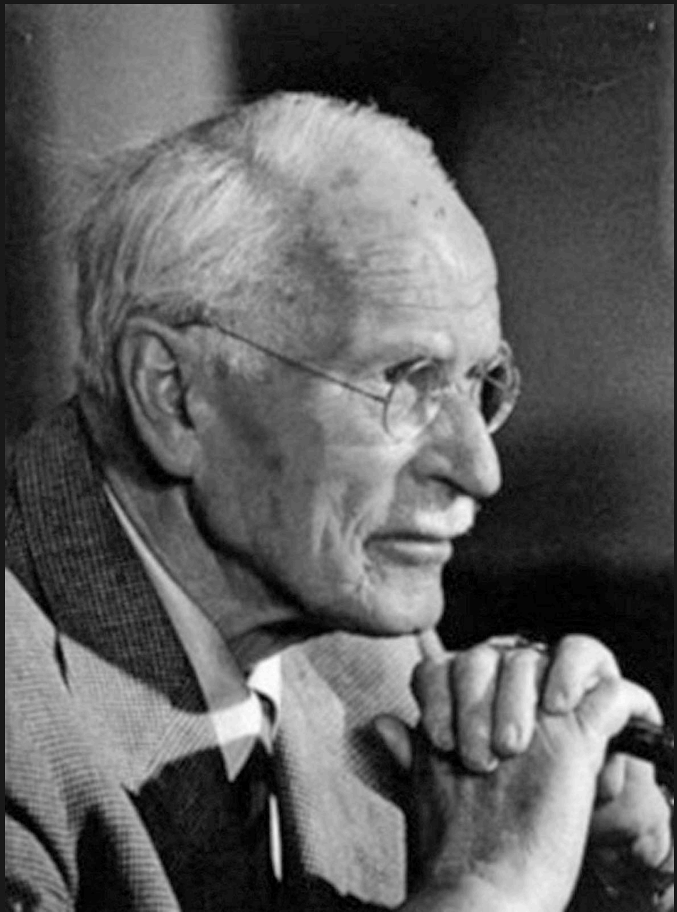
On Benders decomposition
(and on Jung's individuation process)

Andreas Ernst
Alysson M. Costa

May 29, 2023
OPTIMA GR Conference

The individuation process



“The only meaningful life is a life that strives for the individual realization — absolute and unconditional— of its own particular law ... To the extent that a person who is untrue to the law of his being ... has failed to realise his own life's meaning.”

— C.G. Jung

Jung's word association Test

Jung's Word Association Test Form

NAME ________________________________ DATE_______________

INTERVIEWER:______________________________

(ANSWER AS QUICKLY AS POSSIBLE WITH THE FIRST WORD THAT OCCURS TO YOUR MIND)

Word	RT	Response	Word	RT	Response

1. Transmission of sound to the ear of the recipient.
2. Neural conduction to the auditory centre.
3. Word-recognition (primary identification).
4. Word-comprehension (secondary identification).
5. Evocation of the associated image, i.e., pure association.
6. Naming of the idea evoked.
7. Excitation of the motor speech-apparatus or the motor-centre of the hand when measurement is made by means of a Morse telegraph key.
8. Neural conduction to the muscle.

Jung's word association
Test

Jung's word association
Test

Head

Jung's word association

Test

Green

Jung's word association

Test

Water

To pierce

Jung's word association
Test

Angel

Jung's word association
Test

To hit

Jung's word association

Test

Law

Jung's word association Test

Jung's Word Association Test Form

NAME ______________________ DATE ______________

INTERVIEWER: ______________________

(ANSWER AS QUICKLY AS POSSIBLE WITH THE FIRST WORD THAT OCCURS TO YOUR MIND)

Word	RT	Response	Word	RT	Response
head			frog		
green			to part		
water			hunger		
to sing			white		
dead			child		
long			to take care		
ship			pencil		
pay			sad		
window			plum		
friendly			to marry		
to cook			house		
to ask			sweatheart		
cold			glass		
stem			to quarrel		

1. Transmission of sound to the ear of the recipient.
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8. Neural conduction to the muscle.

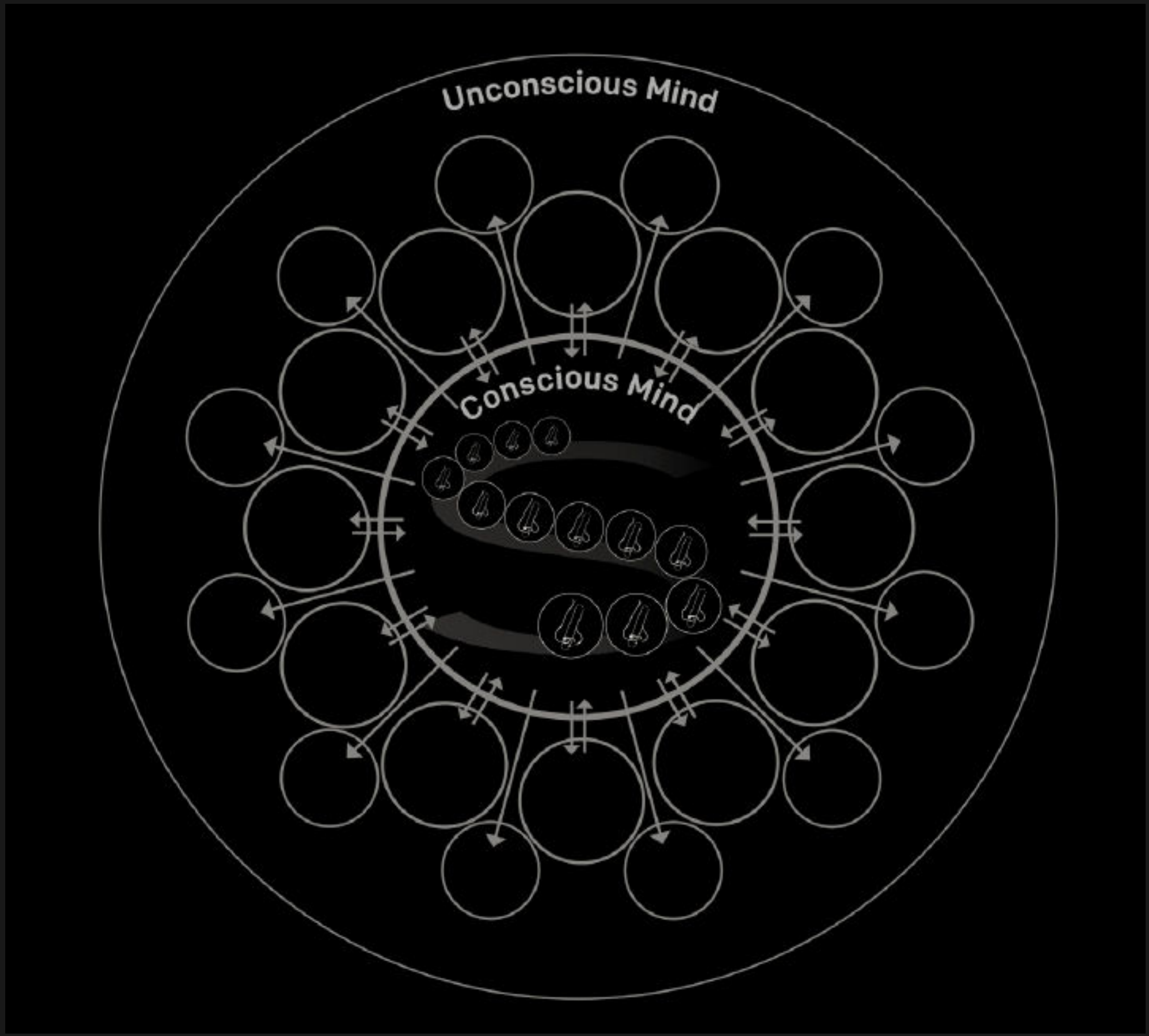
Jung's word association Test

1. head	-scarf	1.0
2. green	grass	0.8
3. water	-fall	1.0
4. to pierce	to cut	0.8
5. angel	-heart	0.8

