D Burgers' equation. For the three cases, the series converges when 44*44, 217*217 and 1862*1862 terms are used, respectively. The numerical results are given in Tables 1–3. In order to indicate the fact that the analytical solution is capable of describing shock wave, the profile of the solutions are given in Figs. 2–4, which shows the shock wave phenomena for $R_e = 100$, 1000 and 10,000. Note that Tables 1–3 and Figs. 2–4 confirm the property (18) of the solution of the 2D Burgers' equation.

Table 1.. The numerical results for the 2D Burgers' equation with $Re = 100$.

| $x$ | $y$ | $t = 0.25$ | $t = 0.5$ | $t = 0.75$ | $t = 1$ |
|---|---|---|---|---|---|
| **0.25** | 0.25 | 3.935490117704355E−1 | 2.911828920816955E−1 | 2.273774661403168E−1 | 1.85 |
| **0.5** | 0.25 | 6.861822403861596E−1 | 5.605467081571704E−1 | 4.479960634879613E−1 | 3.68 |
| **0.75** | 0.25 | 3.935490117704355E−1 | 2.911828920816955E−1 | 2.273774661403168E−1 | 1.85 |
| **0.25** | 0.5 | 2.619506158413131E−1 | 2.619829318741635E−1 | 2.180447995061038E−1 | 1.81 |
| **0.5** | 0.5 | 3.433140255652116E−70 | 6.648346088881589E−70 | 2.301061067978292E−70 | 7.78 |
| **0.75** | 0.5 | −2.619506158413131E−1 | −2.619829318741635E−1 | −2.180447995061038E−1 | −1.8 |
| **0.25** | 0.75 | −3.935490117704355E−1 | −2.911828920816955E−1 | −2.273774661403168E−1 | −1.8 |
| **0.5** | 0.75 | −6.861822403861596E−1 | −5.605467081571704E−1 | −4.479960634879613E−1 | −3.6 |

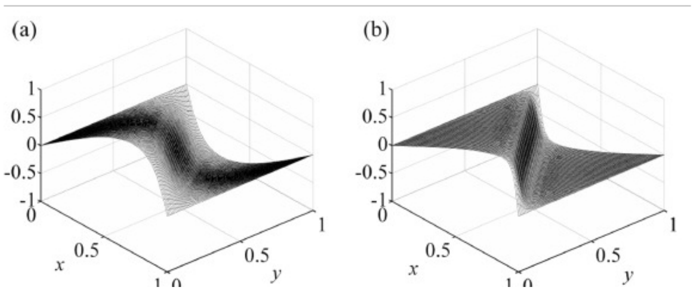
| | | | | | |
|---|---|---|---|---|---|
| **0.75** | 0.75 | −3.935490117704355E−1 | −2.911828920816955E−1 | −2.273774661403168E−1 | −1.8 |
| **0.745** | 0.245 | 4.368877982047649E−1 | 3.883368793402158E−1 | 3.112575694927583E−1 | 2.46 |
| **0.495** | 0.495 | 4.333878643432935E−2 | 9.715398725852026E−2 | 8.388010335244156E−2 | 6.05 |
| **0.245** | 0.745 | −3.502102253361062E−1 | −1.940289048231752E−1 | −1.434973627878752E−1 | −1.2 |

Table 2.. The numerical results for the 2D Burgers' equation with $Re = 1000$.

| $x$ | $y$ | $t = 0.25$ | $t = 0.5$ | $t = 0.75$ | $t$ |
|---|---|---|---|---|---|
| **0.25** | 0.25 | 4.021515879367378E−1 | 2.967292504970852E−1 | 2.309232890910549E−1 | 1 |
| **0.5** | 0.25 | 7.129950304486519E−1 | 5.760121963335145E−1 | 4.566709915854906E−1 | 3 |
| **0.75** | 0.25 | 4.021515879367378E−1 | 2.967292504970852E−1 | 2.309232890910549E−1 | 1 |
| **0.25** | 0.5 | 2.820447431102185E−1 | 2.728815964721490E−1 | 2.236246458200203E−1 | 1 |
| **0.5** | 0.5 | 3.253199921494413E−141 | 9.774207511273605E−136 | 1.442736003379429E−140 | 9 |
| **0.75** | 0.5 | −2.820447431102185E−1 | −2.728815964721490E−1 | −2.236246458200203E−1 | - |
| **0.25** | 0.75 | −4.021515879367378E−1 | −2.967292504970852E−1 | −2.309232890910549E−1 | - |
| **0.5** | 0.75 | −7.129950304486519E−1 | −5.760121963335145E−1 | −4.566709915854906E−1 | - |
| **0.75** | 0.75 | −4.021515879367378E−1 | −2.967292504970852E−1 | −2.309232890910549E−1 | - |
| **0.745** | 0.245 | 4.681541347261701E−1 | 7.847751377838084E−1 | 6.542200827551034E−1 | 5 |
| **0.495** | 0.495 | 6.600254678943236E−2 | 4.880458872867232E−1 | 4.232967936640485E−1 | 3 |
| **0.245** | 0.745 | −3.361490411473054E−1 | 1.913166367896380E−1 | 1.923735045729937E−1 | 1 |

