q-DEFORMED D-MODULES

To do:

(1) check 7.1.4, the relation between D-modules on $X_{\mathbf{q}}$ (or X_Φ) with D-modules on X/Φ.

(2) What should theorem 7.1.7 say?

Contents

1. Introduction

1.1. **Motivation.**

1.1.1. The main point of this document is to understand a q-deformation of the WZW vertex algebra.

The approximate physics picture of the relevant spaces and their (Koszul dual to) local operators is:

$$\begin{array}{cccc} \underbrace{\underline{\mathbf{C}\times\mathbf{C}}}_{\hbar} & \underbrace{\underline{\mathbf{C}\times\mathbf{C}}}_{\hbar}\times\mathbf{R} & \mathrm{Rep}_{VA}\, V^k_\hbar(\mathfrak{g}) & \mathrm{Rep}_{alg}\, Y_\hbar(\hat{\mathfrak{g}})^1,\ \mathrm{Rep}_{VA}\, \mathcal{W}^{c,\mu}_{1+\infty}(\mathfrak{g}) \\ \mathbf{C} & \mathbf{R}\times\mathbf{R}\times\mathbf{R} & \mathrm{Rep}_{VA}\, V^k(\mathfrak{g}) & \mathrm{Rep}_{alg}\, U_\hbar(\mathfrak{g}) \end{array}$$

where the underlined spaces are *noncommutative*, i.e. are a deformation over $\mathbf{C}[\hbar]$ given by the Moyal product, a.k.a. differential operators $\mathcal{D}_\hbar(\mathbf{C})$. In physics terminology this is called Nekrasov's "Ω-background". See [Co] for details about the theory on the top right. For details on the two-parameter deformation see [Li] and [GRZ].

2. Background: vector fields and D-modules

2.1. **Vector fields.**

Recall that a tangent vector is a map

$$\xi : \mathbf{D}_2 \to X$$

from the second order infinitesimal neighbourhood of the origin in the formal disk $\mathbf{D}$. Likewise we get the notion of n-jet for any $n = 1, 2, \cdots, \infty$, and stronger still we could ask for a map

$$\xi : \mathbf{G}_a \to X.$$

A vector field induces a map on functions

$$\mathcal{O}(X) \to \mathbf{C}[\epsilon]/\epsilon^2,$$

and the ϵ coeffient is the *derivative* of the function in the direction of the vector field.

2.1.1. *Multiplicative and elliptic jets.* We make the following redundant definition. If G is a one-dimensional algebraic group, a *G-jet* is a map

$$\xi : \mathbf{D}^G \to X$$

from the formal neighbourhood of the identity in G. Of course, all of these are non-canonically isomorphic and so this is the same thing as an ordinary jet. Let χ_G be a left-invariant vector field on G, then

$$\mathbf{D}_2^G = \mathbf{D}_2 \cdot \chi_G.$$

However, when we pass to the quantum versions of the above definitions, the definitions for different G will seperate.

2.1.2. *Vector fields.* A *vector field* is a map over X

$$\xi : X \times \mathbf{D}_2 \to X.$$

Proposition 2.1.3. *The sheaf $\mathcal{T}_X$ of vector fields is the Lie algebra of the group $\mathrm{Aut}(X)$ over X.*

Proof. A tangent vector inside $\mathrm{Aut}(X)$ is a map

$$\psi : \mathbf{D}_2 \to \mathrm{Aut}(X)$$

which by adjunction is the same as a map

$$\mathbf{D}_2 \times X \to X.$$

The condition that ψ needs to be a tangent vector at the unit $\mathrm{id} \in \mathrm{Aut}(X)$ is equivalent to this map being over X. □

In exactly the same way, an n-jet field on X is the same as an n-jet at the identity of $\mathrm{Aut}(X)$.

2.2. **Koszul dual picture.** If X is a smooth scheme, we have a Koszul duality of sheaves of algebras over X,

$$\mathrm{KD}(\mathcal{D}_X) \;\simeq\; \Omega_X$$

where Ω_X is the de Rham complex. The equivalence is given by a bimodule, the de Rham complex $\mathcal{D}_X \otimes \Omega_X$ equipped with a differential which intertwines the factors.

Thus, if we define q-deformed D-modules on X as D-modules on a noncommutative space $Y = Y_{\mathbf{q}}$, it is natural to expect that it be Koszul dual to the noncommutative de Rham complex Ω_Y, if it is defined.

2.3. **Jets.** In the above, we considered jets, and moreover, the de Rham stack $X_{dR} \;\simeq\; X/\mathcal{G}$ is the quotient by

$$\mathcal{G} \;=\; \exp(\mathcal{T}_X) \;\simeq\; \mathcal{J}_\infty X$$

the formal group scheme over X given by formal jets. In particular, below when we will want to define q-D modules on X as D-modules on a certain noncommutative space $Y = Y_{\mathbf{q}}$, we will need to define the jet space $\mathcal{J}_\infty Y_{\mathbf{q}}$, and

$$Y_{\mathrm{dR}} \;=\; Y/\mathcal{J}_\infty Y.$$

For this we will use the machinery developed by Majid and Simao in [MS].

3. Quantum analogues

3.1. **q-vector fields.** Now let $\mathbf{G}_m$ act on our smooth scheme X. This makes $\mathcal{O}_X$ into a $\mathbf{Z}$-graded sheaf, so we can define the sheaf $\mathcal{T}_X^q \subseteq \mathcal{E}nd(\mathcal{O}_X)$ of q-vector fields consisting of endomorphisms ∂ with

$$\partial(fg) \;=\; \partial(f)g \;+\; q^{|f|}f\partial(g)$$

for all pairs of homogenous functions $f, g \in \Gamma(\mathcal{O}_X)$.