60. to hit	marksman	1.2
61. law	not set	4.8

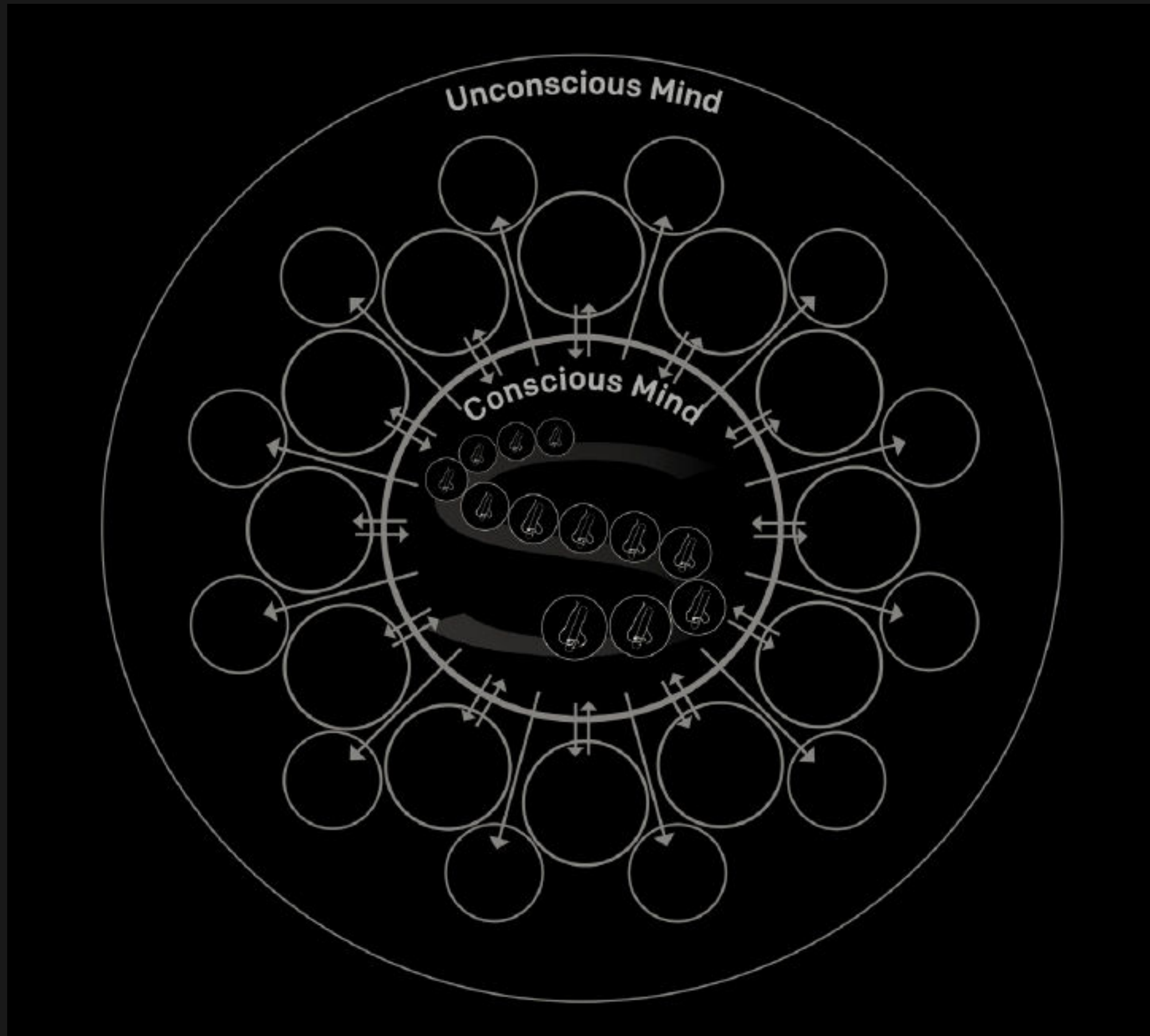
A model of the mind

A model of the mind



The mind illuminated
Culadasa (John Yates, PhD)

And where is Optimisation ?



The mind illuminated
Culadasa (John Yates, PhD)

Network design

Network design problems are central to a large number of contexts.

(transportation, telecommunications, power systems)

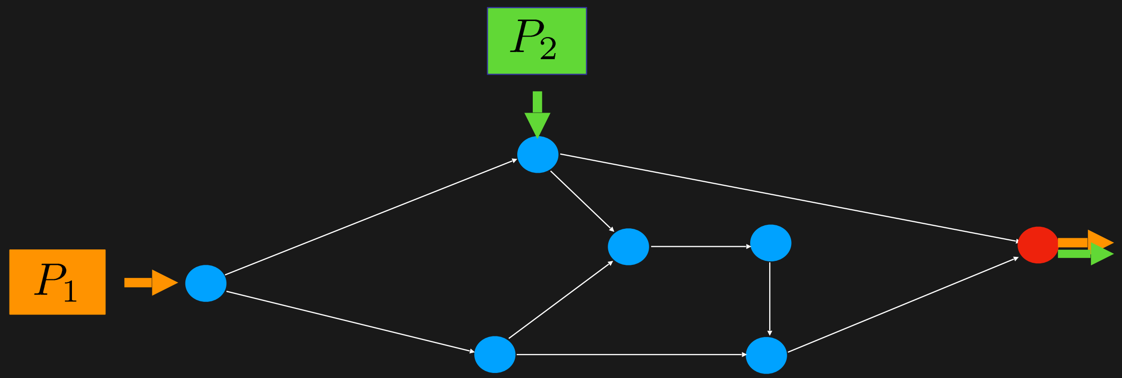
The idea is to establish a network of links

(roads, optical fibres, electric lines)

to enable the flow of commodities

(people, data packets, electricity)

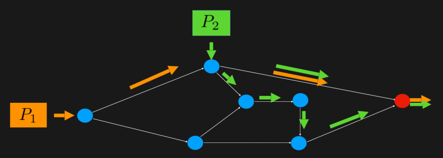
Network design



💰💰💰 fixed cost to build a link (road, fibre, power line)

💰 variable cost to move commodity

Network design



$$\text{Minimize} \quad \sum_{(i,j)\in A} \left(f_{ij} y_{ij} + \sum_{k \in K} c_{ij}^k x_{ij}^k \right)$$

subject to

$$\sum_{j \in N_i^+} x_{ij}^k - \sum_{j \in N_i^-} x_{ji}^k = \begin{cases} d_k, & i = O(k), \\ 0, & i \notin \{O(k), D(k)\}, \\ -d_k, & i = D(k), \end{cases} \quad \forall i \in N, \quad \forall k \in K,$$

$$\begin{aligned} \sum_{k \in K} x_{ij}^k &\leq u_{ij} y_{ij}, && \forall (i,j) \in A, \\ x_{ij}^k &\geq 0, && \forall (i,j) \in A, \quad \forall k \in K, \\ y_{ij} &\in \{0,1\}, && \forall (i,j) \in A, \end{aligned}$$

where

$N_i^+ = \{j | (i,j) \in A\}$ and $N_i^- = \{j | (j,i) \in A\}$.

Branch-and-bound

ECONOMETRICA

VOLUME 28 July, 1960 NUMBER 3

AN AUTOMATIC METHOD OF SOLVING DISCRETE PROGRAMMING PROBLEMS

BY A. H. LAND AND A. G. DOIG

In the classical linear programming problem the behaviour of continuous, nonnegative variables subject to a system of linear inequalities is investigated. One possible generalization of this problem is to relax the continuity condition on the variables. This paper presents a simple numerical algorithm for the solution of programming problems in which some or all of the variables can take only discrete values. The algorithm requires no special techniques beyond those used in ordinary linear programming, and lends itself to automatic computing. Its use is illustrated on two numerical examples.

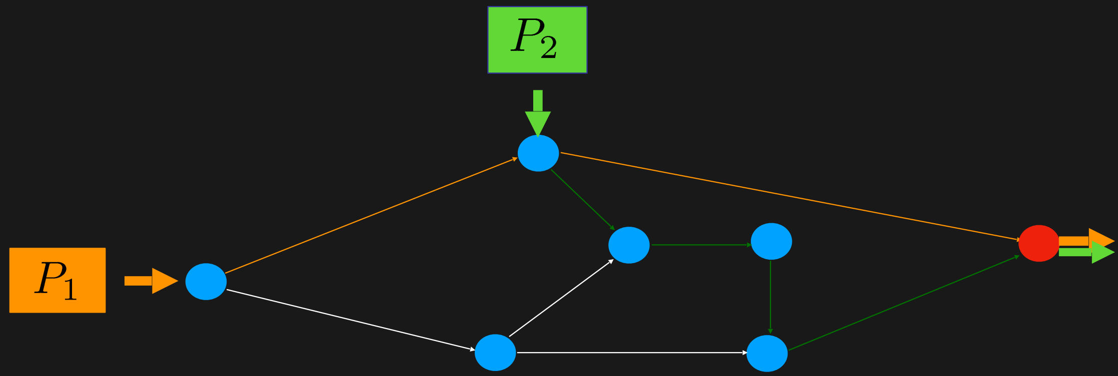
1. INTRODUCTION

THERE IS A growing literature [**1**, **3**, **5**, **6**] about optimization problems which could be formulated as linear programming problems with additional constraints that some or all of the variables may take only integral values. This form of linear programming arises whenever there are indivisibilities. It is not meaningful, for instance, to schedule $3-7/10$ flights between two cities, or to undertake only $1/4$ of the necessary setting up operation for running a job through a machine shop. Yet it is basic to linear programming that the variables are free to take on any positive value,[1] and this sort of answer is very likely to turn up.

