OPTIMA Graduate Research Workshop
On Benders decomposition

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Mixed-Integer Programming (MIP) is a highly effective method for solving optimisation problems. The remarkable progress made by MIP solvers such as CPLEX and GUROBI in the last three decades has resulted in the widespread use of the technique as the approach of choice for tackling many industrial applications.

The complexity of some industrial applications exceeds the capabilities of these solvers. In order to advance the use of MIP, decomposition techniques are a popular approach. These methods divide the problem into smaller subproblems that fall into the solvers reach. Benders decomposition is one of the most successful techniques in this regard.

In this workshop, we provide a gentle introduction to the core concepts of Benders decomposition. This includes a hands-on tutorial on implementing an algorithm for a toy problem. In the second part of the workshop, we will discuss recent advances in the field and exemplify how the technique can be applied to solve large-scale industrial problems.

Participants are requested to bring their laptops and register for an account at Google Colaboratory (https://colab.research.google.com/) in order to participate in the hands-on tutorial.

Mixed-Integer Programming

$$
\begin{aligned}
\text{Maximise} \quad & c^t x + d^t y \\
\text{s.t.} \quad & Ax + By \leq b, \\
& Dy \leq e, \\
& x \in \mathbb{R}^n, \\
& y \in \mathbb{Z}^n.
\end{aligned}
$$

Example

$$
\begin{aligned}
\text{Maximise } & x_1 - x_2 + y_1 + y_2 \\
\text{s.t.} \quad & x_1 + x_2 + y_2 \leq 2, \\
& -x_1 - x_2 + y_1 \leq -1, \\
& x_1, x_2 \geq 0, \\
& y_1, y_2 \in \{0, 1\}.
\end{aligned}
$$

Implementation

```python
# !pip install gurobipy
import gurobipy as gp
from gurobipy import GRB

m = gp.Model("A mixed integer program")


#variables

indices = [1, 2]
x = m.addVars(indices , name = 'x',)
y = m.addVars(indices , name = 'y', vtype = GRB.BINARY)

#objective
m.setObjective( x[1] - x[2] + y[1] + y[2] , GRB.MAXIMIZE)

#constraints
cons1 = m.addConstr( x[1] + x[2] + y[2] <= 2)
cons2 = m.addConstr( -x[1] - x[2] + y[1] <= -1)

#solve
m.optimize()
```


Landoig method

ECONOMETRICA

VOLUME 28 | July, 1960 | NUMBER 3

AN AUTOMATIC METHOD OF SOLVING DISCRETE PROGRAMMING PROBLEMS

BY A. H. LAND AND A. G. DOIG

In the classical linear programming problem the behaviour of continuous, nonnegative variables subject to a system of linear inequalities is investigated. One possible generalization of this problem is to relax the continuity condition on the variables. This paper presents a simple numerical algorithm for the solution of programming problems in which some or all of the variables can take only discrete values. The algorithm requires no special techniques beyond those used in ordinary linear programming, and lends itself to automatic computing. Its use is illustrated on two numerical examples.

Land & Doig's Branch and Bound [Land and Doig, 1960].

```
Username
Academic license - for non-commercial use only - expires 2023-11-04
Gurobi Optimizer version 9.5.0 build v9.5.0rc5 (mac64[x86])
Thread count: 4 physical cores, 8 logical processors, using up to 8 threads
Optimize a model with 2 rows, 4 columns and 6 nonzeros
Model fingerprint: 0x252cfa2e
Variable types: 2 continuous, 2 integer (2 binary)
Coefficient statistics:
  Matrix range     [1e+00, 1e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [1e+00, 2e+00]
Presolve removed 2 rows and 4 columns
Presolve time: 0.00s
Presolve: All rows and columns removed

Explored 0 nodes (0 simplex iterations) in 0.00 seconds (0.00 work units)
Thread count was 1 (of 8 available processors)

Solution count 1: 3

Optimal solution found (tolerance 1.00e-04)
Best objective 3.000000000000e+00, best bound 3.000000000000e+00, gap 0.0000%
x[1] =  2.0
x[2] =  0.0
y[1] =  1.0
y[2] =  0.0
Optimal solution value: 3.0
[Finished in 0.5s]
```

Benders decomposition

Numerische Mathematik 4, 238—252 (1962)

Partitioning procedures for solving mixed-variables programming problems*

By

J. F. BENDERS**

I. Introduction

In this paper two slightly different procedures are presented for solving mixed-variables programming problems of the type

$$\max\{c^T x + f(y) \mid A x + F(y) \leq b,\ x \in R_p,\ y \in S\}, \tag{1.1}$$

where $x \in R_p$ (the p-dimensional Euclidean space), $y \in R_q$, and S is an arbitrary subset of R_q. Furthermore, A is an (m, p) matrix, $f(y)$ is a scalar function and $F(y)$ an m-component vector function both defined on S, and b and c are fixed vectors in R_m and R_p, respectively.

