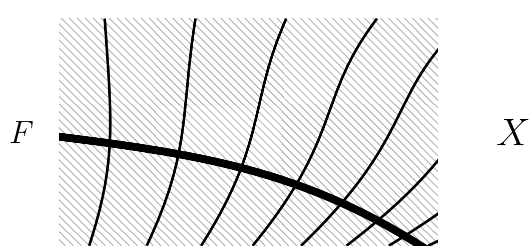
STABLE ENVELOPES

ALEXEI LATYNTSEV

Contents

1. Stable envelopes

Here we explain the stable envelope construction of spectral R-matrices due to Maulik and Okounkov [MO].



Let X be a space acted on by torus A. Pictured above is the induced flow under character $\lambda \in \mathfrak{a}$.

1.1. **Toy model of stable envelopes.**

For F a component of the fixed locus, take

$$\begin{tikzcd} & N_F^+ \arrow[dl, "q"'] \arrow[dr, "\exp"] & \\ F & & X \end{tikzcd} \tag{1}$$

where N_F^+ is the subspace of the normal bundle to F on which λ acts by positive weight. If the image of the exponential map is closed in X, it is proper and so push-pulling along these maps give

$$\mathrm{Stab}_\lambda \;:\; \mathrm{H}^{\mathrm{BM}}_\bullet(F) \;\to\; \mathrm{H}^{\mathrm{BM}}_{\bullet+2\mathrm{rk}N_F^+}(X).$$

All the maps in (1) are equivariant with respect to the torus action, so Stab_λ upgrades to a map on A-equivariant Borel-Moore homology. Note that by construction:

- The image of $N_F^+ \to X$ lies in the attracting locus of F.
- (push-pull is related to Euler class via excess intersection formula), thus
- Stab_λ is an isomorphism on equivariant cohomology, after removing enough planes in $\mathrm{Spec}\,\mathrm{H}^\bullet_A(\mathrm{pt})$ so that (such that the weights of N_F^+ are invertible in the localisation)

We now want to show compatibility if we restrict to a subtorus $A' \subseteq A$. (For this, we need to modify N_F^+ a little bit, corresponding to condition (iii) in [MO]; i.e. compatibility with respect to strata)

1.1.1. *Spectral R-matrices.* The spectral R-matrix is then given by the composition

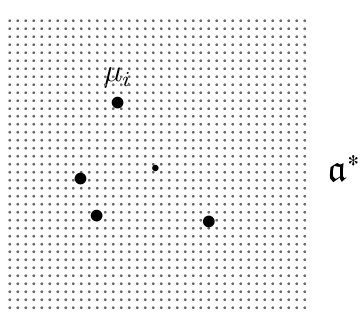
$$R_\lambda(z) \;:\; \mathrm{H}^{\mathrm{BM}}_\bullet(F/A) \;\overset{\mathrm{Stab}_\lambda}{\to}\; \mathrm{H}^{\mathrm{BM}}_{\bullet+2\mathrm{rk}N_F^\pm}(X) \;\overset{\mathrm{Stab}_{-\lambda}}{\leftarrow}\; \mathrm{H}^{\mathrm{BM}}_\bullet(F/A)$$

where $z \in \mathrm{Spec}\,\mathrm{H}^\bullet_A(\mathrm{pt}) \simeq \mathfrak{a}^*$. Here we see that in order for it to have cohomological degree zero, we must have $\mathrm{rk}N_F^+ = \mathrm{rk}N_F^-$.

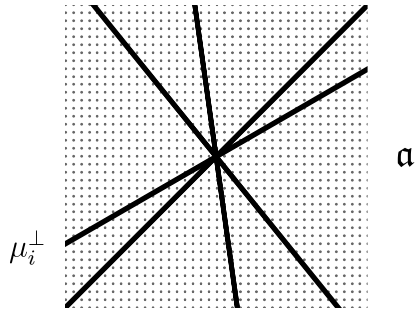
1.1.2. *Varying the character.* The normal bundle decomposes into torus weights $N_F = \bigoplus_{\mu\in\mathfrak{a}^*} N_F^\mu$. Thus as λ varies, we get a decomposition

$$N_F \;=\; N_F^- \oplus N_F^0 \oplus N_F^+$$

according to the sign of $\mu(\lambda)$.



In particular, so long as $\lambda \in \mathfrak{a}$ avoids the codimension one walls cut out by the $\mu_i^\perp$, the map Stab_λ will be constant inside each chamber.



Moreover, the singularities of $R_\lambda(z)$ will be along the $\mu_i^\perp$. In coordinates, if $\Lambda = \mathbf{Z}\{z_1, \ldots, z_n\} \subseteq \mathfrak{a}^*$ is the cocharacter lattice and

$$\mu = a_1(\mu)z_1 + \cdots + a_n(\mu)z_n$$

then $R_\lambda(z_1, \ldots, z_n)$ will have singularities along the hyperplanes $a_1(\mu)z_1 + \cdots + a_n(\mu)z_n = 0$ as μ varies in the normal bundle.

1.1.3. *On the walls.* Note that the walls $\mu_i^\perp$ correspond to the characters λ for which

$$N_F^0 \neq 0.$$

1.2. **Actual definition of stable envelopes.** [MO, Prop 3.5.1]

2. Summary

2.1. What structure does Stab have? We have for every $\mu \in \mathfrak{a}$ maps

$$\begin{tikzcd} \mathrm{H}^\bullet(X^A) \arrow[rr, "\mathrm{Stab}_\mu"] \arrow[dr, "\mathrm{Stab}_{\mu/\mu'}"'] & & \mathrm{H}^\bullet(X) \\ & \mathrm{H}^\bullet(X^{A'}) \arrow[ur, "\mathrm{Stab}_{\mu'}"'] & \end{tikzcd}$$

which says that Stab is *associative* (modulo choice of μ, μ').

For instance, when A has rank three we have

$$\begin{tikzcd} & \mathrm{H}^\bullet(X \times X) \arrow[dr] & \\ \mathrm{H}^\bullet(X \times X \times X) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathrm{H}^\bullet(X) \\ & \mathrm{H}^\bullet(X \times X) \arrow[ur] & \end{tikzcd}$$

Moreover, we get

$$R_\mu \ : \ \mathrm{H}^\bullet(X^\mu) \overset{\mathrm{Stab}_\mu}{\to} \mathrm{H}^\bullet(X) \overset{\mathrm{Stab}_{-\mu}}{\leftarrow} \mathrm{H}^\bullet(X^{-\mu})$$

so that Stab_μ is some sort of "factorisation product" that becomes braided-cocommutative after localisation.

Compare this with when A is a braided-commutative strong algebra, so that the R-matrix is

$$R \ : \ A \otimes A \overset{m}{\to} A \overset{m\otimes\sigma}{\leftarrow} A \otimes A.$$

2.2. Tannakian version of the FRT construction. Let us say that by hook or by crook we have invented a braided monoidal category $(\mathcal{C}, \beta)$ with its associated R-matrix $R : c_1 \otimes c_2 \to c_2 \otimes c_1$. We assume that there is a monoidal map to a background symmetric monoidal category $(\mathcal{E}, \sigma)$. Then we get

$$\begin{tikzcd} & Y\text{-Mod}(\mathcal{E}, \sigma) \arrow[d] \\ (\mathcal{C}, \beta) \arrow[ur, dashed] \arrow[r] & (\mathcal{E}, \sigma) \end{tikzcd}$$

constructed as follows. $Y = Y_c$ depends on an element $c \in \mathcal{C}$. It is the subalgebra of $\tilde{Y} = \mathrm{End}(\mathrm{oblv})$ generated by

$$\mathrm{tr}\left(c \otimes W \overset{m\otimes\mathrm{id}}{\to} c \otimes W \overset{R}{\to} c \otimes W\right) \ = \ \left(W \overset{\mathrm{tr}_{c,R}}{\to} W\right)$$

where $m \in \mathrm{End}(c)$ and $W \in \mathcal{C}$.

There is a map

$$T^\otimes\left(\bigoplus_{c_i \in \mathbf{O}\mathcal{C}} \mathrm{End}(c_i)\right) \ \twoheadrightarrow \ Y_c.$$

For formal reasons there is a coproduct, the *standard coproduct*, on Y; this is because Y-Mod$(\mathcal{E}, \sigma)$ is braided monoidal. It is braided cocommutative for R.

2.3. **Stability and factorisation parameters.**

We have

$$\begin{array}{ccccc} \mathrm{H}^\bullet_T(X^A) & \xrightarrow{\mathrm{Stab}_\lambda} & \mathrm{H}^\bullet_T(X) & \xleftarrow{\mathrm{Stab}_{-\lambda}} & \mathrm{H}^\bullet_T(X^A) \\ \wr\downarrow\mathrm{Fact} & & & & \wr\downarrow\mathrm{Fact} \\ \mathrm{H}^\bullet_T(X\times\cdots\times X) & & & & \mathrm{H}^\bullet_T(X\times\cdots\times X) \end{array}$$

The horizontal and vertical should be viewed as two different algebra structures.