3.1.1. One way to axiomatise this is the following. Extend $\mathcal{O}(X)$ by adding the variable $\mathbf{q}$ with commutation relations

$$\mathbf{q}f \;=\; q^{|f|}f\mathbf{q}$$

for homogeneous elements, where $q \in k$ is central. Then

$$\mathbf{q}\partial(fg) \;=\; \mathbf{q}\partial(f)g \;+\; f\mathbf{q}\partial(g)$$

and so $\mathbf{q}\partial$ defines an honest vector field on $\langle \mathcal{O}(X), \mathbf{q}\rangle$. Thus a q-vector field induces an algebra map

$$\langle \mathcal{O}(X), \mathbf{q}\rangle \;\to\; \langle \mathbf{C}[\epsilon]/\epsilon^2, \mathbf{q}\rangle, \qquad\qquad f \;\mapsto\; f + \mathbf{q}\partial(f)\epsilon,$$

where $\mathbf{q}$ and ϵ commute. We now turn to the question of what this algebra $\langle \mathcal{O}(X), \mathbf{q}\rangle$ is.

Proposition 3.1.2. *$\langle \mathcal{O}(X), \mathbf{q}\rangle[q, q^{-1}]$ is a $\mathbf{Z}[q, q^{-1}]$-quantisation of $\mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})[q, q^{-1}]$ with the grading given by a $\mathbf{G}_m$-action on $X \times \mathbf{G}_{m,\mathbf{q}}$.*

For instance, if every function on X has degree zero, then $\langle \mathcal{O}(X), \mathbf{q}\rangle = \mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})$.

3.1.3. We are now in place to define q-vector field. To begin, we need to *choose* a quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \to G$ of $X \times \mathbf{G}_{m,\mathbf{q}}$ over G. Then,

Definition 3.1.4. A *q-vector field* on X is a vector field

$$\xi \;:\; \mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) \;\to\; (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}})$$

on the noncommutative space $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$, i.e. a map as above, over $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$.

We have immediately

Lemma 3.1.5. *The restriction of a q-vector field to X is a vector field.*

Proof. We take the pullback squares

$$\begin{tikzcd}
\mathbf{D}_2 \times X \arrow[r, "\xi_1"] \arrow[d] & X \arrow[r] \arrow[d] & 1 \arrow[d] \\
\mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) \arrow[r, "\xi"] & (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) \arrow[r] & \mathbf{G}_m
\end{tikzcd}$$

which gives an ordinary vector field on X. □

Thus loosely speaking, a q-vector field is a quantised vector field on X.

3.1.6. *Example:* $X = \mathbf{A}^1$. The operator $\partial(x^n) = n_q x^{n-1}$, where n_q is the nth q-integer,

$$n_q \;=\; 1 + q + \cdots + q^{n-1}, \qquad\qquad (-n)_q \;=\; q^{-1} + q^{-2} + \cdots + q^{-n}$$

which satisfies $(n+m)_q = n_q + q^n m_q$. In particular, $\partial(x^{n+m}) = n_q x^n \cdot x^m + q^n x^n \cdot m_q x^m$, and so this defines a q vector field.

3.1.7. *Remark.* We could also just as well replace $\mathbf{G}_{m,\mathbf{q}}$ by $E_{\mathbf{q},\tau}$ or $G_{\mathbf{q}}$ any one-dimensional algebraic group.

Thus, let X and $G_{\mathbf{q}}$ be viewed as constant schemes over G. Then we *choose* a quantisation $X \tilde{\times} G_{\mathbf{q}} \to G$ over G. In this case, a G*-jet* is a map

$$\xi \;:\; \mathbf{D}_n^G \times (X \tilde{\times} G_{\mathbf{q}}) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

over $X \tilde{\times} G_{\mathbf{q}}$. Over a point $x \in X$ we get

$$\xi_x \;:\; \mathbf{D}_n^G \times (G_{\mathbf{q}} \times G) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

and so we get a map

$$\xi_x \;:\; \mathcal{O}(X \tilde{\times} G_{\mathbf{q}}) \;\to\; \mathcal{O}(\mathbf{D}_n^G \times G_{\mathbf{q}}) \otimes \mathcal{O}(G).$$

For instance, our ordinary notion of q-vector field corresponds to $G = \mathbf{G}_m$. We can define an $\hbar$-adic version by taking $G = \mathbf{G}_a$.

When dealing with elliptic curves, we may also require a compatible family of $\mathbf{G}_m$- and E_τ-jets which glue over $\overline{\mathcal{M}}_{1,1}$.

3.2. h and Ω-backgrounds.

3.2.1. As a motivating computation, let us formally add in the logarithm $\mathbf{h}$, $\hbar$ and $\log f$ of $\mathbf{q}$, q and f respectively. Assume that $[\hbar, \log f]$ is central for every f. Then the commutation relations

$$\mathbf{q} f \;=\; q^{|f|} f \mathbf{q}$$

become by the Baker-Campbell-Hausdorff formula

$$\mathbf{q} f \mathbf{q}^{-1} f^{-1} \;=\; e^{[\mathbf{h}, \log f]} \;=\; e^{|f|\hbar}.$$

This gives us

$$[\mathbf{h}, \log f] \;=\; |f|\hbar.$$

Taking this relation as a definition,[1] one then verifies that an appropriate completion of the $\mathbf{C}[\hbar]$-algebra $\langle \log \mathcal{O}(X), \hbar \rangle$ indeed contains a copy of $\langle \mathcal{O}(X), \mathbf{q}^{\pm} \rangle$, where we have taken logarithms of every homogeneous function.

[1]In particular, the assumption that the commutator is central is now a definition.

3.2.2. If we take $X = \mathbf{C}^\times$, then this algebra is $\langle x, \mathbf{h}\rangle$ with the relation $[\mathbf{h}, x] = \hbar$. In other words, it is the Weyl algebra of differential operators, or the Moyal product on $\mathbf{C}^2$. Here, we have taken x the logarithm of a coordinate on X.

3.3. q-**cotangent bundles.** The cotangent bundle over X is given by taking the relative spectrum of the sheaf of vector fields.