In some cases, notably those which can be expressed as transport problems, the linear programming solution will itself yield discrete values of the variables. In other cases the percentage change in the maximand[2] from common sense rounding of the variables is sufficiently small to be neglected. But there remain many problems where the discrete variable constraints are significant and costly.

Until recently there was no general automatic routine for solving such

Network design



💰💰💰 fixed cost to build a link (road, fibre, power line)

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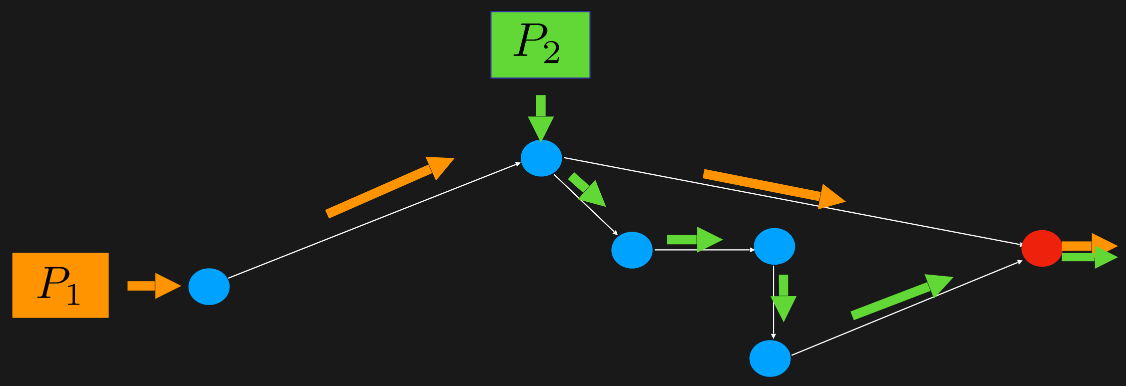
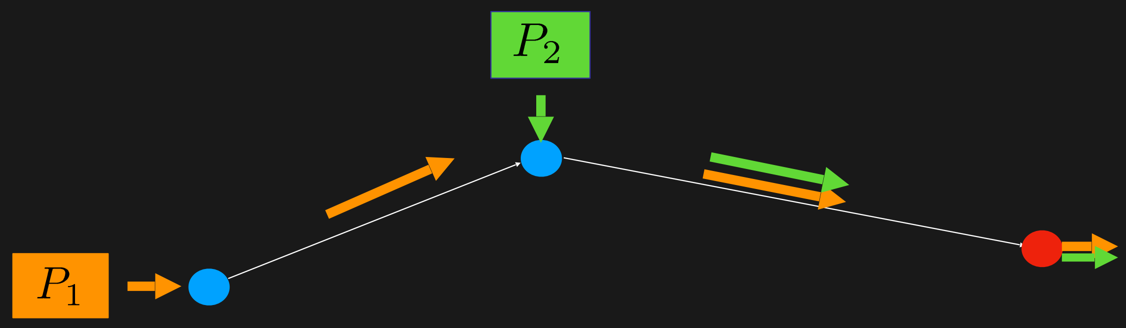
Network design

