KZ WORKING GROUP 2026

ALEXEI LATYNTSEV, XINXANG TANG, AND WHOEVER ELSE WANTS TO JOIN

1. Plan

1.1. **Setting up definitions.** Write down

(1) (a) the KZ equation (over any surface Σ) [ES; Fe; Iv],
  (b) the boundary KZ equation,
  (c) the dynamical KZ equation, [FMTV; LX]

(2) (a) the qKZ equation (additive, multiplicative and heat versions [FV]),
  (b) dynamical qKZ?
  (c) the dynamical qKZ equation [LX],

(3) the (q, t)KZ equation [AKMMMOZ; AKMMSZ].

Then write, without proof, a lot of Theorems of the form

Theorem A. *The X-limit of equation Y is equation Z.*

1.2. **Finite quantum groups and associators.** Prove

Theorem B. [ES, §15.2], [EK, §3] *For simple $\mathfrak{g}$, the KZ equations*

(1) induce a Drinfeld associator on $\mathrm{Rep}_{k[[\hbar]]}U(\mathfrak{g})[[\hbar]] \to \mathrm{Vect}$,

(2) induce a braided monoidal structure on $\mathrm{Rep}_{k[[\hbar]]}U(\mathfrak{g})[[\hbar]] \to \mathrm{Vect}$,

(3) hence a quasitriangular Hopf algebra structure on $U(\mathfrak{g})[[\hbar]]$.

1.2.1. *Axiomatisation of the above Theorem.* Define constructible sheaves, then write a 1-categorical version of [Lu, 5.5.4.10]:

Theorem C. *There is a functor*

ψ : {factorisable constructible sheaves of categories on $\mathrm{Ran}\mathbf{R}^2$} $\to$ {braided monoidal categories},

there is a constructible sheaf of categories $\mathcal{E}$ with

$$\psi(\mathcal{E}) = \mathrm{Rep}_{k[[\hbar]]}U(\mathfrak{g})[[\hbar]]$$

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and interpret the KZ associator and Drinfeld twist and braiding in terms of ψ.

1.3. Multiple zeta values.

Conjecture D. [ES] *The KZ associator is (given in terms of? how?) multiple zeta values.*

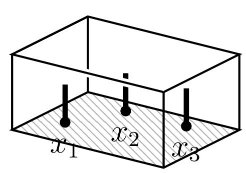
discuss relation to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ iff it's relevant to KZ?

1.4. KZ from QFT.

1.4.1. *Mathematical formalisation.*

Define a Atiyah-style QFT as a functor $\mathcal{Z} : \mathrm{Cob}_n \to \mathrm{Vect}_n$ with an *enriched* structure. Define spacetime symmetry of $\mathcal{Z}$. Give example of holomorphic-topological theories.

Theorem E. *Given an n-dimensional $X \times \mathbf{R}$, the category $\mathcal{Z}(S_x^{n-2})$ of* line operators *naturally forms a (topological/holomorphic/. . .) factorisation category over X:*



