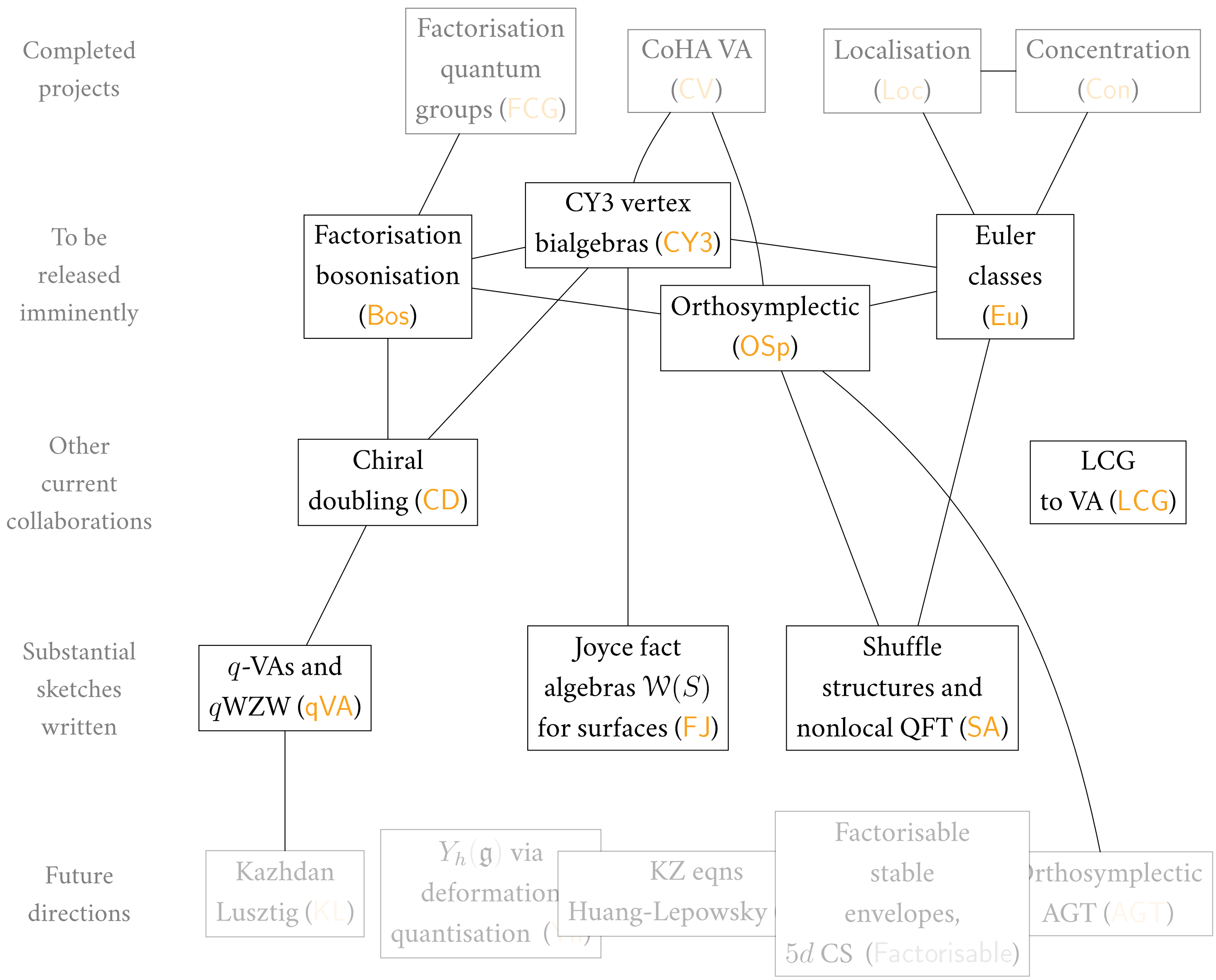
RESEARCH PLANS

ALEXEI LATYNTSEV

This is *extremely* under construction!

See the following sections (with clickable links*) for explanations of the projects and connections between them.*



Contents

1. Summary of projects

Our **objectives** are as follows (see §3 for a more detailed summary):

Project	Description
CV	Define *factorisable* $\mathcal{W}$-algebras, CoHAs, and stable envelopes

In **O1** we will define *factorisable* versions of Joyce's vertex algebras for *dimension zero coherent sheaves* over canonical bundles of arbitrary algebraic *surfaces* S[1] to give a sheaf over S of S-*vertex algebras* which are Morita equivalent on intersections, and relate this to existing presentations of *cohomological Hall W-algebras*;[2] the more *conceptual* (i.e. operadic) nature of this *novel* approach to constructing vertex structures for *non-Calabi-Yau* surfaces will allow for easier generalisation, e.g. to *multiplicative/elliptic* cases, or to more general CY *threefolds*, as it makes visible structure not accessible to the explicit generators-and-relations approach.

Objective **O2** is to generalise key objects in geometric representation theory to live on *Dynkin* spacetimes, then use this as a method to prove new relations between these objects. I will extend my previous work on orthosymplectic CoHAs[3] to *arbitrary* Dynkin-like spacetimes, and prove/contruct analogues of *Chen's Theorem*[4] on the cohomology of loop spaces and *multiple zeta values* (**MZVs**), variants of *vertex*

[1] *B. Davison*, The integrality conjecture and the cohomology of preprojective stacks J. Reine Angew. Math. 804, 105–154 (2023)

[2] *A. Mellit, A. Minets, O. Schiffmann,* and *E. Vasserot*, "Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras," Preprint, arXiv:2311.13415 [math.AG] (2023).

[3] *deHority, S.* and *Latyntsev, A.*, *Orthosymplectic boundary cohomological Hall algebras*, in preparation.

[4] Chen, K.T., 1973. Iterated integrals of differential forms and loop space homology. Annals of Mathematics, 97(2), pp.217-246.

algebras and *boundary KZ equations* (generalising/help understanding *Drinfeld's conjecture*[5] on the relation to MZVs), and *Nakajima quiver varieties*, and Maulik-Okounkov's[6] *stable envelopes* and *Yangians*; simultaneously generalising these topics will make new *connections* betewen them more apparent. Finally, we will prove an analogue of *Kontsevich's formality* theorem, with $\mathbf{E}_n$-algebras replaced by factorisation algebras over Dynkin spacetimes.

