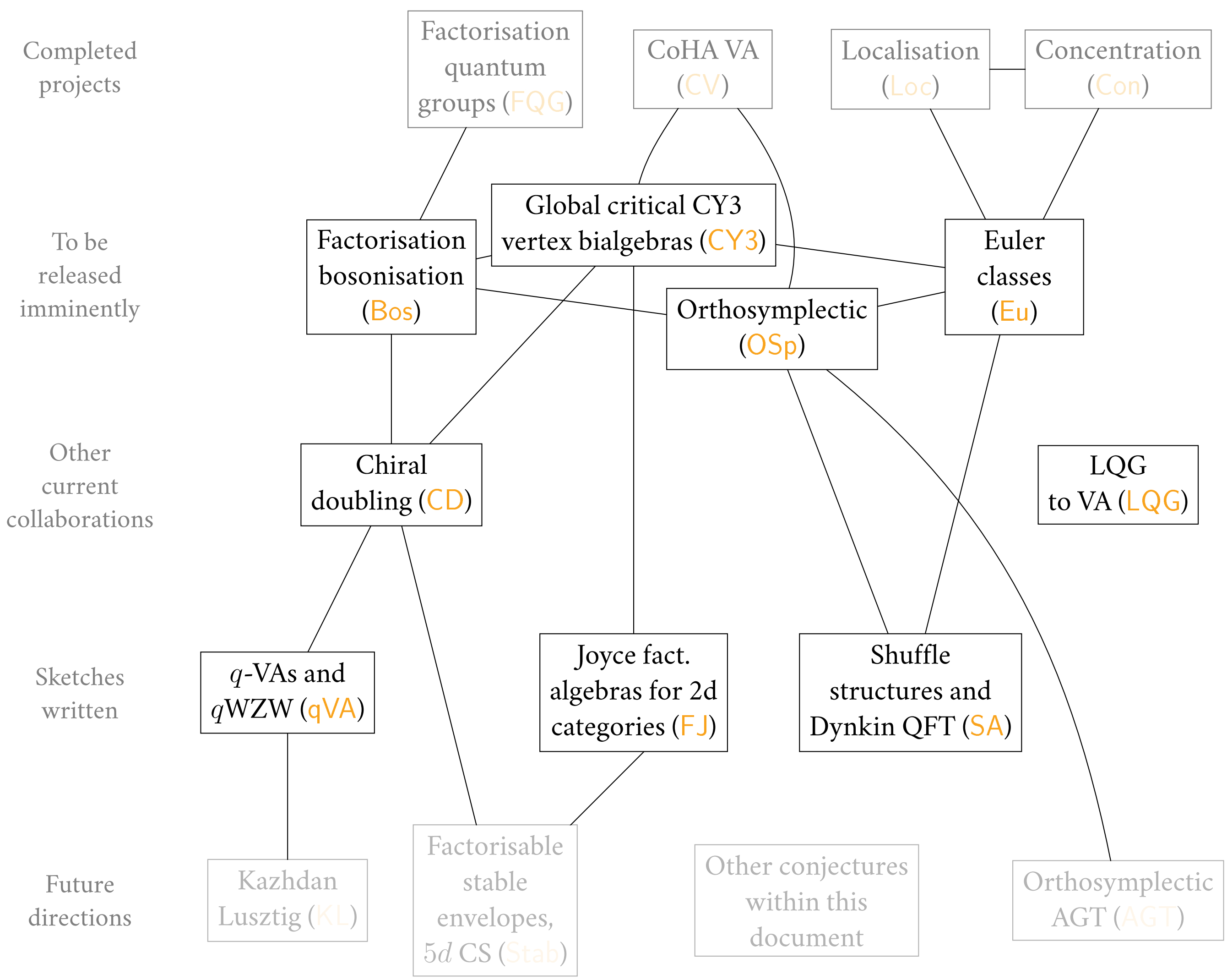
RESEARCH PLANS - ABRIDGED

ALEXEI LATYNTSEV

For a detailed (13 pp.) version, including details/evidence on how we plan to prove the conjectures, see https://alyoshalatyntsev.github.io/plan/plan.pdf.



1. Research statement

I am a mathematician working in geometric representation theory.

