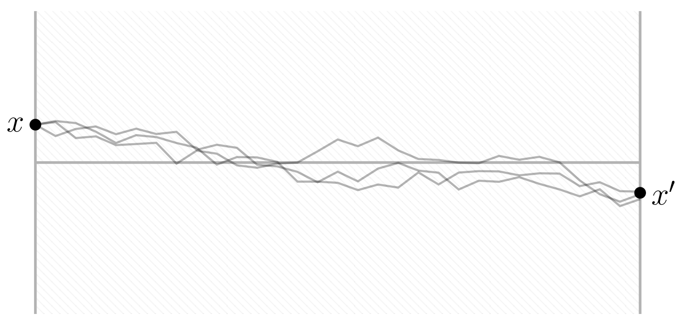
NOTES ON QFT

ALEXEI LATYNTSEV

1. Brownian motion as summing over paths



One can think of Brownian motion on X as a one-dimension quantum field theory. Given a point $x \in X$ (think of it as being a particle at x), the probability of it moving to point x' after time t is

$$\mathbf{P}(x \to x') \;=\; \frac{\#\{\text{Brownian walks } x \to x'\}}{\#\{\text{Brownian walks } x \to ?\}}.$$

In the discrete setting where X is a weighted graph, this formula is literally correct, and if X is a Riemannian manifold we need to replace each count by an integral over paths, and replace x' by an arbitrary measurable subset.

In the above case, the probability is a normal distribution with mean x and variance t.

1.1. The general structure of the above is we have a correspondence

$$\begin{tikzcd} & \{\text{paths } [0,t] \xrightarrow{\gamma} X\} \arrow[dl, "\mathrm{Res}_0"'] \arrow[dr, "\mathrm{Res}_t"] & \\ X & & X \end{tikzcd}$$

of measurable spaces, with relative measures on the restriction maps given in this case by the Brownian motion measure. Calling these correspondences C_{t_1,t_2}, we require them to be compatible in the obvious sense. This allows us to push-and-pull functions,

$$\begin{tikzcd} & \mathcal{O}\left(\mathrm{Maps}([0,1],X)\right) \arrow[dr, "\int_{\mathrm{Res}_t}"] & \\ \mathcal{O}(X) \arrow[ur, "\mathrm{Res}_0^*"] \arrow[rr, "\mathcal{Z}_t"] & & \mathcal{O}(X) \end{tikzcd}$$

and the compatibility condition thus gives us an action of $\mathbf{R}_{\geqslant 0}$ on $\mathcal{O}(X)$.[1] If we impose enough smoothness requirements, it must take the form $\mathcal{Z}_t = e^{Ht}$ for an endomorphism H of $\mathcal{O}(X)$.

1.1.1. In the above context, it might be useful to think of functions $f \in \mathcal{O}(X)$ also as "random points on X" if they have norm one, or as "the wavefunction of a particle on X". If we work with a function space that contains the Dirac delta δ_x, this should be thought of as a non-random point, although typically applying $\mathcal{Z}_t$ will not give back a delta function.

1.1.2. We can write the above definition in symbols as

$$\mathcal{Z}_t \;:\; f \;\mapsto\; \int_{\mathrm{Res}_0^{-1}(f)} \Psi \, d\mathcal{B}$$

where $\mathcal{B}$ is the Brownian measure on the set of paths and we have integrated along the fibres of Res_t. In other words, the coefficient of an element $\alpha \in \mathcal{O}(X)^*$ in the above is

$$\langle \alpha, \mathcal{Z}_t f \rangle \;=\; \int_{\Psi \,:\, \Psi|_0 = f} \langle \alpha, \Psi|_t \rangle \, d\mathcal{B}.$$

1.2. The compatibility condition between the correspondences is that we have a pullback

$$\begin{tikzcd}[column sep=tiny]
& & (C_{t_1,t_3}, d\mathcal{B}_{t_1,t_3}) \arrow[dl] \arrow[dr] & & \\
& (C_{t_1,t_2}, d\mathcal{B}_{t_1,t_3}) \arrow[dl] \arrow[dr] & & (C_{t_2,t_3}, d\mathcal{B}_{t_2,t_3}) \arrow[dl] \arrow[dr] & \\
X & & X & & X
\end{tikzcd}$$

of measure spaces. In other words, the Brownian measure is compatible under cutting up of the time interval; this is also called the *Markov* property of the measure.

Thus if we modify the measure to $e^S d\mathcal{B}$, it is consistent in the above sense if and only if the function

$$S_{0,t} \;\in\; \mathcal{O}(\mathrm{Maps}([0,t], X))$$

is memoryless, i.e. satisfies the cocycle condition

$$S_{t_1,t_3} \;=\; S_{t_1,t_2} \;+\; S_{t_2,t_3}$$

where all three are viewed as functions on $\mathrm{Maps}([t_1, t_3], X)$ by restriction. The set of such functions (modulo functions supported on the measure zero set measure zero $\{t_2\} \times X$) is closed under addition and multiplication.

