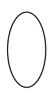
VERTEX STRUCTURES ON CRITICAL COHAS, DRINFELD COPRODUCTS, AND FACTORISATION

ALEXEI LATYNTSEV (JOINT WORK WITH S. KAUBRYS AND S. JINDAL)

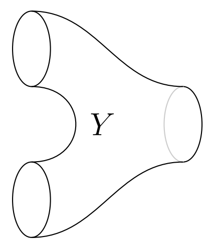
Abstract: Talk for the VBAC conference 2025, https://vbac-2025.ncag.info/.

1. Part 0: What is a vertex algebra?

Atiyah 1988: a (*topological/conformal/…*) *quantum field theory* $\mathcal{T}$ is an assignnment

X  $\rightsquigarrow \mathcal{T}(X) \in$ **Vect**

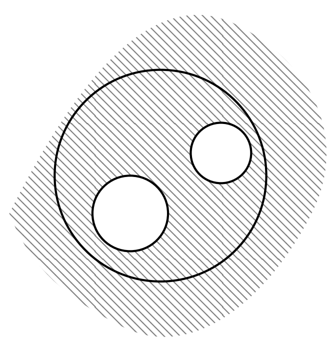
(topological/conformal/…) manifold

X_1  X_2 $\rightsquigarrow \mathcal{T}(X_1) \xrightarrow{\mathcal{T}(Y)} \mathcal{T}(X_2)$

(topological/conformal/…) cobordism

arranging into a symmetric monoidal functor Cob → Vect.[1]

We may restrict the cobordisms to lie inside a fixed manifold Z.



Z

$Z = \mathbf{R}^1, \mathbf{R}^n$ and $\mathbf{C}$ gives forgetful functors

1d TQFT	$\to$	$\mathbf{E}_1$-algebra	$\simeq$	associative algebras
nd TQFT	$\to$	$\mathbf{E}_n$-algebra		
2d CFT	$\to$	$\mathbf{E}_2^{\mathrm{hol}}$-algebra	$\to$	vertex algebras

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[1] n.b. the nontrivial part is to *define* the category Cob, which is not done for many types of manifolds.

Thus - a *vertex algebra* is a vector space A with a map

$$A \otimes A \;\to\; A((z_1 - z_2))$$

satisfying a variant of commutavity, associativity and with a unit $|0\rangle \in A$.

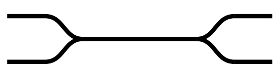
2. Part I: vertex product

$\mathcal{M}$ - ***any*** abelian/derived moduli stack $\mathcal{M} = \mathrm{Crit}(W)$ inside a smooth moduli stack $\mathcal{M}^{sm}$.

Theorem A. *The critcal cohomology* $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ *is a* ***vertex coalgebra****.*

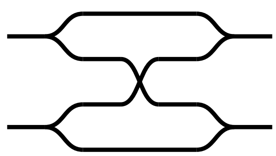
Theorem B. *With the CoHA it forms a* ***vertex bialgebra****.*

2.0.1. *Remark.* The (vertex) bialgebra axiom is



$\Delta(\alpha \cdot \beta)$

$\|$



$\Delta(\alpha) \cdot \Delta(\beta)$

where the swap is the braiding in a (factorisation/meromorphic) braided monoidal category $\mathcal{C}$.

2.0.2. *Examples.*

- Locally - ***any*** CY3 moduli stack,
- $\mathcal{M} = \mathrm{Crit}(\mathrm{tr}W) \subseteq \mathcal{M}^{sm} = \mathcal{M}_{\mathrm{Rep}Q}$,
- Semistable Higgs bundles/local curves e.g. $\mathrm{Perf}(K_{T^*C})$ (Kinjo-Matsuda '22 Thm 5.6, Pădurariu-Toda '24 §2.2).
- More generally *deformed 3CY completions*

$$\begin{array}{ccc} \mathcal{M}_{\Pi_3(\mathcal{C},c)} = \mathrm{Crit}(W) & \longrightarrow & \mathcal{M}_{\mathcal{C}} \\ \downarrow & & \downarrow dW \\ \mathcal{M}_{\mathcal{C}} & \longrightarrow & T^*\mathcal{M}_{\mathcal{C}} \end{array}$$

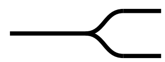
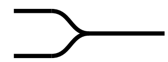
  where $\mathcal{C}$ has finite type, and $c \in \mathrm{HH}_0(\mathcal{C})$.

