Math 440L-111 Numerical Analysis Project

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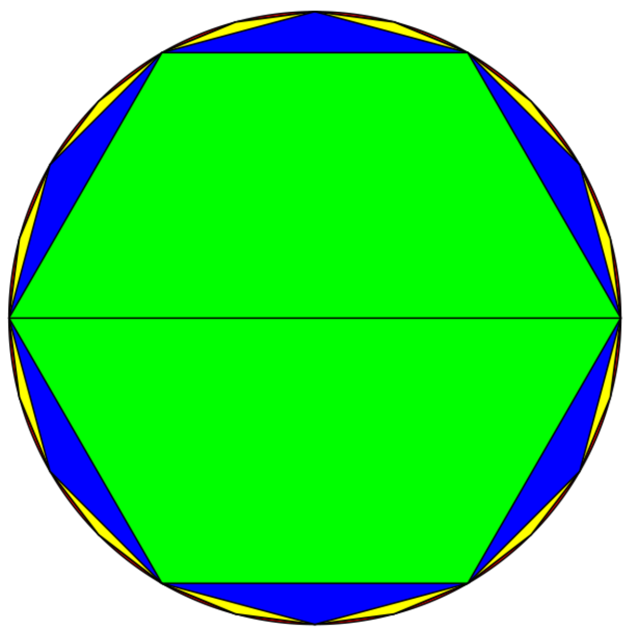
I INTRODUCTION

Liu Hui's Method is a way to approximate π which involves calculating the area of a polygon inscribed in a circle. As the number of sides of the polygon increases, its area becomes closer and closer to the area of the circle, as in the image below.



By increasing the number sides n of the the polygon inscribed in a circle with a radius 1, the area should converge to π.

II BRIEF HISTORY

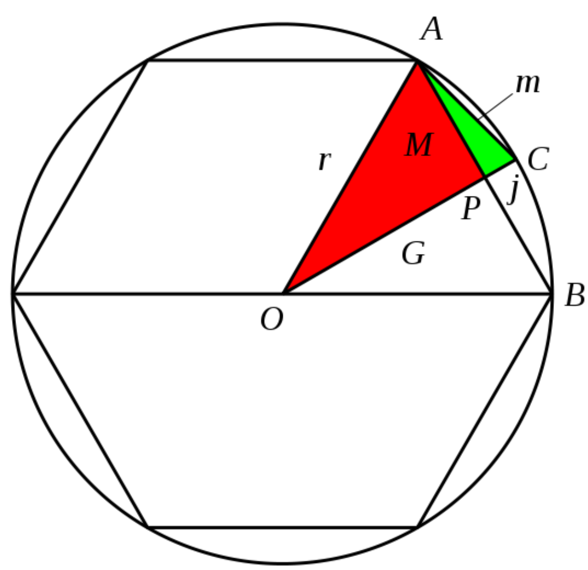
This method was developed in the 3rd Century by Liu Hui, who lived in the state of Cao Wei in China during the Three Kingdoms period of Chinese history, corresponding to current Zixiang township of Shandong province in north-central China[1]. Liu Hui expressed all of his mathematical results in the form of decimal fractions, yet the later Yang Hui (c. 1238-1298 AD) expressed his mathematical results in full decimal expressions[4]. Very little information is known about the influence of the time period on Liu Hui's work or even his life in general except he lived in poverty and read many books.

However we do know that he provided commentary on the original text "The Nine Chapters on the Mathematical Art," which included methematical ideas like fractions, area, square root, volume, and applying them to land, business, engineering, tax, and also abstract topics like equations and logic[5]. This work simply "gave methods to solve various problems, but the methods were merely prescriptions without justification"[4]. Liu Hui's commentary explained the concepts in a more mathematical way with just enough detail to convince a reader that a formal proof could be constructed for each idea. The first chapter involved the approximation of π.

III DERIVING THE FORMULA

Since Liu Hui had already realized the concept of limits, he concluded that $\lim_{n\to\infty}$ (Area of polygon with n sides) = Area of circle. So if he started with circle of radius 1, he should be able to approximate π.

Liu Hui used the Pythagorean Theorem, or what he knew as Gou Gu's Theorem to determine sides of a $2n$ polygon from sides of an n polygon. He used the diagram below[3]:



Assume there is a circle with an inscribed regular hexagon. As the picture shows, connect the endpoint of one side of the hexagon and their mid point on the circle, A, B and C, to the center O. AB crosses OC at P, side OP= G, side PC= j, AB= M, AP= $\frac{M}{2}$, AC= m.

He first used Go Gou's Theorem in triangle AOP to say:

$$G^2 = r^2 - (\frac{M}{2})^2$$

Therefore:

$$G = \sqrt{r^2 - (\frac{M}{2})^2}$$

He then let j be the difference between r and G to get:

$$j = r - G = r - \sqrt{r^2 - (\frac{M}{2})^2}$$

He used Go Gou's Theorem again, this time in triangle APC to say:

$$m^2 = (\frac{M}{2})^2 + j^2$$

$$m^2 = (\frac{M}{2})^2 + (r - \sqrt{r^2 - (\frac{M}{2})^2})^2$$

He simplified his iteration even further and showed for a circle of radius 1:

$$2 - m^2 = \sqrt{2 + (2 - M^2)}$$

Where m is the length of one side of the next order polygon bisected from M. If we substitute $k_n = 2 - M^2$, and $k_{2n} = 2 - m^2$:

$$k_{2n} = \sqrt{2 + k_n}$$

Now to find the side length, S_{2n}, needed to calculate the area of the polygon, $k_{2n} = 2 - m^2$ must be solved in terms of m:

$$S_{2n} = m = \sqrt{2 - k_{2n}}$$

Since a polygon of 6 sides forms equilateral triangles, if it is inscribed in a circle with radius 1, then each side length will also be 1. Therefore we start with:

$$S_6 = 1 = \sqrt{2 - k_6}$$

Therefore our base is:

$$k_6 = 1$$

The formula for the area of a regular polygon is $A = \frac{1}{2}nS_na$ where n is the number of sides, S_n is the side length, and a is the apothem. Since our polygon is inscribed in a circle of radius 1, $\lim_{n\to\infty} a = 1$. Therefore we just let $a = 1$ and the formula becomes $P_n = \frac{1}{2}nS_n$ where S_n is the side length of a polygon with n sides.

