Lesson 17: Vector AutoRegressive Models

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Vector AutoRegressive models

The extension of ARMA models into a multivariate framework leads to

- **V**ector **A**uto**R**egressive (**VAR**) models
- **V**ector **M**oving **A**verage (**VMA**) models
- **V**ector **A**utoreg**R**egressive **M**oving **A**verage (**VARMA**) models

The Vector AutoRegressive (VAR) models ,

made famous in Chris Sims's paper Macroeconomics and Reality, Econometrica, 1980,

are one of the most applied models in the empirical economics.

Let

$$\{y_t = (y_{1t}, ..., y_{Kt})'; t \in \mathbb{Z}\}$$

be a K-variate random process.

Vector AutoRegressive models

We say that the process $\{y_t; t \in \mathbb{Z}\}$ follows a vector autoregressive model of order p, denoted VAR(p) if

$$y_t = \nu + A_1 y_{t-1} + ... + A_p y_{t-p} + u_t, \quad t \in \mathbb{Z}$$

- p is a positive integer,
- A_i are fixed $(K \times K)$ coefficient matrices,
- $\nu = (\nu_1, ..., \nu_K)'$ is a fixed $(K \times 1)$ vector of intercept terms,
- $u_t = (u_{1t}, ..., u_{Kt})'$ is aK-dimensional white noise with covariance matrix Σ_u.

The covariance matrix Σ_u is assumed to be nonsingular.

Example: Bivariate VAR(1)

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

or

$$y_{1t} = \nu_1 + a_{11} y_{1t-1} + a_{12} y_{2t-1} + u_{1t}$$

$$y_{2t} = \nu_1 + a_{21} y_{1t-1} + a_{22} y_{2t-1} + u_{2t}$$

where $cov(u_{1t}, u_{2s}) = \sigma_{12}$ for $t = s$, 0 otherwise

Advantages of VAR models

1. They are easy to estimate.
2. They have good forecasting capabilities.
3. The researcher does not need to specify which variables are endogenous or exogeneous. All are endogenous.
4. In a VAR system is very easy to test for Granger non-causality.

Problems with VARs

1. So many parameters. If there are K equations, one for each K variables and p lags of each of the variables in each equation, $(K + pK^2)$ parameters will have to be estimated.
2. VARs are a-theoretical, since they use little economic theory. Thus VARs cannot used to obtain economic policy prescriptions.

Estimation

In this lesson, the estimation of a vector autoregressive model is discussed.

Consider the VAR(1)

$$y_{1t} = \nu_1 + a_{11} y_{1t-1} + a_{12} y_{2t-1} + u_{1t}$$

$$y_{2t} = \nu_1 + a_{21} y_{1t-1} + a_{22} y_{2t-1} + u_{2t}$$

where

$$\text{cov}(u_{1t}, u_{2s}) = \sigma_{12} \text{ for } t = s, \;\; 0 \text{ otherwise.}$$

Estimation

The model corresponds to 2 regressions with different dependent variables and identical explanatory variables.

We could estimate this model using the ordinary least squares (OLS) estimator computed separately from each equations.

Estimation

It is assumed that a time series

$$y_1 = [y_{11}, y_{21}]', ..., y_T = [y_{1T}, y_{2T}]'$$

of the y variables is available.
In addition, a presample value

$$y_0 = [y_{10}, y_{20}]'$$

is assumed to be available.

Estimation

Consider the first equation

$$y_{1t} = \nu_1 + a_{11}y_{10} + a_{12}y_{2t-1} + u_{1t}; \quad t = 1, ..., T$$

$$y_{11} = \nu_1 + a_{11}y_{10} + a_{12}y_{20} + u_{11}$$

$$y_{12} = \nu_1 + a_{11}y_{11} + a_{12}y_{21} + u_{12}$$

$$\vdots$$

$$y_{1T} = \nu_1 + a_{11}y_{1T-1} + a_{12}y_{2T-1} + u_{1T}$$

Estimation

We define

$$\mathbf{y}_1 = [y_{11}, ..., y_{1T}]',$$

$$\mathbf{X}_1 = \begin{bmatrix} 1 & y_{1,0} & y_{2,0} \\ 1 & y_{1,1} & y_{2,1} \\ \vdots & \vdots & \vdots \\ 1 & y_{1,T-1} & y_{2,T-1} \end{bmatrix},$$