Objective **O3** is focussed on developing the machinery of *q-vertex algebras*, then applying it to prove *Kazhdan-Lusztig equivalences*. I will give a definition of q-vertex algebras, generalising factorisation algebras to live on *noncommtutative* spacetimes; note that such factorisation algebras are *new*; this requires giving a sufficiently functorial modern definition of *q-D-modules*.[7] I will then use it to give a new proof of the Kazhdan-Lusztig equivalence and recent generalisations,[85] giving an *uniform* explanation.

In summary, the *state of the art* and proposed *extensions* of it is as follows:

[5]*Etingof, P.I.* and *Schiffmann, O.*, 1998. Lectures on quantum groups.

[6]*D. Maulik* and *A. Okounkov*, Quantum groups and quantum cohomology. Paris: Société Mathématique de France (SMF) (2019)

[7]Majid, S. and Simão, F., 2023. Quantum jet bundles. Letters in Mathematical Physics, 113(6), p.120.

	State of the art	Beyond state of the art
O1	The Jordan moduli stack $\mathcal{M}^f_{\mathbf{A}^2}$ insantiating Davison's localised coproduct;[8] generators-and-relations definition of W-algebra $\mathcal{W}(S)$ for algebraic surfaces S; [9] cohomological Hall algebras as factorisation algebras over the configuration space[10]	factorisation stacks $\mathcal{M}^f_S$ over the canonical bundle K_S of more general algebraic surfaces; show its critical cohomology forms a *S-vertex algebra*; *configuration-to-Ran space* comparison, obtaining vertex algebra structures
O2	Shuffle algebra formulas for CoHAs;[11] *orthosymplectic* $4d$ Chern-Simons and twisted Yangians[59]; orthosymplectic Joyce vertex bialgebras; boundary KZ equations	*Operadic* definition of ordinary shuffle algebras, extending to arbitrary '**Dykin**' systems of *Kac-Moody groups*; define *Dynkin vertex algebras* and give examples (type F, G, multiplicative, elliptic); *Dynkin* shuffle structure on loop spaces and Dynkin multiple zeta values; producing examples using deformation quantisation of on *orbifolds*
O3	quantum jet spaces[7] and *de Rham* definition of D-modules via *crystals*[12]; vertex algebras as factorisation/chiral algebras; non-operadic definition of deformed vertex algebras[13] | KZ equations and fusion product of vertex modules[14] | Chen-Fu's proof of Kazhdan-Lusztig equivalence;[83] new Kazhdan-Lusztig equivalences from $3d$ *mirror symmetry* and new *quantum groups*/vertex algebras[15]	de Rham definition of q-D-modules and their functoriality; q-vertex algebras as *factorisation algebras* on *noncommutative schemes*; *q-affine* and *q-Virasoro* factorisation algebras | factorisation category explanation of KZ equations, *Zhu algebra* and *fusion product* | proving a Zhu/$q \to 1$ correspondence to obtain Chen-Fu's proof from q-affine vertex algebras; generalising to give a blanket proof of the new Kazhdan-Lusztig equivalences

Methods and challenges. The main **technical methods** (**TM**), **challenges** (**C**) and **solutions** (**S**) to these challenges are:

WP1 **TM**: *free field realisations*[16] for producing actions of W-algebras in proving a Dynkin AGT Theorem, the theory of *Coxeter groups* to organise our combinatorial definitions,[17] the theory of *qKZ* and *KZB*[18] equations which we hope to generalise in the multiplicatie/elliptic case, and *quiver varieties*[19]. **C**: The good moduli spaces are *no longer smooth*. **S**: Use *intersection homology*,[20] adapt the *fixed point* techniques in my upcoming collaboration[1] which resolves these issues in the orthosymplectic case.

[16]Frenkel, E. and Ben-Zvi, D., 2004. Vertex algebras and algebraic curves (No. 88). American Mathematical Soc..

[17]Björner, A. and Brenti, F., 2005. Combinatorics of Coxeter groups (Vol. 231, pp. xiv+-363). New York: Springer.

[18]*G. Felder*, in: Quantum symmetries/ Symétries quantiques. Proceedings of the Les Houches summer school (1995)

[19]Hiraku Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. Duke Math. J. 76 (1994)

[20]Goresky, M. and MacPherson, R., 1983. Intersection homology 11. Inc. Mat, 71, pp.77-129.

WP2 **TM**: *virtual torus localisation*[21][22] for cohomological computations, the *stable envelope* construction to produce factorisation quantum groups. **C**: $\mathcal{M}_S^f$ is *not a global critical locus* over $\mathrm{Ran}K_S$, and so the results of[23] will not apply. **S**: It will only be a vertex algebra *relative* to S: we will get a *sheaf* of factorisation categories; alternatively, use techniques in[43].

WP3 **TM**: *quantum factorisation groups*,[24] the theory of *D-modules*[25] which we will q-deform.

In all projects, *derived algebraic geoemtry*,[26] *stable ∞-categories*[27], *higher algebra* and the theory of $\mathbf{E}_n$-*algebras*,[28] *sheaves of categories*[29], the *Ran space* formulation of vertex algebras to make our definitions operadically,[30][31] and topological *factorisation homology*[32] will be used: these are (some of) the background tools in the subject.

[21] Atiyah, M.F. and Bott, R., 1984. The moment map and equivariant cohomology. Topology, 23(1), pp.1-28.

[22] *Aranha, D., Khan, A.A., Latyntsev, A., Park, H. and Charanya, R.*, 2022. Virtual localization revisited. arXiv preprint arXiv:2207.01652.

[23] *Kaubrys, S., Jidnal, S., Latyntsev, A.*, Vertex bialgebras for Calabi-Yau-three categories. *In preparation*

[24] Latyntsev, A., 2023. Factorisation quantum groups. arXiv preprint arXiv:2312.07274.

[25] Hotta, R. and Tanisaki, T., 2007. D-modules, perverse sheaves, and representation theory (Vol. 236). Springer Science & Business Media.

[26] Toën, B., 2014. Derived algebraic geometry. EMS Surveys in Mathematical Sciences, 1(2), pp.153-240.

[27] Lurie, J., 2009. Derived algebraic geometry I: stable ∞-categories. Preprint.

[28] *Lurie, J.*, Higher Algebra.

[29] *D. Gaitsgory*, Contemp. Math. 643, 127–225 (2015)

[30] Francis, J. and Gaitsgory, D., 2012. *Chiral koszul duality*. Selecta Mathematica, 18(1), pp.27-87.

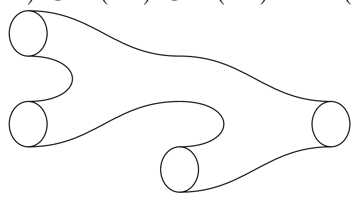
[31] Gaitsgory, D. and Lurie, J., 2014. Weil's conjecture for function fields. preprint.