1.1. Algebraic structures attached to CY3s (CV, CY3 [1,2], FJ [1], Stab)

Cohomological Hall algebras are associative algebras attached to Calabi-Yau varieties or categories [Da; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZ] (e.g. quantum groups [MO]), hence are a key object in geometric representation theory/enumerative invariants.

Global critical CY3 vertex quantum groups. Let $\mathcal{C}$ be the CY3 category $\mathrm{Rep}(Q, W)$, the representations of a Jacobi algebra of a quiver with potential, or $\mathrm{Coh}_0(K_{T^*C})$, zero dimensional coherent sheaves on a local curve [KK], or more generally a *deformed CY3* completion.

Theorem. [CY3] *The critical cohomology*[3] *of the moduli stack of objects $\mathcal{M}$ has a vertex coproduct*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M},\varphi)\,\hat{\otimes}\,\mathrm{H}^\bullet(\mathcal{M},\varphi)((z^{-1}))$$

compatible with the CoHA: it forms a vertex quantum group *(see* FQG*).*

Theorem. [CY3] *There is a functor Φ from Q-localised bialgebras [Da] to vertex bialgebras. The former is equivalent to the category of* factorisation bialgebras *on the coloured configuration space* $\mathrm{Conf}_{Q_0}\mathbf{A}^1$.

Theorem. [CY3; CV for $W=0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}})^{ext} \overset{[\mathrm{BD}]}{\simeq} Y_\hbar(\mathfrak{b}_Q)$ is identified by Φ with the Davison/Yang-Zhao localised coproduct* [Da; YZ]*, and (when defined) Drinfeld's meromorphic coproduct* [Dr; GLW].

This generalises **Joyce-Liu's vertex coproduct** [Joa; Li], shows that all these well-known coproducts on the Yangian are **equal**, and gives a **geometric** definition using moduli stacks and configuration spaces. Finally, Theorem 1.1 points to a new Ran space definition of vertex-vertex *bi*algebras.

Lift to factorisation algebra and relation to stable envelopes. To move towards *arbitrary* CY3s, we first interpret the above in terms of factorisation algebras; second, we relate to the $\mathcal{W}$-algebras for surfaces of [MMSV], thus understanding the structure for K_S. Let Q be a quiver with torus $T=\prod T_d$.

Conjecture. [FJ] *Given $\mathcal{M}^f = \{(m,\lambda)\;:\;\lambda\in\mathfrak{t},\ m\in\mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d) = \mathrm{Ran}_{Q_0}\mathbf{A}^1$, its relative BM-homology $\mathcal{A} = \pi_*\omega$ is a factorisation algebra over the coloured Ran space. Its restriction to the colour-diagonal recovers the vertex quantum group structure on the nilpotent CoHA* [SV].

Conjecture. [FJ] *For S a smooth algebraic surface, there is a braided factorisation category* $\mathrm{Rep}\mathcal{W}$ *over $\mathrm{Ran}_S K_S$ (c.f.* FQG*). Applying* Bos/CD *allows us to construct $\mathcal{W}(S)^{\geqslant 0}$ and $\mathcal{W}(S)$ from* [MMSV]*'s $\mathcal{W}(S)^{>0}$.*

The definition of $\mathcal{M}^f$ is clearly reminiscent of [MO, §5.1.1]'s stable envelope construction, and suggests a resolution to their question: *"we view this definition as provisional; perhaps a better set of axioms will emerge later"*. Let $\mathbf{w}$ be a multidimension vector of quiver Q and $M(\mathbf{w})$ the quiver variety.

Conjecture. [Stab] *There is a factorisation space $\pi_{\mathbf{w}} : M(\mathbf{w})^f \to \mathrm{Ran}_Q\mathbf{A}^1$ and the factorisation category $\mathcal{E}$ spanned by $\pi_{\mathbf{w},*}\omega$ is acted on by $\mathcal{A}=\pi_*\omega$. Applying chiral Tannakian reconstruction $\mathcal{E}\simeq \mathrm{Rep}\mathbf{D}\mathcal{A}$ gives the double of $\mathcal{A}$ with its (two) coproducts.*

The above is a Drinfeld-Kohno Theorem for [MO]'s Yangians $Y_\hbar(\mathfrak{g}_Q)$ (see qVA for relations to qKZ).