Notice that

$$\mathrm{H}^\bullet_T(X^\lambda)_\mu \;=\; \mathrm{H}^\bullet_T(X^\lambda)\otimes_{\mathrm{H}^\bullet(\mathrm{B}T)} \mathrm{H}^\bullet(\mathrm{B}T)/\mu$$

where $\lambda, \mu \in \mathfrak{t} = \operatorname{Spec}\mathrm{H}^\bullet(\mathrm{B}T)$ are prime ideals in $\mathrm{H}^\bullet(\mathrm{B}T)$.

2.4. **Which torus?**

The torus acting on $\mathcal{M}(w)$ is $T_w = \prod T_{w_i}$, and so we get a map to the *coloured* Ran space $\mathrm{Ran}^{|Q|}_{st}\,\mathbf{C}$.

2.4.1.

We can define *coloured* vertex algebras. There is a forgetful map

$$\mathrm{Ran}^{|Q|}\,\mathbf{C} \;\to\; \mathrm{Ran}\,\mathbf{C}$$

taking a coloured finite subset of $\mathbf{C}$ and forgetting its colour. In particular, every vertex algebra is a coloured vertex algebra (whose structure does not depend on the colour).

2.5. **What you need to localise.**

To get

$$\mathrm{H}^\bullet_T(X^\lambda)_\mu \overset{i_*}{\underset{\sim}{\to}} \mathrm{H}^\bullet_T(X)_\mu$$

an isomorphism, we need that $\mathrm{H}^\bullet(X \smallsetminus X^\lambda)_\mu = 0$, which happens if μ is off the λ-diagonal, i.e. we need that

$$X^\mu \;\hookrightarrow\; X^\lambda$$

because then $X \smallsetminus X^\lambda \subseteq X \smallsetminus X^\mu$, which indeed has $\mathrm{H}^\bullet(X\smallsetminus X^\mu)_\mu = 0$.

Conclusion:

$$\mathrm{H}^\bullet(X^\lambda) \overset{\mathrm{Stab}_\mu}{\to} \mathrm{H}^\bullet(X) \overset{\mathrm{Stab}_{-\mu}}{\leftarrow} \mathrm{H}^\bullet(X^{-\lambda})$$

is an isomorphism if μ is off the λ-diagonals.

2.6. **Torus localisation as factorisation of $\mathrm{H}^\bullet(Y^f/T)$.** Let $A \subseteq T$ be a subtorus, and Y^f be a factorisation space over $\mathfrak{t}_\infty$.

Note that $\mathrm{H}^\bullet_T(Y)$ is a quasicoherent sheaf over $\mathfrak{t} = \operatorname{Spec} \mathrm{H}^\bullet(\mathrm{B}T)$. Moreover, we have that

$$Y^A/A \to Y/A$$

satisfies torus localisation, and likewise $Y^\lambda/A \to Y/A$ for $\lambda \in \mathfrak{a}$ away from the diagonal; it is true on the diagonals too as Y^λ will only get bigger that way.

Conclusion:

$$\mathrm{H}^\bullet(Y^\lambda/A)_\lambda \xrightarrow[\sim]{i_*} \mathrm{H}^\bullet(Y/A)_\lambda$$

is an isomorphism for all $\lambda \in \mathfrak{a}$.

2.7. **Space-level stable envelopes as exit path functors.** The toy model for stable envelopes is

$$\begin{tikzcd} & N^+_{F,\lambda} = \mathrm{Stab}_\lambda \arrow[dl] \arrow[dr] & \\ F = X^T = X^\lambda & & X \end{tikzcd}$$

where λ is an off-diagonal element of $\mathfrak{t}$. We we view this as an analogue of

$$X \otimes^{ch} X \to X.$$

For a subtorus $T' \subseteq T$, we get a *relative* stable envelope construction

$$\begin{tikzcd} & \mathrm{Stab}_{\lambda \bmod \mathfrak{t}'} \arrow[dl] \arrow[dr] & \\ F = X^T = X^\lambda & & X^{T'} = X^{\lambda'} \end{tikzcd}$$

where λ' is an off-diagonal element of $\mathfrak{t}'$.

2.7.1. *Viewing this as a perverse sheaf.* Recall that a constructible sheaf on $\operatorname{Ran} \mathfrak{t}_\infty$ with respect to the diagonal stratification consists of a vector space $\mathcal{E}_\lambda$ for every λ, along with monodromy data and a map

$$i : \mathcal{E}_\lambda \to \mathcal{E}_{\lambda'}$$

whenever λ' sits on a smaller diagonal. i.e. it is a functor from the exit path category to Vect. As part of this data is the monodromy[1]

$$R_\lambda : \mathcal{E}_\lambda \xrightarrow{\sim} \mathcal{E}_\lambda$$

which is *topological* in λ, i.e. locally constant.

If $\mathcal{X}$ is a space-valued functor from the exit path category and the maps i are closed embeddings, then $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{X})$ defines a perverse sheaf. Note that $\mathcal{X}$ glues to a space over $\operatorname{Ran} \mathfrak{t}_\infty$. (check)

[1]Around a given diagonal, which we have omitted from the notation, and on the rank two torus we can do this without loss of information.

2.7.2. We now upgrade this to the holomorphic setting, i.e. where R_λ and the rest of the data depends holomorphically, not just topologically, on the base.

Let us include T-equivariance.

Lemma 2.7.3. *Let $\pi : Y \to \mathfrak{t}$ be a T-equivariant map of spaces for the adjoint (trivial) action on $\mathfrak{t}$. Then we have a map*

$$\overline{\pi} : Y/T \to \mathfrak{t}$$

which factors as

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "\pi"'] & Y/T \arrow[d, "\overline{\pi}"] \\ & \mathfrak{t} \end{tikzcd}$$

The fibres of $\overline{\pi}_*\omega$ are $\mathrm{H}^{\mathrm{BM}}_{T,\bullet}(Y^\lambda)$.

Now let $\pi : Y \to \mathfrak{t}$ be a T-equivariant map, where the (nontrivial) action of T on $\mathfrak{t}$ is by multiplication. Then we have maps

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\pi"] & Y/T \arrow[d, "\overline{\pi}"] \\ \mathfrak{t} \arrow[r, "\overline{f}"] & \mathfrak{t}/T \end{tikzcd}$$

and if $\lambda \in \mathfrak{t}$ is generic, the fibre of $\overline{\pi}_*\omega$ over the point $T\cdot\lambda/T$ is $\mathrm{H}^{\mathrm{BM}}_\bullet(Y^{T\cdot\lambda} = Y^\lambda)$. However, if λ' lies on the diagonal and is stabilised by $T'' = T/T'$, then if we consider

$$\begin{tikzcd} Y' \arrow[r, "f"] \arrow[d, "\pi"] & Y'/T \arrow[d, "\overline{\pi}"] \\ \mathfrak{t}' \arrow[r, "\overline{f}"] & \mathfrak{t}'/T' \end{tikzcd}$$

then the fibre of $\overline{\pi}_*\omega$ over the point $T'\cdot\lambda/T'$ is $\mathrm{H}^{\mathrm{BM}}_{\bullet,T''}(Y^{T'\lambda} = Y^{\lambda'})$.

Thus in the above, the equivariant parameters attached to a diagonal $\mathfrak{t}' \subseteq \mathfrak{t}$ correspond to the *off-diagonal* weights in $\mathfrak{t}/\mathfrak{t}'$. This is consistent with the general principle of things having poles along the diagonals.

Our setup is the following: we have

$$(\mathrm{Ran}\,\mathfrak{t}_1)/T_\infty = \mathrm{colim}\,(\mathfrak{t}/T_\infty)$$

and we can decompose $\mathfrak{t}/T_\infty = \mathfrak{t}/T \times \mathrm{B}(T_\infty/T)$. We then want to be taking the T_∞/T-equivariant homology of the fixed locus above $\mathfrak{t}$.

2.7.4. (still need to get a R_λ depending holomorphically on λ)

2.8. Factorisation Tannakian FRT construction. Consider the coloured factorisation monoidal category spanned by $\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{M}^{\mathrm{Nak}}_w)$. The stable envelope

$$R_{st}(z) \qquad \rightsquigarrow \qquad \Delta_{st}(z) \text{ on } Y$$

braided cocommutative for $R_{st}(z)$. Note that Δ_{st} is holomorphic.

Thus for arbitrary quivers Q, we get a category $\mathcal{C}$ with a factorisation monoidal structure,

$$\mathcal{C} \ \in \ \mathrm{FactCat}(\mathrm{Ran}^{|Q|}_{st}\,\mathbf{C})$$

and a Tannakian construction giving

$$\begin{tikzcd} & \mathcal{Y}\text{-FactMod} \arrow[d] \\ (\mathcal{C},\beta) \arrow[r] \arrow[ur, dashed] & (\mathcal{E},\sigma) = \mathcal{D}\text{-Mod}(\mathrm{Ran}^{|Q|}_{st}\,\mathbf{C}), \otimes^{ch} \end{tikzcd}$$

where the bottom map is $\mathbf{E}_1$-factorisation monoidal.

2.8.1. When $Q = Q_0^{trip}$ there is *another* factorisation monoidal structure on $\mathcal{M}^{\mathrm{Nak}}_w$, coming from the eigenvalue of the tripled quiver.

