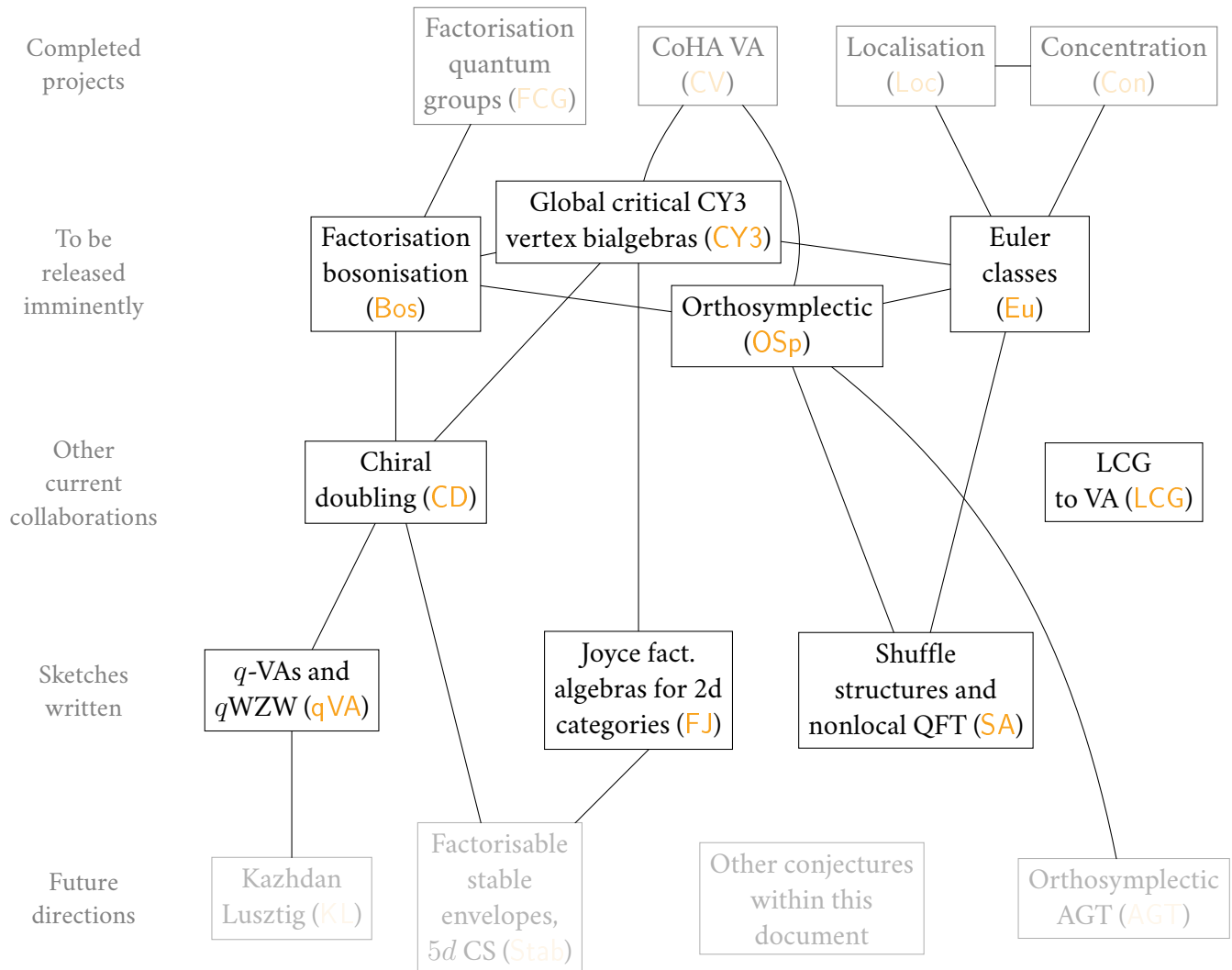


# RESEARCH PLANS - ABRIDGED

ALEXEI LATYNTSEV

See <https://alyoshalatyntsev.github.io/plan/plan.pdf> for a more detailed research statement.



## 1. Research statement

I am a mathematician working in geometric representation theory.

### 1.1. Algebraic structures attached to CY3s (CV, CY3, FJ, Stab)

*Cohomological Hall algebras* are associative algebras attached to Calabi-Yau varieties or categories [Daa; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZb] (e.g. quantum groups [MO]), hence are a key object in geometric representation theory/enumerative invariants.

*Global critical CY3 vertex quantum groups.* Let  $\mathcal{C}$  be the CY3 category  $\text{Rep}(Q, W)$ , the representations of a Jacobi algebra of a quiver with potential, or  $\text{Coh}_0(K_{T^*C})$ , zero dimensional coherent sheaves on a local curve [KK], or more generally a *deformed CY3 completion*.

**Theorem A.** [CY3] *The critical cohomology<sup>1</sup> of the moduli stack of objects  $\mathcal{M}$  has a vertex coproduct*

$$H^\bullet(\mathcal{M}, \varphi) \rightarrow H^\bullet(\mathcal{M}, \varphi) \hat{\otimes} H^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

*compatible with the CoHA: it forms a vertex quantum group (see FQG for a definition).*

**Theorem B.** [CY3; CV for  $W = 0$ ] *For any quiver  $Q$ , the vertex coproduct on the preprojective CoHA  $H_{\mathbf{G}_m}^\bullet(\mathcal{M}_{Q^{(3)}}, \varphi_{W^{(3)}}) \simeq Y_h(\mathbf{n}_Q)$  agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) Drinfeld's meromorphic coproduct.*

*Vertex coalgebras from configuration spaces.* For Theorem B to be well-defined, we need to be able to compare localised and vertex coproducts. Recall from [Daa] that a *localised bialgebra* is a factorisation bialgebra over the configuration space, i.e. sheaf of algebras  $\mathcal{A}$  with a map

$$\begin{array}{ccc} (\text{Conf}\mathbf{A}^1 \times \text{Conf}\mathbf{A}^1)_{\circ} & & \\ \swarrow j & \searrow \cup j & \\ \text{Conf}\mathbf{A}^1 \times \text{Conf}\mathbf{A}^1 & & \text{Conf}\mathbf{A}^1 \end{array} \quad (\cup j)^* \mathcal{A} \xrightarrow{\Delta} j^*(\mathcal{A} \boxtimes \mathcal{A})$$

i.e. there are localised terms  $1/(x_i - x_j)$  in the image of the coproduct. Now,

**Theorem C.** *There is a functor from localised bialgebras to vertex bialgebras.*

The functor is given by taking Talyor expansions  $\frac{1}{x+z} = \frac{1}{z} \left( \frac{x}{z} - \left(\frac{x}{z}\right)^2 + \dots \right)$  where  $z \in H^\bullet(\mathbf{BG}_m)$  comes from the action of  $\mathbf{BG}_m$  on BGL, see SA. This construction suggests a resolution to the issue that so far there is no factorisable definition of vertex coalgebras:

**Conjecture D.** *Configuration-to-Ran construction. One may define a colimit  $\text{Ran}_{\mathbf{P}}\mathbf{A}^1$  of projective spaces  $\mathbf{P}^n$ , and factorisation coalgebras  $\mathcal{A}$  on it induce vertex coalgebras. The construction above then lifts to a functor to these factorisation coalgebras  $\mathcal{A}$ .*

<sup>1</sup>i.e.  $\mathcal{M} = \text{Crit}(W)$  is a critical locus inside a *smooth* moduli stack; we take the vanishing cycle sheaf  $\varphi = \varphi_W$ .