3.3.1. Having chosen a quantisation $\tilde{X} = X \tilde{\times} G_{\mathbf{q}}$, the *quantum cotangent bundle* is

$$\tilde{\mathbf{T}}^*_{\tilde{X}} = \mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}}\times G}.$$

(define this, i.e. show that we get a quantisation)

Lemma 3.3.2. *This is a quantisation of the cotangent bundle of X times $G_{\mathbf{q}} \times G$, i.e.*

$$\mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}}\times G} = \mathbf{T}^*_X \tilde{\times} G_{\mathbf{q}}.$$

For instance, if $X = \mathbf{A}^1$ and $G = \mathbf{G}_m$, then we can take

$$\tilde{X} = \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}^{\pm}, q^{\pm}\rangle$$

where q is central, and

$$\mathbf{T}^q_{\tilde{X}} = \operatorname{Spec} \mathbf{C}\langle x, p, \mathbf{q}^{\pm}, q^{\pm}\rangle$$

is a twisted product of $T^*\mathbf{A}^1$ and $\mathbf{G}_{m,\mathbf{q}} \times \mathbf{G}_m$, where $p = \partial_x$, and so we have that $\mathbf{q}p = q^{-1}p\mathbf{q}$. Notice that we get a closed subscheme

$$\mathbf{A}^2_q = \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}p\rangle$$

which is the quantum affine plane, since writing $y = \mathbf{q}p$, we get the defining relations $xy = qyx$.

3.4. q-**differential operators.** The q-differential operators $\mathcal{D}_q$ will be a filtered quantisation of

$$\operatorname{Spec} \operatorname{Sym}_{\tilde{X}} \tilde{\mathbf{T}}^*_{\tilde{X}}.$$

Notice that the role of q and the q-quantisation is orthogonal to the role of the filtration and the filtered quantisation. We define it as usual: it is the sheaf of differential operators on $\tilde{X}$, i.e. it is the sheaf of subalgebras

$$\tilde{\mathcal{D}}_{\tilde{X}} \subseteq \mathcal{E}\mathrm{nd}_{\tilde{X}}(\mathcal{O}_{\tilde{X}})$$

generated by the q-vector fields and $\mathcal{O}_{\tilde{X}}$.

Notice that by the definition,

Lemma 3.4.1. *$\tilde{\mathcal{T}}_{\tilde{X}}$ forms a sheaf of Lie algebras over $\tilde{X}$.*

This allows us to give a Grothendieck definition of the sheaf of quantum differential operators:

Lemma 3.4.2. *$\tilde{\mathcal{D}}_{\tilde{X}} = \cup_{n\geqslant 0}\tilde{\mathcal{D}}_{\tilde{X},n}$, where the zeroeth term is $\tilde{\mathcal{O}}_{\tilde{X}}$, and above that*

$$\tilde{\mathcal{D}}_{\tilde{X},n} = \text{(recursive definition)}.$$

To summarise, we have the following

$$\begin{array}{cc} \mathrm{gr}\,\mathcal{D}_X & \mathrm{gr}\,\tilde{\mathcal{D}}_{\tilde{X}} \\ & \\ \mathcal{D}_X & \tilde{\mathcal{D}}_{\tilde{X}} \end{array}$$

and the sheaves on the left are given by pulling back the sheaves on the right along $1 \to G$.

3.5. **Relation to automorphisms of X.** Recall that one may define a D-module on X to be a quasicoherent sheaf which is equivariant for the action of the formal group $\exp(\mathcal{T}_X)$; this is the parallel transport map. Likewise, if Φ is an automorphism of X, one possible definition of quantum D-module is a Φ-equivariant quasicoherent sheaf.

How does this relate to the above definition?

To begin with, what has this to do with the quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$? Let us consider the case when the quantisation and the automorphism both come from the same source: a single $\mathbf{G}_m$ action:

$$\begin{tikzcd} & \mathbf{G}_m \text{ action on } X \arrow[dl, squiggly] \arrow[dr, squiggly] & \\ \text{automorphism } \Phi_g \text{ for any } g \in \mathbf{G}_m & & \text{quantisation } X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \end{tikzcd}$$

A quasicoherent sheaf on $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$ is the same as a quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(X)$ with a compatible action of $\mathbf{C}[\mathbf{q}^{\pm}]$, i.e. we have

$$\mathbf{q}_x : \mathcal{M}_x \xrightarrow{\sim} \mathcal{M}_x$$

for every point $x \in X$, and we have

$$\mathbf{q}_x f(x) = q^{|f|} f(x)\mathbf{q}_x$$

as automorphisms of $\mathcal{M}_x$. In particular, this has nothing to do with comparing $\mathcal{M}_x$ and $\mathcal{M}_{\Phi_g \cdot x}$, so *it is unlikely the definitions are related*.

The automorphism definition of quantum D-module is related to

$$\mathbf{Z} \xrightarrow{\Phi} \mathrm{Aut}(X) \leftarrow \exp(\mathcal{T}_X)$$

whereas the q-deformed D-module changes the underlying space,

$$\exp(\tilde{\mathcal{T}}_{\tilde{X}}) \to \exp(\mathcal{T}_X).$$

One expects that it might be possible to quantise both ways simultaneously.

3.6. **Relation to difference equations.** If instead we are to take $\tilde{X}_\hbar = X \tilde{\times} \mathbf{G}_a$, then we get (show how to get difference equations, might need to take $\mathbf{C}[[\hbar]]$)

3.7. **Relation to Beilinsorn-Bernstein.** Let $\lambda : \mathbf{G}_m \to G$ be a character with $\lambda B \lambda^{-1} = B$. Then we get an induced $\mathbf{G}_m$ action on the flag variety G/B, and can form the quantisation.

