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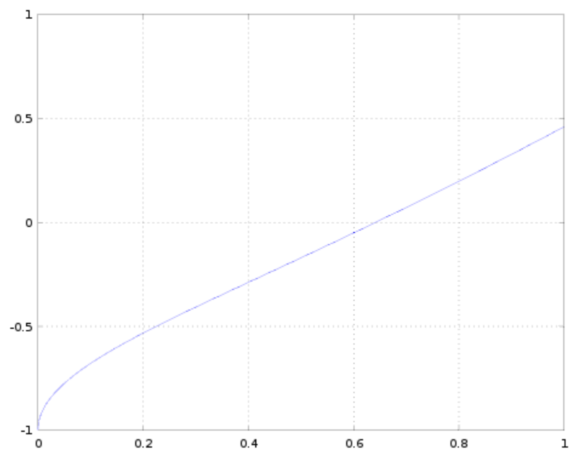
Math 440
Test 1

1.

- Graph of $f(x) = \sqrt{x} - cos(x)$ on $[0, 1]$:



From the graph we can see $f(0)$ is negative while $f(1)$ is positive, and $f`(x)$ is positive over the interval. This points to there being only one root in $[0, 1]$.

- Here is a table of the first four iterations of the bisection method:

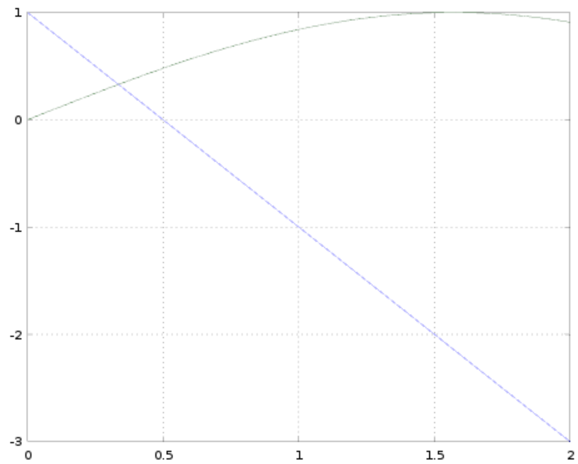
i	a	b	c	$f(c)$	error
0	0	1	$\frac{1}{2}$	-0.17048	0.17048
1	$\frac{1}{2}$	1	$\frac{3}{4}$	0.13434	0.13434
2	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{8}$	-0.020394	0.020394
3	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{11}{16}$	0.056321	0.056321

- The number of steps n needed to obtain an accuracy within 6 decimal places:
$\frac{b-a}{2^{n+1}} \leq \frac{1}{2}10^{-P}$
$\frac{1-0}{2^{n+1}} \leq \frac{1}{2}10^{-6}$
$\frac{1}{2^{n+1}} \leq \frac{1}{2}10^{-6}$
$\frac{1}{2^n} \leq 10^{-6}$
$2^n \leq 10^6$
$n \geq \frac{\ln(10^6)}{\ln(2)}$

$n \geq 19.932$
$n = 20$

2.

- Graph of $y = 1 - 2x$ and $y = sin(x)$ on the interval $[0, 2]$:



- To prove $y = 1 - 2x$ and $y = sin(x)$ intersect at exactly one point, first look at $y = sin(x)$. The range of $y = sin(x)$ is between 1 and -1, and the only times the range of $y = 1 - 2x$ is similar is when $0 \leq x \leq 1$. Looking at $x = 0$, $y = 1 - 2(0) = 1$ and $y = sin(0) = 0$. Looking at $x = 1$, $y = 1 - 2(1) = -1$ and $y = sin(1) \geq 0$. This shows there are some number of points in $0 \leq x \leq 1$ where $y = 1 - 2x$ and $y = sin(x)$ intersect. To know how many points look at their derivatives, $y` = -2$ and $y` = cos(x)$ respectively. Because the derivative of $y = 1 - 2x$ is always negative and the derivative of $y = sin(x)$ is positive in the interval $0 \leq x \leq 1$, the two function share only one intersection point.