$X \times \mathbf{R}$

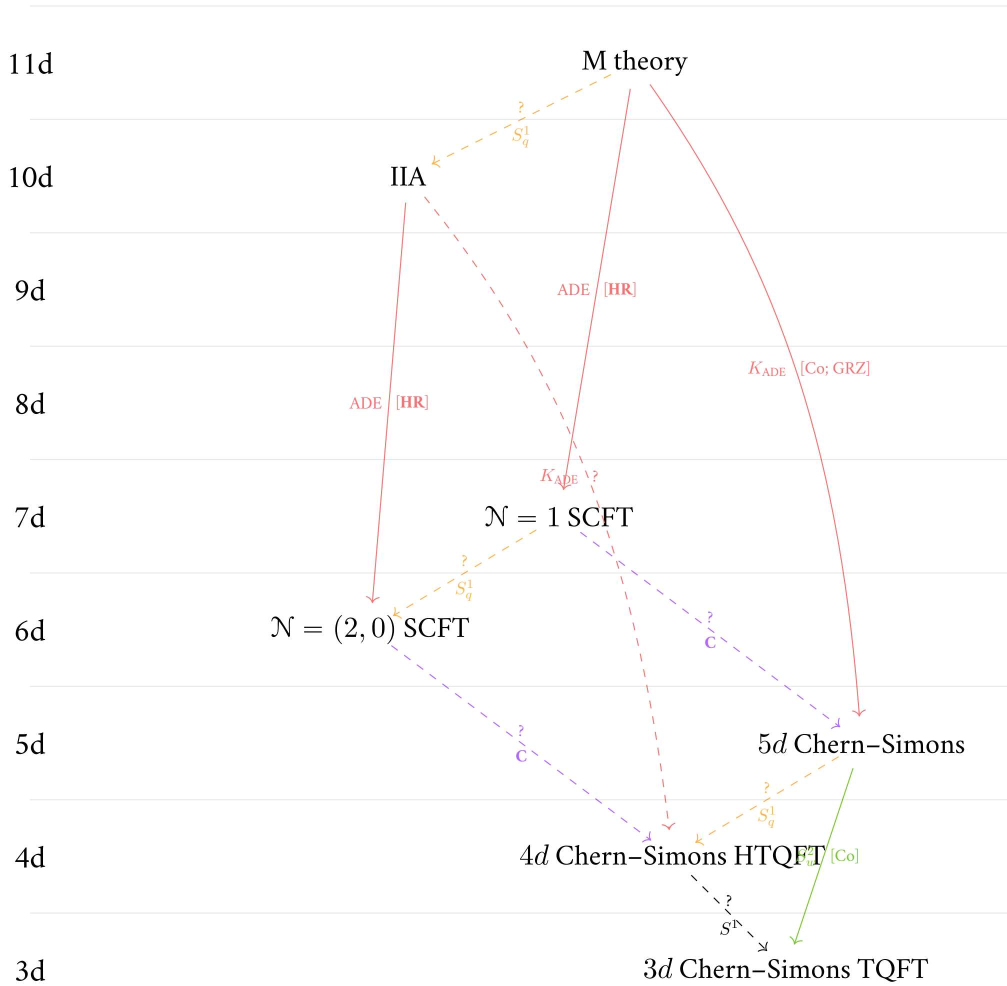
Conclude that for any sections $V_1, \ldots, V_n \in \mathcal{Z}(S^{n-2})$ as x varies, we get a vector bundle (with connection of the appropriate type given by spacetime symmetries) over X^n with fibres $V_1 \otimes \cdots \otimes V_n$.

1.4.2. *To prepare for Kazhdan–Lusztig.*

Give a definition of topological QFT $\mathcal{Z}$ with holomorphic boundary $\partial\mathcal{Z}$, show that there is a "Riemann–Hilbert" functor $\mathcal{Z}(S^{n-2}) \to \partial\mathcal{Z}(S^{n-3})$.

1.4.3. *Physics.*

Discuss the Physics-Theorem due to [Co] that M-theory on a CY3 Y gives a 5d QFT on noncommutative spacetime. Sketch Kevin's example where $Y = K_{\mathrm{ADE}}$:



Discuss how $4d$ Chern–Simons can only be valued on an algebraic group [CWY].

Theorem F. *Assuming X properties about 3, 4 and 5 Chern–Simons theory, we get the KZ, qKZ and (q, t)-KZ equations. The different variables q, t, etc. correspond to Y, Z, etc. in the above geometry.*

1.5. **Relation to Coulomb branches.** Firstly, using quasimaps

$$\begin{tikzcd} & \mathrm{QMaps}^d_{p_1,p_2}(\Sigma, M) \arrow[dl, "\mathrm{ev}_2"'] \arrow[dr, "\mathrm{ev}_1"] & \\ M & & M \end{tikzcd}$$

prove Okounkov's quaisimaps solutions to KZ via *capping operators*,

Theorem 1.5.1. [Ok, Thm. 8.1.16, 8.2.20] *When $\Sigma = \mathbf{P}^1$ and $M = G/B$, the capping operator*

$$J(z) \;=\; \sum_{d \in \mathrm{H}_2(M,\mathbf{Z})} q^d \cdot (\mathrm{ev}_1 \times \mathrm{ev}_2)_* \widehat{\mathcal{O}}_{\mathrm{vir}} \;\in\; \mathrm{K}_{G \times \mathbf{G}_{m\,q}}(M)^{\otimes 2}_{\mathrm{loc}} \otimes \mathbf{Q}[[q^d]]$$

1.5.2. *KZ as a Gauss–Manin connection on quantum cohomology.* Let $\mathcal{M}_C$ denote the BFN Coulomb branch.