A decomposition process

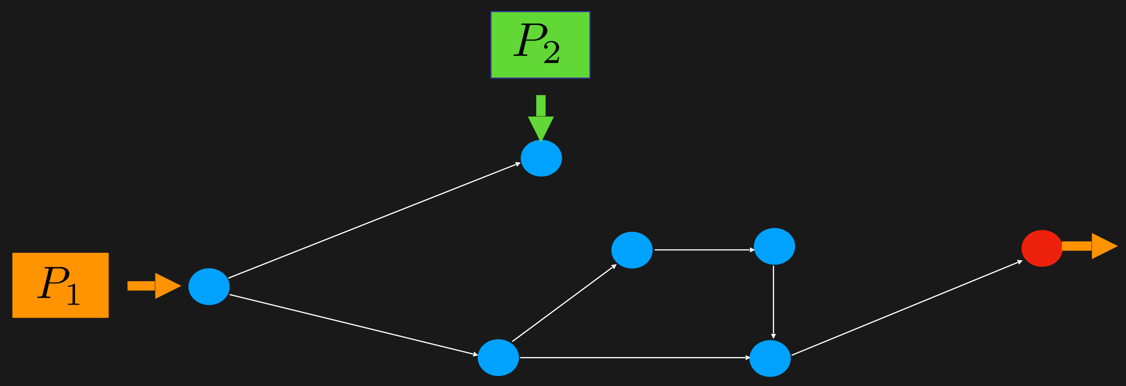
Network design



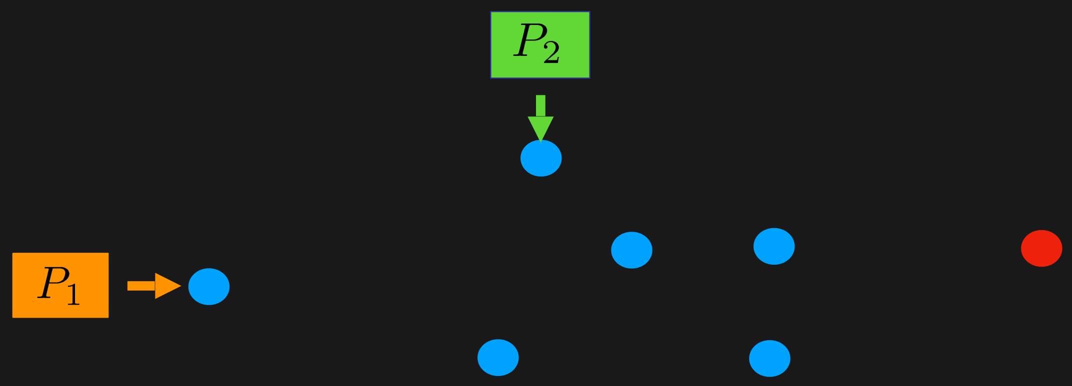
Network design



Network design

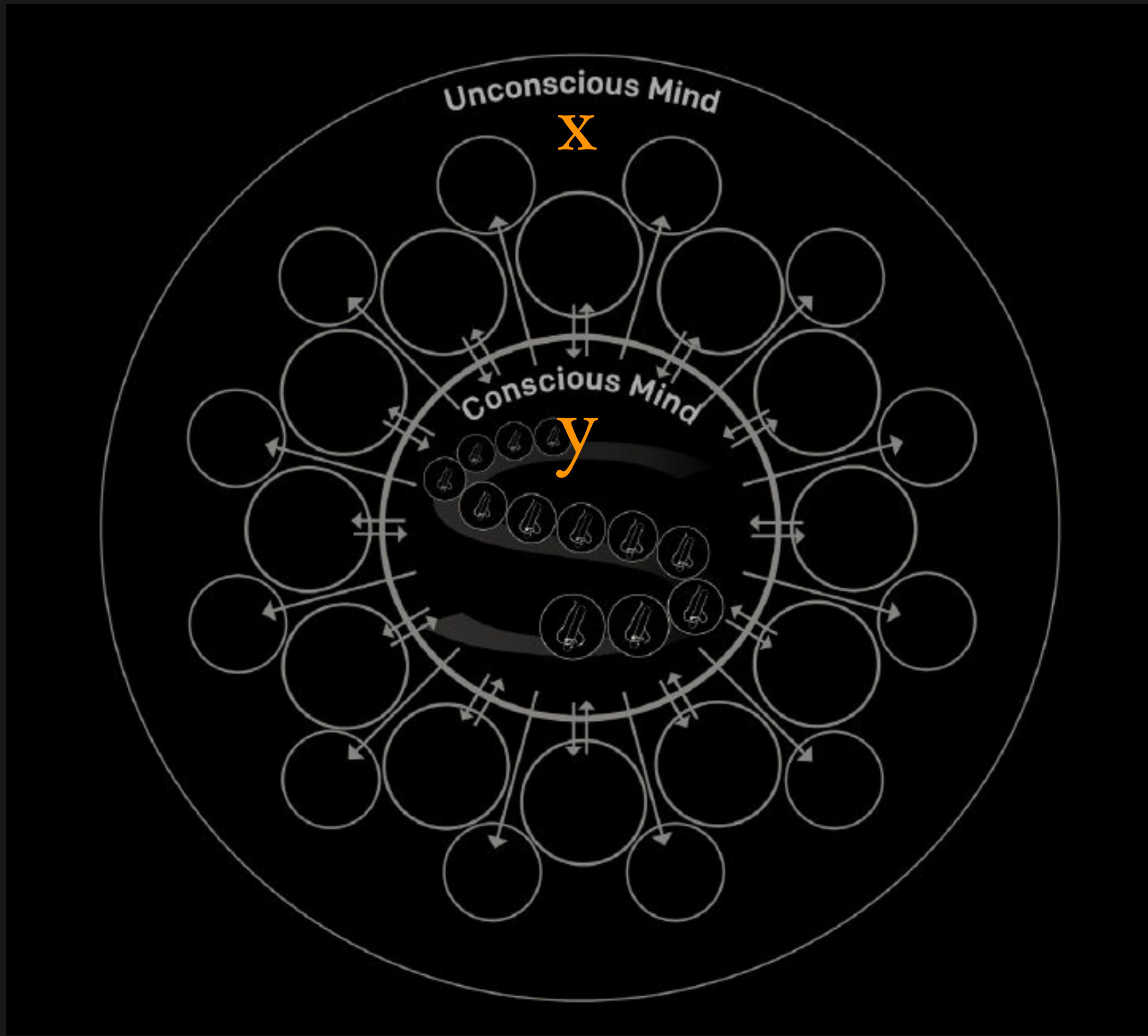


Network design



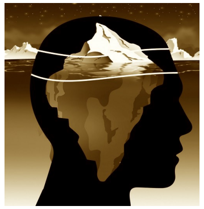
Naïve decomposition is not individuated.

The conscious Y has failed to realise the true natural law of X.



Numerische Mathematik 4, 238—252 (1962)

Partitioning procedures for solving mixed-variables programming problems*



By

J. F. BENDERS**

I. Introduction

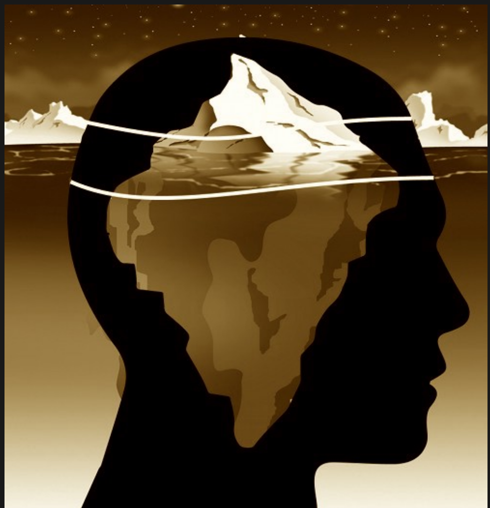
In this paper two slightly different procedures are presented for solving mixed-variables programming problems of the type

$$\max\{c^T x + f(y) \mid A x + F(y) \leqq b,\ x \in R_p,\ y \in S\}, \qquad (1.1)$$

where $x \in R_p$ (the p-dimensional Euclidean space), $y \in R_q$, and S is an arbitrary subset of R_q. Furthermore, A is an (m, p) matrix, $f(y)$ is a scalar function and $F(y)$ an m-component vector function both defined on S, and b and c are fixed vectors in R_m and R_p, respectively.

An example is the mixed-integer programming problem in which certain variables may assume any value on a given interval, whereas others are restricted to integral values only. In this case S is a set of vectors in R_q with integral-valued components. Various methods for solving this problem have been proposed by BEALE [1], GOMORY [9] and LAND and DOIG [11]. The use of integer variables, in particular for incorporating in the programming problem a choice from a set of alternative discrete decisions, has been discussed by DANTZIG [4].

Benders decomposition



Branch-and-bound was computationally much ahead of its time.

Benders even more.

ECONOMETRICA

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2005 C&OR Survey

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A survey on benders decomposition applied to fixed-charge network design problems

Alysson M. Costa*

Canada Research Chair in Distribution Management, HEC Montréal, 3000 Chemin de la Côte-Sainte-Catherine, Montréal, Canada H3T 2A7

Abstract

Network design problems concern the selection of arcs in a graph in order to satisfy, at minimum cost, some flow requirements, usually expressed in the form of origin–destination pair demands. Benders decomposition methods, based on the idea of partition and delayed constraint generation, have been successfully applied to many of these problems. This article presents a review of these applications.

Keywords: Benders decomposition; Network design; Fixed charge

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4. Summary and conclusions

Numerous practical applications can be formulated as network design problems. In these problems, the idea is to obtain a least cost network in order to satisfy some flow constraints, commonly expressed in the form of origin–destination demands. We have presented a review on Benders decomposition methods applied to network design. Formulations for these problems usually contain one set of integer variables associated with the selection of the arcs in the network, and one set of continuous variables associated with commodity flows. This structure offers a natural framework for the decomposition approach which consists of isolating the integer variables in the master problem and the flow variables in the auxiliary subproblem. Moreover, the relative ease of solving the auxiliary subproblem in network design formulations make of Benders decomposition one of the most appropriate approaches. Indeed, in most of the surveyed articles (see Table 1 for a summary), validations tests have indicated that Benders decomposition is an efficient method for solving network design problems, and may outperform traditional techniques such as Branch-and-Bound or Lagrangian relaxation. Efficient solution methodologies have also been obtained by combining Benders decomposition with other techniques, as proposed by Magnanti et al. [24]. A rich variety of these successful hybrid approaches is available in the literature.

In spite of this success, Benders decomposition has been mostly ignored for many years, not only for network design problems but for some of the other applications mentioned in Section 2. We believe that this tendency is slowly changing, given the increasing number of researchers using this technique, as shown in this article.

2005 C&OR Survey

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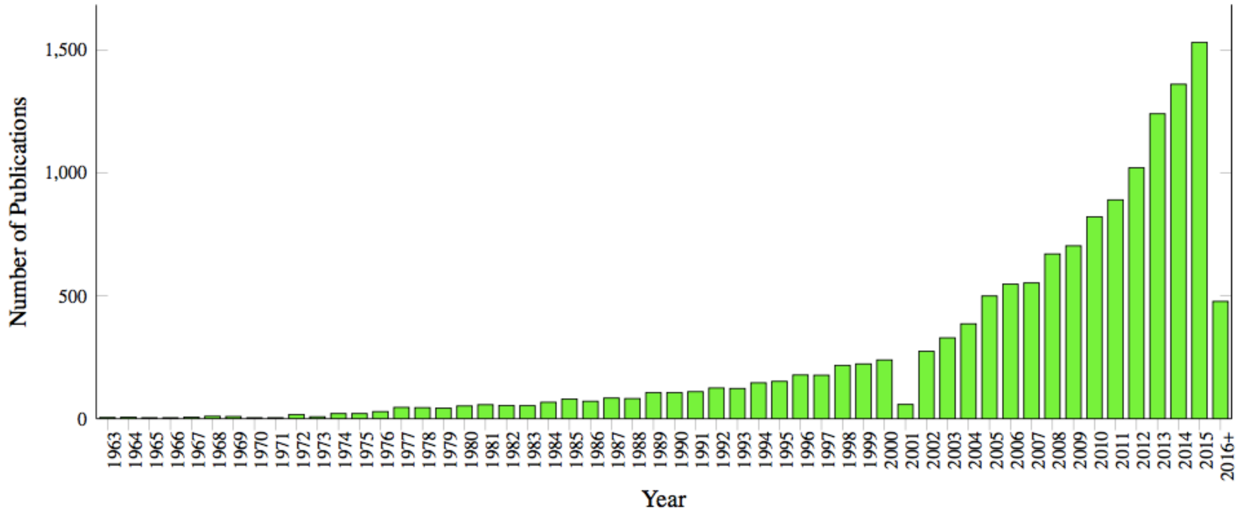

Figure 1: Publication–year distribution of BD research according to https://scholar.google.ca/.

