# RESEARCH PLANS - ABRIDGED

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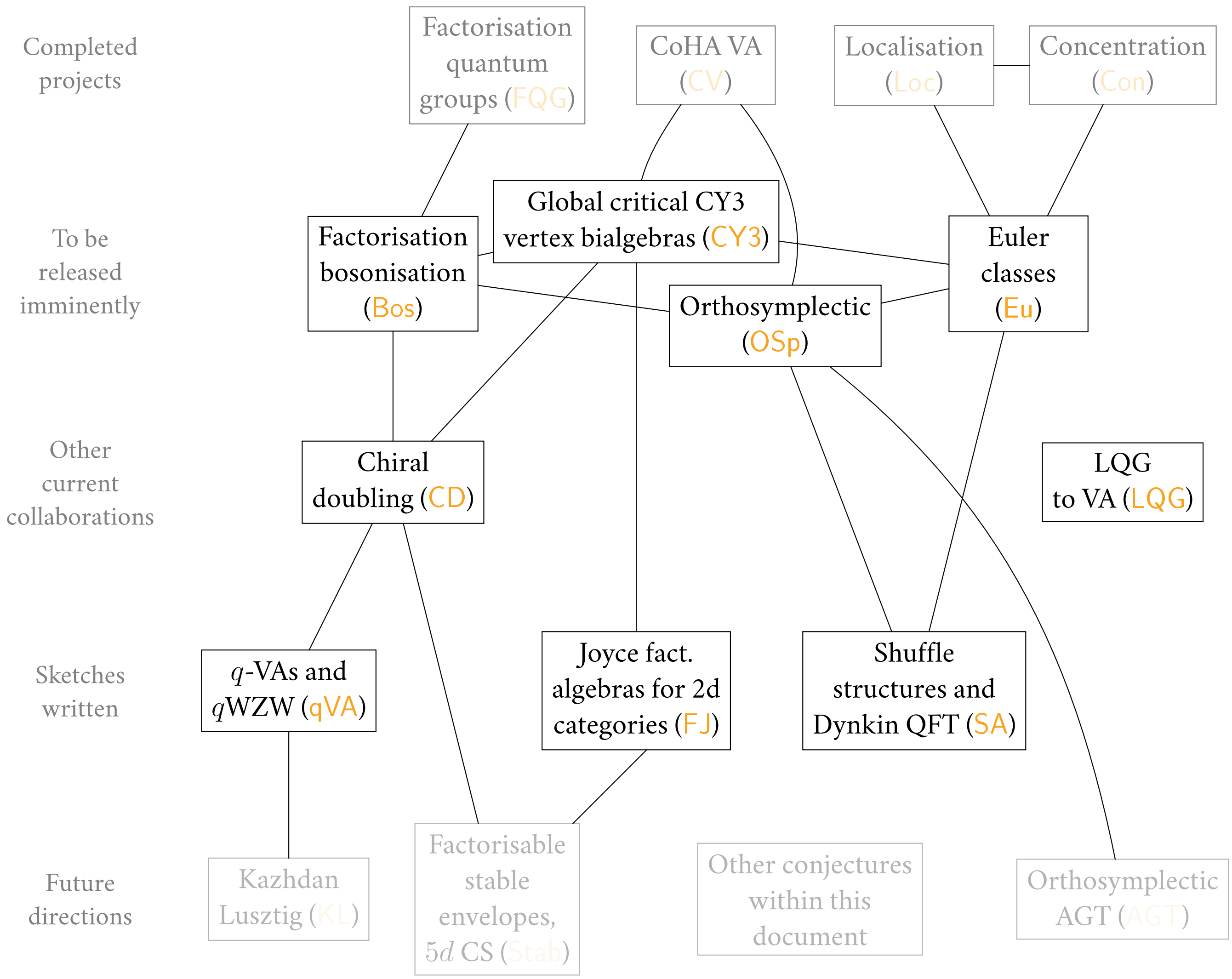
*For a detailed (13 pp.) version, including details/evidence on how we plan to prove the conjectures, see* https://alyoshalatyntsev.github.io/plan/plan.pdf.



# 1. Research statement

I am a mathematician working in geometric representation theory.

## 1.1. Algebraic structures attached to CY3s (CV, CY3 [1][2], FJ [1], Stab)

*Cohomological Hall algebras* are associative algebras attached to Calabi-Yau varieties or categories [Da; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZ] (e.g. quantum groups [MO]), hence are a key object in geometric representation theory/enumerative invariants.