2.0.3. *Remark.* The vertex structure depends on $\mathcal{M}^{sm}$ $\Rightarrow$ expect a *sheaf* of braided monoidal categories on the *Darboux stack* $\mathcal{M}^{\mathrm{Dar}}$, parametrising all local presentations of $\mathcal{M}$ as a critical locus.

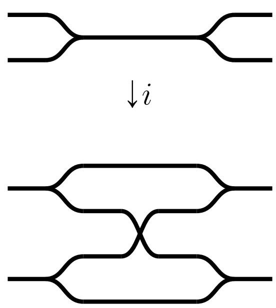
Proof of Theorems A and B. All structures on $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ are downsteam of the structures on $\mathcal{M}$

$$\begin{tikzcd}
 & \mathcal{M}\times\mathcal{M} \arrow[dl] \arrow[dr, equal] & & & \mathrm{SES} \arrow[dl] \arrow[dr] & \\
\mathcal{M} & & \mathcal{M}\times\mathcal{M} & \mathcal{M}\times\mathcal{M} & & \mathcal{M}
\end{tikzcd}$$

$\oplus$ SES

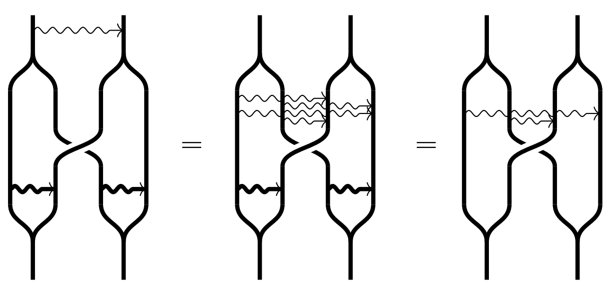
Now $(\mathcal{M}, W)$ forms a *lax* bialgebra: there is a (2-)*map*



$$\begin{tikzcd}
 & & \mathrm{SES}\times\mathrm{SES} \arrow[dll, bend right] \arrow[d, dashed, "i"] \arrow[dr, bend left] & \\
 & & \mathrm{SES}\times_{\mathcal{M}}\mathcal{M}^2 \arrow[dl] \arrow[dr] & \\
\mathcal{M}^2\times\mathcal{M}^2 \arrow[d] & \mathrm{SES} \arrow[dl] \arrow[dr, "p"] & & \mathcal{M}^2 \arrow[dl] \\
\mathcal{M}\times\mathcal{M} & & \mathcal{M} &
\end{tikzcd}$$

If i were an *isomorphism*, $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ would be a bialgebra for $\oplus^*$ in $\mathcal{C} = \mathrm{Vect}$ (defined using functoriality/Thom-Sebastiani for vanishing cycles).

In actuality, by torus localisation the diagram fails to commute up to

$e(\mathrm{N}_i)_{\mathrm{loc}} =$  $\in \mathrm{H}^\bullet(\mathrm{SES}\times\mathrm{SES})_{\mathrm{loc}}$

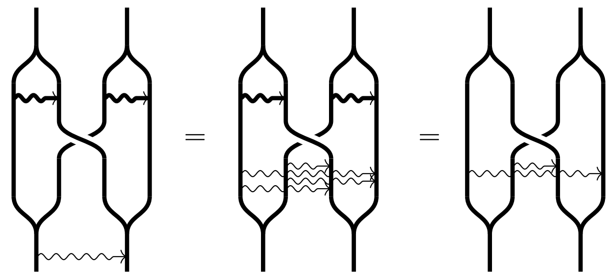
where $\rightsquigarrow = e(\mathrm{Ext}) = e(\mathrm{T}_p)$ and bold its inverse.