2005 C&OR Survey

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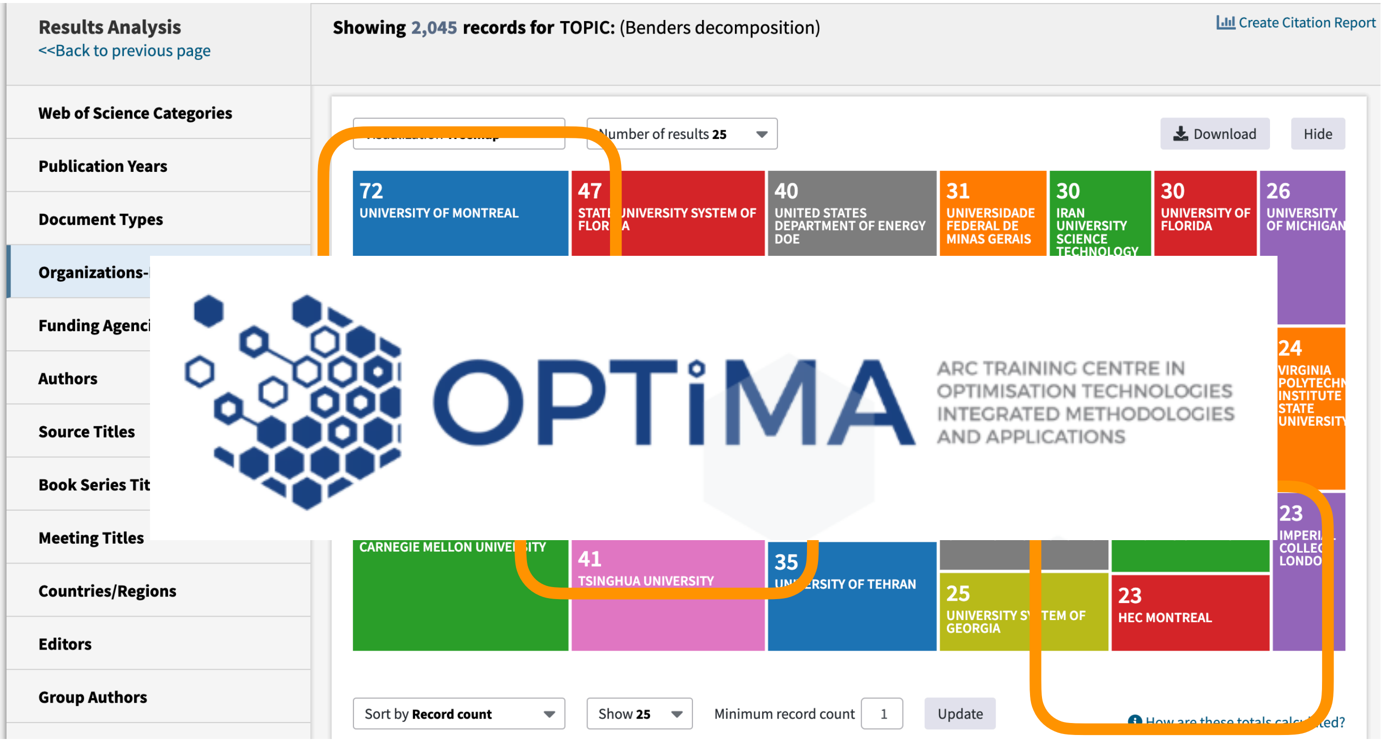


Benders reformulation

A generic MIP

$$\begin{aligned} \text{Minimize} \quad & cx + dy \\ \text{subject to} \quad & Ax + By \geqslant b, \\ & Dy \geqslant e, \\ & x \geqslant 0, \quad y \geqslant 0 \quad \text{and integer.} \end{aligned}$$

Benders reformulation

A generic MIP

$$\min_{\bar{y}\in Y}\{d\bar{y} + \min_{x\geqslant 0}\{cx : Ax \geqslant b - B\bar{y}\}\},$$

$$\begin{aligned}
\text{Minimize} \quad & cx + dy \\
\text{subject to} \quad & Ax + By \geqslant b, \\
& Dy \geqslant e, \\
& x \geqslant 0, \quad y \geqslant 0 \quad \text{and integer.}
\end{aligned}$$

Benders reformulation

The ace in the hole / ‘the jump of the cat’

$$\min_{\bar{y}\in Y}\{d\,\bar{y} + \min_{x\geqslant 0}\{cx : Ax \geqslant b - B\bar{y}\}\},$$

Benders reformulation

$$\min_{\bar{y} \in Y} \{ d\bar{y} + \min_{x \geqslant 0} \{ cx : Ax \geqslant b - B\bar{y} \} \},$$

$$\max_{u \geqslant 0} \{ u(b - B\bar{y}) : uA \leqslant c \}.$$

But how ?

Theorem 1 (Fundamental Theorem of Duality)

With regard to the primal and dual linear programming problems, exactly one of the following statements is true.

1. Both possess optimal solutions $\mathbf{x}^*$ and $\mathbf{w}^*$ with $\mathbf{cx}^* = \mathbf{w}^*\mathbf{b}$.
2. One problem has unbounded objective value, in which case the other problem must be infeasible.
3. Both problems are infeasible.

From this theorem we see that duality is not completely symmetric. The best we can say is that (here optimal means finite optimal, and unbounded means having an unbounded optimal objective):

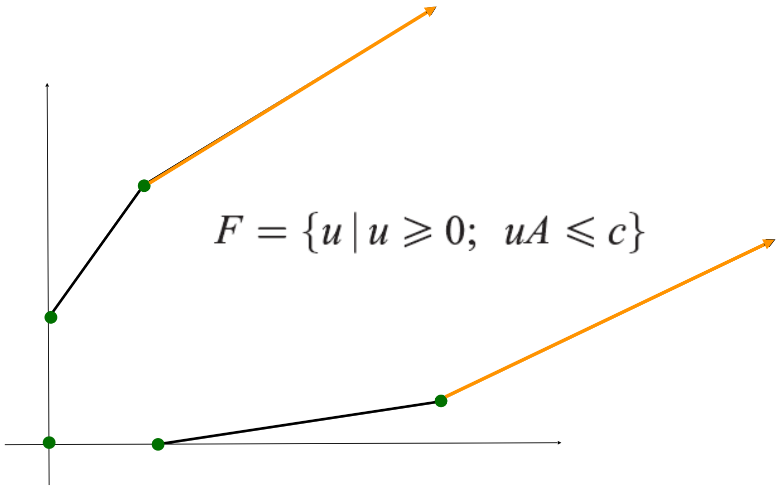
P	OPTIMAL	$\Leftrightarrow$	D	OPTIMAL
P	UNBOUNDED	$\Rightarrow$	D	INFEASIBLE
D	UNBOUNDED	$\Rightarrow$	P	INFEASIBLE
P	INFEASIBLE	$\Rightarrow$	D	UNBOUNDED OR INFEASIBLE
D	INFEASIBLE	$\Rightarrow$	P	UNBOUNDED OR INFEASIBLE

From Bazaraa - Linear Programming and Network flows

Benders reformulation

$$\min_{\bar{y}\in Y}\{d\bar{y}+\min_{x\geqslant 0}\{cx : Ax \geqslant b - B\bar{y}\}\},$$

$$\max_{u\geqslant 0}\{u(b-B\bar{y}) : uA \leqslant c\}.$$



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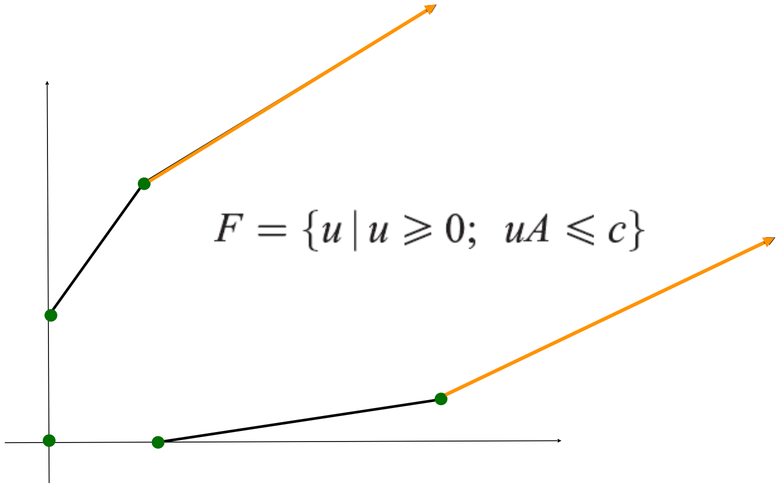


$$F = \{u \mid u \geqslant 0;\; uA \leqslant c\}$$

extreme points

$$\mathbf{F} \begin{cases} \boxed{u^p \;\; (\text{for } \; p = 1, \ldots, P)} \\ \boxed{r^q \;\; (\text{for } \; q = 1, \ldots, Q)} \end{cases}$$