For instance, we may take a function $s \in \mathcal{O}(X)$ and integrate it over $[0,t]$ to get a function $S_{0,t}(\gamma) = \int_{[0,t]} \gamma^* s$ on the path space. Taking the constant function gives for instance $S_{0,t}(\gamma) \;=\; kt$. We could also take a covector field $\xi \in \mathcal{O}(T^*X)$ and evaluate it on the derivative of the path to get $S_{0,t}(\gamma) = \int_{[0,1]} \langle \xi, d\gamma \rangle$. Taking higher order differential forms gives more examples. A non-example is evaluating the path at a particular point.

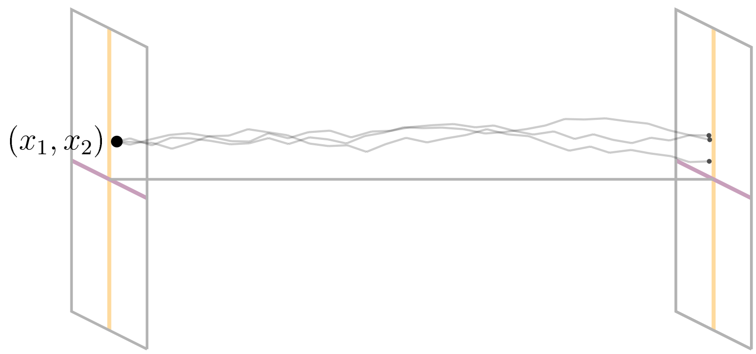
[1] This includes the condition that $\mathcal{Z}_0 = \mathrm{id}$ and that $\mathcal{Z}_{t+t'} = \mathcal{Z}_t \cdot \mathcal{Z}_{t'}$.

A popular choice is

$$S_{0,t}(\gamma) \;=\; \int_{[0,t]} (\gamma', \gamma') dt$$

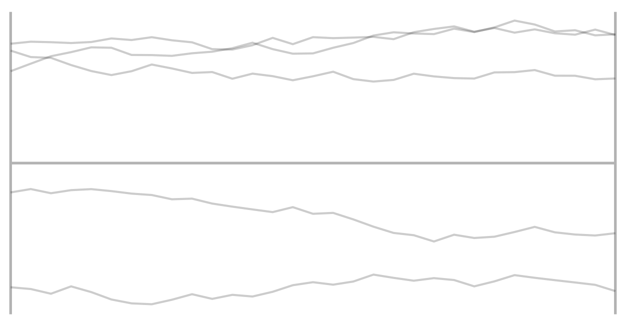
where we have used the Riemannian metric on X.

1.3. There are variants one could consider. For instance, one could consider *coloured* points on X, which is then equivalent to Brownian motion on $X^{\#\text{colours}}$,

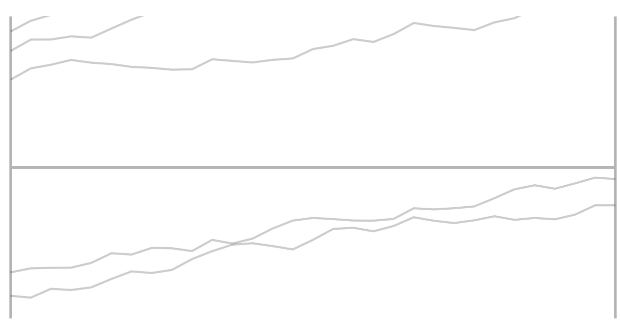


If we want to allow the colours to interact, then we need to change the metric on $X^{\#\text{colours}}$, adding off-diagonal terms. This will mean the Laplacian and hence Brownian motion will have off-diagonal terms.

1.3.1. Another variant is Brownian motion with drift. If a random sample of paths with respect to the Brownian motion measure looks like



then Brownian motion with drift will look like



The Brownian motion with drift $\mathcal{D}$ satisfies the following stochastic differential equation:

$$d\mathcal{D} \;=\; d\mathcal{B} \;+\; k\,dt$$

where real number k is the drift term. Or, viewing $\mathcal{D}$ and $\mathcal{B}$ as random paths $[0,t] \to X$, we have

$$\mathcal{D} \;=\; \mathcal{B} \;+\; k\,t.$$

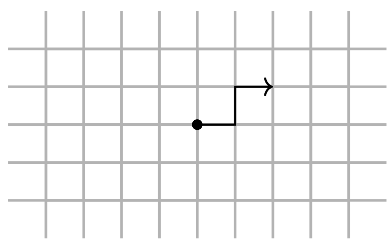
On a general space X with one parameter family of automorphisms $\varphi_t \;=\; e^{tv}$ where here v is a vector field on X, we can likewise define Brownian motion with drift as

$$\mathcal{D}_t \;=\; \varphi_t^* \mathcal{B}_t.$$

Taking the translation vector field on the real line gives back ordinary Brownian motion with drift. We can view the above as changing the projection map:

$$\begin{tikzcd}
& \{\text{paths } [0,t] \xrightarrow{\gamma} X\} \arrow[dl, "\mathrm{Res}_0"'] \arrow[dr, "\varphi_t \cdot \mathrm{Res}_t"] & \\
X & & X
\end{tikzcd} \tag{1}$$

