KZ EQUATIONS

ALEXEI LATYNTSEV

Let $V_1, ..., V_n$ be representations of a finite dimensional simple Lie algebra $\mathfrak{g}$. Pick extra data $\Omega \in \mathrm{Sym}^2\mathfrak{g}$ and $\hbar$ a formal parameter. Then the **KZ equations** are the following n many differential operators

$$\partial_{z_i} + \hbar \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

on the trivial vector bundle with fibre $V_1 \otimes \cdots \otimes V_n$ living over the space

$$(\mathbf{C}^n)_\circ = \{(z_1, \ldots, z_n) : z_i \neq z_j\}.$$

They define a vector bundle with connection on this space.

These notes are about the KZ equations, their many generalisations, and connections to various areas of mathematics and physics.

Contents

1. Motivational Introduction

1.1. **KZ equations from quantum field theory**.

1.1.1. Let $\mathcal{Z}$ be a topological quantum field theory.

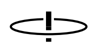
In other words, let $\mathrm{Cob}(n)$ be the n-category of cobordisms of topological manifolds, and

$$\mathcal{Z} \;:\; \mathrm{Cob}(n) \;\to\; \mathrm{Vect}(n)$$

be a dualisable symmetric monoidal n-functor. The dg category

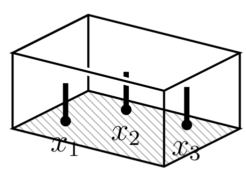
$$\mathcal{Z}(S^{n-2}) \;=\; \Omega^{n-1}\mathcal{Z}(\mathrm{pt}) \;\in\; \Omega^{n-2}\mathrm{Vect}(n) \;=\; \mathrm{dgCat}$$

is called the category of *line operators* of $\mathcal{Z}$.



$\mathcal{Z}(S^{n-2})$

Here we have taken a conormal sphere at a point to a line element $\ell \subseteq \mathbf{R}^n$. We can consider arbitrarily many line elements:



$\mathcal{X} \times \mathbf{R}$

Theorem 1.1.2. *For any topological $(n-1)$-manifold $\mathcal{X}$, we have a factorisable constructible sheaf of categories $\mathcal{C}^{\otimes}$ over $\mathrm{Ran}\mathcal{X}$, with fibre $\mathcal{Z}(S^{n-2})$ over each point.*

Proof. Note that a (category-valued) *constructible sheaf* on stratified space $\mathcal{Y} = \sqcup_{i\in\mathcal{S}}\mathcal{Y}_i$ is equivalent to a functor

$$\mathcal{F} \;:\; \mathrm{ExitPath}(\mathcal{Y};\mathcal{S}) \;\to\; \mathrm{Cat}$$

given by (vanishing cycle) parallel transport $\gamma : \mathcal{F}_x \to \mathcal{F}_y$ along paths which $\gamma : x \rightsquigarrow y$ nondecrease strata level (same is true replacing Cat with anything).

For a finite subset $\{x_1, \ldots, x_k\} \subseteq \mathcal{X}$, we define the fibre of $\mathcal{C}^{\otimes}$ to be

$$\mathcal{F}(\{x_1, \ldots, x_k\}) \;=\; \mathcal{C}^{\otimes}_{\{x_1,\ldots,x_k\}} \;=\; \mathcal{Z}(S^{n-1}_{x_1} \sqcup \cdots \sqcup S^{n-1}_{x_k})$$

where S^{n-1}_x is a small sphere around $x \in \mathcal{X}$.

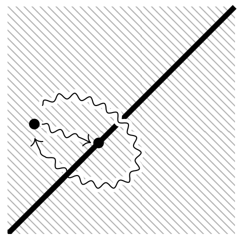
Likewise, if γ

□

1.1.3. *Remark.* For formal reasons, $\mathcal{Z}(S^{n-2})$ naturally carries an $\mathbf{E}_{n-1}$-algebra structure, and so by [Lub] we get the above Theorem for $\mathcal{X} = \mathbf{R}^{n-1}$.

1.1.4. *Remark.* We translate the above Theorem. The parallel transport to the diagonal gives a local system $\mathcal{C}_{\mathcal{X}}$, a monoidal structure on $\mathcal{C} \simeq \mathcal{C}_x$, and the associated $\beta \in \pi_{n-1}(\mathcal{X} \times \mathcal{X} \smallsetminus \Delta)$ gives the (higher) braiding

$\mathcal{X} \times \mathcal{X}$



$$\begin{tikzcd} \mathcal{C} \otimes \mathcal{C} \arrow[r, bend left, "\otimes_{\mathcal{C}}"] \arrow[r, bend right, "\otimes_{\mathcal{C}} \cdot \sigma_{\mathrm{Cat}}"'] \arrow[r, phantom, "\wr\Downarrow\beta"] & \mathcal{C} \end{tikzcd}$$

1.1.5. *Chern–Simons.* 3d Chern–Simons on $\mathbf{C} \times \mathbf{R}$, the above gives the braided monoidal structure on $\mathcal{C} = U_\hbar(\mathfrak{g})$-Mod (on $\mathbf{C}^\times \times \mathbf{R}$, need additional equivariance with respect to $\hbar \mapsto \hbar + 2\pi i \Rightarrow U_q(\mathfrak{g})$-Mod? $E \times \mathbf{R}$?)

1.2. Riemann–Hilbert.

1.2.1. Assume that $\mathcal{X} = |\mathcal{Y}|$ is the topological space underlying a complex manifold $\mathcal{Y}$. Scholze [Sc, p. II.3] has lifted the *Riemann-Hilbert* equivalence

$$\mathcal{D}\text{-Mod}_{qc}^{rh}(\mathcal{Y}) \ \xrightarrow{\sim}\ \mathrm{Sh}^{const}(\mathcal{X}), \qquad (E, \nabla) \ \mapsto\ E^{\nabla}$$

to an equivalence $\mathcal{Y}_{dR} \simeq \mathcal{Y}_B$ of analytic prestacks, by applying $\mathcal{D}_{qc}(-)$. Let us now *assume* there is such a result for sheaves of categories.

Getting KZ. Then if $\tilde{\mathcal{E}} = (\mathcal{E}, \nabla)$ is a sheaf of categories on $\mathcal{Y}$ with connection

$$\begin{tikzcd} & \tilde{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ \mathrm{QCoh}_{\mathcal{Y}} \arrow[ur, "E"] \arrow[r, dashed] & \mathcal{E} \arrow[r] & \mathcal{Y}_{dR} = \mathcal{Y}/e^{T_{\mathcal{Y}}} \end{tikzcd}$$

i.e. equivariance data $\Phi : a^*\mathcal{E} \simeq \mathrm{QCoh}_{\mathcal{Y}} \boxtimes \mathcal{E}$ with respect to the action of $a : e^{T_{\mathcal{Y}}} \times \mathcal{Y} \to \mathcal{Y}$.

A section $E \in \Gamma(\mathcal{Y}, \tilde{\mathcal{E}})$ is *flat*, i.e. is induced from a section of $\mathcal{E}$, if there is an isomorphism $\varphi : \Phi(a^*E) \simeq \mathcal{O}_{\mathcal{Y}} \boxtimes E$.

Proposition 1.2.2. *If* $\tilde{\mathcal{E}} = \mathrm{QCoh}_{\mathcal{Y}} \otimes_{\mathrm{Vect}} \mathcal{C}$ *(for a constant dg category* $\mathcal{C}$*), a flat connection on* $E = \mathcal{O}_{\mathcal{Y}} \boxtimes V$ *is equivalent to a map*

$$\nabla \ :\ \mathcal{T}_{\mathcal{Y}} \boxtimes V \ \to\ \mathcal{O}_{\mathcal{Y}} \boxtimes V$$

with flatness condition $[\nabla_{\xi_1}, \nabla_{\xi_2}] = \nabla_{[\xi_1, \xi_2]}$.

Thus, if the Riemann–Hilbert equivalence for sheaves of categories on Ran$\mathbf{C}$ holds, from the braided monoidal category Rep$U_\hbar(\mathfrak{g})$ we can apply the above argument to get a sheaf of categories with connection over $(\mathbf{C}^n)_\circ \to \mathrm{Ran}\mathbf{C}$

$$\tilde{\mathcal{E}} \ =\ \mathrm{QCoh}_{(\mathbf{C}^n)_\circ} \otimes_{\mathrm{Vect}} (\mathrm{Rep}U_\hbar(\mathfrak{g})^{\otimes n})$$

a flat structure on whose section $\mathcal{O}_{(\mathbf{C}^n)_\circ} \otimes V_1 \otimes \cdots \otimes V_n$ is precisely a connection, which presumably picks out the KZ equation uniquely up to isomorphism.

Defining wildly ramified Chern-Simons.

Question 1.2.3. *Porta–Teyssier* [PTa; PTb] *have defined ∞-category of Stokes exit paths. Is the data of dynamical KZ equivalent to extending the above to a factorisable functor*

$$\mathcal{F} \ : \ \mathrm{ExitPath}^{\mathrm{Stokes}}(\mathrm{Ran}\Sigma) \ \to \ \mathrm{Cat}$$

with $\mathcal{F}_x \simeq \mathrm{Rep}U_\hbar(\mathfrak{g})$?

2. The KZ equations

Let $V_1, \dots, V_n$ be representations of a finite dimensional simple Lie algebra $\mathfrak{g}$. Pick extra data $\Omega \in \mathrm{Sym}^2\mathfrak{g}$ and $k - k_{crit} \in \mathbf{C}$. Then the **KZ equations** are the following n many differential operators

$$(k - k_{crit})\partial_{z_i} \; + \; \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

on the trivial vector bundle with fibre $V_1 \otimes \cdots \otimes V_n$ on the space $(\mathbf{C}^n)_\circ$.

2.1. **Warmup computation.** If we consider the differential operators

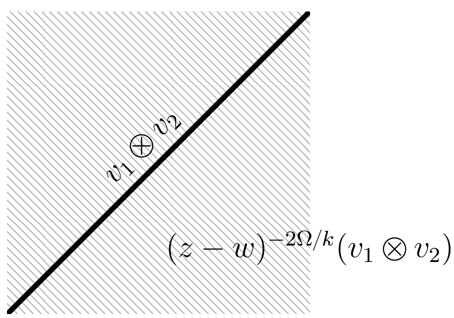
$$k\partial_z \; + \; \frac{\Omega_{12}}{z - w} \qquad\qquad k\partial_w \; + \; \frac{\Omega_{21}}{w - z}$$

then as Ω is symmetric solving these equations is equivalent to $\partial_{z+w} = 0$ and $\partial_{z-w} = \Omega/k(z - w)$. A solution to this is given by

$$v(z, w) \;=\; (z - w)^{-2\Omega/k}(v_1 \otimes v_2) \tag{1}$$

for any $v_i \in V_i$. In particular, the monodromy of this solution is given by $q^\Omega = e^{-\pi i \Omega/k}$.

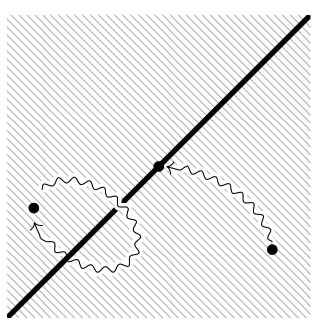
2.1.1. For the above solution (1), for (z, w) off the diagonal we would get the element $(z - w)^{-2\Omega/k}(v_1 \otimes v_2)$, and anywhere on the diagonal we would get $v_1 \otimes v_2$:



The monoidal structure

$$\otimes \;:\; \mathrm{Rep}U(\mathfrak{g}) \otimes \mathrm{Rep}U(\mathfrak{g}) \;\to\; \mathrm{Rep}U(\mathfrak{g})$$

looks like



Its braiding is given by monodromy around the diagonal; note that the braid group is

$$\mathfrak{B}_n \;=\; \pi_1((\mathbf{C}^n)_\circ).$$

Formal KZ equation. Note that the above only converges for $1/k$ small. Thus, we will consider the differential operators

$$\partial_{z_i} + \hbar \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

on the trivial vector bundle with fibre $V_1 \otimes \cdots \otimes V_n[[\hbar]]$ on the space $(\mathbf{C}^n)_\circ$.

Theorem. (Kohno-Drinfeld) *The induced associator and braiding gives* $\mathrm{Rep}U(\mathfrak{g})$ *a different braided monoidal structure, equivalent to* $\mathrm{Rep}U_\hbar(\mathfrak{g})$.

Proof. Given a solution to the KZ equations, we can take:

- take its value away from the diagonals,
- take its residue along a diagonal $z_i = z_j$ avoiding the other diagonals, i.e. take the coefficient of $(z_i - z_j)^{-2\Omega_{ij}/(k-k_{crit})}$,
- take its residue along two diagonals $z_i = z_j$ and $z_k = z_\ell$, avoiding the other diagonals, i.e. take the coefficient of $(z_i - z_j)^{-2\Omega_{ij}/(k-k_{crit})}(z_k - z_\ell)^{-2\Omega_{k\ell}/(k-k_{crit})}$, (need the Ω_{ij}s to commute)
- and so on,

to get an element of $V_1 \otimes \cdots \otimes V_n[[\hbar]]$ attached to every point of $\mathbf{C}^n$. This will be an algebraic function on each locally closed stratum. We may parallel transport between these, since in a neighbourhood of a diagonal there is a unique function on $(\mathbf{C}^n)_\circ$ with that as residue. (check)

We now endow $\mathrm{Rep}U(\mathfrak{g})$ with the same monoidal structure, but choose a different associator $(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$, given by parallel transport from the $z_1 = z_2$ diagonal to the $z_2 = z_3$ diagonal.[1]

□

Remark. More generally, we consider

$$(k - k_{crit})\partial_{z_i} + \sum_{i>j} r_{ij}(z_i - z_j) - \sum_{j<i} r_{ji}(z_j - z_i).$$

$r(z)$ satisfies the classical Yang Baxter equation, i.e. $R(z) = e^{\hbar r(z)}$ satisfies the spectral Yang Baxter equation, if and only if these differential operators commute.

Remark. (check) Note that for any permutation $\sigma \in \mathfrak{S}_n$ acting on $\mathcal{D}_{(\mathbf{C}^n)_\circ}$ preserves the above set of differential operators. However, (might be possible actually, check [GL]) we cannot arrange the above to form a D module on $\mathrm{Ran}\mathbf{A}^1$.

2.2. As a sheaf of categories.

[1]Since we have $\partial_{z_1+z_2} = \partial_{z_2+z_3} = 0$, this does not depend on where on the diagonals we pick. (check)

2.2.1. We have a constructible sheaf of categories $\mathcal{C}$ over Ran$\mathbf{C}$ by Lurie, with fibre

$$\mathcal{C}_{\{x_1,\dots,x_n\}} \;\simeq\; (\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\otimes n}.$$

Let us write $\mathring{\mathcal{C}}_n$ for its restriction to $(\mathbf{C}^n)_\circ$, which is a local system of categories or equivalently a crystal of categories $\mathring{\mathcal{C}}_n \in \mathrm{ShvCat}((\mathbf{C}^n)_{\circ,dR})$.

Conjecture 2.2.2. *$\mathring{\mathcal{C}}_n$ is a trivial sheaf of categories with a nontrivial connection,*

$$\mathring{\mathcal{C}}_n \;\simeq\; (\mathrm{QCoh}_{(\mathbf{C}^n)_\circ} \otimes (\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\otimes n}, \nabla_{\mathcal{C}}).$$

Assume this conjecture. For any section $s : \mathrm{triv} \to \mathrm{QCoh}_{(\mathbf{C}^n)_\circ} \otimes (\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\otimes n}$, which corresponds to

$$s \;=\; \mathcal{F} \otimes V_1 \otimes \cdots \otimes V_n,$$

we can ask whether it lifts along $\mathrm{ShvCat}((\mathbf{C}^n)_{\circ,dR}) \to \mathrm{ShvCat}((\mathbf{C}^n)_\circ)$ to a section

$$\tilde{s} \;:\; (\mathrm{triv}, 0) \;\to\; \mathring{\mathcal{C}}_n.$$

Conjecture 2.2.3. *A choice of lift $\tilde{s}$ corresponds to choosing a connection*

$$(\mathcal{F} \otimes V_1 \otimes \cdots \otimes V_n, 0 \boxplus \nabla_{V_i})$$

which is compatible with $\nabla_{\mathcal{C}}$, i.e. whose monodromy induces induces the braiding in $\mathrm{Rep}U_\hbar(\mathfrak{g})$. *The only such choice is*

$$\nabla_{V_i} \;\simeq\; \nabla_{\mathrm{KZ}}.$$

In other words - to ask if a section of a sheaf $s \in \Gamma(\mathcal{F})$ is flat is a *condition*, and to ask if a section $s \in \Gamma(\mathcal{C})$ of a sheaf of categories if flat is *data*.

3. What is parallel transport?

3.1. The R-matrix.

3.1.1. Let $\mathcal{C}$ be a perverse factorisation category over $\mathrm{Ran}X$. If $X \simeq \mathbf{R}^d$, then $\mathcal{C}_1$ is trivial, and therefore $\mathcal{C}_n|_{(X^n)_\circ}$ is trivial also, therefore its monodromy is trivial.

As $\mathcal{C}$ is a sheaf of categories on $\mathrm{Ran}X$, we have an isomorphism $\varphi_\sigma : \Delta_\sigma^* \mathcal{C}_2 \xrightarrow{\sim} \mathcal{C}_2$, and taking fibres gives

$$\begin{tikzcd} \mathcal{C}_{2,x,y} \arrow[rr, "\varphi_{\sigma,x,y}", "\sim"'] \arrow[dr, "\mathrm{Par}_\gamma"'] & & \mathcal{C}_{2,y,x} \arrow[dl, "\mathrm{Par}_{\overline{\gamma}}"] \\ & \mathcal{C}_{1,z} & \end{tikzcd}$$

where γ is a path from (x, y) to z, and in addition a two-isomorphism making the above diagram commute.

Note that if $\gamma : (0, 1] \to Y$ is a path and $\mathcal{C}$ a sheaf of categories on Y, then the parallel transport map is defined by the recollement map

$$\mathrm{Par}_\gamma \ = \ i^* j_* \ : \ (\gamma^*\mathcal{C})_{(0,1)} \ \to \ (\gamma^*\mathcal{C})_1$$

where j and i are the inclusion of the open disk and end of the disk, and we have taken global sections of the pulled back categories.

In the above Ran space example, we have that $\mathrm{Par}_\gamma = \otimes_{\mathcal{C}}$.

3.1.2. In the holomorphic case, we consider as in [AMR] the recollement map attached to $\gamma : \mathbf{C} \to Y$ as

$$\mathrm{Par}_\gamma \ : \ i^* j_* \ : \ (\gamma^*\mathcal{C})_{\mathbf{C}\setminus 0} \ \to \ (\gamma^*\mathcal{C})_0$$

where $(\gamma^*\mathcal{C})_0 = i^*(\gamma^*\mathcal{C})$ is the restriction to the completion $i : X_Z^\wedge \to X$. An example of this is [AMR, §1.1] the map $\mathrm{QCoh}(U) \to \mathrm{QCoh}(X_Z^\wedge)$. For intance, if $\gamma^*\mathcal{C} = \mathcal{E} \otimes_{\mathrm{Vect}} \mathrm{QCoh}_{\mathbf{C}}$ then the above is

$$\mathcal{E}_{\mathbf{C}\setminus 0} \ = \ \mathcal{E} \otimes_{\mathrm{QCoh}(\mathbf{C})} \mathrm{QCoh}(\mathbf{C} \setminus 0) \ \to \ \mathcal{E}_0[[z]] \ := \ \mathcal{E} \otimes_{\mathrm{QCoh}(\mathbf{C})} \mathrm{QCoh}(\mathbf{C}_0^\wedge)$$

which in the QCoh example, sends

$$i^* j_* \ : \ \mathrm{QCoh}(\mathbf{C} \setminus 0) \ \to \ \mathrm{QCoh}(\mathbf{C}_0^\wedge), \qquad\qquad V[z, z^{-1}] \ \mapsto \ V((z)).$$

In the above Ran space example, we have

$$\begin{tikzcd} \Gamma(\mathcal{C}_1 \boxtimes \mathcal{C}_1) \arrow[r, "\varphi_\sigma", "\sim"'] \arrow[d] & \Gamma(\mathcal{C}_1 \boxtimes \mathcal{C}_1) \arrow[d] \\ \Gamma(\mathcal{C}_{2,U}) \arrow[r, "\varphi_{\sigma,U}", "\sim"'] \arrow[d, "\mathrm{Par}_\gamma"'] & \Gamma(\mathcal{C}_{2,U}) \arrow[d, "\mathrm{Par}_{\overline{\gamma}}"] \\ \Gamma(\mathcal{C}_{2,X_Z^\wedge}) \arrow[r, "\varphi_{\sigma,X_Z^\wedge}"] & \Gamma(\mathcal{C}_{2,X_Z^\wedge}) \end{tikzcd}$$

In examples, we expect that $\mathcal{C}_{2,X^\wedge_Z} \simeq \mathcal{C}_1[[z]] := \mathcal{C}_Z \otimes_{\mathrm{QCoh}_Z} \mathrm{QCoh}_{X^\wedge_Z}$, where

$$\varphi_{\sigma,X^\wedge_Z} \;=\; \mathrm{id} \otimes (z \mapsto -z) \;:\; \mathcal{C}_1[[z]] \;\xrightarrow{\sim}\; \mathcal{C}_1[[z]].$$

In the Ran space and $\mathcal{C} = \mathrm{QCoh}_{\mathrm{Ran}\mathbf{A}^1}$ example, we have

$$i^*j_* \;:\; \mathrm{QCoh}(\mathbf{A}^2 \smallsetminus \Delta) \;\to\; \mathrm{QCoh}(\mathbf{A}^{2\;\wedge}_{\Delta}), \qquad (V[z] \boxtimes W[w])[(z-w)^{-1}] \;\mapsto\; (V \otimes W)[z]((z-w)).$$

and restricting to $\gamma : \mathbf{C} \to \mathbf{A}^2$ the antidiagonal gives

$$i^*j_* \;:\; \mathrm{QCoh}(\mathbf{C} \smallsetminus 0) \;\to\; \mathrm{QCoh}(\mathbf{C}^\wedge_0), \qquad (V \boxtimes W)[z, z^{-1}] \;\mapsto\; (V \otimes W)((z)).$$

We expect $\mathrm{Par}_\gamma = \otimes_{\mathcal{C},z}$.

3.1.3. *Remark.* Note that if we remove the data of the φ_σ isomorphisms for $\sigma \in \mathfrak{S}_n$, i.e. we consider factorisation categories over $\mathrm{Ran}^{\mathrm{ord}}\mathbf{R}^d$, we lose the data of the R-matrix, as it should be.

3.1.4. *Noncommutative case.* One expects that one can use [AMR] to define parallel transport maps for sheaves of categories over noncommutative spaces.

3.2. **Setup.**

3.2.1. The braided monoidal structure on $\mathrm{Rep}U_q(\mathfrak{g})$ is packaged into the exodromy of a constructible factorisation sheaf of categories $\mathcal{C}_{\mathrm{fin}}$ on $\mathrm{Ran}\mathbf{C}^\times$.

