q-DEFORMED D-MODULES

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1. Introduction

1.1. **Vector fields.** Recall that a tangent vector is a map

$$\xi \;:\; \mathbf{D}_2 \;\to\; X$$

from the second order infinitesimal neighbourhood of the origin in the formal disk $\mathbf{D}$. Likewise we get the notion of n-jet for any $n = 1, 2, \cdots, \infty$, and stronger still we could ask for a map

$$\xi \;:\; \mathbf{G}_a \;\to\; X.$$

A vector field induces a map on functions

$$\mathcal{O}(X) \;\to\; \mathbf{C}[\epsilon]/\epsilon^2,$$

and the ϵ coeffient is the *derivative* of the function in the direction of the vector field.

1.1.1. *Multiplicative and elliptic jets.* We make the following redundant definition. If G is a one-dimensional algebraic group, a *G-jet* is a map

$$\xi \;:\; \mathbf{D}^G \;\to\; X$$

from the formal neighbourhood of the identity in G. Of course, all of these are non-canonically isomorphic and so this is the same thing as an ordinary jet. Let χ_G be a left-invariant vector field on G, then

$$\mathbf{D}_2^G \;=\; \mathbf{D}_2 \cdot \chi_G.$$

However, when we pass to the quantum versions of the above definitions, the definitions for different G will seperate.

1.1.2. *Vector fields.* A *vector field* is a map over X

$$\xi \;:\; X \times \mathbf{D}_2 \;\to\; X.$$

Proposition 1.1.3. *The sheaf $\mathcal{T}_X$ of vector fields is the Lie algebra of the group $\operatorname{Aut}(X)$ over X.*

Proof. A tangent vector inside $\operatorname{Aut}(X)$ is a map

$$\psi \;:\; \mathbf{D}_2 \;\to\; \operatorname{Aut}(X)$$

which by adjunction is the same as a map

$$\mathbf{D}_2 \times X \;\to\; X.$$

The condition that ψ needs to be a tangent vector at the unit $\mathrm{id} \in \operatorname{Aut}(X)$ is equivalent to this map being over X. $\square$

In exactly the same way, an n-jet field on X is the same as an n-jet at the identity of $\operatorname{Aut}(X)$.

1.2. **Ordinary $\mathcal{D}$ modules.** Consider the category Sh_X of sheaves of abelian groups on smooth scheme X. We have a functor

$$\mathcal{O}_X\text{-Mod} \;\to\; \operatorname{Sh}_X$$

which is lax monoidal, i.e. we have a map $\mathcal{M}\otimes\mathcal{M}' \to \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}'$ for any $\mathcal{O}$-modules $\mathcal{M}, \mathcal{M}'$. If in addition $\mathcal{O}_X$ forms a bialgebra in Sh_X, then we may ask that $\otimes, \otimes_{\mathcal{O}}$ form a lax braided monoidal structure on $\mathcal{O}$-Mod,

$$(\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2) \otimes (\mathcal{M}_3 \otimes_{\mathcal{O}} \mathcal{M}_4) \xrightarrow{\beta} (\mathcal{M}_1 \otimes \mathcal{M}_3) \otimes_{\mathcal{O}} (\mathcal{M}_2 \otimes \mathcal{M}_4)$$

for all $\mathcal{O}$-modules $\mathcal{M}_i$.

1.2.1. *Example: $X = \mathbf{A}^n$.* The sheaf $\mathcal{O}_{\mathbf{A}^n}$ has a natural coalgebra structure in which the coordinates x_i are primitive. Moreover, this bialgebra structure is graded with respect to (any) linear action of $\mathbf{G}_m$ on $\mathbf{A}^n$.

1.2.2. In particular, we can define $\otimes$-$\otimes_{\mathcal{O}}$ *bilalgebras* $\mathcal{A}$, which are $\mathcal{O}$-modules equipped with maps

$$\mathcal{A} \otimes \mathcal{A} \;\to\; \mathcal{A}, \qquad\qquad \mathcal{A} \;\to\; \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}$$

which are compatible as such:

$$\begin{array}{ccccc}
\mathcal{A} \otimes \mathcal{A} & \longrightarrow & (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \otimes (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) & \xrightarrow{\beta} & (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{O}} (\mathcal{A} \otimes \mathcal{A}) \\
\downarrow & & & & \downarrow \\
\mathcal{A} & \longrightarrow & & & \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}
\end{array}$$

commute as a diagram in Sh_X, and finally $\mathcal{A}$ has a unit and counit which are compatible with each other and the above data.