$$\boldsymbol{\pi}_1 = [\nu_1, a_{11}, a_{12}]',$$

$$\mathbf{u}_1 = [u_{11}, ..., u_{1T}]',$$

Thus

$$\mathbf{y}_1 = \mathbf{X}\boldsymbol{\pi}_1 + \mathbf{u},$$

The OLS estimator $\boldsymbol{\pi}_1$ is given by

$$\hat{\boldsymbol{\pi}}_1 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_1$$

Consider the second equation

$$y_{2t} = \nu_2 + a_{21}y_{10} + a_{22}y_{2t-1} + u_{2t}; \quad t = 1, ..., T$$

$$y_{21} = \nu_2 + a_{21}y_{10} + a_{22}y_{20} + u_{21}$$

$$y_{22} = \nu_2 + a_{21}y_{11} + a_{22}y_{21} + u_{22}$$

$$\vdots$$

$$y_{2T} = \nu_2 + a_{21}y_{1T-1} + a_{22}y_{2T-1} + u_{2T}$$

Estimation

We define

$$\mathbf{y}_2 = [y_{21}, ..., y_{2T}]',$$

$$\boldsymbol{\pi}_2 = [\nu_2, a_{21}, a_{22}]',$$

$$\mathbf{u}_2 = [u_{21}, ..., u_{2T}]',$$

Estimation

Thus

$$\mathbf{y}_2 = \mathbf{X}\boldsymbol{\pi}_2 + \mathbf{u}_2,$$

The OLS estimator $\boldsymbol{\pi}_2$ is given by

$$\hat{\boldsymbol{\pi}}_2 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_2$$

OLS estimators

$$\hat{\boldsymbol{\pi}}_1 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_1 \;\Rightarrow\; \boldsymbol{\pi}_1 = [\nu_1, a_{11}, a_{12}]'$$

$$\hat{\boldsymbol{\pi}}_2 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}_2 \;\Rightarrow\; \boldsymbol{\pi}_2 = [\nu_2, a_{21}, a_{22}]'$$

OLS Estimation

Are the OLS estimators for $\boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_2$ efficient estimators?

Seemingly Unrelated Regressions Estimation

We remember that

$$\mathrm{cov}(u_{1t}, u_{2t}) = \sigma_{12} \neq 0$$

When contemporaneous correlation exists, it may be more efficient to estimate all equations jointly, rather than to estimate each one separately using least squares.

The appropriate joint estimation technique is the GLS estimation

Seemingly Unrelated Regressions Equations

Let as consider the following set of two equations

$$y_{1t} = \beta_{10} + \beta_{11} x_{1t} + \beta_{12} z_{1t} + u_{1t}$$

$$y_{2t} = \beta_{10} + \beta_{11} x_{2t} + \beta_{12} z_{2t} + u_{2t}$$

The two equation can be written in the usual matrix algebra notation as

$$\mathbf{y}_1 = \mathbf{X}_1\beta_1 + \mathbf{u}_1$$
$$\mathbf{y}_2 = \mathbf{X}_2\beta_2 + \mathbf{u}_2$$

where

$$\mathbf{y}_i = [y_{i1}, ..., y_{iT}]' \quad i = 1, 2$$

$$\mathbf{X}_i = \begin{bmatrix} 1 & x_{i,1} & z_{i,1} \\ 1 & x_{i,2} & z_{i,2} \\ \vdots & \vdots & \vdots \\ 1 & x_{i,T} & z_{i,T} \end{bmatrix} \quad i = 1, 2,$$

$$\boldsymbol{\beta}_i = [\beta_{i0}, \beta_{i1}, \beta_{i2}]', \quad i = 1, 2,$$