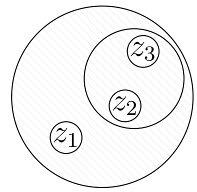
[32] Ayala, D. and Francis, J., 2015. Factorization homology of topological manifolds. Journal of Topology, 8(4), pp.1045-1084.

2. Background

Formalising quantum field theory: factorisation algebras. The task of *axiomatising* topological QFTs was completed by Atiyah[33], as a functor from a *cobordism* category,

$$\mathcal{T}(S^1) \otimes \mathcal{T}(S^1) \otimes \mathcal{T}(S^1) \to \mathcal{T}(S^1)$$





Next, the theory of *vertex* and *chiral algebras* were developed by Borcherds[34] and Beilinson-Drinfeld[35] to axiomatise $2d$ *conformal* QFTs, where the spacetimes above have *holomorphic* structure, the former earning Borcherds a Fields medal and resolving the Moonshine Conjecture on modularity of monster group representations. In recent years, there has been a flurry[36][37] of activities related to *factorisation algebras* and *factorisation homology* as a way to understand *local operators* in a quantum field theory; formed by considering only cobordisms contained *within* a *fixed* manifold M; for instance, this was used to prove[38] a geometric analogue of *Weil's conjecture* for function fields.

However, despite recent progress on *axiomatising* quantum field theories, very few *examples* of (non-topological) quantum fields theories beyond dimension two have been constructed; mathematicians still must rely on (nonrigorous) QFT computations by physicists (e.g. based on *string theory*), which are turned into *provable conjectures*. Much of our proposed work concerns *extending* the range of rigorous mathematics further into physics; some of it concerns *proving* relations between mathematical structures (e.g. *CoHAs*, *vertex algebras*) conjectured by physics.

Cohomological Hall algebras and W-algebras. Cohomological Hall algebras (**CoHAs**) are a mathematical shadow of four-dimensional supersymmetric QFTs $\mathcal{T}$; as these QFTs are not yet rigorously defined, this is currently one of the best handles we have on their structure.

In *physics*, the study of CoHAs began with the space of *BPS states* of $\mathcal{T}$ was shown[39] to carry an *associative algebra* structure. Examples of $\mathcal{T}$ are given by compatifying an 11-dimensional string theory on

[33]Atiyah, M.F., 1988. Topological quantum field theory. Publications Mathématiques de l'IHÉS, 68, pp.175-186.

[34]Borcherds, R. (1986), "Vertex algebras, Kac-Moody algebras, and the Monster", Proceedings of the National Academy of Sciences of the United States of America.

[35]*A. Beilinson* and *V. Drinfeld*, Chiral algebras. Providence, RI: American Mathematical Society.

[36]Lurie, J., 2008. On the classification of topological field theories. Current developments in mathematics, 2008(1), pp.129-280.

[37]Costello, K. and Gwilliam, O., 2021. Factorization algebras in quantum field theory (Vol. 2). Cambridge University Press.

[38]Gaitsgory, D. and Lurie, J., 2014. Weil's conjecture for function fields. preprint.

[39]Harvey, J.A. and Moore, G., 1998. On the algebras of BPS states. Communications in Mathematical Physics, 197, pp.489-519.

a torically-compact Calabi-Yau threefold X, relating the subject to mirror symmetry and the *Geometric Langlands program*.[40]

Kontsevich-Soibelman[41] then discovered an algebra structure on the *critical cohomology*

$$\mathrm{H}^*(\mathcal{M}_{\mathcal{A}}, \varphi)$$

of certain moduli stacks $\mathcal{M}_{\mathcal{A}}$ of CY3 categories (specifically, Jacobi algebras of quivers with potential), which locally models coherent sheaves on CY3s[42], and related their graded dimensions to *Donaldson-Thomas* enumerative invariants. Recently, **Safronov** co-authored a breakthough paper[43] constructing cohomological Hall algebras for *arbitrary* CY3 categories, which will lead to a flurry of research activity in the near future.

Instantons and AGT. The breakthroughs of Grojnowski[44] and Nakajima[45] proved that the *Hilbert scheme* of points on a smooth surface S carries an action of the *Heisenberg vertex algebra* on its cohomology. Later generalisations were conjectured by Alday-Gaiotto-Tachikawa[46] and proved by Braverman-Finkelberg-Nakajima[47] [48] [49] to arbitrary surfaces and gauge groups with an action of $\mathcal{W}$-*vertex algbebras*, which were then realised as quotients of cohomological Hall algebras[50]. This begun the connection between cohomological Hall algebras, *vertex algebras* and *quantum groups*.

[40]Witten, E., 2009. Geometric Langlands from six dimensions. arXiv:0905.2720. (2009)

[41]*M. Kontsevich* and *Y. Soibelman*, Commun. Number Theory Phys. 5, No. 2, 231–352 (2011)

[42]Ben-Bassat, Oren; Brav, Christopher; Bussi, Vittoria; Joyce, Dominic A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications. Geom. Topol. 19, No. 3, 1287-1359 (2015).

[43]injo, T., Park, H. and Safronov, P., 2024. Cohomological Hall algebras for 3-Calabi-Yau categories. arXiv preprint arXiv:2406.12838.

[44]Grojnowski, I., 1997. Instantons and affine algebras. I. The Hilbert scheme and vertex operators, Math. Res. Lett. 3 (1996)

[45]Nakajima, H., 1997. Heisenberg algebra and Hilbert schemes of points on projective surfaces. Annals of mathematics, 145(2), pp.379-388.

[46]Alday, L.F., Gaiotto, D. and Tachikawa, Y., 2010. Liouville correlation functions from four-dimensional gauge theories. Letters in Mathematical Physics,

[47]Nakajima, H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories,I, Adv. Theor. Math. Phys. (2016)

[48] Braverman, A., Finkelberg, M., and Nakajima, H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II, Adv. Theor. Math. Phys. (2018)

[49]Braverman, A., Finkelberg, M. and Nakajima, H., 2014. Instanton moduli spaces and $\mathcal{W}$-algebras. arXiv preprint arXiv:1406.2381.

[50]Rapcák, M., Soibelman, Y., Yang, Y. and Zhao, G., Cohomological Hall algebras, vertex algebras, and instantons, in Comm. Math. Phys.

Quantum groups and the Kazhdan-Lusztig equivalence. The theory of *quantum groups* (**QGs**) was preceded in the statistical physics literature by studies of *integrable systems*[51] and *spin chains*, e.g. studying formation of ice crystals.[52] In the 1986 ICM address Drinfeld[??] developed the mathematical theory of *quasitriangular Hopf algebras* to formalise this, and proved a fundamental result about *existence-uniqueness* of QGs $U_q(\mathfrak{g})$ deforming Lie bialgebras $\mathfrak{g}$.