Question 3.2.2. *What is the analogous statement for* $\mathrm{Rep}U_q(\widehat{\mathfrak{g}})$*?*

We expect the data to be packaged into a quasicoherent–constructible sheaf of categories $\mathcal{C}$ on $\mathrm{Ran}(\mathbf{R} \times_{\mathbf{Z}} \mathbf{C}^\times)$, where $\mathbf{Z}$ acts by $\log|q| +$ on the left and $q\cdot$ on the right. In particular, we get

- a constructible factorisation sheaf of categories $\mathcal{C}_{\mathbf{R},z} = \mathcal{C}|_{\mathrm{Ran}\mathbf{R}}$ on the Ran space of $\mathbf{R} \times_{\mathbf{Z}} q^{\mathbf{Z}} z \simeq \mathbf{R}$,
- a quasicoherent factorisation sheaf of categories $\mathcal{C}_{\mathbf{C}^\times,r} = \mathcal{C}|_{\mathrm{Ran}\mathbf{C}^\times}$ on the Ran space of $(r + \mathbf{Z} \cdot \log|q|) \times_{\mathbf{Z}} \mathbf{C}^\times \simeq \mathbf{C}^\times$.

Note that the maps $\mathbf{R}, \mathbf{C}^\times \to \mathbf{R} \times \mathbf{C}^\times$ are $\mathbf{Z}$-equivariant, where we consider the antidiagonal action on the product, so we get maps $\mathbf{R}/\mathbf{Z}, \mathbf{C}^\times/\mathbf{Z} \to \mathbf{R} \times_{\mathbf{Z}} \mathbf{C}^\times$. It follows that $\mathcal{C}_{\mathbf{R},z}$ and $\mathcal{C}_{\mathbf{C}^\times,r}$ are factorisably $\mathbf{Z}$-equivariant.[2]

Assuming a relative version of Dunn additivity for stratified factorisation categories conjectured in [Be], we expect

$$\mathrm{FactCat}(\mathbf{R} \times_{\mathbf{Z}} \mathbf{C}^\times) \;=\; \mathrm{FactAlg}(\mathbf{R}/\mathbf{Z} \times_{\mathrm{B}\mathbf{Z}} \mathbf{C}^\times/\mathbf{Z}; \mathrm{Cat}) \;\overset{?}{\simeq}\; \mathrm{FactAlg}_{\mathrm{B}\mathbf{Z}}(\mathbf{R}/\mathbf{Z}; \mathrm{FactCat}(\mathbf{C}^\times/\mathbf{Z}))$$

where on the right we consider factorisation algebras with base in the category of spaces with an action of $\mathrm{B}\mathbf{Z}$.

[2] i.e. are pullbacks of factorisable sheaves of categories on $\mathrm{Ran}(\mathbf{R}/\mathbf{Z})$ and $\mathrm{Ran}(\mathbf{C}^\times/\mathbf{Z})$.

Question 3.2.3. *Does* $\mathrm{Rep}U_q(\widehat{\mathfrak{g}})$ *carry a crystal structure when factorised over* $\mathrm{Ran}(\mathbf{C}/\mathbf{Z})$*? If so, is the monodromy around the diagonal equal to* $R^0(z)$*?* ***Answer:*** No!

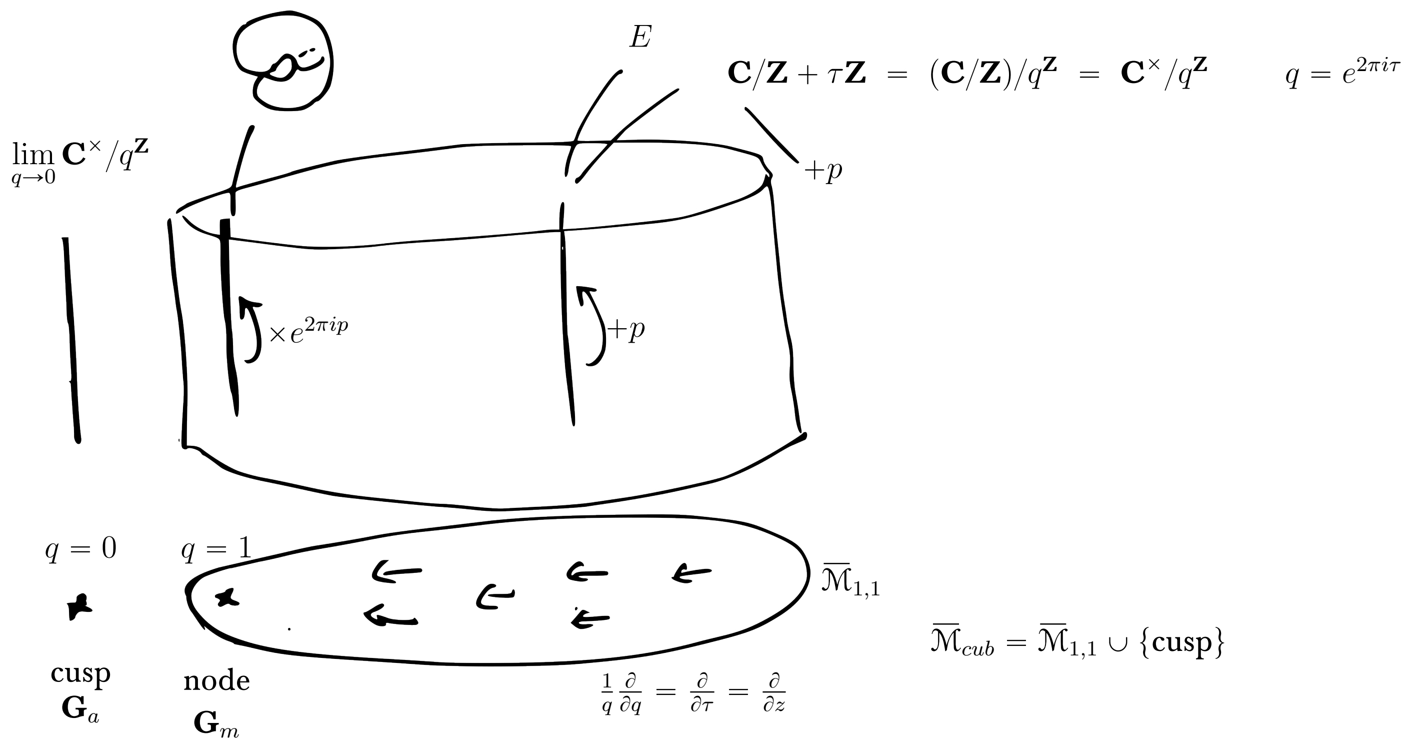
In more basic terms, we are asking about whether the qKZ equation is related to a differential equation, or at least something you can define the monodromy around the diagonal of.

3.2.4. *Other questions.*

Question 3.2.5. *What is the precise relation between abelian qKZ and full qKZ? Something to do with the Gauss decomposition* $R(z) = R^-(z)R^0(z)R^+(z)$ *of the* R*–matrix.*

3.3. **Answers.**

4. The additive, multiplicative, and heat qKZ equations



$$\mathbf{P}(4,6)_z = \overline{\mathcal{M}}_{1,1} = (\mathbf{H}_\tau \cup \mathbf{P}(\mathbf{Q}))/\mathrm{PSL}(2,\mathbf{Z})$$

The variants of the qKZ equations are all equivariance with respect to various automorphisms of the universal curve $\mathcal{E}$. The vector fields ∂_z (should be $\bar{\partial}_z$?) are *gauge fields* coming from the fact that the HT theory depends holomorphically on the curve's moduli. The discrete automorphism by $\mathbf{Z}$ is given by the fact that we are evaluating the HT theory on the *quotient* of the HT manifold

$$(\mathbf{R}\times\mathcal{E})/\mathbf{Z} \;\to\; \overline{\mathcal{M}}_{cub}.$$

The $q=1$ limit gives the multiplicative KZ equation, and the exponential map

$$\exp \;:\; \mathbf{C} \;\to\; \mathbf{C}^\times \;\subseteq\; E_{\text{node}}$$

gives the additive qKZ equation.

In physics, $(p, e^{2\pi ip})$ is usually written $(\hbar, q)$, but this is not the same as the q above so we do not to avoid confusion. In particular, there are *four* limits one can consider

	$q\to 0$	$q\to 1$
$\hbar\to 0$		
$\hbar\to\infty$		

where the q limits correspond to $\mathbf{G}_a$ and $\mathbf{G}_m$ formal group laws;[3] note that we have a family of formal group laws over $\overline{\mathcal{M}}_{cub}$, induced by the relative pointed curve $\mathcal{E}$.

(mild correction - in [FVb] we take *difference* equations in τ (and *this* is called the qKZB heat equation, the difference equation $+p$ is just called qKZ))

[3] See https://www.math.wustl.edu/~matkerr/436/ch17.pdf

5. Varying the combinatorial data

5.1. **Boundary KZ and other singularities.** In this section we will show how to get variants of the KZ equations by chaging the combinatorial data of how points are allowed to merge. This data is encoded in a Dynkin diagram (or more generally, a Coxeter diagram) S, and we can define the S-*Ran space*

$$\mathrm{Ran}_S\mathbf{C}$$

for any such, and

- when we take the collection of linear A_n Dynkin diagrams, we get back the usual Ran space $\mathrm{Ran}\mathbf{C}$,
- when we take the odd orthogonal B_n Dynkin diagrams, we get the orbifold Ran space $\mathrm{Ran}(\mathbf{C}/\pm)$,
- similar examples for other classical types,
- when we take the affine linear $\hat{A}_n$ Dynkin diagram, we get approximately[4] $\mathrm{Ran}(\mathbf{C}/\mathbf{Z})$, where $\mathbf{Z}$ acts by translation on $\mathbf{C}$,
- when we take double affine Kac-Moody groups, get approximately $\mathrm{Ran}(\mathbf{C}/\mathbf{Z}\oplus\tau\mathbf{Z})$ where τ lies in the upper half plane.

The last two will be related to changing the base curve of the Ran space, considered in the previous section. (I think?)

5.2. **Examples.**

5.2.1. The boundary KZ equation looks like

$$\partial_{z_i} \; + \sum_{i\neq j}\frac{\Omega_{ij}}{z_i - z_j} \; + \; \frac{k_i}{z_i}.$$

Moreover, we have other KZ equations, with poles at:

- $z_i = z_j$, as usual,
- $z_i = z_j$ and $z_i = 0$, as above,
- $z_i = \pm z_j$,
- $z_i = \pm z_j$ and $z_i = 0$,

which depends on a choice of root system. These should arise from factorisation algebras living over:

- $\mathrm{Ran}\Sigma$, as usual,
- $\mathrm{Ran}(\Sigma \setminus 0)$,

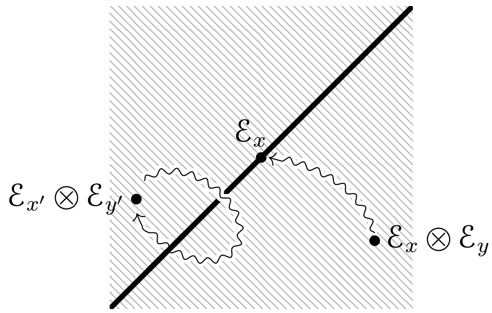
[4] Its category of quasicoherent sheaves will agree with $\mathrm{QCoh}(\mathrm{Ran}(\mathbf{C}/\mathbf{Z}))$ at least.

- $\mathrm{Ran}(\Sigma/(\mathbf{Z}/2))$,
- $\mathrm{Ran}((\Sigma \setminus 0)/(\mathbf{Z}/2))$.

6. Topological picture: perverse sheaves of categories

How to think of all this structure? (maybe we need to consider $\mathrm{Conf}(\mathbf{C})$ instead of $\mathrm{Ran}\mathbf{C}$?)

The answer is as a constructible sheaf of categories over $\mathrm{Ran}\mathbf{C}$, which factorises.



Here, the Ran space is endowed with the stratification by diagonals, and we have a constructible sheaf of categories on each $\mathbf{C}^n$, i.e. a functor

$$\mathcal{E}|_{\mathbf{C}^n} \; : \; \mathrm{Exit}_{\mathbf{C}^n} \; \to \; \mathrm{dgCat}$$

from the category of paths staying within the same strata except at the endpoints. Over $n = 1$ this just gives a category $\mathcal{E}_1$, and considering $n = 2$ gives it a braided monoidal structure, and then $n \geqslant 3$ corresponds to higher homotopy data.

6.1. **Drinfeld-Kohno Theorem**.

6.1.1. Recall from Lurie that there is an equivalence of categories due to Lurie [Lua, §A.6]

$$\mathbf{E}_n\text{-Alg}(\mathcal{C}) \;\simeq\; \mathrm{FactAlgCoSh}_{const}(\mathrm{Ran}\mathbf{R}^n, \mathcal{C})$$

between $\mathbf{E}_n$-algebras in ambient symmetric monoidal category $\mathcal{C}$, and commutative factorisation algebras in the category of $\mathcal{C}$-valued constructible cosheaves on the Ran space of $\mathbf{R}^n$.

When $\mathcal{C} = \mathrm{dgCat}$, there is in [CF, p. 6.3.3] an explicitly constructed functor

$$\mathrm{Fact} \; : \; \mathbf{E}_2\text{-Alg}(\mathrm{dgCat}) \; \to \; \mathrm{FactCat}_{const}(\mathrm{Ran}\mathbf{C}).$$

6.1.2. We may now restate the Drinfeld-Kohno Theorem as:

Lemma 6.1.3. *There is a* ***Drinfeld-Kohno*** *constructible sheaf of categories* $\mathrm{FactRep}U_\hbar(\mathfrak{g})$ *which has the following properties:*

- *Its fibre over* $(z_1, ..., z_n)$ *is spanned as a dg category by tuples* $V_1 \boxtimes \cdots \boxtimes V_n$ *of element of* $(\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\otimes n}$*, where* V_i *are representations of* $\mathfrak{g}$.[5]
- *The exit path sending* $z_i \to z_j$ *is sent to the functor* $\mathrm{Rep}U_\hbar(\mathfrak{g})_i \otimes \mathrm{Rep}U_\hbar(\mathfrak{g})_j \to \mathrm{Rep}U_\hbar(\mathfrak{g})$ *given by* $V_i \boxtimes V_j \mapsto V_i \otimes V_j$.

[5]Note that $\mathrm{Rep}U(\mathfrak{g}) \simeq \mathrm{Rep}U_\hbar(\mathfrak{g})$ as categories if we forget the braided monoidal structure.

- *The monodromy around the diagonal $z_i = z_j$ is given by the endomorphism of* $(\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\otimes n}$ *given by swapping the two factors* $V_i \boxtimes V_j \mapsto V_j \boxtimes V_i$.
- *The contractible two-cell bounded by a loop around $z_i = z_j$ and two exit paths is the natural transformation*

$$\begin{tikzcd} \mathrm{Rep}U_\hbar(\mathfrak{g}) \otimes \mathrm{Rep}U_\hbar(\mathfrak{g}) \arrow[dr, bend left, "\otimes"] \arrow[d, "\wr\downarrow\sigma"'] & \\ \mathrm{Rep}U_\hbar(\mathfrak{g}) \otimes \mathrm{Rep}U_\hbar(\mathfrak{g}) \arrow[r, bend right, "\otimes"'] \arrow[ur, phantom, "\wr\Uparrow"] & \mathrm{Rep}U_\hbar(\mathfrak{g}) \end{tikzcd}$$

given on objects by the endomorphism $R = e^{\hbar\Omega} : V_i \otimes V_j \to V_j \otimes V_i$.

Likewise, it relates to the KZ equations as follows:

- *a flat section* $v_1(z) \otimes \cdots \otimes v_n(z) : \mathrm{triv} \to \mathrm{FactRep}U_\hbar(\mathfrak{g})$ *over an open set* $U \subseteq (\mathbf{C}^n)_\circ$ *is precisely a solution to the KZ equations for* $V_1 \otimes \cdots \otimes V_n$ *on* U.

Proof. This follows from the braided monoidal structure on $\mathrm{Rep}U_\hbar(\mathfrak{g})$. □

6.1.4. Note that if we were to consider other base curves, the restriction $\mathcal{E}_1$ becomes interesting. Whereas over $\mathbf{C}$ it only has the structure of a category, over $\mathbf{C}^\times$ and E it has one and two commuting automorphisms, which the structures we discuss above must respect. For instance, writing T for such an automorphism, we have

$$T(V \otimes V') \;=\; T(V) \otimes T(V')$$

respects the monoidal structure, and likewise the braiding.

If $\mathcal{E}_E$ is any such constructible factorisation category on an elliptic curve, we have functors

$$\mathcal{E}_1 \;=\; \Gamma(\mathbf{C}, \mathcal{E}_{\mathbf{C},1}) \;\overset{\exp^*}{\leftarrow}\; \Gamma(\mathbf{C}^\times, \mathcal{E}_{\mathbf{C}^\times,1}) \;\overset{\pi^*}{\leftarrow}\; \Gamma(E, \mathcal{E}_{E,1}).$$

Moreover, one expects a Galois correspondence between subcategories of $\mathcal{E}_1$ and subgroups of $\pi_1(E)$, and the above we expect is equal to

$$\mathcal{E}_1 \;\overset{\exp^*}{\leftarrow}\; \mathcal{E}_1^{\mathbf{Z}} \;\overset{\pi^*}{\leftarrow}\; \mathcal{E}_1^{\mathbf{Z}^2}.$$

For instance, the deck cover group of $\exp$ is generated by $\hbar \mapsto \hbar + 2\pi i$, so this conjecture is saying that

$$(\mathrm{Rep}U_\hbar(\mathfrak{g}))^{\mathbf{Z}} \;\overset{?}{\overset{\sim}{\leftarrow}}\; \mathrm{Rep}U_q(\mathfrak{g}).$$

This should extend to the entire constructible sheaves of categories, however we note that $\mathrm{Ran}\mathbf{C} \to \mathrm{Ran}\mathbf{C}^\times$ is not a $\mathbf{Z}$-covering map. We do not know the definition of $\mathrm{Rep}U_{q,t}(\mathfrak{g})$, but presumably if the above is correct it should be $\mathbf{Z}^2$-invariants inside $\mathrm{Rep}U_\hbar(\mathfrak{g})$.

The action of $\mathbf{Z}$ on the category $\mathcal{E}_{\mathbf{C},z} \simeq \mathcal{E}_{\mathbf{C}^\times,e^z}$ is given by the monodromy of the trigonometric KZ equation, computed in [EG, Thm. 3.2] to be

$$\tau \;=\; e^{\hbar(s+m(r))}m(R) \;=\; q^{s+m(r)}m(R)$$

where we have contracted $r = \Omega$ using the multiplication m in $U_\hbar(\mathfrak{g})$ and s is any even element with $[\Delta(s), \Omega] = 0$.

6.1.5. *Remark.* The inclusion[6] $U_q(\mathfrak{g}) \hookrightarrow U_\hbar(\mathfrak{g})$ allows us to form[7]

$$\mathrm{Ind} \;:\; \mathrm{Rep}U_q(\mathfrak{g}) \;\leftrightarrows\; \mathrm{Rep}U_\hbar(\mathfrak{g}) \;:\; \mathrm{Res} \;=\; \exp^* .$$

We do not know how to interpret Ind in terms of the constructible sheaf of categories.

6.1.6. *Partial inverses to* $\exp$ *and* π. Given a branch of the logarithm, i.e. a partially defined section $\log : \mathbf{C}^\times \to \mathbf{C}$ to the exponential map, we can consider the function on $(\mathbf{C}^\times)^n_\circ$

$$\log^*(z_i - z_j) \;=\; \log(z_i) - \log(z_j) \;=\; \log(z_i/z_j) \;=\; (1 - z_i/z_j) + \frac{1}{2}(1 - z_i/z_j)^2 + \cdots$$

thus $1/\log^*(z_i - z_j)$ is gauge-equivalent to $1/(1 - z_i/z_j)$. Likewise, $\log_*(z\partial_z) = \partial_z$.[8]

6.1.7. *Remark: affine analogue.* Why couldn't we have just applied the above section to $\mathfrak{g}$ an arbitrary Kac-Moody Lie algebra?

One answer is that we can of course define the equations, but since $\mathrm{Rep}U_\hbar(\mathfrak{g})$ is factorisation braided rather than braided, the Drinfeld-Kohno and Gaitsgory-Lysenko constructions cannot have applied in their usual forms.

6.2. **Comparison to [GL].**

6.2.1. Gaitsgory and Lysenko have studied quantum groups arising from perverse factorisation algebras on the configuration space $\mathrm{Conf}_\Lambda(\mathbf{C})$, as opposed to the Ran space $\mathrm{Ran}(\mathbf{C})$.

6.2.2. In [Ga], one considers the configuration space $\mathrm{Conf}_\Lambda(\mathbf{C})$ of ordered points labelled by nonnegative roots Λ.

One constructs a factorisable $\mathrm{B}\mathbf{G}_m$ gerbe $\mathcal{G}$ on $\mathrm{Conf}_\Lambda(\mathbf{C})$, and consider $\mathcal{G}$-twisted sheaves. Moreover (in [GL] somewhere) we have

$$\mathrm{Rep}_q(T) \;\simeq\; \mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C})).$$

The three integral forms of $U_q(\mathfrak{b})$ are constructed as pushforwards of constant sheaves from the open locus.

In [GL] one defines $u_q(N)$ inside $\mathrm{Rep}_q(T)$, then takes the relative Drinfeld double of $u_q(N)$-Mod($\mathrm{Rep}_q(T)$) to get $u_q(\mathfrak{g})$-Mod. We get the (baby) renormalised version of this if we take the ind-completion with respect to finite dimensional modules (resp. before taking the Drinfeld double).

[6]As $\mathbf{Z}[q, q^{-1}] \hookrightarrow \mathbf{Z}[[\hbar]]$-algebras.

[7]Here $\mathrm{Ind}V = V \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Z}[[\hbar]]$; in particular Ind might send non-isomorphic representations V, V' to isomorphic ones. We have an embedding $V \hookrightarrow V \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Z}[[\hbar]] = \mathrm{ResInd}V$.

[8]This again follows from the chain rule, $z\partial_z = \partial_{\log(z)}$.

Then as in [GL, p202], we apply Lurie's construction of a factorisation algebra $\Omega_B \in \mathcal{D}\text{-Mod}(\text{Ran}\mathbf{A}^1)$ attached to any $\mathbf{E}_2$-algebra B in braided monoidal category, with

$$\Omega_B\text{-FactMod}(\text{Gr}_{T,\mathbf{A}^1}) \;\simeq\; B\text{-Mod}_{\mathbf{E}_2}.$$

We apply this to $B = \text{Aug}(\text{inv}_{u_q(\mathfrak{n})})$ being the augmentation ideal of the invariants functor

$$\text{inv}_{u_q(\mathfrak{n})} \;:\; u_q(\mathfrak{n})\text{-Mod} \;\to\; \text{Rep}_q(T)$$

which for general reasons has $B\text{-Mod}_{\mathbf{E}_2} \simeq Z_{\mathbf{E}_1}(u_q(\mathfrak{n})\text{-Mod}^{ren})$. On the other side, we have by a Riemann-Hilbert argument that factorisation modules over $\text{Gr}_{T,\mathbf{A}^1}$ are equivalent to configuration factorisation modules over $\mathbf{C}$, and under this equivalence we have $\Omega_B\text{-FactMod}(\text{Gr}_{T,\mathbf{A}^1}) \simeq \Omega_q^{sm}\text{-FactMod}(\text{Conf}_\Lambda(\mathbf{C}))$.