We have three different spaces with the same fibre (or the nilpotent version in the $eval$ case):

$$\begin{tikzcd} \mathcal{M}^{\mathrm{Nak}}_w & \mathcal{M}^{1,\mathrm{Nak}}_w \arrow[d] & \mathcal{M}^{2,\mathrm{Nak}}_w \arrow[d] \\ & \mathrm{Ran}^{|Q|}_{st}\,\mathbf{C} & \mathrm{Ran}^{|Q|}_{st}\,\mathbf{C} \times \mathrm{Ran}_{eval}\,\mathbf{C} \end{tikzcd}$$

Note that Δ_{eval} will be holomorphic but *not* cocommutative - it is $e(\mathbf{N}_s)\oplus^*$ in the moduli stack case.

We have

$$R_{eval}(z) \ = \ \mathrm{act}^*\, e(\mathbf{N}_s)/\,\mathrm{act}^*\, e(\sigma^*\mathbf{N}_s) \qquad \rightsquigarrow \qquad \Delta_{eval} \text{ on } Y$$

which is braided cocommutative for $R_{eval}(z)$.

Thus the above contruction for a general quiver gives a category $\mathcal{C}$ with two factorisation structures,

$$\mathcal{C} \ \in \ \mathrm{FactCat}(\mathrm{Ran}^{|Q|}_{st}\,\mathbf{C} \times \mathrm{Ran}_{eval}\,\mathbf{C})$$

giving

$$\begin{tikzcd} & \mathcal{Y}\text{-FactMod} \arrow[d] \\ (\mathcal{C},\beta) \arrow[r] \arrow[ur, dashed] & (\mathcal{E},\sigma) \end{tikzcd}$$

where $\mathcal{Y}$ has two coproducts, the standard coproduct $\Delta_{st}(z)$ and the evaluation coproduct $\Delta_{eval}(z)$.

2.8.2. Note that to get $\mathrm{H}_T^\bullet(\mathcal{M}^{\mathrm{Nak}})$, we need to take

$$\begin{array}{c} \mathcal{M}_w^{1,\mathrm{Nak}}/T_\infty \\ \downarrow \\ \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} = \mathfrak{t}_\infty \end{array}$$

Note that

2.9. **Constructing $\mathcal{C}$.** We want that for every dimension vector w,

$$\mathrm{H}_T^\bullet(\mathcal{M}_w^{1,\mathrm{Nak}}) \;\in\; \Gamma(\mathrm{Ran}_{st}^{|Q|}\,\mathbf{C}, \mathcal{C}).$$

Note that a priori we only have

$$\mathrm{H}_T^\bullet(\mathcal{M}_w^{1,\mathrm{Nak}}) \;\in\; \Gamma(\mathrm{Ran}_{st}^{|Q|}\,\mathbf{C}, \mathrm{QCoh}),$$

so to construct $\mathcal{C}$ as an $\mathbf{E}_1$ factorisation category we are going to take a sum of $\mathrm{QCoh}\,|_{\mathrm{Supp}_w}$.

2.9.1. (potentially delete what follows)

2.9.2. We recall what a holomorphic factorisation space X is. We have

$$\cup \;:\; \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} \times \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} \;\to\; \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C}$$

and

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ X \times X & & \cup^* X \end{tikzcd}$$

For instance, its restriction over

$$\mathbf{C} \times \mathbf{C} \;\to\; \mathbf{C}^2$$

is

$$\begin{tikzcd} & \mathrm{isom} \sqcup \mathrm{Stab} \arrow[dl] \arrow[dr] & \\ X_1 \times X_1 & & X_2 \sqcup X_1|_\Delta \end{tikzcd}$$

where isom is the ordinary factorisation structure and Stab is the stable envelope structure. Fibrewise, this is

2.9.3. A factorisation space structure on $\mathcal{M}_w^{1,\mathrm{Nak}}$ is given by

$$\begin{tikzcd} & \mathrm{Stab} \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathcal{M}^{1,\mathrm{Nak}} \times \mathcal{M}^{1,\mathrm{Nak}} \arrow[d] & (\mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} \times \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C})_\circ \arrow[dl] \arrow[dr] & \mathcal{M}^{1,\mathrm{Nak}} \arrow[d] \\ \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} \times \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} & & \mathrm{Ran}_{st}^{|Q|}\,\mathbf{C} \end{tikzcd}$$

and the fibres of the above correspondence are

$$\begin{tikzcd}[column sep=small]
& \mathrm{Stab}_{\lambda_1,\lambda_2} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{M}^{1,\mathrm{Nak}}_{\lambda_1} \times \mathcal{M}^{1,\mathrm{Nak}}_{\lambda_2} \arrow[d] & (\lambda_1,\lambda_2) \arrow[dl] \arrow[dr] & \mathcal{M}^{1,\mathrm{Nak}}_{\lambda_1 \boxplus \lambda_2} \arrow[d] \\
(\lambda_1,\lambda_2) & & \lambda_1 \boxplus \lambda_2
\end{tikzcd}$$

and $\mathrm{Stab}_{\lambda_1,\lambda_2} = \mathrm{Stab}_{\lambda_1-\lambda_2}$ only depends on the difference of $\lambda_i \in \mathfrak{t}_\infty$, i.e. $\lambda_i : \mathbf{G}_m \to T_\infty$.

When $\lambda_i \in \mathfrak{t}_1$, this is

$$\begin{tikzcd}
& \mathrm{Stab}_{\lambda_1,\lambda_2} \arrow[dl] \arrow[dr] & \\
X \times X & & X^{\lambda_1 \boxplus \lambda_2}
\end{tikzcd}$$

3. Relations to vertex quantum groups

3.1. We often have the following structure on moduli spaces $\mathcal{M}$. We have a PGL_n action on $\mathcal{M}$ so that

$$\mathcal{M}^{A_n} = \mathcal{M} \times \cdots \times \mathcal{M}$$

where $A_n \subseteq \mathrm{GL}_n$ is the maximal torus and there are n multiplicands.[2] Moreover, for a generic character $\lambda \in \mathfrak{a}_n^*$, we also expect

$$\mathcal{M}^{\lambda} = \mathcal{M} \times \cdots \times \mathcal{M}$$

which implies that perhaps we should work on $\mathfrak{a}_n^*$. This suggests forming the space of pairs

$$\mathcal{M}^f = \{(m, \lambda) : \lambda \in \mathfrak{a}^*, m \in \mathcal{M}^{\lambda}\}$$

where A is any torus acting on $\mathcal{M}$. There is a natural projection map $\mathcal{M}^f \to \mathfrak{a}^*$, this *factorises* in the sense that if Δ denotes the union of the walls, then

$$\mathcal{M}_n^f|_{\mathfrak{a}_n^* \setminus \Delta} \simeq (\mathcal{M}_1^f \times \cdots \times \mathcal{M}_1^f)|_{\mathfrak{a}^* \setminus \Delta}.$$

Note that

Lemma 3.1.1. *If a representation of A_n extends to an action of PGL_n, then the weights μ appearing in the representations are direct sums of $z_i - z_j$ for $i \neq j$.*

It follows that $\Delta \subseteq \bigcup \Delta_{ij}$ is contained in the union of hyperplanes $z_i - z_j = 0$, since if $e^{z_i - z_j} \in \mathbf{C}[\mathfrak{a}^*]$ is inverted, then so too is any product of them $e^{\sum a_{ij}(z_i - z_j)}$.

Also note that for $\lambda = (1, ..., 1)$ corresponding to the character of $\mathbf{C}^\times \subseteq \mathrm{GL}_n$, we have

$$\mathcal{M}^f|_{\mathbf{C}\cdot\lambda} \simeq \mathcal{M}^f|_0 \times \mathbf{C}.$$

Note that for this character, $N_F = N_F^0$ since the action on $\mathcal{M}$ is in fact trivial. Moreover,

Lemma 3.1.2. *The space $\mathcal{M}^f \to \mathfrak{a}^*$ is $\mathbf{G}_a$-equivariant for the translation action sending*

$$t : (m, \lambda) \mapsto (m; \lambda + t(1, ..., 1)).$$

3.1.3. Consider the Nakajima quiver variety

$$\mathcal{M} = \bigcup \mathcal{M}(v, w)$$

where $w \in \mathbf{N}^I$. The component $\mathcal{M}(v, w)$ is acted on by $G_w = \prod \mathrm{GL}_{w_i}$. Thus, we have an action of

$$G = \prod G_w$$

on all of $\mathcal{M}$. Writing $\mathcal{M}(w) = \bigcup_v \mathcal{M}(v, w)$, we have compatible inclusions

$$\mathbf{C}^{w_1} \oplus \mathbf{C}^{w_2} \xrightarrow{\sim} \mathbf{C}^{w_1 + w_2}, \qquad \mathcal{M}(w_1) \times \mathcal{M}(w_2) \hookrightarrow \mathcal{M}(w_1 + w_2), \qquad G_{w_1} \times G_{w_2} \hookrightarrow G_{w_1 + w_2}.$$

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[2]For future reference, we need to work with $\mathrm{GL}_n = \mathrm{PGL}_n \times \mathbf{C}^\times$ where the last factor acts trivially on $\mathcal{M}$.