Table 3.. The numerical results for the 2D Burgers' equation with $Re = 10{,}000$.

| $x$ | $y$ | $t = 0.25$ | $t = 0.5$ | $t = 0.75$ |
|---|---|---|---|---|
| **0.25** | 0.25 | 4.029885559386962E−1 | 2.972484863669425E−1 | 2.312490379561160E−1 |
| **0.5** | 0.25 | 7.155574420811523E−1 | 5.773807143927942E−1 | 4.574352773892073E−1 |
| **0.75** | 0.25 | 4.029885559386962E−1 | 2.972484863669425E−1 | 2.312490379561160E−1 |
| **0.25** | 0.5 | 2.839488238668665E−1 | 2.738151985892581E−1 | 2.240998204654204E−1 |
| **0.5** | 0.5 | −2.627366528880526E−817 | −1.842165244157496E−760 | −2.401126660728666E−80 |
| **0.75** | 0.5 | −2.839488238668665E−1 | −2.738151985892581E−1 | −2.240998204654204E−1 |
| **0.25** | 0.75 | −4.029885559386962E−1 | −2.972484863669425E−1 | −2.312490379561160E−1 |
| **0.5** | 0.75 | −7.155574420811523E−1 | −5.773807143927942E−1 | −4.574352773892073E−1 |
| **0.75** | 0.75 | −4.029885559386962E−1 | −2.972484863669425E−1 | −2.312490379561160E−1 |
| **0.745** | 0.245 | 4.743108210190779E−1 | 7.965620559375256E−1 | 6.666894850719867E−1 |
| **0.495** | 0.495 | 7.132226508038170E−2 | 4.993135695705831E−1 | 4.354404471158707E−1 |
| **0.245** | 0.745 | −3.316662908583145E−1 | 2.020650832036407E−1 | 2.041914091597548E−1 |



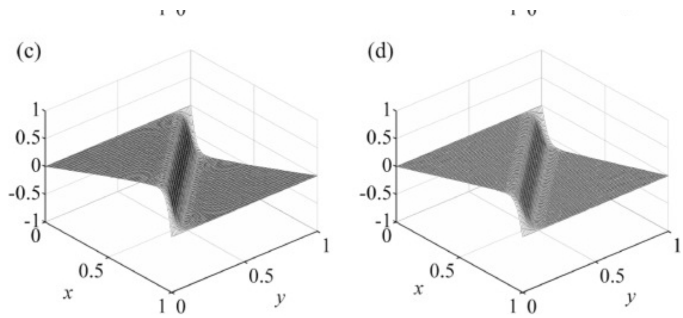



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Fig. 2. The solution of the 2D Burgers' equation for $Re = 100$ at (a) $t = 0.25$; (b) $t = 0.5$; (c) $t = 0.75$; (d) $t = 1$.

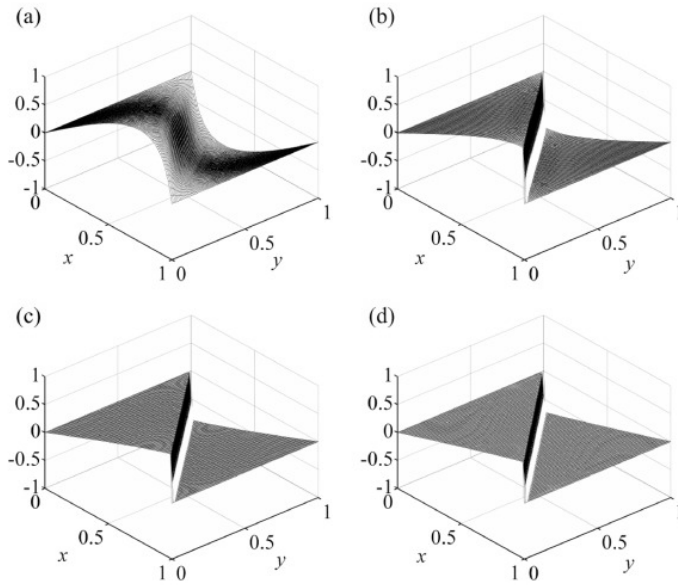


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Fig. 3. The solution of the 2D Burgers' equation for $Re = 1000$ at (a) $t = 0.25$; (b) $t = 0.5$; (c) $t = 0.75$; (d) $t = 1$.