1.3.2. Brownian motion on $\mathbf{R}^n$ are a limit of random walks on $r\mathbf{Z}^n$, taking the limit $r \to 0$.

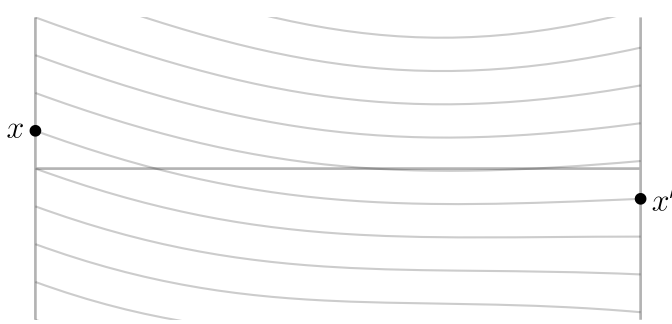


If we use discrete time, we can replace $X = \mathbf{Z}^n$ with any Markov chain, and define $\mathcal{Z}_t = P^t$ in terms of the Markov transition matrix P; since the transition matrix is orthogonal it is diagonalisable and so we may define $\mathcal{Z}_t$ for any real t. Note that we can also write $\mathcal{Z}_t = e^{t \log P}$.

1.3.3. In dynamical systems or ergodic theory, one often considers one-parameter families of automorphisms $\Phi_t : X \xrightarrow{\sim} X$. This gives a map

$$X \;\to\; \mathrm{Maps}([0,t], X)$$

and we can take the pushforward of the usual measure. In other words, given an initial starting point x the only point with nonzero probability it goes to is $\Phi_t(x)$.



Or in the previous notation, the only path γ restricting at 0 to x is the path $\gamma(-) = \Phi_-(x)$,

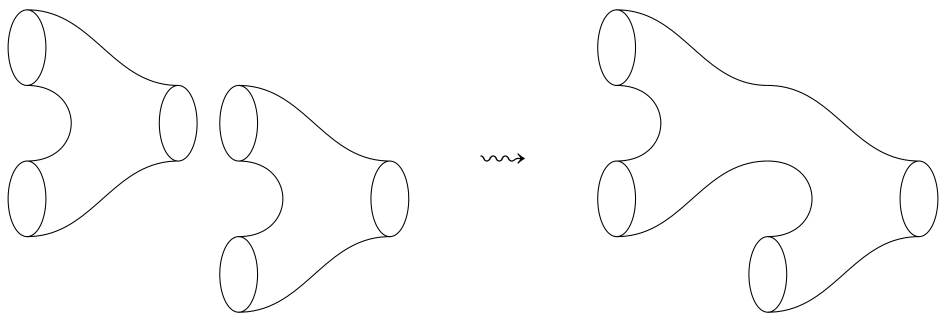
$$\begin{tikzcd} & \gamma \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & \Phi_t(x) \end{tikzcd}$$

Thus this is a "classical" example. To get non-classical examples, one needs to consider *random* dynamical systems, see for instance [Ar].

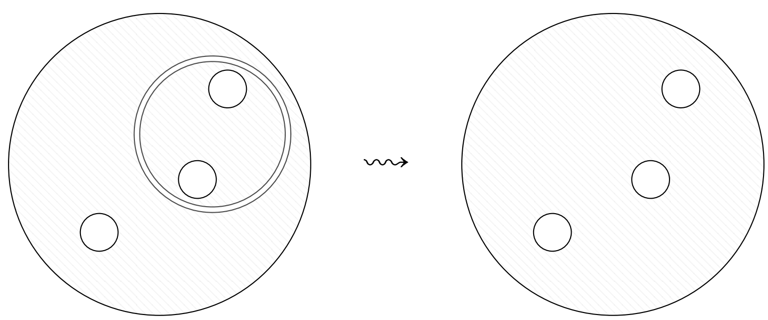
2. Two dimensions

We now replace $[0, t]$ with a two-dimensional surface Σ, i.e. consider two-dimensional quantum field theories. Thus we consider "particles on two-dimensional spacetime". The Markov compatibility condition, which previously had to do with gluing intervals:

will now be replaced with the Markov *domain* property, which has to do with gluing surfaces:



or in other words, it is a Markov property for splitting up a region using codimension one walls:



The basic example is the Gaussian free field. (explain) The Markov property follows from the Markov property for Laplacians,

$$\Delta_{\Sigma_1 \sqcup \Sigma_2} \;=\; \Delta_{\Sigma_1} + \Delta_{\Sigma_2},$$

and hence for the eigenfunctions. (check)

2.1. The general structure is (cobordism stuff)

2.2. We now consider variants.

2.2.1. What is the two-dimensional analogue of Brownian motion with drift? To begin with we need to understand the role of $\mathbf{R}_{\geqslant 0}$ in the one-dimensional case: we identify it as

$$\mathbf{R}_{\geqslant 0} \;\simeq\; \mathrm{Hom}_{\mathrm{Cob}_1}(\mathrm{pt}, \mathrm{pt}) \;=\; \mathrm{Cob}_1(\mathrm{pt}, \mathrm{pt}).$$