We now define

$$\mathcal{M}(\infty) \;=\; \operatorname{colim}_{w\in\mathbf{N}^I}\left(\mathcal{M}(0) \hookrightarrow \cdots \hookrightarrow \mathcal{M}(w) \hookrightarrow \cdots\right),$$

$$G_\infty \;=\; \operatorname{colim}_{w\in\mathbf{N}^I}\left(G_0 \hookrightarrow \cdots \hookrightarrow G_w \hookrightarrow \cdots\right)$$

where the colimit is over all I-coloured finite sets and surjections between them. Here the inclusions are given by taking direct sums with elements $\mathbf{C}^{w'} \in \mathcal{M}(0,w')$. The action

$$G_\infty \times \mathcal{M}(\infty) \;\to\; \mathcal{M}(\infty)$$

is induced by the maps

$$G_{w_1} \times \mathcal{M}(w_2) \;\hookrightarrow\; G_{w_1+w_2} \times \mathcal{M}(w_1+w_2) \;\to\; \mathcal{M}(w_1+w_2)$$

where the last map is the action map we started with. Likewise, G_∞ is an group.

3.1.4. In what follows, we will work with $\mathcal{M}(\infty)$. However, $\mathcal{M}$ itself would work equally well, since we have an action of T_∞ and a map from $\mathcal{M}^f$ to $\mathfrak{t}_\infty = \operatorname{Ran}^{|Q|}_{st}\mathbf{C}$ on that too.

3.1.5. Notice that

Lemma 3.1.6. *The Lie algebra of*

$$T_\infty \;=\; \operatorname{colim}(T(\mathrm{GL}_1) \hookrightarrow T(\mathrm{GL}_2) \hookrightarrow \cdots)$$

where we also take limits over permutation maps (choose better notation), is[3]

$$\mathfrak{t}_\infty \;=\; \operatorname{Lie} T_\infty \;\simeq\; \operatorname{Ran}\mathbf{C}.$$

Proof. We have compatible maps $\operatorname{Lie} T(\mathrm{GL}_n) \to \operatorname{Lie} T_\infty$ and so a map

$$\operatorname{Ran}\mathbf{C} \;\to\; \operatorname{Lie} T_\infty.$$

Depending on how we define Lie for group prestacks, we may show this is an isomorphism (do it) □

Note that any $\lambda \in \mathfrak{t}_\infty$ defines a homomorphism $\lambda : \mathbf{C}^\times \to T_\infty$. We can now define a pair space

$$\mathcal{M}^f \;=\; \left\{\lambda \in \mathfrak{t}_\infty^I\,,\; m \in \mathcal{M}(\infty)^\lambda\right\}.$$

Proposition 3.1.7. $\mathcal{M}^f$ *defines a factorisation space over* $(\operatorname{Ran}\mathbf{A}^1_{\mathbf{Z}})^I$.

[3] Sadly, if $\mathfrak{g}$ is a Lie algebra and H is a Lie group, we have

$$\operatorname{Maps}(\mathfrak{g}, \operatorname{Lie} H) \;=\; \operatorname{Maps}(\exp\mathfrak{g}, H)$$

and so Lie is a right adjoint, hence preserves limits.

Proof. The projection to $(\operatorname{Ran}\mathbf{A}^1)^I$ is given by $(m,\lambda)\mapsto\lambda$. Given $w\in\mathbf{N}^I$, consider the restriction along $\mathbf{A}^w\to(\operatorname{Ran}\mathbf{A}^1)^I$,

$$\mathcal{M}^f|_{\mathbf{A}^w}.$$

If $w=w_1+w_2$, we write $\Delta_{w_1,w_2}\subseteq\mathbf{A}^{w_1}\times\mathbf{A}^{w_2}=\mathbf{A}^w$ for the union of the hyperplanes $z_1=z_2$ where z_i is any coordinate on $\mathbf{A}^{w_i}$. Then

$$\mathcal{M}^f|_{\mathbf{A}^w\backslash\Delta_{w_1,w_2}}\ \simeq\ \left(\mathcal{M}^f|_{\mathbf{A}^{w_1}}\times\mathcal{M}^f|_{\mathbf{A}^{w_2}}\right)|_{\mathbf{A}^w\backslash\Delta_{w_1,w_2}}.$$

(prove this) This follows because for any coweight $\lambda\in\mathfrak{gl}_{w_1+w_2}$ away from the diagonals, being fixed implies the quiver representation does not mix the w_1 and w_2 representations. (make clearer) □

Note that

Proposition 3.1.8. *$\mathcal{M}^f$ is equivariant for the translation action of $\mathbf{G}_a$. Above $\mathbf{A}^1_i\hookrightarrow(\operatorname{Ran}\mathbf{A}^1)^I$ the fibre is $\mathcal{M}^f|_{\mathbf{A}^1_i}\simeq\mathcal{M}(\infty)\times\mathbf{A}^1$.*

It follows that

Theorem 3.1.9. *The Borel-Moore homology*

$$\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(\infty))$$

carries an I-coloured vertex algebra structure; the definition is in the proof below.

Proof. Just as for any factorisation space, projection $\pi_*\omega_{\mathcal{M}^f}$ along $\pi:\mathcal{M}^f\to(\operatorname{Ran}\mathbf{A}^1)^I$ carries a commutative factorisation algebra structure for the coloured chiral product $\otimes^{ch}_I$ given by

$$\begin{tikzcd} & \mathbf{A}^{w_1}\times\mathbf{A}^{w_2}\backslash\Delta_{w_1,w_2} \arrow[dl] \arrow[dr] & \\ \mathbf{A}^{w_1}\times\mathbf{A}^{w_2} & & \mathbf{A}^{w_1+w_2} \end{tikzcd}$$

One can show this is equivalent to a vertex algebra V_i for each $i\in I$, corresponding to the restriction along $(\operatorname{Ran}\mathbf{A}^1)_i\hookrightarrow(\operatorname{Ran}\mathbf{A}^1)^I$, along with intertwining operators (is that right?) $V_i\otimes V_j\to V_{ij}((z_i-z_j))$, where V_{ij} is the vertex algebra coming from the ijth diagonal inclusion $(\operatorname{Ran}\mathbf{A}^1)_{ij}\hookrightarrow(\operatorname{Ran}\mathbf{A}^1)^I$ (write full definition) □

Note that there are compatible maps

$$\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(w))\ \to\ \mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(\infty))$$

for every $w\in\mathbf{N}^I$, because the inclusions $\mathcal{M}(w)\hookrightarrow\mathcal{M}(w+w')$ are all closed embeddings, Thus

$$\operatorname{colim}\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(w))\ \to\ \mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(\infty)).$$

(Check just in case!! is this Borel-Moore homology trivial?) Note for instance that

$$\operatorname{colim} \mathbf{C}[x_1, ..., x_n] \;=\; \mathbf{C}[x_1, x_2, ...\,]^{\mathfrak{S}_\infty}$$

where we take the colimit over all surjections. (in general, we expect this limit to be the Cartan part of the Yangian; this is by Kirwan surjectivity saying that the Borel-Moore homology of the stack is generated by tautological classes)

3.1.10. *Remark.* The above suggests one method to extend to q-deformations, by considering the action of $\mathrm{GL}_{n,q}$ (define) rather than GL_n.

3.1.11. *Stable envelopes.* In the above context, we may consider (keeping the notation of the toy section) the universal coherent sheaf

$$N^{+,f} \;\to\; \mathcal{M}^f$$

whose fibre above $\lambda \in (\operatorname{Ran} \mathbf{A}^1)^I$ is

$$N(\infty)^{+,\lambda} \;\to\; \mathcal{M}(\infty)^\lambda.$$

Here we note that we have $N(w) \hookrightarrow N(w + w')$ for all w, w', so that we can take the colimit $N(\infty)$. (check both $N(\infty)$ and $N(\infty)^{+,\lambda}$ are well defined) The defining correspondence (1) upgrades to the map

$$\tilde{N}^{+,f} \;\to\; ((\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet})^I$$

where $(\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet}$ parametrises flags $S \subseteq S'$ of finite subsets of $\mathbf{A}^1$, and the fibre above $\lambda \leqslant \lambda'$ recovers

$$\begin{tikzcd} & N^+_{\lambda,\lambda'} \arrow[dl, "q"'] \arrow[dr, "\exp"] & \\ \mathcal{M}(\infty)^\lambda & & \mathcal{M}(\infty)^{\lambda'} \end{tikzcd}$$

where in (1) we used $X = \mathcal{M}(\infty)^{\lambda'}$ and $F = \mathcal{M}(\infty)^\lambda$.

Lemma 3.1.12. *The correspondence*

$$\begin{tikzcd} & (\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet\subseteq\bullet} \arrow[dl] \arrow[dr] & \\ (\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet} \times (\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet} & & (\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet} \end{tikzcd}$$

makes $(\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet}$ *into a monoid. An algebra in* $\mathcal{D}\left((\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet}\right)$ *is precisely (describe it)*

Proposition 3.1.13. $N^{+,f}$ *is a decomposition space over* $((\operatorname{Ran} \mathbf{A}^1)^{\bullet\subseteq\bullet})^I$.

