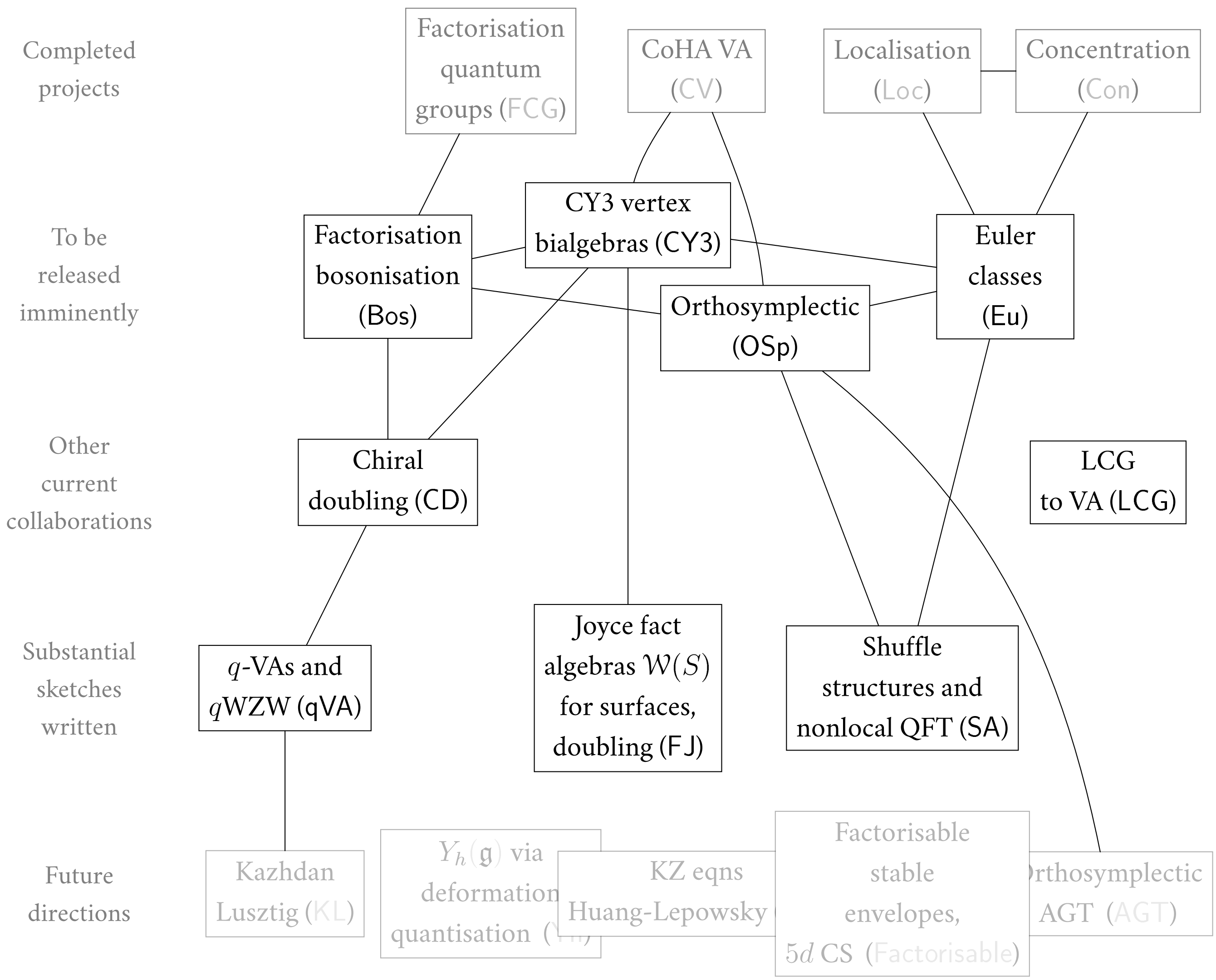
RESEARCH PLANS

ALEXEI LATYNTSEV

This is *extremely* under construction!

See the following sections (with clickable links*) for explanations of the projects and connections between them.*



Summary

0.1. Completed projects.

(1) **Quantum factorisation algebras**. Develop a theory of chiral $\mathbf{E}_n$-categories, which (should) recover previous notions of meromorphic tensor products. In this framework, build a theory of quantum factorisation (/vertex) algebras, spectral R-matrices.