*Global critical CY3 vertex quantum groups.* Let $\mathcal{C}$ be the CY3 category $\mathrm{Rep}(Q,W)$, the representations of a Jacobi algebra of a quiver with potential, or $\mathrm{Coh}_0(K_{T^*C})$, zero dimensional coherent sheaves on a local curve [KK], or more generally a *deformed CY3* completion.

**Theorem.** [CY3] *The critical cohomology*[3] *of the moduli stack of objects* $\mathcal{M}$ *has a vertex coproduct*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M},\varphi)\,\hat{\otimes}\,\mathrm{H}^\bullet(\mathcal{M},\varphi)((z^{-1}))$$

*compatible with the CoHA: it forms a* vertex quantum group *(see* FQG*).*

**Theorem.** [CY3] *There is a functor* $\Phi$ *from* $Q$*-localised bialgebras [Da] to vertex bialgebras. The former is equivalent to the category of* factorisation bialgebras *on the coloured configuration space* $\mathrm{Conf}_{Q_0}\mathbf{A}^1$.

**Theorem.** [CY3; CV for $W=0$] *For any quiver* $Q$*, the vertex coproduct on the preprojective CoHA* $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}})^{ext} \overset{[\mathrm{BD}]}{\simeq} Y_\hbar(\mathfrak{b}_Q)$ *is identified by* $\Phi$ *with the Davison/Yang-Zhao localised coproduct* [Da; YZ]*, and (when defined) Drinfeld's meromorphic coproduct* [Dr; GLW].

This generalises **Joyce-Liu's vertex coproduct** [Joa; Li], shows that all these well-known coproducts on the Yangian are **equal**, and gives a **geometric** definition using moduli stacks and configuration spaces. Finally, Theorem 1.1 points to a new Ran space definition of vertex-vertex *bi*algebras.

*Lift to factorisation algebra and relation to stable envelopes.* To move towards *arbitrary* CY3s, we first interpret the above in terms of factorisation algebras; second, we relate to the $\mathcal{W}$-algebras for surfaces of [MMSV], thus understanding the structure for $K_S$. Let $Q$ be a quiver with torus $T=\prod T_d$.

**Conjecture.** [FJ] *Given* $\mathcal{M}^f = \{(m,\lambda)\;:\;\lambda\in\mathfrak{t},\; m\in\mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d) = \mathrm{Ran}_{Q_0}\mathbf{A}^1$*, its relative BM-homology* $\mathcal{A}=\pi_*\omega$ *is a factorisation algebra over the coloured Ran space. Its restriction to the colour-diagonal recovers the vertex quantum group structure on the nilpotent CoHA* [SV].

**Conjecture.** [FJ] *For* $S$ *a smooth algebraic surface, there is a braided factorisation category* $\mathrm{Rep}\mathcal{W}$ *over* $\mathrm{Ran}_S K_S$ *(c.f.* FQG*). Applying* Bos*/*CD *allows us to construct* $\mathcal{W}(S)^{\geqslant 0}$ *and* $\mathcal{W}(S)$ *from* [MMSV]*'s* $\mathcal{W}(S)^{>0}$.

The definition of $\mathcal{M}^f$ is clearly reminiscent of [MO, §5.1.1]'s stable envelope construction, and suggests a resolution to their question: *"we view this definition as provisional; perhaps a better set of axioms will emerge later"*. Let $\mathbf{w}$ be a multidimension vector of quiver $Q$ and $M(\mathbf{w})$ the quiver variety.

**Conjecture.** [Stab] *There is a factorisation space* $\pi_{\mathbf{w}} : M(\mathbf{w})^f \to \mathrm{Ran}_Q\mathbf{A}^1$ *and the factorisation category* $\mathcal{E}$ *spanned by* $\pi_{\mathbf{w},*}\omega$ *is acted on by* $\mathcal{A}=\pi_*\omega$*. Applying chiral Tannakian reconstruction* $\mathcal{E}\simeq\mathrm{Rep}\mathbf{D}\mathcal{A}$ *gives the double of* $\mathcal{A}$ *with its (two) coproducts.*

The above is a Drinfeld-Kohno Theorem for [MO]'s Yangians $Y_\hbar(\mathfrak{g}_Q)$ (see qVA for relations to $q$KZ).