An example is the mixed-integer programming problem in which certain variables may assume any value on a given interval, whereas others are restricted to integral values only. In this case S is a set of vectors in R_q with integral-valued components. Various methods for solving this problem have been proposed by BEALE [1], GOMORY [9] and LAND and DOIG [11]. The use of integer variables, in particular for incorporating in the programming problem a choice from a set of alternative discrete decisions, has been discussed by DANTZIG [4].

For many years, Benders decomposition [Benders, 1962] was a brilliant theoretical result with little applicability in practice.

Industrial applications

It took 12 years for Benders decomposition to be first used on a practical application [Geoffrion and Graves, 1974].

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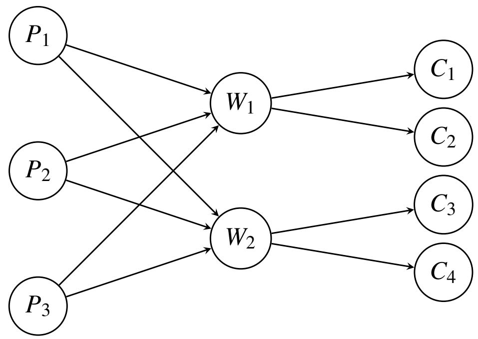
MULTICOMMODITY DISTRIBUTION SYSTEM DESIGN BY BENDERS DECOMPOSITION*†

A. M. GEOFFRION AND G. W. GRAVES§

University of California, Los Angeles

A commonly occurring problem in distribution system design is the optimal location of intermediate distribution facilities between plants and customers. A multicommodity capacitated single-period version of this problem is formulated as a mixed integer linear program. A solution technique based on Benders Decomposition is developed, implemented, and successfully applied to a real problem for a major food firm with 17 commodity classes, 14 plants, 45 possible distribution center sites, and 121 customer zones. An essentially optimal solution was found and proven with a surprisingly small number of Benders cuts. Some discussion is given concerning why this problem class appears to be so amenable to solution by Benders' method, and also concerning what we feel to be the proper professional use of the present computational technique.



The Benders reformulation

Consider again a generic MIP model:

$$\begin{aligned} \text{Maximise} \quad & c^t x + d^t y \\ \text{s.t.} \quad & Ax + By \leq b, \\ & Dy \leq e, \\ & x \geq 0, y \geq 0 \text{ and integer.} \end{aligned}$$

This problem can be rewritten as:

$$\text{Maximise}_{y \in Y} \{\, d^t y + \text{Maximise}_{x \geq 0} \{c^t x : Ax \leq b - By\} \,\},$$

with $Y = \{y \mid Dy \leq e,\ \ y \geq 0 \text{ and integer}\}$.

For fixed values of the y variables, the inner model is a linear program and is known as the Benders decomposition subproblem. Consider a tentative solution $\overline{y}$. Associating dual variables u to constraints $Ax \geq b - B\overline{y}$, we can write the dual version of subproblem as

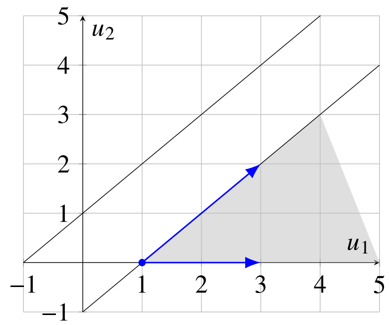
$$\text{Minimise}_{u \geq 0} \{u^t (b - B\overline{y}) : u^t A \geq c\}.$$

Let $\mathbb{F} = \{u \mid u \geq 0\ ,\ u^t A \geq c\}$. We assume that $\mathbb{F}$ is not empty for it would correspond to a primal problem either infeasible or unbounded.

