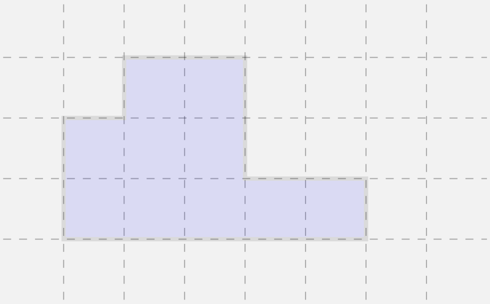
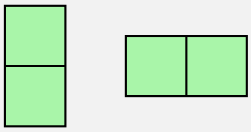
1. Domino tilings

1 July

Definition. A *domino tiling* of a region is a way of covering it completely using 1×2 or 2×1 dominoes, without overlaps or gaps.





dominoes

1. Counting domino tilings.

   a. How many domino tilings are there of the rectangles: 2×2, 2×3, 2×4, 2×5?

   b. Show that the number of domino tilings of a $2 \times n$ rectangle is the nth Fibonacci number F_n. Conclude that

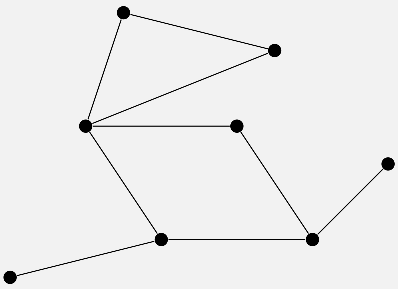
   $$F_{n+m} = F_n F_m + F_{n-1} F_{m-1}.$$

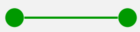
   c. How many domino tilings are there of a $3 \times n$ rectangle ($n = 2, 4, 7, 10$)? *Try to do the bigger numbers in a non-bashy way.*

   d. How many tilings are there of the Aztec diamond with only one row of maximal length?

   e. How many tilings are there of the Aztec diamond with base length $2, 4, 6$?

Definition. A *domino tiling* of a *graph* is a way of covering the vertices completely using dominoes, without overlaps or gaps.

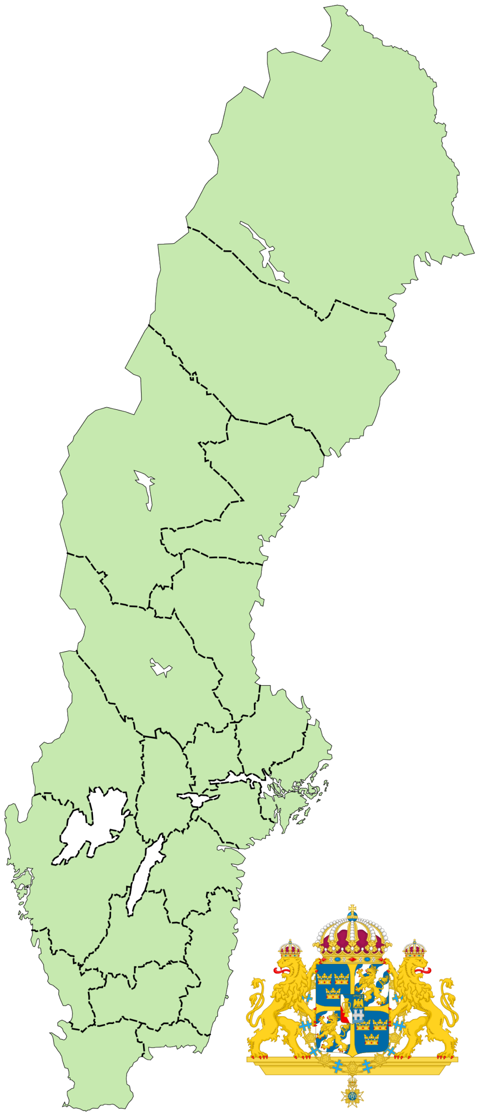




domino

Show that when the graph is a square grid graph, this is the same as the first definition of domino tiling.

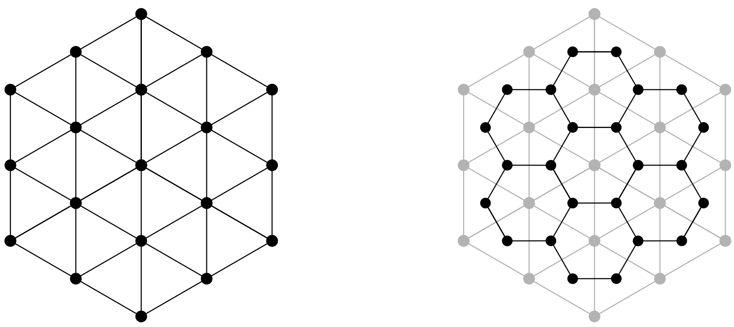
2. Consider the map of Sweden's Län



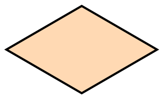
a. Draw its *dual graph*, which has one vertex for each region, and an edge whenever two regions share a 1d border (not just touch at a point).

3. Examples coming from 3d geometry.

a. Consider the hexagon graph and its dual



Draw three possible domino tilings of the dual graph, and show that they correspond to colouring in the regions of the hexagonal graph in by *lozenges*

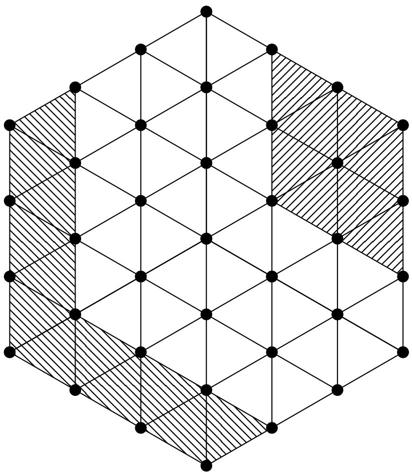


lozenge

b. How many domino tilings are there of the duals to the $2 \times 2 \times 2$ and $3 \times 3 \times 3$ hexagonal graphs?

c. Give a 3D geometric description of such a tiling.

d. If we remove the following region, how many domino tilings are there of the dual graph?