Thus, in line with (1) given a functor $F : \mathrm{Cob}_1 \to \mathrm{Sp}^{\mu}$ out from Cob_1 to the category of spaces equipped with a measure, for each correspondence

$$\begin{tikzcd} & C \simeq [t_1,t_2] & \\ \{t_1\} \arrow[ur] & & \{t_2\} \arrow[ul] \end{tikzcd}$$

we can ask for an action $\varphi_C : F(\{t_2\}) \to F(\{t_2\})$. In the case of Brownian motion with drift this will be just be $\varphi_{t_2-t_1}$. We want this to be compatible in the the sense that the composition (pullback) of

$$\begin{tikzcd} & F(C) \arrow[dl, "\mathrm{Res}_{t_1}"'] \arrow[dr, "\varphi_C\cdot\mathrm{Res}_{t_2}"] & & F(C') \arrow[dl, "\mathrm{Res}_{t_2}"'] \arrow[dr, "\varphi_{C'}\cdot\mathrm{Res}_{t_3}"] & \\ \{t_1\} & & \{t_2\} & & \{t_3\} \end{tikzcd}$$

is equal to

$$\begin{tikzcd} & F(C \sqcup_{\{t_2\}} C') \arrow[dl, "\mathrm{Res}_{t_1}"'] \arrow[dr, "\varphi_{C\sqcup_{\{t_2\}}C'}\cdot\mathrm{Res}_{t_3}"] & \\ \{t_1\} & & \{t_3\} \end{tikzcd}$$

In other words, whenver C, C' are composable cobordisms (this is always true in the one dimensional case), we have the cocycle condition

$$\varphi_{C\cup C'} \;=\; \varphi_C \cdot \varphi_{C'}.$$

Equivalently, note that $\mathrm{Mor}\,\mathrm{Cob}_1$ is a groupoid over Cob_1, i.e. we have head and tail maps

$$\begin{tikzcd} & \mathrm{Mor}\,\mathrm{Cob}_1 \arrow[dl] \arrow[dr] & \\ \mathrm{Cob}_1 & & \mathrm{Cob}_1 \end{tikzcd}$$

and φ may be viewed as an action of this groupoid on F. (write details)

2.2.2. In particular, in the two-dimensional case we will need an action of (write explicitly)

As a consequence, we can ask for an action of the semigroup $\mathcal{A}$ of a parametrised annuli (or, monoid of thin annuli) on $F(S^1)$. To be explicit, it is

$$\mathcal{A} \;=\; \{A \subseteq \mathbf{C} \text{ an annulus},\; S^1 \sqcup S^1 \xrightarrow{\sim} \partial A\}/\Delta S^1,$$

see [Se], which as a topological space is homeomorphic to

$$\mathcal{A} \;\simeq\; (0,1) \times (\mathrm{Aut}^+(S^1) \times \mathrm{Aut}^+(S^1))/\Delta S^1,$$

given by the ratio of the two annulus radii, and automorphisms of the parametrisations.[2] For thin annuli, we (presumably) use $\mathbf{R}_{\geqslant 0}$ instead of $(0,1)$. One step up, for each pair of Riemannian pants we have an action on $F(S^1)$, and this action is compatible with the semigroup of annuli action. Likewise we have compatible data for other surfaces.

(maybe we want to act on $F(S^1 \sqcup S^1)$ also?)

[2]It will follow from this that (we expect) vertex algebras coming from CFTs will always have an action of the Virasoro, i.e. be vertex *operator* algebras.

2.2.3. What is the two-dimensional analogue of Markov chains?

(maybe we just need a representation of the Lie algebra $\mathfrak{a}$ of $\mathcal{A}$? The analogue of $\log P$)

For ordinary Markov chains, we use that $r\mathbf{N}$ is a discrete analogue of $\mathbf{R}_{\geqslant 0}$, which in some sense converges to $\mathbf{R}_{\geqslant 0}$ as $r \to 0$. Thus, we need to construct a discrete analogue of the category of cobordisms. (or something like that?)

To begin with, we find a discrete analogue of $\mathcal{A}$. We have $r \cdot \mathbf{N}_{\geqslant 0}$ a discrete analogue of $\mathbf{R}_{\geqslant 0}$, and a discrete analogue of $\mathrm{Aut}^+(S^1)$ is (what? use the root lattice of $\mathfrak{aut}^+(S^1)$ and exponentiate it) This is

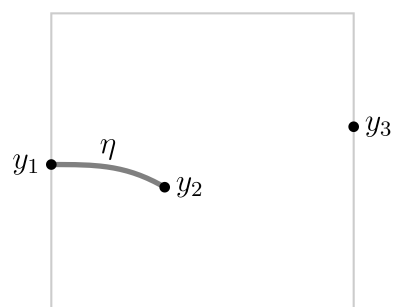
$$\mathbf{N}_{\geqslant 0} \times \Lambda$$

where $\Lambda \subseteq \mathfrak{witt}$ is a lattice inside the Witt Lie algebra of vector fields on S^1, closed under the bracket. For instance, in the holomorphic case we can take

$$\mathbf{N}_{\geqslant 0} \times \mathbf{Z}[z^n \partial_z].$$

2.3. The loop-erased random walk does *not* give an example, however it in some sense lies between the dimension one and two cases.