Since then QGs have taken a central place in mathematics: they were connected to *Chern-Simons* and *knot theory* by Witten,[53] which predicted the famous *Kazhdan-Lusztig equivalence*[54]

$$(\mathrm{Rep}_k \hat{\mathfrak{g}})^{G(\mathcal{O})} \;\simeq\; \mathrm{Rep}\, U_q(\mathfrak{g})$$

relating representations of quantum groups to integrable representations of *vertex algebras* via the *KZ equations*,[5] more generally they relate to $3d$ *TQFTs* and *mirror symmetry*,[8][55] generalisations appear as *Yangians* and *affine/elliptic quantum groups* in Maulik-Okounkov's seminal work,[6] and more recently as *cohomological Hall algebras*.[56][57] The modern physics explanation is that QGs representations give *line operators* for certain QFTs;[60,61] thus the task of understanding/organising these different structures is crucial to understanding QFT and string theory.

There is a long *historical* connection between geometric representation theory and physics sketched in §**??**, two decades-long examples of the two-way exchange includes the Geometric Langlands programme[58] and Mirror Symmetry.

All three of our work projects are efforts to bridge the divide between mathematics and physics. In **WP1**, we use work[59] on $4d$ Chern-Simons on orbifolds. Our results on quantum factorisation algebras for **WP3** are informed by work on $4d$ and $5d$ Chern-Simons theory[60][61] In **WP3** is related to physics-informed conjectures on the q-Langlands correspondence[62], and new *holomorphic-topological*

[51]Yang, C.N., 1967. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. Physical Review Letters, 19(23), p.1312.

[52]Lieb, E.H., 1967. Exact solution of the problem of the entropy of two-dimensional ice. Physical Review Letters, 18(17), p.692.

[53]Witten, E., 1989. Quantum field theory and the Jones polynomial. Communications in Mathematical Physics, 121(3), pp.351-399.

[54]David Kazhdan and George Lusztig. "Tensor structures arising from affine Lie algebras. I-IV". In: Journal of the American Mathematical Society 6.4 (1993-1994).

[55]Creutzig, T., Lentner, S. and Rupert, M., 2021. Characterizing braided tensor categories associated to logarithmic vertex operator algebras. arXiv preprint arXiv:2104.13262.

[56]Yang, Y. and Zhao, G., 2018. The cohomological Hall algebra of a preprojective algebra. Proceedings of the London Mathematical Society, 116(5), pp.1029-1074.

[57]Latyntsev, A., 2021. Cohomological Hall algebras and vertex algebras. arXiv preprint arXiv:2110.14356.

[58]Witten, E., 2009. Geometric Langlands from six dimensions. arXiv preprint arXiv:0905.2720.

[59]*R. Bittleston* and *D. Skinner*, J. High Energy Phys. 2019, No. 5, Paper No. 195, 53 p. (2019; Zbl 1416.81106)

[60]Costello, K., Witten, E. and Yamazaki, M., 2017. Gauge theory and integrability, I. arXiv preprint arXiv:1709.09993.

[61]Costello, K., Witten, E. and Yamazaki, M., 2018. Gauge theory and integrability, II. arXiv preprint arXiv:1802.01579.

[62]*Aganagic, M., Frenkel, E. and Okounkov, A.*, 2018. Quantum q-Langlands correspondence. Transactions of the Moscow Mathematical Society, 79, pp.1-83.

structures we wish to define will be informed by physics papers[63][64] on wide generalisations of Kontesevich's deformation quantisation. The deliverable on q-vertex algebras will be informed by Costello's[65] application of Nekrasov's Ω-background to $5d$ Chern-Simons theory.

Background. A main theme of geometric representation theory/enumerative geometry is: attached to certain Calabi-Yau-threefolds Y or categories $\mathcal{C}$, it has long been conjectured [KS] (now proven [**KPS**]) a "cohomological Hall" algebra structure on

$$\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{C}}, \mathcal{P}) \tag{1}$$

where $\mathcal{P}$ is Joyce's DT sheaf (reference), and

- structure thing one
- two

From the physics perspective, the algebra structure is explained by (1) arising from an 11-dimensional "M" theory compactified on Y, which gives a $5d$ theory, then taking its algebra of BPS states [**Mo**] gives a q-deformed algebra structure. The other structures then arise from varying Y, to get an Alg-valued factorisation algebra over it; the analogy in the trivial toy model where Y is a $6d$ topological manifold is

$$\begin{array}{ccc} \mathrm{TQFT}_{11d} & \overset{\int_{\mathbf{R}^4\times S^1}}{\dashrightarrow} & \mathrm{TQFT}_{6d}(\mathrm{Alg}) \\ \downarrow\int_Y & & \downarrow\int_Y \\ \mathrm{TQFT}_{5d} & \xrightarrow{\int_{\mathbf{R}^4\times S^1}} & \mathrm{Alg} \end{array}$$

The motivating example is when $Y = K_S$ for a smooth algebraic surface S; then in FJ we expect a vertex algebra structures in the fibres of $K_S \to S$; this is proven in some 2CY cases in CY3.

. Explanation: standard and nonstandard coproduct on $Y_\hbar(\mathfrak{g}_Q)$.

. Define CoHAs

[63]*Gaiotto, D., Kulp, J. and Wu, J.*, 2024. Higher Operations in Perturbation Theory. arXiv preprint arXiv:2403.13049.

[64]*Balduf, P.H. and Gaiotto, D.*, 2024. Combinatorial proof of a Non-Renormalization Theorem. arXiv preprint arXiv:2408.03192.

[65]*Costello, K.*, 2016. M-theory in the Omega-background and 5-dimensional non-commutative gauge theory. arXiv preprint arXiv:1610.04144.