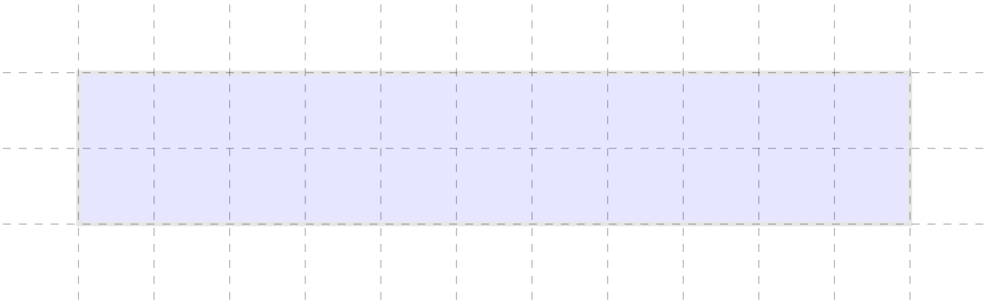
2. Random domino tilings

2 July

Definition. A *random domino tiling* of a finite region is a probability distribution (on the set of all tilings) where each domino tiling T is equally likely

$$\mathbf{P}(\text{tiling } T) = \frac{1}{\#\{\text{all tilings } T'\}}.$$

1. Random domino tilings of $2 \times n$ rectangle.



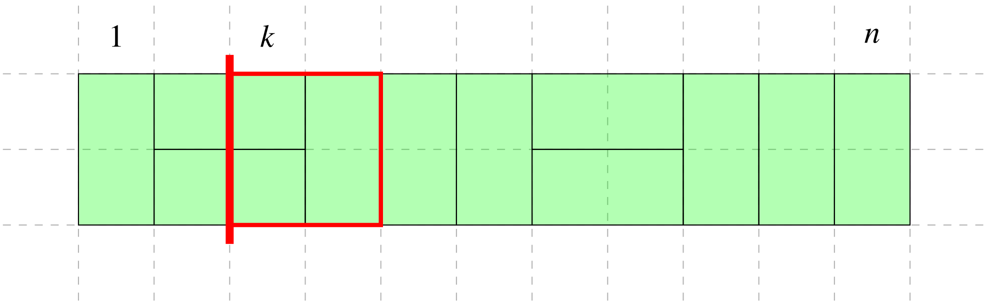
a. Draw all domino tilings of the 2×5 rectangle. Compute the probability that the leftmost domino is vertical.

b. What is the answer for the $2 \times n$ rectangles?

c. If the leftmost domino of an $2 \times n$ rectangle is vertical, what is the probability that the rightmost domino is vertical?

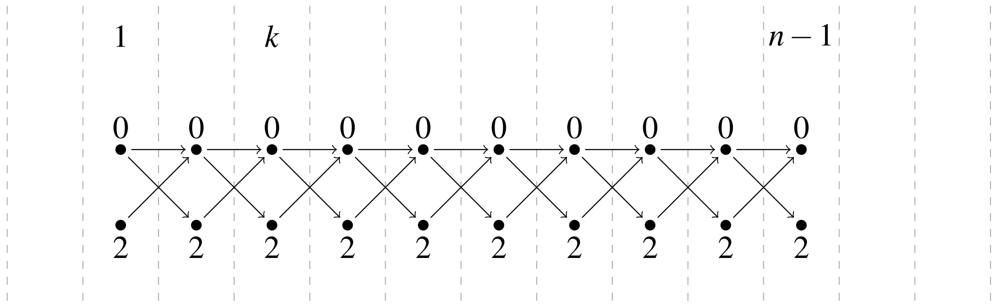
d. What is the limit of b. and c. as $n \to \infty$?

2. Take a domino tiling, and consider the sequence of boxes, labelled by $k = 1, \ldots, n-1$.



a. Let $f : \{0, 1, \ldots, n\} \to \{0, 2\}$ be the function counting the number of horizontal dominoes completely contained in the kth box. Show that f uniquely determines the domino tiling.

b. Give a bijection between domino tilings of the $2 \times n$ rectangle and maximal paths on the following graph:



c. Give another proof that this is the nth Fibonacci number.

d. *Attempt if you know matrices.* Define the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that the number of domino tilings of the $2 \times n$ rectangle is equal to the top left entry a_n of $M^n = M \times \cdots \times M$.

e. *Attempt if you know matrices.* Show that

$$M^2 = M + 1$$

hence show that $a_{n+2} = a_{n+1} + a_n$.

f. *Attempt if you know matrices.* Consider the top-left entry $b_n \in \mathbf{Z}[x,y]$ of N^n, for the matrix

$$N = \begin{pmatrix} x & y \\ y & 0 \end{pmatrix}.$$

Compute b_1, b_2, b_3, b_4. What is the interpretation of the $x^i y^j$-coefficient of b_n? Show that $b_{n+2} = x b_{n+1} + y^2 b_n$. Find a formula for b_n of the form

$$b_n = c_1 \cdot \varphi_1^n + c_2 \cdot \varphi_2^n.$$

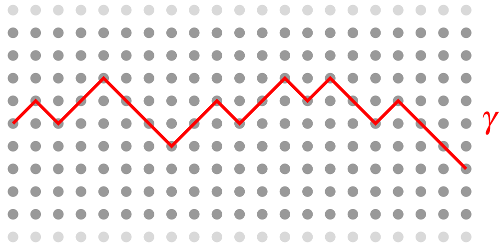
Hint: find two solutions of the form φ_i^n, to the equation, then use b_1, b_2 to find c_1, c_2.

Definition. A *walk* on a graph G is a sequence of vertices $v_0, v_1, \ldots, v_t$, where each vertex is adjacent to the next. A *random walk* is a probability distribution on the set of all walks, where each walk is equally likely,

$$\mathbf{P}(\text{walk } v_0, \ldots, v_t) = \frac{1}{\#\{\text{all walks }\}}.$$

One can similarly define a *random walk starting at a fixed vertex v*, or a random walk with some other set of properties.

3. Random walks. Consider a random walk on $\mathbf{Z}$ of length t, which we draw as a path in $\mathbf{Z} \times \{0, 1, \ldots, t\}$.



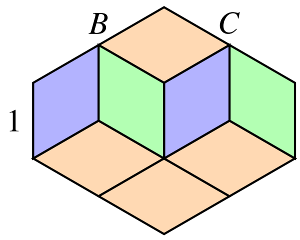
a. How many walks of length t are there?

b. What is the probability that a random walk γ has $\gamma(t) \geq 0$?

c. What is $\mathbf{E}(\gamma(t))$?

d. What is $\mathbf{E}(\gamma(t)^2)$? In plain English, approximately how far away from the origin should we expect γ to be after t steps?