(2) **CoHA and vertex algebras**. Show that dimension one cohomological Hall product and Joyce vertex algebra structures agree. Give new methods for computing CoHAs, giving new formulas.

(3) **Localisation**. Proved a localisation formula for arbitrary quasismooth derived schemes, relating the pushforward and pullback to a closed substack to the virtual Euler class.

(4) **Concentration**. Gave a sufficient condition for the Chow homology to be concentrated on a closed substack.

0.2. Imminently completing projects.

(1) **Orthosymplectic**. Work out how to take group-invariants of constructions in geometric representation theory. Define orthosymplectic moduli stacks, CoHAs, and orbifold vertex algebras (which we develop the theory of). Gives shuffle formulas. Show that we get twisted Yangians.

(2) **Euler classes**. Define Gysin maps, localisation formulas and virtual Euler classes for non-quasismooth closed embeddings between quasismooth spaces, by using a version of the exponential map to relate to the normal complex. Give localisation formulas for arbirary sheaf cohomologies. Develops a general theory of shuffle algebras.

(3) **CY3 vertex bialgebras**. For a large class of CY3 categories (local curves, quiver with potential), we define a Joyce-Liu vertex bialgebra structure on their critical cohomology. Show that Drinfeld's meromorphic and Davison's localised coproducts are examples of Joyce vertex coproducts. Develops a theory of "vertex bosonisation" and applies them to Yangians.

(4) **Chiral centres**. Work out how to take Drinfeld centres of chiral categories. Recovers notions of doubling chiral bialgebras, bubble Grassmannians (when applied to Rep$\mathfrak{g}(\mathcal{O})$), Yangians. Generalises BZFN's derivd loop spaces and centres construction.

0.3. Next stage projects.

(1) **Orthosymplectic AGT**. Proves AGT for the orthosymplectic CoHAs, i.e. that the action of the CoHA on intersection homology of good moduli spaces is the action on orthosymplectic affine $\mathcal{W}$-algebras.

(2) **q-WZW and q-vertex algebras via q-$\mathcal{D}$ modules**. Develop a theory of $\mathcal{D}_q$-modules, and define q-vertex algebras as factorisation algebras over "noncommutative spacetime". Now the ordinary definitions of Virasoro and affine vertex algebras via factorisation algebras carry over

immediately to the noncommutative setting. The q-affine vertex algebra lives on $\mathbf{C}^2$ with a Nekrasov's Ω-background.

(3) **Shuffle algebras**. Develops an operadic theory of shuffle algebras, giving new algebraic structures attached to any sequence of reductive groups, e.g. G_2 vertex algebras and their KZ equations, G_2 iterated loop spaces, and G_2 multiple zeta values. This is more general than the theory of ordinary (chiral) operads.

(4) **Factorisation algebras attached to a surface**. Shows that $\mathcal{W}(S)$ locally in (certain) S forms a sheaf of factorisation algebras over K_S, i.e. "S-vertex algebras", which are Morita equivalent on intersections. Gives an example of the M2-M5 brane construction.

0.4. **Prospective projects.**

(1) **Kazhdan-Lusztig**. Give a new proof of the Kazhdan-Lusztig equivalence which generalises to the new structures defined by Creutzig, Dimofte et al. Use the qWZW affine vertex algebra.

(2) **Stable envelopes factorisably**. Give a "Ran space" version of Maulik-Okounkov construction that includes all generalisations, e.g. the dynamical R-matrices.

(3) **Chiral deformation quantisation**. Make Gaiotto's recent paper rigorous, proving a formality result for chiral $\mathbf{E}_n$-algebras. Use it to recover the construction of Yangians and affine quantum groups.

1. Summary of projects

Contents

1.1. **Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ).**

Background. A main theme of geometric representation theory/enumerative geometry is: attached to certain Calabi-Yau-threefolds Y or categories $\mathcal{C}$, it has long been conjectured [KS] (now proven [**KPS**]) a "cohomological Hall" algebra structure on

$$\mathrm{H}^\bullet(\mathcal{M}_{\mathcal{C}}, \mathcal{P}) \tag{1}$$

where $\mathcal{P}$ is Joyce's DT sheaf (reference), and

- structure thing one
- two

From the physics perspective, the algebra structure is explained by (1) arising from an 11-dimensional "M" theory compactified on Y, which gives a $5d$ theory, then taking its algebra of BPS states [**Mo**] gives a q-deformed algebra structure. The other structures then arise from varying Y, to get an Alg-valued factorisation algebra over it; the analogy in the trivial toy model where Y is a $6d$ topological manifold is

