WHAT STRUCTURE DOES $\mathcal{W}_{1+\infty}$-Mod AND $Y_\hbar$-Mod HAVE?

ALEXEI LATYNTSEV

The answer is that $Y_\hbar$-Mod is a factorisation $\mathbf{E}_2$-category over $\mathrm{Ran}^{ch}\mathbf{C}$. By Dunn additivity, this is equivalent to there being two factorisation monoidal structures $\otimes_1$ and $\otimes_2$ along with an equivalence

$$(\otimes_1) \otimes_2 (\otimes_1) \ \xrightarrow{\sim}\ (\otimes_2) \otimes_1 (\otimes_2)$$

where we have suppressed the implicit σ_{23}. This is an equivalence of functors of sheaves of categories over

$$((\mathrm{Ran}\mathbf{C} \times \mathrm{Ran}\mathbf{C})_\circ \times (\mathrm{Ran}\mathbf{C} \times \mathrm{Ran}\mathbf{C})_\circ)_\circ \ \simeq\ (\mathrm{Ran}\mathbf{C})^4_\circ.$$

Restricting to $(\varnothing \times \mathrm{Ran}\mathbf{C} \times \mathrm{Ran}\mathbf{C} \times \varnothing)_\circ$ gives

$$\otimes_2 \ \xrightarrow{\sim}\ \otimes_1$$

with σ suppressed again.

1.1. In other words, for every $M_2, M_3 \in \mathcal{A}$-Mod, where $\mathcal{A}$-Mod is the putative factorisation $\mathbf{E}_2$-category and $\mathcal{A}$ an $\otimes$-algebra, we have

$$M_2 \otimes_1 M_3 \ \xrightarrow{\sim}\ M_3 \otimes_2 M_2$$

as sections of $(\cup j)^*(\mathcal{A}\text{-Mod})$.

1.1.1. One way of getting such an isomorphism is to multiply by an element

$$R_{12} \ = \ \sigma R_{21}^{-1} \sigma \ \in \ \Gamma((\mathrm{Ran}\mathbf{C})^2_\circ, \mathcal{A} \boxtimes \mathcal{A})$$

which we should require to be invertible to give a natural *isomorphism* above. We expect to be able to use a Barr-Beck-Lurie argument to show that this gives all possible $\mathbf{E}_2$ structures.

1.1.2. Likewise, we have

$$R_{21} \ : \ \otimes_1 \ \xrightarrow{\sim}\ \otimes_2$$

and moreover,

$$\begin{array}{ccc} M_2 \otimes_1 M_3 & \xrightarrow[\sim]{R_{21}} & M_3 \otimes_2 M_2 \\ \wr\downarrow R_1 & & \wr\downarrow R_2 \\ M_3 \otimes_1 M_2 & \xrightarrow[\sim]{R_{12}} & M_2 \otimes_2 M_3 \end{array}$$

a commuting diagram in $\Gamma(\mathrm{Ran}\mathbf{C}, (\cup j)^*(\mathcal{A}\text{-Mod}))$, where $R_{12} = R_{21}^{-1}$.

1.1.3. Note that a factorisation monoidal structure $\otimes_i$ is equivalent to giving a factorisation coproduct

$$\Delta_i \ : \ (\cup j)^* \mathcal{A} \ \to \ j^*(\mathcal{A} \boxtimes \mathcal{A})$$

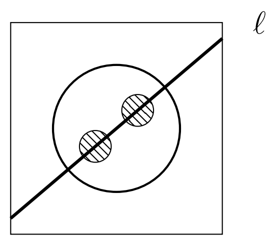
compatible with the algebra structure. The above then says that

$$R_{21} \cdot \Delta_2(a) \ = \ \Delta_1(a) \cdot R_{21}$$

and likewise, $R_2 \cdot \Delta_2(a) \ = \ \Delta_2^{op}(a) \cdot R_2$ and $R_1 \cdot \Delta_1(a) \ = \ \Delta_1^{op}(a) \cdot R_1$,[1] and

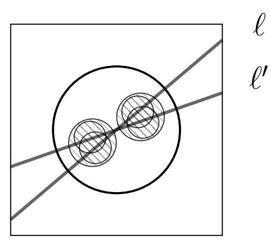
$$R_{12} \cdot \Delta_1^{op}(a) \ = \ \Delta_2^{op}(a) \cdot R_{12}$$

1.2. Recall that an $\mathbf{E}_2$-object is equivalent to a factorisation algebra over $\mathbf{R}^2$. In particular, if we give an $\mathbf{E}_2$-structure to $\mathcal{A}$-Mod, we get a set of coproducts on $\mathcal{A}$ parametrised by a line ℓ:



$$\Delta_\ell(z) \ : \ (\cup j)^* \mathcal{A} \ \to \ j^*(\mathcal{A} \boxtimes \mathcal{A})$$

We have an identification given by $R_{\ell,\ell'}(z)$ of any pair of coproducts, given by including them into a bigger circle:



$$R_{\ell,\ell'}(z) \cdot \Delta_\ell(z) \ = \ \Delta_{\ell'}(z) \cdot R_{\ell,\ell'}(z)$$

In the notation of [GTW] this is $R_{\ell,\ell'}(z) \ = \ A(z + \theta\hbar)$, where $\theta \ = \ \theta(\ell, \ell')$, and only countably many are nontrivial, with

$$R_1 \ = \ \prod_{n \geqslant 0} A(z + \theta_n \hbar)$$

where $\theta_n \ = \ \theta(\ell_n, \ell_{n+1})$. The obvious question is:

Question. *Can these $A(z)$ be expressed as Stokes factors?*

[1](The *op* might be the wrong way round here.)

References

[GTW] Gautam, S., Laredo, V.T. and Wendlandt, C., 2021. *The meromorphic R-matrix of the Yangian.* In Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin (pp. 201-269). Cham: Springer International Publishing.