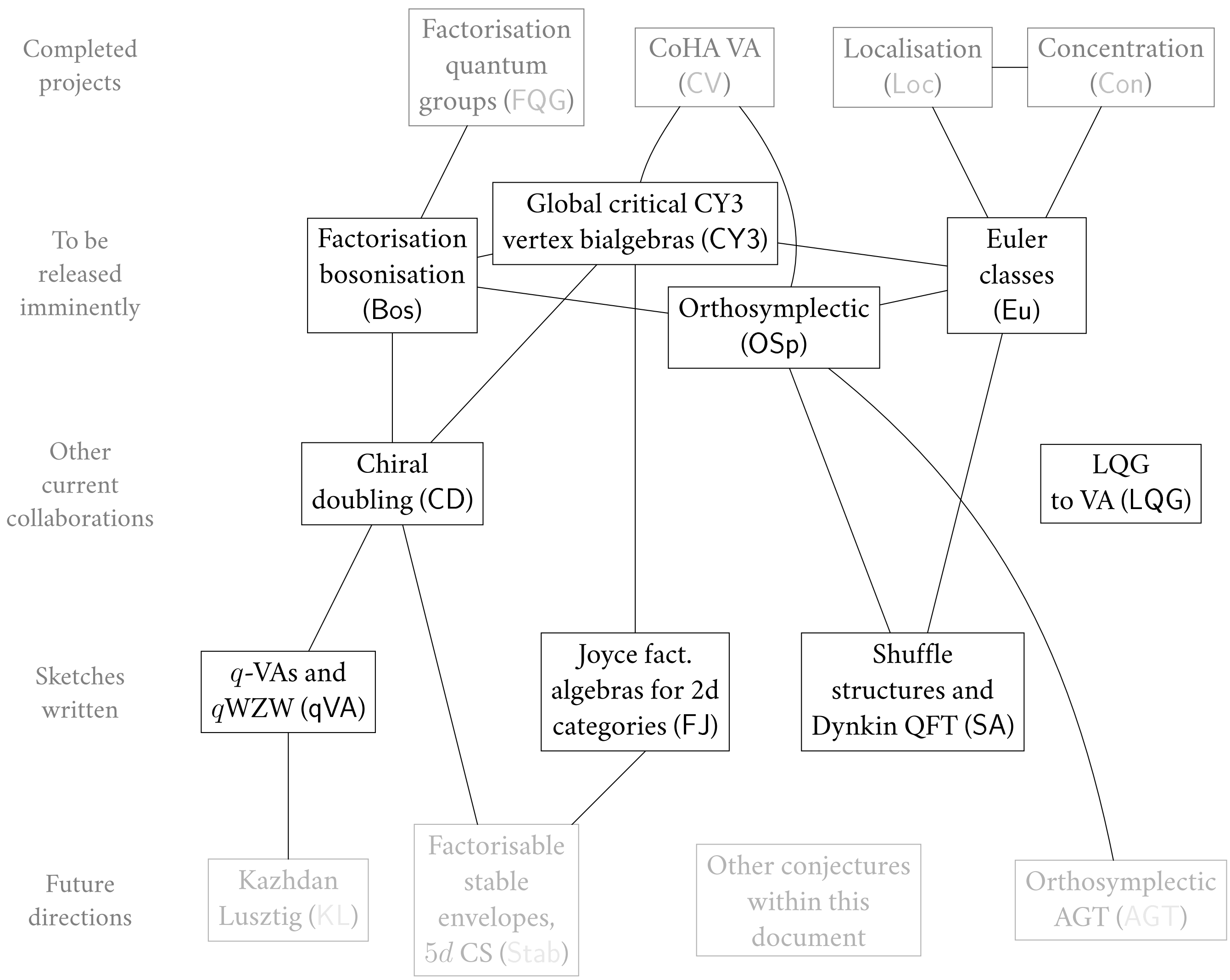
RESEARCH PLANS - ABRIDGED

ALEXEI LATYNTSEV

*For a **full** (13 pp.) version, see* https://alyoshalatyntsev.github.io/plan/plan.pdf.

*For a non-technical **summary** (2 pp.), see* https://alyoshalatyntsev.github.io/plansummary/plansummary.pdf.



Note: CY3,FJ *is joint with S. Kaubrys,* CY3 *with S. Jindal,* CD *with W. Niu,* OSp, Bos, SA *with S. de Hority,* Loc, Con *with A. Khan, D. Aranha, H. Park, and C. Ravi, and* LQG *with V. Giri.*

1. Research statement

I am a mathematician working in geometric representation theory.