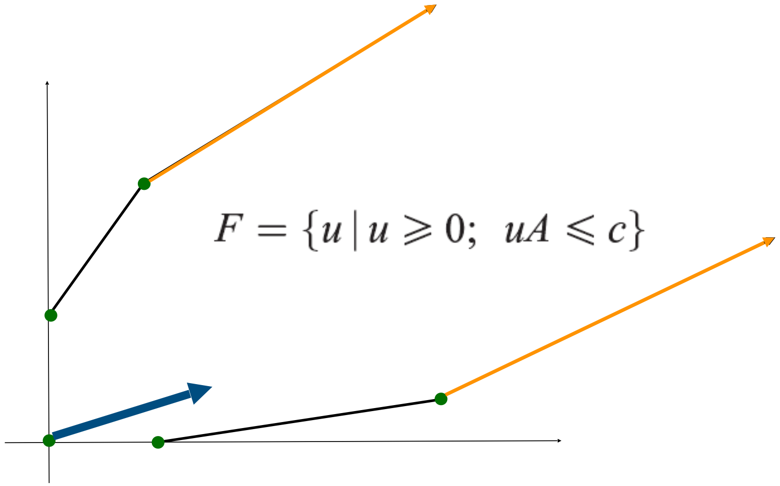
extreme rays

Benders reformulation

A generic MIP

$$\min_{\bar{y}\in Y}\{d\bar{y} + \min_{x\geqslant 0}\{cx : Ax \geqslant b - B\bar{y}\}\},$$

$$\max_{u\geqslant 0}\{\boxed{u(b - B\bar{y})} : uA \leqslant c\}.$$



$F = \{u \mid u \geqslant 0;\ uA \leqslant c\}$

F

extreme points

$\boxed{u^p \text{ (for } p = 1,\ldots,P)}$

$\boxed{r^q \text{ (for } q = 1,\ldots,Q)}$

extreme rays

Benders reformulation

A generic MIP

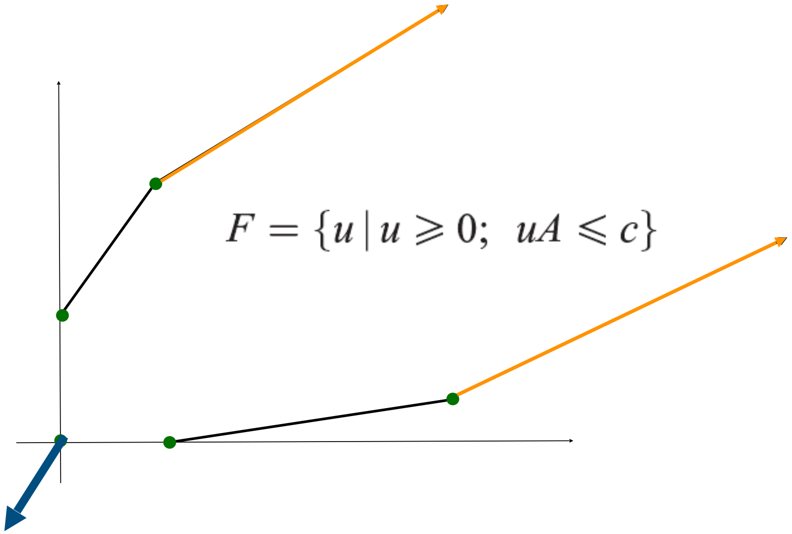
$$\max_{u\geqslant 0}\{u(b-B\bar{y}) : uA \leqslant c\}.$$

$$Z$$

$$\min_{\bar{y}\in Y}\{d\bar{y} + \cancel{\min_{x\geqslant 0}\{cx : Ax \geqslant b - B\bar{y}\}}\},$$

$$\text{s.t} \quad r^q(b-B\bar{y}) \leqslant 0, \qquad q = 1,\ldots,Q,$$

$$z \geqslant u^p(b-B\bar{y}), \qquad p = 1,\ldots,P,$$



$$F = \{u \mid u \geqslant 0;\ uA \leqslant c\}$$

extreme points

F

$\boxed{u^p \ (\text{for } p = 1,\ldots,P)}$

$\boxed{r^q \ (\text{for } q = 1,\ldots,Q)}$

extreme rays

Benders reformulation

$$
\begin{aligned}
\text{Minimize} \quad & dy + z \\
\text{subject to} \quad & z \geqslant u^p(b - B\bar{y}), \qquad p = 1, \ldots, P, \\
& r^q(b - B\bar{y}) \leqslant 0, \qquad q = 1, \ldots, Q, \\
& y \in Y, \quad z \geqslant 0.
\end{aligned}
$$

The number of constraints stresses me.

Benders algorithm

Relax

$$\text{Minimize} \quad d\,y + z$$

$$\text{subject to} \quad \cancel{z \geqslant u^p(b - B\bar{y}), \quad p = 1,\dots,P,}$$

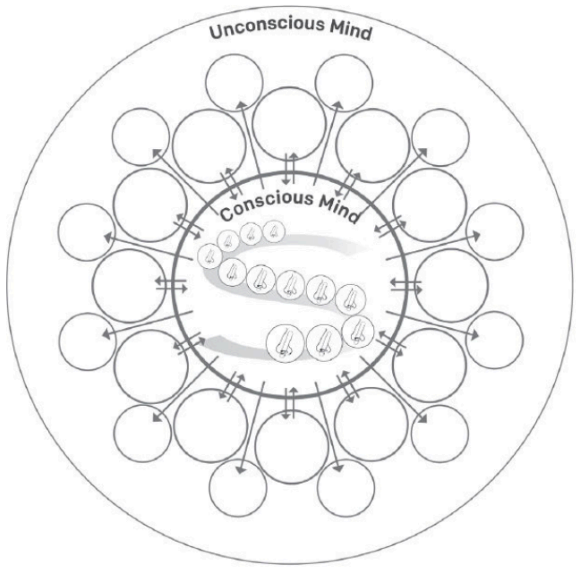
$$\cancel{r^q(b - B\bar{y}) \leqslant 0, \quad q = 1,\dots,Q,}$$

$$y \in Y, \quad z \geqslant 0.$$

$\boxed{u^p \ \text{(for } p = 1,\dots,P)}$

$\boxed{r^q \ \text{(for } q = 1,\dots,Q)}$

feasibility/optimality
cut



Benders algorithm

Convergence

Benders

Master: LB

Feasible Subproblem: UB

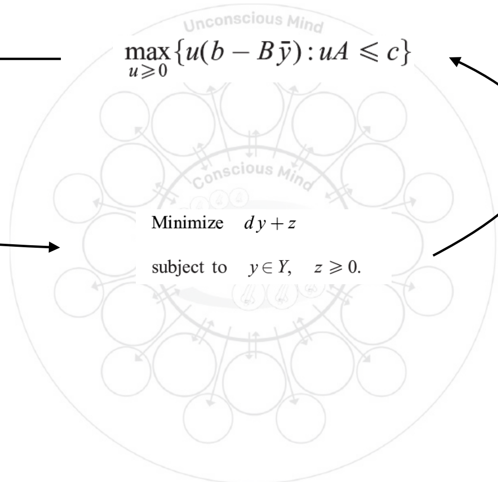
Jung

Individuation

$\boxed{u^p \ (\text{for } \ p = 1, \ldots, P)}$

$\boxed{r^q \ (\text{for } \ q = 1, \ldots, Q)}$

$$\max_{u \geqslant 0} \{u(b - B\bar{y}) : uA \leqslant c\}$$



$$\begin{array}{ll} \text{Minimize} & dy + z \\ \text{subject to} & y \in Y, \quad z \geqslant 0. \end{array}$$

$\bar{y}$

feasibility/optimality
cut

Example

What else you should know?