Conjecture 3.7.1. *We have a surjection* $\tilde{\mathcal{D}}_{\tilde{G/B}} \twoheadrightarrow U_q(\mathfrak{g})$*.*

3.8. **Relation to quantum groups.** We are going to give a *different* relation to quantum groups, where

$$X = \operatorname{Spec} U_q(\mathfrak{g}), \qquad G = T.$$

Note that here we may be using a group of dimension greater than one. If $\mathbf{q}_\lambda$ corresponds to $\lambda \in \mathcal{O}(T) \subseteq \mathfrak{t}^*$, then we set

$$x\mathbf{q}_\lambda = q^{\lambda(x)}\mathbf{q}_\lambda x$$

for all $x \in \mathfrak{g} \subseteq U_q(\mathfrak{g})$.

Conjecture 3.8.1. *We have*

$$\tilde{\mathcal{D}}_{\tilde{X}} = U_q(\mathfrak{g} \oplus_{\mathfrak{t}} \mathfrak{g}^*)$$

is the Takiff algebra.

4. Functoriality

4.1. In the above we defined the category of D-modules over $\operatorname{Spec} A$ for any (check Majid?) noncommutative algebras A as an element of $\operatorname{QCoh}(\operatorname{Spec} A)$ which is equivariant for the action of the formal jet group $\mathcal{J}_\infty \operatorname{Spec} A$. (what about in the non-affine case)

4.1.1. Let $f : X \to Y$ be a map of noncommutative spaces. We then have functor

$$f^\dagger \;:\; \mathcal{D}\text{-Mod}(Y) \;\to\; \mathcal{D}\text{-Mod}(X)$$

induced by pullback of quasicoherent sheaves (i.e. restriction of modules) and functoriality of $\mathcal{J}_\infty$.

4.1.2. Now assume that f is (noncommutative schematic and quasi-compact??). Then we have a pushforward functor

$$f_{dR,*} \;:\; \mathcal{D}\text{-Mod}(X) \;\to\; \mathcal{D}\text{-Mod}(Y)$$

defined by (pushforward on QCoh?). To be explicit, it acts on modules as

$$M \;\mapsto\; f_*(M \otimes \Omega_{X/Y})$$

where $\Omega_{X/Y}$ is the noncommutative de Rham complex of Majid and Simao [MS].

5. Quantum vertex algebras

If ordinary vertex algebras are meant to axiomatise two-dimensional chiral conformal field theory on a complex curve Σ, then $\mathbf{q}$-vertex algebras axiomatise the theory on *noncommutative* curves $\tilde{\Sigma}_{\mathbf{q}}$.

In physics terms, these should be two-dimensional CFTs on $\Sigma \times S^1$, which are compatified along a *nontrivial* S^1 action. (check)

One common way to get noncommutative curves is to quantise curves inside cotangent bundles

$$\Sigma \subseteq T^*C \qquad \rightsquigarrow \qquad \tilde{\Sigma} \subseteq \operatorname{Spec} \mathcal{D}_C$$

where if Σ is the vanishing locus of the symbol σP of differential operator P, then the quantisation has ring of functions $\mathcal{D}_C/P$. If we want this to be an algebra over $k[[\hbar]]$, we may in the above take the $\hbar$-adically completed sheaf of D-modules $\mathcal{D}_{C,\hbar}$. There is a relation to opers, see for instance section 2 of [CPT].

5.1. **Motivation.**

5.1.1. Recall the standard construction of the category of vertex algebras:

(1) We build the Ran space $\operatorname{Ran} \mathbf{A}^1$ parametrising finite subsets of $\mathbf{A}^1$.

(2) We define the category of D-modules over $\operatorname{Ran} \mathbf{A}^1$.

(3) We define a *decomposition structure* on the Ran space, using the open locus $(\mathbf{A}^n \times \mathbf{A}^m)_\circ$ where the set of first n and last m points are required to be disjoint. This allows us to define factorisation algebras over $\operatorname{Ran} \mathbf{A}^1$.

(4) We define an action of $\mathbf{G}_a$ on $\mathbf{A}^1$, and extend it to act on the Ran space.

(5) We define *vertex algebras* to be the (weakly) $\mathbf{G}_a$-equivariant commutative factorisation algebras over $\operatorname{Ran} \mathbf{A}^1$.

(6) Finally, we prove that this is equivalent to the data of a pointed vector space with endomorphism $(V, |0\rangle, T)$ along with a map $Y : V \otimes V \to V((z))$ satisfying conditions.

In defining q-deformed vertex algebras, we will follow the same steps, but with the following modifications:

(1) We use $\mathbf{A}^1_{\mathbf{q}}$ instead of $\mathbf{A}^1$, and define the Ran space as usual (as a noncommutative space).

(2) We define the category of $\mathcal{D}_{\mathbf{q}}$-modules over $\operatorname{Ran} \mathbf{A}^1_{\mathbf{q}}$.

(3) We define a *decomposition structure* on the Ran space as before. The space of $\mathbf{q}$-diagonals naturally appears here, even though we do not insert this into the definition by hand.

(4) We define $\mathbf{Z}$ *many* actions of $\mathbf{G}_{a,\mathbf{q}}$ on $\mathbf{A}^1_{\mathbf{q}}$, and extend it to act on the Ran space. The integer is called the *weight* of the action and the function $x_1 - \mathbf{q}^w x_2$ giving the $\mathbf{q}^w$-diagonal is invariant under the weight w action.

We say that a $\mathcal{D}_{\mathbf{q}}$-module is $\mathbf{G}_{a,\mathbf{q}}$-equivariant if it is a direct sum of $\mathcal{D}_{\mathbf{q}}$-modules which are equivariant for the weight w action.

(5) We define $\mathbf{q}$-*vertex algebras* to be the (weakly) $\mathbf{G}_{a,\mathbf{q}}$-equivariant commutative factorisation algebras over $\operatorname{Ran}\mathbf{A}^1_{\mathbf{q}}$.

(6) Finally, we prove that this is equivalent to concrete data in Theorem 5.3.12 below.