1.2. The structure of factorisation quantum groups (FQG, Bos [4], CD [5])

Historically the definitions of (double) affine quantum groups $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ were (ingeniously) made very explicitly [Dr; MO], and still much of the modern (e.g. CoHA) literature is based on explicit shuffle

[1] Joint with S. Kaubrys.

[2] Joint with S. Jindal.

[3] i.e. $\mathcal{M}=\mathrm{Crit}(W)$ is a critical locus inside a *smooth* moduli stack; we take the vanishing cycle sheaf $\varphi=\varphi_W$.

[4] Joint with S. de Hority.

[5] Joint with W. Niu.

computations, e.g. [MMSV; SV; YZ]. This series of projects *axiomatises* (operadically) these structures, allowing us to import techniques from the theory of ordinary *quantum groups*,[6] to recover the above formulas as a *consequence* of these definitions.

Factorisation quantum groups. In FQG we develop a theory of $\mathbf{E}_n$*-factorisation categories* over *factorisation spaces* X (including ordinary groups G, configuration spaces $\mathrm{Conf}_{Q_0}\mathbf{A}^1$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$). We first give basic structure results for braided factorisation categories $\mathcal{C}$:

Theorem. [FQG] *Let $\mathcal{A}$ is a factorisation algebra in $\mathcal{C}$ over X, a (braided) factorisation structure on $\mathcal{A}$-*FactMod$(\mathcal{C})$ *induces a* factorisation bialgebra *structure on $\mathcal{A}$ (and a* factorisation R-**matrix** $R : \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A}$*).*

Theorem. [FQG] *When $X = \mathrm{Ran}\mathbf{A}^1$ (resp. $\mathrm{Conf}\mathbf{A}^1$), Theorem 1.2 recovers classical notions* [EK; FR] *of **quantum vertex algebras** (resp. localised algebras) and their R-matrices $R(z)$ satisfying the spectral YBE.*

For instance, the structure [GLW] for **Yangians** (e.g. two (meromorphic) coproducts and R-matrices $R^-, R^{0,\epsilon}, R^\epsilon$ relating them) are equivalent to: $Y_\hbar(\mathfrak{g})$-Mod is a lax braided factorisation category. This makes rigorous the physics claim [CWY] that $Y_\hbar(\mathfrak{g})$-Mod from $4d$ Chern-Simons is a topological-holomorphic factorisatsion category over $\mathbf{R}\times\mathbf{C}$. The above may help understand **affine Yangians** (e.g. [GRZ]; qVA for relation to q-WZW) and the **new KL equivalances** [BCDN]. Finally, in principle we may get lots of examples from:

Theorem. [FQG] *A generalisation of Borcherds' twist construction* [Bo] *to arbitrary decomposition algebra.*

Factorisation bosonisation. In the CoHA literature, a lot of algebraic effort needs to be expended each time [Da; RSYZ; YZ], [CY3; OSp] to *add in the Cartan piece* $\mathrm{H}^\bullet(\mathcal{M},\varphi) \rightsquigarrow \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$, e.g. to obtain Yangians of Borels $Y_\hbar(\mathfrak{b}_Q)$. In the finite groups case, a clean construction is Tannakian reconstruction

$$U_q(\mathfrak{n}) \rightsquigarrow U_q(\mathfrak{n}) \qquad \text{via} \qquad U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

c.f. [Ga; Maa; Mab]; in Bos we apply the same to vertex and factorisation bialgebras to obtain:

Theorem. [Bos, in preparation] *There is a "factorisation" **Tannakian reconstruction** functor from (braided) factorisation categories $\mathcal{C}$ to (quasitriangular) factorisation quantum groups $\mathcal{A}$. In the preprojective case of Theorem 1.1,*

$$Y_\hbar(\mathfrak{b}_Q)\text{-Mod} \simeq Y_\hbar(\mathfrak{n}_Q)\text{-Mod}(Y_\hbar(\mathfrak{t}_Q)\text{-Mod})$$

we Tannakian reconstruct $Y_\hbar(\mathfrak{b}_Q) \simeq \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$ and its localised/vertex bialgebra structure.