Corollary 3.1.14. *For a subset* $\lambda_\infty = \{\lambda_i\} \subseteq \mathfrak{t}_\infty$ *which is closed under addition, we get a algebra structure* $\mathrm{Stab}_{\lambda_\infty}$ *on*

$$\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(\infty)^{\lambda_\infty}) \;=\; \bigoplus \mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(\infty)^{\lambda_i}).$$

Moreover,

Proposition 3.1.15. *On localised G_∞-equivariant cohomology, the map $\mathrm{Stab}_{\lambda_\infty}$ is an isomorphism. In particular, we may define*

$$R_{\lambda_\infty}(z) \;:\; \mathrm{H}^{\mathrm{BM}}_{G_\infty,\bullet}(\mathcal{M}(\infty)^{\lambda_\infty})_{\mathrm{loc}} \xrightarrow{\sim} \mathrm{H}^{\mathrm{BM}}_{G_\infty,\bullet}(\mathcal{M}(\infty)^{\lambda_\infty})_{\mathrm{loc}}$$

where $z \in (\mathfrak{t}_\infty)^I = (\mathrm{Ran}\,\mathbf{A}^1)^I$. For (a certain subset of choices of λ_∞) this gives a spectral R-matrix

$$R_{\lambda_\infty}(z) \;:\; \mathrm{H}^{\mathrm{BM}}_{G_\infty,\bullet}(\mathcal{M}(\infty) \times \mathcal{M}(\infty)) \;\to\; \mathrm{H}^{\mathrm{BM}}_{G_\infty,\bullet}(\mathcal{M}(\infty) \times \mathcal{M}(\infty))((z))$$

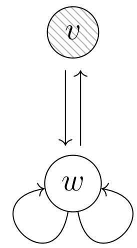
(but an I-coloured version)

3.1.16. *Moduli stacks case.* We can likewise explain the lax decomposition bialgebra structure on moduli stacks $\mathcal{M}$, given by SES and $\oplus$, respectively.

(we have $e(\mathbf{N}_i) = e(\mathbf{N}_i^0)/e(\mathbf{N}_i^1)$; we need to construct an action acting with positive weight on $\mathbf{N}_i^0$ and negative weight on $\mathbf{N}_i^1$. This way we get a geometric explanation of Joyce's torus localisation R-matrices)

4. Example

4.1. Let us take the ADHM quiver, i.e. the framed double Jordan quiver



where $v, w \geqslant 0$ are the dimension vectors. The quiver variety

$$\mathcal{M}(v,w) \;\simeq\; \mathrm{Inst}_v^{\mathrm{GL}_w}(\mathbf{P}^2)$$

is isomorphic to the space of framed instantons with rank w and instanton number v. The limit is given by adding on $\mathcal{O}$:

$$\mathrm{Inst}_v^{\mathrm{GL}_w}(\mathbf{P}^2) \;\hookrightarrow\; \mathrm{Inst}_v^{\mathrm{GL}_{w+1}}(\mathbf{P}^2) \;\hookrightarrow\; \mathrm{Inst}_v^{\mathrm{GL}_{w+2}}(\mathbf{P}^2) \;\hookrightarrow\; \cdots$$

The torus $T_w \subseteq \mathrm{GL}_w$ acts by scaling the framing.

4.2. **Factorisation structure.** Let group G act on X. We then get a space

$$X^f \;\to\; \mathfrak{g}$$

defined as the space of pairs (x, v) with $e^{tv}x = x$, i.e. if we are in a setting where the exponential map $\mathfrak{g} \to G$ is defined we can take the pullback

$$\begin{tikzcd}
X^f \arrow[rr] \arrow[d] & & X \arrow[d, "\Delta"] \\
X \times \mathfrak{g} \arrow[r, hook, "\Delta\times\exp"] & X \times X \times G \arrow[r, "\mathrm{id}\times\mathrm{act}"] & X \times X
\end{tikzcd}$$

If the exponential map is not defined, we still have a map $\mathfrak{g}_X \to \mathcal{T}_X$ of Lie algebra bundles, where $\mathfrak{g}_X = \mathfrak{g} \times X$, and can take the pullback

$$\begin{tikzcd}
X^f \arrow[r] \arrow[d] & X \arrow[d] \\
\mathfrak{g} \times X \arrow[r] & \mathcal{T}_X
\end{tikzcd}$$

where the right vertical map is the zero section.

Note that there is a group action

$$G \times X^f \;\to\; X^f, \qquad\qquad (g, x, v) \;\mapsto\; (gx, gvg^{-1})$$

making the map $X^f \to \mathfrak{g}$ equivariant for the adjoint action on the base. In the special case that the group is abelian, the adjoint action is trivial and so the fibres will have actions of G. This is the case below.

4.2.1. In particular, we have a map

$$\mathrm{Inst}_v^{T_{w'}}(\mathbf{P}^2)^f \;\to\; \mathfrak{t}_w$$

for every $w \leqslant w'$ and $T_w \hookrightarrow T_{w'}$. The above construction is functorial in X and G, so we get a map

$$\mathrm{Inst}_v^{\mathrm{GL}}(\mathbf{P}^2)^f \;\to\; \mathrm{colim}\,\mathfrak{t}_w \;\simeq\; \mathrm{Ran}\,\mathbf{A}^1.$$

4.2.2. Since the main diagonal torus $\mathbf{G}_m \subseteq T_w$ acts trivially on framed quiver representations, the restriction to the main diagonal $\mathbf{A}^1 \subseteq \mathrm{Ran}\,\mathbf{A}^1$ is[4]

$$\mathbf{A}^1 \times \mathrm{Inst}_v^{\mathrm{GL}}(\mathbf{P}^2).$$

4.2.3. Likewise, the data of a two-torus $\mathbf{G}_m^2 \hookrightarrow T_w$ corresponds to a decomposition $w = w_1 + w_2$, and so the restriction to $\mathbf{A}^2 \smallsetminus \mathbf{A}^1$ is

$$(\mathbf{A}^2 \smallsetminus \mathbf{A}^1) \times \bigcup_{v=v_1+v_2} \left(\mathrm{Inst}_{v_1}^{\mathrm{GL}}(\mathbf{P}^2) \times \mathrm{Inst}_{v_2}^{\mathrm{GL}}(\mathbf{P}^2)\right).$$

In particular, $\mathrm{Inst}^{\mathrm{GL}}(\mathbf{P}^2)^f$ factorises.

4.3. Borel-Moore homology.

It follows that $\mathrm{H}_\bullet^{\mathrm{BM}}(\mathrm{Inst}^{\mathrm{GL}}(\mathbf{P}^2))$ is a vertex algebra.

Recall from [BFN, §1.9] that $\mathrm{H}^\bullet_{\mathrm{GL}_w \times \mathbf{G}_m^2}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2), \psi)$ is a Verma module for $\mathcal{W}(\mathfrak{gl}_w)$, where ψ is the intersection cohomology sheaf; there is a version for $\mathrm{H}^\bullet_{T_w \times \mathbf{G}_m^2}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2), \Phi\psi)$ where Φ is the hyperbolic localisation functor, it is also a Verma module.

Lemma 4.3.1. *These actions of $\mathcal{W}(\mathfrak{gl}_r)$ are compatible as (what??)*

Proof. We define as in [BFN] actions of

$$\mathcal{V}^{k+h^\vee}(\mathfrak{t}_w), \qquad k \;=\; -h^\vee + \epsilon_2/\epsilon_1$$

on the homologies, by parabolic induction. We then define an action

$$\mathcal{V}^{k+h^\vee}(\mathfrak{t}_w) \;\to\; \mathrm{End}\left(\mathrm{H}^\bullet_{\mathrm{GL}_{w'} \times \mathbf{G}_m^2}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2), \psi)\right)$$

for every $w \leqslant w'$ and $\mathfrak{t}_w \hookrightarrow \mathfrak{t}_{w'}$, by restricting the ordinary action along $V^{-\epsilon_2/\epsilon_1}(\mathfrak{t}_w) \to V^{-\epsilon_2/\epsilon_1}(\mathfrak{t}_{w'})$. Note that this action is induced by push-pull along

$$\begin{tikzcd} & \mathrm{SES}^{\mathrm{GL}_w} \arrow[dl] \arrow[dr] & \\ \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2) & & \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2) \end{tikzcd}$$

[4] Here we have written $\cup\, \mathrm{Inst}_v^{\mathrm{GL}_w}(\mathbf{P}^2) = \mathrm{Inst}_v^{\mathrm{GL}}(\mathbf{P}^2)$.

and the adjoint operators, where $\mathrm{SES}^{\mathrm{GL}_w} \subseteq \mathbf{A}^2 \times \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2) \times \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)$ is the Lagrangian of triples $(\mathcal{E}_1, \mathcal{E}_2, x)$ with $\mathcal{E}_1/\mathcal{E}_2$ supported at $x \in \mathbf{A}^2$. Thus the operators are labelled by homology classes of $\mathbf{A}^2$.