5.2. Appearance of q-diagonals.

5.2.1. We now consider what the diagonal inside $X\tilde{\times}\mathbf{G}_{m\mathbf{q}}$ looks like.

5.2.2. To begin, for a map $A \to B$ of algebras, note that the relative diagonal is given by the map

$$B\otimes_A B \twoheadrightarrow B, \qquad b\otimes b' \mapsto bb'.$$

5.2.3. For instance, let $X = \mathbf{A}^1 = \operatorname{Spec} k[x]$. Then the quantum diagonal is given by the ideal

$$\tilde{\Delta} : \tilde{X} \to \tilde{X}\times\tilde{X}$$

given by the ideal

$$I_\Delta \subseteq \langle k[x_1], \mathbf{q}_1^{\pm}, k[x_2], \mathbf{q}_2^{\pm}\rangle \twoheadrightarrow \langle k[x], \mathbf{q}^{\pm}\rangle, \qquad x_1, x_2 \mapsto x, \quad \mathbf{q}_i \mapsto \mathbf{q}$$

and where in the domain x_1, x_2 commute, and

$$\mathbf{q}_i x_j = q x_j \mathbf{q}_i$$

for *every* i, j. This is necessary so that the above defines an algebra map. For instance, the ideal of the diagonal contains the element

$$x_1 - x_2(\mathbf{q}_2\mathbf{q}_1^{-1})^n$$

for every integer $n \in \mathbf{Z}$.

5.3. q-additive group.

5.3.1. We consider the group structure,

$$m : \tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}} \to \tilde{\mathbf{A}}^1_{\mathbf{q}}$$

which is the unique map of noncommutative schemes so that

$$m^*\mathbf{q} = \mathbf{q}\otimes\mathbf{q}, \qquad m^*x = x\otimes 1 + \mathbf{q}\otimes x$$

for an integer $w \in \mathbf{Z}$ called the *weight*. This is well-defined, since

$$\begin{aligned} m^*(\mathbf{q}x) &= \mathbf{q}x \otimes \mathbf{q} + \mathbf{q}^2 \otimes \mathbf{q}x \\ &= q(x\mathbf{q} \otimes \mathbf{q} + \mathbf{q}^2 \otimes x\mathbf{q}) \\ &= q \cdot m^*(x\mathbf{q}). \end{aligned}$$

Denote this algebraic group $\mathbf{G}_{a\mathbf{q}}$.

5.3.2. Likewise, we have an action for every integer w

$$m_w^* \mathbf{q} = \mathbf{q} \otimes \mathbf{q}, \qquad m_w^* x = x \otimes 1 + \mathbf{q}^w \otimes x$$

giving a group law as above.

5.3.3. If we write points of $\mathbf{G}_{a\mathbf{q}}$ as z, then the above group law we will write as $(z_1, z_2) \mapsto z_1 + \mathbf{q}_1 z_2$.

5.3.4. Given a representation of $\mathbf{G}_{a\mathbf{q}}$, i.e.

$$V \to V \otimes \langle k[x], \mathbf{q}^{\pm} \rangle,$$

then the invariants are the elements v sent to

$$v \mapsto v \otimes 1.$$

5.3.5. What are the $\mathbf{G}_{a\mathbf{q}}^{\ w}$-invariants of $\mathcal{O}(\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}})$? Note that the coaction is given by

$$m^* \mathbf{q}_i = \mathbf{q}_i \otimes \mathbf{q}, \qquad m^* x_i = x_i \otimes 1 + \mathbf{q}_i^w \otimes x,$$

where the right hand side tensor multiplicand lies in $\mathcal{O}(\mathbf{G}_{a\mathbf{q}})$, and so

$$\begin{aligned} m^*(x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n) &= (x_1 \otimes 1 + \mathbf{q}_1^w \otimes x) - (x_2 \otimes 1 + \mathbf{q}_2^w \otimes x)\,((\mathbf{q}_2/\mathbf{q}_1)^n \otimes 1) \\ &= (x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n) \otimes 1 + (\mathbf{q}_1^w - \mathbf{q}_2^w(\mathbf{q}_2/\mathbf{q}_1)^n) \otimes x. \end{aligned}$$

In particular, $(x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n)$ is invariant with respect to the $\mathbf{G}_{a\mathbf{q}}$-action of weight $w = -n$. Thus we get

Proposition 5.3.6. *For any integer $w \in \mathbf{Z}$, the functions on the complement of the main quantum diagonal which are invariant with respect to the weight w action are*

$$\mathcal{O}((\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}})_{\mathbf{q},\circ})^{\mathbf{G}_{a\mathbf{q}}^{\ w}} = \langle (x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^w) \rangle_{k[\mathbf{q}_1^{\pm}, \mathbf{q}_2^{\pm}]},$$

which is spanned as a vector space by $\mathbf{q}_1^a \mathbf{q}_2^b (x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^w)^c \mathbf{q}_1^d \mathbf{q}_2^e$.

5.3.7. We now ask the question: what is the category of D-modules on $\tilde{\mathbf{A}}^1_{\mathbf{q}}$ which are weakly equivariant with respect to the weight w action of $\mathbf{G}_{a\mathbf{q}}$? Recall that without the $\mathbf{q}$ the answer was it is the category of a vector space (the invariant sections) with endomorphism (the action of ∂_z).

(write)

5.3.8. Notice that the Ran space of $\tilde{\mathbf{A}}^1_{\mathbf{q}}$ is still a symmetric factorisation space,

$$\begin{tikzcd}
 & (\operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}})_\circ \arrow[dl, bend right] \arrow[dr, bend left] \arrow[d, "\sigma", "\wr"'] & \\
 & (\operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}})_\circ \arrow[dl] \arrow[dr] & \\
\operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}} & & \operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}}
\end{tikzcd}$$

because for instance in $\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}}$ functors on the left and right factors commute, so the swap map is indeed a map of noncommutative schemes; considering higher powers of the quantum affine plane induces the symmetric factorisation structure σ considered above.

