Anders Lykkehoy
April 8, 2017
Math 440
Test 2

1.
a. Lagrange Interpolation:
$L_1(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-2.5)(x-4)}{(2-2.5)(2-4)} = \frac{x^2-6.5x+10}{1} = x^2 - 6.5x + 10$
$L_2(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{x^2-6x+8}{-0.75} = \frac{-4}{3}x^2 + 8x - \frac{32}{3}$
$L_3(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \frac{x^2-4.5x+5}{3} = \frac{1}{3}x^2 - 1.5x + \frac{5}{3}$
$P(x) = y_1L_1(x) + y_2L_2(x) + y_3L_3(x)$
$P(x) = 0.5(x^2 - 6.5x + 10) + 0.4(\frac{-4}{3}x^2 + 8x - \frac{32}{3}) + 0.25(\frac{1}{3}x^2 - 1.5x + \frac{5}{3})$
$P(x) = 0.05x^2 - 0.425x + 1.15$

b. Newton Interpolation:

| x | y | $f[x_1]$ | $f[x_1x_2]$ | $f[x_1x_2x_3]$ |
|---|---|---|---|---|
| 2 | 0.5 | 0.5 | $\frac{0.4-0.5}{2.5-2} = -0.2$ | $\frac{-0.1-(-0.2)}{4-2} = 0.05$ |
| 2.5 | 0.4 | 0.4 | $\frac{0.25-0.4}{4-2.5} = -0.1$ | |
| 4 | 0.25 | 0.25 | | |

$P(x) = f[x_1] + f[x_1x_2](x - x_1) + f[x_1x_2x_3](x - x_1)(x - x_2)$
$P(x) = 0.5 + (-0.2)(x - 2) + (0.05)(x - 2)(x - 2.5)$
$P(x) = 0.05x^2 - 0.425x + 1.15$

c. Because the reduced Lagrange Interpolation Polynomial is the same as the reduced Newton Interpolation, the 2 polynomials are identical.

2.
a.
$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 3(\frac{e-1}{1} - 1) \\ 3(\frac{e^2-e}{1} - \frac{e-1}{1}) \\ 3(\frac{e^3-e^2}{1} - \frac{e^2-e}{1}) \\ 3(e^3 - \frac{e^3-e^2}{1}) \end{bmatrix} = \begin{bmatrix} 3e - 6 \\ 3e^2 - 6e + 3) \\ 3e^3 - 6e^2 + 3e) \\ 3e^2 \end{bmatrix}$$
Therefore $c_1 = 1.6068, c_2 = -1.0588, c_3 = 9.3682, c_4 = 6.3995$.

For the $d$'s:
$d_1 = \frac{-1.0588-1.6068}{3(1)} = \frac{-2.6656}{3} = -0.88853$
$d_2 = \frac{9.3682-(-1.0588)}{3(1)} = \frac{10.427}{3} = 3.4757$
$d_3 = \frac{6.3995-9.3682}{3(1)} = \frac{-2.9687}{3} = -0.98957$

For the $b$'s:
$b_1 = \frac{e-1}{1} - \frac{1}{3}(2(1.6068) + -1.0588) = 1$

$b_2 = \frac{e^2-e}{1} - \frac{1}{3}(2(-1.0588) + 9.3682) = 2.2539$
$b_3 = \frac{e^3-e^2}{1} - \frac{1}{3}(2(9.3682) + 6.3995) = 4.3178$

Putting everything together we get:

$$S(x) = \begin{cases} S_0(x) = 1 + 1(x-0) + 1.6068(x-0)^2 + (-0.88853)(x-0)^3, & \text{on } [0,1] \\ S_1(x) = e + 2.2539(x-1) + (-1.0588)(x-1)^2 + 3.4757(x-1)^3, & \text{on } [1,2] \\ S_3(x) = e^2 + 4.3178(x-2) + 9.3682(x-2)^2 + (-0.98957)(x-2)^3, & \text{on } [2,3] \end{cases}$$

b.