We have then, assuming throughout that $\mathcal{O}_X$ is a bialgebra,

Proposition 1.2.3. *For any Lie algebroid $\mathcal{L}$, its universal enveloping algebra $U(\mathcal{L})$ is a bialgebra.*

Proof. (write, should be abstract nonsense) $\square$

Examples of Lie algebroids include tangent bundles and relative tangent bundles. Thus,

Corollary 1.2.4. *The sheaf $\mathcal{D}_X$ forms a bialgebra.*

As a consequence,

Corollary 1.2.5. *The symmetric monoidal structure $\otimes_{\mathcal{O}}$ has a canonical lift along $\mathcal{D}_X$-*Mod$^{\otimes} \to \mathcal{O}_X$-Mod.

1.2.6. *Example:* $X = \mathbf{A}^1$. In this case, the coalgebra structure on $\mathcal{D}_{\mathbf{A}^1}$, which we identify with $k\langle x, \partial_x\rangle$, is

$$\Delta(\partial_x) \;=\; \partial_x \otimes 1 \,+\, 1 \otimes \partial_x.$$

Note that by the coalgebra axioms and $\mathcal{O}$-linearity, $\Delta(1) = 1 \otimes 1$ and

$$\Delta(x^n \partial_x^m) \;=\; x^n(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^m$$

are forced, likewise if we are to ask that it be a bialgebra (how to define bialgebra?) this forces

$$\Delta(x^{n_1}\partial_x^{m_1} \cdots x^{n_k}\partial_x^{m_k}) \;=\; x^{n_1}(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^{m_1} \cdots x^{n_k}(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^{m_k}.$$

Note that

$$\Delta([x, \partial_x]) \;=\; x(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x) - (\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)x \;=\; 1 \otimes 1 \;=\; \Delta(1).$$

In particular, we have $\mathrm{Prim}(\mathcal{D}_{\mathbf{A}^1}) = \mathcal{T}_{\mathbf{A}^1}$.

1.3. The tangent bundle as a Lie bialgebroid.

The tangent sheaf $\mathcal{T}$ is naturally a Lie algebroid.

1.3.1. *Lie bialgebroids.* We can now define *Lie bialgebroids* over X as a sheaf $\mathcal{L} \in \mathcal{O}$-Mod with a Lie algebra structure in Sh_X

$$[\,,\,] \,:\, \mathcal{L} \otimes \mathcal{L} \;\to\; \mathcal{L}$$

which is $\mathcal{O}$-linear, in the sense that there is a map $\rho : \mathcal{L} \to \mathcal{T}_X$ with $[\ell, f\ell'] = (\rho(\ell)f)\ell' + f[\ell, \ell']$, and a Lie coalgebra structure in $\mathcal{O}_X$-Mod

$$\delta \,:\, \mathcal{L} \;\to\; \mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}$$

such that the usual axiom of a Lie bialgebra holds:

$$\delta([\ell, \ell']) \;=\; (\mathrm{ad}_\ell \otimes_{\mathcal{O}} \mathrm{id} \,+\, \mathrm{id} \otimes_{\mathcal{O}} \mathrm{ad}_{\ell'})\delta(\ell) \;-\; (\mathrm{ad}_{\ell'} \otimes_{\mathcal{O}} \mathrm{id} \,+\, \mathrm{id} \otimes_{\mathcal{O}} \mathrm{ad}_\ell)\delta(\ell'),$$

the relation viewed as a map $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}$.

1.3.2. *Example:* $X = \mathbf{A}^1$. In this case, we identify $\mathcal{T}_{\mathbf{A}^1}$ with the free $\mathcal{O}(\mathbf{A}^1)$-module $k[x]\partial_x$. Then as for any Lie algebroid, $\delta = 0$ defines a Lie bialgebroid structure on $\mathcal{T}_{\mathbf{A}^1}$.