$$\begin{array}{ccc} \mathrm{TQFT}_{11d} & \overset{\int_{\mathbf{R}^4\times S^1}}{\dashrightarrow} & \mathrm{TQFT}_{6d}(\mathrm{Alg}) \\ \downarrow\int_Y & & \downarrow\int_Y \\ \mathrm{TQFT}_{5d} & \xrightarrow{\int_{\mathbf{R}^4\times S^1}} & \mathrm{Alg} \end{array}$$

The motivating example is when $Y = K_S$ for a smooth algebraic surface S; then in FJ we expect a vertex algebra structures in the fibres of $K_S \to S$; this is proven in some 2CY cases in CY3.

. Explanation: standard and nonstandard coproduct on $Y_\hbar(\mathfrak{g}_Q)$.

. Define CoHAs

CoHAs as vertex quantum groups. One aim of **project** $\mathsf{CY3}$[1] and **project** Bos[2] is to push the analogy between CoHAs and finite quantum groups:

$\mathrm{Rep}_q T$	$U_q(\mathfrak{n})$	$U_q(\mathfrak{b})$	$U_q(\mathfrak{g})$
$\mathrm{RepH}^\bullet_{\mathbf{G}_m}(\mathcal{M})$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}$	c.f. CD

To begin with, whereas $U_q(\mathfrak{n})$ a braided cocommutative bialgebra inside the braided monoidal category $\mathrm{Rep}_q T$,

Theorem A. [$\mathsf{CY3}$] *For any deformed CY3 category (e.g. coherent sheaves on local curve, quiver with potential) there is a vertex coproduct on the CoHA*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M},\varphi)\,\hat{\otimes}\,\mathrm{H}^\bullet(\mathcal{M},\varphi)((z^{-1}))$$

making it into a braided colocal vertex bialgebra inside the braided factorisation category $\mathrm{Rep}(\mathrm{H}^\bullet(\mathcal{M}),\cup)$.[3]

We sanity-check that this is an interesting structure:

Theorem B. [$\mathsf{CY3}$; CV for $W=0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA* $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}}) \simeq Y_\hbar(\mathfrak{n}_Q)$ *agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) Drinfeld's meromorphic coproduct.*

Next, $U_q(\mathfrak{b})$ is constructed by Tannakian reconstruction on $U_q(\mathfrak{b})\text{-Mod}(\mathrm{Rep}_q T)$, and in Bos we develop a factorisable analogue of this. This results in a vertex bialgebra structure on the extended CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos} = \mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M})$,

$$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}\text{-Mod} \;=\; \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)\text{-Mod}(\mathrm{RepH}^\bullet_{\mathbf{G}_m}(\mathcal{M}))$$

which in the preprojective case recovers (Soibelman-Rapcak)-Yang-Zhao's construction on $Y_\hbar(\mathfrak{b}_Q)$. This "automates" difficult generating-series definitions of CoHA extensions: they follow as a consequence of factorisable Tannakian reconstruction. (give more evidence/details)

Lift to factorisation algebra. To finish the analogy with [Ga], it remains to construct the Yangian factorisably:

Conjecture C. *Construction of $Y_\hbar(\mathfrak{g})$ factorisably. (finish)*

[1]Joint with S. Jidnal and S. Kaubrys.
[2]Joint with S. de Hority.
[3]The formalism of braided factorisation categories is developed in FQG.

Project FJ[4] aims to define factorisable lift of the above structures. In the case of quivers Q, we have an action of the torus $T_d = \prod T_{d_i}$ on the stack of representations, and

$$\mathcal{M}^f \;=\; \{(m,\lambda) \;:\; \lambda \in \mathfrak{t}^*,\; m \in \mathcal{M}^\lambda\} \;\xrightarrow{\pi}\; \mathrm{colim}(\mathfrak{t}_d)$$

defines a factorisation space over the Q_0-coloured Ran space.

Conjecture D. *The relative critical cohomology $\mathcal{A} = \pi_*\varphi_W$ defines a $\mathbf{G}_a^{Q_0}$-equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\mathrm{Ran}\mathbf{A}^1 \;\subseteq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1$$

recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{nilp})$ *of* [SV].