A central problem of geometric representation theory and enumerative geometry is understanding structures attached to *Calabi-Yau threefolds*: Donaldson-Thomas invariants, mirror symmetry, quantum groups, ... Most of my research is in this area, making physics-inspired rigorous geometric constructions of such structures and proving new relations between them.

1.1. Algebraic structures attached to CY3s (CV, CY3, FJ, Stab)

Cohomological Hall algebras are associative algebras attached to Calabi-Yau varieties or categories [Da; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZ] (e.g. quantum groups [MO]), wall crossing, 3d mirror symmetry, instantons, skeins, ... Thus the following question is important and useful: what are all the algebraic structures on CoHAs?

Global critical CY3 vertex quantum groups. Let $\mathcal{C}$ be the CY3 category $\mathrm{Rep}(Q, W)$, the representations of a Jacobi algebra of a quiver with potential, or $\mathrm{Coh}_0(K_{T^*C})$, zero dimensional coherent sheaves on a local curve [KK], or more generally a *deformed CY3* completion.

Theorem. [CY3] *The critical cohomology*[1] *of the moduli stack of objects $\mathcal{M}$ has a vertex coproduct*

$$\mathrm{H}^\bullet(\mathcal{M}, \varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M}, \varphi) \,\hat{\otimes}\, \mathrm{H}^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

compatible with the CoHA: it forms a vertex quantum group *(see* FQG*).*

Theorem. [CY3] *There is a functor Φ from Q-localised bialgebras [Da] to vertex bialgebras. The former is equivalent to the category of* factorisation bialgebras *on the coloured configuration space* $\mathrm{Conf}_{Q_0}\mathbf{A}^1$.

Theorem. [CY3; CV for $W = 0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}}, \varphi_{W^{(3)}})^{ext} \overset{[\mathrm{BD}]}{\simeq} Y_\hbar(\mathfrak{b}_Q)$ is identified by Φ with the Davison/Yang-Zhao localised coproduct* [Da; YZ]*, and (when defined) Drinfeld's meromorphic coproduct* [Dr; GLW].

This generalises **Joyce-Liu's vertex coproduct** [Joa; Li], shows that all these well-known coproducts on the Yangian are **equal**, and gives a **geometric** definition using moduli stacks and configuration spaces. Finally, Theorem 1.1 points to a new Ran space definition of vertex-vertex *bi*algebras.

Lift to factorisation algebra and relation to stable envelopes. To move towards *arbitrary* CY3s, we first interpret the above in terms of factorisation algebras; second, we relate to the $\mathcal{W}$-algebras for surfaces of [MMSV], thus understanding the structure for K_S. Let Q be a quiver with torus $T = \prod T_d$.

Conjecture. [FJ] *Given $\mathcal{M}^f = \{(m, \lambda) \;:\; \lambda \in \mathfrak{t},\; m \in \mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d) = \mathrm{Ran}_{Q_0}\mathbf{A}^1$, its relative BM-homology $\mathcal{A} = \pi_*\omega$ is a factorisation algebra over the coloured Ran space. Its restriction to the colour-diagonal recovers the vertex quantum group structure on the nilpotent CoHA* [SV].

Conjecture. [FJ] *For S a smooth algebraic surface, there is a braided factorisation category* $\mathrm{Rep}\mathcal{W}$ *over* $\mathrm{Ran}_S K_S$ *(c.f.* FQG*). Applying* Bos/CD *allows us to construct $\mathcal{W}(S)^{\geqslant 0}$ and $\mathcal{W}(S)$ from* [MMSV]*'s $\mathcal{W}(S)^{>0}$.*

The definition of $\mathcal{M}^f$ is clearly reminiscent of [MO, §5.1.1]'s stable envelope construction, and suggests a resolution to their question: *"we view this definition as provisional; perhaps a better set of axioms will emerge later"*. Let $\mathbf{w}$ be a multidimension vector of quiver Q and $M(\mathbf{w})$ the quiver variety.