## 1.2. The structure of factorisation quantum groups (FQG, Bos [4], CD [5])

Historically the definitions of (double) affine quantum groups $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ were (ingeniously) made very explicitly [Dr; MO], and still much of the modern (e.g. CoHA) literature is based on explicit shuffle

[1] Joint with S. Kaubrys.

[2] Joint with S. Jindal.

[3] i.e. $\mathcal{M}=\mathrm{Crit}(W)$ is a critical locus inside a *smooth* moduli stack; we take the vanishing cycle sheaf $\varphi=\varphi_W$.

[4] Joint with S. de Hority.

[5] Joint with W. Niu.

computations, e.g. [MMSV; SV; YZ]. This series of projects *axiomatises* (operadically) these structures, allowing us to import techniques from the theory of ordinary *quantum groups*,[6] to recover the above formulas as a *consequence* of these definitions.

*Factorisation quantum groups.* In FQG we develop a theory of $\mathbf{E}_n$-*factorisation categories* over *factorisation spaces* $X$ (including ordinary groups $G$, configuration spaces $\mathrm{Conf}_{Q_0}\mathbf{A}^1$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$). We first give basic structure results for braided factorisation categories $\mathcal{C}$:

**Theorem.** [FQG] *Let $\mathcal{A}$ is a factorisation algebra in $\mathcal{C}$ over $X$, a (braided) factorisation structure on $\mathcal{A}$-*$\mathrm{FactMod}(\mathcal{C})$ *induces a* factorisation bialgebra *structure on $\mathcal{A}$ (and a* factorisation $R$-**matrix** $R : \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A}$*).*

**Theorem.** [FQG] *When $X = \mathrm{Ran}\mathbf{A}^1$ (resp. $\mathrm{Conf}\mathbf{A}^1$), Theorem 1.2 recovers classical notions* [EK; FR] *of **quantum vertex algebras** (resp. localised algebras) and their $R$-matrices $R(z)$ satisfying the spectral YBE.*

For instance, the structure [GLW] for **Yangians** (e.g. two (meromorphic) coproducts and $R$-matrices $R^-$, $R^{0,\epsilon}$, $R^\epsilon$ relating them) are equivalent to: $Y_\hbar(\mathfrak{g})$-Mod is a lax braided factorisation category. This makes rigorous the physics claim [CWY] that $Y_\hbar(\mathfrak{g})$-Mod from $4d$ Chern-Simons is a topological-holomorphic factorisatsion category over $\mathbf{R}\times\mathbf{C}$. The above may help understand **affine Yangians** (e.g. [GRZ]; qVA for relation to $q$-WZW) and the **new KL equivalances** [BCDN]. Finally, in principle we may get lots of examples from:

**Theorem.** [FQG] *A generalisation of Borcherds' twist construction* [Bo] *to arbitrary decomposition algebra.*

*Factorisation bosonisation.* In the CoHA literature, a lot of algebraic effort needs to be expended each time [Da; RSYZ; YZ], [CY3; OSp] to *add in the Cartan piece* $\mathrm{H}^\bullet(\mathcal{M},\varphi) \rightsquigarrow \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$, e.g. to obtain Yangians of Borels $Y_\hbar(\mathfrak{b}_Q)$. In the finite groups case, a clean construction is Tannakian reconstruction

$$U_q(\mathfrak{n}) \rightsquigarrow U_q(\mathfrak{n}) \qquad \text{via} \qquad U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

c.f. [Ga; Maa; Mab]; in Bos we apply the same to vertex and factorisation bialgebras to obtain:

**Theorem.** [Bos, in preparation] *There is a "factorisation" **Tannakian reconstruction** functor from (braided) factorisation categories $\mathcal{C}$ to (quasitriangular) factorisation quantum groups $\mathcal{A}$. In the preprojective case of Theorem 1.1,*

$$Y_\hbar(\mathfrak{b}_Q)\text{-Mod} \ \simeq\ Y_\hbar(\mathfrak{n}_Q)\text{-Mod}(Y_\hbar(\mathfrak{t}_Q)\text{-Mod})$$

*we Tannakian reconstruct $Y_\hbar(\mathfrak{b}_Q) \simeq \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$ and its localised/vertex bialgebra structure.*

Applying this to $\mathrm{H}^\bullet(\mathcal{M},\varphi)\text{-Mod}(\mathrm{H}^\bullet(\mathcal{M})\text{-Mod}^\cup)$ **automates** the process of extending CoHAs.