Only the last part is nontrivial. This would be interesting for the following reasons:

- This should relate to Yang-Zhao's proof [YZ] that CoHAs form a localised factorisation bialgebra over $\mathrm{Conf}_\Lambda(E)$. We expect that the relation should be a factorisation space version of the Conf-to-Ran construction in $\mathsf{CY3}$.
- This should relate to Maulik-Okounkov's stable envelope construction [MO] of Yangians.
- This construction makes the role of the torus $\mathfrak{t}_d$ clear, and therefore in (ref) we may generalise it to arbitrary Kac-Moody groups.

Crucially, having repackaged the vertex bialgebra structure as a factorisation algebra, we can consider applying them to more general CY3 categories.

Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work), and the above should be an algebraic analogue of their construction.

Relation to $\mathcal{W}$-algebras.

Conjecture E. *When $\mathcal{A}$ is the category of zero-dimensional coherent sheaves on a surface S, $\mathcal{A}$ is equivalent to the factorisation bialgebra $\mathcal{W}^+(S)$ of [**MMSV**].*

This could give a coceptual explanation for the "off-local" terms in [**MMSV**]

$$\text{(write)},$$

i.e. $\mathcal{A}$ will be braided colocal for the factorisation category $(\mathcal{B},\cup)$-Mod, where $\mathcal{B} = \pi_* k$. Moreover, one might expect that the techniques in CD may explain how to form doubles of these algebras.

Relation to stable envelopes.

[4]Joint with S. Kaubrys.

Vertex coalgebras from configuration spaces. (recall what localised (bi)algebras are!) To compare localised and vertex coproducts in CY3, we introduce a *Ran-to-Conf* construction: taking localised terms $1/x$, pulling back by a $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$-coaction and taking a power seires expansion in z^{-1}

$$\frac{1}{x+nz} = \frac{1}{nz}\left(\frac{x}{nz} - \left(\frac{x}{nz}\right)^2 + \cdots\right)$$

defines a functor from localised coalgebras to vertex coalgebras.

Conjecture F. *The Ran-to-Conf construction lifts to a functor* $\mathrm{FactCoAlg}(\mathrm{Conf}\mathbf{A}^1) \to \mathrm{VertexCoAlg}$.

We notice as an aside that the

Conjecture G. (Properadic vertex algebra-coalgebras) *Vertex coalgebras from factorisation algebras*

1.1.1. (trim!!)

Note that the map $\mathcal{M} \to \mathrm{BGL}$ makes $\mathrm{H}^\bullet(\mathcal{M}, \varphi) \in \mathrm{QCoh}(\mathrm{Conf}(\mathbf{A}^1))$, moreover it is equivariant for the action (actually the $\mathbf{A}^1$ is acting on $\mathrm{Conf} \times \mathrm{Conf}$)

$$\mathbf{A}^1 \times \mathrm{Conf}(\mathbf{A}^1) \xrightarrow{a} \mathrm{Conf}(\mathbf{A}^1) \quad \text{induced by applying cohomology to} \quad \mathrm{B}\mathbf{G}_m \times \mathrm{BGL} \to \mathrm{BGL}.$$

The map a carries a flat connection, (we also need π_1 to, no?) so we may take

$$\pi_{1,*}a^*\mathrm{H}^\bullet(\mathcal{M}, \varphi) \in \mathcal{D}\text{-}\mathrm{Mod}(\mathbf{A}^1).$$

(just say I want to understand this structure operadically, be honest)

Thus, if $\mathcal{A}$ is a localised coalgebra,

$$\begin{tikzcd} & (\mathrm{Conf}\mathbf{A}^1 \times \mathrm{Conf}\mathbf{A}^1)_\circ \arrow[dl] \arrow[dr] & \\ \mathrm{Conf}\mathbf{A}^1 \times \mathrm{Conf}\mathbf{A}^1 & & \mathrm{Conf}\mathbf{A}^1 \end{tikzcd} \qquad \Delta : \cup^*\mathcal{A} \to j^*(\mathcal{A} \boxtimes \mathcal{A})$$

Then the fibre over the antidiagonal (doesn't preserve disjointness)

$$D_\infty^\times \to \mathbf{A}^1 \xleftarrow{p} \mathbf{A}^1 \times \mathrm{Conf}\mathbf{A}^1 \times \mathrm{Conf}\mathbf{A}^1 \xrightarrow{(a\otimes a)(-\Delta\otimes\mathrm{id})} \mathrm{Conf}\mathbf{A}^1 \times \mathrm{Conf}\mathbf{A}^1$$

has that $\Delta_{z^{-1}}$ defines a vertex coalgebra structure.