4. Show that the set of tilings of the $1 \times B \times C$ hexagon by lozenges



is equivalent to a certain set of walks in $\mathbf{Z}$ (which you should define precisely).

a. Picking a random tiling (or equivalently, a random walk), what is the expected number of 3d cubes in a random $1 \times B \times C$ lozenge tiling?

b. Given a tiling T on the hexagonal graph G, we define the *height function*

$$h_T : V(G) \to \mathbf{N}$$

whose value is the number of 3d boxes under ($\downarrow$) that vertex. Describe the walk attached to T in terms of h_T.

c. *If you're interested in continuing, try the lattice paths long problem.*

3. Random dominoes 3

3 July

Probability.

Definition. A *probability space* is a countable (i.e. finite or has as many elements as $\mathbf{N}$) set Ω together with a function

$$\mathbf{P} \ : \ \{\text{subsets of } \Omega\} \ \to \ [0,1]$$

such that

1. $\mathbf{P}(\Omega) = 1$,
2. if $A_1, A_2 \ldots$ are disjoint subsets of Ω, then $\mathbf{P}(A_1 \cup A_2 \cup \cdots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \cdots$.

If $X : \Omega \to \mathbf{R}$ is a function, its *expectation* is defined to be

$$\mathbf{E}(X) \ = \ \sum_{a \in \Omega} \mathbf{P}(\{a\}) \cdot X(a).$$

A function X is called a *random variable*, or a *random real number*.

1. Consider one white and one black 6-sided dice, both with faces numbered $1,2,3,4,5,6$.

a. Draw the set Ω of possible outcomes of rolling the two dice (in a 6×6 table).

b. Let $W, B : \Omega \to \mathbf{R}$ denote the values of the white and black dice, respectively. What are the expectations

$$\mathbf{E}(W), \ \mathbf{E}(B), \ \mathbf{E}(10W + 2B)?$$

What are *variances*

$$\mathbf{E}((W - \mathbf{E}(W))^2), \ \mathbf{E}((B - \mathbf{E}(B))^2), \ \mathbf{E}((10W + 2B - \mathbf{E}(10W + 2B))^2)?$$

Write down on paper a plain English/Swedish description of what this means.

c. The *correlator*

$$\mathbf{E}\left((W - \mathbf{E}(W))(B - \mathbf{E}(B))\right)$$

measures how much information you get about the value $W(a)$ if you know the value $B(a)$, and vice versa. Draw the values of the function

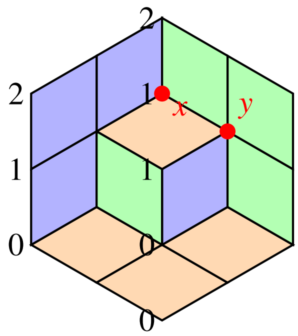
$$(W - \mathbf{E}(W))(B - \mathbf{E}(B)) \ : \ \Omega \ \to \ \mathbf{R}$$

on the 6×6 table. What is the value of this correlator?

d. The dice are now enchanted, so that the rolls are never more than 4 apart. Other than that all pairs of rolls are equally likely. What is the new answer to a., b. and c.?

Back to random domino tilings.

2. Correlators. In the following Ω will be the set of all lozenge tilings of a certain region, and the value of $\mathbf{P}$ is the same for each tiling.



a. Show that there are 20 lozenge tilings of the $2 \times 2 \times 2$ hexagon.

b. Write down the *height function* $h_T : V(G) \to \mathbf{N}$ of two tilings T. Here $V(G)$ is the vertex set of the hexagonal graph, some heights are given above for the above tiling.

c. What are the expected values $\mathbf{E}(h_T(x))$ and $\mathbf{E}(h_T(y))$ for a random tiling T?

d. What is the value of the *correlator*

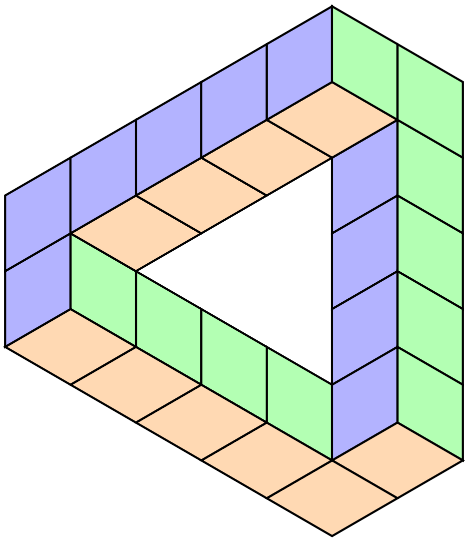
$$C(x,y) \;=\; \mathbf{E}(h_T(x)h_T(y)) - \mathbf{E}(h_T(x))\mathbf{E}(h_T(y))$$

for the two points x, y?

4. Random dominoes 4

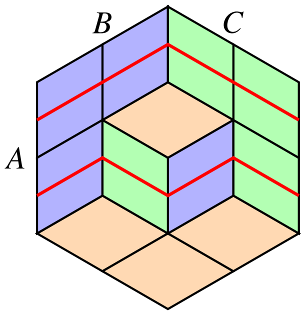
4 July 2025

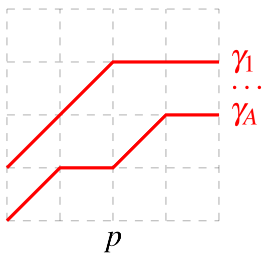
We'll begin with a proof that lozenge tilings of the $A \times B \times C$ hexagonal graph biject with 3d cube stackings of the $A \times B \times C$ 3d box. Be careful - it's not true if for certain subgraphs of the hexagonal graph!



1. Hexagonal paths.

   a. Give a bijection between tilings of the $A \times B \times C$ hexagon and certain a certain set of paths in $\mathbf{Z}^2$ (which you should precisely specify).





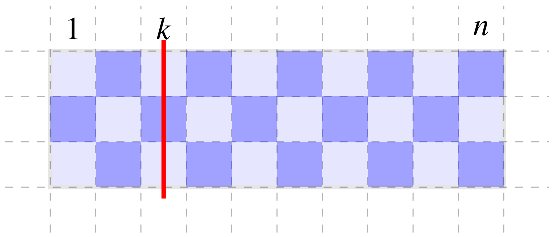
   b. What are the graphs corresponding to the lozenge tilings with the maximum and minimum number of 3d boxes?

   c. How can you see the number of 3d boxes from the paths?

