(BORCHERDS) TWISTS OF QFTS

ALEXEI LATYNTSEV

Let Cob be one's favourite category of cobordisms, e.g. whose objects are topological manifolds of specific dimension and morphisms cobordisms between them. Let $\mathcal{Q}$ be a *quantum field theory*, i.e. a functor

$$\mathcal{Q} \;:\; \mathrm{Cob} \;\to\; \mathrm{Vect}$$

which is symmetric monoidal. If we weaken to just monoidal, e.g. as we might do if we use the category Cob^{ord} of topological manifolds with a labelling of their connected components by some poset, this is called a *nonlocal* QFT, or choose another target category $\mathcal{E}$ and just ask that $\mathcal{Q}$ is braided-monoidal, this is called a *braided-local* QFT, etc.

1.1. A *Borcherds twist* $J_{\mathcal{Q}}$ of $\mathcal{Q}$ is a pair of endomorphisms

$$J_{\mathcal{Q}}(M)^- \;:\; \mathcal{Q}(N) \;\to\; \mathcal{Q}(N), \qquad\qquad J^+_{\mathcal{Q}}(M) \;:\; \mathcal{Q}(N') \;\to\; \mathcal{Q}(N')$$

for every morphism $M : N \to N'$ in Cob, such that given a composition of two cobordisms

$$N_1 \stackrel{M}{\to} N_2 \stackrel{M'}{\to} N_3,$$

the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{Q}(N_1) \arrow[r, "\mathcal{Q}(M)"] & \mathcal{Q}(N_2) \arrow[r, "J^+_{\mathcal{Q}}(M)"] & \mathcal{Q}(N_2) \arrow[r, "J^-_{\mathcal{Q}}(M')"] & \mathcal{Q}(N_2) \arrow[r, "\mathcal{Q}(M')"] & \mathcal{Q}(N_2) \arrow[dr, "J^+_{\mathcal{Q}}(M')"] & \\
\mathcal{Q}(N_1) \arrow[ur, "J^-_{\mathcal{Q}}(M)"] \arrow[rr, "J^-_{\mathcal{Q}}(M\cdot M')"'] & & \mathcal{Q}(N_1) \arrow[rr, "\mathcal{Q}(M\cdot M')"'] & & \mathcal{Q}(N_3) \arrow[rr, "J^+_{\mathcal{Q}}(M\cdot M')"'] & & \mathcal{Q}(N_3)
\end{tikzcd}$$

Moreover, we ask that if $M = M_+ \sqcup M_-$ is a disjoint union of two cobordisms (inducing the monoidal structure in Cob is) then

$$J_{\mathcal{Q}}(M_- \sqcup M_+) \;\simeq\; J_{\mathcal{Q}}(M_-) \otimes J_{\mathcal{Q}}(M_+).$$

If $\mathcal{Q}$ is (braided) local, we require the above isomorphisms to respect the (braided) monoidal structure of $\mathcal{Q}$. The definition is made so that

Theorem 1.1.1. *The twist*

$$\tilde{\mathcal{Q}} \;=\; \mathcal{Q} \cdot J_{\mathcal{Q}}$$

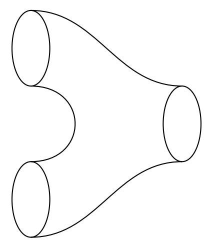
defines a new QFT, which is equal to $\mathcal{Q}$ on objects and with $\tilde{\mathcal{Q}}(M) = \mathcal{Q}(M) \cdot J_{\mathcal{Q}}(M)$ on morphisms. The set Borc *of all Borcherds twists forms a groupoid, which acts on* $\mathrm{QFT} = \mathrm{Fun}(\mathrm{Cob}, \mathrm{Vect})$ *by the above.*

1.1.2. An simple case is when we have invertible endomorphisms

$$J_{\mathcal{Q}}(N) \,:\, \mathcal{Q}(N) \xrightarrow{\sim} \mathcal{Q}(N)$$

for every object N of Cob; this gives a Borcherds twist using $J^-_{\mathcal{Q}}(M) = J_{\mathcal{Q}}(N)^{-1}$ and $J^+_{\mathcal{Q}}(M) = J_{\mathcal{Q}}(N')$. (it doesn't satisfy the monoidal condition?)

1.1.3. For instance, if $\mathcal{A}$ is a two-dimensional TQFT, i.e. a commutative Frobenius algebra, the pair of pants Σ



we can compose by an endomorphism

$$J \,:\, \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

and more generally if there is a cobodisms $M : D^I \to D^J$, we precompose and postcompose by $\prod_{i,i' \in I} J_{i,i'}$ and $\prod_{j,j' \in J} J_{j,j'}$ respectively. This defines a Borcherds twist of $\mathcal{A}$ so long as (write conditions)

1.1.4. *Question.* If Cob is the category of open sets inside n-dimensional Euclidean space, then $\mathrm{Fun}^{\otimes}(\mathrm{Cob}, \mathrm{Vect}) \simeq \mathbf{E}_n\text{-Alg}(\mathrm{Vect})$, and if $n \geqslant 2$ this is equivalent to the category of algebras over the Kontevich-Tamarkin graph operad. What interpretation does the Borcherds twist have in this context?