1.2. The structure of factorisation quantum groups (FCG, Bos, CD).

. At this point, the theory of quantum groups $U_q(\mathfrak{g})$ is well-developed:

(1) There are basis-free constructions [Ga] of $U_q(\mathfrak{g})$-Mod,

  (a) by working in the category $\mathrm{Perv}(\mathrm{Conf}_\Lambda(\mathbf{A}^1))$ of perverse sheaves on the configuration spaces,

  (b) by double-bosonisation [Ma].

(2) There is a "geometric" proof [CF] of the Kazhdan-Lusztig equivalence $U_q(\mathfrak{g})\text{-}\mathrm{Mod}^{ren} \simeq \hat{\mathfrak{g}}\text{-}\mathrm{Mod}_k^I$

In this series of projects, we develop the theory where vertex-algebraic analogues of quantum groups can be reasoned about in a basis-free category-theory way just as for finite quantum groups.

Factorisation quantum groups. It is well-known that the representation categories of $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, quantum vertex algebras, etc., should be controlled by "spectral" analogues $R(z)$ of R-matrices.

There have been many (inequivalent) attempts to axiomatise this (ref ref ref). The paper **project FCG** developed answered the following: they are *braided factorisation categories*.

Theorem H. *If $\mathcal{A}$ is a factorisation bialgebra, braided factorisation structures on* $\mathcal{A}$-Mod *are equivalent to factorisation R-matrices $R : A \otimes_2 A \to A \otimes_2 A$. (fix notation)*

Moreover, we show that we indeed recover standard structures:

Theorem I. *In the case of* $\mathrm{Ran}\mathbf{A}^1$*, a factorisation R-matrix induces an endomorphism*

$$R(z) \;:\; V \otimes V((z)) \;\to\; V \otimes V((z))$$

satisfying the spectral hexagon relations.

Likewise, in the case of $\mathrm{Conf}\mathbf{A}^1$ where factorisation bialgebras are precisely localised bialgebras V of (ref), we get a factorisation R-matrix induces an endomorphism

$$R \;:\; (V \otimes V)_{\mathrm{loc}} \;\to\; (V \otimes V)_{\mathrm{loc}}$$

of the localised bialgebra V, satisfying the hexagon relations.

For instance, (give example of $Y_\hbar(\mathfrak{g})$ with two coproducts) (ref GTW)

Factorisation bosonisation. In **project Bos**[5]

Factorisation Drinfeld doubling. In **project CD**[6]

1.3. Orthosymplectic vertex algebras (OSp, SA).

. In **project OSp**[7]

Work out how to take group-invariants of constructions in geometric representation theory. Define orthosymplectic moduli stacks, CoHAs, and orbifold vertex algebras (which we develop the theory of). Gives shuffle formulas. Show that we get twisted Yangians.

Theorem J. *Definition of CoHAM, compatible with CoHA*

(give shuffle formula)

Conjecture K. *(this is equal to twisted Yangian)*

[5] Joint with S. de Hority.

[6] Joint with W. Niu.

[7] Joint with S. de Hority.

Nonlocal QFT and shuffle structures. **Project SA** begun by noticing the following interesting pattern in structures considered project OSp.

$$\mathrm{BGL} \rightsquigarrow \mathrm{BSp}, \qquad \mathrm{Conf}(\mathbf{A}^1) \rightsquigarrow \mathrm{Conf}(\mathbf{A}^1), \qquad \mathrm{VA} \rightsquigarrow \mathrm{OSpVA}, \qquad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties, (KZ equations)) simultaneously generalise - this points towards this being a shadow of a more general theory.

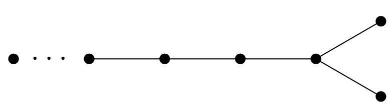
The starting observation is this - the definition (ref) a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL} \to \mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n \geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

$$\begin{tikzcd} & P_{n,m}(\sigma) \arrow[dl] \arrow[dr] & & & \\ \mathrm{GL}_n \times \mathrm{GL}_m & & \mathrm{GL}_{n+m} & \overset{A}{\mapsto} & A_n \otimes A_m \xrightarrow{m(\sigma)} A_{n+m} \end{tikzcd}$$

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n,m)$.