(Part 2)

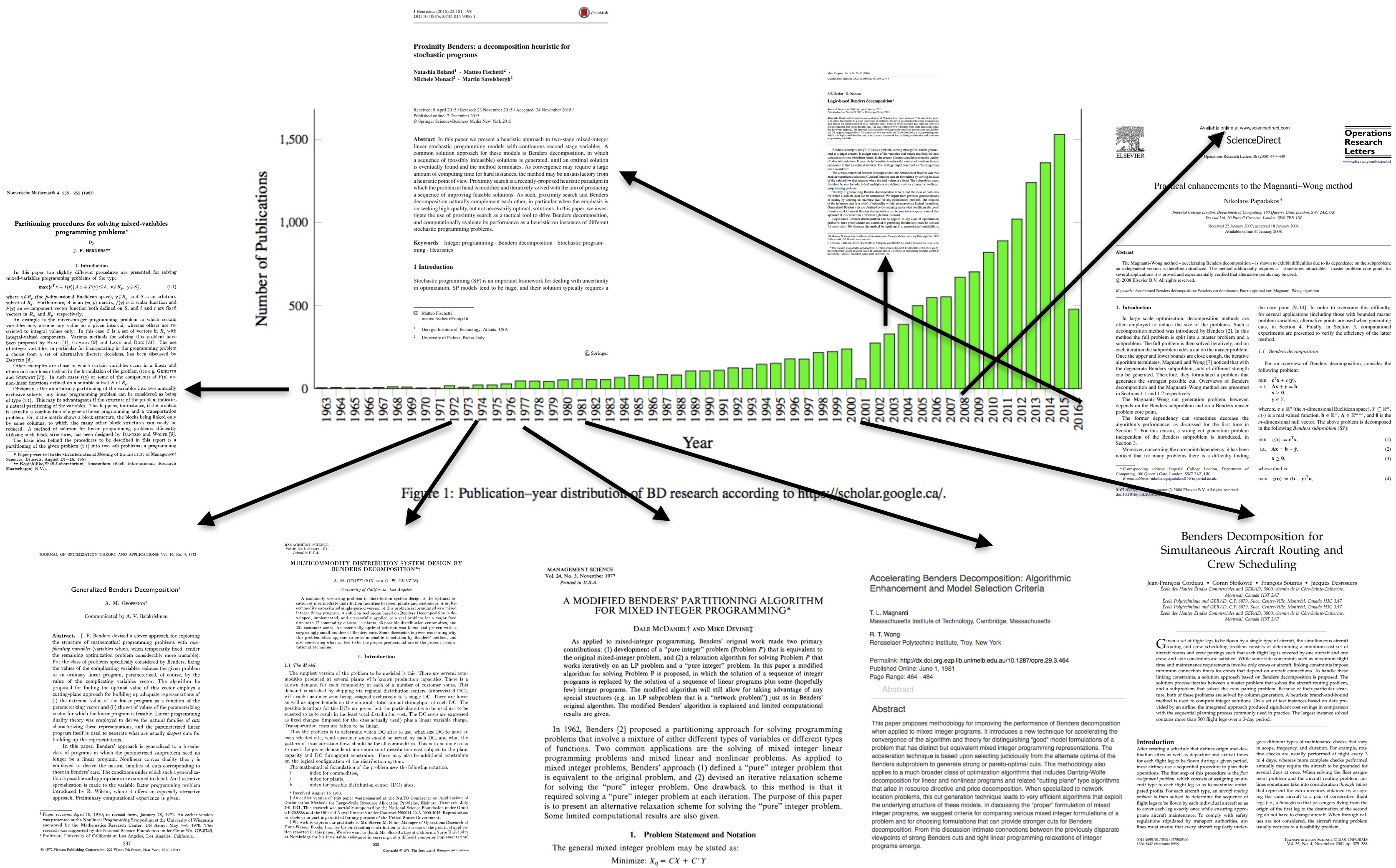
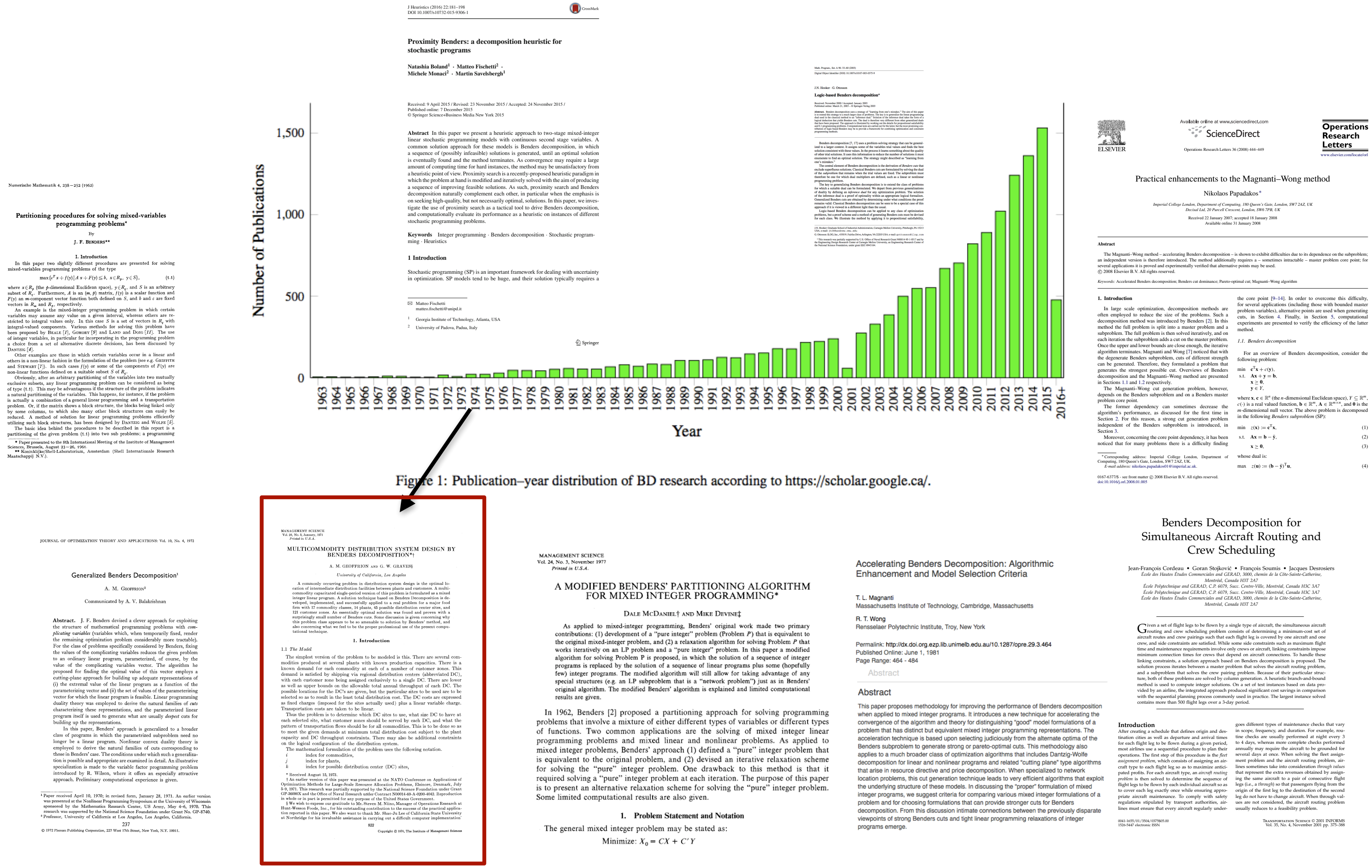


Figure 1: Publication–year distribution of BD research according to https://scholar.google.ca/.

What else you should know.

Clear applications



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What else you should know.

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Printed in U.S.A.