*Lift to factorisation algebra.* An interesting question is: does the above vertex coproduct factorise? In other words, is there a natural geometric structure over  $\text{Ran}Y$  which induces the vertex coproduct of Theorem A? Answering this is the content of [FJ](#).

If we had an answer to this question,

- It may make it clearer how to generalise away from the case that  $\mathcal{M}_c$  is given as a global critical locus, to more general CY3s.
- It gives us new understanding of the [MMSV] of W-algebras attached to surfaces  $S$ .
- We could push the analogy with quantum groups further, by constructing a Yangian  $Y_h(\mathfrak{g})$  factorisably, mirroring [Ga].

In the case of quivers  $Q$ , we have an action of the torus  $T_d = \prod T_{d_i}$  on the stack of representations, and

$$\mathcal{M}^f = \{(m, \lambda) : \lambda \in \mathfrak{t}, m \in \mathcal{M}^\lambda\} \xrightarrow{\pi} \text{colim}(\mathfrak{t}_d) \quad (1)$$

defines a factorisation space over the  $Q_0$ -coloured Ran space.

**Conjecture E.** [\[FJ\]](#) *The relative critical cohomology  $\mathcal{A} = \pi_* \varphi_W$  defines a translation equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\text{Ran} \mathbf{A}^1 \subseteq \text{Ran}_{Q_0} \mathbf{A}^1$$

*recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA  $H_\bullet^{\text{BM}}(\mathcal{M}_{\text{nilp}})$  of [SV].*

This should relate to Yang-Zhao's proof [YZa] that CoHAs form a localised factorisation bialgebra over  $\text{Conf}(\Sigma)$  by the Configuration-to-Ran construction D. Finally, this construction makes the role of the torus  $\mathfrak{t}_d$  clear, and therefore in [SA](#) we may generalise it to arbitrary Kac-Moody groups.

*Remark.* Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work, see also [Dac]), and the above should be an algebraic analogue of their construction.

*Relation to  $\mathcal{W}$ -algebras.* In the same way, for any surface  $S$  we may define the stack  $\pi : \mathcal{M}_c^f \rightarrow \text{Ran}_S K_S$  parametrising a finite subset of points  $x_i : S \rightarrow K_S$  and a dimension zero coherent sheaf supported on their joint image.

This gives the following new structure on the algebras  $W^+(S)$  of [MMSV], which e.g. after completing [CD](#) may allow us to take its double.

**Conjecture F.** [\[FJ\]](#) *We have a global sheaf of categories over  $\text{Ran}_S K_S$  which have a braided factorisation structure along the fibres of  $K_S \rightarrow S$ .*

In general  $\mathcal{M}_c^f$  this will *not* be a global critical locus in a smooth factorisation space  $\mathcal{M}^f$ , only locally over an open cover  $S = \bigcup U_i$  do we have such an  $\mathcal{M}_i^f$ . In particular, by applying the construction of [CY3](#)<sup>2</sup> we get a vertex quantum group  $\mathcal{V}_i$  inside  $H^\bullet(\mathcal{M}_i)\text{-Mod}$ , which will be Morita equivalent on the intersections  $U_i \cap U_j$ ,<sup>3</sup> hence we expect  $\mathcal{V}_i\text{-Mod}(H^\bullet(\mathcal{M}_i)\text{-Mod})$  to glue over  $\text{Ran}_S K_S$ .

<sup>2</sup>We assume that  $K_S$  has a cover e.g. by local curves so that we can apply [CY3](#), such as for instance if  $S = \mathbf{P}^2$ .

<sup>3</sup>Indeed, they are two different vertex algebra structures on the same vector space  $H^\bullet(\mathcal{M}_{U_i \cap U_j}, \varphi)$ .

**Relation to stable envelopes.** The definition (1) is clearly reminiscent of the definition of stable envelopes and the structure [MO, §5.1.1] on Nakajima quiver varieties used to define  $Y_h(\mathfrak{g}_Q)$  via stable envelopes; we plan to study this once **FJ** is finished. This suggests the following answer to Maulik-Okounkov’s question about how to interpret their data: for every multidimension vector  $w$ , define a factorisable version  $\pi : M(w)^f \rightarrow \text{Ran}\mathbf{A}^1$  of the quiver variety,<sup>4</sup> let  $\mathcal{C}$  be the factorisation category generated by the relative Borel-Moore homology of  $\pi_*\omega_{M(w)^f}$ , then apply Tannakian reconstruction to get the definition [MO, Def. 5.2.6] of  $Y_h(\mathfrak{g})$ .<sup>5</sup>

Thus, just as the Drinfeld-Kohno Theorem reconstructed  $U_h(\mathfrak{g})$  with its  $R$ -matrix from  $U_q(\mathfrak{g})\text{-Mod}$  (viewed [CF] as a perverse sheaf of categories over  $\text{Ran}\mathbf{R}^2$ ), we expect to do the same to  $\mathcal{C} \simeq Y_h(\mathfrak{g})\text{-Mod}$  over  $\text{Ran}\mathbf{C} \times \mathbf{R}$ .

## 1.2. The structure of factorisation quantum groups (FCG, Bos, CD)

*History.* A collection of structures all loosely called “quantum groups” have been some of the main objects in mathematical physics and geometric representation theory since the 80s:

- (1) It is well-known that the representation categories of  $U_q(\hat{\mathfrak{g}})$ ,  $Y_h(\mathfrak{g})$ ,  $\mathcal{E}_{h,\tau}(\mathfrak{g})$  should be controlled by “spectral” analogues  $R(z)$  of  $R$ -matrices [CWY; GLW].
- (2) The affine case has recently been understood much better: the algebras  $Y_h(\hat{\mathfrak{g}})$ ,  $\mathcal{W}_{1+\infty}(\mathfrak{g})$  in [GRZ; RSYZ]
- (3) In [MO] Maulik-Okounkov define a bialgebra  $Y_h(\mathfrak{g}_Q)$  attached to *any* quiver  $Q$ .

Historically these definitions were (ingeniously) made very explicitly using formulas: generators and relations/RTT definitions, e.g. [Dr; MO], still much of the modern (e.g. CoHA) literature is based on explicit shuffle computations, e.g. [MMSV; SV; YZb].

The point of this series of projects is to first give a more conceptual definition of these structures, second recover the above formulas as a *consequence* of these definition, and third to generalise them to more general structures.

**Factorisation quantum groups.** In **FCG** we develop a theory of *braided factorisation categories*. The basic idea is to take Lurie’s [Lu] result

$$\mathbf{E}_2\text{-dgCat} \simeq \mathbf{E}_2\text{-Alg}(\text{Sh}(\text{Ran}\mathbf{R}^2, \text{dgCat}))$$

that a braided monoidal category  $\mathcal{C}$  is equivalent to<sup>6</sup> a factorisable constructible sheaf of categories over the Ran space of  $\mathbf{R}^2$ , and replace  $\text{Ran}\mathbf{R}^2$  with an arbitrary *factorisation space*  $X$ .<sup>7</sup>

$$\begin{array}{ccc} & (\text{Ran}\mathbf{R}^2 \times \text{Ran}\mathbf{R}^2)_\circ & \\ \swarrow & & \searrow \\ \text{Ran}\mathbf{R}^2 \times \text{Ran}\mathbf{R}^2 & & \text{Ran}\mathbf{R}^2 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & M_X & \\ \swarrow & & \searrow \\ X \times X & & X \end{array}$$

<sup>4</sup>To get this, one expects that one would have to interpret stable envelopes  $\text{Stab}_\lambda : H_T^\bullet(M^\lambda) \rightarrow H_T^\bullet(M) \simeq H_T^\bullet(M^0)$  as a kind of factorisation structure.