3. Details of projects

3.1. Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ)

CoHAs as vertex quantum groups. One aim of project CY3[66] and project Bos[67] is to push the analogy between CoHAs and finite quantum groups:

$\mathrm{Rep}_q T$	$U_q(\mathfrak{n})$	$U_q(\mathfrak{b})$	$U_q(\mathfrak{g})$
$\mathrm{Rep}\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M})$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}$	c.f. CD

To begin with, whereas $U_q(\mathfrak{n})$ a braided cocommutative bialgebra inside the braided monoidal category $\mathrm{Rep}_q T$,

Theorem A. [CY3] *For any deformed CY3 category (e.g. coherent sheaves on local curve, quiver with potential) there is a vertex coproduct on the CoHA*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M},\varphi)\,\hat{\otimes}\,\mathrm{H}^\bullet(\mathcal{M},\varphi)((z^{-1}))$$

making it into a braided colocal vertex bialgebra inside the braided factorisation category $\mathrm{Rep}(\mathrm{H}^\bullet(\mathcal{M}),\cup)$.[68]

We sanity-check that this is an interesting structure:

Theorem B. [CY3; CV for $W = 0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA* $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}}) \simeq Y_\hbar(\mathfrak{n}_Q)$ *agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) Drinfeld's meromorphic coproduct.*

Next, $U_q(\mathfrak{b})$ is constructed by Tannakian reconstruction on $U_q(\mathfrak{b})$-Mod$(\mathrm{Rep}_q T)$, and in Bos we develop a factorisable analogue of this. This results in a vertex bialgebra structure on the extended CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos} = \mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M})$,

$$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}\text{-Mod} \;=\; \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)\text{-Mod}(\mathrm{Rep}\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}))$$

which in the preprojective case recovers (Soibelman-Rapcak)-Yang-Zhao's construction on $Y_\hbar(\mathfrak{b}_Q)$. This "automates" difficult generating-series definitions of CoHA extensions: they follow as a consequence of factorisable Tannakian reconstruction. (give more evidence/details)

Vertex coalgebras from configuration spaces. (recall what localised (bi)algebras are!) To compare localised and vertex coproducts in CY3, we introduce a *Ran-to-Conf* construction: taking localised terms $1/x$, pulling back by a $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$-coaction and taking a power seires expansion in z^{-1}

$$\frac{1}{x+nz} \;=\; \frac{1}{nz}\left(\frac{x}{nz} - \left(\frac{x}{nz}\right)^2 + \cdots\right)$$

defines a functor from localised coalgebras to vertex coalgebras.

[66] Joint with S. Jidnal and S. Kaubrys.

[67] Joint with S. de Hority.

[68] The formalism of braided factorisation categories is developed in FQG.

Conjecture C. *The Ran-to-Conf construction lifts to a functor* $\mathrm{FactCoAlg}(\mathrm{Conf}\mathbf{A}^1) \to \mathrm{VertexCoAlg}$.

We notice as an aside that the

Conjecture D. (Properadic vertex algebra-coalgebras) *Vertex coalgebras from factorisation algebras*

Lift to factorisation algebra. To finish the analogy with [Ga], it remains to construct the Yangian factorisably, which we plan to do in FJ

Conjecture E. *Construction of* $Y_\hbar(\mathfrak{g})$ *factorisably. (finish)*

project FJ[69] will aim to define factorisable lift of the above structures. In the case of quivers Q, we have an action of the torus $T_d = \prod T_{d_i}$ on the stack of representations, and

$$\mathcal{M}^f \;=\; \{(m,\lambda)\;:\;\lambda\in\mathfrak{t}^*,\; m\in\mathcal{M}^\lambda\}\;\xrightarrow{\pi}\;\mathrm{colim}(\mathfrak{t}_d)$$

defines a factorisation space over the Q_0-coloured Ran space.

Conjecture F. *The relative critical cohomology* $\mathcal{A} = \pi_*\varphi_W$ *defines a* $\mathbf{G}_a^{Q_0}$*-equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\mathrm{Ran}\mathbf{A}^1 \;\subseteq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1$$

recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{nilp})$ *of* [SV].

Only the last part is nontrivial. This would be interesting for the following reasons:

- This should relate to Yang-Zhao's proof [YZ] that CoHAs form a localised factorisation bialgebra over $\mathrm{Conf}_\Lambda(E)$. We expect that the relation should be a factorisation space version of the Conf-to-Ran construction in CY3.
- This should relate to Maulik-Okounkov's stable envelope construction [MO] of Yangians.
- This construction makes the role of the torus $\mathfrak{t}_d$ clear, and therefore in (ref) we may generalise it to arbitrary Kac-Moody groups.

Crucially, having repackaged the vertex bialgebra structure as a factorisation algebra, we can consider applying them to more general CY3 categories.

Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work), and the above should be an algebraic analogue of their construction.

[69] Joint with S. Kaubrys.

Relation to $\mathcal{W}$-algebras.

Conjecture G. *When $\mathcal{A}$ is the category of zero-dimensional coherent sheaves on a surface S, $\mathcal{A}$ is equivalent to the factorisation bialgebra $\mathcal{W}^+(S)$ of [MMSV].*

This could give a coceptual explanation for the "off-local" terms in [MMSV]

$$(\text{write}),$$

i.e. $\mathcal{A}$ will be braided colocal for the factorisation category $(\mathcal{B}, \cup)$-Mod, where $\mathcal{B} = \pi_* k$. Moreover, one might expect that the techniques in CD may explain how to form doubles of these algebras.

Shows that $\mathcal{W}(S)$ locally in (certain) S forms a sheaf of factorisation algebras over K_S, i.e. "S-vertex algebras", which are Morita equivalent on intersections. Gives an example of the M2-M5 brane construction.

Joyce factorisable $\mathcal{W}(S)$-algebras. Define *factorisable* moduli stacks of *coherent sheaves* over canonical bundles of algebraic curves and a sheaf of critical charts[70] (**M1.1**), glue Joyce-Liu's vertex algebras[71][72] *factorisably* over *canonical bundles* of algebraic surfaces (**M1.2**), give a *new* construction of W-algebras[73] $\mathcal{W}(S)$ (**M1.3**);

Factorisable stable envelopes. Give a *Tannakian* (*factorisation category*) reformulation of the stable envelope construction over the Ran space (**M2.1**), obtain give a *vertex bialgebra* action of $\mathcal{W}(S)$ and *factorisation bialgebra* structure on the *nilpotent CoHA*[74][75] (**M2.2**), generalise to the *elliptic/multiplicative* case (**M2.3**).

Relation to stable envelopes. (write)

3.2. The structure of factorisation quantum groups (FCG, Bos, CD)

At this point, the theory of quantum groups $U_q(\mathfrak{g})$ is well-developed:

(1) There are basis-free constructions [Ga] of $U_q(\mathfrak{g})$-Mod,
   (a) by working in the category $\mathrm{Perv}(\mathrm{Conf}_\Lambda(\mathbf{A}^1))$ of perverse sheaves on the configuration spaces,
   (b) by double-bosonisation [Ma].