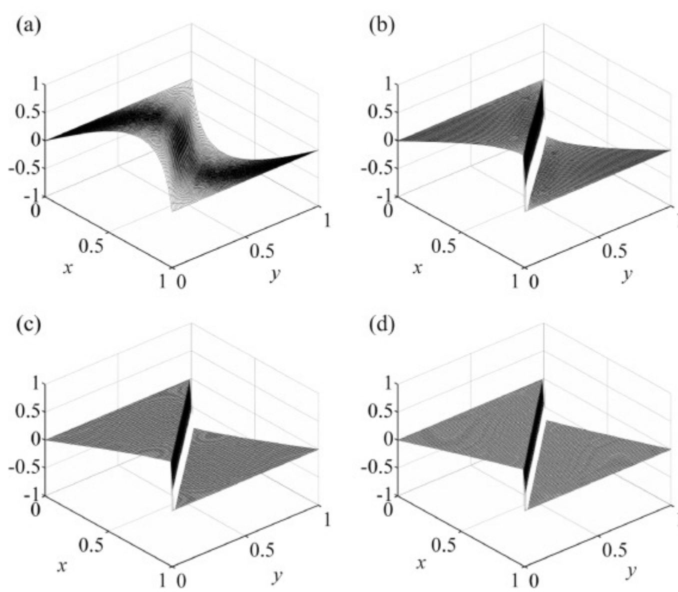

Fig. 4. The solution of the 2D Burgers' equation for $Re = 10{,}000$ at (a) $t = 0.25$; (b) $t = 0.5$; (c) $t = 0.75$; (d) $t = 1$.

### 4.2. Numerical results for the 3D Burgers' equation

For testing the validation of the analytical formula for the 3D Burgers' equation, the analytical solutions are compared with numerical solutions. The solution of the 3D Burgers' equation is given by the Hopf–Cole transformation (41). Therefore, if the solution (46) of the heat conduction problem is correct, the solution of the Burgers' equation is also correct. Let the Reynolds number $R_e$ be 10. A combination of a finite element method (FEM) in space domain and a Crank–Nicolson scheme with time step 0.001(s) in time domain is used to compute the numerical solutions of Eqs. (42)–(45). Table 4 gives the solution $\phi$ of the heat conduction problem at $t = 0.1$(s) obtained from the analytical formula (46) and the numerical method with different FEM grids. Table 6 shows that the numerical solutions with more FEM grids are more close to the analytical solutions, which shows the validation of the proposed analytical formula.

Table 4.. The solution of the heat conduction problem at $t = 0.1$(s) for $Re = 10$.

| $x$ | $y$ | $z$ | $\phi$ | | | |
|---|---|---|---|---|---|---|
| | | | **Analytical solution** | **FEM with 40*40*40 meshes** | **FEM with 100*100*100 meshes** | **FEM w 200*20 meshes** |
| **1/4** | 0 | 0 | 5.121139094268445E−1 | 5.123835274630919E−1 | 5.121574025971580E−1 | 5.12116 |
| **1/4** | 1/4 | 0 | 4.021754605524777E−1 | 4.022569056961954E−1 | 4.021832308799626E−1 | 4.02178 |
| **1/4** | 1/4 | 1/4 | 3.332558679440735E−1 | 3.332967989703344E−1 | 3.332624499504631E−1 | 3.33259 |
| **1/4** | 1/2 | 0 | 2.221335878372145E−1 | 2.220632032495130E−1 | 2.221195126283573E−1 | 2.22131 |
| **1/4** | 1/2 | 1/4 | 2.145599507332575E−1 | 2.145455604143418E−1 | 2.145578143852998E−1 | 2.14561 |
| **1/4** | 3/4 | 0 | 1.232224673074507E−1 | 1.231631542198841E−1 | 1.232115116009095E−1 | 1.2322( |
| **1/2** | 0 | 0 | 2.350307713064107E−1 | 2.348337634508464E−1 | 2.349996700538997E−1 | 2.3502( |
| **1/2** | 1/4 | 0 | 2.221335878372145E−1 | 2.220630388109582E−1 | 2.221195160032975E−1 | 2.22131 |
| **1/2** | 1/4 | 1/4 | 2.145599507332575E−1 | 2.145454933723427E−1 | 2.145578324085150E−1 | 2.14561 |
| **3/4** | 0 | 0 | 9.757645937131027E−2 | 9.752507440940110E−2 | 9.756832939249252E−2 | 9.75734 |
| **3/4** | 1/4 | 0 | 1.232224673074507E−1 | 1.231630205852527E−1 | 1.232114495876140E−1 | 1.2322( |
| **Maximum Relative error** | | | – | 8.38E−4 | 1.32E−4 | 4.55E−5 |