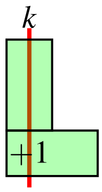
   d. What is the expected value of $\gamma_1(p), \dots, \gamma_A(p)$ for the $2 \times 2 \times 2$ hexagon?

In the next problem we will find an analogous set of paths for the domino tilings of the $m \times n$ rectangle.

2. Flux.



Colour the squares of an $n \times m$ rectangle in a chessboard pattern. The *flux* $\Phi(k)$ *through the kth slice* of a domino tiling is the sum of ± 1 for each domino perpendicular to the slice



where it is $+1$ or -1 depending on the colour of the square. (*The word "flux" means "flow", like the amount of water flowing through a pipe.*)

a. Pick an arbitrary domino tiling of a $3 \times n$ and a $4 \times n$ square, and draw $\Phi(k)$ for each.

b. What do you notice about the values of $\Phi(k)$? Form a conjecture, and prove it.

c. For a random domino tiling of the $2 \times n$ and $3 \times n$ rectangle, what is the expected value of the flux Φ?

5. Domino tilings: long problems

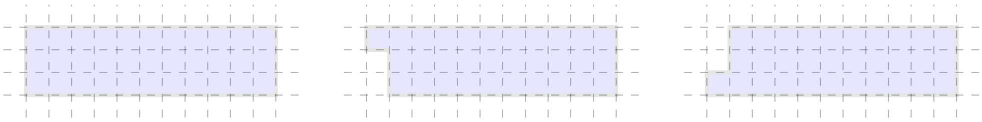
All of Mattekollo

- *Attempt problems in any order.*
- *Sometimes problems will use definitions intruduced in later classes. Feel free to use one problem you've solved to solve another, but if it trivialises the problem, see if you can find another proof as well!*
- *More long problems, or if necessary, hints, will be added a bit each lesson.*

Formulas.

1. Height three rectangles.

   a. Find a recurrence relation relating the numbers A_n, B_n, C_n of domino tilings of

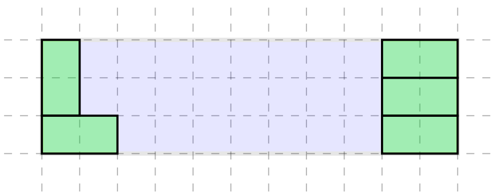
   where the base has length n.

   b. Use it to find a formula for A_n in terms of $A_{n-1}, A_{n-2}, \ldots, A_1$.

   c. Show that $A_n, B_n, C_n = 0$ if n is odd.

   d. Compute A_n for $n = 1, 2, \ldots, 10$. Can you find a formula for the number of domino tilings of the $3 \times n$ rectangle?

2. There are three possibilities for what a domino tiling looks of the $3 \times n$ rectangle at the left (or right) boundary, e.g. one left and one right boundary is



Let B_L, B_R be boundary states on the left and right. For each choice of B_L, B_R, what is the probability

$$\mathbf{P}(B_L \to B_R)$$

that a random domino tiling with state B_L on the left has state B_R on the right?

Make sure you do the $2 \times n$ case first!

3. Prove that the number of tilings of the $n \times n$ Aztec diamond is $2^{n(n-1)/2}$.

One method (no guarantee it works!): can you solve this by finding a set S of size $\binom{n}{2} = n(n-1)/2$ and a bijection

$$\{\text{domino tilings of the Aztec diamond}\} \simeq \{\text{finite subsets of } S\}?$$

4. Prove that the number of domino tilings of a dual $A \times B \times C$ hexagonal graph is

$$\prod_{a=1}^{A} \prod_{b=1}^{B} \prod_{c=1}^{C} \frac{a+b+c-1}{a+b+c-2}.$$

5. What is the expected number of 3d boxes in (the 3d interpretation of) a domino tiling of the dual $A \times B \times C$ hexagonal graph?

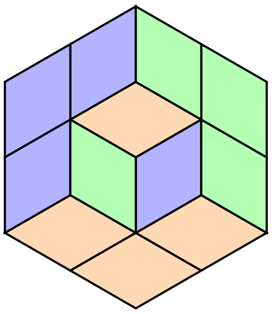
6. What is the number of domino tilings of the $m \times n$ rectangle?

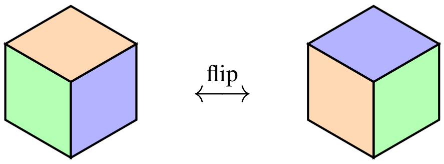
Lattice paths.

7. Count the number of tilings of the $3 \times n$ rectangle in terms of the number of paths on a certain graph.

   a. *Attempt if you know matrices.* Define a matrix M_3 such that the top left entry of M_3^n is the number of domino tilings of the $3 \times n$ rectangle.

8. Show that any tiling of the hexagonal $A \times B \times C$ graph by lozenges can be obtained from any other by a sequence of *flips*





   a. What does the hexagonal flip $T \leftrightarrow T'$ do to the height function? i.e. how is $h_{T'}$ related to h_T?

   b. What about the $m \times n$ rectangle? What about any square graph (subgraph of $\mathbf{Z}^2$)? Define a height function for the $m \times n$ rectangle which has an analogous property to a.

   c. Can you define analogous paths in the $m \times n$ rectangle attached to a tiling T?

Determinants.

9. Colour the faces of the hexagonal graph in alternating white and black colours. Write N for the number of white or black squares (which we assume are equal, otherwise there are no domino tilings). Let K be the $N \times N$ matrix of numbers

$$K_{i,j} = \begin{cases} 1 & \text{if the } i\text{th white vertex is adjacent to the } j\text{th black} \\ 0 & \text{otherwise.} \end{cases}.$$

Show that $|\det K|$ is the number of domino tilings of the hexagonal graph by lozenges.

Can you find a matrix $\tilde{K}$ that works for any planar bipartite graph?

Random walks through the graph of all tilings.

How do you in practice find a random lozenge tiling (there are *so many!*)? *Just start with any lozenge tiling, and randomly flip!*

Theorem 1. Let n be an integer and T_0 any lozenge tiling. Pick

- $t \in \{0, 1, \ldots, n\}$ randomly (uniformly),
- a random sequence of t many flips $T_0 \to T_1 \to \cdots \to T_t$, at each step choosing all allowed flips equally likely.