We have the following commuting diagram

$$\begin{tikzcd} & \mathrm{SES}^{\mathrm{GL}_w} \arrow[dl, "q_w"'] \arrow[d] \arrow[dr, "p_w"] & \\ \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2) \arrow[d] & \mathrm{SES}^{\mathrm{GL}_{w'}} \arrow[dl] \arrow[dr] & \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2) \times \mathbf{A}^2 \arrow[d] \\ \mathrm{Inst}^{\mathrm{GL}_{w'}}(\mathbf{P}^2) & & \mathrm{Inst}^{\mathrm{GL}_{w'}}(\mathbf{P}^2) \times \mathbf{A}^2 \end{tikzcd} \tag{2}$$

sending

$$\begin{tikzcd} & (\mathcal{E}_1 \supseteq \mathcal{E}_2, x) \arrow[dl, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & \\ \mathcal{E}_2 \arrow[d, mapsto] & (\mathcal{E}_1 \oplus \mathcal{O}^{w'-w} \supseteq \mathcal{E}_2 \oplus \mathcal{O}^{w'-w}, x) \arrow[dl, mapsto] \arrow[dr, mapsto] & (\mathcal{E}_1, x) \arrow[d, mapsto] \\ \mathcal{E}_2 \oplus \mathcal{O}^{w'-w} & & (\mathcal{E}_1 \oplus \mathcal{O}^{w'-w}, x) \end{tikzcd}$$

In particular, because $\mathcal{E}_2$ is the kernel of

$$(\mathcal{E}_2 \oplus \mathcal{O}^{w'-w}) \to (\mathcal{E}_1 \oplus \mathcal{O}^{w'-w})/\mathcal{E}_1$$

the left square is a pullback, i.e. the data of $(\mathcal{E}_1, \mathcal{E}_2, x)$ can be recovered from the rest of the data in the square.

We have a $\mathfrak{t}_w = \mathrm{H}_{\bullet, T_w}(\mathbf{A}^2)$-Heisenberg action on $\mathrm{H}^{\mathrm{BM}}_{T_w}(\mathrm{Inst}^{\mathrm{GL}_w(\mathbf{P}^2)})$ by

$$p_{w,*}q_w^* = \sum_{n \geqslant 0} b_{-n} : \mathrm{H}^{\mathrm{BM}}_{\bullet,T_w}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)) \to \mathrm{H}^{\mathrm{BM}}_{\bullet,T_w}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)) \otimes \mathrm{H}^{\mathrm{BM}}_{\bullet,T_w}(\mathbf{A}^2) \simeq \mathrm{H}^{\mathrm{BM}}_{\bullet,T_w}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)) \otimes \mathfrak{t}_w^*$$

where the decomposition into modes is given by decomposition into rank. We get an action of positive modes by taking adjoint. (is there a doubling way to get this?)

Note that the vertical arrows in (2) are closed embeddings, so we may take pushforwards. Note secondly that the diagram in (2) is equivariant for $T_{w'}$, where the action of $T_{w'-w} \simeq T_{w'}/T_w$ on the top correspondence is trivial. Thus by base change applied to (2) we have we have a commuting diagram

$$\begin{tikzcd} \mathrm{H}^{\mathrm{BM}}_{\bullet,T_{w'}}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)) \arrow[r, "p_{w*}q_w^*"] \arrow[d, "i_*"] & \mathrm{H}^{\mathrm{BM}}_{\bullet,T_{w'}}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)) \otimes \mathfrak{t}_{w'}^* \arrow[d, "i_*"] \\ \mathrm{H}^{\mathrm{BM}}_{\bullet,T_{w'}}(\mathrm{Inst}^{\mathrm{GL}_{w'}}(\mathbf{P}^2)) \arrow[r, "p_{w'*}q_{w'}^*"] & \mathrm{H}^{\mathrm{BM}}_{\bullet,T_{w'}}(\mathrm{Inst}^{\mathrm{GL}_{w'}}(\mathbf{P}^2)) \otimes \mathfrak{t}_{w'}^* \end{tikzcd}$$

where the top row is $p_{w*}q_w^*$ tensor the identity on $\mathfrak{t}^*_{w'-w}$. Here the equivariance is

$$\begin{tikzcd} & \mathrm{SES}^{\mathrm{GL}_w}/T_w \times T_w \arrow[dl] \arrow[dr] & \\ \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)/T_w & & \mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2)/T_w \times \mathbf{A}^2/T_w \end{tikzcd} \tag{3}$$

(in the above we need to replace T_w by $T_w \times \mathbf{G}_m^2$ so that $\mathbf{A}^2$ is equivariantly compact and use torus localisation to define p_{w*}.)

We have shown

Lemma 4.3.2. *There is an action of* $V(\mathfrak{t}_\infty)$ *on* $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Inst}^{\mathrm{GL}}(\mathbf{P}^2)) = \operatorname{colim} \mathrm{H}^{\mathrm{BM}}_\bullet(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2))$.

We then show that this induces an action of $\mathcal{W}(\mathfrak{gl}_\infty)$. (do it! What is a free field realisation for this?) □

4.3.3. *Remark.* Note that $\bigoplus_v \mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}(v,w))$ is a Verma module for $\mathcal{W}^{k_w}(\mathfrak{gl}_w)$, and thus their limit as $w \to \infty$ has the same size as the Verma module for $U(\mathcal{W}_{1+\infty})$, i.e. as the vertex algebra $\mathcal{W}_{1+\infty}$. The weight of this Verma is a formal variable valued in $\mathrm{H}^\bullet_{T_\infty}(\mathrm{pt})$. (check)

Conjecture 4.3.4. *The union* $\operatorname{colim} \mathrm{H}^{\mathrm{BM}}_{\bullet,T_w}(\mathrm{Inst}^{\mathrm{GL}_w}(\mathbf{P}^2))$ *with the factorisation structure defined above is isomorphic to* $\mathcal{W}_{1+\infty}$.

5. Relation to CoHAs

5.1. Recall that we have $D\mathrm{H}^{\mathrm{BM}}_{\mathbf{G}_m,\bullet}(\mathcal{M}_Q) = Y_\hbar(\mathfrak{g})$ as localised bialgebras,[5] and by [YZ, YZ2] this localised bialgebra structure comes from

$$Y_\hbar(\mathfrak{g}) \in \mathrm{FactAlg}_{tw}(\mathrm{QCoh}(\mathrm{Conf}_\Lambda(\mathbf{C})))$$

after taking global (meromorphic[6]) sections [YZ, p11]. Recall also that $Y_\hbar(\mathfrak{g})$ has a coalgebra and non-local vertex coalgebra structure.

This algebra acts on $V = \mathrm{H}^{\mathrm{BM}}_{\mathbf{G}_m,\bullet}(M(\infty))$ faithfully.[7]

We have shown that $M(\infty)$ factorises over $\mathbf{C}^I$.

5.1.1. Compare [Ga, 3.4.2] or [GL, 29.5]

$$\mathrm{Rep}_q(T) = \mathrm{FactAlg}(\mathrm{Perv}_{tw}(\mathrm{Conf}_\Lambda(\mathbf{C})))$$

in which lives $u_q(\mathfrak{n})$ and its Koszul dual, Ω_q. The equivalence[8] is given by taking fibre over the main diagonal, or projecting along $\mathrm{Conf}_\Lambda(\mathbf{C}) \to \Lambda$.

5.1.2. We have that $\mathrm{H}^{\mathrm{BM}}_{\mathbf{G}_m,\bullet}(\mathcal{M}_Q)$ is a quasicoherent sheaf over $\mathrm{Conf}_\Lambda(\mathbf{C})$, with global sections $\mathrm{H}^{\mathrm{BM}}_{\mathbf{G}_m,\bullet}(\mathcal{M}_Q)$.

5.2. The following is based on the summary [YZ2] of [YZ].

5.2.1. *Maps we do and do not have.* For Q a quiver, not that we do *not* have the following maps: $\mathcal{M}_Q \to \mathrm{BGL}, \mathrm{B}T$ or $\mathcal{M}_{Q,d} \to \mathrm{BGL}_d, \mathrm{B}T_d$. We *can* define the stack

$$\mathcal{M}^f_{Q,d,\mathfrak{t}} = \mathrm{Rep}(Q,d)^{f,\mathfrak{t}}/\,\mathrm{GL}_d$$

where $\mathrm{Rep}(Q,d)^{f,\mathfrak{t}} \subseteq \mathrm{Rep}(Q,d) \times \mathfrak{t}_d$ is the subset of pairs (V,ξ) with $\xi \cdot V = V$. This is a stack over $\mathfrak{t}_d$ with fibre over ξ the quotient $\mathrm{Rep}(Q,d)^\xi/C(\xi,\mathrm{GL}_d)$. However, this does *not* define a map $\mathcal{M}^f_{Q,d} \to \mathfrak{t}_d$ or indeed $\mathcal{M}^f_Q \to \mathrm{Conf}_{\Lambda_Q}(\mathfrak{t}_1)$. However, it *does* define a map $\mathcal{M}^f_{Q,d,\mathfrak{t}} \to \mathfrak{t}_d/\,\mathrm{GL}_d$. We can likewise define a space

$$\mathcal{M}^f_{Q,d} = \mathrm{Rep}(Q,d)^f/\,\mathrm{GL}_d$$

where we define the subset $\mathrm{Rep}(Q,d)^f \subseteq \mathrm{Rep}(Q,d) \times \mathfrak{gl}_d$ is the subset of pairs (V,ξ) with $\xi \cdot V = V$. We have maps

$$\begin{array}{ccccc}
\tilde{\mathcal{M}}^f_{Q,d} & \to & \mathfrak{t}_d/\mathfrak{S}_d & \to & \mathfrak{t}_d/\!/\mathfrak{S}_d \\
\downarrow & & \downarrow & & \downarrow\wr \\
\mathcal{M}^f_{Q,d} & \to & \mathfrak{gl}_d/\,\mathrm{GL}_d & \to & \mathfrak{gl}_d/\!/\,\mathrm{GL}_d
\end{array}$$

---

[5] $\mathcal{M}$ is the moduli stack.