Applying this to $\mathrm{H}^\bullet(\mathcal{M},\varphi)$-Mod$(\mathrm{H}^\bullet(\mathcal{M})$-Mod$^\cup)$ **automates** the process of extending CoHAs.

Factorisation Drinfeld doubling. An active problem is how the structures in CY3 relate to **wall crossing** and stability conditions, see [Br; Job]. As a first attempt, we use FQG to understand **doubling**, where the CoHA of heart $\mathcal{A}$ and its opposite $\mathcal{A}[1]$ are glued, in a similar way to Bos:

Conjecture. [CD] *There is a "factorisation" **Drinfeld centre** construction $Z_{\mathbf{E}_1}(\mathcal{C})$ of a chiral factorisation category $\mathcal{C}$, which carries compatible chiral and ordinary monoidal structures, Tannakian reconstruction gives*

$$Z_{\mathbf{E}_1}^{Y_\hbar(\mathfrak{t}_Q)\text{-Mod}}(Y_\hbar(\mathfrak{b}_Q)\text{-Mod}) \simeq Y_\hbar(\mathfrak{g}_Q)\text{-Mod},$$

and likewise we recover the Takiff algebra double construction of [AN].

1.3. Orthosymplectic structures (OSp [4], SA [4], AGT [4])

Orthosymplectic CoHAs. We define *orthosymplectic moduli stacks* $\mathcal{M}^{\mathrm{OSp}}$, including **perverse-coherent sheaves** with a sympletic/orthogonal bilinear form, and quivers and potential with involution.

Theorem. [OSp] *For $\mathcal{M}$ as in CY3 the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M},\varphi)$ acts on $\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$, i.e.*

1. *there is a left module action a of the **CoHA** respecting the involution, i.e. it **factorises** over $\mathbf{R}/\pm$, compatible with*
2. *a **symplectic vertex algebra** structure (factorisation coalgebra over symplectic Ran space $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colimt}_{\mathfrak{sp}_{2n}}$, see SA). The defining R-/K-matrices satisfy the **Cherednik reflection equation**.*

[6] Namely, when $X = \mathrm{Ran}\mathbf{R}^2$ in the below, via Lurie [Lu].

Theorem. [OSp] *There is an action of $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M})$ of the CoHA of zero-dimensional coherent sheaves on surface S on $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}_{\sigma\text{-ss}})$ the BM homology of a compactification of the stack of $G \in \{\mathrm{Sp}, \mathrm{O}\}$-bundles.*

This gives a rigorous definition of the objects coming from boundary $4d$ Chern-Simons theory [BS], should relate to work in ι**quantum groups** [LW] and Finkelberg-Hanany-Nakajima's ongoing work on **orthosymplectic Coulomb branches** (see also AGT). In the quiver with potential case (resp. in the conjecture, type A_{2n+1} preprojective), we have:

Theorem. [OSp] *An explicit shuffle formula for the CoHA action and vertex coaction on $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}_{Q,W}, \varphi^{\mathrm{OSp}})$.*

Conjecture. *The preprojective orthosymplectic CoHA*[7] *is isomorphic to the* ***twisted Yangian*** *$Y_\hbar(\mathfrak{gl}_n)^{tw}$ from* [BR].

Dynkin QFT and shuffle structures. The structures (2) in OSp are defined over the *symplectic configuration space*[8] $\mathrm{Conf}_{\mathrm{Sp}}(\mathbf{A}^1) = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$, with singularities on root hyperplanes; alternatively over $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\,\mathfrak{t}_{\mathfrak{sp}_{2n}}$. To give examples:

Theorem. [OSp] *Restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$ gives an* ***invariants*** *functor $\iota : \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \to \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1)$. n.b. we expect the structures in OSp come from applying ι to $\mathcal{M}^f$ (see FJ).*

In SA, we **generalise** the above to systems of **arbitrary Kac-Moody groups** G_i [Ku, §V]; this forms a category K_{KM} with morphisms the parabolics. Write $K_{\mathrm{GL}}, K_{\mathrm{Sp}}, \ldots$ for the appropriate subcategories.