and

$$\mathbf{u}_i = [u_{i1}, ..., u_{iT}]', i = 1, 2,$$

Seemingly Unrelated Regressions Equations

Further, we assume that

1. $E[u_{it}] = 0 \qquad i = 1, 2 \qquad t = 1, 2 ... T$
2. $\text{var}(u_{it}) = \sigma_i^2 \qquad i = 1, 2 \qquad t = 1, 2 ... T;$
3. $E[u_{it} u_{jt}] = \sigma_{ij} \qquad i, j = 1, 2;$
4. $E[u_{it} u_{js}] = 0 \qquad \text{for } t \neq s \qquad \text{and} \qquad i, j = 1, 2;$

Now we write the two equations as the folllowing ‘super model’

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

or

$$\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$$

The first point to note is that it can be shown that least squares applied to this system is identical to applying least squares separately to each of the two equations.

Note that the covariance matrix of the joint disturbace vector $\mathbf{u}$ is given by

$$\mathbf{\Phi} = \begin{bmatrix} \sigma_1^2 \mathbf{I} & \sigma_{12} \mathbf{I} \\ \sigma_{12} \mathbf{I} & \sigma_2^2 \mathbf{I} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \otimes \mathbf{I}_T = \mathbf{\Sigma} \otimes \mathbf{I}_T$$

The disturbance covariance matrix $\mathbf{\Phi}$ is of dimension $(2T \times 2T)$.An important point to note is that $\mathbf{\Phi}$ cannot be written as scalar multiplied by a $2T$-dimensional identity matrix. Thus the best linear unbiased estimator for β is given by the generalized least squares estimator

$$\hat{\beta}_{GLS} = (\mathbf{X}'\mathbf{\Phi}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{\Phi}^{-1}\mathbf{y} = [\mathbf{X}'(\mathbf{\Sigma}^{-1} \otimes \mathbf{I}_T)\mathbf{X}]^{-1}\mathbf{X}(\mathbf{\Sigma}^{-1} \otimes \mathbf{I}_T)\mathbf{y}$$

It has lower variance than the least squares estimators for β because it takes into account the contemporaneous correlation between the disturbances in different equation.

Seemingly Unrelated Regressions Equations

There are two conditions under the which least squares is identical to generalized least squares.

1. The first is when all contemporaneus covariances are zero, $\sigma_{12} = 0$.
2. The second is when the explanatory variables in each equation are identical, $\mathbf{X}_1 = \mathbf{X}_2$

The use of generalized least squares estimator does not lead to a gain in efficiency when each equation contains the same explanatory variables.

Thus the autoregressive cooefficients of our VAR(1) model can be estimated, without loss of estimation efficiency, by ordinary least squares. This in turn is equivalent to estimating each equation separately by OLS

Estimation of A VAR model

The (2×2) unknown covariance matrix Σ may be consistent estimated by $\hat{\Sigma}$ whose elements are

$$\hat{\sigma}_{ij} = \frac{\hat{\mathbf{u}}_i' \hat{\mathbf{u}}_j}{T - 2p - 1} \quad \text{for} \quad i, j = 1, 2$$

where $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{X}\hat{\boldsymbol{\pi}}_i$

Estimation of A VAR model

In general, the autoregressive cooefficients of a K-dimensional VAR(p) model

$$y_t = \nu + A_1 y_{t-1} + ... + A_p y_{t-p} + u_t$$

can be estimated, without loss of estimation efficiency, by ordinary least squares.In the first equation, we have to run the regression

$$y_{1t} \;\; \text{on} \;\; y_{1t-1}, ..., y_{Kt-1}, ..., y_{1t-p}, ..., y_{Kt-p},$$

in the second equation, we regress

$$y_{12} \;\; \text{on} \;\; y_{1t-1}, ..., y_{Kt-1}, ..., y_{1t-p}, ..., y_{Kt-p},$$

and so on.

Conclusion

Since the VAR(p) model is just a Seemingly Unrelated Regression (SUR) model where each equation has the same explanatory variables, each equation may be estimated separately by ordinary least squares without losing efficiency relative to generalized least squares.