5.3.9. In particular, this means we should consider the categories

$$\bigoplus_{w\in\mathbf{Z}} \mathcal{D}\text{-Mod}(\operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}})^{\mathbf{G}_{a\mathbf{q},w}}$$

of D-modules which are weakly equivariant respect to some weight w. (how to combine these together more naturally?) Notice that

Proposition 5.3.10. *For each weight w, the w summand upgrades to a symmetric factorisation category $\mathcal{D}\text{-Mod}^{\mathbf{G}_{a\mathbf{q},w}}$ over $\operatorname{Ran}\tilde{\mathbf{A}}^1_{\mathbf{q}}$.*

5.3.11. We can finally define a $\mathbf{q}$-*vertex algebra* to be a strong factorisation algebra in this category.

Theorem 5.3.12. *A $\mathbf{q}$-vertex algebra is equivalent to a direct sum of vector spaces (or $k[\mathbf{q}^{\pm}]$-comodules?)*

$$V \;=\; \bigoplus_{w\in\mathbf{Z}} V_w$$

along with a map of $\mathcal{D}(\tilde{\mathbf{A}}^1_{\mathbf{q}})$-modules (how should this interact with the weight w?)

$$Y \;:\; V\otimes V \;\to\; V((\{z_1 - \mathbf{q}^n z_2\}))$$

satisfying (a commutativity and associativity condition), and equipped with a vector $|0\rangle \in V_0$ and (whatever data is equivalent to a $\mathcal{D}(\tilde{\mathbf{A}}^1_{\mathbf{q}})$-module)

6. How to construct the **q**-affine vertex algebra

In this section, we recall the construction of the affine vertex algebra, then show how to deform it to the **q**-affine vertex algebra.

6.1. **General picture.**

6.1.1. Let X be an algebraic curve and Y a prestack with maps

$$\operatorname{Ran} X \xrightarrow{s} Y \xrightarrow{p} \operatorname{Ran} X$$

of factorisation spaces. Moreover, assume that the latter map admits a *connection*, i.e. comes from a pullback

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \operatorname{Ran} X \arrow[d] \\ \overline{Y} \arrow[r] & \operatorname{Ran} X_{dR} \end{tikzcd} \tag{1}$$

Moreover,

Lemma 6.1.2. *The map $Y \to Y_{dR}$ factors as*

$$\begin{tikzcd} & Y \arrow[dl] \arrow[d] \\ Y_{dR} & \overline{Y} \arrow[l, dashed] \end{tikzcd}$$

Proof. Apply the functor $(-)_{dR}$ (which is a right adjoint, hence preserves limits) to diagram (1). This gives pullback

$$\begin{tikzcd} Y_{dR} \arrow[r] \arrow[d] & \operatorname{Ran} X_{dR} \arrow[d] \\ \overline{Y}_{dR} \arrow[r] & \operatorname{Ran} X_{dR} \end{tikzcd}$$

hence $Y_{dR} = \overline{Y}_{dR}$, and the Lemma follows by functoriality of $(-)_{dR}$. $\square$

Thus we have commuting diagram

$$\begin{tikzcd} & \operatorname{QCoh}(Y) \arrow[r, "p_*^{\operatorname{QCoh}}"] & \operatorname{QCoh}(\operatorname{Ran} X) \\ \mathcal{D}\text{-Mod}(Y) \arrow[ur] \arrow[r, dashed] & \operatorname{QCoh}(\overline{Y}) \arrow[u] \arrow[r] & \mathcal{D}\text{-Mod}(\operatorname{Ran} X) \arrow[u] \end{tikzcd} \tag{2}$$

We stress that the pushforward $\mathcal{D}\text{-Mod}(Y) \to \mathcal{D}\text{-Mod}(\operatorname{Ran} X)$ is *not* the ordinary $\mathcal{D}$-module pushforward p_*. Rather, the above says that any quasicoherent sheaf pushforward of a $\mathcal{D}$-module, which usually has no reason to be a $\mathcal{D}$-module, in this case always carries a natural $\mathcal{D}$-module structure.

If in addition the maps in (1) are maps of factorisation spaces, it not not hard to show that

Lemma 6.1.3. *The functors in (2) are functors of symmetric monoidal categories.*

In particular, this implies

Corollary 6.1.4. *The quasicoherent sheaf pushforward p_*^{QCoh} extends to a functor on factorisation algebras*

$$p_*^{\mathrm{QCoh}} : \mathrm{CommAlg}(\mathcal{D}\text{-}\mathbf{Mod}(Y), \otimes_Y^{ch}) \to \mathrm{CommAlg}(\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Ran}\, X), \otimes^{ch}).$$

In particular, if $\mathcal{A}$ is a factorisation algebra on $\mathrm{Ran}\, X$, then so $s_*\mathcal{A}$ is a factorisation algebra on Y, and hence we get a new factorisation algebra $p_*^{\mathrm{QCoh}}(s_*\mathcal{A})$ on $\mathrm{Ran}\, X$.

6.2. **Affine Grassmannian.** We apply this to the Beilinson-Drinfeld Grassmannian

$$\mathrm{Ran}\, X \overset{\mathrm{triv}}{\to} \mathrm{Gr}_{G,X} \to \mathrm{Ran}\, X$$

and the constant factorisation algebra $\mathcal{A} = \omega_{\mathrm{Ran}\, X}$.