- Matlab program using the Bisection method to find where the equations intersect:

problem_2.m

```matlab
a = 0;
b = 1;
%calculate the number of steps needed for the error to be less than 10^-4
n = ceil(log(10^4)/log(2));

f1 = @(x) 1 - 2.*x;
f2 = @(x) sin(x);
f3 =  @(x) f1(x) - f2(x);

bisect_ForLoop(f3, a, b, n);
```

Command Window

```
>> problem_2
step | a | f(a) | c | f(c) | b | f(b)
0 | 0.000000 | 1.000000 | 0.500000 | -0.479426 | 1.000000 | -1.841471
1 | 0.000000 | 1.000000 | 0.250000 | 0.252596 | 0.500000 | -0.479426
2 | 0.250000 | 0.252596 | 0.375000 | -0.116273 | 0.500000 | -0.479426
3 | 0.250000 | 0.252596 | 0.312500 | 0.067561 | 0.375000 | -0.116273
4 | 0.312500 | 0.067561 | 0.343750 | -0.024520 | 0.375000 | -0.116273
5 | 0.312500 | 0.067561 | 0.328125 | 0.021481 | 0.343750 | -0.024520
6 | 0.328125 | 0.021481 | 0.335938 | -0.001529 | 0.343750 | -0.024520
7 | 0.328125 | 0.021481 | 0.332031 | 0.009973 | 0.335938 | -0.001529
8 | 0.332031 | 0.009973 | 0.333984 | 0.004221 | 0.335938 | -0.001529
9 | 0.333984 | 0.004221 | 0.334961 | 0.001346 | 0.335938 | -0.001529
10 | 0.334961 | 0.001346 | 0.335449 | -0.000092 | 0.335938 | -0.001529
11 | 0.334961 | 0.001346 | 0.335205 | 0.000627 | 0.335449 | -0.000092
12 | 0.335205 | 0.000627 | 0.335327 | 0.000268 | 0.335449 | -0.000092
13 | 0.335327 | 0.000268 | 0.335388 | 0.000088 | 0.335449 | -0.000092
14 | 0.335388 | 0.000088 | 0.335419 | -0.000002 | 0.335449 | -0.000092
>> |
```

From the print out we can see after the 14th step we have an error of 0.000088 when computing $f(0.335388)$.

3.

- Matlab program using Newton's method:

problem_3_a.m | problem_3_b.m | problem_3_c.m

```matlab
f = @(x) x.^3 - 5.*x.^2 + 3.*x - 7;
fs = @(x) 3.*x.^2 - 10.*x + 3;

n = 10;
x0 = 5;

newtonMethod(f, fs, x0, n);
```

Command Window

```
>> problem_3_a
1 | 5.00000000 | 4.71428571 | 8.00000000 | -0.28571429
2 | 4.71428571 | 4.67908903 | 0.79300292 | -0.03519669
3 | 4.67908903 | 4.67857362 | 0.01128263 | -0.00051541
4 | 4.67857362 | 4.67857351 | 0.00000240 | -0.00000011
5 | 4.67857351 | 4.67857351 | 0.00000000 | -0.00000000
6 | 4.67857351 | 4.67857351 | 0.00000000 | -0.00000000
7 | 4.67857351 | 4.67857351 | 0.00000000 | 0.00000000
8 | 4.67857351 | 4.67857351 | 0.00000000 | 0.00000000
9 | 4.67857351 | 4.67857351 | 0.00000000 | 0.00000000
10 | 4.67857351 | 4.67857351 | 0.00000000 | 0.00000000
>>
```

From the table we can see there is a root at $x = 4.67857351$. The root was found after 5 steps.

- A formula for the fifth root of any positive real number can be found by: $\sqrt[5]{A} = x$
  $A = x^5$
  $0 = x^5 - A$
  If $f(x) = x^5 - A$ and $f'(x) = 5x^4$, we can create the equation for the Newton method, $x_{i+1} = x_i - \frac{x_i^5 - A}{5x_i^4}$. Here is a Matlab program using these equations to compute $\sqrt[5]{3}$:

problem_3_a.m | problem_3_b.m | problem_3_c.m

```
A = 3;

f = @(x) x.^5 - A;
fs = @(x) 5.*x.^4;

n = 10;
x0 = 1;

newtonMethod(f, fs, x0, n);
```