Compensate by *Borcherds twisting*:

$$\Delta_{\text{Joyce-Liu}} \ : \ \mathrm{H}^\bullet(\mathcal{M},\varphi) \overset{\oplus^*}{\to} \mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M},\varphi) \overset{\cdot e(\mathrm{Ext})}{\to} (\mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M},\varphi))_{\mathrm{loc}}$$

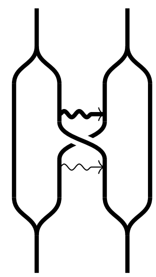
where $(-)_{\mathrm{loc}} = (-)[e(\mathrm{Ext})^{\pm}]$. The extra factor contributes

J =



Thus the difference is precisely multiplication by $R = \sigma^* e(\mathrm{Ext})/e(\mathrm{Ext})$:

$J/e(\mathbf{N}_i)$ =



$\square$

2.0.4. *Remark.* Here

$$\mathcal{C} \;=\; \mathrm{H}^\bullet(\mathcal{M})\text{-Mod}$$

with (factorisation) braiding $R\cdot\sigma$ of (factorisation) monoidal structure $\otimes$ induced by coproduct $\oplus^*$ on $\mathrm{H}^\bullet(\mathcal{M})$.

3. Part II: Coloured by cohomology generators

Question: How does this give a vertex algebra? Where's the z?

Question: Geometric interpretation of above structure?

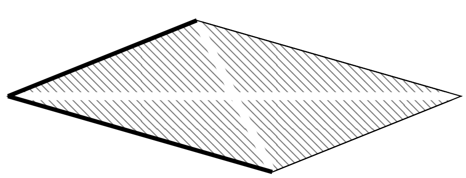
Answer: $\mathrm{H}^\bullet(\mathcal{M},\varphi)$ has the cup product action of $\mathrm{H}^\bullet(\mathcal{M})$ $\Rightarrow$ is a quasicoherent *sheaf* $\mathcal{A}$ over

$$\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1 \;=\; \coprod_{\lambda\in\pi_0(\mathcal{M})} \mathrm{Spec}\mathrm{H}^\bullet(\mathcal{M}_\lambda).$$

$\oplus \Rightarrow$ union of coloured points

$$\text{——} \times \text{——} \;\overset{\cup}{\to}\; \text{——} \qquad\qquad (\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1)^2 \;\to\; \mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1$$

$e(\mathrm{Ext}) \in (\mathrm{H}^\bullet(\mathcal{M})\otimes \mathrm{H}^\bullet(\mathcal{M}))_{\mathrm{loc}} \Rightarrow$ hyperplane arrangement, with complement



$$\overset{\circ}{\hookrightarrow}\; (\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1)^2$$

Theorem C. $\mathcal{A} = \mathrm{H}^\bullet(\mathcal{M}, \varphi)$ *is a factorisation bialgebra in* $\mathrm{QCoh}(\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1)$*, i.e.*

$$\Delta \;:\; \cup^*\mathcal{A}|_\circ \;\to\; (\mathcal{A} \boxtimes \mathcal{A})|_\circ,$$

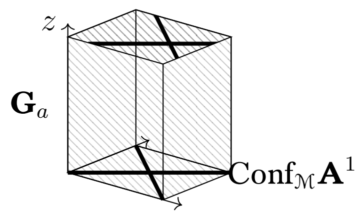
compatible with the CoHA up to $R \in \mathcal{O}((\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1)^2_\circ)$.

3.0.1. *Remark.* $\Leftrightarrow$ bialgebra in an appropriate category.

3.0.2. *Remark.* Quiver case $\Rightarrow$ agrees with "localised bialgebra" in the sense of Davison.[2]

Theorem D. (Conf-to-Ran) *There is a functor from translation-equivariant factorisation coalgebras in* $\mathrm{QCoh}(\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1)$ *to vertex coalgebras.*

This comes from the action $\mathbf{G}_a \curvearrowright \mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1$



Expanding in the punctured disk around $z = \infty \Rightarrow$ vertex coproduct.