$\mathbb{F}$ is therefore composed of extreme points u^p (for $p = 1 \ldots P$) and extreme rays r^q (for $q = 1 \ldots Q$).[1]

Example

$$\begin{aligned} \text{Minimise} \quad & (2 - \overline{y}_2)u_1 + (-1 - \overline{y}_1)u_2 \\ & u_1 - u_2 \geq 1, \\ & u_1 - u_2 \geq -1, \\ & u_1, u_2 \geq 0. \end{aligned}$$



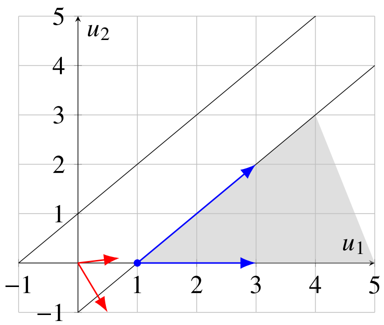
[1]To simplify the notation in the following, assume that r^q and u^p are row vectors.

Using duality theory, the primal and dual formulations can be interchanged.

$$\text{Maximise}_{y \in Y} \{ d^t \overline{y} + \text{Minimise}_{u \geq 0} \{ u^t (b - B) y : u^t A \geq c \} \}.$$

The feasible space of the dual subproblem does not depend on the choice of y.

The objective function of the subproblem depends on the choice $\overline{y}$ and can be either bounded or unbounded.



In the first case, we obtain an extreme point u^p. In the latter situation, there is a direction r^q for which $r^q(b - B\overline{y}) < 0$. This direction is either extreme or a convex combination of extreme directions [Costa et al., 2009].

All tentative solutions $\overline{y}$ that provide an unbounded solution to the dual subproblem imply an infeasible subproblem and must be avoided. This is done by adding constraints:

$$r^q(b - By) \geq 0, \qquad q = 1 \ldots Q. \tag{1}$$

Adding constraints (1) to the formulation.

$$\begin{aligned} \text{Maximise}_{y \in Y} \ & d^t y + \{\text{Minimise}_{u \geq 0} \{ u(b - By) : uA \geq c \} \\ \text{s.t.} \quad & r^q(b - By) \geq 0, \qquad q = 1 \ldots Q. \end{aligned}$$

Now the solution must be one of the extreme points $u_p, \ p = 1, \ldots, P$.

$$\begin{aligned} \text{Maximise}_{y \in Y} \ & d^t y + \{\text{Minimise}\{ u^p(b - By) : p = 1, \ldots, P \} \\ \text{s.t.} \quad & r^q(b - By) \geq 0, \quad q = 1, \ldots, Q. \end{aligned}$$

Which can be linearised with the use of a continuous variable z as:

$$\begin{aligned} \text{Maximise}_{y \in Y} \ & d^t y + z \\ \text{s.t.} \quad & r^q(b - By) \geq 0, \quad q = 1, \ldots, Q, \\ & z \leq u^p(b - By), \quad p = 1, \ldots, P. \end{aligned}$$

This is the **Benders reformulation**.

Benders decomposition algorithm

The idea of the **Benders decomposition algorithm** is to ignore most of the initial constraints of the Benders reformulation and generate them as needed.

We start with a relaxed version of the reformulation called the *relaxed master problem.*

$$\begin{aligned} \text{Maximise}_{y \in Y} \; & d^t y + z \\ \text{s.t.} \quad & z \geq -M. \end{aligned}$$

This gives us tentative values for the integer variables, $\overline{y}$, that are sent to the subproblem:

$$\begin{aligned} \text{Minimise}_{u \geq 0} \; & u^t (b - B\overline{y}) \\ \text{s.t.} \quad & u^t A \geq c. \end{aligned}$$

which returns us an extreme ray r^q or an extreme point u^p that can be used to generate a feasibility or optimality cut for the master problem (and the method iterates).

Convergence

Every time the Master problem is run, we have a dual bound for the problem (i.e., we have solved a relaxation of the problem). Every time the subproblem finds an extreme point, we have found a primal bound for the problem (i.e.,we have found a feasible solution).

The method converges when the last dual bound found is equal (up to a tolerance) to the best primal bound found during the process.

Improving convergence

There are many ways to fasten the convergence of a Benders decomposition algorithm [McDaniel and Devine, 1977, Magnanti and Wong, 1981, Papadakos, 2008, Costa et al., 2012, Crainic et al., 2021]

See [Costa, 2005, Rahmaniani et al., 2017] for surveys.