<sup>5</sup>Notice how Maulik-Okounkov’s definition essentially defines  $Y_h(\mathfrak{g})$  as something reminiscent of a subalgebra of “factorisation endomorphisms” of  $\mathcal{C}$ .

<sup>6</sup>To see a precise statement and of this, see [CF, Prop 7.0.2].

<sup>7</sup>Space here means derived prestack.

Examples include: ordinary groups  $G$ , configuration spaces  $\text{Conf}_\Lambda(\mathbf{A}^1)$ , and algebraic-topological Ran spaces  $\text{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$ , or any space  $X$  with the inclusion of the diagonal.

We first check that this is a reasonable definiton: it recovers  $R$ -matrices,

**Theorem G.** [FCG] *One may define factorisation categories, algebras, etc., over  $X$ . If  $\mathcal{A}$  is a factorisation bialgebra in factorisation category  $(\mathcal{C}, \otimes_{\mathcal{C}, X})$  over  $X$ , each braided factorisation structure on  $\mathcal{A}\text{-Mod}(\mathcal{C})$  induce a factorisation  $R$ -matrix  $R : \mathcal{A} \otimes_{\mathcal{C}, X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C}, X} \mathcal{A}$  for which  $\mathcal{A}$  is quasitriangular.*

In the vertex algebra case we recover classical notions [EK; FR] of quantum vertex algebras:

**Theorem H.** [FCG] *When  $X = \text{Ran}\mathbf{A}^1$  and  $\mathcal{A}$  is a nonlocal vertex bialgebra  $V$ , a factorisation  $R$ -matrix is an endomorphism  $R(z) : V \otimes V((z)) \xrightarrow{\sim} V \otimes V((z))$  satisfying the spectral hexagon relations, such that  $V$  is  $R(z)$ -twisted local (fields satisfy  $R(z)$ -twisted OPEs).*

For instance, this allows us to interpret result [GLW] on the structure of  $\mathcal{A} = Y_h(\mathfrak{g})$  as follows. It is a lax quasitriangular factorisation bialgebra with laxly compatible  $\otimes^*$ - and  $\otimes^{ch}$ -coproducts on the factorisation category of equivariant D-modules on  $X = \text{Ran}\mathbf{A}^1$ , corresponding to the standard and meromorphic coproducts on  $Y_h(\mathfrak{g})$ . The various  $R$ -matrices of [GLW]

$$\mathcal{M}_1 \otimes^* \mathcal{M}_2 \xrightarrow{R^-(z)} \mathcal{M}_2 \otimes^{ch} \mathcal{M}_1, \quad \mathcal{M}_1 \otimes^{ch} \mathcal{M}_2 \xrightarrow{R^{0,\epsilon}} \mathcal{M}_2 \otimes^{ch} \mathcal{M}_1, \quad \mathcal{M}_1 \otimes^* \mathcal{M}_2 \xrightarrow{R^\epsilon} \mathcal{M}_2 \otimes^* \mathcal{M}_1$$

then induce lax compatibility maps between the two coproducts on  $\mathcal{A}\text{-Mod}$ . So: we have lifted  $Y_h(\mathfrak{g})\text{-Mod}$  to a topological-holomorphic factorisation algebra over  $\mathbf{R} \times \mathbf{C}$  rigorously.

Theorem G specialises to the ordinary theory of  $R$ -matrices and quantum groups (e.g. for  $U_q(\mathfrak{g})$ ) when  $X = \text{Ran}\mathbf{R}^2$ . Likewise, when  $X = \text{Conf}\mathbf{A}^1$  a factorisation bialgebra  $\mathcal{A}$  is equivalent to the localised bialgebras  $B$  of [Daa] (the main examples being CoHAs), a factorisation  $R$ -matrix is an endomorphism

$$R : (B \otimes B)_{\text{loc}} \xrightarrow{\sim} (B \otimes B)_{\text{loc}}$$

of the localised bialgebra  $B$ .

These techniques are fairly flexible and seem to be able to prove chiral analogues of many structure results in the theory of quantum groups, e.g. FCG ends by constructing a generalisation of the Borchers twist construction [Bo].

**Factorisation bosonisation.** In Bos, we automate the construction of adding in the Cartan:

$$U_q(\mathfrak{n}) \xrightarrow{[\text{Maa}; \text{Mab}]} U_q(\mathfrak{b}), \quad Y_h(\mathfrak{n}) \xrightarrow{[\text{Dr}]} Y_h(\mathfrak{b}), \quad H^\bullet(\mathcal{M}, \varphi) \xrightarrow{[\text{Daa}; \text{RSYZ}; \text{YZb}]} H^\bullet(\mathcal{M}, \varphi)^{\text{ext}}.$$

Indeed, in both CY3 and OSp we had to do complicated generating series manipulations to extend the CoHA (i.e. add a Cartan part) as a vertex bialgebra. In the finite case [Maa; Mab], the bialgebra structure on  $U_q(\mathfrak{b})$  can be written in terms of  $U_q(\mathfrak{n})$ 's using Tannakian reconstruction (“bosonisation”):

$$U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\text{Rep}_q T),$$

and we want to do the same for vertex algebras.

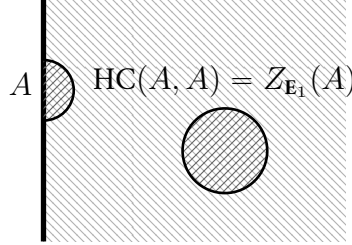
**Theorem 1.2.1.** [Bos, in preparation] *A Tannakian reconstruction for factorisation categories, which when applied to*

$$Y_h(\mathfrak{n})\text{-Mod}(Y_h(\mathfrak{t})\text{-Mod}) \simeq H^\bullet(\mathcal{M}, \varphi)\text{-Mod}(H^\bullet(\mathcal{M})\text{-Mod})$$

*gives the vertex bialgebra  $Y_h(\mathfrak{b}) \simeq H^\bullet(\mathcal{M}, \varphi)^{\text{ext}}$ .*

The analogues of Majid's [Maa; Mab] formulas recovers e.g. the localised bialgebra structure on the extended CoHA from [YZb, §1.3] or [RSYZ].

**Factorisation Drinfeld doubling.** The Drinfeld centre (see [Lu]) of an  $E_n$ -algebra by Kontsevich's conjecture [Th] gives an algebra over the Swiss cheese operad:



In **CD** the main aim is to replace  $E_n$ -algebras with *chiral*  $E_n$ -algebras:

**Conjecture I.** *There is a construction  $Z_{E_1}(\mathcal{C})$  of a chiral factorisation category over  $\text{Ran}\mathbf{A}^1$ , which carries compatible chiral and ordinary monoidal structures. Applying this to the category of modules for the affine vertex algebra  $V^k(\mathfrak{g})$  and the (dual) Yangian  $Y_h(\mathfrak{b})^*$  give category of modules over the quantisation  $[AN]$  of  $U(T^*\mathfrak{g}[t])$  and  $Y_h(\mathfrak{g})^*$ , respectively.*

The first part has followed easily from the results in **FCG**, and we are currently working on the second.

### 1.3. Orthosymplectic structures (**OSp**, **SA**, **AGT**)

*Physics heuristic.* In project **OSp** we make a mathematical theory of *boundary* 4d Chern-Simons [BS] on  $\mathbf{R} \times (\mathbf{R} \times \mathbf{C})/\pm$ , for instance our structures satisfy *boundary Yang-Baxter/Cherednik reflection equations*. More generally, we define boundary versions of compactifications of 4d SCFTs  $\int_Y \mathcal{M}$  attached to a CY3  $Y$  - at least, those for which non-boundary versions have been defined. It should relate to Finkelberg-Hanany-Nakajima's ongoing work on orthosymplectic Coulomb branches (see **AGT**).