Then if

$$P(T) \;:\; \{\text{lozenge tilings}\} \;\to\; [0,1]$$

is the probability that $T_t = T$, we have

$$P(T) \;\to\; \frac{1}{|\{\text{lozenge tilings}\}|}$$

as $n \to \infty$.

10. *Good to practice linear algebra skills.* Let's prove Theorem 1.

a. Let G be an undirected graph, with set of vertices denoted by $V(G)$. Let

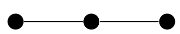
$$p_t(x) \;:\; V(G) \times \mathbf{N} \;\to\; [0,1] \tag{5.1}$$

be the probability that, at time $t = 0, 1, 2, \ldots$, a random walk starting at vertex i is at a given vertex x. Show that p_t satisfies

$$p_{t+1}(x) \;=\; \sum_{y \sim x} \frac{1}{\deg(y)} p_t(y) \tag{5.2}$$

and denote by $A : \mathbf{C}^{V(G)} \to \mathbf{C}^{V(G)}$ the associated linear map.

b. Compute (and graph) for low values of t what p_t is for the graph



if we start at the middle, or edge. Conjecture what happens as $t \to \infty$.

c. Show that the function $\lambda^t \cdot p_0$, where $p_0 : V(G) \to \mathbf{C}$ is a function independent of t and $\lambda \in \mathbf{C}$, satisfies (5.2) if and only if $\lambda = 0$, or

$$\lambda \cdot p_0(x) \;=\; A \cdot p_0(x).$$

In the following, you may assume that $p_t(x)$ is a sum of solutions of this form.

d. Assume from now on that G is connected. Pick N such that there is a path of length at most N between any two vertices, and let

$$B = \frac{1}{N}(I + A + \cdots + A^{N-1}).$$

Show that if $Av = \lambda v$, that

$$B \cdot v = \frac{1}{N}(1 + \lambda + \cdots + \lambda^{N-1})v.$$

e. Show that if $Bv = \mu v$ for some nonzero v, $|\mu| < 1$ unless for $\mu = 1$, in which there is a unique solution

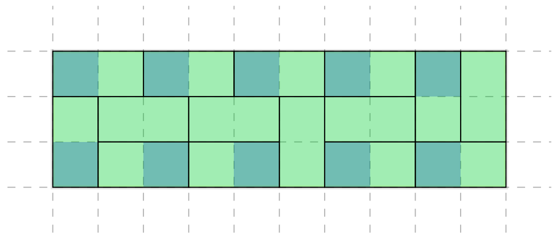
$$v = \frac{1}{|V(G)|}.$$

f. Show that this implies Theorem 1.

Spanning trees.

11. Temperley bijection.

a. Consider a certain square graph G within the $n \times m$ rectangle.



When $n = m$, give a bijection between domino tilings of the rectangle minus a corner and spanning trees of the graph G.

b. Can you find a similar result if you do not remove a corner?

c. What about if you remove an arbitrary subset of the boundary?

d. Can you get this to work for other n, m?

Dual graphs.

12. Let G be a graph drawn on the sphere.

a. Show that the *dual graph* $\widehat{G}$ is also a graph inside the sphere, and that $\widehat{\widehat{G}} = G$.

b. Show that G has no cycles if and only if $\widehat{G}$ is connected, and that G is connected if and only if $\widehat{G}$ has no cycles.

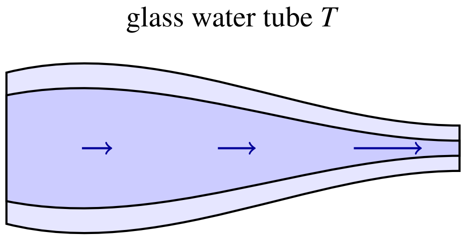
c. The *Euler characteristic* of a graph G with faces is defined as $\chi(G) = |V(G)| - |E(G)| + |F(G)|$, where $V(G)$ is the set of vertices, $E(G)$ is the set of edges, and $F(G)$ is the set of faces. Show that $\chi(G) = \chi(\widehat{G})$.

Now let G be a graph on the donut with $g > 0$ many holes. Is a. - c. still true? Prove it or find a counterexample.

6. Random graphs : networks

6 July

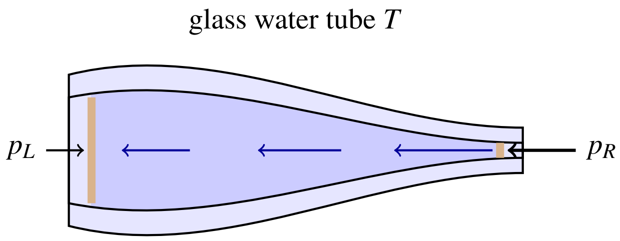
Imagine a water flowing through a glass tube



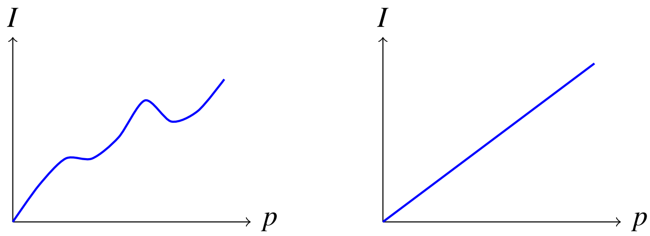
In the above, the speed of individual water molecules is marked.

> **Axiom.** **(Water is incompressible)** The volume per second I of water flowing through each vertical slice in the tube is the same.

Now place two frictionless wooden sheets at each end, and push different amounts on each end



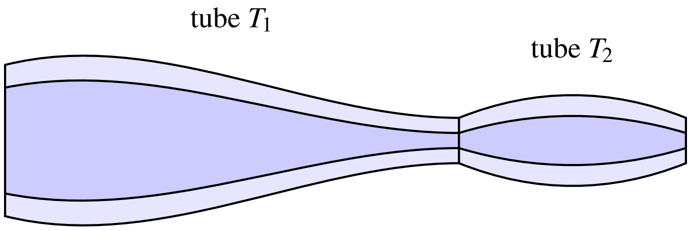
The flow of water I through the tube in response to the difference $p = p_L - p_R$ in pressure forms some graph, e.g.