Conjecture. [Stab] *There is a factorisation space $\pi_{\mathbf{w}} : M(\mathbf{w})^f \to \mathrm{Ran}_Q\mathbf{A}^1$ and the factorisation category $\mathcal{E}$ spanned by $\pi_{\mathbf{w},*}\omega$ is acted on by $\mathcal{A} = \pi_*\omega$. Applying chiral Tannakian reconstruction $\mathcal{E} \simeq \mathrm{Rep}\mathbf{D}\mathcal{A}$ gives the double of $\mathcal{A}$ with its (two) coproducts.*

The above is a Drinfeld-Kohno Theorem for [MO]'s Yangians $Y_\hbar(\mathfrak{g}_Q)$ (see qVA for relations to qKZ).

[1] i.e. $\mathcal{M} = \mathrm{Crit}(W)$ is a critical locus inside a *smooth* moduli stack; we take the vanishing cycle sheaf $\varphi = \varphi_W$.

1.2. The structure of factorisation quantum groups (FQG, Bos, CD)

Historically the definitions of (double) affine quantum groups $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ were (ingeniously) made very explicitly [Dr; MO], and still much of the modern (e.g. CoHA) literature is based on explicit shuffle computations, e.g. [MMSV; SV; YZ]. This series of projects *axiomatises* (operadically) these structures, allowing us to import techniques from the theory of ordinary *quantum groups*,[2] to recover the above formulas as a *consequence* of these definitions.

Factorisation quantum groups. In FQG we develop a theory of $\mathbf{E}_n$*-factorisation categories* over *factorisation spaces* X (including ordinary groups G, configuration spaces $\mathrm{Conf}_{Q_0}\mathbf{A}^1$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$). We first give basic structure results for braided factorisation categories $\mathcal{C}$:

Theorem. [FQG] *Let $\mathcal{A}$ is a factorisation algebra in $\mathcal{C}$ over X, a (braided) factorisation structure on $\mathcal{A}$-*FactMod$(\mathcal{C})$ *induces a* factorisation bialgebra *structure on $\mathcal{A}$ (and a* factorisation R-**matrix** $R : \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A}$*).*

Theorem. [FQG] *When $X = \mathrm{Ran}\mathbf{A}^1$ (resp. $\mathrm{Conf}\mathbf{A}^1$), Theorem 1.2 recovers classical notions* [EK; FR] *of **quantum vertex algebras** (resp. localised algebras) and their R-matrices $R(z)$ satisfying the spectral YBE.*

For instance, the structure [GLW] for **Yangians** (e.g. two (meromorphic) coproducts and R-matrices $R^-, R^{0,\epsilon}, R^\epsilon$ relating them) are equivalent to: $Y_\hbar(\mathfrak{g})$-Mod is a lax braided factorisation category. This makes rigorous the physics claim [CWY] that $Y_\hbar(\mathfrak{g})$-Mod from $4d$ Chern-Simons is a topological-holomorphic factorisatsion category over $\mathbf{R}\times\mathbf{C}$. The above may help understand **affine Yangians** (e.g. [GRZ]; qVA for relation to q-WZW) and the **new KL equivalances** [BCDN]. Finally, in principle we may get lots of examples from:

Theorem. [FQG] *A generalisation of Borcherds' twist construction* [Bo] *to arbitrary decomposition algebra.*

Factorisation bosonisation. In the CoHA literature, a lot of algebraic effort needs to be expended each time [Da; RSYZ; YZ], [CY3; OSp] to *add in the Cartan piece* $\mathrm{H}^\bullet(\mathcal{M},\varphi) \rightsquigarrow \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$, e.g. to obtain Yangians of Borels $Y_\hbar(\mathfrak{b}_Q)$. In the finite groups case, a clean construction is Tannakian reconstruction

$$U_q(\mathfrak{n}) \rightsquigarrow U_q(\mathfrak{n}) \qquad \text{via} \qquad U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

c.f. [Ga; Maa; Mab]; in Bos we apply the same to vertex and factorisation bialgebras to obtain:

Theorem. [Bos, in preparation] *There is a "factorisation" **Tannakian reconstruction** functor from (braided) factorisation categories $\mathcal{C}$ to (quasitriangular) factorisation quantum groups $\mathcal{A}$. In the preprojective case of Theorem 1.1,*

$$Y_\hbar(\mathfrak{b}_Q)\text{-Mod} \;\simeq\; Y_\hbar(\mathfrak{n}_Q)\text{-Mod}(Y_\hbar(\mathfrak{t}_Q)\text{-Mod})$$

we Tannakian reconstruct $Y_\hbar(\mathfrak{b}_Q) \simeq \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$ and its localised/vertex bialgebra structure.