It satisfies the domain Markov property in the sense that if we have a loop-erased random η walk on Y,



then the loop-erased random walk conditional on starting at η is equivalent to the loop-erased random walk from y_2 to y_3. In other words, it has to do with gluing

$$Y_1 = \eta, \; Y_2 = (Y \setminus \eta) \qquad \rightsquigarrow \qquad Y.$$

We thus define a category whose objects are dimension zero and one manifolds with boundary, and morphisms are *cobordisms*, i.e. manifolds with boundary N with submanifolds

$$Y_1, Y_2 \hookrightarrow N$$

such that the complement has no boundary and (what?)

For instance, the above picture represents two cobordisms

$$\{y_1\} \xrightarrow{\eta} \{y_2\} \xrightarrow{Y \setminus \eta} \{y_3\}.$$

(maybe instead we should consider manfiolds with defect?)

maybe instead we need to consider

$$\eta \xrightarrow{Y} \{y_3\}$$

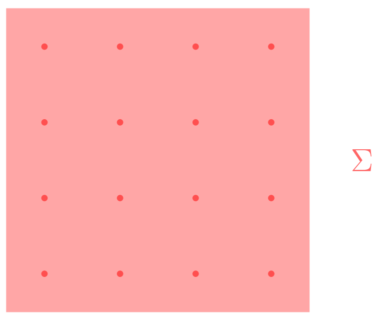
2.3.1. One can likewise consider loop-erased Markov chains, see [La].

2.4. **Thermodynamic examples.**

2.4.1. Another class of examples comes from statistical mechanics, we will write some two-dimensional examples here.

2.4.2.

2.4.3. *Ising model.* Consider a finite graph Λ of particles on a Riemann surface, each can be in two states $\{\pm 1\}$, called *spin up* or *spin down*. Pick a positive real number T called the *temperature*.



Something that is close to (but not) a classical CFT is:

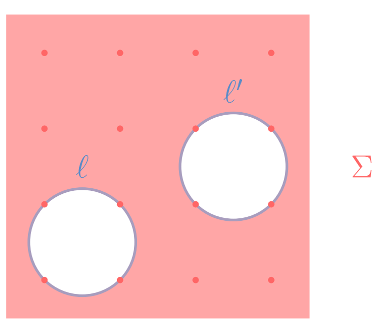
$$\tilde{\mathcal{Z}}(\Sigma) \;=\; \mathrm{Fun}\,(\Lambda, \{\pm 1\})$$

with the probability measure given by

$$\mu(\sigma) \propto \exp\left(-\frac{1}{T}\sum_{\lambda\sim\lambda'}\sigma(\lambda)\sigma(\lambda')\right)$$

where the sum is called the *energy*.

Now given one-manifolds



then setting $\mathcal{Z}(\ell) = \mathrm{Fun}\,(\Lambda \cap \ell, \{\pm 1\})$, we have restriction maps

$$
\begin{tikzcd}
 & \tilde{\mathcal{Z}}(\Sigma) \arrow[dl] \arrow[dr] & \\
\mathcal{Z}(\ell) & & \mathcal{Z}(\ell')
\end{tikzcd}
$$

which we can pull-push along using the measure:

$$\mathcal{Z}(\ell) \to \mathcal{Z}(\ell') \qquad\qquad f \mapsto \sum_{\sigma\,:\,\sigma|_\ell = f} \mu(\sigma)\cdot\sigma|_{\ell'}.$$

Thus the functions $\sigma \in \tilde{\mathcal{Z}}(\Sigma)$ that contribute the most to this map are the low energy ones.

(cut the following?) One can compute that (for square lattices in **C**),

$$\mathbf{E}\left(\sigma(\lambda_1)\sigma(\lambda_2)\right) \approx \begin{cases} \log|\lambda_1-\lambda_2| & |\lambda_1-\lambda_2| \ll L \\ e^{-|\lambda_1-\lambda_2|/L}\cdot|\lambda_1-\lambda_2|^{1/2} & |\lambda_1-\lambda_2| \gg L \end{cases}$$

Where the so called *length scale* L is a function of T that has a single pole at T_c, the *critical temperature* (see [To]). Thus,

- away from critical temperature, a generic σ will have blobs of the same spin, with most blobs of radius approximately L,
- at critical temperature there are blobs of all sizes, and the correlation between the value of $\sigma(\lambda_1)$ and $\sigma(\lambda_2)$ is to leading order $\log|\lambda_1-\lambda_2|$.