Command Window

```
>> problem_3_b
1 | 1.00000000 | 1.40000000 | -2.00000000 | 0.40000000
2 | 1.40000000 | 1.27618492 | 2.37824000 | -0.12381508
3 | 1.27618492 | 1.24715013 | 0.38507307 | -0.02903479
4 | 1.24715013 | 1.24573417 | 0.01712765 | -0.00141597
5 | 1.24573417 | 1.24573094 | 0.00003885 | -0.00000323
6 | 1.24573094 | 1.24573094 | 0.00000000 | -0.00000000
7 | 1.24573094 | 1.24573094 | 0.00000000 | 0.00000000
8 | 1.24573094 | 1.24573094 | 0.00000000 | 0.00000000
9 | 1.24573094 | 1.24573094 | 0.00000000 | 0.00000000
10 | 1.24573094 | 1.24573094 | 0.00000000 | 0.00000000
>>
```

From the table we can see $\sqrt[5]{3} = 1.24573094$ was found after 6 steps.

- A formula for the reciprocal for any non-zero positive real number can be found by:
  $\frac{1}{A} = x$
  $\frac{A^2}{A} = A^2x$
  $A = A^2x$
  $0 = A^2x - A$
  So if $f(x) = A^2x - A$ and $f'(x) = A^2$, we can create the equation for the Newton method, $x_{i+1} = x_i - \frac{A^2x - A}{A^2}$. Here is a Matlab program using these equations to compute $\frac{1}{2016}$:

problem_3_a.m problem_3_b.m problem_3_c.m

```
A = 2016;

f = @(x) (A.^2).*x - A;
fs = @(x) A.^2;

n = 10;
x0 = 1;

newtonMethod(f, fs, x0, n);
```

Command Window

```
>> problem_3_c
1 | 1.00000000 | 0.00049603 | 4062240.00000000 | -0.99950397
2 | 0.00049603 | 0.00049603 | -0.00000000 | 0.00000000
3 | 0.00049603 | 0.00049603 | 0.00000000 | -0.00000000
4 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
5 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
6 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
7 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
8 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
9 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
10 | 0.00049603 | 0.00049603 | 0.00000000 | 0.00000000
>>
```

From the table we can see $\frac{1}{2016} = 0.00049603$ was found after only 2 steps.

4.

-

- If $g(x) = (x^2 - 1)/3$ has a fixed point in $[-1, 1]$ then there must be a point where $g(r) = (r^2 - 1)/3 = r$. This can be rewritten as $r^2 - 3r - 1 = 0$. Testing our end points for r we can see that when $r = -1$ the function is equal to 3, but when $r = -1$ the function is equal to -3. Knowing that $g(x)$ is differentiable on $[-1, 1]$ and there is a sign change from our end points, we know there is some point r that completes our function $r^2 - 3r - 1 = 0$. Therefore the function $g(x) = (x^2 - 1)/3$ does have a fixed point on $[-1, 1]$.

- Here is a Matlab program using the fixed point iteration formula:

problem_4.m

```
g = @(x) (x.^2 - 1) ./ 3;
x0 = 0;

fpi(g, x0, 6);

```

Command Window

```
>> problem_4
i|  x_i     | g(x_i) ||x(i)-g(x(i))|
1|  0.00000000    | -0.33333333 | 0.33333333
2|  -0.33333333    | -0.29629630 | 0.03703704
3|  -0.29629630    | -0.30406950 | 0.00777321
4|  -0.30406950    | -0.30251391 | 0.00155559
5|  -0.30251391    | -0.30282844 | 0.00031453
6|  -0.30282844    | -0.30276498 | 0.00006347
>> |
```

From the table we can see the fixed point $x = -0.30276498$ with $x_i - |g(x_i)| < 10^{-4}$.

5.

- If $g_1(r) = r$ then:
  $r = \frac{1}{2}(10 - r^3)^{\frac{1}{2}}$
  $(2r)^2 = 10 - r^3$
  $4r^2 + r^3 - 10 = 0$
  Since the right hand side of the equation is $f(r)$ we end up with $f(r) = 0$.

  If $g_2(r) = r$ then:
  $r = (\frac{10}{4+x})^{\frac{1}{2}}$

$r^2 = \frac{10}{4+x})$
$r^2(4+r) = 10$
$r^3 + 4r^2 = 10$
$r^3 + 4r^2 - 10 = 0$
Since the right hand side of the equation is $f(r)$ we end up with $f(r) = 0$.