*Factorisation Drinfeld doubling.* An active problem is how the structures in CY3 relate to **wall crossing** and stability conditions, see [Br; Job]. As a first attempt, we use FQG to understand **doubling**, where the CoHA of heart $\mathcal{A}$ and its opposite $\mathcal{A}[1]$ are glued, in a similar way to Bos:

**Conjecture.** [CD] *There is a "factorisation" **Drinfeld centre** construction $Z_{\mathbf{E}_1}(\mathcal{C})$ of a chiral factorisation category $\mathcal{C}$, which carries compatible chiral and ordinary monoidal structures, Tannakian reconstruction gives*

$$Z_{\mathbf{E}_1}^{Y_\hbar(\mathfrak{t}_Q)\text{-Mod}}(Y_\hbar(\mathfrak{b}_Q)\text{-Mod}) \ \simeq\ Y_\hbar(\mathfrak{g}_Q)\text{-Mod},$$

*and likewise we recover the Takiff algebra double construction of* [AN].

### 1.3. Orthosymplectic structures (OSp [4], SA [4], AGT [4])

*Orthosymplectic CoHAs.* We define *orthosymplectic moduli stacks* $\mathcal{M}^{\mathrm{OSp}}$, including **perverse-coherent sheaves** with a sympletic/orthogonal bilinear form, and quivers and potential with involution.

**Theorem.** [OSp] *For $\mathcal{M}$ as in CY3 the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M},\varphi)$ acts on $\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$, i.e.*

1. *there is a left module action $a$ of the **CoHA** respecting the involution,*[7] *compatible with*
2. *a **symplectic vertex algebra** structure (factorisation coalgebra over symplectic Ran space $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colimt}_{\mathfrak{sp}_{2n}}$, see SA). The defining $R$-/$K$-matrices satisfy the **Cherednik reflection equation**.*

[6] Namely, when $X = \mathrm{Ran}\mathbf{R}^2$ in the below, via Lurie [Lu].

[7] i.e. the left action $a$ and the right action $a \cdot (\mathrm{id} \otimes \tau)$ commute, where $\tau$ is the involution.

**Theorem.** [OSp] *There is an action of* $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M})$ *of the CoHA of zero-dimensional coherent sheaves on surface* $S$ *on* $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}_{\sigma\text{-ss}})$ *the BM homology of a compactification of the stack of* $G \in \{\mathrm{Sp}, \mathrm{O}\}$*-bundles.*

This gives a rigorous definition of the objects coming from boundary $4d$ Chern-Simons theory [BS], should relate to work in $\iota$**quantum groups** [LW] and Finkelberg-Hanany-Nakajima's ongoing work on **orthosymplectic Coulomb branches** (see also AGT). In the quiver with potential case (resp. in the conjecture, type $A_{2n+1}$ preprojective), we have:

**Theorem.** [OSp] *An explicit shuffle formula for the CoHA action and vertex coaction on* $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}_{Q,W}, \varphi^{\mathrm{OSp}})$.

**Conjecture.** *The preprojective orthosymplectic CoHA*[8] *is isomorphic to the* ***twisted Yangian*** $Y_\hbar(\mathfrak{gl}_n)^{tw}$ *of* [BR].

*Dynkin QFT and shuffle structures.* The structures (2) in OSp are defined over the *symplectic configuration space*[9] $\mathrm{Conf}_{\mathrm{Sp}}(\mathbf{A}^1) = \mathrm{Spec}\,\mathrm{H}^\bullet(B\mathrm{Sp})$, with singularities over the root hyperplanes; alternatively over $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\mathfrak{t}_{\mathfrak{sp}_{2n}}$.

**Theorem.** [OSp] *To give examples: restricting along* $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$ *gives an* ***invariants*** *functor* $\iota : \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \to \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1)$. *n.b. we expect the structures in* OSp *come from applying* $\iota$ *to* $\mathcal{M}^f$ *(see* FJ*).*

In SA, we **generalise** the above to systems of **arbitrary Kac-Moody groups** $G_i$ [Ku, §V]; this forms a category $K_{\mathrm{KM}}$ with morphisms the parabolics. Write $K_{\mathrm{GL}}, K_{\mathrm{Sp}}, \ldots$ for the appropriate subcategories.