[70] *B. Davison*, The integrality conjecture and the cohomology of preprojective stacks J. Reine Angew. Math. 804, 105–154 (2023)

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[74] *O. Schiffmann* and *E. Vasserot*, J. Reine Angew. Math. 760, 59–132 (2020; Zbl 1452.16017)

[75] *Y. Yang* and *G. Zhao*, "Quiver varieties and elliptic quantum groups", Preprint, arXiv:1708.01418 [math.RT] (2017)

(2) There is a "geometric" proof [CF] of the Kazhdan-Lusztig equivalence $U_q(\mathfrak{g})\text{-Mod}^{ren} \simeq \hat{\mathfrak{g}}\text{-Mod}_k^I$

In this series of projects, we develop the theory where vertex-algebraic analogues of quantum groups can be reasoned about in a basis-free category-theory way just as for finite quantum groups.

Factorisation quantum groups. It is well-known that the representation categories of $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, quantum vertex algebras, etc., should be controlled by "spectral" analogues $R(z)$ of R-matrices.

There have been many (inequivalent) attempts to axiomatise this (ref ref ref). The paper project FCG developed answered the following: they are *braided factorisation categories*.

Theorem H. *If $\mathcal{A}$ is a factorisation bialgebra, braided factorisation structures on $\mathcal{A}$-*Mod *are equivalent to factorisation R-matrices $R : A \otimes_2 A \to A \otimes_2 A$. (fix notation)*

Moreover, we show that we indeed recover standard structures:

Theorem I. *In the case of* Ran$\mathbf{A}^1$*, a factorisation R-matrix induces an endomorphism*

$$R(z) \;:\; V \otimes V((z)) \;\to\; V \otimes V((z))$$

satisfying the spectral hexagon relations.

Likewise, in the case of Conf$\mathbf{A}^1$ where factorisation bialgebras are precisely localised bialgebras V of (ref), we get a factorisation R-matrix induces an endomorphism

$$R \;:\; (V \otimes V)_{\text{loc}} \;\to\; (V \otimes V)_{\text{loc}}$$

of the localised bialgebra V, satisfying the hexagon relations.

For instance, (give example of $Y_\hbar(\mathfrak{g})$ with two coproducts) (ref GTW)

Factorisation bosonisation. In project Bos[76]

Factorisation Drinfeld doubling. In project CD[77]

Work out how to take Drinfeld centres of chiral categories. Recovers notions of doubling chiral bialgebras, bubble Grassmannians (when applied to Rep$\mathfrak{g}(\mathcal{O})$), Yangians. Generalises BZFN's derivd loop spaces and centres construction.

Stable envelopes. Give a "Ran space" version of Maulik-Okounkov construction that includes all generalisations, e.g. the dynamical R-matrices.

[76]Joint with S. de Hority.

[77]Joint with W. Niu.

3.3. Orthosymplectic structures (OSp, AGT, SA)

Physics heuristic. In project OSp[78] we make a mathematical theory of *boundary $4d$ Chern-Simons* [BS] on $\mathbf{R} \times (\mathbf{R} \times \mathbf{C})/\pm$, for instance our structures structures satisfy *boundary Yang-Baxter/Cherednik reflection equations*. More generally, we define boundary versions of compactifications of $4d$ SCFTs $\int_Y \mathcal{M}$ attached to a CY3 Y - at least, those for which non-boundary versions have been defined. It should relate to Finkelberg-Hanany-Nakajima's ongoing work on orthosymplectic Coulomb branches (see AGT).

Details. Attached to an abelian category $\mathcal{A}$,[79] we construct the *orthosymplectic moduli stack* $\mathcal{M}_{\mathcal{A}}^{\mathrm{OSp}}$: a fixed point stack whose points are objects with a symmetric pairing $a \simeq a^*$.

Theorem J. [OSp] *For $\mathcal{A}$ in CY3 or examples below, the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ "acts" on $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}, \varphi^{\mathrm{OSp}})$:*

(1) *there is a left module action a of the CoHA respecting the involution,[80] compatible with*

(2) *its* ***symplectic vertex algebra*** *structure: it is a factorisation coalgebra over symplectic Ran space $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\,\mathfrak{t}_{\mathfrak{sp}_{2n}}$ (coming from a localised structure over $\mathrm{Conf}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$).*

Points (1) and (2) is equivalent to being a topological and holomorphic factorisation algebra over $\mathbf{R}/\pm$ and $\mathbf{C}/\pm$, respectively. We give an equivalent definition of symplectic vertex algebra in terms of fields $V \otimes M \to M((z))$, etc. To give examples, we construct an **invariants** functor by restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$

$$\iota \;:\; \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \;\to\; \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1), \qquad\qquad (\mathcal{A}, \tau) \;\mapsto\; (\mathcal{A}, \mathcal{A}^\tau)$$

where $\mathcal{A}$ is a factorisation algebra with involution τ; we expect Theorem J may also be proved by applying ι to the factorisable moduli stack $\mathcal{M}^f$ from FJ. See also the link to stable envelopes (ref), and:

Conjecture K. *The* ***boundary KZ equations*** *may be derived by applying ι to the BD Grassmannian, taking distributions supported at the identity, and taking conformal blocks over $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1$.*

Examples include $\mathrm{B}\mathbf{Z}/2$ orbifold quivers with potential,[81] or orthosymplectic perverse-coherent sheaves on surfaces, e.g. orthosymplectic ADHM quiver/perverse-coherent sheaves on $\mathbf{A}^2$:

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[loop above] \arrow[loop below] \arrow[r, shift left] & \mathrm{Sp}(2m) \arrow[l, shift left] \end{tikzcd} \qquad\qquad \mathcal{E} \simeq \mathrm{RHom}(\mathcal{E}, \mathcal{O})$$

[78] Joint with S. de Hority.

[79] More generally abelian category with involution $(\mathcal{A}, \tau)$, e.g. $\tau = (-)^*$.

[80] i.e. the left action a and the right action $a \cdot (\mathrm{id} \otimes \tau)$ commute, where τ is the involution.

[81] i.e. either an ordinary quiver with involution, or an orbifold-valued quiver.

Theorem L. [OSp] *In the quiver with potential case, an explicit shuffle formula for the CoHA action.*

We end with a conjecture:

Conjecture M. *The orthosymplectic CoHA for the "folded" linear quiver A_{2n}[82] is isomorphic to the twisted Yangian $Y_\hbar(\mathfrak{gl}_n)^{tw}$ of [BR].*

3.3.1. *Dynkin QFTs.* Develop the theory of analogues of *Ran space, loop spaces, quiver varieties, MZVs, vertex algebras* and *KZ equations* attached to Coxeter/Kac-Moody data (**M3.1**), give *affine* examples of associated *vertex algebras, quantum groups* (using a variant of *Kontsevich formality*??) and *Yangians* (**M3.2**); compute the *generalised KZ equations* on their conformal blocks, formulate analogue of *Drinfeld's conjecture* (**M3.3**).