**Details.** Attached to an abelian category  $\mathcal{A}$ ,<sup>8</sup> we construct the *orthosymplectic moduli stack*  $\mathcal{M}_{\mathcal{A}}^{\text{OSp}}$ : a fixed point stack whose points are objects with a symmetric pairing  $a \simeq a^*$ .

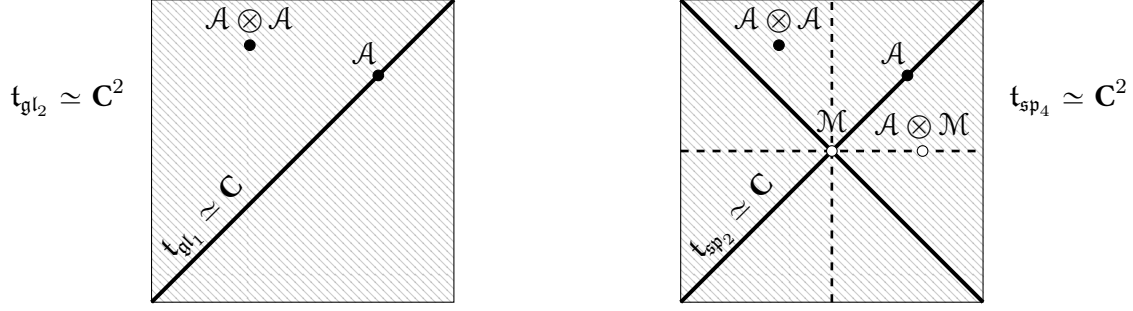
**Theorem J.** [**OSp**] *For  $\mathcal{A}$  in **CY3** or examples below, the vertex quantum group  $H^\bullet(\mathcal{M}, \varphi)$  “acts” on  $H^\bullet(\mathcal{M}^{\text{OSp}}, \varphi^{\text{OSp}})$ :*

- (1) *there is a left module action  $a$  of the CoHA respecting the involution,<sup>9</sup> compatible with*
- (2) *its symplectic vertex algebra structure: it is a factorisation coalgebra over symplectic Ran space  $\text{Ran}_{\text{Sp}}\mathbf{A}^1 = \text{colim}_{\text{Sp}_{2n}} (\text{coming from a localised structure over } \text{Conf}_{\text{Sp}}\mathbf{A}^1 = \text{Spec } H^\bullet(\text{BSp}))$ .*

The data (1) and (2) is equivalent to a topological and holomorphic factorisation algebra over  $\mathbf{R}/\pm$  and  $\mathbf{C}/\pm$ , respectively. We give an equivalent vertex algebra style definition of the latter in terms of fields  $A \otimes M \rightarrow M((z))$ . In pictures, a factorisation and symplectic factorisation algebra are:

<sup>8</sup>More generally abelian category with involution  $(\mathcal{A}, \tau)$ , e.g.  $\tau = (-)^*$ .

<sup>9</sup>i.e. the left action  $a$  and the right action  $a \cdot (\text{id} \otimes \tau)$  commute, where  $\tau$  is the involution.



To give examples, we construct an **invariants** functor involving restricting along  $t_{sp_{2n}} \hookrightarrow t_{gl_{2n}}$

$$\iota : \text{FactAlg}_{\text{GL}}(\mathbf{A}^2) \rightarrow \text{FactAlg}_{\text{Sp}}(\mathbf{A}^1), \quad (\mathcal{A}, \tau) \mapsto (\mathcal{A}, \mathcal{A}^\tau)$$

where  $\mathcal{A}$  is a factorisation algebra with involution  $\tau$ ; we expect Theorem J may also be proved by applying  $\iota$  to the factorisable moduli stack  $\mathcal{M}^f$  from FJ. See also the link to stable envelopes [Stab](#), and:

**Conjecture K.** *The **boundary KZ equations** may be derived by applying  $\iota$  to the BD Grassmannian, taking distributions supported at the identity, and taking conformal blocks over  $\text{Ran}_{\text{Sp}} \mathbf{A}^1$ .*

Examples include  $\mathbf{Z}/2$ -equivariant quivers with potential,<sup>10</sup> or orthosymplectic perverse-coherent sheaves on surfaces, e.g. orthosymplectic ADHM quiver/perverse-coherent sheaves on  $\mathbf{A}^2$ :

$$\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \text{SO}(n) \rightleftharpoons \text{Sp}(2m) \quad \mathcal{E} \simeq \text{RHom}(\mathcal{E}, \mathcal{O})$$

**Theorem L.** [OSp] *In the quiver with potential case, an explicit shuffle formula for the CoHA action and vertex coaction on  $H^\bullet(\mathcal{M}^{\text{OSp}}, \varphi^{\text{OSp}})$ .*

Using techniques from [Eu](#), we can give a geometric interpretation of this. We end with a conjecture:

**Conjecture M.** *The orthosymplectic CoHA for the “folded” linear quiver  $A_{2n}$ <sup>11</sup> is isomorphic to the twisted Yangian  $Y_h(\mathfrak{gl}_n)^{tw}$  of [BR].*

[Nonlocal QFT and shuffle structures.](#) Project [SA](#) begun by noticing the following interesting pattern in structures considered project [OSp](#).

$$\text{BGL} \rightsquigarrow \text{BSp}, \quad \text{Conf}(\mathbf{A}^1) \rightsquigarrow \text{Conf}(\mathbf{A}^1), \quad \text{VA} \rightsquigarrow \text{OSpVA}, \quad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see [AGT](#)) action on Nakajima quiver varieties and the KZ equation, simultaneously generalise - this points towards this being a shadow of a more general theory.

<sup>10</sup>n.b. we can view any quiver with involution, as orbifold-valued quiver with vertices  $Q_0/\mathbf{Z}/2$  and edges  $Q_1/\mathbf{Z}/2$ ; it is natural to ask if we can generalise away from global quotients.

<sup>11</sup>i.e. with the involution being reflection in the linear direction.

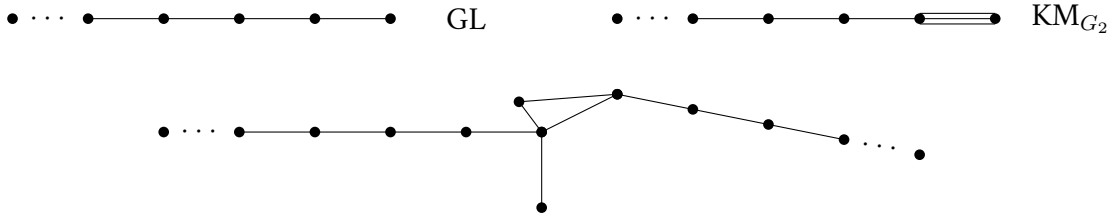


The starting observation is this - the definition [KS; Gr] a shuffle algebra is equivalent to a monoidal functor  $A : \text{GL} \rightarrow \text{Vect}$  from the category GL whose objects are finite products of the groups  $\text{GL}_n$  for  $n \geq 0$ , and the morphisms are parabolics between them. Indeed, the parabolics

$$\begin{array}{ccc} & P_{n,m}(\sigma) & \\ \swarrow & & \searrow \\ \text{GL}_n \times \text{GL}_m & & \text{GL}_{n+m} \end{array} \xrightarrow{A} A_n \otimes A_m \xrightarrow{m(\sigma)} A_{n+m}$$

are labelled by shuffles  $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \text{Sh}(n, m)$ .

The motivating idea of **SA** is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dynkin diagrams, e.g.