2. Quantum analogues

2.1. q-**vector fields.** Now let $\mathbf{G}_m$ act on our smooth scheme X. This makes $\mathcal{O}_X$ into a $\mathbf{Z}$-graded sheaf, so we can define the sheaf $\mathcal{T}_X^q \subseteq \mathcal{E}\mathrm{nd}(\mathcal{O}_X)$ of q-vector fields consisting of endomorphisms ∂ with

$$\partial(fg) \;=\; \partial(f)g \;+\; q^{|f|}f\partial(g)$$

for all pairs of homogenous functions $f, g \in \Gamma(\mathcal{O}_X)$.

2.1.1. One way to axiomatise this is the following. Extend $\mathcal{O}(X)$ by adding the variable $\mathbf{q}$ with commutation relations

$$\mathbf{q}f \;=\; q^{|f|}f\mathbf{q}$$

for homogeneous elements, where $q \in k$ is central. Then

$$\mathbf{q}\partial(fg) \;=\; \mathbf{q}\partial(f)g \;+\; f\mathbf{q}\partial(g)$$

and so $\mathbf{q}\partial$ defines an honest vector field on $\langle \mathcal{O}(X), \mathbf{q}\rangle$. Thus a q-vector field induces an algebra map

$$\langle \mathcal{O}(X), \mathbf{q}\rangle \;\to\; \langle \mathbf{C}[\epsilon]/\epsilon^2, \mathbf{q}\rangle, \qquad\qquad f \;\mapsto\; f + \mathbf{q}\partial(f)\epsilon,$$

where $\mathbf{q}$ and ϵ commute. We now turn to the question of what this algebra $\langle \mathcal{O}(X), \mathbf{q}\rangle$ is.

Proposition 2.1.2. *$\langle \mathcal{O}(X), \mathbf{q}\rangle[q, q^{-1}]$ is a $\mathbf{Z}[q, q^{-1}]$-quantisation of $\mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})[q, q^{-1}]$ with the grading given by a $\mathbf{G}_m$-action on $X \times \mathbf{G}_{m,\mathbf{q}}$.*

For instance, if every function on X has degree zero, then $\langle \mathcal{O}(X), \mathbf{q}\rangle = \mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})$.

2.1.3. We are now in place to define q-vector field. To begin, we need to *choose* a quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \to G$ of $X \times \mathbf{G}_{m,\mathbf{q}}$ over G. Then,

Definition 2.1.4. A *q-vector field* on X is a vector field

$$\xi \;:\; \mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) \;\to\; (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}})$$

on the noncommutative space $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$, i.e. a map as above, over $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$.

Notice that we can take the pullback squares

$$\begin{array}{ccccc}
\mathbf{D}_2 \times X & \xrightarrow{\;\;\xi_1\;\;} & X & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) & \xrightarrow{\xi} & (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) & \longrightarrow & \mathbf{G}_m
\end{array}$$

which gives an ordinary vector field on X. Thus loosely speaking, a q-vector field is a quantised vector field on X.

2.1.5. *Example:* $X = \mathbf{A}^1$. The operator $\partial(x^n) = n_q x^{n-1}$, where n_q is the nth q-integer,

$$n_q \;=\; 1 + q + \cdots + q^{n-1}, \qquad\qquad (-n)_q \;=\; q^{-1} + q^{-2} + \cdots + q^{-n}$$

which satisfies $(n+m)_q = n_q + q^n m_q$. In particular, $\partial(x^{n+m}) = n_q x^n \cdot x^m + q^n x^n \cdot m_q x^m$, and so this defines a q vector field.

2.1.6. *Remark.* We could also just as well replace $\mathbf{G}_{m,\mathbf{q}}$ by $E_{\mathbf{q},\tau}$ or $G_{\mathbf{q}}$ any one-dimensional algebraic group.