Applying this to $\mathrm{H}^\bullet(\mathcal{M},\varphi)\text{-Mod}(\mathrm{H}^\bullet(\mathcal{M})\text{-Mod}^\cup)$ **automates** the process of extending CoHAs.

Factorisation Drinfeld doubling. An active problem is how the structures in CY3 relate to **wall crossing** and stability conditions, see [Br; Job]. As a first attempt, we use FQG to understand **doubling**, where the CoHA of heart $\mathcal{A}$ and its opposite $\mathcal{A}[1]$ are glued, in a similar way to Bos:

Conjecture. [CD] *There is a "factorisation" **Drinfeld centre** construction $Z_{\mathbf{E}_1}(\mathcal{C})$ of a chiral factorisation category $\mathcal{C}$, which carries compatible chiral and ordinary monoidal structures, Tannakian reconstruction gives*

$$Z_{\mathbf{E}_1}^{Y_\hbar(\mathfrak{t}_Q)\text{-Mod}}(Y_\hbar(\mathfrak{b}_Q)\text{-Mod}) \;\simeq\; Y_\hbar(\mathfrak{g}_Q)\text{-Mod},$$

and likewise we recover the Takiff algebra double construction of [AN].

1.3. Orthosymplectic structures (OSp, SA, AGT)

What happens if we apply the above techniques to orbifolds? What do boundary Chern-Simons, outer automorphisms of Lie algebras, twisted Yangians, boundary KZ equations, orthosymplectic quiver varieties, . . . have in common?

[2] Namely, when $X = \mathrm{Ran}\mathbf{R}^2$ in the below, via Lurie [Lu].

Orthosymplectic CoHAs. We define *orthosymplectic moduli stacks* $\mathcal{M}^{\mathrm{OSp}}$, including **perverse-coherent sheaves** with a sympletic/orthogonal bilinear form, and quivers and potential with involution.

Theorem. [OSp] *For $\mathcal{M}$ as in* CY3 *the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M},\varphi)$ acts on $\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$, i.e.*

(1) *there is a left module action a of the **CoHA** respecting the involution, i.e. it **factorises** over $\mathbf{R}/\pm$, compatible with*
(2) *a **symplectic vertex algebra** structure (factorisation coalgebra over symplectic Ran space $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\mathfrak{t}_{\mathfrak{sp}_{2n}}$, see* SA*). The defining R-/K-matrices satisfy the **Cherednik reflection equation**.*

Theorem. [OSp] *There is an action of $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M})$ of the CoHA of zero-dimensional coherent sheaves on surface S on $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}_{\sigma\text{-ss}})$ the BM homology of a compactification of the stack of $G\in\{\mathrm{Sp},\mathrm{O}\}$-bundles.*

This gives a rigorous definition of the objects coming from boundary $4d$ Chern-Simons theory [BS], should relate to work in ι**quantum groups** [LW] and Finkelberg-Hanany-Nakajima's ongoing work on **orthosymplectic Coulomb branches** (see also AGT). In the quiver with potential case (resp. in the conjecture, type A_{2n+1} preprojective), we have:

Theorem. [OSp] *An explicit shuffle formula for the CoHA action and vertex coaction on $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}_{Q,W},\varphi^{\mathrm{OSp}})$.*

Conjecture. *The preprojective orthosymplectic CoHA*[3] *is isomorphic to the **twisted Yangian** $Y_\hbar(\mathfrak{gl}_n)^{tw}$ from* [BR].

Dynkin QFT and shuffle structures. The structures (2) in OSp are defined over the *symplectic configuration space*[4] $\mathrm{Conf}_{\mathrm{Sp}}(\mathbf{A}^1) = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$, with singularities on root hyperplanes; alternatively over $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\mathfrak{t}_{\mathfrak{sp}_{2n}}$. To give examples:

Theorem. [OSp] *Restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$ gives an **invariants** functor $\iota : \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \to \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1)$. n.b. we expect the structures in* OSp *come from applying ι to $\mathcal{M}^f$ (see* FJ*).*

In SA, we **generalise** the above to systems of **arbitrary Kac-Moody groups** G_i [Ku, §V]; this forms a category K_{KM} with morphisms the parabolics. Write $K_{\mathrm{GL}}, K_{\mathrm{Sp}},\dots$ for the appropriate subcategories.