To summarise, we expect to define KM analogues of the following:

- *Shuffle algebras*, likewise analogues localised and vertex algebras living over new configuration and Ran spaces

$$\text{Conf}_{\text{KM}}(\mathbf{A}^1) = \coprod_G \text{Spec } H^*(BG), \quad \text{Ran}_{\text{KM}}(\mathbf{A}^1) = \text{colim}_G \mathfrak{t}_G,$$

where  $\mathfrak{t}_G$  is the Cartan of Kac-Moody group  $G$ . Topological sheaves on  $\text{Ran}_{\text{KM}} \mathbf{C}$  gives analogues of  $\mathbf{E}_2$ -algebras/braided monoidal categories.

- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups

$$\begin{array}{ccccc} \text{GL}_3 & U_{3,5} & \text{GL}_5 & U_{5,4} & \text{GL}_4 \\ \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc \end{array}$$

where  $P_{n,m} \rightarrow U_{n,m}$  is a unipotent. We can define the stack of KM-quiver representations as  $\mathcal{M}_Q = \coprod [U_e/G_i]$  the product over all maps  $(G_i) : Q_0 \rightarrow \text{KM}$  and  $U_e$  is a choice of unipotent for each edge  $e$ . Analogue of stable envelope construction **Stab**, e.g. giving [MO]-analogue construction of **OSp** CoHAs.

- Iterated integrals. Chen's [Ch] shuffle structure on cochains  $C^\bullet(LX)$  of the loop space may be deduced from a shuffle structure on the spaces  $L_n X = \text{Maps}(\Delta^n, X)$ , where  $\Delta^n = T^n/\mathfrak{S}_n$ ; in the general case we may replace this with the quotient  $\Delta_G = T_G/\mathfrak{W}_G$  by the Weyl group of  $G$ . Understand the relation to Dynkin/ $q$ -analogues of multiple zeta values [KMT; Mi].

For instance, the structures in **OSp** (e.g.  $\pm$ -equivariant factorisation algebras on  $\mathbf{C}$ ) are obtained from  $\text{KM}_{\text{SO}(2n), \text{Sp}(2n), \text{SO}(2n+1)}$ ; so too let us consider  $K_{G_2}$  - factorisation algebras consist of ordinary factorisation algebras but where for any *triple* of points  $z_1, z_2, z_3$  there is in addition equivariance with



respect to

$$\begin{aligned}\tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3) \\ \tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1) \\ \tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2),\end{aligned}$$

a square root of (231) generating  $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$ . Just as in **OSp** we show  $K_{\mathrm{Sp}_{2n}}$ -analogues of (factorisation) braided monoidal categories give Cherednik's reflection equation, we expect to obtain the  $G_2$ -reflection equation [Ku]. For another example, considering the groups  $\hat{\mathfrak{gl}}_n$  of affine type  $A$  gives  $\mathrm{Ran}_{\widehat{\mathrm{GL}}} \mathbf{A}^1$ ,  $\mathcal{D}$ -modules on which are related to  $\mathcal{D}$ -modules on  $\mathbf{A}^1/\mathbf{Z}$ , so we expect this should relate to trigonometric KZ equations.

Just as in **OSp** we used that  $C_n$  is obtained by *folding*  $A_{2n}$ ,<sup>12</sup> we expect to be able to produce  $G_2$  structures by taking  $\mathbf{Z}/3$ -invariants of type  $D$  structures.

**A twisted AGT correspondence.** After **OSp**, one natural next step (project **AGT**) is to construct a boundary version [AGT; BFN]:

**Conjecture N.** [AGT] *The equivariant intersection homology of the invariant locus  $\mathcal{U}_{\mathbf{p}^2, \mathrm{GL}_n}^{\mathbf{Z}/2}$  in the Uhlenbeck space is a Verma module for an orthosymplectic analogue of the vertex  $W$ -algebra  $\mathcal{W}^k(\mathfrak{gl}_n)$ .*

Likewise, we expect a version for an arbitrary smooth projective surface  $S$ . We expect the proof of the above should proceed in much the same way as in [BFN], but with the parabolic induction data replaced by

$$\begin{array}{ccc} & \mathrm{BP} & \\ \swarrow & & \searrow \\ \mathrm{BGL}_n \times \mathrm{BSp}_{2m} & & \mathrm{BSp}_{2n+2m} \end{array}$$

as in **OSp**; i.e. we expect a **SA**-type analogue of free field realisations. Likewise, we expect a generalisation of [RSYZ] for instantons on  $\mathbf{A}^3$ :

**Conjecture O.** [AGT] *There is vertex algebra structure on the the orthosymplectic CoHA of the Jordan quiver, which acts on  $\mathrm{IH}_T^\bullet(M^{\mathbf{Z}/2})$ , the equivariant intersection cohomology of the invariant locus in the quiver variety.*

Just as the CoHA  $\mathcal{W}_{1+\infty}^+$  of the Jordan quiver is by [Dab] the universal enveloping algebra of positive half of differential operators on  $\mathbf{C}^\times$  and admits the  $W$ -algebras of [BFN] as quotients, we expect the above to be a universal enveloping on differential operators on  $\mathbf{C}^\times/\pm$ , and admit the above  $W$ -algebras  $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$  as quotients.

#### 1.4. $q$ -vertex algebras (**qVA**, **KL**)

**$q$ -vertex algebras.** The main goal of project **qVA** is to develop the machinery of  $q$ -vertex algebras and define  $q$ -affine vertex algebras. In **KL** we hope to use it to produce the  $q$ KZ equations and relate to Kazhdan-Lusztig equivalences.

A natural first guess at a definition is to take the usual definition of vertex algebra but using  $\mathcal{D}_q$ -modules in place of  $\mathcal{D}$ -modules. The first observation is that a  $q$ -difference operator  $\partial_x$  on  $\mathbf{A}^1$  induces

<sup>12</sup>i.e. the invariants construction discussed in **OSp**.

a derivation  $y\partial_x$  on the noncommutative plane<sup>13</sup>  $\mathbf{A}_q^2$ , and indeed the physics heuristic below points towards  $\mathcal{D}$ -modules on  $\mathbf{A}_q^2$  (e.g. via [MS]) as the correct setting for  $q$ -vertex algebras:

**Conjecture P.** [qVA] *There is a factorisation category over the noncommutative space  $\mathbf{A}_q^2$ , such that any  $\mathcal{A} \in \text{FactAlg}^{st}(\mathcal{D}\text{-Mod}_{\text{Ran}\mathbf{A}_q^2})$  defines a  $q$ -vertex algebra (generalising e.g. [FR]).*

To construct this category precisely, one needs to develop the theory of  $\mathcal{D}$ -modules (e.g. functoriality) over noncommutative spaces. We propose using work [FMW; MS] on jet spaces of noncommutative schemes to give a “ $q$ -crystal/de Rham” definition.

*Physics heuristic.* Our guiding heuristic from physics is the following: much as  $V^k(\mathfrak{g})$  and  $U_h(\mathfrak{g})$  have module categories giving line operators for “3d Chern-Simons with boundary” on

$$\mathbf{C} \times \mathbf{R}_{\geq 0}$$

or more cleanly, on the suspension  $S(\mathbf{CP}^1)$ , so then module categories for  $V_h^k(\mathfrak{g})$  and  $Y_h(\hat{\mathfrak{g}})$  should define line operators for “5d Chern-Simons theory with boundary” on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geq 0}$$

where  $\mathbf{A}_q^2 = (\mathbf{C} \times \mathbf{C})_{nc}$  is the noncommutative plane with ring of functions  $\mathbf{C}[x, y]/(xy - qyx)$ , see [GRZ] or particularly Costello’s [Co] work.