**Fact.** [SA] *A shuffle algebra* [KS; Gr] *is equivalent to a monoidal functor* $K_{\mathrm{GL}} \xrightarrow{A} \mathrm{Vect}$.

There are KM analogues of *vertex algebra* and *localised bialgebra*, *quiver representation* with moduli stack $\mathcal{M}_Q = \sqcup \mathfrak{u}_e/G_i$, *quiver varieties*, and *(factorisation) braided monoidal categories* (generalising the $G_2$-**reflection equation** [Ku]). We generalise[10] **Chen's Theorem** [Ch] on the shuffle structure on cohomology $\mathrm{H}^\bullet(LX)$ of loop spaces; leading to the

**Question.** [SA] *Can we recover Dynkin/*$q$*-analogues of* ***multiple zeta values*** [KMT; Mi] *using this* KM*-shuffle structure?*

We produce $G_2, C_n, \ldots$ structures by **folding** $D_4, A_{2n+1}, \ldots$ structures by outer automorphisms $\mathbf{Z}/3, \mathbf{Z}/2, \ldots$.

**Conjecture.** [SA] *We may recover* $\{\varnothing$*,**boundary**,* $G_2$*-, ...*$\}$ *spherical (or trigonometric)* ***KZ equations*** [ES] *by constructing a* $K_{\mathrm{GL}}, K_{\mathrm{Sp}}, K_{G_2} \ldots$ *(or* $K_{\widehat{\mathrm{GL}}}, \ldots$*)* ***affine vertex algebra*** *via BD Grassmannians, and taking conformal blocks.*

**Question.** [SA] *Can we recover Dynkin/*$q$*-analogues of* ***stable envelopes*** *(see* [MO], Stab*)?*

*A twisted AGT correspondence.* A foundational result in geometric representation theory is the **AGT correspondence** linking $\mathcal{W}$-vertex algebras and surfaces, first by Grojnowski and Nakajima [Gr; Na] then in [AGT; BFN] and [RSYZ].

**Conjecture.** [AGT] *The equivariant intersection homology of the invariant locus* $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ *in the Uhlenbeck space is a Verma module for an orthosymplectic analogue* $\mathcal{W}^k(\mathfrak{gl}_n)^{\mathrm{OSp}}$ *of* $\mathcal{W}^k(\mathfrak{gl}_n)$ *(proof sketch: use* SA*-techniques on* [BFN]*).*

**Conjecture.** [AGT] *The dimension zero CoHA of* $\mathbf{A}^3$ *(which is* $U_\hbar(\mathcal{D}(\mathbf{C}/\pm))$ *and admits* $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ *as quotients) acts on* $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$*, the equivariant intersection cohomology of* $\mathbf{A}^3$*-instantons. Likewise for any quiver with potnetial.*

## 1.4. $q$-vertex algebras (qVA, KL)

*$q$-vertex algebras.* There have been many attempts [FJW; FR; EK] to define $q$-affine vertex algebras; none are conceptual enough to apply factorisation techniques [CF]/physics heuristics [Co; GR; Wi] to relate to $q$**KZ connections** or the (conjecturally, **affine**) **Kazhdan-Lusztig equivalence**, or new KL equivalences [BCDN].

**Conjecture.** [qVA] *There is a factorisation category* $\mathcal{D}$-Mod *over the noncommutative plane*[11] $\mathbf{A}^2_q$*, a factorisation algebra in which induces a* $q$*-**vertex algebra*** *(i.e. ordinary vertex algebra with poles on the* $q$*-diagonals, e.g.* [FR], [EK] *for* $\mathfrak{g} = \mathfrak{sl}_n$*).*

**Conjecture.** [qVA] *There is an analogue of the BD Grassmannian* $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$ *(this induces a* $q$*-affine VOA* $V^k_q(\mathfrak{g})$*).*

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[8] i.e. the image of the CoHA in $\mathrm{End}(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}))$.

[9] As opposed to the ordinary configuration space $\mathrm{Conf}\mathbf{A}^1 = \sqcup \mathbf{A}^n/\!/\mathfrak{S}_n = \mathrm{Spec}\,\mathrm{H}^\bullet(B\mathrm{GL})$.

[10] Replacing the $n$-simplex $\Delta^n$ with $\Delta_G = T_G/W_G$.

[11] Its with ring of functions $\mathbf{C}\langle x, y, q\rangle/(yx - xyq)$ with $q$ central.