Thus, let X and $G_{\mathbf{q}}$ be viewed as constant schemes over G. Then we *choose* a quantisation $X \tilde{\times} G_{\mathbf{q}} \to G$ over G. In this case, a G*-jet* is a map

$$\xi \;:\; \mathbf{D}_n^G \times (X \tilde{\times} G_{\mathbf{q}}) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

over $X \tilde{\times} G_{\mathbf{q}}$. Over a point $x \in X$ we get

$$\xi_x \;:\; \mathbf{D}_n^G \times (G_{\mathbf{q}} \times G) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

and so we get a map

$$\xi_x \;:\; \mathcal{O}(X \tilde{\times} G_{\mathbf{q}}) \;\to\; \mathcal{O}(\mathbf{D}_n^G \times G_{\mathbf{q}}) \otimes \mathcal{O}(G).$$

For instance, our ordinary notion of q-vector field corresponds to $G = \mathbf{G}_m$. We can define an $\hbar$-adic version by taking $G = \mathbf{G}_a$.

When dealing with elliptic curves, we may also require a compatible family of $\mathbf{G}_m$- and E_τ-jets which glue over $\overline{\mathcal{M}}_{1,1}$.

2.2. q**-cotangent bundles.** The cotangent bundle over X is given by taking the relative spectrum of the sheaf of vector fields.

2.2.1. Having chosen a quantisation $\tilde{X} = X \tilde{\times} G_{\mathbf{q}}$, the *quantum cotangent bundle* is

$$\tilde{\mathbf{T}}^*_{\tilde{X}} \;=\; \mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}} \times G}.$$

(define this, i.e. show that we get a quantisation)

Lemma 2.2.2. *This is a quantisation of the cotangent bundle of X times $G_{\mathbf{q}} \times G$, i.e.*

$$\mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}} \times G} \;=\; \mathbf{T}^*_X \tilde{\times} G_{\mathbf{q}}.$$

For instance, if $X = \mathbf{A}^1$ and $G = \mathbf{G}_m$, then we can take

$$\tilde{X} \;=\; \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}^{\pm}, q^{\pm} \rangle$$

where q is central, and

$$\mathbf{T}^q_{\tilde{X}} \;=\; \operatorname{Spec} \mathbf{C}\langle x, p, \mathbf{q}^{\pm}, q^{\pm} \rangle$$

is a twisted product of $T^*\mathbf{A}^1$ and $\mathbf{G}_{m,\mathbf{q}} \times \mathbf{G}_m$, where $p = \partial_x$, and so we have that $\mathbf{q}p = q^{-1}p\mathbf{q}$. Notice that we get a closed subscheme

$$\mathbf{A}^2_q \;=\; \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}p \rangle$$

which is the quantum affine plane, since writing $y = \mathbf{q}p$, we get the defining relations $xy = qyx$.

2.3. **q-differential operators.** The q-differential operators $\mathcal{D}_q$ will be a filtered quantisation of

$$\operatorname{Spec}\operatorname{Sym}_{\tilde{X}} \tilde{\mathbf{T}}^*_{\tilde{X}}.$$

Notice that the role of q and the q-quantisation is orthogonal to the role of the filtration and the filtered quantisation. We define it as usual: it is the sheaf of differential operators on $\tilde{X}$, i.e. it is the sheaf of subalgebras

$$\tilde{\mathcal{D}}_{\tilde{X}} \subseteq \mathcal{E}\mathrm{nd}_{\tilde{X}}(\mathcal{O}_{\tilde{X}})$$

generated by the q-vector fields and $\mathcal{O}_{\tilde{X}}$.

Notice that by the definition,

Lemma 2.3.1. *$\tilde{\mathcal{T}}_{\tilde{X}}$ forms a sheaf of Lie algebras over $\tilde{X}$.*

This allows us to give a Grothendieck definition of the sheaf of quantum differential operators:

Lemma 2.3.2. *$\tilde{\mathcal{D}}_{\tilde{X}} = \cup_{n\geqslant 0}\tilde{\mathcal{D}}_{\tilde{X},n}$, where the zeroeth term is $\tilde{\mathcal{O}}_{\tilde{X}}$, and above that*

$$\tilde{\mathcal{D}}_{\tilde{X},n} = \text{(recursive definition)}.$$

To summarise, we have the following

$$\begin{array}{cc} \operatorname{gr}\mathcal{D}_X & \operatorname{gr}\tilde{\mathcal{D}}_{\tilde{X}} \\ \\ \mathcal{D}_X & \tilde{\mathcal{D}}_{\tilde{X}} \end{array}$$

and the sheaves on the left are given by pulling back the sheaves on the right along $1 \to G$.