*Examples.* The above definition will have been correct if we may answer

**Question Q.** *Is there an analogue of the Beilinson-Drinfeld Grassmannian  $\text{Gr}_{G,q} \rightarrow \text{Ran}\mathbf{A}_q^2$ ?*

Such a factorisation space would for free by Conjecture P define for us a  $q$ -vertex algebra  $V_q^k(\mathfrak{g})$ , by the same construction as for the affine WZW vertex algebra: taking distributions supported at the identity. We expect that  $V_q^k(\mathfrak{g})$  should be a  $q$ -deformation of the affine vertex algebra and should agree with Etingof-Kazhdan’s RTT construction in [EK] when  $\mathfrak{g} = \mathfrak{sl}_n$ .

**Kazhdan-Lusztig.** Conditional on having defined a factorisation algebra  $V_q^k(\mathfrak{g})$  as in Question Q, many interesting questions follow. To begin with, for formal reasons just as for ordinary vertex algebras, given  $V_q^k(\mathfrak{g})$ -modules  $M_1, \dots, M_n$  we expect to obtain a  $\mathcal{D}_q$ -module of *conformal blocks*  $\mathbf{C}^\bullet(M_1, \dots, M_n)$  on  $(\mathbf{A}_q^2)^n$ .

**Question R.** [KL] *Is its restriction to  $(\mathbf{A}^1)_\circ^n$  equal to the  $q$ KZ connection?*

It has been long expected that one may define an affine analogue of the Kazhdan-Lusztig equivalence, and answering the above question would be a first step in understanding whether the geometric proof [CF] might be generalised to the affine setting.<sup>14</sup>

Orthogonally to this, we can try to understand ordinary Kazhdan-Lusztig better. First, we ask whether there is a lift of the *Zhu algebra* functor to the  $q$ -setting, fitting into a commuting square

<sup>13</sup>Its with ring of functions  $\mathbf{C}\langle x, y, q \rangle / (yx - xyq)$  with  $q$  central.

<sup>14</sup>Specifically, one wants a Riemann-Hilbert type functor “RH :  $\text{FactCat}(\mathbf{A}_q^2) \rightarrow \text{FactCat}^{\text{QCoh}}(\mathbf{C}_q^2)$ ”, which sends the category  $V_q^k(\mathfrak{g})\text{-Mod}$  to  $Y_h(\mathfrak{g})\text{-Mod}^{fd}$ .

$$\begin{array}{ccc} V_q^k(\mathfrak{g}) & \xrightarrow{q \rightarrow 1} & V^k(\mathfrak{g}) \\ \Downarrow \text{Zhu} & & \Downarrow \text{Zhu} \\ U_q(\mathfrak{g}) & \xrightarrow{q \rightarrow 1} & U(\mathfrak{g}) \end{array}$$

Noting the appearance of both objects  $U_q(\mathfrak{g})$  and  $V^k(\mathfrak{g})$  appearing in the Kazhdan-Lusztig equivalence, having done this we then ask whether these are the special and general fibres of a structure on  $\mathbf{C} \times \mathbf{R}_{\geq 0}$ :

**Question S.** [KL] Does  $V_q^k(\mathfrak{g})$  induce a topological-holomorphic factorisation algebra  $\mathcal{A}$  on  $\mathbf{C} \times \mathbf{R}_{\geq 0}$ , whose restriction to  $\mathbf{C}$  is  $V^k(\mathfrak{g})$  and whose restriction to  $\mathbf{C} \times \mathbf{R}_{>0}$  is  $U_q(\mathfrak{g})$ ?

One would then hope to interpret the fact that [CF]’s RH functor sends  $V^k(\mathfrak{g})$ -FactMod to  $\text{KD}(U_q(\mathfrak{g}))$ -FactMod as some sort of flatness statement for  $\mathcal{A}$ -FactMod over  $\mathbf{R}_{\geq 0}$ .<sup>15</sup> This may give a new way to understand the recent Kazhdan-Lusztig equivalences [BCDN] coming from 3d mirror symmetry.

### 1.5. Sheaf methods (Con, Loc, Eu)

*Localisation methods.* Torus localisation is one of the main methods in enumerative geometry, and projects Con and Loc were concerned with extending these techniques to the Artin moduli stacks appearing in enumerative geometry. Given a closed Artin substack

$$\mathcal{Z} \hookrightarrow \mathcal{X}$$

not necessarily quasicompact,

**Theorem T.** [Conc] If  $\mathcal{L}_i$  are a collection of line bundles such that at least one of them vanishes on each geometric point  $x \in \mathcal{X} \setminus \mathcal{Z}$ , then

$$\mathbf{C}_{\bullet}^{\text{BM}}(\mathcal{X} \setminus \mathcal{Z})_{\text{loc}} = 0,$$

so then the cohomology of  $\mathcal{X}$  is “concentrated” on  $\mathcal{Z}$ : we have  $i_* : \mathbf{C}_{\bullet}^{\text{BM}}(\mathcal{Z}) \xrightarrow{\sim} \mathbf{C}_{\bullet}^{\text{BM}}(\mathcal{X})$ .

Here we have localised with respect to  $c_1(\mathcal{L}_i)$ , for instance we show the condition holds if  $\mathcal{Z}_0/T \hookrightarrow \mathcal{X}_0/T$  is an inclusion of quotient stacks with  $\dim \text{Stab}_x(T)$  non-maximal for all  $x \in \mathcal{X}_0 \setminus \mathcal{Z}_0$ , and we take for  $\bigoplus \mathcal{L}_i$  the tautological  $T$ -bundle.

**Theorem U.** [Loc] If  $i : \mathcal{X}^T \hookrightarrow \mathcal{X}$  is the inclusion of the homotopy fixed points of a torus action on quasismooth dg scheme  $\mathcal{X}$ , there is a **Gysin pullback** map  $i^! : \mathbf{C}_{T,\bullet}^{\text{BM}}(\mathcal{X})_{\text{loc}} \rightarrow \mathbf{C}_{T,\bullet}^{\text{BM}}(\mathcal{X}^T)_{\text{loc}}$  satisfying Atiyah-Bott and Graber-Pandharipande formulas:

$$\text{id} = i_* \frac{i^!(-)}{e(N_{\text{vir}})}, \quad [\mathcal{X}]^{\text{vir}} = i_* \frac{[\mathcal{X}^T]^{\text{vir}}}{e(N_{\text{vir}})}, \quad (2)$$

relating to pushforward and fundamental classes.

This recovers the usual torus localisation results when  $\mathcal{Z} = X^T/T$  and  $\mathcal{X} = X/T$  are quotients of smooth finite-type schemes by tori.

<sup>15</sup>By means of extra evidence, it seems plausible that ordinary Riemann-Hilbert  $\mathcal{D}\text{-Mod}^{rh} \xrightarrow{\sim} \text{Perv}$  may be interpreted this way, where we consider  $\mathcal{A}$  a sheaf of algebras generated by  $\mathcal{O}_{\Sigma \times \mathbf{R}_{\geq 0}}$  and the Lie algebra  $\mathcal{T}_{\Sigma \times \mathbf{R}_{\geq 0}}$  of infinitesimal automorphisms of  $\Sigma \times \mathbf{R}_{\geq 0}$  whose restriction to the boundary is antiholomorphic.

*Virtual Euler classes and shuffle structures.* In [Eu](#), we strengthen the above results in [Con](#) and [Loc](#) until:

- they give a general geometric method to output *shuffle products* for CoHAs,
- and show CoHAs are compatible with Davison/Yang-Zhao localised/Joyce vertex coproducts.