Theorem 6.2.3. *(prove this) The constructible factorisation category over* $\text{Conf}_\Lambda(\mathbf{C})$

$$u_q(\mathfrak{g})\text{-Mod}^{\textit{baby ren}} \;\simeq\; B\text{-Mod}_{\mathbf{E}_2} \;\simeq\; \Omega_B\text{-FactMod}(\text{Gr}_{T,\mathbf{A}^1}) \simeq \Omega_q^{sm}\text{-FactMod}(\text{Conf}_\Lambda(\mathbf{C}))$$

has sections being collections of $V_1, ..., V_n$ *together with their KZ equations over* $\mathbf{C}^n$.

Proof. The equivalences follow by the above discussion □

There is a factorisable version of (a completion of) $u_q(\mathfrak{g})$-Mod over $\text{Conf}_\Lambda(\mathbf{C})$. It equivalent to factorisation modules over Ω_q.

6.2.4. (copy the Conf-Ran section here)

6.3. Relation to Riemann-Hilbert.

6.3.1. All the above is on the topological side; we now talk about how to pass to the algebraic side. As explained (where? 6.3.3 doesn't do it) there is a functor

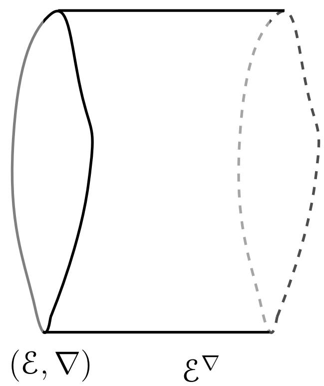
$$\mathbf{E}_2\text{-Cat} \;\to\; \text{FactCat}(\text{Ran}\mathbf{A}^1)$$

compatible with the global sections functor (check). It sends (find reference)

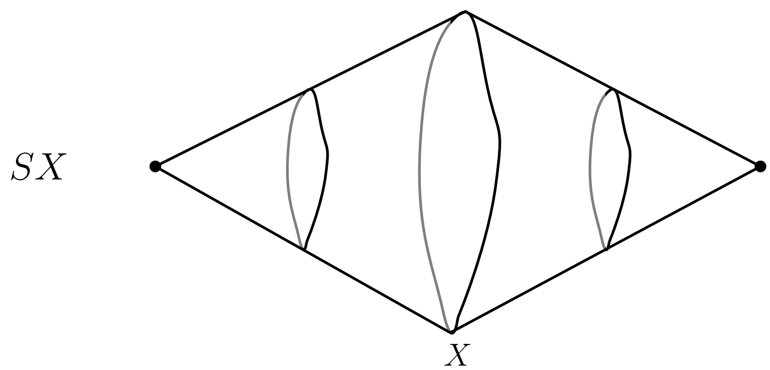
$$U_q^{Lus}(\mathfrak{g})\text{-Mod} \;\mapsto\; \hat{\mathfrak{g}}\text{-Mod}^{G(\mathcal{O})}, \qquad\qquad \text{Rep}_q(G)^{mxd} \;\mapsto\; \hat{\mathfrak{g}}\text{-Mod}^{I}.$$

This is used crucially in the proof of [CF].

6.3.2. Loosely speaking, Riemann-Hilbert should be viewed as passing to the bulk:



6.3.3. In particular, we would like to view the Riemann-Hilbert construction on a complex manifold X as somehow living over its suspsension SX, whose endpoints lose the topological-holomorphic structure: there is no compatible codimension one foliation of SX by complex submanifolds.



We propose that we should consider the perfect complex on $X \times \mathbf{R}$[9]

$$\tilde{\mathcal{E}} \;=\; \left(\mathcal{E} \overset{r\cdot\nabla}{\to} \mathcal{E} \otimes \Omega^1_X\right) \boxtimes \mathbf{C}_{\mathbf{R}}$$

where $r \in \mathbf{R}$. Note that when $r = 0$, the zero cohomology sheaf is $\mathcal{E}$, whereas when $r \neq 0$, the zero cohomology sheaf is $\mathcal{E}^{\nabla}$.

Lemma 6.3.4. *This descends to an element of* $\mathrm{QCoh}(SX)$*, which we also denote by* $\tilde{\mathcal{E}}$*. In particular, it restricts to a holomorphic perfect complex on* X*, and a local system on the endpoints.*

(prove)

6.3.5. To the extent that D-modules correspond to holomorphic loops $\gamma : \mathbf{C}^\times \to X$, (e.g. we have that $\mathcal{D}_X$ and $\mathcal{O}_{\Omega_X}$ are Koszul dual, as are $k[\epsilon]/\epsilon^2$ and $k[t]$) one might imagine that the above corresponds to holomorphic-topological maps from the suspension $S\mathbf{C}^\times$ into either X or SX.

[9] Here, we view SX as a derived stack. Recall that every topological space T is a derived stack, and $\mathrm{QCoh}(T)$ is the category of local systems on T.

6.3.6. There is an analogous but different place a similar construction appears in mathematics. The solutions of Hitchin's equations give λ-*connections* on a curve C, i.e. maps
$$\alpha \ : \ \mathcal{E} \ \to \ \mathcal{E} \otimes \Omega^1_C$$
satisfying
$$\alpha(fe) \ = \ f\alpha(e) \ + \ \lambda \cdot edf$$
which when $\lambda = 0$ is a Higgs bundle and otherwise $\lambda^{-1}\alpha$ is a vector bundle with connection. (make precise)

6.4. **Remark: doubling and bosonisation.**

6.4.1. One might wonder how the above fits with the construction of quantum groups as a Drinfeld double of a bosonisation.

6.4.2. Recall the following picture:
$$U_q(\mathfrak{n}) \ \in \ \mathrm{BiAlg}(\mathrm{Rep}_q\mathfrak{t}) \quad \overset{\text{Bosonisation}}{\rightsquigarrow} \quad U_q(\mathfrak{b}) \ \in \ \mathrm{BiAlg}(\mathrm{Vect})$$
$$U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q\mathfrak{t}) \ = \ \underbrace{U_q(\mathfrak{b})\text{-Mod}(\mathrm{Vect})}_{\otimes} \quad \overset{\mathcal{Z}^{\mathbf{E}_1}}{\rightsquigarrow} \quad \underbrace{U_q(\mathfrak{g})\text{-Mod}(\mathrm{Vect})}_{\otimes_{\mathbf{E}_2}}$$
where the braiding on $\mathrm{Rep}_q\mathfrak{t}$ is given by $q^{\kappa(\lambda,\mu)} \in q^{\mathbf{R}} = \mathbf{C}[[\hbar]]$. Note that we need to use this instead of Rep_qT if we are to get an algebra $U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \rtimes U_q(\mathfrak{t})$, since $\mathrm{Rep}_q\mathfrak{t}$ is a category of modules, for $U_q(\mathfrak{t})$.

The factorisaton story only works with the unbosonised $U_q(\mathfrak{n})$, rather than $U_q(\mathfrak{b})$.

6.4.3. For Yangians, we expect to have
$$Y_\hbar(\mathfrak{n}) \ \in \ \mathrm{BiAlg}_{ch,*}(\mathrm{Rep}Y_\hbar(\mathfrak{t})) \quad \overset{\text{Bosonisation}}{\rightsquigarrow} \quad Y_\hbar(\mathfrak{b}) \ \in \ \mathrm{BiAlg}_{ch,*}(\mathrm{Vect})$$
$$Y_\hbar(\mathfrak{n})\text{-Mod}(\mathrm{Rep}Y_\hbar(\mathfrak{t})) \ = \ \underbrace{Y_\hbar(\mathfrak{b})\text{-Mod}(\mathrm{Vect})}_{\otimes^{ch}} \quad \overset{\mathcal{Z}^{\mathbf{E}_1,\otimes}}{\rightsquigarrow} \quad \underbrace{Y_\hbar(\mathfrak{g})\text{-Mod}}_{\otimes,\otimes^{ch}}.$$
Note that $Y_\hbar(\mathfrak{t})$ has a chiral and standard coproduct, so its category of representations has $\otimes$ and $\otimes^{ch}$.

Thus, we expect that $Y_\hbar(\mathfrak{n})$ has a chiral coproduct inside $\mathrm{Rep}Y_\hbar(\mathfrak{t})$, and its double $Y_\hbar(\mathfrak{g})$ has a chiral and standard coproduct. Notice that the formula in [GTb, §3.1] for the standard coproduct involves the Killing form (β, α_i), which is a smoking gun of it arising from a doubling construction.

6.4.4. In particular, we need to construct analogues to

$\mathrm{Conf}_\Lambda(\mathbf{C})$	$\mathcal{G}$	$\mathrm{Rep}_qT \simeq \mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C})_{x\cdot\infty})$
?	?	$\mathrm{Rep}Y_\hbar(\mathfrak{t})^{T(\mathcal{O})}$

where we have taken the category of $Y_\hbar(\mathfrak{t})$-modules with integral eigenvalues for the action of t_i, where $t \in \mathfrak{t}$ and $i \geqslant 0$.

6.5. **Relation to Chern-Simons**. Consider Chern Simons on $\Sigma \times \mathbf{R}_{\geqslant 0}$ with line operators $V_i \in \mathrm{Rep}U_\hbar(\mathfrak{g})$ living on $\{z_i\} \times \mathbf{R}_{\geqslant 0}$. Its value is

$$\mathrm{LocSys}_G^{(V_i,z_i)}\Sigma$$

where we consider local systems on $\Sigma \smallsetminus \{z_i\}$ valued in $V_1 \otimes \cdots \otimes V_n$ whose monodromy around z_i is given by the action of the representation V_i. There is a quantisation of this

$$\mathcal{O}(\mathrm{LocSys}_G^{(V_i,z_i)}\Sigma) \quad \rightsquigarrow \quad \mathcal{O}_\hbar(\mathrm{LocSys}_G^{(V_i,z_i)}\Sigma) \ = \ C^0((V_i, z_i)),$$

is the space of conformal blocks. Note that varying z_i makes $\mathrm{LocSys}_G^{(V_i,z_i)}\Sigma$ into a family of spaces. This gives the structure of a vector bundle with connection on conformal blocks,

$$C^0((V_i, -)) \ \to \ (\mathbf{C}^n)_\circ.$$

7. KZ from vertex algebras: conformal blocks

If $\mathcal{V}$ is a factorisation algebra, its space of *conformal blocks* is the factorisation homology

$$\mathrm{Conf}(\Sigma) \;=\; \mathrm{H}^\bullet(\mathrm{Ran}\Sigma, \mathcal{V}).$$

More generally, if $\mathcal{M}_{x_1,...,x_n}$ is a family of $\mathcal{V}$-modules concentrated at arbitrary points $x_1, ..., x_n \in \Sigma$, then we get the D-module of *conformal blocks*

$$\mathrm{Conf}(\Sigma, \mathcal{M}) \;\to\; \mathrm{Ran}^{\leqslant n}\Sigma,$$

which is a vector bundle over each stratum. If we are given such compatible data for all n, we get a D-module over the Ran space. When $\mathcal{M} = \mathcal{V}$ and take the fibre of this sheaf over $\varnothing \in \mathrm{Ran}\Sigma$, we get back the first definition.

7.1. **Summary**.

7.1.1. See [FB] for a non-factorisation summary of conformal blocks of vertex algebras.

7.1.2. What is the relation to the KZ equations? Assume V is a vertex operator algebra, so that we may induce representations the Zhu algebra (with conditions) of V to get representations of V,

$$\mathrm{Zhu} \;:\; \mathrm{Zhu}(V)\text{-Mod} \;\to\; V\text{-Mod}$$

which upgrades to a family of vertex modules over arbitrary points of Σ. (check)

Then, if we take conformal blocks of $M_1, ..., M_n \in \mathrm{Zhu}(V)$-Mod, the fibres of this vector space will be related to $M_1 \otimes \cdots \otimes M_n$, e.g. perhaps

$$\mathrm{Conf}(\mathbf{P}^1, z_i, \mathrm{Zhu}(M_i))$$

is a subquotient of $M_1 \otimes \cdots \otimes M_n$. In particular, we get braiding data on these subquotients.

7.1.3. To be more precise,

Definition 7.1.4. Let $\mathcal{A}$ be a weakly $\mathbf{G}_m \rtimes \mathbf{G}_a$-equivariant factorisation algebra on $\mathbf{A}^1$, i.e. a $\mathbf{Z}$-graded vertex algebra. Its $\mathbf{Z}$-graded *algebra of modes* is

$$U(\mathcal{A}) \;=\; \Gamma(\mathbf{D}_0^\times, \mathcal{A})$$

and its *Zhu algebra* is

$$\mathrm{Zhu}(\mathcal{A}) \;=\; \Gamma(\mathbf{D}_0^\times, \mathcal{A})^{\mathbf{G}_m}/I$$

where I is the ideal given by sums of $\alpha_{-n} \cdot \beta_n$, elements in degrees $-n$ and n, for $n \leqslant -1$.

See [FZ] for this definition. Note that we have

$$\mathcal{A}\text{-Mod} \;\to\; U(\mathcal{A})\text{-Mod}_{\mathbf{E}_1} \;\xrightarrow{\mathrm{Res}}\; U(\mathcal{A})^{\mathbf{G}_m}\text{-Mod}_{\mathbf{E}_1}$$

and one expects that for a module in the image, the ideal I acts trivially.

7.1.5. Let $V_{\lambda_i,k}$ be representations of $V^k(\mathfrak{g})$ induced by highest weights λ_i of $\mathfrak{g}$. Then we can by [FB, p. 13.3.5] define a vector bundle of conformal blocks

$$C^0(\mathbf{P}^1, \infty, z_1, ..., z_n)$$

over $(\mathbf{C}^n)_\circ$, with fibres $V_{\lambda_1}\otimes\cdots\otimes V_{\lambda_n}$ the tensor product of finite dimensional representations of $\mathfrak{g}$. It is a subbundle

$$C^0(\mathbf{P}^1, \infty, z_1, ..., z_n) \subseteq (\boxtimes \mathcal{V}_{\lambda_i,k})^*$$

where $\mathcal{V}_{\lambda_i,k}$ is the vector bundle attached to the $V(\mathfrak{g})$-module

Proposition 7.1.6. [FB, Lem. 13.3.7] *The differential operator $\partial_{z_i} + T_i$ on $(\otimes\mathcal{V}_{\lambda_i,k})^*$ preserves the conformal blocks.*

It is (expected?) that there are differential operators for all VOAs and modules. For instance the Virasoro and the *BPZ equations*.

7.1.7. *Remark.* Conformal blocks (three points at $0, z, \infty$) are called *intertwining operators*.

7.2. **Conformal blocks**. If $\mathcal{V}$ is a factorisation algebra over Riemann surface Σ, its chiral homology $\mathrm{H}^\bullet(\mathrm{Ran}\Sigma, \mathcal{V}) = \mathrm{Conf}(\Sigma)$ is also called its *conformal blocks*.

Note that as a vector bundle, $V|_{(\Sigma^n)_\circ} \simeq \mathcal{O}\otimes V^{\otimes n}$ is trivial. In particular, the restriction of a conformal block

$$\mathrm{H}^\bullet(\mathrm{Ran}\Sigma, \mathcal{V}) \;\to\; \mathrm{H}^\bullet((\Sigma^n)_\circ, \mathcal{V}), \qquad\qquad \Phi \;\mapsto\; \Phi|_{(\Sigma^n)_\circ}$$

is a $V^{\otimes n}$-valued function

$$\Phi|_{(\Sigma^n)_\circ} \;:\; (\Sigma^n)_\circ \;\to\; V^{\otimes n},$$

satisfying some differential equation given by the connection on $\mathcal{V}$. Moreover, as $z_i \to z_j$ this function satisfies

$$\Phi|_{(\Sigma^n)_\circ} \;\to\; Y_{ij}(z_i - z_j)\cdot\Phi|_{(\Sigma^{n-1})_\circ}, \tag{2}$$

where we have applied the vertex operator to the ijth entries, and $\Sigma^{n-1}\subseteq\Sigma^n$ is the diagonal $z_i = z_j$. We have shown

Proposition 7.2.1. *A conformal block is the same data as a collection of $V^{\otimes n}$-valued functions on $(\Sigma^n)_\circ$ satisfying:*

- *they satisfy the differential equations given by $\mathcal{V}$,*
- *they are $\mathfrak{S}_n$-invariant,*
- *they satisfy the operator product expansion (2) as $z_i \to z_j$.*

7.2.2. *Remark.* Often conformal blocks are presented after taking elements $\alpha_1, ..., \alpha_n \in V^*$, and then using notation

$$\langle\alpha_1(z_1)\cdots\alpha_n(z_n)\rangle_\Phi \;:=\; (\alpha_1\otimes\cdots\otimes\alpha_n)\Phi|_{(\Sigma^n)_\circ}(z_1, ..., z_n).$$

This is now a $\mathbf{C}$-valued function on $(\Sigma^n)_\circ$ satisfying the same properties as above.

7.2.3. *Remark.* We have

$$\langle \alpha_1(z_1)\cdots(T\alpha_i)(z_i)\cdots\alpha_n(z_n)\rangle_\Phi \;=\; \partial_{z_i}\langle\alpha_1(z_1)\cdots\alpha_n(z_n)\rangle_\Phi,$$

and so it follows together with the structure of the $z_i \to z_j$ limit that a conformal block is determined by its values for $\{\alpha_i\}$ varying over (duals of) generating fields of V.

7.2.4. *Example.* For instance, when we take the Heisenberg vertex algebra a conformal block Φ consists of functions over $(\Sigma^n)_\circ$ denoted

$$\langle h^{(1)}(z_1)\cdots h^{(n)}(z_n)\rangle_\Phi \;\in\; \mathcal{O}((\Sigma^n)_\circ)$$

which as n vary are compatible according to the operator product expansion of the Heisenberg vertex algebra:

$$\langle h^{(1)}(z_1)\cdots h^{(n)}(z_n)\rangle_\Phi \;=\; \frac{1}{(z-w)^2}\langle h^{(1)}(z_1)\cdots\widehat{h^{(i)}(z_i)}\cdots\widehat{h^{(j)}(z_j)}\cdots h^{(n)}(z_n)\rangle_\Phi \;+\; \mathcal{O}(1).$$

as $z_i \to z_j$.

7.3. **Preliminaries on pair Ran spaces.**

7.3.1. To begin with, we recall what it means for strong factorisation algebras $\mathcal{A},\mathcal{H}$ for $\mathcal{A}$ to act on $\mathcal{H}$. The definition is essentially the same as for what it means for an ordinary algebra to act on another, but we spell it out for clarity.

To begin with, an $\mathcal{A}$-module at a subset $T_1 \subseteq X$ is a D-module on $\mathrm{Ran}_T X$ equipped with identifications

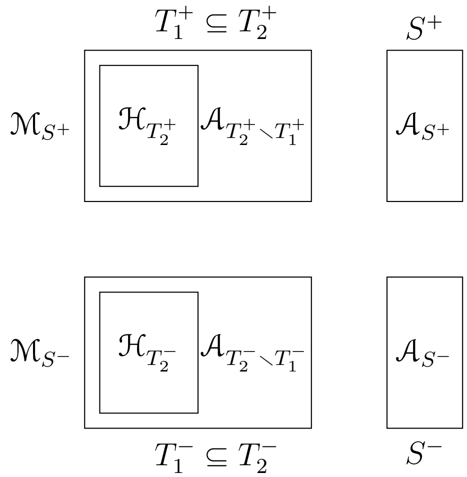
$$a_{S,T_2} \;:\; \mathcal{A}_S \otimes \mathcal{M}^{T_2}_{T_1} \;\simeq\; \mathcal{M}^{T_2}_{S\sqcup T_2}$$

for all disjoint finite subsets $(S,T_2) \in (\mathrm{Ran}X \times \mathrm{Ran}_{T_1}X)_\circ$, where we have $T_1 \subseteq T_2$. If $\mathcal{M}^T_T = \mathcal{H}_T$ is the !-fibre of a strong factorisation algebra, we can ask in addition for isomorphism

$$m_{\mathcal{M}} \;:\; \mathcal{M}^{T_2^-}_{T_1^-} \otimes \mathcal{M}^{T_2^+}_{T_2^+} \;\simeq\; \mathcal{M}^{T_2}_{T_1}$$

for every partition of the flag $T_1 \subseteq T_2$ into two disjoint parts, $T_1^\pm \subseteq T_2^\pm$. Restricting to the case that $T_2 = T_1$ gives back the factorisation product on $\mathcal{H}$.

In pictures, these structures and the compatibility conditions are:



(3)

where we have six finite subsets of X with various inclusion and disjointness assumptions. The algebraic structures a and $m_{\mathcal{M}}, m_{\mathcal{A}}$ correspond to the factorisation structures in the horizontal and vertical directions, respectively.

We now define the decomposition spaces to describe these structures. To begin, $\mathcal{M}$ as above will naturally live over the prestack of ***flags*** of finite subsets

$$\mathrm{Ran}_{\mathrm{SES}} X \ = \ \mathrm{colim}_{I_1 \subseteq I_2} X^{I_2}$$

where we have taken the colimit over all length-two flags of finite sets $I_1 \subseteq I_2$, with surjections between them preserving the flags. It has the following structures:

- For each $i = 1, 2$ the are natural maps

  $$\mathrm{triv}_i \ : \ (\mathrm{Ran} X)_i \ \leftrightarrows \ \mathrm{Ran}_{\mathrm{SES}} X \ : \ \mathrm{oblv}_i$$

  where $\mathrm{oblv}_i(T_1 \subseteq T_2) = T_i$, and $\mathrm{triv}_i(T_i)$ is the constant flag with value T_i. There is also the map

  $$\iota_2 \ : \ (\mathrm{Ran} X)_2 \ \to \ \mathrm{Ran}_{\mathrm{SES}} X$$

  sending $\iota_2(I_2) \mapsto (\varnothing \subseteq I_2)$.

- It is a commutative decomposition space, with decomposition product

  $$\begin{tikzcd} & (\mathrm{Ran}_{\mathrm{SES}} X \times \mathrm{Ran}_{\mathrm{SES}} X)_\circ \arrow[dl, "j"'] \arrow[dr, "j\cup"] & \\ \mathrm{Ran}_{\mathrm{SES}} X \times \mathrm{Ran}_{\mathrm{SES}} X & & \mathrm{Ran}_{\mathrm{SES}} X \end{tikzcd}$$

  corresponding to vertical composition in the diagram ((3)).

- It follows that it is a decomposition module over $(\mathrm{Ran} X)_i$ for $i = 1, 2$,

$$\begin{tikzcd} & (\mathrm{Ran}X \times \mathrm{Ran}_{\mathrm{SES}}X)_\circ \arrow[dl, "j"'] \arrow[dr, "j\cup_i"] & \\ \mathrm{Ran}X \times \mathrm{Ran}_{\mathrm{SES}}X & & \mathrm{Ran}_{\mathrm{SES}}X \end{tikzcd}$$

which when $i = 2$ corresponds to the horizontal composition in the diagram (3). Here $\cup_i = \cup \cdot (\mathrm{oblv}_i \times \mathrm{id})$.