Lemma 6.2.1. *The map $\mathrm{Gr}_{G,X} \to \mathrm{Ran}\, X$ admits a connection.*

Proof. (find proof in literature) □

Specfically, consider

$$\begin{tikzcd}
 & \mathrm{QCoh}(\mathrm{Gr}_{G,X}) \arrow[r, "p_*^{\mathrm{QCoh}}"] & \mathrm{QCoh}(\mathrm{Ran}\, X) \\
\mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Gr}_{G,X}) \arrow[ur] \arrow[r, dashed] & \mathrm{QCoh}(\overline{\mathrm{Gr}_{G,X}}) \arrow[u] \arrow[r] & \mathcal{D}\text{-}\mathbf{Mod}(\mathrm{Ran}\, X) \arrow[u]
\end{tikzcd} \tag{3}$$

which allows us to define the affine WZW factorisation algebra $\mathcal{A}_{\mathfrak{g}}$ as

$$\begin{tikzcd}
 & i_*\omega \arrow[r, mapsto, "p_*^{\mathrm{QCoh}}"] & \Gamma(i_*\omega) & \\
\omega \arrow[ur, mapsto] \arrow[r, dashed, mapsto] & i_*\omega \arrow[u, mapsto] \arrow[r, mapsto] & \Gamma(i_*\omega) \arrow[u, mapsto] & = \mathcal{A}_{\mathfrak{g}}
\end{tikzcd} \tag{4}$$

Lemma 6.2.2. *As a vector space, the underlying vertex algebra is*

$$V = \mathrm{Sym}(t^{-1}\mathfrak{g}[t^{-1}]).$$

Proof. This is the vector space corresponding to the D-module pushforward along the affine Grassmannian

$$i_0 : 0 \to \mathrm{Gr}_G .$$

Specifically, the normal bundle to i_0 is the vector space $t^{-1}\mathfrak{g}[t^{-1}]$, and so the pushforward is the symmetric algebra on this. □

6.3. **q-analogue.**

6.3.1. The above construction generalises.

7. Variant: Quantum vertex algebras via q-automorphisms

There are two (inequivalent?) notions of q-D-module in the literature: one in terms of q-derivations, which we covered in section 3, and another in terms of automorphism-invariant quasicoherent sheaves. In this section we relate the two notions.

7.1. G-**D-modules.**

7.1.1. Note that if G is a discrete group acting on X, then

$$(X/G)_{dR} \;=\; X_{dR}/G,$$

in other words, the category of D-modules on X/G is equivalent to weakly G-equivariant D-modules on X.

7.1.2. Let Φ be the automorphism of $\mathbf{A}^1$ acting by $t \mapsto qt$. Then we have

$$\Phi f(x)\cdot g(x) \;=\; f(qx)g(qx), \qquad\qquad f(x)\Phi\cdot g(x) \;=\; f(x)g(qx).$$

In particular, if f is homogeneous of degree $|f|$ then we have

$$\Phi f(x) \;=\; q^{|f|}f(x)\Phi.$$

Note that acting on functions, $\Phi = e^{\log(q)x\partial_x} = q^{x\partial_x}$.

7.1.3. Thus, Φ takes the role of $\mathbf{q}$ in section 3. To make this analogy stronger, the analogue of $\mathbf{A}^1_{\mathbf{q}}$ is the product

$$\mathbf{A}^1_\Phi \;=\; \mathbf{A}^1\tilde{\times}\mathbf{Z} \;=\; \mathbf{A}^1\tilde{\times}\{\Phi^n\}_{n\in\mathbf{Z}},$$

on which Φ acts diagonally. More generally, for any space X we can form

$$X_{\mathrm{Aut}\,X} \;=\; X\tilde{\times}\,\mathrm{Aut}(X),$$

which when X is affine is defined as the subalgebra of $\mathrm{End}\,\mathcal{O}(X)$ generated by $\mathcal{O}(X)$ and $\mathrm{Aut}(X)$. If one considers only infinitesimal automorphisms, this recovers $\mathcal{D}(X)$.

7.1.4. Note that the usual definition of q-derivation may be written as

$$\partial(fg) \;=\; \partial(f)g \;+\; \Phi(f)\partial(g).$$

7.1.5. One can define (check) the action map $X\tilde{\times}\,\mathrm{Aut}(X) \to X$, and so we can define

$$\begin{array}{ccc} X\tilde{\times}\,\mathrm{Aut}(X) & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X\tilde{/}\,\mathrm{Aut}(X) \end{array}$$

We expect that $X\tilde{/}\,\mathrm{Aut}(X) \;=\; X/\,\mathrm{Aut}(X)$ (unsure). In particular, we expect that D-modules on X_Φ act as descent data for gluing D-modules on X to D-modules on X/Φ.

7.1.6. One definition of q-D-module is simply a quasicoherent sheaf on X/Φ.

Theorem 7.1.7. *There is (a functor?) of categories over $\mathbf{C}[q^{\pm}]$:*

$$\mathrm{QCoh}(X/\Phi) \xrightarrow{??} \mathcal{D}_q\text{-Mod}(X) \;=\; \mathrm{QCoh}(X_{\mathbf{q}}/\exp \mathcal{T}_{X_{\mathbf{q}}}).$$

(check the $q \to 1$ limit to make this an equivalence) If ξ is the vector field inducing Φ, then

$$\mathrm{QCoh}(X/\Phi) \xrightarrow{??} \mathrm{QCoh}(X_{\mathbf{q}}/\exp(\mathcal{O}_{X_{\mathbf{q}}} \cdot \xi))$$

is an equivalence.

7.2. G-factorisation algebras.

7.2.1. Let G act on a space X. We will now define a version of factorisation algebra on subsets $\{x_1, \ldots, x_n\} \subseteq X$, where two subsets are equivalent if they differ by a G-shift. This necessitates the use of G-diagonals.

The factorisation space to consider is then $(\mathrm{Ran}\, X)/G$, with factorisation structure

$$\begin{tikzcd}
& (\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{G,\circ}/G \arrow[dl] \arrow[dr] & \\
(\mathrm{Ran}\, X)/G \times (\mathrm{Ran}\, X)/G & & (\mathrm{Ran}\, X)/G
\end{tikzcd}$$

where $(\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{G,\circ}$ is the open subset of (S, S') with $gS \cap S' = \varnothing$ for all $g \in G$. The left map is the composition

$$(\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{G,\circ}/G \;\to\; (\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{G,\circ}/G \times G \;\to\; (\mathrm{Ran}\, X \times \mathrm{Ran}\, X)/G \times G.$$

7.2.2. *Remark.* Note that for the above to work the open subset $(\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{G,\circ}$ must be a $G \times G$-invariant, which is why we made this definition.