To find the error we take $|actual - extimate|$ where the $actual = \pi$ and the $extimate = \frac{1}{2}nS_n$. This means $E_n = |\pi - \frac{1}{2}nS_n|$.

To find the rate of convergence, let us divide the error of P_{12} by P_6.

$$\begin{aligned}
P_{12} &= \frac{1}{2}(12)\sqrt{2-\sqrt{2+1}} = 3.106 \\
E_{12} &= \pi - 3.106 = 0.03559 \\
P_6 &= \frac{1}{2}(6)\sqrt{2-1} = 3 \\
E_6 &= \pi - 3 = 0.14159 \\
\frac{E_{12}}{E_6} &= \frac{0.03559}{0.14159} = 0.25136
\end{aligned}$$

IV EXAMPLE

Here is an example of the first 5 steps of Liu Hui's Algorithm:

Sides	k_n	S_n	Area	Error
6	1	1	3	0.141593
12	$\sqrt{2+1}$	$\sqrt{2-\sqrt{3}}$	3.1058285	0.0357641
24	$\sqrt{2+\sqrt{3}}$	$\sqrt{2-\sqrt{2+\sqrt{3}}}$	3.1326286	0.00896404
48	$\sqrt{2+\sqrt{2+\sqrt{3}}}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$	3.1393502	0.00224245
96	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	3.141032	0.000560703

V ADVANTAGES OF USE

There are certain advantages to using this method over others. Fist is it an efficient algorithm, second it held the record as the most accurate approximation of π using 96-gon and 192-gon until the 18th century. Using his algorithm π was calculated to eight decimal places.

Initial guesses are not a problem, since the method should always start with a hexagon. Calculating the next iteration is simple, requiring basic arithmetic.

VI MATLAB

Here is matlab code[2] for calculating Liu Hui's Pi algorithm for any number of steps. The input, n, is the number of iterations you want to take. The output is a table with the number of sides, area, error, and rate of convergence.

```
%input n is the number of iterations
%output is a table of the sides, area, error, and rate of convergence
function x = luiHuiPi(n)
printf("sides | area | error | rate of convergence\n");
```

```
%Starting with a hexagon with side length of 1
k(1) = 1;
s = sqrt(2 - k(1));
pn = (6 ./ 2) * s;
error(1) = pi - pn;
printf("%d | %d | %d\n", 6, pn, error(1));

for i = 2:n
  k(i) = sqrt(2 + k(i - 1));
  s = sqrt(2 - k(i));
  pn = ((2 .^ (i - 1) .* 6) ./ 2) .* s;
  error(i) = pi - pn;
  convergence = error(i) ./ error(i - 1);
  printf("%d | %.8d | %d | %d\n", (2 .^ (i - 1) .* 6), pn, error(i), converge
end
```

Calling the function with n = 10, we get the following table as output:

Sides	Area	Error	rate of convergence
6	3	0.141593	
12	3.1058285	0.0357641	0.252585
24	3.1326286	0.00896404	0.250643
48	3.1393502	0.00224245	0.250161
96	3.141032	0.000560703	0.25004
192	3.1414525	0.000140181	0.25001
384	3.1415576	3.50457e-05	0.250003
768	3.1415839	8.76144e-06	0.250001
1536	3.1415905	2.19035e-06	0.249999
3072	3.1415921	5.47547e-07	0.249981

VII CONCLUSION

This report included some background about Liu Hui's method of approximating π and explained his method by introducing the concept of limits and geometry knowledge. It is possible to implement his method into a computer iteration algorithm, which was shown in Matlab. The error generated by this algorithm will get smaller and smaller distinctly as we increase the number of iterations.

As it was stated in this report, it is a efficient algorithm. It held the record as the most accurate approximation using 96-gon and 192-gon until the 18th century. Today there exists more complex methods developed specifically with the speed of computer computation in mind. As a geometrical solution, Liu Hui's method is still popular and used by many people.

VIII IN-CLASS QUESTIONS

1. Try the first step of iteration by hand.
2. What is the limit of this algorithm?
3. What if we use a regular pentagon as the initial shape?
4. If we had a shape with 10 sides how many sides would the next shape have?

IX Works Cited

[1] Liu Hui J J O'Connor and E F Robertson, Dec. 2003. Web. 06 May 2017

[2] "Liu Hui's Algorithm for Calculating Pi." Lucky's Notes. 12 July 2010. Web. 06 May 2017.

[3] "Liu Hui's Algorithm." Wikipedia. Wikimedia Foundation, 07 Apr. 2017. Web. 06 May 2017.

[4] Needham, Joseph (1986). Science and Civilization in China: Volume 3, Mathematics and the Sciences of the Heavens and the Earth. Taipei: Caves Books, Ltd.

[5] Nine Chapters on the Mathematical Art. J J O'Connor and E F Robertson, Dec. 2003. Web. 06 May 2017