Note that

Lemma 7.3.2. *Let $(\mathcal{M}, \mathcal{A})$ be a D-module on $\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$ such that its restriction $\mathcal{A}$ to $\mathrm{Ran}_2\mathbf{A}^1$ is a translation invariant strong factorisation algebra. Then the !-fibre $\mathcal{M}_x$ above any point is a module for the vertex algebra A attached to $\mathcal{A}$.*

We want to form the factorisation category $\mathcal{C}$ (a version of $\mathcal{A}$-FactMod) over $\mathrm{Ran}_1\mathbf{A}^1$, and we would like $\mathcal{M}_x \in \Gamma(x, \mathcal{C})$.

We define

$$\mathcal{C}_{\mathcal{A}} \;=\; \mathrm{triv}_1^*\mathcal{D}_{\mathcal{A}}$$

where $\mathrm{oblv}_1 : \mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1 \to \mathrm{Ran}_1\mathbf{A}^1$, and $\mathcal{D}$ is the sheaf of categories over $\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$ with sections classifying:

- a D-module $\mathcal{B} = (\mathcal{M}, \mathcal{A})$ over $\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$,
- an equivalence $\mathcal{A} \simeq \mathrm{triv}_2^!\mathcal{B}$, thus giving it a strong factorisation algebra structure,
- a strong action of $\mathcal{A}$ on $\mathcal{B}$ (living over the action of $\mathrm{Ran}_2\mathbf{A}^1$ on $\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$).

Note that we do not put any conditions on $\mathcal{M}$, e.g. that it must factorise over $\mathrm{Ran}_1\mathbf{A}^1$. Thus, a section of $\mathcal{C}$ can be thought of as a section of $\mathcal{D}$ in a small neighbourhood of $\mathrm{Ran}_1\mathbf{A}^1 \xrightarrow{\mathrm{triv}_1} \mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$, tensored as $(-) \otimes_{\mathrm{QCoh}(\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1)} \mathrm{QCoh}(\mathrm{Ran}_1\mathbf{A}^1)$. (make sure this isn't trivial!)

As an analogy, consider the category $\mathcal{C}$ of quasicoherent sheaves of algebras A on X with a module at $x \in X$, i.e. an action $A_x \otimes M \to M$. A quasicoherent sheaf F_X acts as $F_X \cdot (A, M) = (F_X \otimes A, F_{X,x} \otimes M)$. Then $\mathcal{C}_x$ is equivalent to the algebras on x with a module. However, if we fix A, then the fibre $\mathcal{C}_{A,x}$ of the associated category is the set of modules of A at x.

Before continuing, recall how to make $\mathrm{FactAlg}_{C'}(\mathcal{E})$ into a C-factorisation category if $\mathcal{E}$ is a factorisation category for C and C'. Take

$$\begin{tikzcd} & C \arrow[dl, "q"'] \arrow[dr, "p"] & \\ Y \times Y & & Y \end{tikzcd}$$

we then show that $\otimes_{\mathcal{E}}$ lifts:

$$\begin{tikzcd} q^*(\mathrm{FactAlg}_{C'}(\mathcal{E}) \boxtimes \mathrm{FactAlg}_{C'}(\mathcal{E})) \arrow[r, dashed] \arrow[d, "\mathrm{oblv}"] & p^*\mathrm{FactAlg}_{C'}(\mathcal{E}) \arrow[d, "\mathrm{oblv}"] \\ q^*(\mathcal{E} \boxtimes \mathcal{E}) \arrow[r, "\otimes_{\mathcal{E}}"] & p^*\mathcal{E} \end{tikzcd}$$

by a diagram chase and compatibility between C and C'. (write down proof)

Lemma 7.3.3. *$\mathcal{D}_{\mathcal{A}}$ is a decomposition category over* $\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1$.

Proof. Note that $\mathcal{D}\text{-Mod}_{\mathrm{Ran}_{\mathrm{SES}}\mathbf{A}^1}$ has a natural decomposition structure, so it remains to show that this decompisition structure respects the other data parametrised by $\mathcal{D}_{\mathcal{A}}$. (should follow easily once it's ironed out how to make FactAlg($\mathcal{E}$) into a factorisation category) □

Corollary 7.3.4. *$\mathcal{C}_{\mathcal{A}}$ is a decomposition category over* $\mathrm{Ran}_1\mathbf{A}^1$.

Proof. This is true since pullbacks preserve factorisation structures on sheaves of categories. □

(does this imply that $\mathcal{A}$ has a coproduct?)

In other words, we have equivalences

$$\mathcal{C}_{\mathcal{A},S\sqcup T} \;\simeq\; \mathcal{C}_{\mathcal{A},S}\otimes\mathcal{C}_{\mathcal{A},T}$$

for all disjoint subsets S,T of $\mathbf{A}^1$, and $\mathcal{C}_S \simeq \mathcal{A}\text{-FactMod}_S$.

7.4. Conformal blocks for modules.

7.4.1. *Warning.* Let us begin with the *wrong* definition of factorisation module $\mathcal{M}$ over $\mathcal{V}$. If we ask

$$j^*(\mathcal{V}\otimes\mathcal{M}) \;\xrightarrow{\sim}\; (\cup j)^*\mathcal{M}$$

then if we are working with unital Ran spaces, we get $\mathcal{V}\xrightarrow{\sim}\mathcal{M}$ by taking the restriction of the above map to

$$\begin{tikzcd}
 & (\mathrm{Ran}X\times\{\varnothing\})_\circ \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\mathrm{Ran}X\times\{\varnothing\} & & \mathrm{Ran}X
\end{tikzcd}$$

Thus, this definition never gives an interesting example.

7.4.2. *Insertions.* We now give the correct definition. Instead, let us pull back along f_x : $\mathrm{Ran}_xX\to\mathrm{Ran}X$, the prestack of finite subsets containing $x\in X$, and form

$$\begin{tikzcd}
 & (\mathrm{Ran}X\times\mathrm{Ran}_xX)_\circ \arrow[dl, "j_x"'] \arrow[d] \arrow[dr, "\cup_x j_x"] & \\
\mathrm{Ran}X\times\mathrm{Ran}_xX \arrow[d] & (\mathrm{Ran}X\times\mathrm{Ran}X)_\circ \arrow[dl] \arrow[dr] & \mathrm{Ran}_xX \arrow[d] \\
\mathrm{Ran}X\times\mathrm{Ran}X & & \mathrm{Ran}X
\end{tikzcd} \tag{4}$$

where the left square is a pullback.

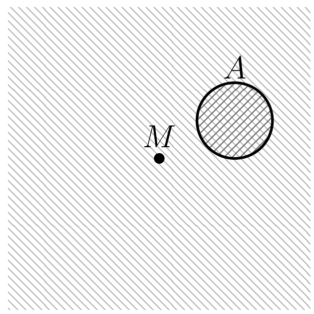
Definition 7.4.3. A *$\mathcal{V}$-module at $x\in X$* is a factorisation $\mathcal{V}$-module $\mathcal{M}$ on Ran_xX.

Said explicitly, it consists of a sheaf $\mathcal{M}$ along with structure map

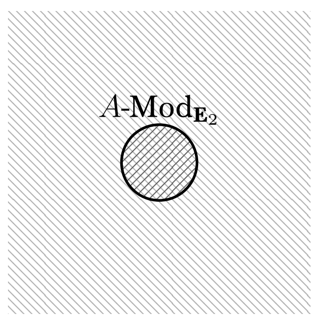
$$j_x^*(\mathcal{V} \boxtimes \mathcal{M}^x) \ \xrightarrow{\sim}\ (\cup_x j_x)^* \mathcal{M}^x$$

which is linear over $\mathcal{V}$. This should be viewed as the $2d$ CFT analogue of an associative algebra and a bimodule over it, i.e. a module over the two-sided Swiss cheese operad:

or rather the codimension two version of this, of a braided commutative algebra along with an $\mathbf{E}_2$-module for it:



We will now talk about the analogue of the fact that A-$\mathrm{Mod}_{\mathbf{E}_2}$ is itself braided monoidal, i.e. factorises over $\mathbf{R}^2$:



Note that if $\mathcal{M}_1, ..., \mathcal{M}_n$ are modules at $x \in X$ then there is no obvious way that $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ is also a $\mathcal{V}$-module at x. Instead, the category $\mathcal{V}$-Mod_x of such will itself form a factorisation category over X.

Definition 7.4.4. For any subset $S : T \to \mathrm{Ran}X$, the category $\mathcal{V}$-Mod_S of **factorisation modules at** S is the category of $\mathcal{M} \in \mathcal{D}$-$\mathrm{Mod}(\mathrm{Ran}_S X_T)$ along with structure map

$$j_S^*(\mathcal{V} \boxtimes \mathcal{M}) \ \xrightarrow{\sim}\ (\cup_S j_S)^* \mathcal{M} \tag{5}$$

linear over $\mathcal{V}$.

Here as before, we have correspondence of prestacks over T:

$$\begin{tikzcd} & (\mathrm{Ran}X_T \times \mathrm{Ran}_S X_T)_\circ \arrow[dl, "j_x"'] \arrow[dr, "\cup_S j_S"] & \\ \mathrm{Ran}X_T \times \mathrm{Ran}_S X_T & & \mathrm{Ran}_S X_T \end{tikzcd} \tag{6}$$

The structure map (5) is in the category of D-modules on $(\mathrm{Ran}X_T \times \mathrm{Ran}_S X_T)_\circ$. If $\mathcal{E}$ is any quasicoherent sheaf on T, then extending (5) linearly gives $\mathcal{M} \otimes \mathcal{E}$ the structure of a factorisation module at S.

Notice that

$$S \;\mapsto\; \mathcal{V}\text{-}\mathrm{Mod}_S$$

defines a sheaf of categories over $\mathrm{Ran}X$, which we also defote by $\mathcal{V}$-Mod. We now show that it factorises (check)

Proposition 7.4.5. $\mathcal{V}$-Mod *forms a factorisable sheaf of categories over* $\mathrm{Ran}X$.

Proof. To prove that $\mathcal{V}$-Mod factorises, we need to give an equivalence

$$\otimes_{\mathcal{V}} \;:\; j^*(\mathcal{V}\text{-}\mathrm{Mod} \boxtimes \mathcal{V}\text{-}\mathrm{Mod}) \;\xrightarrow{\sim}\; (\cup j)^*(\mathcal{V}\text{-}\mathrm{Mod}) \tag{7}$$

of sheaves of categories over $(\mathrm{Ran}X \times \mathrm{Ran}X)_\circ$.

To understand this statement fibrewise, let us understand the data of $\mathcal{M} \in \mathcal{V}\text{-}\mathrm{Mod}_S$. This consists of

$$\mathcal{M} \;\in\; \Gamma(\mathrm{Ran}_S X, \mathcal{D}\text{-}\mathrm{Mod}), \qquad\qquad \varphi \;:\; j^*(\mathcal{V} \boxtimes \mathcal{M}) \;\xrightarrow{\sim}\; (\cup j)^*\mathcal{M},$$

or more explicitly, for every subset $T \supseteq S$ and T_V such that T_V, T are disjoint, we have a vector space $\mathcal{M}_T$ and linear map $\varphi_{T_V,T} : \mathcal{V}_{T_V} \otimes \mathcal{M}_T \xrightarrow{\sim} \mathcal{M}_{T_V \sqcup T}$.

Now, let us consider two disjoint subsets (S_1, S_2) and let

$$(\mathcal{M}_1, \mathcal{M}_2) \;\in\; (\mathcal{V}\text{-}\mathrm{Mod} \boxtimes \mathcal{V}\text{-}\mathrm{Mod})_{S_1,S_2},$$

which consits of a pair of data as above. To define the factorisation structure, we will use this to define

$$\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2 \;\in\; \mathcal{V}\text{-}\mathrm{Mod}_{S_1 \sqcup S_2}.$$

To begin with, if $T \supseteq S_1 \sqcup S_2$ and T_V are disjoint, we define

$$(\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_T \;:=\; \bigoplus_{T_1, T', T_2} (\mathcal{M}_1)_{T_1} \otimes \mathcal{V}_{T'} \otimes (\mathcal{M}_2)_{T_2}$$

the sum taken over partitions $T = T_1 \sqcup T' \sqcup T_2$ into disjoint subsets such that $T_1 \supseteq S_1$, $T_2 \supseteq S_2$.

We can give a less ad-hoc definition as follows. We define $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2$ as a D-module on $\mathrm{Ran}_{S_1 \sqcup S_2} X$ using a colimit as follows. On the stack

$$(\mathrm{Ran}_{S_1} X \times \mathrm{Ran}X \times \mathrm{Ran}_{S_2} X)_\circ$$

we define the D-module

$$\mathcal{M}_1 \tilde{\otimes}_{\mathcal{V}} \mathcal{M}_2 \;=\; \mathrm{colim}\,(j_3^*(\mathcal{M}_1 \boxtimes \mathcal{V} \boxtimes \mathcal{M}_2) \to j_3^*(\cup \times \mathrm{id})^*\mathcal{M}_1 \otimes \mathcal{M}_2,\; j_3^*(\mathrm{id} \times \cup)^*\mathcal{M}_1 \otimes \mathcal{M}_2)$$

of the two maps given by the two actions φ_1 and φ_2, with $\mathcal{V}$ acting on the left and right respectively. We then project to

$$\cup_3 j_3 \;:\; (\mathrm{Ran}_{S_1}X \times \mathrm{Ran}X \times \mathrm{Ran}_{S_2}X)_\circ \;\to\; \mathrm{Ran}_{S_1 \sqcup S_2}X$$

and define $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2 = \cup_{3,*} j_{3,*}(\mathcal{M}_1 \tilde{\otimes}_{\mathcal{V}} \mathcal{M}_2)$. This has fibres given as above.

We now endow $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2$ with an action of $\mathcal{V}$. Take a subset $T \supseteq S_1 \sqcup S_2$ and T_V disjoint. We will define

$$\varphi_{T,T_V} \;:\; \mathcal{V}_{T_V} \otimes (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_T \;\to\; (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_{T \sqcup T_V}$$

by the factorisation structure on $\mathcal{V}$,

$$\mathcal{V}_{T_V} \otimes ((\mathcal{M}_1)_{T_1} \otimes \mathcal{V}_{T'} \otimes (\mathcal{M}_2)_{T_2}) \;\xrightarrow{\sim}\; (\mathcal{M}_1)_{T_1} \otimes \mathcal{V}_{T_V \sqcup T'} \otimes (\mathcal{M}_2)_{T_2}.$$

To define this globally as a map over $(\mathrm{Ran}X \times \mathrm{Ran}_{S_1 \sqcup S_2}X)_\circ$, (do it; maps out of a colimit are easy)

□

We draw what the above proof is doing: (draw it)

Moreover,

Proposition 7.4.6. *The category* $\mathcal{V}$-Mod *is holomorphic braided monoidal, i.e. there is a factorisation product*

$$\otimes^*_{\mathcal{V}} \;:\; \mathcal{V}\text{-Mod} \boxtimes \mathcal{V}\text{-Mod} \;\to\; \cup^! \mathcal{V}\text{-Mod}.$$

Proof. (check, but presumably same proof should work) □

Notice that the formula for the fibre $(\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_T$ can in this case be quite complicated if S_1 and S_2 are not disjoint.

Corollary 7.4.7. *The category* $\mathcal{V}$-Mod *has a fibrewise monoidal structure.*

Proof. We restrict along the diagonal decomposition structure,

$$\begin{tikzcd}
 & \mathrm{Ran}X \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & (\mathrm{Ran}X \times \mathrm{Ran}X) \arrow[dl, equal] \arrow[dr, "\cup"] & \\
\mathrm{Ran}X \times \mathrm{Ran}X & & \mathrm{Ran}X
\end{tikzcd}$$

giving that $\mathcal{V}$-Mod is fibrewise monoidal. □

(why is this not just symmetric monoidal? Need to check that the factorisation structure on $\mathcal{V}$-Mod is just $\mathbf{E}_2$)

(log stuff has not shown up yet!)

(*this* is how to study intertwining operators, the braided monoidal structure on V-Mod, the fusion product, etc.)

7.5. **Comparison to Huang-Lepowsky.**

7.5.1. A good review is [ALSW].

7.5.2. There is a kind of coproduct induced on V from the monoidal structure $\otimes_{P(z)}$ on V-Mod, which is given in [HL, Eqn. 13.6] using residues, and a third point x_0. This gives the action $Y_{P(z)}(v, w)$ of V on $M_1 \otimes_V M_2$.

7.5.3. Recall that if V is a vertex *operator* algebra with modules M_1, M_2. Then [HL] constructs a map (power series stuff not quite right)

$$\tau(z) \ : \ V \otimes \mathbf{C}[w, w^{-1}, (z^{-1} - w)^{-1}] \ \to \ \mathrm{End}\,((M_1 \otimes M_2)^*)$$

defined as in [HL, p. 13.2] by (approximately)

$$\tau(z)\delta((w-z)/u)Y(w) \ = \ \delta((w-u)/z)(Y(u)e^{wL_{-1}}w^{-2L_0} \otimes \mathrm{id}) \ + \ \delta((z-w)/u)(\mathrm{id} \otimes Y(w)).$$

Notice that we only use the first modes L_0, L_{-1} of the Virasoro. It involves:

- a translation by w: $\exp(wL_{-1})$,
- a scaling by $-2\log w$: w^{-2L_0}.

Then by Theorem [HL, Cor. 13.11]:

Theorem 7.5.4. *If there is a V-module $M_1 \otimes_V M_2$ corepresenting intertwining operators, it takes the form of*

$$M_1 \otimes_V M_2 \ = \ (S)^* \ \twoheadleftarrow \ (M_1 \otimes_k M_2)^*$$

where S is the subspace of elements satisfying a dimension condition and $\tau(z)\delta(z-w)Y(w) = \delta(z-w)\tau(z)Y(w)$, see [HL, p.26]. Moreover, it exists if and only if τ makes S into a V-module.

In this case, maps out of $M_1 \otimes_V M_2$ are *intertwining operators* from $M_1 \otimes M_2$.

7.5.5. See for instance [ALSW, p. 2.27], where the braiding on $M_1 \otimes_{\mathcal{V}} M_2$ is given by

$$\beta_{M_1,M_2} Y_{M_1,M_2}(m_1, z)m_2 \ = \ e^{zT} Y_{M_2,M_1}(m_2, -z)m_1$$

where

$$Y_{M_1,M_2} \ : \ M_1 \otimes_{\mathcal{V}} M_2 \ \to \ M_1 \otimes M_2\{z\}[\log z]$$

is an intertwining operator, and $-z = e^{i\pi} z$.

7.6. **Relation to Chern-Simons.**

7.6.1. Recall the physics story of Chern-Simons theory: given a Riemannian three-manifold M with boundary and $P = G \times M \to M$ the trivial G-bundle, we take the sheaf

$$\mathrm{Conn}'(P) \ \to \ M$$

of smooth $\mathfrak{g}$-connections

$$\nabla \ : \ \mathcal{T}_M \to \mathrm{End}(\mathrm{ad}P),$$

such that for a normal vector ξ along $\partial M \subseteq M$, we have that the ***boundary condition*** that $\nabla(\xi) = 0$ as an element of $\mathrm{End}(\mathrm{ad}P)|_{\partial M}$ (moreover, we need to ask that it vanishes to which order?); see the discussion around [Wi, Eqn. 3.1].

One can define a function on $\mathrm{Conn}'(P)$ by

$$\nabla \ \mapsto \ \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

which physics-defines a $3d$ QFT, which Witten physics-showed does not depend on the metric (i.e. is topological) if $\partial M = 0$, and is independent under rescaling of the metric (i.e. is conformal) in a neighbourhood of ∂M. Note that ∂M has measure zero, so we do not need to worry about whether the integrand is well-defined on the boundary.

7.6.2. *Classical version.* There is a classical version of this, instead taking the sheaf of sections of P itself:

$$P \ \to \ M.$$

Given a section $\gamma : U \to P$, we can take the differential 2- and 3-forms $\alpha_2, \alpha_3 \in \Omega^\bullet(G)$, and define the function

$$\gamma \ \mapsto \ \int_U \gamma^* \alpha_3 \ + \ 3k \int_{\partial U} \gamma^* \alpha_2.$$

Note that α_2 is given by the Killing form κ, and α_3 is given by $\kappa(-, [-,-])$.

Note that a function $\gamma : U \to G$ induces a map $\mathcal{T}_U \to \mathcal{T}_G \twoheadrightarrow \mathfrak{g}$. (is this how we get the connection above?)

7.6.3. *Line defects.* To add in line operators, mathematically one considers instead ***parabolic*** G-bundles, i.e. those equipped with a flag plus weights. Given any complex structure on ∂M, one can geometrically quantise this moduli stack using the level line bundle $\mathcal{L}$, giving

$$\mathrm{Bun}_G^{\mathrm{Par}}(\mathcal{E}_{\partial M}) \qquad \rightsquigarrow \qquad V_{\partial M, G} \ \to \ \mathcal{M}_{\partial M, n}$$

a vector bundle over the moduli stack of complex structures on ∂M. Here $\mathcal{E}$ is the universal curve over $\mathcal{M}_{\partial M, n}$.

One can show that this is the bundle of conformal blocks for $V(\mathfrak{g})$,[10] and has a KZ connection ∇_{KZ}.

One can *also* construct a so-called "Hitchin connection" ∇_{Hitch}, which *projectively* flat, is different from the KZ connection (but is projectively equivalent to KZ).

[10]This is called the *Pauly isomorphism*.

One can show that the vector space *Chern-Simons theory* attaches to a surface is

$$\partial M \qquad \rightsquigarrow \qquad \Gamma_{\nabla_{\text{Hitch}}\text{-flat}}(\mathcal{M}_{\partial M}, V_{\partial M,G}) \overset{?}{(=)} \Gamma_{\nabla_{\text{KZ}}\text{-flat}}(\mathcal{M}_{\partial M}, V_{\partial M,G}) \overset{?}{(=)} \mathbf{Conf}(V(\mathfrak{g}), \partial M_\sigma)$$

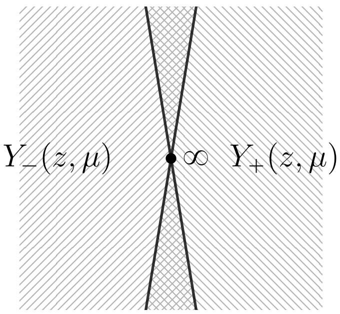
where σ is here a complex structure.

8. Stokes phenomena and dynamical KZ

8.1. One can consider the *dynamical* KZ equation

$$\mu_i \ + \ (k - k_{crit})\partial_{z_i} \ + \ \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

where $\mu \in \mathfrak{t}^{reg} \subseteq \mathfrak{g}$ acts on the ith factor, see [Xu]. This picks up an ***irregular*** singularity at $z_i = \infty$, around which there is a unique formal solution $Y(z, \mu)$ but on different sectors in the z_i-plane around $z_i = \infty$ there are ***different*** holomorphic solutions:



which are unique if we prescribe behaviour $Y(z, \mu) \to z^{\hbar\Omega} e^{z\mu_1} \mathcal{O}(1)$ as $z \to \infty$ along any sector. The ***Stokes matrix*** is

$$S_+ \ = \ Y_+(z, \mu)/Y_-(z, \mu) \ \in \ U(\mathfrak{g}) \otimes U(\mathfrak{g})[[\hbar]]$$

where the z independence is due to [LX, §4].