Specifically, we prove analogues of Theorems T and U for the *vanishing cycle* (or any sheaf) cohomology of arbitrary closed embeddings  $\mathcal{Z} \hookrightarrow \mathcal{X}$  which is quasismooth other a common base, and concentrated with respect to a multiplicative subset  $\mathcal{S} \subseteq H^\bullet(\mathcal{X})$ . As a result,

**Theorem V.** [\[Eu\]](#) For any “split locus” map  $\pi : \mathcal{M}^s \rightarrow \mathcal{M}$ , we get a diagram

$$\begin{array}{ccc}
 C^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) & \xrightarrow{-1/e(N_{i, \text{vir}})} & C^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) & \xrightarrow{p_*^s q^{s,*}} & C^\bullet(\mathcal{M}^s, \varphi^s) \\
 (\pi \times \pi)^* \uparrow & & & & \uparrow \pi^* \\
 C^\bullet(\mathcal{M} \times \mathcal{M}, \varphi \boxtimes \varphi) & \xrightarrow{p_* q^*} & & & C^\bullet(\mathcal{M}, \varphi)
 \end{array} \tag{3}$$

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Here  $p, p^s$  (proper) and  $q, q^s$  (quasismooth) are

$$\begin{array}{ccccc}
 & & \text{SES}^s & & \\
 & q^s \swarrow & \downarrow & \searrow p^s & \\
 \mathcal{M}^s \times \mathcal{M}^s & & \text{SES} & & \mathcal{M}^s \\
 & \downarrow q & \swarrow p & \searrow p & \downarrow \\
 \mathcal{M} \times \mathcal{M} & & & & \mathcal{M}
 \end{array}$$

for instance  $\mathcal{M}$  is *smooth* moduli stack containing as a critical locus  $\text{Crit}W$  the deformed CY3 moduli stacks considered in [CY3](#), and  $\varphi = \varphi_W$ , and we apply localisation to  $i : \text{SES}^s \rightarrow \text{SES} \times_{\mathcal{M}^s} \mathcal{M}$ .

Two consequences of this are:

- If we take  $\mathcal{M}^s$  to be a *shuffle space*<sup>16</sup> given by products of “simple” moduli stacks, e.g. parametris-ing tuples of rank one quiver representations, then (3) recovers shuffle formulas [Daa; SV; YZb] for CoHAs and localised/vertex coproducts.
- If we take  $\mathcal{M}^s = \mathcal{M} \times \mathcal{M}$  together with its direct sum map to  $\mathcal{M}$ , (3) recovers the compatibility [[CV](#), [CY3](#), [Li](#)] between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

Thus, this turns algebraic properties of stacks (shuffle/bialgebra-type structures) into algebraic properties on their critical cohomology. In [OSp](#) this explains the OSp-shuffle module structure on  $H^\bullet(\mathcal{M}^\tau, \varphi^\tau)$ , and plan to generalise this in [SA](#).

## 1.6. Liouville quantum gravity to vertex algebras (LCG)

*History.* In recent years, probabilists have increasingly understood quantum field theory, giving rigorous definitions of Feynman measures for  $2d$  CFTs, e.g. [CRV; DS; Sh] whose “holomorphic part” are expected to be W-algebras, Virasoro, and Heisenberg vertex algebras.

This approach is very different to the factorisation/vertex algebra/functorial QFT approach in the above projects, e.g. it can directly study level sets of fields as SLE curves [MS; SS], there is a rigorous

<sup>16</sup>i.e. shuffle algebra in the category of spaces, see [SA](#).

connection to combinatorial toy models like the discrete Gasussian Free Field [BPR], and it is able to access the *full* CFT, not just the chiral part as we are in geometric representation theory, e.g. [KRV] proves the DOZZ formula for full OPEs in the Liouville CFT.

However, there is currently not much interaction between the two approaches, and this project aims to build a bridge between the two so that techniques/results/heuristics can move between subjects more easily (then give a simple example of this).

*Goal.* In LCG we aim to define a functor from Segal-style  $2d$  conformal field theories to vertex algebras

$$\text{CFT} \xrightarrow{(-)^{ch}} \text{CFT}^{hol} \xrightarrow{\text{Res}} \text{FactAlg}(\mathbf{C})_{\mathbf{C} \times \mathbf{C}^\times}^{hol} \xrightarrow{[\text{CG}]} \text{VertexAlg}, \quad (4)$$

then show that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs are sent to the Heisenberg and Virasoro vertex algebras, respectively.

*Details.* We will need to upgrade  $\mathcal{Z} \in \text{CFT}$  to a definition that remembers the geometric structure on the category  $\text{Cob}_2$  of conformal cobordisms. Namely, consider a complex vector bundle  $\mathcal{V}$  with connection over the Teichmuller space  $\mathcal{T}_{g,n}$  satisfying a factorisation condition, and with a section  $\psi$ . The fibre of this data over  $\Sigma$  is the vector space  $\mathcal{Z}(\partial\Sigma)$  and  $\mathcal{Z}(\Sigma) : \mathbf{C} \rightarrow \mathcal{Z}(\partial\Sigma)$ .

The induced factorisation algebra over  $\mathbf{C}$  is automatically smoothly translation and rotation equivariant, so if it is *holomorphic* (i.e.  $\partial_{\bar{z}}\psi = 0$ ) then it is by [CG] a vertex algebra; these are the last two maps in (4). The equivariance comes from a  $G$ -action on  $\mathcal{T}_{0,n}$ , since then the Lie algebra  $\mathfrak{g}$  acts on  $\mathcal{V}$  by the connection, e.g. the vertex algebras in the image of (4) will automatically have an action by vector fields on  $\mathbf{P}^1$ , so we expect they are VOAs.

The main task is to define a chiralisation functor  $(-)^{ch}$  to holomorphic CFTs, and prove that [GKRV]'s LQG Segal CFT (upgraded appropriately in the above sense) is sent by (4) to the Virasoro vertex algebra, and relate the DOZZ formula [KRV, (1.12)] to the Virasoro OPE. Having done this, we plan to do the same for the GFF, and finally to give a new example of these methods, construct a probability measure in the domain of (4) recovering the affine vertex algebra, e.g. by using the free field embedding [FB, §11] to a direct sum of Heisenberg algebras.