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3, & \text{on } [0,1] \\ S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3, & \text{on } [1,2] \\ S_3(x) = a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3, & \text{on } [2,3] \end{cases}$$

$\int_0^3 S(x)dx = \int_0^1 S_0(x)dx + \int_1^2 S_1(x)dx + \int_2^3 S_2(x)dx$

$\int_0^1 S_0(x)dx = \int_0^1 a_0 + b_0(x) + c_0(x)^2 + d_0(x)^3 dx =$
$a_0(x) + \frac{1}{2}b_0(x)^2 + \frac{1}{3}c_0(x)^3 + \frac{1}{4}d_0(x)^4\Big|_0^1 = a_0 + \frac{1}{2}b_0 + \frac{1}{3}c_0 + \frac{1}{4}d_0$

$\int_1^2 S_1(x)dx = \int_1^2 a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3 dx =$
$a_1(x-1) + \frac{1}{2}b_1(x-1)^2 + \frac{1}{3}c_1(x-1)^3 + \frac{1}{4}d_1(x-1)^4\Big|_1^2 = a_1 + \frac{1}{2}b_1 + \frac{1}{3}c_1 + \frac{1}{4}d_1$

$\int_2^3 S_2(x)dx = \int_2^3 a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3 dx =$
$a_2(x-2) + \frac{1}{2}b_2(x-2)^2 + \frac{1}{3}c_2(x-2)^3 + \frac{1}{4}d_2(x-2)^4\Big|_2^3 = a_2 + \frac{1}{2}b_2 + \frac{1}{3}c_2 + \frac{1}{4}d_2$

Therefor: $\int_0^3 S(x)dx = a_0+\frac{1}{2}b_0+\frac{1}{3}c_0+\frac{1}{4}d_0+a_1+\frac{1}{2}b_1+\frac{1}{3}c_1+\frac{1}{4}d_1+a_2+\frac{1}{2}b_2+\frac{1}{3}c_2+\frac{1}{4}d_2$. After factoring this becomes: $(a_0+a_1+a_2)+\frac{1}{2}(b_0+b_1+b_2)+\frac{1}{3}(c_0+c_1+c_2)+\frac{1}{3}(c_0+c_1+c_2)+\frac{1}{4}(d_0+d_1+d_2)$

3. First start with $f(x+h)$ and $f(x+2h)$. Using Taylor expansions we can find:
$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f^2(x) + \frac{h^3}{6}f^3(C_1)$
$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f^2(x) + \frac{8h^3}{6}f^3(C_2)$
If we add the two expansions after the top expansion is multiplied by 4 and the bottom expansion is multiplied by $-1$ we get:
$4f(x+h) = 4f(x) + 4hf'(x) + \frac{4h^2}{2}f^2(x) + \frac{4h^3}{6}f^3(C_1)$
$-f(x+2h) = -f(x) - 2hf'(x) - \frac{4h^2}{2}f^2(x) - \frac{8h^3}{6}f^3(C_2)$

---

$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + \frac{4h^3}{6}f^3(C_1) - \frac{8h^3}{6}f^3(C_2)$

This can be simplified down to:
$\frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} = f'(x) + \frac{1h^2}{3}f^3(C_1) - \frac{2h^2}{3}f^3(C_2)$
When $h$ is very small this becomes:
$\frac{-f(x+2h)+4f(x+h)-3f(x)}{2h} = f'(x)$

Therefore the error is: $\frac{1h^2}{3}f^3(C_1) - \frac{2h^2}{3}f^3(C_2)$

4.
a. Simpson's rule with $n = 6$:
$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | 0 | $\frac{1}{6}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | $\frac{2}{3}$ | $\frac{5}{6}$ | 1 |
| y | 1 | 0.75 | 0.6 | 0.5 | $\frac{3}{7}$ | 0.375 | $\frac{1}{3}$ |