The main Theorem [LX, Thm. 4.2] says that

$$R \ = \ e^{i\pi\hbar\Omega} S_+$$

defines the R-matrix for $U_\hbar(\mathfrak{g})$-Mod.

8.1.1. *Why care?* From [Xu, §3], if we play the same game around $z = 0$, we can define $Y^0_\pm(z, \mu)$, and set

$$J_+ \ = \ Y^\infty_+(z, \mu)/Y^0_+(z, \mu)$$

by [LX, Thm. 3.12] kills the associator of $U_\hbar(\mathfrak{g})$-Mod, and so it follows that ***all*** information of $U_\hbar(\mathfrak{g})$ as a braided monoidal 1-category is contained in the $n = 2$ case, unlike when $\mu = 0$, where we need $n \leqslant 3$ to also get the associator. (write/think more precisely)

8.1.2. The above seems to give a factorisable perverse sheaf of categories over (or Ran$\mathbf{P}^1$)

$$\mathrm{Conf}(\mathbf{P}^1).$$

In the elliptic case, we can consider the dynamical KZ equation

$$\mu^i \ + \ \xi_i \ + \ \sum_{i \neq j} \frac{\Omega_{ij}}{(???)}$$

over the universal curve $\overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$, where ξ is a generating vector field. (is that defined over all $\overline{\mathcal{E}}_{1,1}$?) This does not add any more singularities to the KZ equation. (maybe in the $\mathbf{G}_m$ case though?)

Appendix A. Differential and difference equations

Let $\mathcal{G}$ be a group or formal group scheme which acts on X. For instance, we could consider

$$\mathcal{G} \;=\; \exp(\mathcal{T}_X) \;\simeq\; \mathcal{J}_\infty X$$

the exponential of the sheaf of Lie algebras over X given by the tangent bundle, or any subgroup generated by some vector fields. It is identified with the formal jet space of X. (is that right?) We have an action

$$\mathcal{G} \times_X X \;\to\; X.$$

The *de Rham stack* of this action is the quotient stack $X_{dR,\mathcal{G}} = X/\mathcal{G}$. For instance,

Lemma A.0.1. *When $\mathcal{G} = \exp(\mathcal{T}_X)$, we recover the usual notion of the de Rham stack $X_{dR,\mathcal{G}} = X^\wedge_{X^2}$, usually denoted just X_{dR}.*

Proof. (write) □

A.0.2. *Loop spaces.* Note that we have

$$\mathcal{O}(L_{\mathbf{G}_a}X) \;=\; \text{Maps}(\text{B}\mathbf{G}_a, X) \;=\; \wedge^\bullet_X \mathbf{L}_X$$

is the de Rham complex without the differential, and by [BN, Thm. 1.3] the differential is encoded by the translation action of $\text{B}\mathbf{G}_a$. In particular, we have by [BN, Thm. 1.5] that if X is a smooth underived scheme, an equivalence to the category of equivariant sheaves

$$(\wedge^\bullet_X \mathbf{T}^*_X, \text{d}_{dR})\text{-Mod} \;\simeq\; \text{QCoh}_{S^1}(L_{\mathbf{G}_a}X) \;\simeq\; \text{QCoh}_{\text{B}\mathbf{G}_a}(L_{\mathbf{G}_a}X)\,. \tag{8}$$

Given this, we use the Koszul duality

$$(\wedge^\bullet_X \mathbf{T}^*_X, \text{d}_{dR}) \;\simeq\; \mathcal{D}(X)$$

to conclude that all of (8) is equivalent to the category $\mathcal{D}$-Mod(X).

To understand where the Vect(X)-action comes from, note

$$(\text{Sym}^\bullet \mathbf{T}_X, 0)\text{-Mod} \;\overset{\text{KD}}{\simeq}\; (\wedge^\bullet_X \mathbf{T}^*_X, 0)\text{-Mod} \;\simeq\; \text{QCoh}\,(L_{\mathbf{G}_a}X) \tag{9}$$

and the $\mathbf{T}_X$-action on the right turns into the vector field action after we turn on the de Rham differential.

Thus, if we want to consider *difference*, *multiplicative difference* or *kZB heat* equations instead of differential equations, we need to first generalise (9). The right side has been generalised to arbitrary (algebraic) curves C replacing $\mathbf{G}_a$, by [BK]. However the defintion is the same for $\mathbf{G}_a, \mathbf{G}_m, E$ if X is a scheme, so it is expected that we need to consider the analytic curves $\mathbf{G}_a/\mathbf{Z}\hbar, \mathbf{G}_m/q^{\mathbf{Z}}, \mathcal{E}/\mathbf{Z}\times\mathbf{Z}$ where $\mathcal{E}$ is the universal elliptic cure.

Notice that being a module over difference and differential equations on X are all equivariance properties with respect to (formal) subgroups of

$$e^{\mathbf{T}_X},\; \langle e^{\mathbf{T}_X}, \varphi\rangle \;\subseteq\; X \times \text{Aut}(X)$$

where φ is a fixed automorphism. For any subgroup $\mathrm{G} \subseteq X \times \mathrm{Aut}(X)$ over X, we can ask what is

$$\mathrm{G}\text{-Rep}_X \;\simeq\; \mathcal{O}(\mathrm{G})\text{-Mod} \;\overset{\mathrm{KD}}{\simeq}\; ?$$

and *then* ask about noncommutative deformations, as in [BN, §1.6].

A.0.3. *Gauss-Manin connections from loop spaces.* By [BK] one expects a pushforward along $f : X \to Y$ on the level of loop spaces

$$f_* \;:\; \mathrm{QCoh}(L_C X) \;\to\; \mathrm{QCoh}(L_C Y),$$

and we expect that $\mathrm{QCoh}_{S^1}(L_{\mathbf{G}_a} X) \to \mathrm{QCoh}_{S^1}(L_{\mathbf{G}_a} Y)$ is the D-module pushforward, and $\nabla_f = f_*\mathcal{O}$ is the Gauss-Manin connection.

In examples, we will consider stacks $\overline{f} : \mathcal{M} \to \mathrm{BGL}$. We expect that

$$f \;:\; L_C\mathcal{M} \;\to\; L_C\mathrm{BGL} \;\simeq\; \mathrm{Conf}C,$$

note that $\mathrm{Conf}_C \mathrm{B}\mathbf{G}_m^n = C^{\times n}$, and that

$$f_*\mathcal{O} \;=\; \mathrm{H}^{\bullet}_C(\mathcal{M}) \;\in\; \mathrm{QCoh}(\mathrm{Conf}C).$$

(not quite right, ask Emile)

A.0.4. *Motivation.* We should view the de Rham stack as being the pushout

$$\begin{array}{ccc} \mathcal{G} \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/\mathcal{G} \end{array} \qquad \begin{array}{ccc} \{(x, g\cdot x)\} & \longmapsto & g \cdot x \\ \downarrow & & \\ x & & \end{array}$$

For instance, let $v \in \Gamma(X, \mathcal{T}_X)$ be a nonvanishing vector field and $\mathcal{G}$ be the formal group over X it generates. Assume the flow of v is complete, so it exponentiates to an action of $\mathbf{G}_a$. Then we can consider

$$\begin{array}{ccc} X \times \mathbf{A}^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_{\mathrm{dR},v} \end{array} \qquad \begin{array}{ccc} \{(x, e^{tv}\cdot x)\} & \longmapsto & e^{tv} \cdot x \\ \downarrow & & \\ x & & \end{array}$$

Likewise, if all vector fields' flows are complete, we have that $\mathcal{G} \times X = X^2$ (check this) and so we get (probably wrong)

$$\begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} \{(x, e^{tv}\cdot x)\} & \longmapsto & e^{tv} \cdot x \\ \downarrow & & \\ x & & \end{array}$$

Taking the completion gives back X_{dR}.

A.1. We thus make the definition that a $\mathcal{G}$-**differential equation** is a quasicoherent sheaf $M \in \mathrm{QCoh}(X_{dR,\mathcal{G}})$. Explicitly this consists of *parallel transport* isomorphisms

$$\varphi_{g,x} \ : \ M_x \ \xrightarrow{\sim} \ M_{g\cdot x}$$

for every pair of points $g \in \mathcal{G}$ and $x \in X$.

The *solutions* to a $\mathcal{G}$-differential equation are its image under the pushforward

$$\mathrm{QCoh}(X/\mathcal{G}) \ \to \ \mathrm{QCoh}(\mathrm{pt}) \ \simeq \ \mathrm{Vect},$$

which generalises the notion of flat sections (or de Rham cohomology) of a vector bundle with connection.

A.1.1. *Example.* For the ordinary de Rham space this is equivalent to a $\mathcal{D}$-module structure.

For instance, if the vector bundle $\mathcal{V}_X = V \otimes \mathcal{O}_X$ is trivial then we get an isomorphism

$$\varphi \ : \ \mathrm{act}^*\mathcal{V} \ \xrightarrow{\sim} \ \pi_2^*\mathcal{V}$$

where $\mathrm{act}, \pi_2 : \mathcal{G} \times_X X \rightrightarrows X$. In other words, this gives an automorphism of $\mathcal{V}_{\mathcal{G}\times X}$, with the condition that it pull back along X to the trivial automorpism of $\mathcal{V}_X$, plus the cocycle condition. On global sections for $X = \mathbf{A}^1$, this gives (check)

$$\Gamma(X, \mathcal{V}_X) \otimes \mathbf{C}[[t]] \ \xrightarrow{\sim} \ \Gamma(X, \mathcal{V}_X) \otimes \mathbf{C}[[t]].$$

The conditions imply that

Lemma A.1.2. *This map is of the form $e^{\partial \otimes t}$, where ∂ is an $\mathrm{End}(V)$-valued derivation on $\Gamma(X, \mathcal{O}_X)$.*

Proof. (write) $\square$

In other words, we get a derivation ∂. (how do we get higher order ODEs?) The flat sections consist of functions $f(x)$ with

$$e^{t\otimes\partial} f(x) \;=\; f(x) + t\partial f(x) + \cdots \;=\; f(x)$$

which is equivalent to $\partial f(x) = 0$.

A.1.3. *Example.* We can consider $\mathcal{G} = \mathbf{Z}$ acting on X generated by automorphism q, in which case a $\mathcal{G}$-differential equation is just a quasicoherent sheaf M along with compatible automorphisms

$$q^* M \;\simeq\; M.$$

Examples of this are when X it itself a group and the automorphism is action by a point $q \in X$.

For instance, if $\mathcal{V} = V \otimes \mathcal{O}_X$ is the trivial vector bundle, then the sections consist of functions consist of functions $f(x)$ with

$$q \cdot f(x) \;=\; f(qx) \;=\; f(x).$$

A.1.4. *Example.* We can contruct “mixed” examples as follows. Say a two dimensional torus $T \simeq \mathbf{G}_m \times \mathbf{G}_m$ acts on X, and $v = (1,0) \in \mathfrak{t}$ and $q = (1,t) \in T$. Then we can take

$$\mathcal{G} \;=\; \exp(\mathcal{O} \cdot v) \times \mathbf{Z} \cdot q.$$

Loosely speaking, a $\mathcal{G}$-differential equation is a connection along the flowlines of action of the first $\mathbf{G}_m$, and a difference equation along the second.

A.1.5. *Example.* For instance, we may take $X = \mathbf{C}$ and $v = \partial_z$, then renaming $t = \hbar$ the $\mathcal{G}$-differential equation becomes

$$e^{\hbar\partial_z} f(x) \;=\; f(x).$$

Note that we have $e^{\hbar\partial_z} f(x) = f(x + \hbar z)$ by Taylor's Theorem, which under the exponential map $\exp : \mathbf{C} \to \mathbf{C}^\times$ corresponds to multiplication by $q = e^{\hbar z}$,

$$\begin{array}{ccccc} \mathcal{O}(\mathbf{C}) & \xrightarrow{+\hbar z} & \mathcal{O}(\mathbf{C}) & \qquad & f(x) \longmapsto f(x+\hbar z) \\ {\scriptstyle \exp^*}\uparrow & & {\scriptstyle \exp^*}\uparrow & & \\ \mathcal{O}(\mathbf{C}^\times) & \xrightarrow{q} & \mathcal{O}(\mathbf{C}^\times) & & f(x) \longmapsto f(qx) \end{array}$$

where $X = \mathbf{C}^\times$ and $\mathcal{G} = \mathbf{Z}$. (write in a more canonical way)

A.1.6. *Example.* An action of a group G on X gives a map of groups in PreStk

$$G \ \simeq \ \mathrm{Maps}(\mathrm{pt}, G) \ \overset{\mathrm{id}}{\to} \ \mathrm{Maps}(\mathrm{pt}, G) \times \mathrm{Maps}(X, X) \ \to \ \mathrm{Maps}(X, G \times X) \ \overset{\mathrm{act}}{\to} \ \mathrm{Maps}(X, X).$$

Taking the associated map on Lie algebras (i.e. applying $\mathrm{Maps}_*(\mathrm{Spec} k[\epsilon]/\epsilon^2, -)$, where we take pointed maps) gives

$$\mathfrak{g} \ \to \ \Gamma(X, \mathcal{T}_X).$$

In particular, we get $\exp(\mathfrak{g})_X \to \exp(\mathcal{T}_X)$

A.1.7. Note that $\mathrm{LieAut}(X) = \Gamma(X, \mathcal{T}_X)$, thus we can consider

$$X/\exp(\mathcal{T}_X) \qquad \rightsquigarrow \qquad X/\mathrm{Aut}(X).$$

Or likewise,

$$X/\exp(\mathcal{O}_X \cdot v) \qquad \rightsquigarrow \qquad X/e^{\mathbf{C} \cdot v}$$

or take a subgroup $q^{\mathbf{Z}} \subseteq e^{\mathbf{C} \cdot v}$.

A.2. **Elliptic differential equations.** Take the universal curve

$$\pi \ : \ \mathcal{E} \ \to \ \mathcal{M}_{1,1}$$

and consider both:

- an automorphism $+p$ on $\mathcal{E}_\tau$ given by adding a point, (we need to specify a point, i.e. work with $\mathcal{M}_{1,2}$, or quotient by all of $\mathrm{Aut}(\mathcal{E}_\tau)$),
- vector fields on the base, $\mathcal{T}_{\mathcal{M}_{1,1}}$.

In particular, we have that

$$\mathcal{E}_{dR} \ = \ \mathcal{E}/(p^{\mathbf{Z}} \times \exp(\pi^* \mathcal{T}_{\mathcal{M}_{1,1}}))$$

and so $M \in \mathrm{QCoh}(\mathcal{E}_{dR})$ corresponds to a quasicoherent sheaf with an action of differential operators on the base and an automorphism of the fibre. In an important example of M in **[FTV3]**, these are called the *heat equation* and the *qKZB* equation, respectively.

A.2.1. If one considers

$$\overline{\pi} \ : \ \overline{\mathcal{E}} \ \to \ \overline{\mathcal{M}}_{1,1}$$

then we can consider:

- the automorphism $+p$ extends to $\overline{\mathcal{E}}$, and above $\mathcal{E}_\infty$ it becomes multiplication by $q = f(p)$ (which function?)
- an action of differential operators on $\overline{\mathcal{M}}_{1,1}$, which is still smooth.

One can thus define as before

$$\overline{\mathcal{E}}_{dR} \ = \ \overline{\mathcal{E}}/(\overline{p}^{\mathbf{Z}} \times \exp(\overline{\pi}^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}})).$$

Note that (check)

$$\overline{\mathcal{E}}_{dR,\infty} \ = \ E_\infty/q^{\mathbf{Z}}$$

which contains $\mathbf{C}^\times/q^{\mathbf{Z}}$ as an open subset, and its normalisation is $\mathbf{P}^1/q^{\mathbf{Z}}$,

$$\mathbf{C}^\times/q^{\mathbf{Z}} \overset{j}{\hookrightarrow} E_\infty/q^{\mathbf{Z}} \leftarrow \mathbf{P}^1/q^{\mathbf{Z}}.$$

In particular, an element $M \in \mathrm{QCoh}(E_\infty/q^{\mathbf{Z}})$ is equivalent to $M \in \mathrm{QCoh}(\mathbf{P}^1/q^{\mathbf{Z}})$ with a $q^{\mathbf{Z}}$-equivariant identification of $M_0 \simeq M_\infty$, which is (check!) equivalent to an element $M \in \mathrm{QCoh}(\mathbf{A}^1/q^{\mathbf{Z}})$ with (what other data?).

A.2.2. One should probably actually consider

$$\overline{\mathcal{E}}_{dR} = \overline{\mathcal{E}}/(\overline{\mathcal{E}} \times \exp(\pi^*\mathcal{T}_{\overline{\mathcal{M}}_{1,1}}))$$

where $\mathcal{E}$ via the group law on the universal elliptic curve. In particular, this allows us to ***both***:

- pass to the formal completion of the identity in $\mathcal{E}$, ***and***
- pass to the boundary of $\mathcal{M}_{1,1}$.

These give group maps

$$\exp(\mathcal{T}_{\overline{\mathcal{E}}}) \simeq \bar{\mathfrak{e}} \times \exp(\pi^*\mathcal{T}_{\overline{\mathcal{M}}_{1,1}}) \to \overline{\mathcal{E}} \times \exp(\pi^*\mathcal{T}_{\overline{\mathcal{M}}_{1,1}}) \leftarrow (\overline{\mathcal{E}} \times \exp(\pi^*\mathcal{T}_{\overline{\mathcal{M}}_{1,1}}))_\infty$$

interpolating between D-modules, elliptic differential modules, and difference modules.

A.3. **Riemann-Hilbert.** We have defined parallel transport, by definition.

This should be related to ongoing work by Kontsevich and Soibelman [KS].

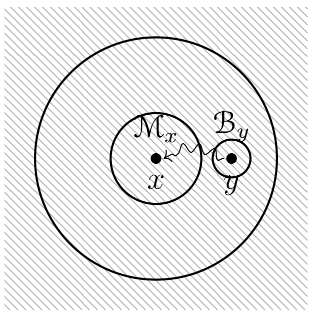
A.3.1. For q-difference modules, Riemann Hilbert was developed in [RSZ]

Appendix B. KZ talk

B.1. **Vertex algebras.** Recall the notion of *vertex algebra* V

algebra A	vertex algebra V
$A \otimes A \to A$	$V \otimes V \to V((z_1 - z_2))$
top. fact alg $\mathcal{A}$ over $\mathbf{R}$	hol. fact alg $\mathcal{V}$ over $\mathbf{C}$
A-BiMod $\simeq \mathcal{A}$-FactMod$_x$	V-VAMod $\simeq \mathcal{V}$-FactMod$_z$

If $\mathcal{B}$ is a factorisation algebra on $\mathcal{X}$, recall that a *factorisation module* at $x \in \mathcal{X}$ is a vector space $\mathcal{M}_x$ with action of $\mathcal{B}_y$:



In the topological case, this is equivalent to:

$$\gamma \ : \ \mathcal{B}_y \otimes \mathcal{M}_x \ \to \ \mathcal{M}_x$$

More generally, it is a sheaf on $\mathrm{Ran}_x\mathcal{X}$ with an action of $\mathcal{B}$.

Expectation is that $\{x_1, \dots, x_n\} \mapsto \mathcal{B}\text{-FactMod}_{\{x_1,\dots,x_n\}}$ is a factorisable sheaf of categories over $\mathrm{Ran}\mathcal{X}$.

Corollary B.1.1. *If V is a vertex algebra, we get a sheaf of* QCoh*-module categories with connection*

$$\mathcal{C} \ \to \ \mathrm{Ran}\mathbf{C}$$

with factorisation condition.

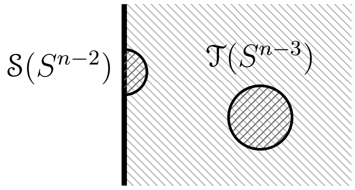
For a general vertex algebra we do *not* expect that $\mathcal{C}$ is the categorical analogue of regular holonomic, so we do *not* get a braided monoidal structure from $\mathcal{C}^\nabla$.

It *is* true given extra assumptions on V, see work [HL] by Huang–Lepowsky, also [ALSW]. The braided monoidal structure on $V\text{-Mod}_{\mathrm{VA}}$ is called the *fusion product* $\otimes_V$. However, not many such V are known, and it is not clear what are the weakest assumptions needed on V.

Remark. If $\mathcal{M}_{x_1}, \dots, \mathcal{M}_{x_n}$ are modules at $x_1, \dots, x_n$, their *conformal blocks* is the cohomology

$$C^\bullet(\mathcal{X}, (x_i); (\mathcal{M}_{x_i})) \ = \ \mathrm{H}^\bullet(\mathrm{Ran}_{\{x_1,\dots,x_n\}}\mathcal{X}, \mathcal{M}_{x_1} \otimes \cdots \otimes \mathcal{M}_{x_n}).$$

B.1.2. *Boundaries.* One can define (extended) *TQFTs with boundaries.* This gives an *action* of line operator categories



$$\mathcal{T}(S^{n-2}) \times \mathcal{S}(S^{n-3}) \;\to\; \mathcal{S}(S^{n-3})$$

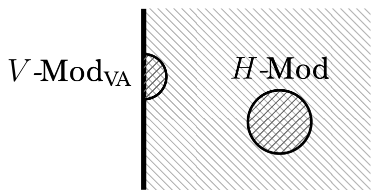
Examples from physics: 3d Chern–Simons is meant to have two *holomorphic* boundary theories, with category of line operators

$$\mathrm{Rep}V_k(\mathfrak{g}), \qquad\qquad \mathrm{Rep}W^k(\mathfrak{g}; e_{prin})$$

and in the first case the action is meant to be the Kazhdan–Lusztig equivalence $\mathrm{Rep}V_k(\mathfrak{g}) \simeq \mathrm{Rep}U_q(\mathfrak{g})$.

Question B.1.3. *What is the vertex algebraic analogue of the extra Stokes strucuture/dynamical KZ equations for* $\mathrm{Rep}U_q(\mathfrak{g})$*?*

Question B.1.4. *There are now many more examples* [BCDN] *of boundaries*



where H is a quasitriangular Hopf algebra and V a vertex algebra. What are the KZ equations for these?

Remark. It is basically unknown how this story is meant to recover qKZ equations, and how this relates to q-vertex algebras. c.f. section on 4d/5d Chern–Simons.

B.2. **Cohomological Hall algebras.** Take a moduli stack $\mathfrak{M}$ of a CY3 category, e.g. $\mathrm{Coh}(K_S)$, $\mathrm{Rep}(Q, W)$, or $\mathrm{Rep}\Pi_Q \otimes \mathrm{Coh}_0(\mathbf{A}^1)$. We get an algebra structure on various cohomologies by pushing and pulling.

e.g. if Q is an ADE quiver,[11] get

$$\begin{tikzcd} & \mathfrak{SES} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathfrak{M} \times \mathfrak{M} & & \mathfrak{M} \end{tikzcd}$$

$\mathbf{F}_q[\pi_0(-)]$	$\mathrm{H}^\bullet_{crit}(-)$	$\mathrm{K}^\bullet_{crit}(-)$	$\mathrm{Ell}^\bullet_{crit?}(-)$
$U_q(\mathfrak{n})$	$Y_\hbar(\mathfrak{n})$	$U_q(\widehat{\mathfrak{n}})$?	$\mathcal{E}_{\hbar,\tau}(\mathfrak{n})$?