Nonlocal QFT and shuffle structures. project SA begun by noticing the following interesting pattern in structures considered project OSp.

$$\mathrm{BGL} \rightsquigarrow \mathrm{BSp}, \qquad \mathrm{Conf}(\mathbf{A}^1) \rightsquigarrow \mathrm{Conf}(\mathbf{A}^1), \qquad \mathrm{VA} \rightsquigarrow \mathrm{OSpVA}, \qquad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties, (KZ equations)) simultaneously generalise - this points towards this being a shadow of a more general theory.

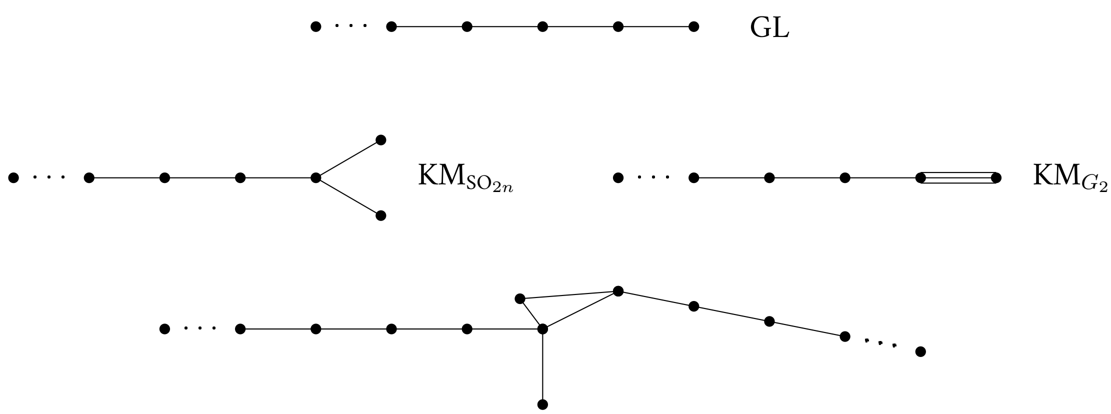
The starting observation is this - the definition (ref) a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL} \to \mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n \geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

$$\begin{array}{ccc} & P_{n,m}(\sigma) & \\ \swarrow & & \searrow \\ \mathrm{GL}_n \times \mathrm{GL}_m & & \mathrm{GL}_{n+m} \end{array} \quad \overset{A}{\mapsto} \quad A_n \otimes A_m \overset{m(\sigma)}{\to} A_{n+m}$$

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n,m)$.

The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.

82 i.e. with the involution being reflection in the linear direction.



To summarise:

- We get analogues of *shuffle algebras*.
- We get new configuration and Ran spaces

$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \coprod_G \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{B}G), \qquad\qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G^*,$$

  where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G, so can define generalised *localised* and *vertex* algebras (and as in CY3 a Conf-to-Ran construction relating them). We expect to recover *boundary KZ* equations by taking conformal blocks (i.e. cohomology over $\mathrm{Ran}_{\mathrm{KM}}\mathbf{A}^1$).

- Topological case - topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras, then by considering $\mathrm{FactAlg}^{top}(\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}, \mathrm{Cat})$ we get analogues of the notion of *braided monoidal categories*.
- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups

$$\underset{\bigcirc}{\mathrm{GL}_3} \xrightarrow{\;U_{3,5}\;} \underset{\bigcirc}{\mathrm{GL}_5} \xrightarrow{\;U_{5,4}\;} \underset{\bigcirc}{\mathrm{GL}_4}$$

  where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as

$$\mathcal{M}_Q \;=\; \coprod \mathfrak{u}_e/G_i$$

  the product over all maps $(G_i) : Q_0 \to \mathrm{KM}$ and U_e is a choice if unipotent for each edge e.

  Relation to orbifolding.

- Stable envelope construction.
- Chen's [Ch] shuffle structure on cochains $\mathrm{C}^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX \;=\; \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n \;=\; T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G.
- Iterated integrals.

For the orthosymplectic example $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$, many of these structures are considered in OSp. Let us consider K_{G_2}

Example: G_2. For K_{G_2}, factorisation algebras consist of ordinary factorisation algebras but for any *triple* of points there is in addition equivariance with respect to the group $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$ acting on $\mathbf{C}^3$, in which the element

$$\begin{aligned} \tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3) \\ \tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1) \\ \tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2) \end{aligned}$$

squares to $\tau^2 = (231)$. Thus for instance a G_2 vertex algebra is a vertex algebra V with an (copy-paste from notes), and a topological G_2 factorisation category is a braided monoidal category $\mathcal{C}$ along with (copy-paste from notes; G_2 reflection equations)

Relation to folding. We expect there to be a folding construction of G_2 structures. (reference conjecture on twisted Yangians)

A twisted AGT correspondence. In the *finite type* case, define an action of *twisted CoHA* on the quiver varieties (**M4.1**), prove an *AGT* result: that this is a *Verma module* for a *twisted affine W-algebra*, which we define (**M4.2**).[47,48]

After OSp, one natural next step (project AGT) is to construct a boundary version [BFN]:

Conjecture N. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a Verma module for an orthosymplectic analogue of the vertex W-algebra $\mathcal{W}^k(\mathfrak{gl}_n)$.*

We expect the proof should proceed in much the same way as in [BFN], but with the parabolic induction data replaced by

(write OSp correspondence)

Likewise, we expect a generalisation of [RSYZ] for instantons on $\mathbf{A}^3$:

Conjecture O. [AGT] *There is vertex algebra structure on the the orthosymplectic CoHA of the Jordan quiver, which acts on the equivariant critical cohomology of $\mathcal{M}^{\mathbf{Z}/2}$, the invariant locus in the quiver variety.*

and likewise for arbitrary quivers with potential. We expect this CoHA should be equal to (W algebra thing), which admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients

(nonabelian stable envelopes)

3.4. q-vertex algebras (qVA, KL)

3.4.1. Kazhdan-Lusztig equivalences from q-vertex algebras.

Kazhdan-Lusztig (KL) equivalences. Uplift the *Zhu algebra* (**M6.1**) and *Huang-Lepowsky fusion product* (**M6.2**) to the level of factorisation and q-vertex algebras, recover *Chen-Fu*'s proof[83] of KL using q-WZW, and extend to *new* examples, e.g.[85] (**M6.3**)

The q-WZW vertex algebra. Build a theory of *q-D modules*/D-modules on noncommutative schemes and prestacks, then apply it to define/prove structural results on *q-vertex algebras* (**M5.1**), use *q-affine Grassmannians* and *q-coordinate bundles* to define q-WZW and q-Virasoro vertex algebras (**M5.2**).