$I \approx \frac{h}{3}(y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)$
$I \approx \frac{\frac{1}{6}}{3}(1 + \frac{1}{3} + 4(0.75 + 0.5 + 0.375) + 2(\frac{1}{3} + \frac{2}{3}) = \frac{1}{18}(\frac{4}{3} + 6 + 2) = \frac{14}{27}$

$Error = \frac{(b-a)h^4}{180}f^4(c) = \frac{(1-2)(\frac{1}{6})^4}{180}f^4(c)$
$f(x) = \frac{1}{2x+1}$
$f^1(x) = \frac{-2}{(2x+1)^2}$
$f^2(x) = \frac{8}{(2x+1)^3}$
$f^3(x) = \frac{-48}{(2x+1)^4}$
$f^4(x) = \frac{384}{(2x+1)^5}$
$Error = \frac{1}{233280}f^4(0) = \frac{384}{233280} \approx 0.001646$

b. $10^{-10} \geq Error$
$10^{-10} \geq \frac{(b-a)h^4}{180}f^4(c)$
$10^{-10} \geq \frac{(1-0)(\frac{1-0}{n})^4}{180}384 = \frac{384}{180n^4}$
$n^4 \geq \frac{384}{180}10^{-10}$
$n \geq 382.18$
$n = 383$

5.
a. Given:

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Using the equation: $X_{i+1} = D^{-1}(b - (L+U)X_i)$ the first iteration is:

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.09091 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.125 \end{bmatrix} \left( \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix} - \left( \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -0 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.18182 \\ -1.1 \\ 1.62500 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.09091 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.125 \end{bmatrix} \left( \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix} - \left( \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -0 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} 0.5 \\ 2.18182 \\ -1.1 \\ 1.62500 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_4^{(2)} \end{bmatrix} = \begin{bmatrix} 1.03818 \\ 1.77500 \\ -0.81932 \\ 0.91932 \end{bmatrix}$$

b. Using the equation: $X_{i+1} = D^{-1}(b - UX_i - LX_{i+1})$:

$$\begin{bmatrix} x_1^{(i+1)} \\ x_2^{(i+1)} \\ x_3^{(i+1)} \\ x_4^{(i+1)} \end{bmatrix} = \begin{bmatrix} \frac{6+x_2^{(i)}-2x_3^{(i)}}{10} \\ \frac{25+x_1^{(i+1)}+x_3^{(i)}-3x_4^{(i)}}{11} \\ \frac{-11-2x_1^{(i+1)}+x_2^{(i+1)}+x_4^{(i)}}{10} \\ \frac{15-3x_2^{(i+1)}+x_3^{(i+1)}}{8} \end{bmatrix}$$

Given:

$$\begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \\ x_4^{(0)} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

First iteration:

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{6+1-2(1)}{10} \\ \frac{25+0.5+1-3(1)}{11} \\ \frac{-11-2(0.5)+\frac{47}{22}+1}{10} \\ \frac{15-3(\frac{47}{22})+\frac{-39}{44}}{8} \end{bmatrix} = \begin{bmatrix} 0.5 \\ \frac{47}{22} \\ \frac{-39}{44} \\ \frac{339}{352} \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_4^{(2)} \end{bmatrix} = \begin{bmatrix} \frac{6+\frac{47}{22}-2\frac{-39}{44}}{10} \\ \frac{25+x_1^{(i+1)}+x_3^{(i)}-3x_4^{(i)}}{11} \\ \frac{-11-2x_1^{(i+1)}+x_2^{(i+1)}+x_4^{(i)}}{10} \\ \frac{15-3x_2^{(i+1)}+x_3^{(i+1)}}{8} \end{bmatrix} = \begin{bmatrix} 0.63636 \\ 1.9873 \\ -0.93223 \\ 1.0132 \end{bmatrix}$$