These are sheaves of (bi)algebras over

$$\mathrm{Spec}(\mathrm{A}^\bullet(\mathfrak{M}), \cup) \;=\; \mathrm{Conf}_Q\Sigma, \qquad \Sigma \;\in\; \{\mathbf{C}, \mathbf{C}^\times, E\}.$$

Add in the Cartan $\mathfrak{t}$ corresponding to tautological cohomology classes, then apply the *Drinfeld centre* construction to get

$$U_q(\mathfrak{g}) \;\Big\|\; Y_\hbar(\mathfrak{g}) \;\Big|\; U_q(\widehat{\mathfrak{g}})? \;\Big|\; \mathcal{E}_{\hbar,\tau}(\mathfrak{g})?$$

[11]If Q is affine ADE, expect an extra loop.

which for formal reasons inherit an extra coproduct.

Proposition B.2.1. *The category* $\mathcal{C}$ *of finite-dimensional modules of the above factorise over* $\mathrm{Ran}(\Sigma \times \mathbf{R})$.

The expectation is that we formally obtain the $R(z)$-matrix and q-difference equations from the above.

Conjecture B.2.2. *There are* q*-difference equations attached to:*

- *zero-dimensional coherent sheaves* $\mathrm{Coh}_0(S)$ *on an algebraic surface,* [MMSV]
- *certain quivers with potential* (Q, W)*,*
- *more generally, any doubled CoHA* $D(A)$ *which has so far been defined.*

Question B.2.3. *What is the correct replacement of* $\mathfrak{M}$ *and cohomology to give dynamical (q)KZ equations? In* [KS]*, the authors suggest* rapid decay *cohomology.*

When Q is affine ADE, we expect to add an extra loop variable to the above, and should get a category factorising over $\mathrm{Ran}(\Sigma \times \Sigma_2)_\hbar$ [GRZ].

B.2.4. *Stable envelopes [MO].* We often get a finite-dimensional *variety* $\mathcal{M}_{\mathbf{w}}$ with an action

$$\mathfrak{M} \times \mathcal{M} \leftarrow C \rightarrow \mathcal{M}$$

and hence a finite dimensional $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathfrak{M})$-module structure $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{\mathbf{w}})$. Often $\mathcal{M}_{\mathbf{w}}$ is defined using the good moduli space of semistable *framed* objects, i.e. representations of the framed quiver or coherent sheaves with a framing at a divisor at infinity:

$$\begin{tikzcd} \square \arrow[d] & \square \arrow[d] & \square \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \mathcal{F},\ \varphi : \mathcal{F}|_D \simeq \mathcal{O}_D^{\oplus n}$$

The torus action $T_{\mathbf{w}}$ on $\mathcal{M}_{\mathbf{w}}$ makes

$$\mathrm{H}^{\mathrm{BM}}_{T_{\mathbf{w}},\bullet}(\mathcal{M}_{\mathbf{w}}) \ \to \ \mathrm{Conf}_Q \mathbf{C}$$

into a quasicoherent sheaf. The $\{\mathrm{H}^{\mathrm{BM}}_{T_{\mathbf{w}},\bullet}(\mathcal{M}_{\mathbf{w}})\}$ arrange into a factorisable sheaf of categories $\mathcal{C}$ over $\mathbf{C} \times \mathbf{R}$, with the factorisation structure over $\mathbf{R}$ is induced by

$$\oplus \ : \ \mathcal{M}_{\mathbf{w}_1} \times \mathcal{M}_{\mathbf{w}_2} \ \to \ \mathcal{M}_{\mathbf{w}_1+\mathbf{w}_2}$$

and over $\mathbf{C}$ induced by the stable envelope construction

$$\mathcal{M}_{\mathbf{w}_1} \times \mathcal{M}_{\mathbf{w}_2} \leftarrow \mathrm{Stab}_{\mathfrak{C}} \rightarrow \mathcal{M}_{\mathbf{w}_1+\mathbf{w}_2}$$

ADE quiver case. Maulik–Okounkov [MO] then apply Tannakian reconstruction by hand to define $Y_\hbar(\mathfrak{g})$ in terms of $\mathcal{C}$. We know that tensor products of $Y_\hbar(\mathfrak{g})$-modules satisfy the additive qKZ equations [GLW].

Question B.2.5. *How does one geometrically see the q-difference structure on $\mathcal{M}_{\mathbf{w}}$?*

If one answers this question, one can then ask try to answer Conjecture 1.4.2 on q-difference equations for general CoHAs. Finally,

Question B.2.6. *What is the analogue of the stable envelope construction for* ***dynamical*** *KZ?*

B.3. **Physics heuristics.** [Co] for any X_{CY3} with G_2-holonomy and symplectic surface Y with $\mathbf{C}_q^\times$-action and a deformation quantisation $Y_\hbar$, we can consider $\mathcal{M}$-theory on

$$X \times Y_\hbar \times \mathbf{R}$$

e.g. $X = K_S$, $Y = T^*\Sigma$, ... to get a 5d QFT on $Y_\hbar \times \mathbf{R}$.

Example. Writing $(\mathbf{C} \times \mathbf{C})_\hbar = \mathcal{D}(\mathbf{C})$,

$$K_{ADE} \times (\mathbf{C} \times \mathbf{C})_\hbar \times \mathbf{R}.$$

Pushing forward $\mathcal{M}$-theory along $K_{ADE} \to \mathrm{pt}$ gives 5d Chern–Simons on $(\mathbf{C}\times\mathbf{C})_\hbar \times \mathbf{R}$ [GRZ].

Further pushing forward along

$$(\mathbf{C} \times \mathbf{C})_\hbar \times \mathbf{R} \;\to\; (\mathbf{R} \times \mathbf{C}) \times \mathbf{R} \;\to\; \mathbf{C} \times \mathbf{R}, \qquad (w, z, r) \;\mapsto\; (z, |(w, r)|)$$

gives 5d, 4d, and 3d Chern–Simons, with line operator categories

$$U_{q,t}(\hat{\hat{\mathfrak{g}}})\text{-Mod} \;\overset{?}{\mapsto}\; U_q(\hat{\mathfrak{g}})\text{-Mod} \;\overset{?}{\mapsto}\; U_q(\mathfrak{g})\text{-Mod}$$

factorising over $(\mathbf{C}\times\mathbf{C})_\hbar(?)$, $\mathbf{R}\times\mathbf{C}$ and $\mathbf{C}_{top}$. Each step ? is something like taking Hochschild homology.

Remark. This gives a physics explanation for why qKZB heat(*?*), qKZ, and KZ equation are (*?*), q-difference, and differential equations: they are all equivariance data of the form

$$\mathcal{G} \times V \;\to\; V$$

on smooth vector bundle V, where $\mathcal{G}$ is the *gauge group* of the theory, e.g.

$$\mathcal{G} \;=\; T_X^{ahol}, T_X^{sm}$$

in the holomorphic and topological cases. When we consider a holomorphic manifold with $\mathbf{C}_q^\times$-action, we get $\langle T_X^{ahol}, \mathbf{C}_q^\times\rangle$. In particular, $V^{\mathcal{G}}$ satisfies a differential equation, is a *holomorphic* vector bundle, and is a holomorphic vector bundle satisfying a q-difference equation, respectively. One expects similar statements on the level of categories.

Question B.3.1. *One can define the gauge group $\mathcal{G}$ of a QFT rigorously. For what $\mathcal{G}$, and which QFTs, does $V^{\mathcal{G}}$ have Stokes data?*

K_{ADE}	**C**	**C**	**R**	
		$*$		KZ base
	$*$	$*$		qKZ base?
		$*$		loop in $\hat{\mathfrak{g}}$ & first loop in $\hat{\hat{\mathfrak{g}}}$
	$*$			second loop in $\hat{\hat{\mathfrak{g}}}$
H^2				root lattice of $\mathfrak{g}$
			$*$	CoHA product
		$*$		framing torus Spec$\mathrm{H}^\bullet_{T_{\mathrm{w}}}(\mathrm{pt})$
		$*$/**R**		MO standard coproduct/stable envelope
	$*$			MO vertex/infinite slope coproduct
		$*$		type of Coulomb branch $\mathcal{M}_C$/base of π
	$*$			cohomology theory applied to $\mathcal{M}_C$
		$*$		quasimap/GW $\mathbf{P}^1$

B.4. **3d Coulomb branches**. The *bubble Grassmannian*, or *Hecke stack*,

$$\mathcal{B}_z \;=\; G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$$

has a factorisation stuctucture over curve Σ_z, and algebra structure

$$\begin{tikzcd} & \mathcal{B}_z \times_{\mathrm{B}\mathcal{G}(\mathcal{O})} \mathcal{B}_z \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{B}_z \times \mathcal{B}_z & & \mathcal{B}_z \end{tikzcd}$$

More generally, we can build a space $\mathcal{B}_z$ with this structure out of any G-representation $\mathbf{N}$, or any quiver Q with representation $\mathbf{N}$. Then the sheaf of de Rham forms $\Omega^\bullet = (\Omega^\bullet, d_{dR})$ has a dg algebra structure

$$m \;=\; \int_p q^*$$

such that its cohomology $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{B}_z)$ is commutative. The **Coulomb branch** is

$$\mathcal{M}_C \;=\; \mathrm{Spec}(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{B}_z), m).$$

See [BFNa; BFNb]

Properties. The commuting subalgebra $\mathrm{H}^\bullet_{G(\mathcal{O})}(\mathrm{pt}) \simeq \mathrm{H}^\bullet_G(\mathrm{pt})$ induces a map

$$\pi \;:\; \mathcal{M}_C \;\to\; \mathfrak{g}/\!/G \;\simeq\; \mathfrak{t}/\!/W \;=\; \mathrm{Conf}_Q\mathbf{C}.$$

$\mathcal{M}_C$ is Poisson, with universal Poisson deformation

$$\begin{array}{ccc} \mathcal{M}_C & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathcal{M}_{C,\tilde{G}}/\!/T_F & \xrightarrow{\mu} & \mathfrak{t}_F \end{array}$$

where $G \to \tilde{G} \to T_F$ is an extension by a torus. $\mathcal{M}_C$ admits a quantisation $\mathcal{A}_\hbar = \mathrm{Spec}\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{B}_z/\mathbf{G}_m)$, a ring of difference operators.

Generalisations.

- $\mathrm{H}^{\mathrm{BM}}_\bullet \rightsquigarrow \mathrm{K}, \mathrm{Ell}$ changes the base curve of π: $\mathbf{C} \rightsquigarrow \mathbf{C}^\times, E$, and μ becomes a multiplicative/elliptic(?) Hamiltonian reduction.
- Can replace $\Omega^\bullet$ with the complexes computing $\mathrm{K}, \mathrm{Ell}(?)$. Can also use *critical cohomology* $(\Omega^\bullet, d_{dR} + df)$. Joyce sheaf?

Zastava. If G is ADE, there is an identification with the *open Zastava space* [BFNa, Thm. 3.1]

$$\mathcal{M}_C \;\simeq\; \mathrm{Maps}_*(\mathbf{P}^1, G/B) \;=\; \mathring{\mathcal{Z}}_G \;\subseteq\; \mathcal{Z}_G.$$

of maps sending $\infty \mapsto B/B$. Pulling back the divisor at infinity $D = \cup_{i \in Q} D_i$ gives the map $\pi : \mathcal{Z}_G \to \mathrm{Conf}_Q \mathbf{A}^1$, and the degree of the map $[\mathbf{P}^1] \in \mathrm{H}^2(G/B) \simeq \mathbf{Z} \cdot Q$ gives the dimension vector of $\mathbf{N}$.

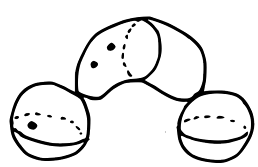
Question B.4.1. *Is there a* ***ramified*** *version of (quasi)maps and Zastava spaces?*

If G is affine ADE, $\mathcal{M}_C \simeq \mathring{\mathcal{Z}}_{\hat{G}}$ is the a partial compactification of a space of G-bundles on $\mathbf{P}^1 \times \mathbf{P}^1$.

Coulomb branches to KZ

- The Gauss-Manin connection (of π or μ?) is expected(?) to agree with the KZ connection,
- quasimaps and capping operators [Ok],
- quantum cohomology of $\mathcal{M}_C$ [Da],

B.4.2. *Quasimaps and* q*KZ.* Recall that a *quasimap* to a GIT quotient M is a map from a marked *pre*stable curve[12] to the quotient *stack*



$$\xrightarrow{f} \mathcal{M}$$

sending the marked points inside the stable locus $\mathrm{M} \simeq \mathcal{M}^s \subseteq \mathcal{M}$, which is identified with the GIT quotient M.

For instance, we can consider M the Nakajima quiver variety and

[12] Over the complex numbers, this is equivalent to having one connected component and only nodal singularities, see 0E6S [St].

$$\begin{tikzcd} & \mathrm{QMap}^d_{p_1,p_2}(C,\mathbf{M}) \arrow[dl, "\mathrm{ev}_2"'] \arrow[dr, "\mathrm{ev}_1"] & \\ \mathbf{M} & & \mathbf{M} \end{tikzcd}$$

for a fixed curve C with marked points p_1, p_2 and $d \in \mathrm{H}_2(\mathrm{M}, \mathbf{Z})$. When $C = \mathbf{P}^1$ and M are acted on by $\mathbf{C}_q^\times = \mathrm{Aut}(\mathbf{P}^1, 0, \infty)$, we define the *capping operator*

$$J(z) \;=\; \sum_{d \in \mathrm{H}^2(\mathrm{M},\mathbf{Z})} q^d \cdot (\mathrm{ev}_1 \times \mathrm{ev}_2)_*(\widehat{\mathcal{O}}_{\mathrm{vir}}) \;\in\; \mathrm{K}_{\mathrm{G}\times\mathbf{C}_q^\times}(\mathbf{M})^{\otimes 2}_{\mathrm{loc}} \otimes \mathbf{Q}[[q^d]]$$

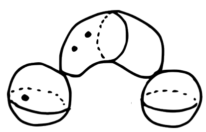
which is a section of a quasicoherent sheaf over $\mathrm{Conf}_w\mathbf{C}^\times = \mathrm{Spec}\mathrm{K}_T(\mathrm{pt})$, which has an action of $\mathbf{Z}^{\mathrm{rk}T}$.

Theorem. [Ok, Thm 8.1.16, 8.2.20] *The $J(z)$ satisfies the qKZ equations.*

B.4.3. *Quantum cohomology of $\mathcal{M}_C$. Quantum cohomology* is

$$\mathrm{QH}_T(\mathcal{M}_C) \;=\; \mathrm{H}^\bullet_T(\mathcal{M}_C)[[q^{\mathrm{H}^2(\mathcal{M}_C,\mathbf{Z})}]]$$

with product $*$ defined using the space $\overline{\mathcal{M}}_{0,3}(\mathcal{M}_C)$ of maps f by pull-pushing along

 $\xrightarrow{f} \mathcal{M}_C$

$$\begin{tikzcd} & \overline{\mathcal{M}}_{0,3}(\mathcal{M}_C) \arrow[dl] \arrow[dr] & \\ \mathcal{M}_C \times \mathcal{M}_C & & \mathcal{M}_C \end{tikzcd}$$

Space's connected components are labelled by the degree $\deg f \in \mathrm{H}^2(\mathcal{M}_C)$, counted by q. Degree zero term $\overline{\mathcal{M}}_{0,3}(\mathcal{M}_C)_0 = \mathcal{M}_C$ gives cup product.

Pick a Weyl chamber $\mathfrak{C} \subseteq \mathfrak{h} \subseteq \mathfrak{g}$. This gives a choice of Casimir Ω, hence of a trigonometric KZ connection

$$\nabla_i^{\mathrm{KZ}} \;=\; z_i\partial_{z_i} + h_i + \hbar\sum_{i\neq j}\frac{z_i\Omega_{ij,\mathfrak{C}} - z_j\Omega_{ji,-\mathfrak{C}}}{z_i - z_j}$$

on $\underbrace{\mathbf{C}^n}_{\mathrm{H}^2_T(\mathrm{pt})} \times \mathfrak{h} \times \mathbf{C}_\hbar$, where $h_i \in \mathfrak{h}$.

Theorem B.4.4. [Da] *For G simply laced the map*

$$(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})_\mu \;\xrightarrow{\sim}\; \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}_C^T) \;\xrightarrow{\mathrm{Stab}_{\mathfrak{C}}}\; \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}_C)_{T-\mathrm{loc}}$$

of vector bundles over $\mathbf{C}^n \times \mathfrak{h}$ intertwines

$$\nabla_i^{\mathrm{KZ}}, \qquad \text{and} \qquad c_1(\mathcal{E}_i)*$$

$$z_i, \qquad \text{and} \qquad q^{e_i}$$

where V_λ is irreducible finite dimensional with highest weight λ.

Here, T is the *framing* torus and H is the *internal* torus.

General principle. We have taken cohomology *twice*,

$$\mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathcal{M}_C) \;=\; \mathrm{H}^{\mathrm{BM}}_{T,\bullet}(\mathrm{Spec}\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{B}_z, m))$$

and therefore we can try to apply any pair of cohomology theories:

$$\mathrm{A}^{\bullet}_{T}(\mathrm{B}^{\bullet}_{H}(\mathcal{B}_z))$$

and recall that cohomology theories are labelled by one-dimensional formal groups:

	$\mathbf{C}$	$\mathbf{C}^{\times}$	E	(Σ)	
$\mathbf{C}$					KZ
$\mathbf{C}^{\times}$					qKZ
E					qKZB heat

There is also *critical cohomology* versions.

Question B.4.5. *What is the correct cohomology theory to take to take into account Stokes data? In [KS], the authors suggest rapid decay cohomology.*

Question B.4.6. *What do we use where instead of a product of curves we use an arbitrary conical symplectic surface Y?*

Appendix C. To be integrated into the main text

C.1. **The a, z variables.** In general, we expect a pair of differential or difference equations on

$$(\Sigma_a)^n_\circ \times (\Sigma'_z)^m_\circ$$

where $\Sigma, \Sigma' \in \{\mathbf{C}, \mathbf{C}^\times, E\}$. This is attached to a finite ADE quiver, i.e. is attached to the associated CY2 category; this gives the KZ equations.

The equation in the z variables will not contain any a terms, but the equation in the a variables will contain z terms. (see [Ko], or [AFO])

(how does this story relate to the story of KZ equations as coming from vertex algebras?)

C.1.1. In general, for X a local CY2 surface, we expect a pair of differential or difference equations on

$$(\Sigma_a)^n_\circ \times (\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}^\times)_\circ$$

where $(\mathrm{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}^\times)_\circ$ is the subspace of $e^{\omega + i\beta}$ where $\omega, \beta \in \mathrm{Pic}(X)$ have ω ample and β arbitrary. Here, $\Sigma_a = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{C}^\times$ is given by the Lie algebra $\mathfrak{a}$ of the *framing torus* of "symmetries of the moduli problem preserving the holomorphic symplectic form on X", e.g. if $\mathcal{M}$ is the moduli stack of instantons, A scales the framing at infinity. n.b. when this is in fact GL_n, this is why we get Ran spacey behaviour.

For instance, when we consider framed representations $\mathcal{M}^{fr}(w)$ of a quiver with framing vector $w \in \mathbf{N}^{Q_0}$, we have $A = \prod A_i \simeq \prod \mathbf{G}_m^{w_i}$. Note that

$$\mathcal{M}^{fr}(w) \;=\; (\text{vector space})/\textstyle\prod \mathrm{GL}_{v_i}$$

and $G = \prod \mathrm{GL}_{w_i}$ acts on this, and its good moduli space $\mathcal{X}^{fr}(w)$. The singularities of the KZ equations on $\mathfrak{a} \subseteq \mathfrak{g}$ will Lie along the locus where $\mathfrak{a}$ has higher than usual dimensonal fixed point locus when acting on $\mathcal{X}^{fr}(w)$.

Note that viewing $\Sigma_a^n = \mathfrak{a}$, the singularities of the KZ equations will lie along the root hyperplanes of the full framing group $\mathfrak{g}$. For instance, for $\mathfrak{sp}_{2n}$ (type C) these are $a_i = \pm a_j$ and $a_i = 0$, for $\mathfrak{so}_{2n}$ (type D) we have $a_i = \pm a_j$ for $i \neq j$, and for $\mathfrak{so}_{2n+1}$ (type B) they are $a_i = \pm a_j$ and $a_i = 0$.[13]

C.1.2. *Remark.* We have that $\pi_1((\mathbf{C}^\times)_\circ)$ is the ***affine*** braid group, so we get an affine braid group action on $V_1 \otimes \cdots \otimes V_n$. See [EG, Lem. 5.5], where the monodromy around $\mathbf{C}^\times$ is given in terms of $q = e^\hbar$.

Likewise, $\pi_1((E^n)_\circ)$ is the elliptic braid group, see [Jo].

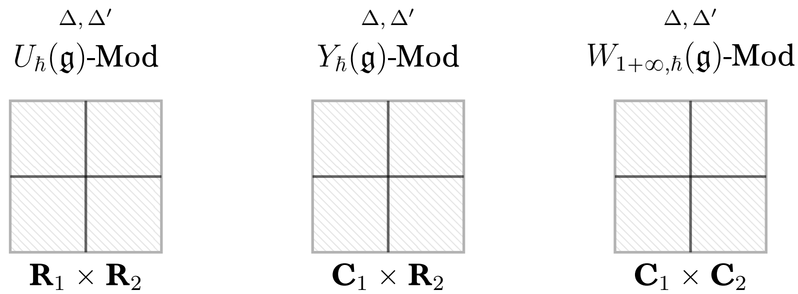
[13] The root hyperplanes are (from Fulton and Harris):

- $D/\mathfrak{so}_{2n}$ are $\pm a_i \pm a_j$ for $i \neq j$,
- $B/\mathfrak{so}_{2n+1}$ are $a_i \pm a_j$ for all $i \neq j$ and $a_i = 0$,
- $C/\mathfrak{sp}_{2n}$ are $\pm a_i \pm a_j$ for $i \neq j$ and $2a_i = 0$.

C.2. Passing to a quantisation of the KZ equation corresponds to Etingof-Kazhdan quantising $r(z) \rightsquigarrow R(z)$.

C.3. Affine KZ equations: Yangians, affine quantum groups, etc.

C.3.1. *Motivation.* We have the following picture: (not quite right, $\mathcal{W}_{1+\infty}$ is a vertex algebra not an algebra)



Each of the three algebras have two compatible coproducts Δ, Δ', hence their module categories are expected to factorise over the marked spaces. See [GRZ] for $W_{1+\infty}$.