Fact. [SA] *A shuffle algebra* [KS; Gr] *is equivalent to a monoidal functor $K_{\mathrm{GL}} \xrightarrow{A} \mathrm{Vect}$.*

There are KM analogues of *vertex algebra* and *localised bialgebra*, *quiver representation* with moduli stack $\mathcal{M}_Q = \sqcup \mathfrak{u}_e/G_i$, *quiver varieties*, and *(factorisation) braided monoidal categories* (generalising the G_2-**reflection equation** [Ku]). We generalise[9] **Chen's Theorem** [Ch] on the shuffle structure on cohomology $\mathrm{H}^\bullet(LX)$ of loop spaces; leading to the

Question. [SA] *Can we recover Dynkin/q-analogues of* ***multiple zeta values*** [KMT; Mi] *using this KM-shuffle structure?*

We produce $G_2, C_n, \ldots$ structures by **folding** $D_4, A_{2n+1}, \ldots$ structures by outer automorphisms $\mathbf{Z}/3, \mathbf{Z}/2, \ldots$.

Conjecture. [SA] *We may recover $\{\varnothing$,**boundary**, G_2-, …$\}$ spherical (or trigonometric)* ***KZ equations*** [ES] *by constructing a $K_{\mathrm{GL}}, K_{\mathrm{Sp}}, K_{G_2} \ldots$ (or $K_{\widehat{\mathrm{GL}}}, \ldots$)* ***affine vertex algebra*** *via BD Grassmannians, and taking conformal blocks.*

Question. [SA] *Can we recover Dynkin/q-analogues of* ***stable envelopes*** *(see* [MO], *Stab)?*

A twisted AGT correspondence. A foundational result in geometric representation theory is the **AGT correspondence** linking $\mathcal{W}$-vertex algebras and surfaces, first by Grojnowski and Nakajima [Gr; Na] then in [AGT; BFN] and [RSYZ].

Conjecture. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a* ***Verma module*** *for an orthosymplectic analogue $\mathcal{W}^k(\mathfrak{gl}_n)^{\mathrm{OSp}}$ of a $\mathfrak{gl}_n$ W-algebra (proof sketch: use SA-techniques on* [BFN]*).*

Conjecture. [AGT] *The dimension zero CoHA of $\mathbf{A}^3$, which is $U_\hbar(\mathcal{D}(\mathbf{C}/\pm))$ and admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients, acts on $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$, the equivariant intersection cohomology of $\mathbf{A}^3$-**instantons** (see* [RSYZ]*). Likewise for any quiver with potential.*

1.4. q-vertex algebras (qVA, KL)

q-vertex algebras. There have been many attempts [FJW; FR; EK] to define q-affine vertex algebras; none are conceptual enough to apply factorisation techniques [CF]/physics heuristics [Co; GR; Wi] to relate to q**KZ connections** or the (conjecturally, **affine**) **Kazhdan-Lusztig equivalence**, or new KL equivalences [BCDN].

Conjecture. [qVA] *There is a factorisation category $\mathcal{D}$-Mod over the noncommutative plane*[10] *$\mathbf{A}^2_q$, a factorisation algebra in which induces a q-**vertex algebra** (i.e. ordinary vertex algebra with poles on the q-diagonals, e.g.* [FR], [EK] *for $\mathfrak{g} = \mathfrak{sl}_n$).*

Conjecture. [qVA] *There is an analogue of the BD Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$ (this induces a q-affine VOA $V^k_q(\mathfrak{g})$).*

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[7] i.e. the image of the CoHA in $\mathrm{End}(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}))$.

[8] As opposed to the ordinary configuration space $\mathrm{Conf}\mathbf{A}^1 = \sqcup \mathbf{A}^n /\!/ \mathfrak{S}_n = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BGL})$.

[9] Replacing the n-simplex Δ^n with $\Delta_G = T_G/W_G$.

[10] Its with ring of functions $\mathbf{C}\langle x, y, q\rangle/(yx - xyq)$ with q central.

The physics heuristic for this is: $V^k(\mathfrak{g})$ comes from $3d$ Chern-Simons with boundary $\mathbf{C}$ [Wi]; $V_q^k(\mathfrak{g})$ comes from $5d$ **Chern-Simons** with noncommutative boundary $\mathbf{C}^2_{nc}$ [Co; GR; GRZ]. The category $\mathcal{D}$-Mod will be related to q-**difference modules**[11] on $\mathbf{A}^1$, so the above directly generalises the usual factorisation definition of vertex algebras.