Example

Write the full Benders reformulation for the following MIP.

$$\begin{aligned} \text{Maximise } & x_1 - x_2 + y_1 + y_2 \\ \text{s.t.} \quad & x_1 + x_2 + y_2 \leq 2, \\ & -x_1 - x_2 + y_1 \leq -1, \\ & x_1, x_2 \geq 0, \\ & y_1, y_2 \in \{0, 1\}. \end{aligned}$$

Implement the Benders decomposition algorithm.

The following templates can be found in file 'code/OPTIMAGR.ipynb'.

Master Problem template:

```
import gurobipy as gp
from gurobipy import GRB

m = gp.Model(#fill)
y = m.addVars(#fill)
z = m.addVar(#fill, ub=10000)
m.setObjective(#fill)
```

Subproblem template:

```
import gurobipy as gp
from gurobipy import GRB

s = gp.Model(#fill)
s.Params.InfUnbdInfo = 1 #allow to get dual ray
x = s.addVars(#fill)
cons1 = s.addConstr(#fill, name="cons1")
cons2 = s.addConstr(#fill, name="cons2")
```

External loop template:

```python
import gurobipy as gp
from gurobipy import GRB

#Benders loop
LB = -100000; UB = 100000

while(UB - LB >= 0.00001):
  #optimize Master
  m.optimize()
  UB = m.objVal

  #update the right hand side of the subproblem
  cons1.rhs = #fill
  cons2.rhs = #fill

  #optimize sub
  s.optimize()

  #generate cuts
  if s.status == 3:
    print("Infeasibility cut")
    # get the dual ray
    u1 = cons1.getAttr('FarkasDual')
    u2 = cons2.getAttr('FarkasDual')
    m.addConstr(#fill)  #add the new cut
  else:
    print("Feasibility cut")
    # get the dual ray
    u1 = cons1.getAttr('Pi')
    u2 = cons2.getAttr('Pi')
    m.addConstr( #fill)

    #fill (update the best lower bound if necessary).
```


Notes:

An implementation using lazy constraints:

```
import gurobipy as gp
from gurobipy import GRB


#Initial Subproblem:
def initial_sub():
  s = gp.Model()
  s.Params.InfUnbdInfo = 1
  s.Params.OutputFlag= 0
  x = s.addVars(2, name="x")
  cons1 = s.addConstr(x[0] + x[1] <= 0, name="cons1")
  cons2 = s.addConstr(-x[0] -x[1] <= 0, name="cons2")
  s.setObjective(x[0] - x[1],GRB.MAXIMIZE)
  return s

#Update subproblem
def update_subproblem(s,y1,y2):
  cons1 = s.getConstrByName("cons1")
  cons2 = s.getConstrByName("cons2")
  cons1.rhs = 2 - y2
  cons2.rhs = -1 - y1

def generate_cuts(model,where):
  if where == GRB.Callback.MIPSOL:
    valsy = model.cbGetSolution(model._y)
    print(valsy)
    update_subproblem(s,valsy[0],valsy[1])
    s.optimize()
    if s.status == 3:
      print("Infeasibility constraint")
      cons1 = s.getConstrByName("cons1")
      cons2 = s.getConstrByName("cons2")
      u1 = cons1.getAttr('FarkasDual')
      u2 = cons2.getAttr('FarkasDual')
      expr = u1*(2 - y[1]) + u2*(-1  - y[0])
      print(expr, ">=", "0")
      model.cbLazy( expr >= 0)
```



```
    else:
      print("Feasibility constraint")
      cons1 = s.getConstrByName("cons1")
      cons2 = s.getConstrByName("cons2")
      u1 = cons1.getAttr('Pi')
      u2 = cons2.getAttr('Pi')
      expr = u1*(2 - y[1]) + u2*(-1  - y[0])
      print(expr, ">=", "z")
      model.cbLazy( expr >= m._z)

s = initial_sub()
s.write("model.lp")

#master
m = gp.Model()
y = m.addVars(2, vtype = GRB.BINARY, name="y")
z = m.addVar(name="z", ub=1000)

m.setObjective(y[0]  + y[1] + z, GRB.MAXIMIZE)

m.Params.LazyConstraints = 1
m._y = y
m._z = z
m.Params.OutputFlag= 0
m.optimize(generate_cuts)

s.write("model.lp")


# Display optimal values of decision variables
for v in m.getVars():
    if v.x > 1e-6:
        print(v.varName, v.x)

# Display optimal solution value
print('Total profit: ', m.objVal)
```

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