2.4. **Relation to automorphisms of X.** Recall that one may define a D-module on X to be a quasicoherent sheaf which is equivariant for the action of the formal group $\exp(\mathcal{T}_X)$; this is the parallel transport map. Likewise, if Φ is an automorphism of X, one possible definition of quantum D-module is a Φ-equivariant quasicoherent sheaf.

How does this relate to the above definition?

To begin with, what has this to do with the quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$? Let us consider the case when the quantisation and the automorphism both come from the same source: a single $\mathbf{G}_m$ action:

$$\begin{array}{ccc} & \mathbf{G}_m \text{ action on } X & \\ \swarrow & & \searrow \\ \text{automorphism } \Phi_g \text{ for any } g \in \mathbf{G}_m & & \text{quantisation } X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \end{array}$$

A quasicoherent sheaf on $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$ is the same as a quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(X)$ with a compatible action of $\mathbf{C}[\mathbf{q}^{\pm}]$, i.e. we have

$$\mathbf{q}_x : \mathcal{M}_x \xrightarrow{\sim} \mathcal{M}_x$$

for every point $x \in X$, and we have

$$\mathbf{q}_x f(x) = q^{|f|} f(x) \mathbf{q}_x$$

as automorphisms of $\mathcal{M}_x$. In particular, this has nothing to do with comparing $\mathcal{M}_x$ and $\mathcal{M}_{\Phi_g \cdot x}$, so *it is unlikely the definitions are related*.

The automorphism definition of quantum D-module is related to

$$\mathbf{Z} \xrightarrow{\Phi} \mathrm{Aut}(X) \leftarrow \exp(\mathcal{T}_X)$$

whereas the q-deformed D-module changes the underlying space,

$$\exp(\tilde{\mathcal{T}}_{\tilde{X}}) \rightarrow \exp(\mathcal{T}_X).$$

One expects that it might be possible to quantise both ways simultaneously.

2.5. **Relation to Beilinsorn-Bernstein.** Let $\lambda : \mathbf{G}_m \to G$ be a character with $\lambda B \lambda^{-1} = B$. Then we get an induced $\mathbf{G}_m$ action on the flag variety G/B, and can form the quantisation.

Conjecture 2.5.1. *We have a surjection* $\tilde{\mathcal{D}}_{\widetilde{G/B}} \twoheadrightarrow U_q(\mathfrak{g})$.

2.6. **Relation to quantum groups.** We are going to give a *different* relation to quantum groups, where

$$X = \operatorname{Spec} U_q(\mathfrak{g}), \qquad G = T.$$

Note that here we may be using a group of dimension greater than one. If $\mathbf{q}_\lambda$ corresponds to $\lambda \in \mathcal{O}(T) \subseteq \mathfrak{t}^*$, then we set

$$x \mathbf{q}_\lambda = q^{\lambda(x)} \mathbf{q}_\lambda x$$

for all $x \in \mathfrak{g} \subseteq U_q(\mathfrak{g})$.

Conjecture 2.6.1. *We have*

$$\tilde{\mathcal{D}}_{\tilde{X}} = U_q(\mathfrak{g} \oplus_{\mathfrak{t}} \mathfrak{g}^*)$$

is the Takiff algebra.

2.6.2. We now consider the analogue of the Lie algebra structure on $\mathcal{T}^q_X$. Let $\mathbf{q}$ be the operator acting as $f \mapsto q^{|f|}f$ on homogenous elements.

Lemma 2.6.3. *The sheaf $\mathbf{q}^{-1}\mathcal{T}^q_X$ is closed under the bracket*

$$[\partial_1, \partial_2]_q \;=\; \partial_1\partial_2 \;-\; q^{|\partial_2|-|\partial_1|}\partial_2\partial_1.$$

(check the q factor, note that $|\mathbf{q}| = 0$)