3.0.3. *Remark.* Quiver case, have a map

$$(\mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M}))_{\mathrm{loc}} \;\to\; \mathrm{H}^\bullet(\mathcal{M}) \otimes \mathrm{H}^\bullet(\mathcal{M})((z^{-1}))$$

$$\frac{1}{x_i \otimes 1 - 1 \otimes x_j} \;\mapsto\; \frac{1}{z + x_i \otimes 1 - 1 \otimes x_j} \;=\; z^{-1} \sum_{k \geqslant 0} \left(\frac{-x_i \otimes 1 + 1 \otimes x_j}{z} \right)^k$$

3.0.4. *Remark.* Expect $\mathrm{K}, \mathrm{Ell} \rightsquigarrow \mathbf{G}_m, E$ configuration spaces.

4. Part III: Bosonisation

Let

- H-Mod braided monoidal ($\Rightarrow$ $(H, \cdot, \Delta, R)$ is quasitriangular bialgebra)
- B - bialgebra in H-Mod.

Tannakian reconstruction:

$$B\text{-Mod}(H\text{-Mod}) \;\xrightarrow{\sim}\; (H \ltimes B)\text{-Mod},$$

where

[2] We have $\mathrm{H}^\bullet(\mathcal{M}_\lambda)$-modules $A_\lambda = \mathrm{H}^\bullet(\mathcal{M}_\lambda, \varphi)$, plus

$$\Delta_{\lambda,\mu} \;:\; A_{\lambda+\mu} \;\to\; (A_\lambda \boxtimes A_\mu)_{\mathrm{loc}}$$

linear over the localised ring.

Theorem. (Majid, Radford) $H \ltimes B = H \otimes B$ *is a bialgebra with*

$$(h \otimes b) \cdot (h' \otimes b') \;=\; hh'_{(1)} \otimes (h'_{(2)} \cdot b)b' \tag{1}$$

$$\Delta(h \otimes b) \;=\; (h_{(1)} \otimes R^{(1)} \cdot b_{(1)}) \otimes (R^{(2)} h_{(2)} \otimes b_{(2)}). \tag{2}$$

Taking

- $H = \mathrm{H}^\bullet(\mathcal{M})_{\mathrm{taut}} \subseteq \mathrm{H}^\bullet(\mathcal{M})$ generated by tautological classes (i.e. by the coefficients of the R-matrix)
- $B = \mathrm{H}^\bullet(\mathcal{M}, \varphi)$

Theorem E. *The above Theorem is true for localised/vertex bialgebras, giving a formula for the* ***extended CoHA***

$$\mathrm{H}^\bullet(\mathcal{M}, \varphi)^{\mathrm{ext}} \;=\; \mathrm{H}^\bullet(\mathcal{M})_{\mathrm{taut}} \otimes \mathrm{H}^\bullet(\mathcal{M}, \varphi)$$

as a localised/vertex bialgebra.

5. Part IV: Yangians

Let $\mathcal{M} = \mathrm{Crit}(W) \subseteq \mathcal{M}_{\mathrm{Rep}Q}$.

Assume $\mathrm{H}^\bullet(\mathcal{M}_{\delta_i}, \varphi) \simeq \mathrm{H}^\bullet(\mathcal{M}_{\delta_i}) = \mathbf{C}[u_i] \cdot x^+_{i,1}$; define *spherical CoHA*

$$\langle x^+_{i,1} \;:\; i \in Q_0 \rangle \;\subseteq\; \mathrm{H}^\bullet(\mathcal{M}, \varphi) \tag{3}$$

Extend by Chern characters $H = \langle h_{i,r} \;:\; i \in Q_0, r \geqslant 0 \rangle$.

Theorem F. *Some explicit formula for* $\Delta^{\mathrm{ext}}(x^+_{i,1}, z)$ *and* $\Delta^{\mathrm{ext}}(h_{i,r}, z)$.