[6] We can instead choose a section of the line bundle $\mathcal{L}$ on $\mathbf{C}$ defining the twisted product $\otimes_{tw}$, and a section of $\sigma^*\mathcal{L}^{-1} \otimes \mathcal{L}$ on $\mathbf{C} \times \mathbf{C}$, see [YZ, p11].

[7] M is the moduli space.

[8] On the level of vector spaces.

where $\tilde{\mathcal{M}}^f_{Q,d} = (\mathrm{Rep}(Q,d)^{f,\mathfrak{t}})/\mathfrak{S}_d$, and hence

$$\pi_d \;:\; \mathcal{M}^f_{Q,d} \;\to\; \mathfrak{gl}_d/\,\mathrm{GL}_d \;\to\; \mathfrak{gl}_d//\,\mathrm{GL}_d \;\simeq\; \mathfrak{t}_d//\mathfrak{S}_d.$$

Notice that we have an equivariant map $(\mathfrak{t}_d,\mathfrak{S}_d) \to (\mathfrak{t}_{d+1},\mathfrak{S}_{d+1})$, and so we *do* get a map

$$\pi \;:\; \coprod_d \mathcal{M}^f_{Q,d} \;\to\; \coprod_d \mathfrak{t}_d//\mathfrak{S}_d \;=\; \mathrm{Conf}(\mathbf{C}).$$

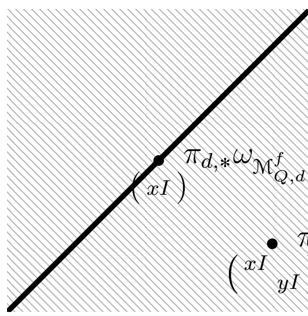
Note that the fibre of π_d over the main diagonal is

$$\mathcal{M}^f_{Q,d} \times_{\mathfrak{t}_d//\mathfrak{S}_d} \mathfrak{t}_1 \;\simeq\; \mathcal{M}_{Q,d} \times \mathbf{C}.$$

In particular, *orthogonally* to (i.e. along the fibre of) π_d we have BGL_d's (along the main diagonal) and $\mathrm{BGL}_{d_1} \times \cdots \times \mathrm{BGL}_{d_n}$'s (along the off-diagonals).

Thus, in addition to being a D-module over $\mathfrak{t}_d//\mathfrak{S}_d$, the relative Borel-Moore homology of π_d is a module over the cohomology of $\mathrm{H}^\bullet(\mathrm{BGL}_d)$:

$\mathfrak{t}_d \simeq \mathbf{C}^d$

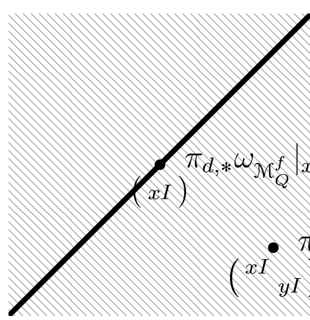


$$\pi_{d,*}\omega_{\mathcal{M}^f_{Q,d}}|_x \;\in\; \mathrm{Mod}\,\mathbf{C}[t_1,\ldots,t_d]^{\mathfrak{S}_d}$$

$$\pi_{d,*}\omega_{\mathcal{M}^f_{Q,d}}|_{x,y} \;\in\; \mathrm{Mod}\,\mathbf{C}[t_1,\ldots,t_d]^{\mathfrak{S}_{d'}\times\mathfrak{S}_{d''}}$$

where we mean that it splits into a sum of modules over $\mathbf{C}[t_1,\ldots,t_d]^L$ for every levi $L \simeq \mathrm{GL}_{d'} \times \mathrm{GL}_{d''}$ of GL_d. Likewise, if $\mathcal{M}^f_Q = \coprod_d \mathcal{M}^f_{Q,d}$ then we get

$\mathrm{Ran}\,\mathbf{C} \;\overset{(?)}{=}\; \mathfrak{t}_\infty/\mathfrak{S}_\infty$



$$\pi_{d,*}\omega_{\mathcal{M}^f_Q}|_x \;\in\; \mathrm{Mod}\,\mathbf{C}[t_1,t_2,t_3,\ldots]^{\mathfrak{S}_\infty}$$

$$\pi_{d,*}\omega_{\mathcal{M}^f_Q}|_{x,y} \;\in\; \mathrm{Mod}\,\mathbf{C}[t_1,t_2,t_3,\ldots]^{\mathfrak{S}_\infty}$$

where we view xI as the sequence $xI_d \in \mathfrak{t}_d$ for all d, likewise $xI \boxplus yI$, etc. Thus

Theorem 5.2.2. *$\pi_*\omega_{\mathcal{M}^f_Q}$ is a factorisation algebra over $\mathrm{Conf}\,\mathbf{C} = \coprod \mathfrak{t}_d//\mathfrak{S}_d$ valued in (check this! In particular, is it Conf or Ran-ny?) $\mathrm{Mod}\,\mathbf{C}[t_1,t_2,t_3,\ldots]^{\mathfrak{S}_\infty} \simeq \mathrm{QCoh}(\mathrm{colim}\,\mathbf{A}^d/\mathfrak{S}_d) = \mathrm{QCoh}(\mathrm{Ran}\,\mathbf{C}')$. Moreover, each fibre factorises, i.e. it actually lies in $\mathrm{QCoh}(\mathrm{Ran}\,\mathbf{C}')$. (probably actually it lies in $\mathrm{QCoh}(\mathrm{Conf}(\mathbf{C}))$)*

Is this in fact the same structure? (Maybe), but note that

- the D-module is the $\mathcal{D}(\mathfrak{t}_d)$-module structure is given by the *space* $\mathfrak{t}_d//\mathfrak{S}_d$ and the map $\pi_d : \mathcal{M}^f_{Q,d} \to \mathfrak{t}_d//\mathfrak{S}_d$,
- the quasicoherent sheaf structure is given the $\mathcal{O}(\text{fibre})$-module structure.

(The above is the case where there is *one* vertex. In general we will need to consider $\Lambda = \mathbf{Z}_{\geqslant 0}Q_0$-coloured factorisation versions.)

(think *carefully* about whether we mean $\operatorname{Ran}\mathbf{C} = \operatorname{colim}\mathbf{C}^n$, colimit taken over all diagonal maps $\mathbf{C}^n \to \mathbf{C}^m$, or $\operatorname{Conf}(\mathbf{C}) = \operatorname{colim}\mathbf{C}^n \simeq \coprod \mathbf{C}^n$, colimit taken over *no* maps)

In particular, it defines something like an element in

$$\text{“}\mathcal{D}\text{-Mod}^{fact}(\operatorname{Conf}\mathbf{C}_1) \times \operatorname{QCoh}^{fact}(\operatorname{Conf}\mathbf{C}_2)\text{”}$$

5.3. (some of the below in this section might be wrong)

5.3.1. Let us do the additive case first. In this case, we have that

$$A = \bigoplus \mathrm{H}^{\mathrm{BM}}_{\bullet,\mathbf{G}_m}(\operatorname{Rep}(\Pi_Q, d)/\operatorname{GL}_d) = \bigoplus \mathrm{H}^{\mathrm{BM}}_{\bullet,\mathbf{G}_m}(\mathcal{M}_d)$$

whose dth summand is a module over $\mathrm{H}^\bullet(\mathrm{B}T_d)$, hence defines a quasicoherent sheaf

$$|A_d| \to \operatorname{Spec}\mathrm{H}^\bullet(\mathrm{B}T_d) \simeq \mathbf{C}^{|d|}.$$

The action is $\mathfrak{S}_d$-symmetric, hence each one is a module over $\operatorname{colim}\mathrm{H}^\bullet(\mathrm{B}T_d)$, hence we have a quasicoherent sheaf

$$|A| \to \operatorname{colim}\operatorname{Spec}\mathrm{H}^\bullet(\mathrm{B}T_d) \simeq \mathbf{C}^{Q_0\cdot\infty} = \operatorname{Conf}_\Lambda(\mathbf{C})$$

where $\Lambda = \mathbf{Z}_{\geqslant 0}Q_0$.