The *resistance* of a tube T is $R(T) = \frac{d(p_L - p_R)}{dI}$. We will make the simplifying **assumption** that all above graphs are linear (which is approximately true for small p, I), so

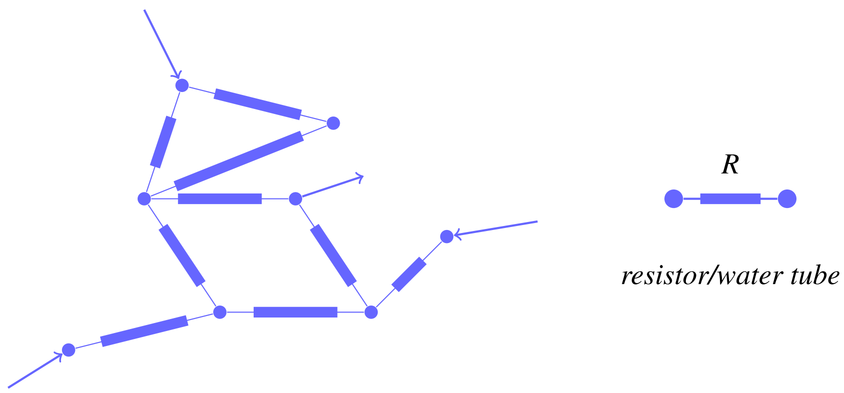
> **Definition.** The *resistance* of a tube T is the ratio
> $$R(T) = \frac{p}{I}.$$

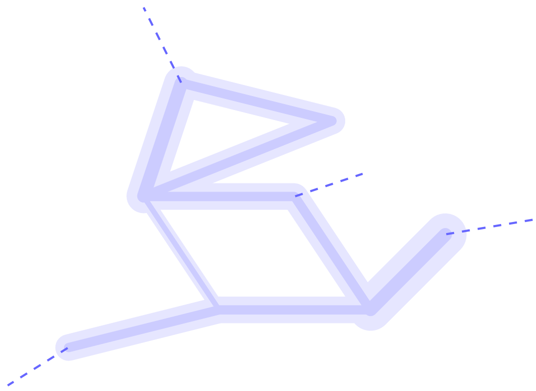
For instance, from this we can physically argue the **addition of resistances**, that putting two tubes T_1 and T_2 after one another adds their resistances:



$$R(T_1 \cdot T_2) \;=\; R(T_1) + R(T_2)$$

We will consider a network of tubes fused together to form a 3d glass surface, which has no holes except at a finite number of points, which we mark as external edges. The internal edges correspond to length of glass tube, and vertices correspond to vertical slices through the tube.





1. Make arguments for the following axioms:

Axiom. A *water network* W is a graph G with the additional data:

- numbers $p_x \in \mathbf{R}$ and $I_x \in \mathbf{R}$ attached to vertices x;
- numbers $R_e \in \mathbf{R}_{>0}$ and $I_e \in \mathbf{R}$ attached to edges $e : x \to y$;

such that the following axioms hold:

a. At each vertex x, we have

$$I_x + \sum_{y \sim x} I_{x,y} \;=\; 0.$$

b. $I_e R_e = p_y - p_x$ for each edge $e : x \to y$.

We call (p_x, I_x, R_e, I_e) the pressure, external flow, resistance and internal flow. We draw an exernal vertex above if $I_x \neq 0$. If $e : x \to y$ is an edge, we often write $R_{x,y}$ for R_e and $I_{x,y}$ for I_e. We write $V(G), E(G)$ for the vertices and edges of G, and $F(G)$ for the faces of G, if it is planar.

2. *Understanding the axioms.* Let W be a water network.

a. If $x_1 \to x_2 \to \cdots \to x_n \to x_1$ is a cycle in G with all $R_e = 1$, then

$$I_{x_1,x_2} + I_{x_2,x_3} + \cdots + I_{x_n,x_1} \;=\; 0.$$

b. Show that if all $I_x = 0$, then all $I_{x,y} = 0$ and p_x is constant on each connected component of G. *If there are no holes - no water moves! Hint:* show it for cyclic graphs and for trees first.

c. Show that given a water network with $p_{x_1} = p_{x_2}$, we get another water network by merging x_1 and x_2 to form a single vertex.

d. Consider the function $p : V(G) \to \mathbf{R}$ sending $t \mapsto p_t$. Show that $I_x = 0$ if and only if:

$$p_x \;=\; \sum_{y \sim x} w_{x,y} p_y$$

where $w_{x,y} = \frac{1/R_{x,y}}{\sum_{x \sim z} 1/R_{x,z}}$.

3. *Group structure of water networks.* Fix a graph G and resistance values $R_e > 0$ for each edge $e \in E(G)$. We denote a water network on (G, R_e) by $W = (p_x, I_e, I_x)$.

a. Show that the operation

$$(p_x, I_e, I_x) + (p'_x, I'_e, I'_x) \;=\; (p_x + p'_x, I_e + I'_e, I_x + I'_x)$$

gives a binary operation on the set of water networks for (G, R_e).

b. What is the unit and inverse to this operation?

c. Show that the set of water networks for (G, R_e) is a group under this operation.

Existence and uniqueness of water networks. Let Γ be a connected graph and let $w_{x,y} > 0$ be any collection of positive real numbers which is symmetric: $w_{x,y} = w_{y,x}$.

Theorem 2. The only functions $f : V(\Gamma) \to \mathbf{R}$ which satisfy the equation

$$f(x) \;=\; \sum_{y \sim x} w_{x,y} f(y), \qquad \text{for all } x \in V(\Gamma)$$

are the *constant functions*

$$f(x) \;=\; c,$$

where $c \in \mathbf{R}$ is a constant.

Now let $\partial\Gamma \subseteq \Gamma$ be a subgraph of Γ, which we calle the *boundary* of Γ.

Theorem 3. For any function $g : V(\partial\Gamma) \to \mathbf{R}$, there exists a unique function $f : V(\Gamma) \to \mathbf{R}$ such that

$$f(x) \;=\; \sum_{y\sim x} w_{x,y} f(y), \qquad \text{for all } x \in V(\Gamma \setminus \partial\Gamma)$$

and $f|_{\partial\Gamma} = g$.

This is a special case of the Dirichlet problem for harmonic functions, read up more on that if you're interested.

4. Use the above Theorems and problem 3 to show that, for a fixed connected graph G, resistance values $R_e > 0$ and external flows I_x, that there is a water network $W = (G, R_e, p_x, I_e, I_x)$, and that it is unique up to addition of a constant

$$p_x \;\mapsto\; p_x + c,$$

for some constant $c \in \mathbf{R}$.