## References

- [AGT] Luis F Alday, Davide Gaiotto, and Yuji Tachikawa. “Liouville correlation functions from four-dimensional gauge theories”. In: *Letters in Mathematical Physics* 91.2 (2010), pp. 167–197.
- [AN] Raschid Abedin and Wenjun Niu. “Yangian for cotangent Lie algebras and spectral  $R$ -matrices”. In: *arXiv preprint arXiv:2405.19906* (2024).
- [BCDN] Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu. “3d mirror symmetry of braided tensor categories”. In: *arXiv preprint arXiv:2304.11001* (2023).
- [BD] Tommaso Maria Botta and Ben Davison. “Okounkov’s conjecture via BPS Lie algebras”. In: *arXiv preprint arXiv:2312.14008* (2023).
- [BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. “Instanton moduli spaces and  $\mathcal{W}$ -algebras”. In: *arXiv preprint arXiv:1406.2381* (2014).
- [Bo] R. E. Borcherds. “Quantum Vertex Algebras”. In: *Advanced Studies in Pure Mathematics*. Vol. 31. Mathematical Society of Japan, 1999, pp. 51–74.
- [BPR] Roland Bauerschmidt, Jiwoon Park, and Pierre-François Rodriguez. “The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature”. In: *The Annals of Probability* 52.4 (2024), pp. 1360–1398.
- [BR] Samuel Belliard and Vidas Regelskis. “Drinfeld J presentation of twisted Yangians”. In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 13 (2017), p. 011.
- [BS] Roland Bittleston and David Skinner. “Gauge theory and boundary integrability”. In: *Journal of High Energy Physics* 2019.5 (2019), pp. 1–53.
- [CF] Li Chen and Yuchen Fu. “An Extension of the Kazhdan-Lusztig Equivalence”. PhD thesis. Harvard University, 2022.
- [CG] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*. Vol. 31. New Mathematical Monographs. Cambridge University Press, Cambridge, 2017, pp. ix+387.
- [Ch] Kuo-tsai Chen. “Iterated integrals of differential forms and loop space homology”. In: *Annals of Mathematics* 97.2 (1973), pp. 217–246.
- [Co] Kevin Costello. “M-theory in the Omega-background and 5-dimensional non-commutative gauge theory”. In: *arXiv preprint arXiv:1610.04144* (2016).
- [CRV] Baptiste Cerclé, Rémi Rhodes, and Vincent Vargas. “Probabilistic construction of Toda conformal field theories”. In: *arXiv preprint arXiv:2102.11219* (2021).
- [CWY] Kevin Costello, Edward Witten, and Masahito Yamazaki. “Gauge theory and integrability, I”. In: *arXiv preprint arXiv:1709.09993* (2017).
- [Daa] Ben Davison. “The critical CoHA of a quiver with potential”. In: *Quarterly Journal of Mathematics* 68.2 (2017), pp. 635–703. arXiv: arXiv:1311.7172 [math.AG].
- [Dab] Ben Davison. “Affine BPS algebras,  $\mathcal{W}$  algebras, and the cohomological Hall algebra of  $\mathbf{A}^2$ ”. In: *arXiv preprint arXiv:2209.05971* (2022).
- [Dac] Ben Davison. “The integrality conjecture and the cohomology of preprojective stacks”. In: *Journal für die reine und angewandte Mathematik* 804 (2023), pp. 105–154.

- [Dr] Vladimir Gershonovich Drinfeld. “Hopf algebras and the quantum Yang–Baxter equation”. In: *Doklady Akademii Nauk*. Vol. 283. 5. Russian Academy of Sciences. 1985, pp. 1060–1064.
- [DS] Bertrand Duplantier and Scott Sheffield. “Liouville quantum gravity and KPZ”. In: *Inventiones mathematicae* 185.2 (2011), pp. 333–393.
- [EK] P. Etingof and D. Kazhdan. “Sel. Math., New Ser. 6, No. 1, 105–130”. In: *Selecta Mathematica, New Series* 6.1 (2000), pp. 105–130.
- [FB] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*. 88. American Mathematical Soc., 2004.
- [FMW] Keegan J Flood, Mauro Mantegazza, and Henrik Winther. “Jet functors in noncommutative geometry”. In: *arXiv preprint arXiv:2204.12401* (2022).
- [FR] Edward Frenkel and Nicolai Reshetikhin. “Towards deformed chiral algebras”. In: *arXiv preprint* (1997). arXiv: q-alg/9706023 [q-alg].
- [Ga] Dennis Gaitsgory. “On factorization algebras arising in the quantum geometric Langlands theory”. In: *Advances in Mathematics* 391 (2021), p. 107962.
- [GKRV] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. “Segal’s axioms and bootstrap for Liouville Theory”. In: *arXiv preprint arXiv:2112.14859* (2021).
- [GLW] Sachin Gautam, Valerio Toledano Laredo, and Curtis Wendlandt. “The meromorphic R-matrix of the Yangian”. In: *Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin*. Springer, 2021, pp. 201–269.
- [Gr] James Alexander Green. *Shuffle algebras, Lie algebras and quantum groups*. Vol. 9. Departamento de Matemática da Universidade de Coimbra, 1995.
- [GRZ] Davide Gaiotto, Miroslav Rapčák, and Yehao Zhou. “Deformed Double Current Algebras, Matrix Extended  $\mathcal{W}_{1+\infty}$  Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection”. In: *arXiv preprint arXiv:2309.16929* (2023).
- [KK] Tasuki Kinjo and Naoki Koseki. *Cohomological  $\chi$ -independence for Higgs bundles and Gopakumar-Vafa invariants*. 2023. arXiv: 2112.10053 [math.AG].
- [KMT] Yasushi Komori, Kohji Matsumoto, and Hirofumi Tsumura. “A study on multiple zeta values from the viewpoint of zeta-functions of root systems”. In: *Functiones et Approximatio Commentarii Mathematici* 51.1 (2014), pp. 43–46.
- [KPS] Tasuki Kinjo, Hyeonjun Park, and Pavel Safronov. “Cohomological Hall algebras for 3-Calabi-Yau categories”. In: *arXiv preprint arXiv:2406.12838* (2024).
- [KRV] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. “Integrability of Liouville theory: proof of the DOZZ formula”. In: *Annals of Mathematics* 191.1 (2020), pp. 81–166.
- [KS] Maxim Kontsevich and Yan Soibelman. “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants”. In: *arXiv preprint arXiv:1006.2706* (2010).
- [Ku] Atsuo Kuniba. “Matrix product solutions to the G 2 reflection equation”. In: *Journal of Integrable Systems* 3.1 (2018), xyy008.
- [Lu] Jacob Lurie. *Higher Algebra*. Preprint, available at <http://www.math.harvard.edu/~lurie>. 2016.



- [Maa] Shahn Majid. “Transmutation theory and rank for quantum braided groups”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 113. 1. Cambridge University Press. 1993, pp. 45–70.
- [Mab] Shahn Majid. “Double-bosonization of braided groups and the construction of  $U_q(\mathfrak{g})$ ”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 125. 1. Cambridge University Press. 1999, pp. 151–192.
- [Mi] Antun Milas. “Generalized multiple q-zeta values and characters of vertex algebras”. In: *arXiv preprint arXiv:2203.15642* (2022).
- [MMSV] Anton Mellit, Alexandre Minets, Olivier Schiffmann, and Eric Vasserot. “Coherent sheaves on surfaces, COHAs and deformed  $\mathcal{W}_{1+\infty}$ -algebras”. In: *arXiv preprint arXiv:2311.13415* (2023).
- [MO] Daveshe Maulik and Andrei Okounkov. “Quantum groups and quantum cohomology”. In: *arXiv preprint arXiv:1211.1287* (2012).
- [MS] Shahn Majid and Francisco Simão. “Quantum jet bundles”. In: *Letters in Mathematical Physics* 113.6 (2023), p. 120.
- [RSYZ] Miroslav Rapčák, Yan Soibelman, Yaping Yang, and Gufang Zhao. “Cohomological Hall algebras, vertex algebras and instantons”. In: *Communications in Mathematical Physics* 376.3 (2020), pp. 1803–1873.
- [Sh] S Sheffield. “Gaussian free fields for mathematicians. preprint”. In: *arXiv preprint math.PR/0312099* (2003).
- [SS] O. Schramm and S. Sheffield. “SS”. In: *Annals of Probability* 33.6 (2005), pp. 2127–2148.
- [SV] Olivier Schiffmann and Eric Vasserot. “On cohomological Hall algebras of quivers: generators”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2020.760 (2020), pp. 59–132.
- [Sz] Balázs Szendroi. “Cohomological Donaldson–Thomas theory”. In: *Proceedings of String-Math* 2015 (2014).
- [Th] J. Thomas. “Kontsevich’s Swiss cheese conjecture”. In: *Geometry & Topology* 20.1 (2016), pp. 1–48.
- [YZa] Yaping Yang and Gufang Zhao. “Quiver varieties and elliptic quantum groups”. In: *arXiv preprint arXiv:1708.01418* (2017).
- [YZb] Yaping Yang and Gufang Zhao. “Cohomological Hall algebras and affine quantum groups”. In: *Selecta Mathematica* 24.2 (2018), pp. 1093–1119.