Conjecture 2.4.4. *If we take some appropriate limit over finer and finer graphs $\Lambda \subseteq \Sigma$, the Ising model at critical temperature gives a $2d$ conformal field theory.*

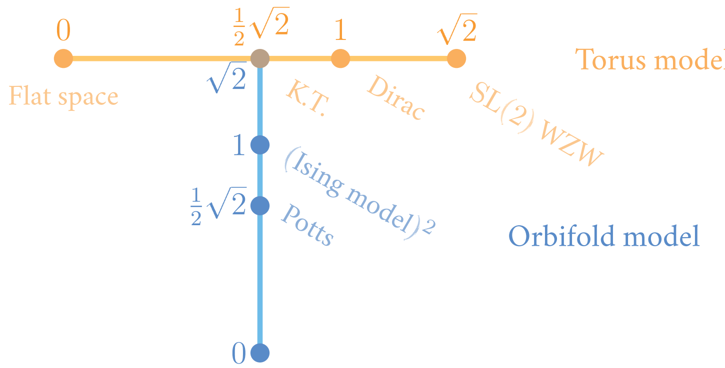
This is related to *conformal nets* in mathematics, see [He].

2.4.5. *Percolation.* (write)

2.5. **Conformal $2d$ theories.** (write)

2.6. **Chiral conformal theories.** (write)

2.7. **All $2d$ CFTs with charge one.** Physicists expect that the only conformal field theories on which the Virasoro acts with central charge $c = 1$ are *torus models* and *orbifold models*, see [DVV]:



The *torus* (or *lattice*) *model* is meant to be a quantisation of maps

$$\mathcal{Z}(S^1) \;=\; \mathcal{O}\left(\mathrm{Maps}(S^1, T)\right) \qquad \text{where} \quad T \;=\; \mathbf{R}/2\pi\frac{R}{2}\mathbf{Z} \,\times\, \mathbf{R}/2\pi\frac{1}{R}\mathbf{Z},$$

with R a positive real number and Maps means homotopy classes of maps. The elements of the mapping space are

$$r \;\mapsto\; \left(\frac{1}{2}nRr, \frac{m}{R}r\right), \qquad n, m \,\in\, \mathbf{Z}.$$

3. Quantum spacetime

3.1. Given the action of S^1 on a torus T, one can consider the subalgebra of

$$\mathcal{O}_q(T) \;\subseteq\; \mathrm{End}(\mathcal{O}(T))$$

generated by the commutative algebra and $\mathcal{O}(T)$ and the endomrophism **q** given by rotation of the ith factor by θ_i. If z_i is a coordinate on the ith factor, we have

$$\mathbf{q} z_i \;=\; e^{2\pi i\cdot\theta_i}\cdot z_i\mathbf{q},$$

and so this subalgeba is noncommutative.

3.1.1. One can consider an infinitesimal version of this, which just requires the action of the Lie algebra $\mathfrak{s}^1$ on T. It is the subalgebra generated by $\mathcal{O}(T)$ and the vector field v generating $\mathfrak{s}^1$.

Indeed, if we let $\mathbf{q} = \mathrm{id} + \hbar v$ be rotation by $\hbar\theta_i$ where $\hbar^2 = 0$, then we have

$$\mathbf{q} z_i \;=\; e^{2\pi i\cdot\hbar\theta_i}\cdot z_i\mathbf{q} \;=\; z_i\mathbf{q} \;+\; \hbar\cdot 2\pi i\theta_i z_i\mathbf{q}.$$

It follows from this that

$$[v, z_i] \;=\; 2\pi i\theta_i z_i.$$

3.1.2. Crucially, it is much less strong a property for a manifold to have an action by $\mathfrak{s}^1$ than the full circle group, and so we can define the noncommutative algebra

$$\langle \mathcal{O}(X), v\rangle \;\subseteq\; \mathcal{D}(X) \;\subseteq\; \mathrm{End}(\mathcal{O}(X)).$$

4. Misc

Brownian motion is a certain random real-valued function on the interval $[0, t]$. In particular, it is a measurable map

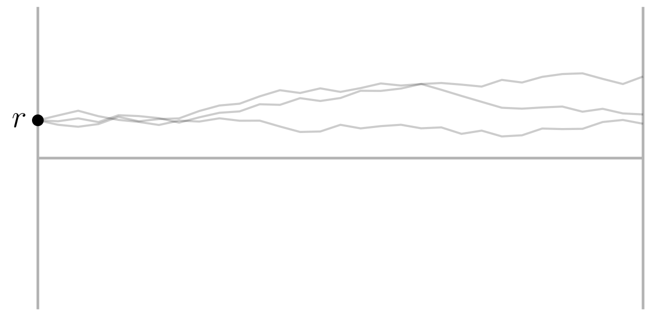
$$B : \Omega \to \mathrm{Fun}([0, t])$$

and so this induces a probability measure on $\mathrm{Fun}([0, t])$. See above for a few functions picked randomly according to this distribution.

For any real number r we can also define a random function on the interval that always begins at r,

$$B_r : \Omega \to \mathrm{Fun}_r([0, t]) \subseteq \mathrm{Fun}([0, t]).$$

Some samples from the induced measure on $\mathrm{Fun}_r([0, t])$:



Taking average endpoint of one of these random functions gives us a linear map:

$$H_t : \mathbf{R} \xrightarrow{\;r \,\mapsto\, \int_{\mathrm{Fun}_r([0,t])} B_r(t)\;} \mathbf{R}$$

Thus H_t is defined as “summing over all paths” to get a transformation. Note that

$$H_t \cdot H_{t'} = H_{t+t'}$$

by the Markov property of Brownian motion. In physics terminology, this gives us a $1d$ quantum field theory. In fact in this case $H_t = \mathrm{id}$, but we will now follow these ideas to get more interesting examples.