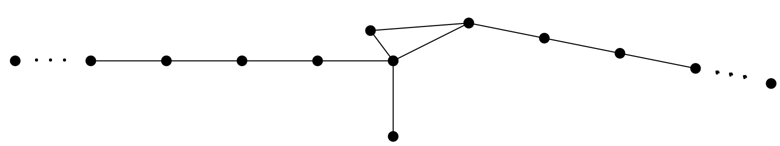
The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.

GL



$\mathrm{KM}_{\mathrm{SO}_{2n}}$

KM_{G_2}



To summarise:

- We get analogues of *shuffle algebras*.
- We get new configuration and Ran spaces

$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \textstyle\coprod_G \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{B}G), \qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G^*,$$

  where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G, so can define generalised *localised* and *vertex* algebras (and as in CY3 a Conf-to-Ran construction relating them). We expect to recover *boundary KZ* equations by taking conformal blocks (i.e. cohomology over $\mathrm{Ran}_{\mathrm{KM}}\mathbf{A}^1$).

- Topological case - topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras, then by considering $\mathrm{FactAlg}^{top}(\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}, \mathrm{Cat})$ we get analogues of the notion of *braided monoidal categories*.

- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups

$$\overset{\mathrm{GL}_3}{\bigcirc} \xrightarrow{U_{3,5}} \overset{\mathrm{GL}_5}{\bigcirc} \xrightarrow{U_{5,4}} \overset{\mathrm{GL}_4}{\bigcirc}$$

  where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as

$$\mathcal{M}_Q \;=\; \coprod \mathfrak{u}_e/G_i$$

  the product over all maps $(G_i) : Q_0 \to \mathrm{KM}$ and U_e is a choice if unipotent for each edge e.

  Relation to orbifolding.

- Stable envelope construction.

- Chen's [Ch] shuffle structure on cochains $\mathrm{C}^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX = \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n = T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G.

- Iterated integrals.

For the orthosymplectic example $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$, many of these structures are considered in OSp. Let us consider K_{G_2}

Example: G_2. For K_{G_2}, factorisation algebras consist of ordinary factorisation algebras but for any *triple* of points there is in addition equivariance with respect to the group $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$ acting on $\mathbf{C}^3$, in which the element

$$\begin{aligned}
\tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3)\\
\tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1)\\
\tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2)
\end{aligned}$$

squares to $\tau^2 = (231)$. Thus for instance a G_2 vertex algebra is a vertex algebra V with an (copy-paste from notes), and a topological G_2 factorisation category is a braided monoidal category $\mathcal{C}$ along with (copy-paste from notes; G_2 reflection equations)

Relation to folding. We expect there to be a folding construction of G_2 structures.

Orthosymplectic AGT. In **AGT**,

1.4. q-vertex algebras (qVA, KL).

. It has been long expected that one may define a q-analogue of the Kazhdan-Lusztig equivalence, but this has been hampered by the lack of a good definition of q-WZW algebras: currently, the available definition is an RTT-style definition from [**EK**].

q-vertex algebras. The main goal of **project qVA** is:

Conjecture L. *There is a factorisation category over the noncommutative space $\mathbf{A}^2_q$, such that any*

$$\mathcal{A} \in \mathrm{FactAlg}^{st}(\mathcal{D}\text{-Mod}_{\mathrm{Ran}\mathbf{A}^2_q})$$

*defines a q-**vertex algebra**.*

Moreover, for any complex finite-dimensional simple Lie algebra $\mathfrak{g}$, we may ask

Question M. *Is there an analogue of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$?*

Such a factorisation space would for free by Conjecture L define for us a q-vertex algebra $V^k_q(\mathfrak{g})$, by the same construction as for the affine WZW vertex algebra (and which we expect it would be is a q-deformation of) and we expect should agree with [**EK**] when $\mathfrak{g} = \mathfrak{sl}_n$. We expect there to be an algebra of modes fucntor $A(-)$, and we propose to finish with a sanity-check of our definitions by showing $A(V^k(\mathfrak{g})) \simeq U_q(\hat{\mathfrak{g}})$.