Proof. $x^+_{i,1}$ is primitive for Δ. Finish by computing $R(z)$. ☐

When $(Q, W) = (\overline{Q}^{(3)}, \overline{W}^{(3)})$, up to localisation (3) is an equivalance and equals the *Yangian* $Y_\hbar(\mathfrak{g}_{\overline{Q}})$.[3]

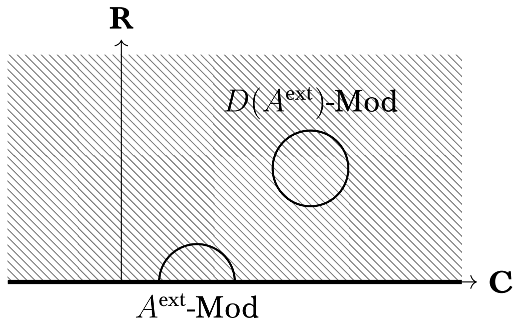
Theorem G. $\Delta^{\mathrm{ext}}(z) = \Delta_{\mathrm{Dr}}(z)$ *is Drinfeld's meromorphic coproduct on* $Y_\hbar(\mathfrak{g}_{\overline{Q}})$ *and (by Theorem D) Davison/Yang-Zhao's localised coproduct.*

[3]Due to Schiffman-Vasserot & Botta-Davison.

6. Part IV: Motivation & future

What's going on here?

Physics $\Rightarrow$ for any appropriate CY3 X we expect



e.g. $Y_\hbar(\mathfrak{g})$-Mod

$Y_\hbar(\mathfrak{b})$-Mod

where A is the CoHA $\Rightarrow$ coproducts $\Delta(z)$, and Δ_{std} on $D(A^{\mathrm{ext}})$ coacting on A^{ext}.

Analogy to finite quantum groups:

6.1. **Questions.**

- How does this relate to KPS?

Appendix A. Bonus: Yangians

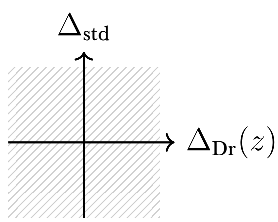
The *Yangian* $Y_\hbar(\mathfrak{g})$ (or more generally, $Y_\hbar(\mathfrak{g}_{Q,W})$ for quiver with potential (Q, W)) is an algebra deformation

$$\mathrm{gr}_\hbar Y_\hbar(\mathfrak{g}_Q) \;\simeq\; U_\hbar(\mathfrak{g}_Q)$$

Three (conjecturally) equivalent definitions (due to Schiffmann-Vasserot, Botta-Davison):

- Generated by coefficients of Maulik-Okounkov R-matrix $R_{\mathrm{MO}}(z)$.
- The bosonised Drinfeld double of the CoHA.
- Generated by coefficients of $x_i^\pm(z) = \hbar \sum_{r\geqslant 0} x_{i,r}^+ z^{-r-1}$, $h_i(z) = 1 + \hbar \sum_{r\geqslant 0} h_{i,r} u^{-r-1}$ for $i \in Q$, modulo relations.

They have *standard* and “*Drinfeld*" vertex/localised coproducts,



$$\Delta \;:\; Y_\hbar(\mathfrak{g}_{Q,W}) \;\to\; Y_\hbar(\mathfrak{g}_{Q,W}) \,\hat{\otimes}\, Y_\hbar(\mathfrak{g}_{Q,W})$$

$$\Delta_{\mathrm{Dr}}(z) \;:\; Y_\hbar(\mathfrak{g}_{Q,W}) \;\to\; Y_\hbar(\mathfrak{g}_{Q,W}) \,\hat{\otimes}\, Y_\hbar(\mathfrak{g}_{Q,W})((z^{-1}))$$

R-matrices intertwining the two coproducts.

Theorem H. *The Joyce-Liu vertex coproduct $\Delta(z)$ we defined before equals the Drinfeld coproduct*

$$\Delta(z) \;=\; \Delta_{\mathrm{Dr}}(z) \;=\; \mathrm{act}^*\Delta_{\mathrm{DYZ}}(z)$$

which also equals the Davison/ Yang-Zhao localised coprouduct.

Proof. A computation on generators using Majid-Radford’s formula. □

A.0.1. *Remark.* $R(z) = e(\mathrm{Ext}, z)$ should be the torus part in the triangular decomposition of $R_{\mathrm{MO}}(z)$.