C.3.2. To be precise, we expect sheaves of categories $\mathcal{C}$ over all three spaces, i.e. $\mathrm{Ran}(\mathbf{R}_1 \times \mathbf{R}_2)$ and so on, whose fibres are the three categories named above.

In addition, we need $\mathcal{C}$ to be endowed with a flat connection, loosely speaking because it comes from a TQFT or a holomorphic QFT and so has an action of $\mathrm{LieDiff}(X) = \Gamma(X, \mathcal{T}_X)$ and $\mathrm{LieConf}(X) = \Gamma(X, \mathcal{T}_X^{hol})$. Flatness corresponds to it being a Lie algebra action.

Thus for instance, we expect a sheaf of categories on

$$(\mathbf{C}_1 \times \mathbf{R}_2)_{dR} \ = \ (\mathbf{C}_1 \times \mathbf{R}_2)/\exp(\mathcal{T}^{hol}_{\mathbf{C}_1} \boxplus \mathcal{T}^{sm}_{\mathbf{R}_2})$$

and likewise over $\mathrm{Ran}(\mathbf{C}_1 \times \mathbf{R}_2)$.

C.3.3. *Remark.* Let us consider the relation between these three. Identifying $\mathbf{C}/S^1 \simeq \mathbf{R}_{\geqslant 0}$, the above is presumably attached to

$$\mathbf{C}_{\theta_1,\theta_2} \longrightarrow \mathbf{C}^\times_{\theta_1,\theta_2} \longrightarrow E_{\theta_1,\theta_2}$$

where here $\mathbf{C}_{\theta_1,\theta_2} \simeq \mathbf{R}_{\theta_1} \times \mathbf{R}_{\theta_2}$ is the universal cover of the angle coordinate circles. Thus if we have analogues:

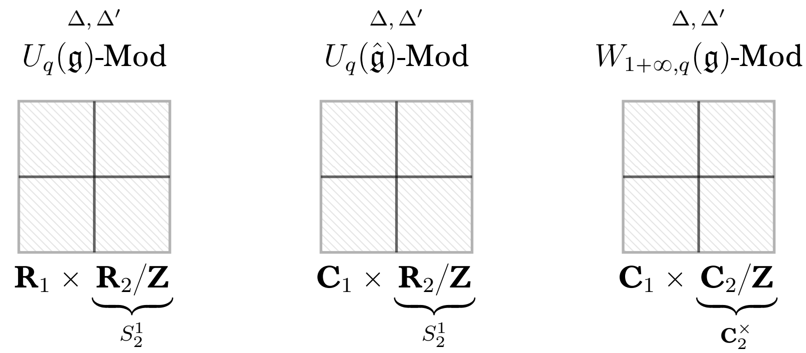
$$\mathbf{R}_{\theta_1} \times \mathbf{R}_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2} \leftsquigarrow S^1_{\theta_1} \times \mathbf{R}_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2} \leftsquigarrow S^1_{\theta_1} \times S^1_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2}$$

$$\mathbf{R}_1 \times \mathbf{R}_2 \longleftarrow \mathbf{C}_1 \times \mathbf{R}_2 \longleftarrow \mathbf{C}_1 \times \mathbf{C}_2$$

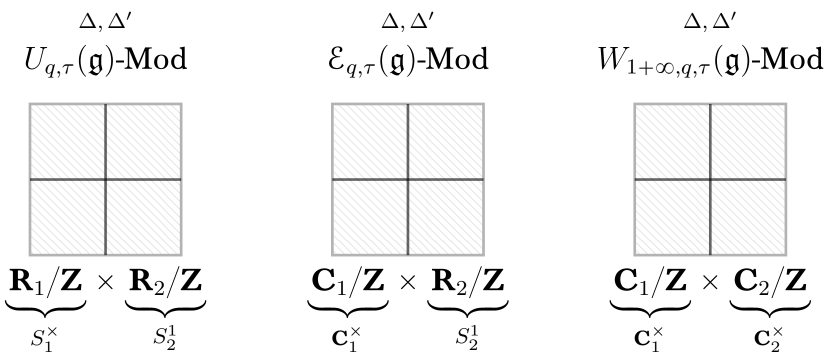
where this analogy matches collapsing an S^1 and taking its universal cover.

C.3.4. *Remark.* The KZ equations for $Y_\hbar(\mathfrak{g})$ are not expected to give the KZ (or qKZ) equations for $U_\hbar(\mathfrak{g})$. Instead, they are meant to be differential equations on valued in representations of $Y_\hbar(\mathfrak{g})$, with Ω replaced by the Casimir element of $Y_\hbar(\mathfrak{g})$. (check, seems dodgy, shouldn't we get [GTb] stuff?)

C.3.5. *Remark.* There should also be multiplicative and elliptic versions of the above. The multiplicative version quotients the first (or second) space by $\mathbf{Z}$:



and the elliptic analogue does it for both:



(maybe there should be other ways of quotienting which give you $E_1 \times \mathbf{C}_2$? This should be the story about taking limits)

C.4. **Gautam and Toledano Laredo's [GTb].** We have an inclusion (of meromorphic tensor categories [GTa])

$$\mathrm{Rep}^{fd} Y_\hbar(\mathfrak{g}) \;\to\; \mathrm{Rep}^{fd} U_q(\hat{\mathfrak{g}})$$

over Vect, whose definition involves choosing a branch of $\log(z)$. It exponentiates the roots of the Drinfeld polynomials $P_i(u)$ of representations, which are defined for $Y_\hbar(\mathfrak{g})$ by

$$\xi_i(u)v \;=\; \frac{P_i(u+d_i\hbar)}{P_i(u)}v$$

where v is a generating vector, $\xi_i(u) = 1 + \hbar\sum_{r\geqslant 0}\xi_{i,r}u^{-r-1}$ and if $\mathfrak{h}$ acts as λ, then $d_i v = \xi_{i,0}/\lambda(\alpha_i^\vee)v$. For $U_q(\hat{\mathfrak{g}})$ they have a similar definition.

This is in some sense a pullback along

$$\log \;:\; \mathbf{C} \smallsetminus \ell \;\to\; \mathbf{C}^\times$$

a section of $\exp : \mathbf{C} \to \mathbf{C}^\times$. The meromorphic tensor structures are given by the Drinfeld coproducts on $Y_\hbar(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$, see [GTa, §2].

C.5. q**-conformal blocks**. Whatever our definition of q-vertex algebra and $V_q^k(\mathfrak{g})$ should recover the qKZ equations. In particular, we would like to have a $\mathcal{V}_q \in \mathcal{D}_q\text{-Mod}(\mathrm{Ran}\Sigma)$ such that $\mathrm{Conf}_q(\Sigma) = \Gamma(\mathrm{Ran}\Sigma, \mathcal{V}_q)$ is a q-conformal block.

Let us consider the restriction

$$\Gamma(\mathrm{Ran}\Sigma, \mathcal{V}_q) \;\to\; \Gamma((\Sigma^n)_\circ, \mathcal{V}_q) \qquad\qquad \Phi \;\mapsto\; \Phi|_{(\Sigma^n)_\circ}.$$

Assume for now that $\mathcal{V}_q$ is trivial as a vector bundle over $(\Sigma^n)_\circ$, so that we again get a function

$$\Phi|_{(\Sigma^n)_\circ} \;:\; (\Sigma^n)_\circ \;\to\; V^{\otimes n}$$

by the factorisation condition. Moreover,

- it is $\mathfrak{S}_n$-invariant,
- it satisfies a q-*difference* equation,
- it satisfies a q-operator product expansion as $z_i \to q^n z_j$ for any $n \in \mathbf{Z}$,
$$\Phi_{(\Sigma^n)_\circ} \;\to\; Y_{ij}^{q^n}(z_i - z_j) \cdot \Phi_{(\Sigma^{n-1})_\circ} \tag{10}$$
where $Y_{ij}^{q^n}$ is (bla) and $\Sigma^{n-1} \subseteq \Sigma^n$ is the q^n-*diagonal* $z_i = q^n z_j$.

Notice that in the above limit (10), only the $(z_i - q^n z_j)$ poles contribute.

(do we consider $\mathrm{Ran}(X_{dR})$ or $(\mathrm{Ran}X)_{dR}$ in the q-case? the above assumes the former)

C.5.1. *Remark*. We expect to have the following story.

$$\begin{tikzcd} & \mathcal{V}_q \arrow[dl, squiggly, "\mathrm{Zhu}"'] \arrow[dr, squiggly, "q\to 1"] & \\ A_q & & \mathcal{V} \end{tikzcd}$$

(and an associated projection functor on their conformal blocks, assuming that A_q has them. The fusion coproduct on $\mathcal{V}_q$, if it exists, should be sent to a braided monoidal product on A_q.)

C.6. **Other versions of KZ - Higher terms**. Whereas the KZ equations have to do with Lie algebra invariants, the higher terms of the KZ equation should correspond to higher Lie algebra cohomology, see [SVa].

Appendix D. Old

D.1. KZ equations on other curves.

D.1.1. Let us consider the sequence of maps

$$\mathbf{C} \overset{\exp}{\to} \mathbf{C}^\times \overset{\pi}{\to} E.$$

We construct analytic D-modules on each of these spaces, pulling back to each other, with the one on $\mathbf{C}$ being the KZ equations.

D.1.2. On $(\mathbf{C}^\times)^n_\circ$, the KZ equations are

$$z_i\partial_{z_i} \;+\; \sum_{i\neq j}\frac{\Omega_{ij}}{1-z_i/z_j} \;+\; \lambda_i$$

on $(\mathbf{C}^\times)^n_\circ$, for some constant λ_i. Thus, the classical r-matrix is $r(z) = \frac{\Omega_{ij}}{1-z_i/z_j}$, and $R(z) = e^{\hbar r(z)}$ satisfies the trigonometric Yang-Baxter equation.

Lemma D.1.3. *This pulls back to the KZ D-module on* $\mathbf{C}$*. In other words, the pulled back differential equation is gauge equivalent to the KZ equation on* $\mathbf{C}$*.*

Proof. Note that indeed under the exponential map we have $\exp_* \partial_z = z\partial_z$,[14] so this matches with our expectation in section 6. Next, we have as functions on $(\mathbf{C}^n)_\circ$ that

$$\exp^*(1-z_i/z_j) \;=\; (1-e^{z_i}/e^{z_j}) \;=\; (1-e^{z_i-z_j}) \;=\; (z_i-z_j) \;+\; \mathcal{O}((z_i-z_j)^2).$$

Thus, the pullback of the KZ equation on $\mathbf{C}^\times$ is gauge equivalent to the KZ equation on $\mathbf{C}$ since the higher order terms of this expansion give holomorphic terms:

$$\frac{1}{1-e^{z_i-z_j}} \;=\; \frac{1}{z_i-z_j} \;+\; \mathcal{O}(1),$$

thus this pullback can be gauged to the KZ equation on $\mathbf{C}$. □

D.1.4. On E, the equation is

$$\xi_i \;+\; \sum_{i\neq j}\frac{\Omega_{ij}}{w_i -_E w_j} \;+\; \mu_i$$

where ξ_i is the generating vector field on E, w_i are (what?) and μ_i are constants.

Lemma D.1.5. *This pulls back to the KZ equations on* $\mathbf{C}^\times$ *and* $\mathbf{C}$*. (check)*

We recall from [FVa] that the elliptic KZ equation are *not* valued in $\mathcal{O}_{E^n}$, but rather in the line bundle $\mathcal{L}$ on $E^n = \mathbf{C}^n/(\Lambda+\tau\Lambda)$ (where here Λ is the coroot lattice of the Lie algebra $\mathfrak{g}$ we are considering and $\mathbf{C}^n = \mathfrak{t}$) given by monodromy

$$\ell(z+\lambda_1+\lambda_2\tau) \;=\; \exp\left(-\pi i\kappa(\lambda_2,\lambda_2) \;-\; 2\pi i\kappa(\lambda_2, z+\lambda_1)\right)\cdot \ell(z). \tag{11}$$

[14] This formula follows from $\partial_z = e^z\partial_{e^z}$, which is an application of the chain rule.

Note that in [FB, §I] they omit the z from this notation. We also assume that it is W_G-symmetric, and ℓ vanishes to a certain order along the coroot hyperplanes. (check)

Note that only degree *zero* line bundles can have connections. In particular, since $\mathcal{O}\cdot\theta \simeq \mathcal{O}\left(0+\frac{1}{2}+\frac{\tau}{2}+\frac{\tau+1}{2}\right)$, the theta line bundle does not have a connection.

Note that if we consider

$$\mathbf{C}^{\times} \ \to \ E$$

then the pullback of the θ line bundle is trivial; since the monodromy of the θ line bundle in the Λ-direction was trivial:

$$\ell(z+\lambda_1) \ = \ \ell(z).$$

The KZ equations on E are now

$$\xi_i \ + \ \sum_{i\neq j}\frac{\Omega_{ij}}{\theta_i-\theta_j}$$

where ξ_i acts on $\mathcal{L}$ as (write! does ∂_z descend to a vector field on $\mathcal{L} \hookrightarrow \mathcal{O}_{mer} = j_*\mathcal{O}$ where $j:\eta\to E$?) If we take the derivative of (11) then we get

$$\ell'(z+\lambda_1+\lambda_2\tau) \ = \ -2\pi i\kappa(\lambda_1,\lambda_2)e^{(-\pi i\kappa(\lambda_2,\lambda_2)\,-\,2\pi i\kappa(\lambda_2,z+\lambda_1))}.\ell(z)+e^{(-\pi i\kappa(\lambda_2,\lambda_2)\,-\,2\pi i\kappa(\lambda_2,z+\lambda_1))}.\ell'(z)$$

D.2. **Explicit equations for qKZ.**

D.2.1. The multiplicative KZ equation are the differential operators

$$(k-k_{crit})z_i\partial_{z_i} \ + \ \sum_{i\neq j}r(z_i/z_j)+\pi_i(\lambda) \tag{12}$$

see [FR, p5], where λ is a weight of $\mathfrak{g}$ and $\pi_i(\lambda)$ denotes action of this weight on the ith representation. Likewise for the elliptic KZ equation,

$$(k-k_{crit})\xi_i \ + \ \sum_{i\neq j}r(z_i -_E z_j)+(corrections?) \tag{13}$$

where ξ_i is the generating vector field on elliptic curve E.

The multiplicative qKZ equations (attached to $V_i\in \mathrm{Rep}U_q(\mathfrak{g})$, [GTa, §8.9]) are the ***difference*** operators

$$q_i^{-2(k-k_{crit})} \ + \ \prod_{i>j}R_{ij}(q^{2(k-k_{crit})}z_i/z_j)\cdot(\overline{R}_{i0}\pi_i(q^{2\rho})\overline{R}_{iN}^{-1})\cdot\prod_{i<j}R_{ij}(z_i/z_j)$$

as in [FR, p. 1.12] and [FR, p33], where $q_i:(z_1,...,z_n)\mapsto(z_1,...,qz_i,...,z_n)$, and both products are taken over j decreasing. Here $\overline{R}_{ij}$ are the R-matrices for $U_q(\mathfrak{g})$, ρ is the sum of the positive roots in $\mathfrak{g}$ and π_i is the action of $\mathfrak{g}$ on the ith factor. (Presumably) the additive qKZ equation (attached to $V_i\in \mathrm{Rep}Y_\hbar(\mathfrak{g})$, [GTa, §2.11]) is of the form

$$q_i^{-2(k-k_{crit})} \ + \ \prod_{i>j}R_{ij}(q^{2(k-k_{crit})}z_i-z_j)\cdot(\overline{R}_{i0}\pi_i(q^{2\rho})\overline{R}_{iN}^{-1})\cdot\prod_{i<j}R_{ij}(z_i-z_j).$$

The elliptic analogue of the qKZ equation by [FVT, §2], are differential operators valued on the vector bundle with value $\mathrm{Fun}_{mer}(\mathbf{A}^1_\lambda, V_1 \otimes \cdots \otimes V_n)$ (is that right? why no periodicity in λ? What is the actual data the elliptic qKZ is attached to?) given by

$$p_i \ + \prod_{i>j} R_{ij}(z_i - z_j + p, \lambda - 2\hbar \sum_{r=1, r\neq i}^{j-1} h^{(r)}) \cdot \Gamma_i \cdot \prod_{i<j} R_{ij}(z_i - z_j)$$

where $p_i : (z_1, ..., z_n) \mapsto (z_1, ..., z_i +_E p, ..., z_n)$, $h^{(i)}$ is a basis of the Cartan, Γ_i translates $\lambda \mapsto \lambda - 2\hbar\mu$ if μ is the eigenvalue of $h^{(i)}$. (finish this definition)

The R matrices $R_{ij}(z, \lambda)$ depend on two complex numbers (z, λ), unlike the additive or multiplicative case (compare [TVa]).

D.2.2. Compare the multiplicative qKZ equations to [GTa, §8.9],

$$\overline{\mathcal{R}}_{V_1,V_2}(q^{2\ell}\zeta) \ = \ \mathcal{A}_{V_1,V_2}(\zeta)\overline{\mathcal{R}}_{V_1,V_2}(\zeta).$$

Here $\mathcal{A}_{V_1,V_2}(\zeta)$ is the monodromy of the difference equation.

D.3. **Affinised analogue**. We can do the above for an arbitrary quiver Q, or replace $\mathfrak{g}$ with an arbitrary Kac-Moody Lie algebra in the above. We should have which are valued on tensor products $V_1(a_1) \otimes \cdots \otimes V_n(a_n)$ of evaluation representations of $Y_\hbar(\mathfrak{g}_Q)$, $U_q(\mathfrak{g}_Q)$ or $\mathcal{E}_{\hbar,\tau}(\mathfrak{g}_Q)$.

D.3.1. There is also a *boundary KZ equation* ∂KZ, which looks like

$$\partial_{z_i} \ + \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j} \ + \ \frac{k_i}{z_i}$$

where $k \in \mathfrak{g}$ is a classical K-matrix.

D.4. **Where does the KZ equation come from?** Before we proceed by listing many differential, difference and heat equations, any many other variants, we should ask: where do the KZ equations come from? Answering this will help organise the rest of the data.

The two answers are:

(1) ($3d$ TQFT) the KZ equations describe the category $\mathrm{Rep}U_q(\mathfrak{g})$ of representations of the quantum group, and
(2) ($2d$ CFT) the KZ equations arise from the factorisation category $\mathrm{Rep}\mathcal{V}^k(\mathfrak{g})$ of the affine WZW vertex algebra, by taking conformal blocks.

Both answers are related, by the Kazhdan-Lusztig equivalence between the two categories where $q = e^{\pi i/rc}$.[15] In physics terms, there is meant to be a holomorphic-topological QFT with boundary on

$$\Sigma \times \mathbf{R}_{\geqslant 0}$$

such that its restriction off the boundary $\Sigma \times \mathbf{R}_{>0}$ is topological.

[15]Here, $c = 2h^\vee \check{\kappa} + h^\vee \check{\kappa}_{min}$, where $\check{\kappa}_{min}(\alpha_i, \alpha_i) = 2$ for the long roots $\alpha_i \in \check{\Lambda}$.

In the following two sections we spell both points out in detail, after discussing more physics relations.

We will later consider analogues of the KZ equations where in place of $U_q(\mathfrak{g})$ we use Yangians or affine Yangians. In physics terms, these will correspond to using four or five dimensional Chern-Simons theories, as in [CWY] and [GRZ] respectively.

D.4.1. Finally, we note that given a $3d$ theory with holomorphic boundary $\mathcal{T}$, we expect to get an action

$$\mathcal{T}(\ell) \circlearrowleft \partial\mathcal{T}(\ell)$$

of monoidal categories of line operators on the bulk on line operators on the boundary, which after acting on the unit object gives a monoidal functor

$$\mathcal{T}(\ell) \ \to \ \partial\mathcal{T}(\ell).$$

For instance, for $3d$ Chern-Simons there are expected at least two boundary conditions: WZW and oper, which are meant to give the functors

$$\mathrm{Rep}U_q(\mathfrak{g}) \ \to \ \mathrm{Rep}V^k(\mathfrak{g}), \mathrm{Rep}W^k(\mathfrak{g}, e_{prin})$$

given by the Kazkdhan-Lusztig equivalence and the Drinfeld-Sokolov reduction functors, respectively.

D.5. **Further relations to physics.**

D.6. Recall that Nakajima quiver varieties are Higgs branches, $X = \mathrm{Spec}\mathcal{Z}(S^2)$, of three dimensional theories. Recall from [BFNa] that the theory are $3d$ $\mathcal{N}=4$ quiver gauge theories, attached to a quiver Q and v, w dimension and framing vectors, with Higgs and Coulomb branches

$$\mathcal{M}_H \ = \ X_Q(v,w), \qquad\qquad \mathcal{M}_C \ = \ \mathrm{SpecH}^{\mathrm{BM}}_{G(\mathcal{O}),\bullet}(\mathcal{R}).$$

These both have quantisations (does $X_Q(v,w)$?).

D.6.1. (what is the analogue of this for an arbitrary CY2 surface as in the previous section?)

D.6.2. Recall that an example of a quiver gauge theory is (a circle reduction of) $4d$ super Yang-Mills theory.

D.7. **Questions.**

(1) $Y(\mathfrak{g}_Q)$ (or its double) is Koszul dual to local operators in what theory (of what dimension)? What does doubling correspond to physically? (Sam's not sure; see Costello and Yagi "unification of integrability"-chapter 6 or something)

(2) X_Q is the Higgs branch of which theory? ($3d$ $\mathcal{N}=4$ dimensionally reduced $4d$ $\mathcal{N}=2$ quiver gauge theory)

(3) Why do we expect asymptotic Higgs branches to have a factorisation structure? (it's probably some $5d$ Chern-Simons $W_{1+\infty}$ or $5d$ SYM thing)

(4) What is the relation between this Coulomb branch stuff and $4d$ Chern Simons (i.e. Yangians)?

(5) Is the trichotomy in a and z orthogonal to the issue of taking double loops? i.e. is the quiver fixed as we vary a, z? If so, what is different when we take double loops, e.g. affine ADE?

(6) Is Kazhdan Lusztig to KZ what double affine Kazhdan Lusztig is to qKZ?

(7) (see Stable envelopes CoHA section) (is there a sense in which Ω_q is over $\mathrm{Conf}_\Lambda(\mathbf{C})$ in the finite ADE case, but there is something over $\mathrm{Conf}_\Lambda(\mathbf{C}\times\mathbf{R})$ in the affine case?) (is this to do with the rational sections stuff in YZ's elliptic quantum groups?)

(8) In the tri$\times$trichotomy, what is the fibre of the vector bundle? I assume something like $\mathrm{Maps}(G, V_1\otimes\cdots\otimes V_n)$ (evaluation reps) for V_i representations of $Y_\hbar(\mathfrak{g}), U_q(\hat{\mathfrak{g}}), \mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, but if so, why are conformal blocks expected to be this?

(9) Continue: KZ, qKZ, ?

(10) What do differential equations, difference equations and elliptic difference equations have to do with $\mathbf{G}_a, \mathbf{G}_m, E$?