We spell out evidence for Conjecture L, first from physics, then give explicit mathematical details.

Physics: $5d$ Chern-Simons. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V^k_\hbar(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

where $\mathbf{A}^2_q = (\mathbf{C} \times \mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x,y]/(xy - qyx)$. Thus by analogy with the $3d$ case, to search for $V^k_q(\mathfrak{g})$ we need to understand factorisation algebras over $\mathbf{A}^2_q$.

Mathematical details. (copy-paste from the notes)

Kazhdan-Lusztig. One ultimate goal of projects FJ and qVA is to give an affine analogue of the factorisable proof [CF] of the Kazhdan-Lusztig equivalence.

Question N. *Is there a Riemann-Hilbert functor* $\mathrm{RH} : \mathrm{FactCat}(\mathbf{A}^2_q) \to \mathrm{FactCat}^{QCoh}(\mathbf{C}^2_q)$*, which sends the category* $V^k_q(\mathfrak{g})$-Mod *to* $Y_\hbar(\mathfrak{g})$-Mod*? (too vague)*

(need to write down what topological factorisation algebras on $\mathbf{C}^2_q$ are)

1.5. **Sheaf methods (Con, Loc, Eu).**

Localisation methods. In **projects Con and Loc**[8]

Virtual Euler classes and shuffle structures. In **project Eu**, we prove Atiyah-Bott torus localisation formulas on vanishing cycle cohomology, for certain non-quasismooth closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ of Artin stacks. This gives a unified torus localisation way to compute cohomological Hall type products. As a result, we recover shuffle descriptions of CoHAs and a new proof of compatibility between them and Davison's localised coproduct

Theorem O. *For any "split locus" map $\mathcal{M}^s \to \mathcal{M}$, we get a diagram*

$$\begin{tikzcd} \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, dashed, "1/e(\mathrm{N}_i)_{\mathrm{loc}}\cdot"] & \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, "p^s_* q^{s,*}"] & \mathrm{C}^\bullet(\mathcal{M}^s, \varphi^s) \\ \mathrm{C}^\bullet(\mathcal{M} \times \mathcal{M}, \varphi \boxtimes \varphi) \arrow[u, "(\pi\times\pi)^*"] \arrow[rr, "p_* q^*"] & & \mathrm{C}^\bullet(\mathcal{M}, \varphi) \arrow[u, "\pi^*"'] \end{tikzcd} \tag{2}$$

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Two consequences of this are:

- If we take $\mathcal{M}^s$ to be a *shuffle space*[9] given by products of "simple" moduli stacks, e.g. rank one quiver representations, then (2) recovers shuffle formulas for CoHAs.
- If we take $\mathcal{M}^s = \mathcal{M} \times \mathcal{M}$ together with its direct sum map to $\mathcal{M}$, (2) recovers the compatibility between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

The first we plan to use in SA to give more general shuffle-style products.

1.6. Louiville quantum gravity to vertex algebras (LCG).

History. In recent years, probabilists have increasingly understood quantum field theory (ref, detail, ref, detail)

However, there is currently very limited interaction between this field and geometric representation theorists, and this project tries to build a bridge between the two.

Goal. In **project LCG**[10] we aim to build a bridge between the two subjects. First, we aim to define a functor from Segal-style $2d$ conformal field theories to vertex algebras

$$\mathrm{CFT} \overset{(-)^h}{\to} \mathrm{CFT}^{hol} \overset{\mathrm{Res}}{\to} \mathrm{FactAlg}(\mathbf{C})^{hol}_{\mathbf{C}\rtimes\mathbf{C}^\times} \overset{[\mathrm{CG}]}{\to} \mathrm{VertexAlg}.$$

Second, we then aim to check that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs of (ref) are sent to the Heisenberg and Virasoro vertex algebras, repectively.

[8]Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.
[9]i.e. shuffle algebra in the category of spaces, see SA.
[10]Joint with V. Giri.

Details. The last two functors are Costello-Gwilliam's factorisation algebra to vertex algebra construction, and restriction to the subcategory $\mathrm{Cob}_{2,/\mathbf{C}} \to \mathrm{Cob}_2$ of cobordisms *inside* **C**, so the main content is the first, which takes *holomophic part*.

To define the category CFT, (copy notes)

References

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