7. Random graphs 2 & 3: walks and trees

7 July & 8 July

Definition. Let G be a connected graph and x, y be two vertices of G. The *effective resistance* between x and y is the positive real number

$$R_G(x,y)$$

defined as follows. *Pick any* water network $W = (G, R_e, p_x, I_e, I_x)$ on the graph G such that all $R_e = 1$, and $I_x = 1$, $I_y = -1$, and $I_z = 0$ for all other vertices z. Then we define $R_G(x,y) = p_x - p_y$.

For this definition to make sense,

- We need *at least one* water network W with the above properties, and
- The value of $R_G(x,y)$ should not depend on the choice of W: if $W^{(1)}, W^{(2)}$ are two such water networks, we should have $p_x^{(1)} - p_y^{(1)} = p_x^{(2)} - p_y^{(2)}$.

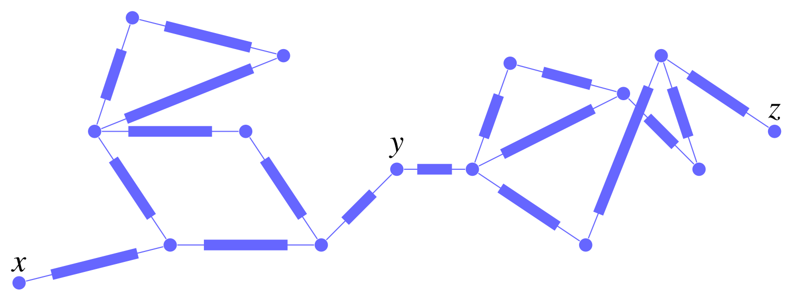
Both parts are true by problem 2f of the 6 July problem set - convince yourself that 2f indeed implies these.

1. Understanding effective resistance.

   a. Show that for a graph

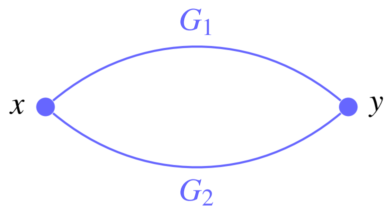
   the effective resistance is $R_G(x,y) = 1$. Note that this agrees with the previous definition of resistance.

   b. Write $G_1 \cdot_y G_2$ for the *join* of two graphs G_1 and G_2 along vertices $y_1 \in G_1$ and $y_2 \in G_2$, which is the graph formed by merging y_1, y_2 to a single vertex y:



   Given water networks W_1 on G_1 and W_2 on G_2, construct a new water network $W_1 \cdot_y W_2$ on $G_1 \cdot_y G_2$. Show that $R_{G_1 \cdot G_2}(x,z) = R_{G_1}(x,y) + R_{G_2}(y,z)$.

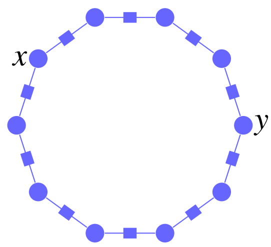
   c. Show that adding two graphs in parallel to form $G_1 + G_2$



gives $\frac{1}{R_{G_1+G_2}(x,y)} = \frac{1}{R_{G_1}(x,y)} + \frac{1}{R_{G_2}(x,y)}$.

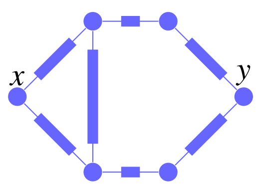
d. Compute the resistance between x and y in the above graph G_1.

2. Consider the circular graph



What is the effective resistance between a pair of points x and y?

3. Consider the graph



What is the effective resistance between a pair of points x and y?

Definition. A *random walk* on a graph G starts at a given vertex x and at each step moves to a neighbouring vertex, with each one chosen equally likely.

Definition. The *expected hitting time* $H(x,y)$ between $x, y \in G$ is the expected number of steps a random walk starting at x takes before first hitting y.

Definition. The *escape probability* $P(x,y)$ is the probability that a random walk starting at x will hit y before returning to x.

The edges of G have resistance 1 unless otherwise specified.

4. Resistances from random walks.

a. Show that the *commute time* $\kappa(x,y) = H(x,y) + H(y,x)$ is the expected number of steps a random walk starting at x takes to hit y, and then return to x. *Hint:* expectation is linear.

b. Show that

$$H(x,y) \;=\; 1 \;+\; \frac{1}{\deg(x)} \sum_{z\sim x} H(z,y).$$

c. Now externally add $\deg(z)$ units of water per second to each vertex z, and remove $\sum \deg(x) = 2|E(G)|$ units of water per second from y. Show that in the resulting water network W, we have

$$H(x,y) \;=\; p_y^W - p_x^W\,.$$

Hint. Show that $p_y - p_x$ satisfies the same relation as $H(x,y)$, then try to conclude using this.

d. Show that

$$R(x,y) \;=\; \frac{1}{2|E(G)|}\kappa(x,y)$$

where $|E(G)|$ is the number of edges in the graph G. *Hint:* Let W, W' and W'' be the water networks in the defintions of $H(x,y), H(y,x)$ and $R(x,y)$ respectively. Can you find a linear relation between these three water networks?

5. Extra problem. Give a new proof of

a. Parts a.-d. of Problem 1.

b. That in the graph of 1b., $R_G(x,y)$ does not change if we remove everything to the right of y.

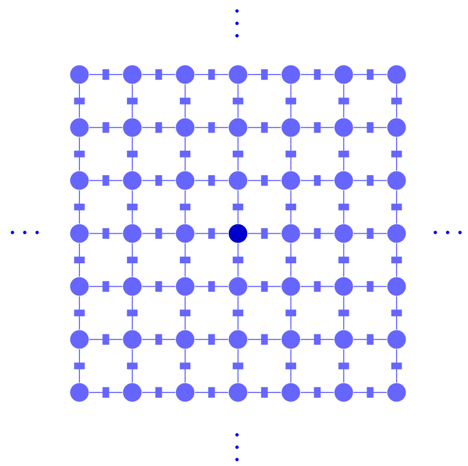
c. That if $\tilde{G}$ is the graph formed from G by removing an edge, $R_{\tilde{G}}(x,y) \geq R_G(x,y)$

8. Random graphs : drunkenness

9 July

Definition. Let A, B be subsets of a graph G. The *escape probability* $P(A,B)$ is the probability that a random walk starting at A will hit B before returning to A.