(11) In just the KZ case, we get a braided monoidal structure $\mathrm{Rep}U_\hbar(\mathfrak{g})$ when the base is $\mathbf{C}$. What structure do we get when the base is $\mathbf{G}_m$ or E? Is the factorisable category on $\mathrm{Conf}_\Lambda(\mathbf{G}_m)$ and $\mathrm{Conf}_\Lambda(E)$ still $\mathrm{Rep}U_\hbar(\mathfrak{g})$? Or it is $\mathrm{Rep}U_q(\mathfrak{g})$? Or is the fibre $\mathrm{Rep}U_\hbar(\mathfrak{g})$, but the global sections are $\mathrm{Rep}U_q(\mathfrak{g})$? (c.f. Vanya's work about monodromy around $\mathbf{C}^\times$ and the trigonometric (i.e. $\mathbf{C}^\times$) KZ equation)

D.7.1. There is a pair of commuting differential equations, one in the a-variables, one in the z-variables.

D.8. **Affine analogue.**

D.8.1. It is natural to ask whether there is a Gaitsgory Lysenko factorisation story when replacing

$$u_q(\mathfrak{n}) \rightsquigarrow Y(\mathfrak{g}_Q) = Y_\hbar(\mathfrak{n})?$$

To solve this question;

- we need to have a Riemann-Hilbert for difference equations, which we do; see [RSZ] or [KS],
- (partial evidence for this: BPS sheaf over $\mathcal{X}$ or rather $\mathrm{Conf}_\Lambda(\mathbf{C})$ should be an analogue of $u_q(\mathfrak{n})$ over $\mathrm{Conf}_\Lambda(\mathbf{C})$)
- (the analogue of $\mathrm{Rep}_q T$ as a factorisation category $\mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C}))$ might be the limit $\lim \mathbf{H}^{\mathrm{BM}}(\mathcal{M}(v,w))$?)

- (unclear how the qKZ relates to the stable envelope, Nakajima quiver variety etc story)

D.8.2. Ignoring ellitpic, we have 2^4 choices,

- a, z are differential or difference (or elliptic?),
- whether a, z lie on $\mathbf{C}$ or $\mathbf{C}^\times$

We can have additive or multiplicative difference equation. We can have additive and multiplicative differential equations.

Ignore a for now (set it to be (??)), so we have 4 choices. The value of V_i are then:

- z differential equation on $\mathbf{C}$, $V_i \in \mathrm{Rep}U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}U(\mathfrak{g}[u])$,
- z differential equation on $\mathbf{C}^\times$, $V_i \in \mathrm{Rep}U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}U(\mathfrak{g}[u^{\pm 1}])$,
- z difference equation on $\mathbf{C}$, $V_i \in \mathrm{Rep}U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}Y_\hbar(\mathfrak{g})$,
- z difference equation on $\mathbf{C}^\times$, $V_i \in \mathrm{Rep}U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}U_q(\hat{\mathfrak{g}})$,

D.8.3. In the affine case, you can replace $\mathfrak{g}$ with any Kac-Moody algebra. These KZ equations aren't well-studied.

D.8.4. We can also consider equivariant BM homology of X_Q, they satisfy differential equations (KZ equations) in the torus-equivariant parameters, a.

D.8.5. Note that for ζ a positive stability condition, there is an action of $\mathcal{M}$ on X_Q tautologically.

D.8.6. There is a completely different curve to the z, a curves; it's the quasimap curve, the curve over which you dimensionally reduce, the one where $\hbar$ is an equivariant parameter on that curve. And this has to do with the asymptotic R-matrices.

D.8.7. The Drinfeld coproduct comes from some the dimensional reduction curve, the $\mathbf{C}$ on

D.8.8. The KZ equations are the ward identity for the conformal transformations.

D.8.9. There are also notions of *twisted* and *coset* KZ equations.

D.8.10. Vanya's equation is a differential equation on $\mathbf{C}^\times/(\mathbf{Z}/2)$ or $\mathbf{C}/(\mathbf{Z}/2)$; this is not anywhere else in the literature. Call it DKZ. (Interesting question: what is the qKZ analogue of this?) The multiplicative DKZ equation look like

$$z_i\partial_{z_i} + \sum_{i \neq j} \frac{\Omega_{ij}}{1 - z_i/z_j}, + \sum_{i \neq j} \frac{\Omega_{ij}^{long}}{1 + z_i/z_j}$$

where $\Omega \in S^2\mathfrak{g}^{long}$ where $\mathfrak{g}^{long} \subseteq \mathfrak{g}$ are the long root Lie subalgebra of a simple Lie algebra $\mathfrak{g}$. For instance, $\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \subseteq \mathfrak{sp}_4$.

If we want to understand orthosymplectic $Y_\hbar(\mathfrak{g})$, we then would have to consider the *difference* DKZ equations.

D.8.11. Read Agaganic Frenkel about quantum q-Langlands, (to get less confused about where all these curves come from; bottom of page 16 or picture on p17)

D.8.12. The KZB equations is the name for KZ equations over E. They are probably

$$\xi_i + \sum_{i\neq j} \frac{\Omega_{ij}}{z_i -_E z_j}$$

where ξ_i is the generating vector field on E.

D.9. **Variants.** We expect the following theories to be attached to

	$\mathbf{C}$	$\mathbf{C}^\times$	E	
$(\mathbf{C})_{\mathrm{top}}$	$U_\hbar(\mathfrak{g})$	$U_{q'}(\mathfrak{g})$	?	1 loop
$(\mathbf{C}^\times)_{\mathrm{top}}$	$Y_\hbar(\mathfrak{g})$	$U_q(\widehat{\mathfrak{g}})$	$\mathcal{E}_{q,\tau,\hbar}(\mathfrak{g})$	2 loops
$(E)_{\mathrm{top}}$	$Y_{\hbar_1,\hbar_2}(\widehat{\widehat{\mathfrak{g}}})$	$U_{q_1,q_2}(\widehat{\widehat{\mathfrak{g}}})$	?	3 loops

Physically, each object is obtained from 5d gauge theory on

$$\mathbf{R} \times \underbrace{(T^*C)_{nc}}_{X} \simeq \mathbf{R} \times (\mathbf{C} \times C)_{nc,\epsilon}$$

for $C = \mathbf{C}, \mathbf{C}^\times, E$. The $\hbar_1, \hbar_2, \hbar_3 = -\hbar_1 - \hbar_2$ scale the CY3 [GRZ, §0]. By [GRZ, Rem. 2.3.3] these are related to the formal variable ϵ, and $w \in \mathrm{H}^\bullet(X)$ in [Co, Thm. 9.0.2].

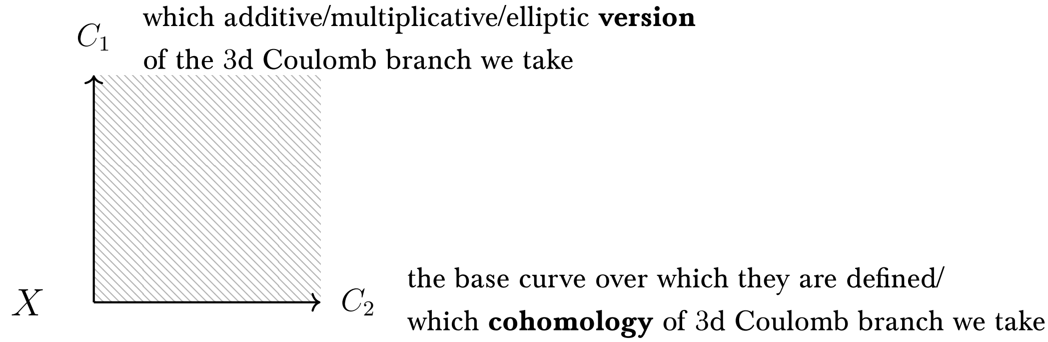
D.9.1. Note that in [AFSSZ] the 4d WZW boundary of 5d Chern Simons is defined!!! See [As] for a 3d version too!

D.10. **Variants.** We have the following table of definitions.

	$\mathbf{C}$	$\mathbf{G}_m$	E	Σ
$\mathrm{H}^{\mathrm{BM}}_\bullet$	KZ [ES], DKZ [FMTV; LX]		KZB [Fe, §2][ES, §6.4][Ha]	KZB [Iv]
$\mathrm{K}^\bullet$	additive qKZ [GTb, (2.1)]	qKZ [FR] DqKZ [TVb]	qKZB [Fe, p. 4.2] qKZB heat [FVb; EV]	
$\mathrm{Ell}^\bullet$		??	(q,t) KZ [AKMMSZ; AKMMMOZ]	

(14)

where the elliptic version should be elliptic Zastava [FMP], and where



and the following table of interesting facts about the corresponding definitions:

Base curve	$\mathbf{C}$	$\mathbf{G}_m$	E	Σ
	Gauss Manin $\mathcal{M}_C$ [SVb; SVa]		Periods [Ha]	
	shift op. [MO, (1.15)]			

D.10.1. *Degenerations.* By degenerating various parameters in the equations, we move ***up*** the table (14), see e.g. [NPT],

D.10.2. (q,t)*KZ equations.* There is also a (q,t)KZ equation [AKMMSZ]. In [AKMMMOZ, p12], it is:

D.10.3. *Further notes.* We remark:

(1) More generally we can replace $\mathfrak{g}$ or $\hat{\mathfrak{g}}$ (for $\mathfrak{g}$ simple finite dimensional) with any Kac-Moody Lie algebra, though this is less well-studied,

(2) The abelian qKZ equation [GTa, (2.1)] is a an additive difference equation for the translation operator (in [GTa] $a = 1$)

$$a \;:\; \mathbf{C} \;\to\; \mathbf{C}, \qquad\qquad z \;\mapsto\; z+a$$

and the original qKZ equation [FR] is a q-difference equation

$$q \;:\; \mathbf{C}^{(\times ?)} \;\to\; \mathbf{C}^{(\times ?)}, \qquad\qquad z \;\mapsto\; q\cdot z.$$

We expect these are related by $q = e^a$. The $q \to 1$ degeneration is expected [FR] to be $U_{q'}(\mathfrak{g})$, where $q' = \exp(2\pi i/(k+g))$ for k the level of the representations.

(3) (add dynamical KZ variable)

(4) In [GTa, §2.11] the relation between the $R(z)$-matrix and monodromy for the qKZ equation is explained.

(5) We expect each entry to correspond to a sheaf of categories over some space, and I expect the fibres to be

Base curve	$\mathbf{C}$	$\mathbf{G}_m$	E	Σ
	$\mathrm{Rep}^{\text{f.d.}} U_\hbar(\mathfrak{g})$	$\mathrm{Rep}^{\text{f.d.}} U_q(\mathfrak{g})$		
	$\mathrm{Rep}^{\text{ev}} Y_\hbar(\mathfrak{g})$	$\mathrm{Rep}^{\text{ev}} U_q(\hat{\mathfrak{g}})$	$\mathrm{Rep}^{\text{ev}} \mathcal{E}_{\hbar,\tau}(\mathfrak{g})$	

(6) See [EFK] to show how $U_q(\hat{\mathfrak{g}})$ sees the level.

(7) [GTb] considers the abelian qKZ, i.e. where $R(z)$ is replaced by $R^0(z)$, the Cartan part. See [MO, §9.4] for details on triangular decompositions of R-matrices.

(8) In [FR] considers both representations of $U_q(\hat{\mathfrak{g}})$ and the double Yangian $DY_\hbar(\mathfrak{g})$.

(9) See [Zh] for constructions of the quantum toroidal algebra via the Dubrovin quantum connection.

D.11. **Relation to Coulomb branches.**

D.11.1. *$3d$ Coulomb branches.* Let us take two cohomology theories

$$\mathrm{A}^\bullet,\ \mathrm{B}^\bullet \ \in\ \{\mathrm{H}^\bullet, \mathrm{K}^\bullet, \mathrm{Ell}^\bullet\}.$$

Then for G a reductive group with representation N, we may consider

$$\mathrm{A}^\bullet_T\left(\mathrm{Spec}\mathrm{B}^\bullet_{G(\mathcal{O})}(\mathcal{R}_{G,N})\right) \ \in\ \mathrm{QCoh}(\mathrm{Spec}\mathrm{A}^\bullet_T(\mathrm{pt})) \ \simeq\ \mathrm{QCoh}(\mathrm{Conf}C_2).$$

where the space in parentheses is called the *Coulomb branch* [BFNb], or [BFNb, Rem 3.9 (3)] for multiplicative, and conjecturally elliptic cases. In various cases $\mathcal{M}_C = \mathbf{M}(v_B, w_A)$ is a quiver variety and $T = T_w$ is the framing torus.

There is a quantisation $\mathrm{Spec}\mathrm{B}^\bullet_{G(\mathcal{O})\rtimes\mathbf{C}^\times}(\mathcal{R}_{G,N})$ of the Coulomb branch [BFNb], which we suspect might live over $(\mathrm{Conf}C_1)_\hbar = \mathrm{Spec}\mathrm{B}^\bullet_{G\rtimes\mathbf{C}^\times}(\mathrm{pt})$. However, how does one make sense of its singular cohomology?

Note that this allows us to generalise the KZ equations in the following way: take the *quantum cohomology* of $\mathcal{M}_C/T_v$, and the associated Dubrovin connection on that. This is called the *quantum KZ equation*. See e.g. [Ag].

D.11.2. *Gauss-Manin connection.* Note that we have a map

$$\mathrm{Spec}\mathrm{B}^\bullet(G(\mathcal{O})\backslash\mathcal{R}_{G,N}) \ \to\ \mathrm{Conf}C_2 \ =\ \mathrm{Spec}\mathrm{B}^\bullet_G(\mathrm{pt})$$

from the Coulomb branch to, in the additive case, $\mathbf{C}^n$. Is this related to the fact that the Gauss-Manin connection of the Coulomb branch gives solutions to KZ?

D.11.3. *Kahler variables.* The *Kahler variables* [Ok, §1.2.4] are

$$z \ =\ z_1^{d_1}\cdots z_n^{d_n} \ \in\ K(\mathrm{BT})^\wedge \ =\ \mathbf{C}[[z]] \ =\ \mathbf{C}[[\mathrm{H}^2(\mathrm{M},\mathbf{Z})]]$$

where for M a Nakajima quiver variety we have

$$T \ =\ \mathrm{H}^2(\mathrm{M},\mathbf{C})/2\pi i\mathrm{H}^2(\mathrm{M},\mathbf{Z})$$

related to the Poisson deformations. We identify $\mathrm{H}^2(\mathrm{M},\mathbf{Z})$ with topological line bundles on M: (the last is topological line bundles)

$$\mathrm{H}^1(\mathrm{M},\mathcal{O}) \ \to\ \mathrm{H}^1(\mathrm{M},\mathcal{O}^\times) \ \xrightarrow{c_1}\ \mathrm{H}^2(\mathrm{M},\mathbf{Z})$$

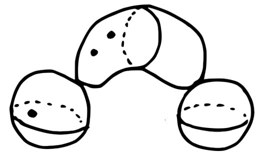
D.11.4. See [AO]. There is a family of qKZ equations parametrised by the *Kahler* variables

$$z \ \in\ \mathbf{Z} \ =\ \{\text{grouplike elements of } U_\hbar(\hat{\mathfrak{g}})\}/\text{centre}$$

which is identified with a torus. There are also *equivariant* variables

$$a \ \in\ \mathbf{A} \ =\ \mathrm{H}^\bullet_A(\mathrm{pt}).$$

D.11.5. *Quasimaps and qKZ.* Recall that a *quasimap* to M is a map from a marked *pre*stable curve[16] to the Nakajima quotient *stack*



$\xrightarrow{f} \mathcal{M}$

sending the marked points to the stable locus $\mathrm{M} \simeq \mathcal{M}^s \subseteq \mathcal{M}$, which is identified with the GIT quotient.

For instance, we can consider

$$\begin{tikzcd} & \mathrm{QMap}^d_{C,p_1,p_2}(\mathbf{M}) \arrow[dl, "\mathrm{ev}_2"'] \arrow[dr, "\mathrm{ev}_1"] & \\ \mathbf{M} & & \mathbf{M} \end{tikzcd}$$

for a fixed curve C with marked points p_1, p_2 and $d \in \mathrm{H}_2(\mathrm{M}, \mathbf{Z})$. When $C = \mathbf{P}^1$ and M are acted on by $\mathbf{C}^\times_q = \mathrm{Aut}(\mathbf{P}^1, 0, \infty)$, we define the *capping operator*

$$J(u,z) \;=\; \sum_d z^d \cdot (\mathrm{ev}_1 \times \mathrm{ev}_2)_*(\widehat{\mathcal{O}}_{\mathrm{vir}}) \;\in\; \mathrm{K}_{\mathrm{G}\times\mathbf{C}^\times_q}(\mathrm{M})^{\otimes 2}_{\mathrm{loc}} \otimes \mathbf{Q}[[z]]$$

where z are the Kahler variables, $u \in \mathrm{G}$ are equivariant variables, and we *define* the pushforward using torus localisation.

In the above, $J(u,z)$ only depends on $u \in \mathrm{A}$ the framing torus (in what sense/ref?), and so geometrically we get a section (J or Stab $\cdot\, J$) of

$$\mathrm{K}_{\mathrm{G}\times\mathbf{C}^\times_q}(\mathrm{M})[[z]] \;\in\; \mathrm{QCoh}(\mathrm{Conf}_w\mathbf{C}^\times) \;=\; \mathrm{QCoh}(\mathrm{K}_{\mathrm{A}}(\mathrm{pt})).$$

The qKZ equations will then be a difference equation for the action of $\mathbf{Z}^{\mathrm{rkA}}$ on $\mathrm{Conf}_w\mathbf{C}^\times$.

The capping operator satisfies qKZ:

Theorem. [Ok, Thm 8.1.16, 8.2.20] *We have*

$$\begin{aligned} J(u,z)\cdot(\mathrm{id}\otimes\mathcal{L}) \;&=\; \mathbf{M}_{\mathcal{L}}(u,z)J(u,z) \\ J(qu,z)E(u,z) \;&=\; \mathrm{S}(u,v)J(u,v) \end{aligned} \tag{qKZ}$$

where we have multiplication in K theory, where $(z_1,\dots,z_n)\cdot q^{\mathcal{L}} = (z_1\cdot q^{m_1},\dots,z_n\cdot q^{m_n})$ *if we have* $\mathcal{L} = \mathcal{L}_1^{\otimes m_1}\otimes\cdots\otimes\mathcal{L}_n^{\otimes m_n}$*, and where* $\mathcal{L}_i = \det\mathcal{V}_i$ *are tautological line bundles on* M. *Finally,*

- *[Ok, (8.1.12)]*

$$\mathbf{M}_{\mathcal{L}}(u,z) \;=\; \frac{z^{\deg f}(\mathrm{ev}_1\times\mathrm{ev}_2)_*\left(\widehat{\mathcal{O}}_{\mathrm{vir}}\cdot\det\mathbf{H}^\bullet(\mathcal{V}_i\otimes\pi^*(\mathcal{O}_{p_1}))\right)}{z^{\deg f}(\mathrm{ev}_1\times\mathrm{ev}_2)_*\left(\widehat{\mathcal{O}}_{\mathrm{vir}}\right)}$$

[16]Over the complex numbers, this is equivalent to having one connected component and only nodal singularities, see 0E6S [St].

*where $\pi : C' \to C$ is a certain stabilisation, and in [*t*]he denominator is called the gluing operator* $\mathbf{G}$.

- $\mathrm{S}(u, v)$ *[Ok, (8.2.18)]*
- $E(u, z)$ *[Ok, (8.2.13)]*

See also [AO, §1.2.4] or [Ok, §7.4].

D.11.6. *Remark.* By [MO, (1.15)] the *shift operator construction* produces a difference-differential equation over $\mathfrak{t} \times \mathrm{H}^2(X)$, vieiwing $\mathcal{O}(\mathrm{H}^2(X)) \simeq \mathbf{C}[q^\beta]$, $\Lambda^\vee$-difference in $\mathfrak{t}$ and differential in $\mathrm{H}^2(X)$, which is equal to the qKZ equation on $\mathbf{C}$.

D.11.7. *Why do Coulomb branches show up?* (interpretation as Zastava spaces)

D.12. **Weyl groups**.

D.12.1. Paper [EV] builds on Felder's construction [Fe] of KZB, to define the *dynamical Weyl group*, which generalises the Weyl group (for differential equations) and q-Weyl group (for q-difference equations).

D.13. **Relation to quiver varieties, quasimaps and stable envelopes**.

D.13.1. Also see [OS], where a quantum dynamical Weyl group is constructed for arbitrary quiver variety.

D.13.2. By [Ok, p. 26.7], the K-theory $K(T^*\mathrm{Gr}(k, n))$ is the weight k subspace of $\mathbf{C}^2(a_1) \otimes \cdots \otimes \mathbf{C}^2(a_n)$, where a_i are the equivariant variables, as a $U_\hbar(\hat{\mathfrak{gl}}_2)$-module where $\mathbf{C}^2(a_i)$ is the evaluation representation.

D.14. **Cohomology theories**.

D.14.1. Stoltz-Teichner [ST] have shown that $\mathrm{H}^\bullet(X), \mathrm{K}^\bullet(X), \mathrm{Ell}^\bullet(X)$ define $0|1$, $1|1$ and $2|1$-dimensional Euclidean field theories, i.e. those valued in $n|1$-dimensional Riemannian manifolds with trivial curvature.

D.14.2. There is a universal cohomology theory $\mathrm{MP}^\bullet(X)$ which is a quasicoherent sheaf over the stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups, see [Lub, §1]. Its pullback to the stacks $\{\mathbf{G}_a\}, \{\mathbf{G}_m\}, \mathcal{M}_{1,1}$ over Spec$\mathbf{Z}$ give $\mathrm{H}^\bullet(X), \mathrm{K}^\bullet(X), \mathrm{Ell}^\bullet(X)$, respectively.

There is also a map $\mathcal{M}_{\mathrm{CY}} \to \mathcal{M}_{\mathrm{FG}}$ from the stack of Calabi-Yau varieties of dimension n, taking Y to its Artin-Mazur formal group Φ_Y: the completion of the group $\mathrm{H}^n(X, \mathbf{G}_m)$ of $\mathrm{B}^{n-1}\mathbf{G}_m$-bundles, see [AM, §II.1]. This gives K-theory when $n = 0$. This is the Picard group when $n = 1$, and gives elliptic cohomology. See [Sz] for the K3 cohomology case $n = 2$. This is a generalised cohomology theory: it satisfies all axioms of an ordinary cohomology theory [Lub] except for the dimension axiom $\mathrm{H}^\bullet(\mathrm{pt}) \simeq \mathbf{Z}[0]$.

D.14.3. *Remark.* Recall the separate fact that $\mathbf{T}_X[-1] = \mathrm{Maps}(\mathrm{B}\mathbf{G}_a, X)$ is the space whose ring of functions is the de Rham complex, and the $\mathrm{B}\mathbf{G}_a$ action is the de Rham differential, see [BN, Thm. 1.3].[17]

[17] This essentially follows from $\mathcal{O}^{\mathrm{der}}(\mathrm{B}\mathbf{G}_a) \simeq \mathrm{H}^\bullet(S^1)$. Note that for G a reductive group the invariants functor is exact so $\mathcal{O}^{\mathrm{der}}(\mathrm{BG}) = k$. Thus $\mathcal{O}^{\mathrm{der}}(\mathrm{B}\mathbf{G}_m) = k$ is uninteresting.

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