Fact. [SA] *A shuffle algebra* [KS; Gr] *is equivalent to a monoidal functor $K_{\mathrm{GL}} \xrightarrow{A} \mathrm{Vect}$.*

There are KM analogues of *vertex algebra* and *localised bialgebra*, *quiver representation* with moduli stack $\mathcal{M}_Q = \sqcup\mathfrak{u}_e/G_i$, *quiver varieties*, and *(factorisation) braided monoidal categories* (generalising the G_2-**reflection equation** [Ku]). We generalise[5] **Chen's Theorem** [Ch] on the shuffle structure on cohomology $\mathrm{H}^\bullet(LX)$ of loop spaces; leading to the

Question. [SA] *Can we recover Dynkin/q-analogues of **multiple zeta values*** [KMT; Mi] *using this* KM*-shuffle structure?*

We produce $G_2, C_n,\dots$ structures by **folding** $D_4, A_{2n+1},\dots$ structures by outer automorphisms $\mathbf{Z}/3, \mathbf{Z}/2,\dots$.

Conjecture. [SA] *We may recover $\{\varnothing,$**boundary,** G_2**-, …**$\}$ spherical (or trigonometric) **KZ equations*** [ES] *by constructing a $K_{\mathrm{GL}}, K_{\mathrm{Sp}}, K_{G_2}\dots$ (or $K_{\widehat{\mathrm{GL}}},\dots$) **affine vertex algebra** via BD Grassmannians, and taking conformal blocks.*

Question. [SA] *Can we recover Dynkin/q-analogues of **stable envelopes** (see* [MO], Stab*)?*

A twisted AGT correspondence. A foundational result in geometric representation theory is the **AGT correspondence** linking $\mathcal{W}$-vertex algebras and surfaces, first by Grojnowski and Nakajima [Gr; Na] then in [AGT; BFN] and [RSYZ].

Conjecture. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a **Verma module** for an orthosymplectic analogue $\mathcal{W}^k(\mathfrak{gl}_n)^{\mathrm{OSp}}$ of a $\mathfrak{gl}_n$ W-algebra (proof sketch: use* SA*-techniques on* [BFN]*).*

Conjecture. [AGT] *The dimension zero CoHA of $\mathbf{A}^3$, which is $U_\hbar(\mathcal{D}(\mathbf{C}/\pm))$ and admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients, acts on $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$, the equivariant intersection cohomology of $\mathbf{A}^3$-**instantons** (see* [RSYZ]*). Likewise for any quiver with potential.*

1.4. q-vertex algebras (qVA, KL)

q-vertex algebras. Our understanding of double affine quantum groups or Kazhdan Lusztig equivalences, qKZ equations, … is seriously hampered by the abscence of a good definition of q-affine vertex algebra.

[3] i.e. the image of the CoHA in $\mathrm{End}(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}))$.
[4] As opposed to the ordinary configuration space $\mathrm{Conf}\mathbf{A}^1 = \sqcup\mathbf{A}^n/\!/\mathfrak{S}_n = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BGL})$.
[5] Replacing the n-simplex Δ^n with $\Delta_G = T_G/W_G$.

The many attempts [FJW; FR; EK] to define q-affine vertex algebras have not yet been conceptual enough to apply factorisation techniques [CF] or physics heuristics [Co; GR; Wi] to relate to q**KZ connections** or the (conjecturally, **affine**) **Kazhdan-Lusztig equivalence**, or the new KL equivalences [BCDN].

Our goal is to build this, inspired by Costello's physics work on deformed spacetimes for 5d Chern-Simons in physics and using techniques of FQG.

Conjecture. [qVA] *There is a factorisation category* $\mathcal{D}$-Mod *over the noncommutative plane*[6] $\mathbf{A}^2_q$*, a factorisation algebra in which induces a* q-***vertex algebra*** *(i.e. ordinary vertex algebra with poles on the* q*-diagonals, e.g.* [FR], [EK] *for* $\mathfrak{g} = \mathfrak{sl}_n$*).*

Conjecture. [qVA] *There is an analogue of the BD Grassmannian* $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$ *(this induces a* q*-affine VOA* $V^k_q(\mathfrak{g})$*).*

The physics heuristic for this is: $V^k(\mathfrak{g})$ comes from $3d$ Chern-Simons with boundary $\mathbf{C}$ [Wi]; $V^k_q(\mathfrak{g})$ comes from $5d$ **Chern-Simons** with noncommutative boundary $\mathbf{C}^2_{nc}$ [Co; GR; GRZ]. The category $\mathcal{D}$-Mod will be related to q-**difference modules**[7] on $\mathbf{A}^1$, so the above directly generalises the usual factorisation definition of vertex algebras.