If $g_r(r) = r$ then:
$r = r - \frac{x^3+4x^2-10}{3x^2+8x}$
$0 = -\frac{x^3+4x^2-10}{3x^2+8x}$
$-(3x^2+8x)0 = x^3 + 4x^2 - 10$
$0 = x^3 + 4x^2 - 10$
Since the right left side of the equation is $f(r)$ we end up with $0 = f(r)$.

- For $g_1(x)$:

problem_5_1.m | problem_5_2.m | problem_5_3.m

```
g1 = @(x) 0.5 .* (10 - x.^3).^(0.5);
x0 = 1.5;
n = 30;

r = 1.36523001;
gs1 = @(x) (-3.*x.^2) ./ (4.*(10 - x.*3).^(0.5));
%gs1(r)

fpi(g1, x0, n);
```

Com

Command Window

```
>> problem_5_1
i|  x_i     | g(x_i) ||x(i)-g(x(i))|
1|  1.50000000    | 1.28695377 | 0.21304623
2|  1.28695377    | 1.40254080 | 0.11558704
3|  1.40254080    | 1.34545837 | 0.05708243
4|  1.34545837    | 1.37517025 | 0.02971188
5|  1.37517025    | 1.36009419 | 0.01507606
6|  1.36009419    | 1.36784697 | 0.00775277
7|  1.36784697    | 1.36388700 | 0.00395996
8|  1.36388700    | 1.36591673 | 0.00202973
9|  1.36591673    | 1.36487822 | 0.00103852
10|  1.36487822    | 1.36541006 | 0.00053184
11|  1.36541006    | 1.36513782 | 0.00027224
12|  1.36513782    | 1.36527721 | 0.00013939
13|  1.36527721    | 1.36520585 | 0.00007136
14|  1.36520585    | 1.36524238 | 0.00003653
15|  1.36524238    | 1.36522368 | 0.00001870
16|  1.36522368    | 1.36523326 | 0.00000958
17|  1.36523326    | 1.36522835 | 0.00000490
18|  1.36522835    | 1.36523086 | 0.00000251
19|  1.36523086    | 1.36522958 | 0.00000128
20|  1.36522958    | 1.36523024 | 0.00000066
21|  1.36523024    | 1.36522990 | 0.00000034
22|  1.36522990    | 1.36523007 | 0.00000017
23|  1.36523007    | 1.36522998 | 0.00000009
24|  1.36522998    | 1.36523003 | 0.00000005
25|  1.36523003    | 1.36523001 | 0.00000002
26|  1.36523001    | 1.36523002 | 0.00000001
27|  1.36523002    | 1.36523001 | 0.00000001
28|  1.36523001    | 1.36523001 | 0.00000000
29|  1.36523001    | 1.36523001 | 0.00000000
30|  1.36523001    | 1.36523001 | 0.00000000
>> |
```

For $g_2(x)$:

problem_5_1.m | problem_5_2.m | problem_5_3.m

```
g2 = @(x) (10 ./ (4 + x)) .^ (0.5);
x0 = 1.5;
n = 30;

r = 1.36523001;
gs2 = @(x) 0.5 .* (10./(4 + x))^(-0.5) .* (-10 ./((4 + x).^2));
%gs2(r)

fpi(g2, x0, n);
```