If the double-precision floating-point number is used, the exact solution (50) cannot give accurate results for large Reynolds numbers. We have to increase the number of significant digits in the floating-point number. For $R_e = 100$ and 1000, the parameter *Digits* in Maple is set to be 30 and 130, respectively, which ensures that the analytical formula gives the accurate solutions. Tables 5 and 6 give the solutions of the 3D Burgers' equation obtained from the analytical formula and Figs. 5 and 6 give the profile of the solutions for $x = 0.25$. It can be observed from Figs. 5 and 6 that, for $R_e = 100$ or 1000, the solution of the Burgers' equation incurs a shock wave. Note that Figs. 5 and 6 confirm the property (82) of the solution of the 3D Burgers' equation.

Table 5.. The numerical results for the 3D Burgers' equation with $Re$= 100.

| $x$ | $y$ | $z$ | $t$ = 0.2 | $t$ = 0.4 | $t$ = 0.6 | $t$ = ( |
|---|---|---|---|---|---|---|
| **1/4** | 0 | 0 | 4.587111814722234E−1 | 3.357636213308880E−1 | 2.644571890480045E−1 | 2.18 |
| **1/4** | 0 | 1/4 | 4.147693533983876E−1 | 3.229466410200492E−1 | 2.594937196235130E−1 | 2.15 |
| **1/4** | 1/4 | 1/4 | 3.622948313446699E−1 | 3.071275624398623E−1 | 2.536018991937781E−1 | 2.13 |
| **1/4** | 1/2 | 0 | 2.309933483466460E−1 | 2.692481504484167E−1 | 2.403105301706379E−1 | 2.07 |
| **1/4** | 1/2 | 1/4 | 1.484490790386645E−1 | 2.323817128404845E−1 | 2.285427423855319E−1 | 2.02 |
| **1/4** | 3/4 | 0 | −4.117008417885243E−1 | −3.193327592389574E−1 | −2.577490656315094E−1 | −2.1 |
| **1/2** | 0 | 0 | 8.284494488258751E−1 | 6.451778237531177E−1 | 5.186084581597956E−1 | 4.31 |
| **1/2** | 1/4 | 0 | 6.885948428912090E−1 | 6.042552867975663E−1 | 5.043734696911075E−1 | 4.25 |
| **1/2** | 1/4 | 1/4 | 5.272625522589685E−1 | 5.439370795696320E−1 | 4.848861978305562E−1 | 4.17 |
| **3/4** | 0 | 0 | 9.127629699489180E−1 | 8.678005148958547E−1 | 7.421711022676433E−1 | 6.31 |
| **3/4** | 1/4 | 0 | 4.117008417885243E−1 | 3.193327592389574E−1 | 2.577490656315094E−1 | 2.14 |

Table 6.. The numerical results for the 3D Burgers' equation with $Re$ = 1000.

| $x$ | $y$ | $z$ | $t$ = 0.2 | $t$ = 0.4 | $t$ = 0.6 | $t$ = ( |
|---|---|---|---|---|---|---|
| **1/4** | 0 | 0 | 4.692977841044507E−1 | 3.439719734426368E−1 | 2.704364871888210E−1 | 2.22 |
| **1/4** | 1/4 | 0 | 4.276600731855017E−1 | 3.329342342130911E−1 | 2.664480799522073E−1 | 2.20 |
| **1/4** | 1/4 | 1/4 | 3.777212055334077E−1 | 3.197109908551749E−1 | 2.618993137233148E−1 | 2.18 |
| **1/4** | 1/2 | 0 | 2.552422997378726E−1 | 2.897573235980928E−1 | 2.521345854192811E−1 | 2.14 |
| **1/4** | 1/2 | 1/4 | 1.683151185940934E−1 | 2.635386885722663E−1 | 2.449330701687437E−1 | 2.12 |
| **1/4** | 3/4 | 0 | −4.273401206609559E−1 | −3.326724629431774E−1 | −2.663661286798494E−1 | −2.2 |
| **1/2** | 0 | 0 | 8.552189677123123E−1 | 6.658130354805496E−1 | 5.328701788651632E−1 | 4.41 |
| **1/2** | 1/4 | 0 | 7.244360759301604E−1 | 6.336717511554823E−1 | 5.225445941025038E−1 | 4.37 |
| **1/2** | 1/4 | 1/4 | 5.639219812115523E−1 | 5.902374529756745E−1 | 5.100825897689267E−1 | 4.32 |
| **3/4** | 0 | 0 | 9.790228301169137E−1 | 9.230811847294103E−1 | 7.745511863202593E−1 | 6.52 |
| **3/4** | 1/4 | 0 | 4.273401206609559E−1 | 3.326724629431774E−1 | 2.663661286798494E−1 | 2.20 |