1. Consider the infinite square grid

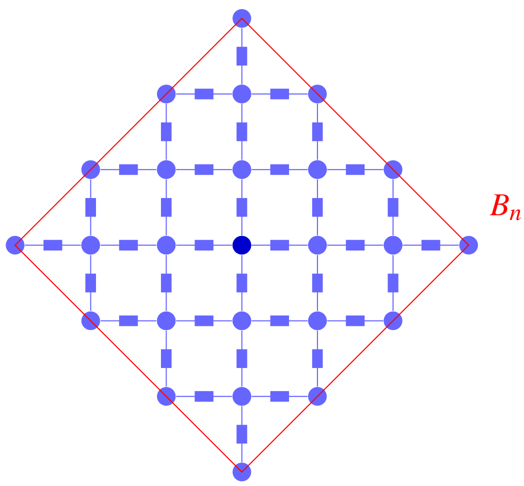


What is the probability P that a drunken person starting at the origin will ever return to the origin? Well, they might just walk left then right, or up then down etc., so it is it at least

$$P \geq \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{1}{4}.$$

But is $P = 1$?

a. Consider the $n \times n$ diamond grid



Show that $1 - P(0, B_n) \leq P$.

b. Argue that the drunk man doesn't return to the origin if and only if, for any n, they hit B_n before returning to the origin. Thus

$$P \;=\; 1 - \lim_{n\to\infty} P(0, B_n).$$

Theorem 4. Let G be a connected graph and $x, y \in V(G)$ be two vertices. Then

$$P(x,y) \;=\; \frac{1}{R_G(x,y)}$$

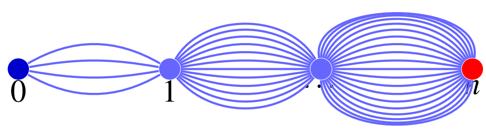
where $R_G(x,y)$ is the effective resistance between x and y in the graph G. Likewise, for $A, B \subseteq V(G)$, we have

$$P(A,B) \;=\; \frac{1}{R_{G/A\cup B}(\bullet_A, \bullet_B)}$$

is the effective resistance between $\bullet_A$ and $\bullet_B$ in the graph formed by crushing A and B into single points $\bullet_A$ and $\bullet_B$.

2. We now compute the resistance between 0 and B_n in the diamond graph, and use this to discover what P is.

a. Show that $R(0, B_n)$ is the same as the effective resistance R_n between 0 and n in the graph (edit this!!)



b. Show that there are $(2i+1)$ edges between the boundaries of the $i \times i$ and $(i+1) \times (i+1)$ diamonds.

c. Thus, compute what R_n is. *Hint:* use the rules for computing effective resistance in the graphs $G_1 \cdot G_2$ and $G_1 + G_2$ from last time.

d. Conclude that

$$\lim_{n\to\infty} R_n \;=\; \frac{1}{4} + \frac{1}{12} + \frac{1}{20} + \cdots \;=\; \sum_{i\geq 0} \frac{1}{4\cdot(2i+1)} \;=\; \infty.$$

e. **The random walker comes back!** Conclude that

$$1 - P \;=\; \lim_{n\to\infty} P(0, B_n) \;=\; \lim_{n\to\infty} \frac{1}{R(0,B_n)} \;=\; \lim_{n\to\infty} \frac{1}{R_n} \;=\; 0$$

so that the random walker starting at the origin will return to the origin with probability 1.

3. **Extra problem**. Generalise the above to the d-dimensional grid graph $\mathbf{Z}^d$. Show that the random walker starting at the origin will return to the origin with probability 1 if and only if $d \le 2$.

4. **Proving Theorem 3**: *gambler's ruin.* Let G be a graph and $A, B \subseteq G$ subsets.

a. Let $P_x = \mathbf{P}_x(\tau_A < \tau_B)$ be the probability that a random walk starting at x hits A before hitting B. Show that

$$P_x = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \in A. \end{cases}$$

b. Show that for $x \notin A \cup B$,

$$P_x = \frac{1}{\deg(x)} \sum_{z \sim x} P_z.$$

c. Set $I_{x,y} = P_x - P_y$. Show that if $x \notin A \cup B$ it satisfies

$$\sum_{y \sim x} I_{x,y} = 0.$$

d. Conclude that there is a water network $W = (G, 1, P_x, I_{x,y}, I_x)$, for a function $I_x : V(G) \to \mathbf{R}$ that you should compute, which satisfies $I_z = 0$ if $z \notin A \cup B$.

e. Show that this is the *unique* water network of the form

$$W = (G, 1, ?, ?, I_x)$$

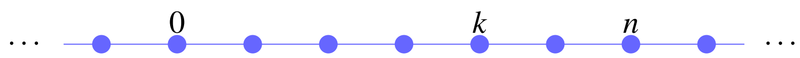
for the function I_x you defined. *Hint:* use 2f of the 6 July problem set.

Thus - we have an interpretation of $\mathbf{P}_x(\tau_A < \tau_B)$ as the pressure at x in a water network.

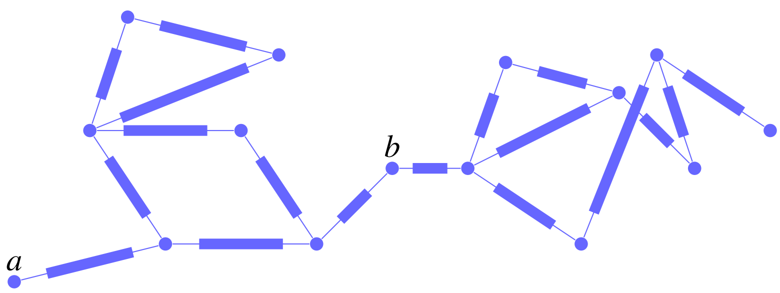
f. Conclude that if $a, b \in G$, (edit this!!)

$$P(a,b) = \frac{1}{\deg(a)} \sum_{z \sim a} P_z = \frac{1}{R_G(a,b)}.$$

5. A drunk man is walking randomly left and right, starting at $k \in \mathbf{Z}$. What is the probability he gets to n before returning to 0?



6. Now take a drunk man walking on



Above each vertex x in the graph, write the value of $\mathbf{P}_x(\tau_a < \tau_b)$, the probability that a random walk starting at x hits a before hitting b.