7.2.3. *Remark.* The above is a colimit of

$$\begin{tikzcd}
& (X^n \times X^m)_{G,\circ}/G \arrow[dl] \arrow[dr] & \\
X^n/G \times X^m/G & & X^{n+m}/G
\end{tikzcd} \tag{5}$$

7.3. G-vertex algebras.

7.3.1. We now give an explicit model for the above. Let $\mathcal{M}$ be a factorisation algebra on $(\mathrm{Ran}\, X)/G$, in the category:

(1) $\mathrm{QCoh}(\mathrm{Ran}\, X/G)$, i.e. $\mathcal{D}_q$-Mod(X) when $G = \mathbf{Z} \cdot q$, or otherwise

(2) $\mathcal{D}$-Mod$(\mathrm{Ran}\, X/G)$.

We consider the restriction of the factorisation map to (5) when $n, m = 1$, in which case we have open and closed complements

$$\Delta_G X/G \xrightarrow{i_G} (X^n \times X^m)/G \xleftarrow{j_G} (X^n \times X^m)_{G,\circ}/G$$

where $\Delta_G X \subseteq X \times X$ consists of points of the form (x, gx) for $g \in G$, and G acts diagonally.

We now consider the cofibre sequence

$$i_G^* \mathcal{M} \to i_G^* j_* j^* \mathcal{M} \to \mathrm{cofib}\,.$$

7.3.2. In the D-module case $\mathrm{cofib} = i_G^! \mathcal{M}[1]$. Assume that $i_G^* \mathcal{M} = V \otimes \mathcal{O}$ and z, w are local coordinates on X, then taking global sections of the cofibre sequence gives

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G}[\{(z - gw)^{\pm 1}\}_{g \in G}] \to V \otimes \mathcal{O}_{X/G}[\{\delta_{z-gw}\}_{g \in G}].$$

Crucially, because we have only quotiented by a single, diagonal, G-action throughout, in the above we have *not* taken G-invariants with respect to the antidiagonal action, which would have killed the $z - gw$ terms.

Lemma 7.3.3. *The data of the above is equivalent to a map*

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G} \to V \otimes \mathcal{O}_{X/G} \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z - gw)).$$

Assuming that X is itself an algebraic group, we take X-invariant sections of the above to get a map

$$Y_G : V^{\otimes 2} \to V((z - g_1 w, z - g_2 w, \cdots)) = V \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z - gw)).$$

7.3.4. In the quasicoherent sheaf case, since the pullback/forgetful functor $\mathcal{D}\text{-Mod}(Z) \to \mathrm{QCoh}(Z)$ is exact, we have that cofib is the same as above, and given the above assumptions, taking global sections of the cofibre sequence gives

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G}[\{(z - gw)^{\pm 1}\}_{g \in G}] \to V \otimes \mathcal{O}_{X/G}[\{\delta_{z-gw}\}_{g \in G}].$$

What is different is that we have only remembered that this is a map inside $\mathrm{QCoh}(X/G)$. However,

Lemma 7.3.5. (check) *When $X = \mathbf{C}$ and $G = \mathbf{Z} \cdot q \simeq \mathbf{Z}$, this is equivalent to a map*

$$Y_G : V^{\otimes 2} \to V((z - g_1 w, z - g_2 w, \cdots)) = V \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z - gw))$$

(with some T action, or rather $q = \exp(\hbar T)$.)

Proof. (The same proof as for the D-module case should work, except that instead of asking that the map commutes with ∂, we ask that it be $\mathbf{Z}$-graded, a $\mathbf{Z}$ acts on $k[x]$ as $x^n \mapsto (qx)^n$.) ☐

7.4. **Exponentiating.**

7.4.1. We expect that we have

$$\begin{array}{ccc} \mathcal{D}\text{-Mod}(\mathbf{C}) & \xrightarrow{\exp_*} & \mathcal{D}\text{-Mod}(\mathbf{C}^*) \\ \downarrow & & \downarrow \\ \mathcal{D}_\hbar\text{-Mod}(\mathbf{C}) & \xrightarrow{\exp_*} & \mathcal{D}_q\text{-Mod}(\mathbf{C}^*) \end{array}$$

and we have likewise the notions of vertex algebras for these, where on the bottom the OPEs have singularities of the form $(z - w - n\hbar)$ and $(z - q^n w)$.

References

[Co] Costello, K., 2016. M-theory in the Omega-background and 5-dimensional non-commutative gauge theory. arXiv preprint arXiv:1610.04144.

[CPT] Coman, I., Pomoni, E. and Teschner, J., 2023. From quantum curves to topological string partition functions. Communications in mathematical physics, 399(3), pp.1501-1548.

[GR] Gaitsgory, D. and Rozenblyum, N., 2017. *A Study in Derived Algebraic Geometry: Volume II: Deformations, Lie Theory and Formal Geometry.* Mathematical surveys and monographs, 221.

[GRZ] Gaiotto, D., Rapčák, M. and Zhou, Y., 2023. Deformed Double Current Algebras, Matrix Extended $\mathcal{W}_\infty$ Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection. arXiv preprint arXiv:2309.16929.

[Li] Linshaw, A.R., 2021. Universal two-parameter $\mathcal{W}_\infty$-algebra and vertex algebras of type $\mathcal{W}(2, 3, \dots, N)$. Compositio Mathematica, 157(1), pp.12-82.

[MS] Majid, S. and Simão, F., 2023. Quantum jet bundles. Letters in Mathematical Physics, 113(6), p.120.