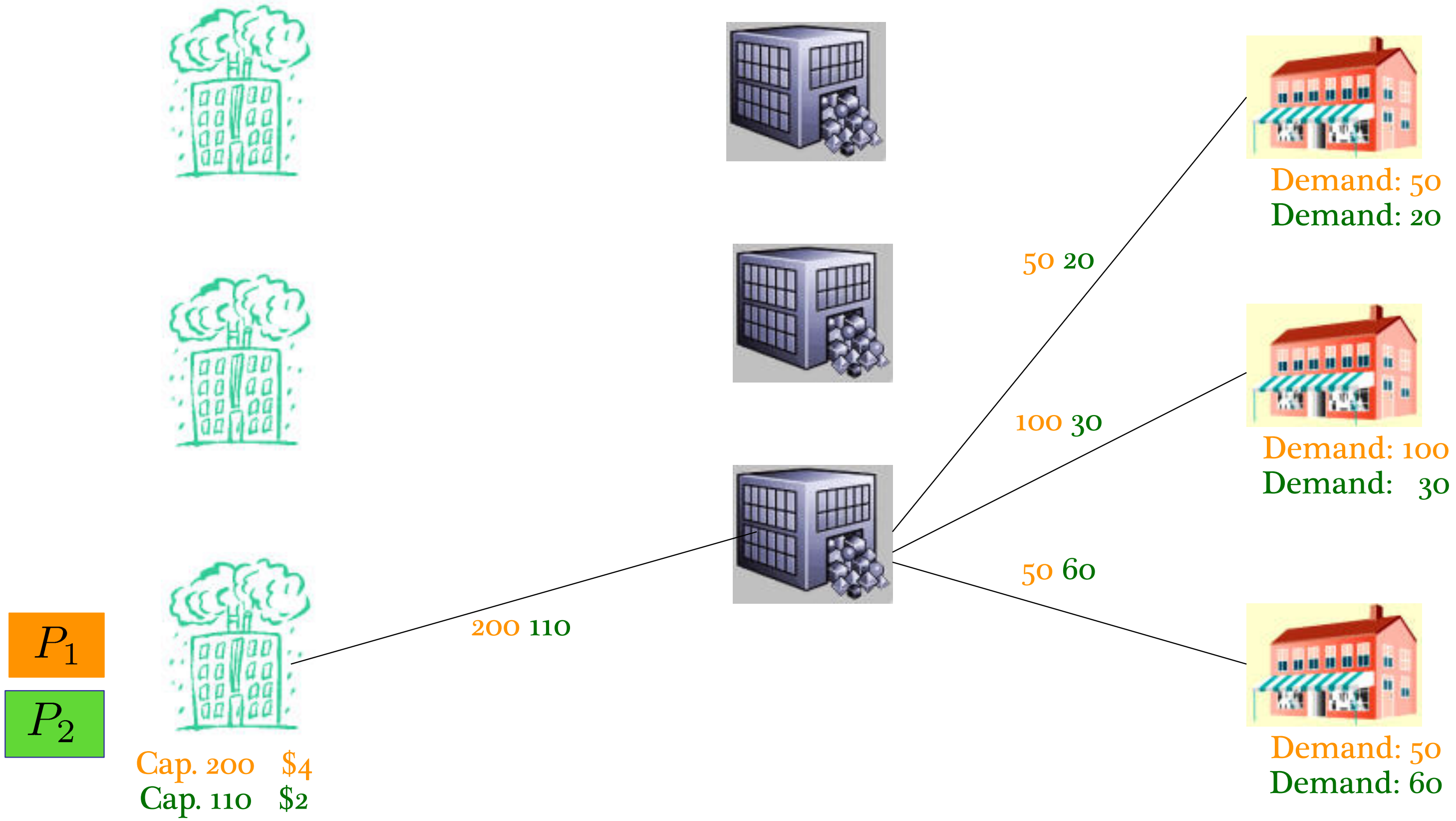
MULTICOMMODITY DISTRIBUTION SYSTEM DESIGN BY BENDERS DECOMPOSITION*†

A. M. GEOFFRION AND G. W. GRAVES§

University of California, Los Angeles

A commonly occurring problem in distribution system design is the optimal location of intermediate distribution facilities between plants and customers. A multicommodity capacitated single-period version of this problem is formulated as a mixed integer linear program. A solution technique based on Benders Decomposition is developed, implemented, and successfully applied to a real problem for a major food firm with 17 commodity classes, 14 plants, 45 possible distribution center sites, and 121 customer zones. An essentially optimal solution was found and proven with a surprisingly small number of Benders cuts. Some discussion is given concerning why this problem class appears to be so amenable to solution by Benders' method, and also concerning what we feel to be the proper professional use of the present computational technique.

What else you should know.



What else you should know.

x_{ijkl}	a variable denoting the amount of commodity i shipped from plant j through DC k to customer zone l,
y_{kl}	a 0–1 variable that will be 1 if DC k serves customer zone l, and 0 otherwise
z_k	a 0–1 variable that will be 1 if a DC is acquired at site k, and 0 otherwise.

(1) $\text{Minimize}_{x \geqq 0; y,z=0,1} \sum_{ijkl} c_{ijkl} x_{ijkl} + \sum_k [f_k z_k + v_k \sum_{il} D_{il} y_{kl}]$

subject to

(2) $\sum_{kl} x_{ijkl} \leq S_{ij}$, all ij

(3) $\sum_j x_{ijkl} = D_{il} y_{kl}$, all ikl

(4) $\sum_k y_{kl} = 1$, all l

(5) $\underline{V}_k z_k \leq \sum_{il} D_{il} y_{kl} \leq \bar{V}_k z_k$, all k

(6) Linear configuration constraints on y and/or z.

What else you should know.

Third, as indicated previously, the LP subproblem (10) is most easily solved by solving an equivalent collection of independent classical transportation problems—one for each commodity. This can be demonstrated by observing that since y^{H+1} satisfies (4), (3) implies

$$x_{ijkl}^{H+1} = 0 \text{ for all } ijkl \text{ with } k \neq \bar{k}(l)$$

where $\bar{k}(l)$ is the k-index for which $y_{\bar{k}l}^{H+1} = 1$. Thus (10) simplifies to

$$\text{Minimize } \textstyle\sum_i \left(\sum_{jl} c_{ij\bar{k}(l)l} x_{ij\bar{k}(l)l}\right)$$

subject to

$$\begin{aligned} \textstyle\sum_l x_{ij\bar{k}(l)l} &\leq S_{ij}, & \text{all } ij \\ \textstyle\sum_j x_{ij\bar{k}(l)l} &= D_{il}, & \text{all } il \\ x_{ij\bar{k}(l)l} &\geq 0, & \text{all } ijl. \end{aligned}$$

This problem obviously separates on i into independent transportation problems of the form (7i). If the optimal value of (7i) is denoted by $T_i(y^{H+1})$, then $T(y^{H+1}) = \sum_i T_i(y^{H+1})$.

The reduction of (10) to independent problems of the form (7i) greatly simplifies Step 2a, but Step 2b then becomes less straightforward. The required optimal dual solution for (10) must be synthesized from the optimal dual solutions of (7i). The relationship between the optimal primal solutions of (10) and (7i) is obvious, but the relationship between the optimal dual solutions requires some analysis. This analysis is as follows.

Convergence

In Benders

Master: LB

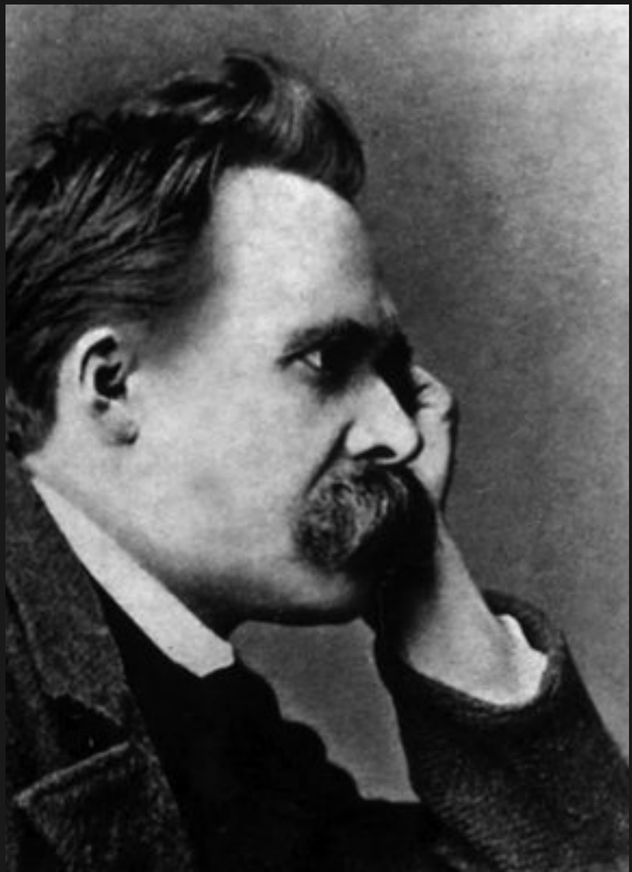
Feasible Subproblem: UB

Solving MIPs e LPs

In the mind

Individuation

Beyond good and evil



"No price is too high to pay for the privilege of owning yourself".

Thank you