Command Window

```
>> problem_5_2
i|  x_i     | g(x_i) ||x(i)-g(x(i))|
1|  1.50000000    | 1.34839972 | 0.15160028
2|  1.34839972    | 1.36737637 | 0.01897665
3|  1.36737637    | 1.36495702 | 0.00241936
4|  1.36495702    | 1.36526475 | 0.00030773
5|  1.36526475    | 1.36522559 | 0.00003915
6|  1.36522559    | 1.36523058 | 0.00000498
7|  1.36523058    | 1.36522994 | 0.00000063
8|  1.36522994    | 1.36523002 | 0.00000008
9|  1.36523002    | 1.36523001 | 0.00000001
10|  1.36523001    | 1.36523001 | 0.00000000
11|  1.36523001    | 1.36523001 | 0.00000000
12|  1.36523001    | 1.36523001 | 0.00000000
13|  1.36523001    | 1.36523001 | 0.00000000
14|  1.36523001    | 1.36523001 | 0.00000000
15|  1.36523001    | 1.36523001 | 0.00000000
16|  1.36523001    | 1.36523001 | 0.00000000
17|  1.36523001    | 1.36523001 | 0.00000000
18|  1.36523001    | 1.36523001 | 0.00000000
19|  1.36523001    | 1.36523001 | 0.00000000
20|  1.36523001    | 1.36523001 | 0.00000000
21|  1.36523001    | 1.36523001 | 0.00000000
22|  1.36523001    | 1.36523001 | 0.00000000
23|  1.36523001    | 1.36523001 | 0.00000000
24|  1.36523001    | 1.36523001 | 0.00000000
25|  1.36523001    | 1.36523001 | 0.00000000
26|  1.36523001    | 1.36523001 | 0.00000000
27|  1.36523001    | 1.36523001 | 0.00000000
28|  1.36523001    | 1.36523001 | 0.00000000
29|  1.36523001    | 1.36523001 | 0.00000000
30|  1.36523001    | 1.36523001 | 0.00000000
>>
```

For $g_3(x)$:

problem_5_1.m   problem_5_2.m   problem_5_3.m

```
g3 = @(x) x - (x.^3 + 4.*x.^2 - 10) ./ (3.*x.^2 + 8.*x);
x0 = 1.5;
n = 30;

r = 1.36523001;
gs3 = @(x) ((6.*x).*(x.^3 + 4.*x.^2 - 10)) ./ (x.^2 .* (3.*x + 8).^2);
%gs3(r)

fpi(g3, x0, n);
```

Command Window

```
>> problem_5_3
i|  x_i    | g(x_i) ||x(i)-g(x(i))|
1|  1.50000000    | 1.37333333 | 0.12666667
2|  1.37333333    | 1.36526201 | 0.00807132
3|  1.36526201    | 1.36523001 | 0.00003200
4|  1.36523001    | 1.36523001 | 0.00000000
5|  1.36523001    | 1.36523001 | 0.00000000
6|  1.36523001    | 1.36523001 | 0.00000000
7|  1.36523001    | 1.36523001 | 0.00000000
8|  1.36523001    | 1.36523001 | 0.00000000
9|  1.36523001    | 1.36523001 | 0.00000000
10|  1.36523001    | 1.36523001 | 0.00000000
11|  1.36523001    | 1.36523001 | 0.00000000
12|  1.36523001    | 1.36523001 | 0.00000000
13|  1.36523001    | 1.36523001 | 0.00000000
14|  1.36523001    | 1.36523001 | 0.00000000
15|  1.36523001    | 1.36523001 | 0.00000000
16|  1.36523001    | 1.36523001 | 0.00000000
17|  1.36523001    | 1.36523001 | 0.00000000
18|  1.36523001    | 1.36523001 | 0.00000000
19|  1.36523001    | 1.36523001 | 0.00000000
20|  1.36523001    | 1.36523001 | 0.00000000
21|  1.36523001    | 1.36523001 | 0.00000000
22|  1.36523001    | 1.36523001 | 0.00000000
23|  1.36523001    | 1.36523001 | 0.00000000
24|  1.36523001    | 1.36523001 | 0.00000000
25|  1.36523001    | 1.36523001 | 0.00000000
26|  1.36523001    | 1.36523001 | 0.00000000
27|  1.36523001    | 1.36523001 | 0.00000000
28|  1.36523001    | 1.36523001 | 0.00000000
29|  1.36523001    | 1.36523001 | 0.00000000
30|  1.36523001    | 1.36523001 | 0.00000000
>>
```

Ranking: $g_3(x)$, $g_2(x)$, $g_1(x)$
To explain why, look at the absolute value of derivatives at the root r.
Where $r \approx 1.36523001$:
$|g_1'(r)| = |\frac{-3r^2}{4(10-r^3)^{\frac{1}{2}}}| = |-0.57529| = 0.57529$
$|g_2'(r)| = |\frac{1}{2}(\frac{10}{4+r})^{\frac{-1}{2}}(\frac{-10}{(4+r)^2})| = |-0.12723| = 0.12723$
$|g_3'(r)| = |\frac{(6r+8)(r^3+4r^2-10)}{r^2(3r+8)^2}| = |-0.0000000016935| = 0.0000000016935$
Therefore $|g_3'(r)| < |g_2'(r)| < |g_1'(r)|$. This shows the order of the function's speed of convergence.