Theorem 1.5.3. [Da] *For G simply laced the map*

$$(V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n})_\mu \xrightarrow{\sim} \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}^T_C) \xrightarrow{\mathrm{Stab}_{\mathfrak{C}}} \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}_C)_{T-\mathrm{loc}}$$

of vector bundles over $\mathbf{C}^n\times\mathfrak{h}$ intertwines the KZ differential ∇^{KZ}_i and quantum multiplication $c_1(\mathcal{E}_i)*$ by the first chern class, where V_λ is irreducible finite dimensional with highest weight λ.[1]

1.5.4. *Question.* As the Coulomb branch is Spec of cohomology of the Hecke stack $\mathcal{B}_z$, we have taken cohomology *twice*,

$$\mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}_C) \;=\; \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathrm{Spec}\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{B}_z,m))$$

and therefore we can try to apply any pair of cohomology theories (labelled by one-dimensional formal groups):

$\mathrm{A}^\bullet_T(\mathrm{B}^\bullet_H(\mathcal{B}_z))$

	$\mathbf{C}$	$\mathbf{C}^\times$	E	(Σ)
$\mathbf{C}$				
$\mathbf{C}^\times$				
E				

KZ
qKZ
qKZB heat

What is the analogue of [Da]'s Theorem in each case? Is there a version for critical cohomology?

1.5.5. *Physics question.* Understand the physics relation between $4d$ Chern–Simons theory and $3d$ $\mathcal{N}=4$ gauge theories whose Higgs/Coulomb branches are Nakajima quiver varieties. Talk to Tianqing about this

1.6. **KZ via the Riemann–Hilbert correspondence: Kazhdan–Lusztig.**

1.6.1. One can view the KZ equations as defining an element of either side of the Riemann–Hilbert correspondence

$$\mathrm{RH}\;:\;\{\text{vector bundle with connection with regular singularity on }\mathbf{C}^n\}\;\xrightarrow{\sim}\;\{\text{perverse sheaf on }\mathbf{C}^n\}.$$

We show that this corresponds to two equivalent ways of viewing the KZ equations: via affine vertex algebras and via quantum groups.

1.6.2. Show vertex algebras equivalent to factorisable sheaves over Ran$\mathbf{C}$, and

Theorem G. FBZ *Let V be a vertex algebra.*

(1) If $M_1,\dots,M_n$ are vertex modules over V, then there is a vector bundle with connection of their conformal blocks

$$\mathrm{C}^0(M_1,\dots,M_n)\;\to\;\mathrm{Conf}_n\mathbf{C}.$$

[1]Here T is the framing torus and H the gauge torus.

(2) There is a natural functor

$$\psi_{\mathrm{hol}} \;:\; \{\text{factorisable sheaves of QCoh-module categories on } \mathrm{Ran}\mathbf{C}\} \;\to\; \{\text{categories}\}$$

and $\mathrm{Rep}_{\mathrm{VA}} V$ *lies in the image.*

1.6.3. Define a Riemann–Hilbert functor

$$\mathrm{RH} \;:\; \{\text{sheaves of QCoh-module categories on } X \text{ with connection}\} \;\to\; \{\text{constructible sheaves of categories on } X\}$$

sending $(\mathcal{C}, \nabla) \mapsto \mathcal{C}^{\nabla}$, see [CF].

Theorem H. [CF, §1] *There is a factorisable sheaf of* QCoh*-module categories with flat connection* $(\mathcal{C}, \nabla)$ *with* $\mathrm{RH}(\mathcal{C}, \nabla) = \mathcal{E}$*.*

1.6.4. *Kazhdan–Lusztig.* Define vertex algebras, and state without proof:

Theorem I. (Kazhdan–Lusztig; [CF]) *There is an equivalence of categories*

$$\widehat{\mathfrak{g}}\text{-}\mathrm{Mod}_{\kappa}^{G(\mathcal{O})} \;\simeq\; U_q(\mathfrak{g})\text{-}\mathrm{Mod}^{\mathrm{f.d.}}$$

between integrable representations of the affine Lie algebra at level κ *and finite-dimensional representations of the quantum group at* $q = \exp(\pi i/(\kappa - h^{\vee}))$*, for all nice enough* κ*.*

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