The physics heuristic for this is: $V^k(\mathfrak{g})$ comes from $3d$ Chern-Simons with boundary $\mathbf{C}$ [Wi]; $V^k_q(\mathfrak{g})$ comes from $5d$ **Chern-Simons** with noncommutative boundary $\mathbf{C}^2_{nc}$ [Co; GR; GRZ]. The category $\mathcal{D}$-Mod will be related to $q$-**difference modules**[12] on $\mathbf{A}^1$, so the above directly generalises the usual factorisation definition of vertex algebras.

**Conjecture.** [KL] *The restriction of **conformal blocks** of $V^k_q(\mathfrak{g})$-modules to $(\mathbf{A}^1)^n_\circ \subseteq (\mathbf{A}^2_q)^n_\circ$ equal to the qKZ connection.*

**Conjecture.** [KL] *There is a **Zhu algebra** functor,* Zhu $: V^k_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g})$*, and the functor* RH $: V^k(\mathfrak{g})$-Mod $\mapsto U_q(\mathfrak{g})$-Mod *in the proof* [CF] *of KL is "parallel transport" along a factorisation category* $V^k_q(\mathfrak{g})$-Mod *on* $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$.

The above would help understand **affine KL** and the **new KL equivalences** [BCDN] from $3d$ mirror symmetry.

## 1.5. Sheaf methods (Con [13], Loc [4], Eu)

*Localisation methods.* We generalise *torus localisation* and *Graber-Panharipande formulas* [GP] (one of the main technique in enumerative geometry) to the Artin moduli stacks appearing in modern enumerative/algebraic geometry.

**Theorem.** [Conc; Loc] *We give conditions for the cohomology of an Artin stack $\mathcal{X}$ to be **concentrated** on a closed substack $\mathcal{Z}$; when $\mathcal{Z} = \mathcal{X}^T$ is fixed points of a quasismooth dg scheme we give **Atiyah-Bott*** $\mathrm{id} = i_*\left(i^!(-)/e(N_{vir})\right)$ *and **Graber-Pandharipande localisation** formulas* $[\mathcal{X}]^{vir} = i_*\left([\mathcal{X}^T]^{vir}/e(N_{vir})\right)$.

**Theorem.** [Eu] *We strengthen both parts of the theorem to **critical cohomology** of **arbitrary** closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ quasismooth over a common base. As a result, for $\mathcal{M}$ as in CY3, we have the following Theorem:*

**Theorem.** [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we have* $\mathrm{CoHA} = \mathrm{CoHA}^s/e(\mathrm{N}_{i,vir})$.

Taking $\mathcal{M}^s$ a *shuffle space*[14], this gives a **universal, geometric** way of producing/interpreting shuffle products for CoHAs [Da; SV; YZ]. Taking $\mathcal{M}^s = \mathcal{M} \times \mathcal{M} \xrightarrow{\oplus} \mathcal{M}$ proves **compatibility** [CV,CY3,Li] between **coproducts** and the CoHA (c.f. CY3). This easily **generalises** to the OSp/Dynkin SA cases.

## 1.6. Liouville quantum gravity to vertex algebras (LQG [15])

Probabilists are beginning construct Feynman measures (GFF [BPR], LQG [KRV]) **rigorously** to understand $2d$ CFTs [CRV; DS; Sh]. There is almost no intereaction with the algebraic geometry/functorial school to QFT, and they understand/have access to techniques (e.g. SLE curves [MS; SS], combinatorial approximations) that the latter school does not. We want to build a bridge:

**Conjecture.** [LQG] *There is a **chiral part** functor* $F^{ch} : \mathrm{CFT} \to \mathrm{VertexAlg}$, *from Segal-style full $2d$ CFTs to vertex algebras. It sends* [GKRV]*'s LQG (resp. GFF) to the Virasoro (resp. Heisenberg) VOA, and* [KRV]*'s DOZZ formula to the Virasoro OPE.*

We will then use **free field embeddings** to produce a probabilistic CFT analogue of the $\mathfrak{g}$-**affine vertex algebra**.

[12] e.g. a $q$-difference operator $\partial_x$ on $\mathbf{A}^1$ induces a derivation $y\partial_x$ on $\mathbf{A}^2_q$.

[13] Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.

[14] i.e. shuffle algebra in the category of spaces, see SA.

[15] Joint with V. Giri.

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