. It has been long expected that one may define a q-analogue of the Kazhdan-Lusztig equivalence, but this has been hampered by the lack of a good definition of q-WZW algebras: currently, the available definition is an RTT-style definition from [**EK**].

q-vertex algebras. The main goal of project qVA is:

Conjecture P. *There is a factorisation category over the noncommutative space $\mathbf{A}^2_q$, such that any*

$$\mathcal{A} \in \mathrm{FactAlg}^{st}(\mathcal{D}\text{-Mod}_{\mathrm{Ran}\mathbf{A}^2_q})$$

*defines a q-**vertex algebra**.*

Moreover, for any complex finite-dimensional simple Lie algebra $\mathfrak{g}$, we may ask

Question Q. *Is there an analogue of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$?*

Such a factorisation space would for free by Conjecture P define for us a q-vertex algebra $V^k_q(\mathfrak{g})$, by the same construction as for the affine WZW vertex algebra (and which we expect it would be is a q-deformation of) and we expect should agree with [**EK**] when $\mathfrak{g} = \mathfrak{sl}_n$. We expect there to be an algebra of modes fucntor $A(-)$, and we propose to finish with a sanity-check of our definitions by showing $A(V^k(\mathfrak{g})) \simeq U_q(\hat{\mathfrak{g}})$.

We spell out evidence for Conjecture P, first from physics, then give explicit mathematical details.

Physics: $5d$ Chern-Simons. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V^k_\hbar(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

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[85] *A. Ballin, T. Creutzig, T. Dimofte, W. Niu*, "3d mirror symmetry of braided tensor categories", Preprint, arXiv:2304.11001 [hep-th] (2023)

where $\mathbf{A}^2_q = (\mathbf{C}\times\mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x,y]/(xy-qyx)$. Thus by analogy with the $3d$ case, to search for $V^k_q(\mathfrak{g})$ we need to understand factorisation algebras over $\mathbf{A}^2_q$.

Mathematical details. (copy-paste from the notes)

Kazhdan-Lusztig. One ultimate goal of projects FJ and qVA is to give an affine analogue of the factorisable proof [CF] of the Kazhdan-Lusztig equivalence.

Question R. *Is there a Riemann-Hilbert functor* $\mathrm{RH} : \mathrm{FactCat}(\mathbf{A}^2_q) \to \mathrm{FactCat}^{QCoh}(\mathbf{C}^2_q)$*, which sends the category* $V^k_q(\mathfrak{g})$-Mod *to* $Y_\hbar(\mathfrak{g})$-Mod*?* *(too vague)*

(need to write down what topological factorisation algebras on $\mathbf{C}^2_q$ are)

3.5. Sheaf methods (Con, Loc, Eu)

Localisation methods. In projects Con and Loc[86]

Localisation. Proved a localisation formula for arbitrary quasismooth derived schemes, relating the pushforward and pullback to a closed substack to the virtual Euler class.

Concentration. Gave a sufficient condition for the Chow homology to be concentrated on a closed substack.

Virtual Euler classes and shuffle structures. In project Eu, we prove Atiyah-Bott torus localisation formulas on vanishing cycle cohomology, for certain non-quasismooth closed embeddings $\mathcal{Z}\hookrightarrow\mathcal{X}$ of Artin stacks. This gives a unified torus localisation way to compute cohomological Hall type products. As a result, we recover shuffle descriptions of CoHAs and a new proof of compatibility between them and Davison's localised coproduct

Theorem S. *For any "split locus" map* $\mathcal{M}^s \to \mathcal{M}$*, we get a diagram*

$$\begin{tikzcd}
\mathrm{C}^\bullet(\mathcal{M}^s\times\mathcal{M}^s, \varphi^s\boxtimes\varphi^s) \arrow[r, dashed, "1/e(\mathrm{N}_i)_{\mathrm{loc}}\cdot"] & \mathrm{C}^\bullet(\mathcal{M}^s\times\mathcal{M}^s, \varphi^s\boxtimes\varphi^s) \arrow[r, "p^s_* q^{s,*}"] & \mathrm{C}^\bullet(\mathcal{M}^s, \varphi^s) \\
\mathrm{C}^\bullet(\mathcal{M}\times\mathcal{M}, \varphi\boxtimes\varphi) \arrow[u, "(\pi\times\pi)^*"] \arrow[rr, "p_* q^*"] & & \mathrm{C}^\bullet(\mathcal{M}, \varphi) \arrow[u, "\pi^*"']
\end{tikzcd} \tag{2}$$

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Two consequences of this are:

- If we take $\mathcal{M}^s$ to be a *shuffle space*[87] given by products of "simple" moduli stacks, e.g. rank one quiver representations, then (2) recovers shuffle formulas for CoHAs.
- If we take $\mathcal{M}^s = \mathcal{M}\times\mathcal{M}$ together with its direct sum map to $\mathcal{M}$, (2) recovers the compatibility between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

[86] Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.

[87] i.e. shuffle algebra in the category of spaces, see SA.

The first we plan to use in SA to give more general shuffle-style products.

3.6. Louiville quantum gravity to vertex algebras (LCG)

History. In recent years, probabilists have increasingly understood quantum field theory (ref, detail, ref, detail)

However, there is currently very limited interaction between this field and geometric representation theorists, and this project tries to build a bridge between the two.

Goal. In project LCG[88] we aim to build a bridge between the two subjects. First, we aim to define a functor from Segal-style $2d$ conformal field theories to vertex algebras

$$\mathrm{CFT} \stackrel{(-)^h}{\to} \mathrm{CFT}^{hol} \stackrel{\mathrm{Res}}{\to} \mathrm{FactAlg}(\mathbf{C})^{hol}_{\mathbf{C}\rtimes\mathbf{C}^\times} \stackrel{[\mathrm{CG}]}{\to} \mathrm{VertexAlg}.$$

Second, we then aim to check that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs of (ref) are sent to the Heisenberg and Virasoro vertex algebras, repectively.

Details. The last two functors are Costello-Gwilliam's factorisation algebra to vertex algebra construction, and restriction to the subcategory $\mathrm{Cob}_{2,/\mathbf{C}} \to \mathrm{Cob}_2$ of cobordisms *inside* $\mathbf{C}$, so the main content is the first, which takes *holomophic part*.

To define the category CFT, (copy notes)

[88] Joint with V. Giri.

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