Conjecture. [KL] *The restriction of* ***conformal blocks*** *of* $V^k_q(\mathfrak{g})$*-modules to* $(\mathbf{A}^1)^n_\circ \subseteq (\mathbf{A}^2_q)^n_\circ$ *equal to the* q*KZ connection.*

Conjecture. [KL] *There is a* ***Zhu algebra*** *functor,* $\mathrm{Zhu} : V^k_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g})$*, and the functor* $\mathrm{RH} : V^k(\mathfrak{g})\text{-Mod} \mapsto U_q(\mathfrak{g})\text{-Mod}$ *in the proof* [CF] *of KL is "parallel transport" along a factorisation category* $V^k_q(\mathfrak{g})$-Mod *on* $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$*.*

The above would help understand **affine KL** and the **new KL equivalences** [BCDN] from $3d$ mirror symmetry.

1.5. Sheaf methods (Con, Loc, Eu)

Localisation methods. One of the main techniques in enumerative geometry are the *torus localisation* and *Graber-Panharipande formulas* [GP]. We generalise these to the Artin moduli stacks appearing in modern enumerative/algebraic geometry.

Theorem. [Conc; Loc] *We give conditions for the cohomology of an Artin stack* $\mathcal{X}$ *to be* ***concentrated*** *on a closed substack* $\mathcal{Z}$*; when* $\mathcal{Z} = \mathcal{X}^T$ *is fixed points of a quasismooth dg scheme we give* ***Atiyah-Bott*** $\mathrm{id} = i_*\left(i^!(-)/e(N_{vir})\right)$ *and* ***Graber-Pandharipande localisation*** *formulas* $[\mathcal{X}]^{vir} = i_*\left([\mathcal{X}^T]^{vir}/e(N_{vir})\right)$*.*

Theorem. [Eu] *We strengthen both parts of the theorem to* ***critical cohomology*** *of* ***arbitrary*** *closed embeddings* $\mathcal{Z} \hookrightarrow \mathcal{X}$ *quasismooth over a common base. As a result, for* $\mathcal{M}$ *as in* CY3*, we have the following Theorem:*

Theorem. [Eu] *For any "split locus" map* $\pi : \mathcal{M}^s \to \mathcal{M}$*, we have* $\mathrm{CoHA} = \mathrm{CoHA}^s/e(\mathrm{N}_{i,vir})$*.*

Taking $\mathcal{M}^s$ a *shuffle space*[8], this gives a **universal, geometric** way of producing/interpreting shuffle products for CoHAs [Da; SV; YZ]. Taking $\mathcal{M}^s = \mathcal{M} \times \mathcal{M} \xrightarrow{\oplus} \mathcal{M}$ proves **compatibility** [CV,CY3,Li] between **coproducts** and the CoHA (c.f. CY3). This easily **generalises** to the OSp/Dynkin SA cases.

1.6. Liouville quantum gravity to vertex algebras (LQG)

Probabilists are beginning construct Feynman measures (GFF [BPR], LQG [KRV]) **rigorously** to understand $2d$ CFTs [CRV; DS; Sh]. There is almost no intereaction with the algebraic geometry/functorial school to QFT, and they understand/have access to techniques (e.g. SLE curves [MS; SS], combinatorial approximations) that the latter school does not. We want to build a bridge:

Conjecture. [LQG] *There is a* ***chiral part*** *functor* $F^{ch} : \mathrm{CFT} \to \mathrm{VertexAlg}$*, from Segal-style full* $2d$ *CFTs to vertex algebras. It sends* [GKRV]*'s LQG (resp. GFF) to the Virasoro (resp. Heisenberg) VOA, and* [KRV]*'s DOZZ formula to the Virasoro OPE.*

We will then use **free field embeddings** to produce a probabilistic CFT analogue of the $\mathfrak{g}$-**affine vertex algebra**.

[6] Its with ring of functions $\mathbf{C}\langle x, y, q\rangle/(yx - xyq)$ with q central.

[7] e.g. a q-difference operator ∂_x on $\mathbf{A}^1$ induces a derivation $y\partial_x$ on $\mathbf{A}^2_q$.

[8] i.e. shuffle algebra in the category of spaces, see SA.

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