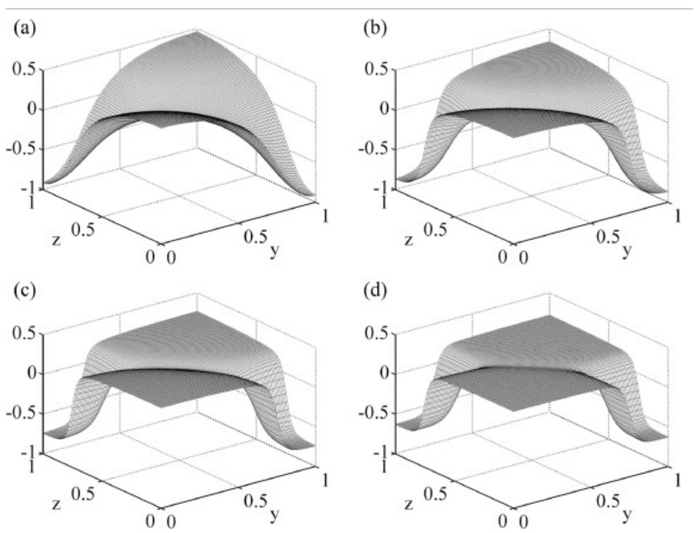

Fig. 5. The solution of the 3D Burgers' equation for $Re = 100$ and $x = 0.25$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$ and (d) $t = 0.8$.

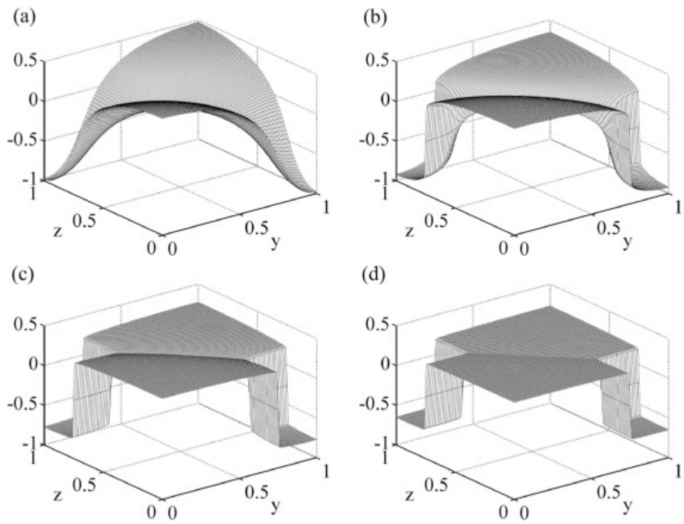




Fig. 6. The solution of the 3D Burgers' equation for $Re = 1000$ and $x = 0.25$ at (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$ and (d) $t = 0.8$.

## 5. Conclusions

In this paper, an analytical solution for the two and three dimensional Burgers' equation with a special set of initial condition and boundary conditions is derived. The main contribution of this paper is that the double and triple integrals presented in the solution of Burgers' equation are simplified in terms of known special functions. The analytical solution for the two dimensional Burgers' equation is given by the quotient of two infinite series which involve Bessel, exponential, and trigonometric functions, and the analytical solution for the three dimensional Burgers' equation is given by the quotient of two infinite series which involve hypergeometric, exponential, trigonometric and power functions.

The analytical formulas derived in the paper have been compared with the numerical results obtained from FEM, which shows the correctness of the proposed analytical formula. Using the proposed analytical formula, the solutions of the Burgers' equation with a particular set of boundary and initial conditions can be obtained accurately and efficiently. Numerical results show that the analytical formula can describe shock wave phenomena accurately, which can be used as a reference for testing numerical methods. Numerical results also confirm the properties Eqs. (18) and (82) of the solutions of 2D and 3D Burgers' equations.

## Acknowledgments

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