Conjecture. [KL] *The restriction of **conformal blocks** of $V_q^k(\mathfrak{g})$-modules to $(\mathbf{A}^1)^n_\circ \subseteq (\mathbf{A}^2_q)^n_\circ$ equal to the qKZ connection.*

Conjecture. [KL] *There is a **Zhu algebra** functor,* $\mathrm{Zhu} : V_q^k(\mathfrak{g}) \mapsto U_q(\mathfrak{g})$*, and the functor* $\mathrm{RH} : V^k(\mathfrak{g})\text{-Mod} \mapsto U_q(\mathfrak{g})\text{-Mod}$ *in the proof* [CF] *of KL is "parallel transport" along a factorisation category* $V_q^k(\mathfrak{g})\text{-Mod}$ *on* $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$.

The above would help understand **affine KL** and the **new KL equivalences** [BCDN] from $3d$ mirror symmetry.

1.5. Sheaf methods (Con [12], Loc [4], Eu)

Localisation methods. We generalise *torus localisation* and *Graber-Panharipande formulas* [GP] (one of the main technique in enumerative geometry) to the Artin moduli stacks appearing in modern enumerative/algebraic geometry.

Theorem. [Conc; Loc] *We give conditions for the cohomology of an Artin stack $\mathfrak{X}$ to be **concentrated** on a closed substack $\mathfrak{Z}$; when $\mathfrak{Z} = \mathfrak{X}^T$ is fixed points of a quasismooth dg scheme we give **Atiyah-Bott*** $\mathrm{id} = i_*\left(i^!(-)/e(N_{vir})\right)$ *and **Graber-Pandharipande localisation** formulas* $[\mathfrak{X}]^{vir} = i_*\left([\mathfrak{X}^T]^{vir}/e(N_{vir})\right)$.

Theorem. [Eu] *We strengthen both parts of the theorem to **critical cohomology** of **arbitrary** closed embeddings $\mathfrak{Z} \hookrightarrow \mathfrak{X}$ quasismooth over a common base. As a result, for $\mathcal{M}$ as in CY3, we have the following Theorem:*

Theorem. [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we have* $\mathrm{CoHA} = \mathrm{CoHA}^s/e(\mathrm{N}_{i,vir})$.

Taking $\mathcal{M}^s$ a *shuffle space*[13], this gives a **universal, geometric** way of producing/interpreting shuffle products for CoHAs [Da; SV; YZ]. Taking $\mathcal{M}^s = \mathcal{M} \times \mathcal{M} \xrightarrow{\oplus} \mathcal{M}$ proves **compatibility** [CV,CY3,Li] between **coproducts** and the CoHA (c.f. CY3). This easily **generalises** to the OSp/Dynkin SA cases.

1.6. Liouville quantum gravity to vertex algebras (LQG [14])

Probabilists are beginning construct Feynman measures (GFF [BPR], LQG [KRV]) **rigorously** to understand $2d$ CFTs [CRV; DS; Sh]. There is almost no intereaction with the algebraic geometry/functorial school to QFT, and they understand/have access to techniques (e.g. SLE curves [MS; SS], combinatorial approximations) that the latter school does not. We want to build a bridge:

Conjecture. [LQG] *There is a **chiral part** functor* $F^{ch} : \mathrm{CFT} \to \mathrm{VertexAlg}$*, from Segal-style full $2d$ CFTs to vertex algebras. It sends* [GKRV]*'s LQG (resp. GFF) to the Virasoro (resp. Heisenberg) VOA, and* [KRV]*'s DOZZ formula to the Virasoro OPE.*

We will then use **free field embeddings** to produce a probabilistic CFT analogue of the $\mathfrak{g}$-**affine vertex algebra**.

[11] e.g. a q-difference operator ∂_x on $\mathbf{A}^1$ induces a derivation $y\partial_x$ on $\mathbf{A}^2_q$.
[12] Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.
[13] i.e. shuffle algebra in the category of spaces, see SA.
[14] Joint with V. Giri.

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