4.1. **General picture.** We can restrict functions on an interval to either endpoint:

$$\begin{tikzcd} & \mathrm{Fun}([0,t]) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbf{R} \simeq \mathrm{Fun}(\{0\}) & & \mathrm{Fun}(\{t\}) \simeq \mathbf{R} \end{tikzcd}$$

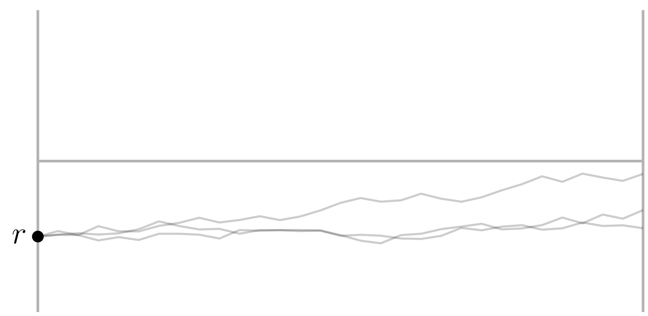
Whenever we have a measure μ_t on $\mathrm{Fun}([0,t])$ plus conditional probability data along p, q, then we get a linear map

$$H_t \ : \ \mathrm{Fun}(\{0\}) \ \to \ \mathrm{Fun}(\{t\}) \qquad\qquad r \ \mapsto \ p_* q^* r \ := \ \int_{F \in q^{-1}(r)} F(t)$$

as before. We need compatibility data to ensure that $H_t \cdot H_{t'} \ = \ H_{t+t'}$.

4.2. **Examples.**

4.2.1. *Brownian motion with drift.* We get that $H_t(r) = r + t$.



4.2.2. *Polynomials.* We can also take polynomials in B, for instance,

$$B^2 + B \ : \ \Omega \ \to \ \mathrm{Fun}([0,t]).$$

All such random functions are bounded below by $-1/2$, i.e. the induced measure on $\mathrm{Fun}([0,t])$ gives measure zero to any measurable set of functions not of this form.

The resulting $H_t : \mathbf{R} \to \mathbf{R}$ will clearly be non-linear. It is easy to compute as $H_t = t + r^2 + r$ since we know the expectation of $B_0(t)^2$ is t since it is a Gaussian distribution.[3] Thus it does not satisfy the Markov property so cannot come from a quantum field theory.

[3]Indeed, $\mathbf{E}(B_r(t)^2 + B_r(t)) = \mathbf{E}((B_0(t) + r)^2 + (B_0(t) + r)) = t + r^2 + r$.

4.2.3. *Remark.* The Markov or *memoryless* property of a random function is related to the fact that physics theories are *local*.

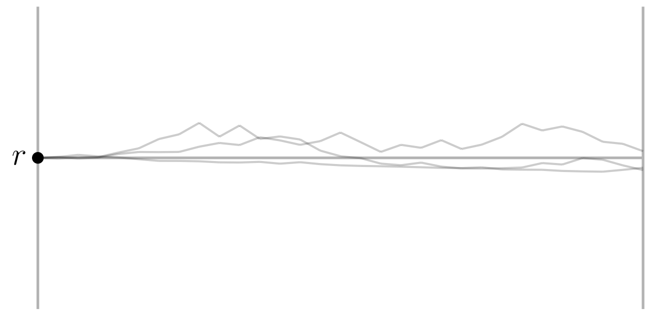
4.2.4. *Ito processes.* To get more examples with the Markov property, note that $B_r(t) = \int_0^r dB_r$, where dB_r is a random one-form. (check) Ito showed that

$$X(t) = \int_0^t f(B)dB$$

is a Markov process for f any L^2 function, and more generally (write). For instance,

$$\int_0^t BdB = \frac{1}{2}(B^2 - t)$$

which still gives $H_t = \mathrm{id}$ since its expectation is zero.



(is this Markov?)

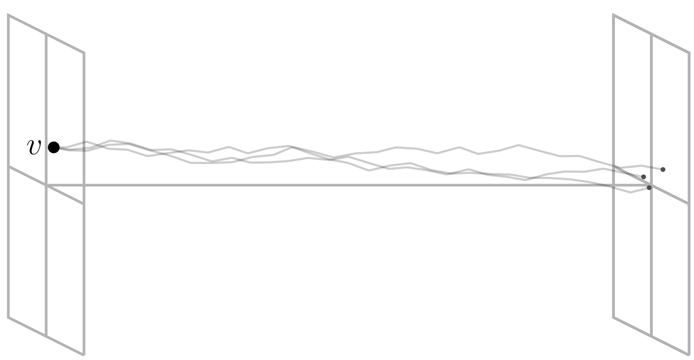
4.2.5. *Brownian motion in* $\mathbf{R}^d$. We can consider Brownian motion valued in a vector field V, which is a random function as before

$$B : \Omega \to \mathrm{Fun}([0,t], V)$$

where V is a vector space.[4] For a vector $v \in V$, we get a random function

$$B_v : \Omega \to \mathrm{Fun}_v([0,t], V) = q^{-1}(v) \subseteq \mathrm{Fun}([0,t], V)$$

as before, some samples of which are:

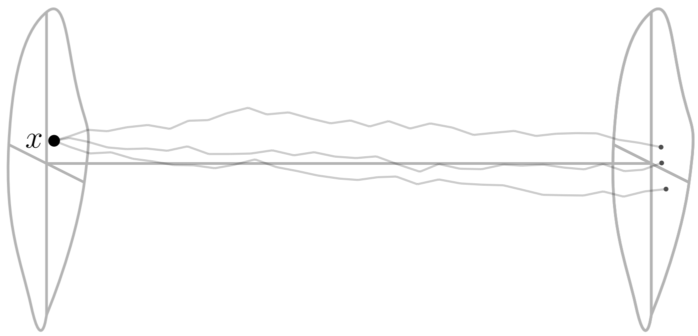


As before, $H_t : V \to V$ is the identity map, but we can take e.g. coordinatewise polynomials in B to get other maps.

[4]To specify B we also need to give a symmetric bilinear form on V giving the covariance of B.

4.2.6. *Brownian motion on general spaces, i.e. sigma models.* For a Riemannian manifold X, we can consider again Brownian motion on X,

$$B : \Omega \to \mathrm{Fun}([0,1], X)$$



Because X does not have a group structure, we are not able to take the average value of $B_x(t)$ like before. As before we can restrict

$$\begin{tikzcd} & \mathrm{Fun}([0,t],X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \simeq \mathrm{Fun}(\{0\},X) & & \mathrm{Fun}(\{t\},X) \simeq X \end{tikzcd}$$

But even if we have a measure on $\mathrm{Fun}([0,t], X)$ with appropriate conditionals defined push-pull only gives a map on *functions*, which if we normalise to have integral one we can think of as a map on *random points*

$$p_* q^* : \mathcal{R}X \to \mathcal{R}X.$$

Here, if Y is a measurable space $\mathcal{R}Y = \mathrm{Maps}(\Omega, Y)$ is the space of measurable maps from a fixed probability space Ω to Y, in other words the random points of Y.

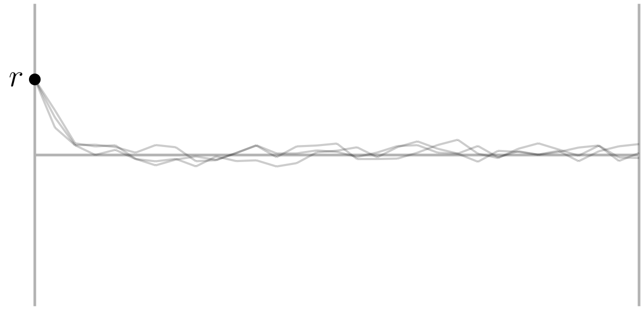
4.2.7. *Remark.* All Markov maps are of the form $\exp(tv) : X \to X$ where v is a vector field on X whose flow is complete.[5] Pushing forward by this map induces $\exp(tv) : \mathcal{R}X \to \mathcal{R}X$, which inherits the Markov property.

4.2.8. *Ornstein-Uhlenbeck process.* We consider an equation

$$dX(t) = -2X(t)dt + dB(t)$$

which is Markov. Some samples of it are

[5]Indeed, if we have a homomorphism $\varphi : \mathbf{G}_a \to \mathrm{Aut}(X)$ then the map on Lie algebras is $\mathbf{C} \to \Gamma(X, \mathcal{T}_X)$, the image of 1 gives a vector field v which exponentiates to φ.



Note that this is Markov in the sense that $H_t \cdot H_{t'} = H_{t+t'}$ as functions $\mathcal{R}X \to \mathcal{R}X$.

4.2.9. *Remark.* If we restrict to random functions $\mathcal{R}X$ which are smooth, this is preserved under $p_* q^*$, and the solution H_t satisfies the *Fokker-Planck* equation.

References

[Ar] Arnold, L., Jones, C.K., Mischaikow, K., Raugel, G. and Arnold, L., 1995. Random dynamical systems (pp. 1-43). Springer Berlin Heidelberg.

[DVV] Dijkgraaf, R., Verlinde, E. and Verlinde, H., 1988. $C = 1$ *conformal field theories on Riemann surfaces.* Communications in Mathematical Physics, 115, pp.649-690.

[He] Henriques, A., 2018. *Conformal nets are factorization algebras.* String-Math 2016, 98, p.229.

[La] Lawler, G.F., 2018. Topics in loop measures and the loop-erased walk.

[Se] Segal, G.B., 1988. The definition of conformal field theory. In Differential geometrical methods in theoretical physics (pp. 165-171). Dordrecht: Springer Netherlands.

[To] Tong, D., 2017. *Statistical field theory*. Lecture Notes.