Proof. If ∂_1, ∂_2 are two q-derivations on $\mathcal{O}_X$, then

$$\begin{aligned}
[\partial_1, \partial_2]_q(fg) &= \partial_1\partial_2(fg) - q^A\partial_2\partial_1(fg) \\
&= \partial_1(\partial_2(f)g + q^{|f|}f\partial_2(g)) \;-\; q^A\partial_2(\partial_1(f)g + q^{|f|}f\partial_1(g)) \\
&= (\partial_1(\partial_2(f))g + q^{|\partial_2(f)|}\partial_2(f)\partial_1(g) + q^{|f|}\partial_1(f)\partial_2(g) + q^{2|f|}f\partial_1(\partial_2(g))) \\
&\qquad - \; q^A(\partial_2(\partial_1(f))g + q^{|\partial_1(f)|}\partial_1(f)\partial_2(g) + q^{|f|}\partial_2(f)\partial_1(g) + q^{2|f|}f\partial_2(\partial_1(g))) \\
&= [\partial_1, \partial_2]_q(f)g + q^{2|f|}f \cdot [\partial_1, \partial_2]_q(g) + \text{(other stuff)}
\end{aligned}$$

(look at the q-Virasoro note to fix this) □

Note that the sheaf $\mathbf{q}^{-1}\mathcal{T}^q_X$ are the endomorphisms ∂ with

$$\partial(fg) \;=\; q^{-|\partial(f)|}\partial(f)g \;+\; q^{|f|-|\partial(g)|}f\partial(g).$$

Definition 2.6.4. A *q-Lie algebra is* (define)

Definition 2.6.5. A *q-Lie algebroid is* (define)

Definition 2.6.6. The *universal enveloping algebra* $U^q(\mathcal{L})$ of a q-Lie algebroid $\mathcal{L}$ is (define)

In particular, we see that

Lemma 2.6.7. *$U^q(\mathcal{L})$ is a bialgebra, and setting $q = 1$ gives $U(\mathcal{L}_0)$.*

Proof. (prove) □

The crucial difference here is that $U^q(\mathcal{L})$ is not cocommutative, even though $U(\mathcal{L}_0)$ is. Thus,

Corollary 2.6.8. *$U^q(\mathcal{L})$-Mod is a monoidal category, which when $q = 1$ becomes symmeteric monoidal.*

We can now ask that $U^q(\mathcal{L})$ be cocommutative, or more generally have an R-matrix.

Proposition 2.6.9. *If $\mathcal{L}$ has (properties), then $U^q(\mathcal{L})$ forms a quantum group. In particular, $U^q(\mathcal{L})$-Mod is braided monoidal.*

.1. q differential operators.

.1.1. Note that unless $q = \pm 1$, the operation $a \otimes b \mapsto q^{|a|-|b|} b \otimes a$ does not define a symmetric monoidal structure on the category of $\mathbf{Z}$-graded vector spaces. Thus, to define "q-Lie algebra", we do not take a Lie algebra in a q-deformed symmetric monoidal structure, but instead manually define q-Lie algebra in the original symmetric monoidal category. (potentially just a certain quadratic operad)

Under the bracket
$$[a, b]_q \;=\; ab - q^{|a|-|b|} ba$$
the analogue of antisymmetry is
$$[a, b]_q \;=\; q^{2(|b|-|a|)} [b, a]_q.$$
Notice also that we have
$$[aa', b]_q \;=\; aa'b - q^{|a|+|a'|-|b|} baa' \;=\; a(a'b - q^{|a'|-|b|} ba') + q^{|a'|-|b|}(ab - q^{|a|} ba)a'$$
which is not quite $[a, b]_q a' + a[a', b]_q$. The analogue of Jacobi is
$$[a, [b, c]_q]_q - [c, [b, a]_q]_q + [b, [c, a]_q]_q \;=\; abc(1 - q^{|b|-|a|} q^{|c|-|b|-|a|}) + acb(-q^{|b|-|c|} + q^{|c|-|a|} q^{|b|-|c|-|a|}) + \cdots$$
which does not vanish. We can instead try
$$[a, [b, c]_q]_q - q^{2|a|-|c|} [c, [b, a]_q]_q + q^{4|a|-2|c|} [b, [c, a]_q]_q \;=\; abc(0) + acb(0) + bca(-q^{|a|-|b|-|c|} + q^{4|a|-2|c|}) + \cdots$$
which is again not zero, so there is no salvage of Jacobi that just adjusts each summand by $q^{f(|a|,|b|,|c|)}$.

References

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