- Looking at $|g_4'(r)|$ at $r \approx 1.36523001$ we find:
  $|g_4'(r)| = |1 - 3r^2 - 8r + 10| = |-5.5134| = 5.5134$
  Since $5.5134 > 1$, $g_4(x)$ does not converge to the root r of $f(x)$.

6.

problem_6.m

```
A = 135000;
P = 12000;
N = 30;

f = @(x) ((P ./ x) .* (1 - (1 + x) .^ (-N))) - A;

bisect_ForLoop(f, 0.001, 0.2, 15);
```

Command Window

```
>> problem_6
step | a | f(a) | c | f(c) | b | f(b)
0 | 0.001000 | 219479.032280 | 0.100500 | -22347.170426 | 0.200000 | -75252.763214
1 | 0.001000 | 219479.032280 | 0.050750 | 47902.737858 | 0.100500 | -22347.170426
2 | 0.050750 | 47902.737858 | 0.075625 | 5866.648032 | 0.100500 | -22347.170426
3 | 0.075625 | 5866.648032 | 0.088063 | -9566.797963 | 0.100500 | -22347.170426
4 | 0.075625 | 5866.648032 | 0.081844 | -2223.086125 | 0.088063 | -9566.797963
5 | 0.075625 | 5866.648032 | 0.078734 | 1722.696541 | 0.081844 | -2223.086125
6 | 0.078734 | 1722.696541 | 0.080289 | -274.204463 | 0.081844 | -2223.086125
7 | 0.078734 | 1722.696541 | 0.079512 | 718.150560 | 0.080289 | -274.204463
8 | 0.079512 | 718.150560 | 0.079900 | 220.460941 | 0.080289 | -274.204463
9 | 0.079900 | 220.460941 | 0.080095 | -27.248329 | 0.080289 | -274.204463
10 | 0.079900 | 220.460941 | 0.079998 | 96.511982 | 0.080095 | -27.248329
11 | 0.079998 | 96.511982 | 0.080046 | 34.608268 | 0.080095 | -27.248329
12 | 0.080046 | 34.608268 | 0.080070 | 3.674083 | 0.080095 | -27.248329
13 | 0.080070 | 3.674083 | 0.080083 | -11.788595 | 0.080095 | -27.248329
14 | 0.080070 | 3.674083 | 0.080077 | -4.057624 | 0.080083 | -11.788595
15 | 0.080070 | 3.674083 | 0.080073 | -0.191862 | 0.080077 | -4.057624
>> |
```

Using midpoint iteration in the interval $[0.001, 0.2]$ for 15 steps, we can see a rate of 8.0073% is the maximum that could be afforded. Since most banks only display to 2 decimal places, the maximum realistic interest rate that could be afforded is 8.00%.

7. Doing some manipulation of $x_{i+1} = \frac{1}{2}x_i + \frac{A}{2x_i}$ we get:

$x_{i+1} = \frac{1}{2}x_i + \frac{A}{2x_i} + \frac{1}{2}x_i - \frac{1}{2}x_i$

$x_{i+1} = x_i + \frac{A}{2x_i} - \frac{1}{2}x_i$

$x_{i+1} = x_i - (\frac{-A}{2x_i} + \frac{1}{2}x_i)$

$x_{i+1} = x_i - (\frac{-A}{2x_i} + \frac{1}{2}x_i(\frac{x_i}{x_i}))$

$x_{i+1} = x_i - (\frac{-A}{2x_i} + \frac{x_i^2}{2x_i})$

$x_{i+1} = x_i - (\frac{x_i^2 - A}{2x_i})$

Since we are given $x_0 > 0$, $i \geq 0$, and this equation is in the form of the Newton Method, $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, where $f(x) = x^2 - A$ and $f'(x) = 2x$, we therefore know the sequence converges to the root of $f(x) = x^2 - A$, $x = \sqrt{A}$.