5.3.2. Note that we have an action of $\mathcal{M}_d$ on $\mathcal{M}_d(w)$ by correspondences or direct sums. Moreover, there is an action on the moduli *space* (Nakajima quiver variety) $\mathfrak{X}_d(w)$ by correspondences. We have a factorisation space

$$\pi_w : \mathfrak{X}^f(\infty) \to \operatorname{Conf}_{\Lambda^{fr}}(\mathbf{C})$$

where $\Lambda^{fr} = \mathbf{Z}_{\geqslant 0}Q^{fr}$ where Q^{fr} are the framing vertices. There is a stacky version of this,

$$\mathcal{M}^f(\infty) \to \operatorname{Conf}_{\Lambda^{fr}}(\mathbf{C}) \times \mathrm{B}T_v^\infty.$$

5.3.3. On the space level, we have

(1) A factorisation space

$$\pi_w : \mathfrak{X}^f \to \operatorname{Conf}_{\Lambda^{fr}}(\mathbf{C}_w)$$

where $\Lambda^{fr} = \mathbf{Z}_{\geqslant 0}Q^{fr}$ for Q^{fr} the framing vertices. Thus,

$$\mathcal{B} = \pi_{w,*}\omega_{\mathfrak{X}^f} \in \mathcal{D}\text{-Mod}^{fact}(\operatorname{Conf}_{\Lambda^{fr}}(\mathbf{C}))$$

(2) We also have

$$\pi_v \;:\; \mathcal{M} \;\to\; \operatorname{colim} BT_v^d \;=\; \mathrm{B}T_v^\infty.$$

Note that $\operatorname{Spec} \mathrm{H}^\bullet(\mathrm{B}T_v^\infty) = \mathrm{Conf}_\Lambda(\mathbf{C})$, so as $\mathrm{H}_\bullet^{\mathrm{BM}}(\mathcal{M})$ is a module over $\mathrm{H}^\bullet(\mathrm{B}T_v^\infty)$, we have

$$\mathcal{A} \;=\; \mathrm{H}_\bullet^{\mathrm{BM}}(\mathcal{M}) \;\in\; \mathrm{QCoh}^{fact}(\mathrm{Conf}_\Lambda(\mathbf{C})).$$

This corresponds to

Theorem 5.3.4. [YZ] *We have:*

(1) M^f forms

(2) A forms a factorisation algebra over $\mathrm{Conf}_\Lambda(\mathbf{C})$, in an appropriately twisted category of sheaves.

5.4. **Elliptic version.** The quantised elliptic CoHA of the preprojective category

$$A \;=\; \bigoplus \mathrm{Ell}_{\mathrm{GL}_d \times \mathbf{G}_m}(\mathrm{Rep}(\Pi_Q, d))$$

is a coherent sheaf over

$$\sqcup \mathrm{Ell}_{\mathrm{GL}_d \times \mathbf{G}_m}(\mathrm{pt}) \;=\; \sqcup(E^{d_1} \times \cdots \times E^{d_{|Q_0|}})/\mathfrak{S}_d \times E \;=\; \mathrm{Conf}_\Lambda(E) \times E$$

where $\Lambda = \mathbf{Z}_{\geqslant 0} Q_0$.[9] Moreover,

Proposition 5.4.1. *A with the CoHA product defines a (gerbe twisted?) factorisation algebra over $\mathrm{Conf}_\Lambda(E) \times E$, i.e. $\oplus_*(A_{d'} \boxtimes A_{d''} \otimes \mathcal{L}_{d',d''}) \to A_{d'+d''}$ where $\oplus : \mathrm{Conf}_\Lambda^{d'}(E) \times \mathrm{Conf}_\Lambda^{d''}(E) \to \mathrm{Conf}_\Lambda^{d}(E)$ is the addition map and $\mathcal{L}_{d',d''} = \mathcal{O}(\mathrm{fac}_1 + \mathrm{fac}_2)$ is a line bundle defined in [YZ, §2.1].*

The localised coproduct defines a factorisation coalgebra, i.e. $\Delta : A_{d'+d''} \to \oplus_*(A_{d'} \boxtimes A_{d''} \otimes \mathcal{L}_{d',d''})$, by [YZ, §3.2], which (is an isomorphism on the open locus)

$$\Delta : A_{d'+d''}|_{\mathrm{Conf}_\circ} \xrightarrow{\sim} \oplus_*(A_{d'} \boxtimes A_{d''} \otimes \mathcal{L}_{d',d''})|_{\mathrm{Conf}_\circ}.$$

(this thus is actually a vertex algebra structure if we apply act*)* [10]

Recall that we can apply the following to get a vertex algebra from this:

Theorem 5.4.2. [CF, 10.8.1] *There is an equivalence between the category of twisted unital (do they show unital?) actorisation algebras on $\mathrm{Conf}_\Lambda(\mathbf{C})$ and on $\mathrm{Gr}_T^{\leqslant 0}$. Moreover, the latter is a factorisation space over $\operatorname{Ran} \mathbf{A}^1$, so pushing forward gives a factorisation algebra on $\operatorname{Ran} \mathbf{A}^1$. (check)*

[9] Note $\mathrm{Conf}_\Lambda(E) = \sqcup \mathrm{Conf}_\Lambda^d(E)$.

[10] (maybe $\mathcal{M} \times \mathcal{M} = \mathcal{M}^{\mathrm{B}\mathbf{G}_m}$ or $\mathcal{M}^{\mathbf{G}_m}$; though I think not, e.g. we want to get $\mathrm{B}\mathbf{G}_m \times \mathrm{pt} \sqcup \mathrm{pt} \times \mathrm{B}\mathbf{G}_m \to \mathrm{B}\mathbf{G}_m$, but $\mathrm{B}\mathbf{G}_m$ only has a trivial $\mathbf{G}_m$ action, with the wrong fixed points $\mathbf{Z} \times \mathrm{B}\mathbf{G}_m$, and I don't see why $\mathrm{B}\mathbf{G}_m^{\mathrm{B}\mathbf{G}_m}$ would be $\mathrm{B}\mathbf{G}_m \sqcup \mathrm{B}\mathbf{G}_m$)

This equivalence is given by the maps

$$\mathrm{Conf}_\Lambda(\mathbf{C}) \overset{(\star)}{\leftarrow} \mathrm{Gr}^{<0}_{\check{T},\mathbf{A}^1} \to \mathrm{Gr}^{\leqslant 0}_{\check{T},\mathbf{A}^1}$$

and one can show that $(\star)$ induces an isomorphism on $\mathrm{B}\mathbf{G}_m$-gerbes and on categories of twisted D modules. Note that $(\star)$ is an isomorphism after sheafifying with respect to finite morphisms. In particular,

Proposition 5.4.3. *(check) The braided factorisation structure on $\mathcal{D}\text{-Mod}(\mathrm{Conf}_\Lambda(\mathbf{A}^1))$ given by (whatever is in [YZ]) is equivalent to (explicit) braided factorisation structure on $\mathcal{D}\text{-Mod}(\mathrm{Gr}^{<0}_{\check{T},\mathbf{A}^1})$.*

5.4.4. *Remark.* Recall that a *twisted sheaf* on X is a map

$$X \to \mathcal{O}\text{-Mod}/\mathrm{B}\mathbf{G}_m$$

where $\mathcal{O}$-Mod is the sheaf of categories over pt representing $\mathcal{O}$-modules (of course, in practice we can replace pt with any base B and $\mathrm{B}\mathbf{G}_m$ with any gerbe $\mathcal{G}_B$). We have pullback

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & \mathcal{O}\text{-Mod} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{O}\text{-Mod}/\mathrm{B}\mathbf{G}_m \end{array}$$

in particular, we have an $\mathcal{O}$-module F on $\mathcal{G}$, such that $\mathcal{G} \to \mathcal{O}\text{-Mod}$ is $\mathrm{B}\mathbf{G}_m$-equivariant. This means that we have a commuting diagram

$$\begin{array}{ccc} \mathrm{B}\mathbf{G}_m \times \mathcal{G} & \xrightarrow{\mathrm{id}\times F} & \mathrm{B}\mathbf{G}_m \times \mathcal{O}\text{-Mod} \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow{\scriptstyle \mathrm{act}} \\ \mathcal{G} & \xrightarrow{F} & \mathcal{O}\text{-Mod} \end{array} \qquad \begin{array}{ccc} (\mathcal{L}, g) & \longmapsto & (\mathcal{L}, F_g) \\ \downarrow & & \downarrow \\ \mathcal{L}\cdot g & \longmapsto & F_{\mathcal{L}\cdot g} = \mathcal{L}\otimes F_g \end{array}$$

This says precisely that under

$$\mathrm{act} \ : \ \mathrm{B}\mathbf{G}_m \times \mathcal{G} \to \mathcal{G}$$

we have $\mathrm{act}^* F = \gamma \boxtimes F$ where γ is the tautological line bundle on $\mathrm{B}\mathbf{G}_m$. We may change the weight of the action of $\mathrm{B}\mathbf{G}_m$ on $\mathcal{O}$-Mod to get the same definition for other weights.

(are Joyce-style moduli stacks examples of gerbe-twisted spaces over $\mathrm{Ran}\,\mathbf{A}^1$?)

Theorem 5.4.5. ([YZ2, Thm B] and [YZ, §0.4]) *The meromorphic sections of the Drinfeld double $D(A)$ over $\mathrm{Conf}_\Lambda(E) \times E$ is the elliptic quantum group.*

5.4.6. *Remark.* We expect this bialgebra structure over $\mathrm{Conf}_\Lambda(E)$ for *any* quiver Q.

5.4.7. *Remark.* Likewise, the CoHA and KHA live over $\mathrm{Conf}_\Lambda(\mathbf{C}) \times \mathbf{C}$ and $\mathrm{Conf}_\Lambda(\mathbf{G}_m) \times \mathbf{G}_m$, where $\mathbf{C} = \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathrm{pt})$ and $\mathbf{C}^\times = \mathrm{K}^\bullet_{\mathbf{G}_m}(\mathrm{pt})$.

5.5. **CoHAs as positive half of quantum groups.** Recall the construction from [Ga] of the three integral forms of $U_q(\mathfrak{n})$. (there should also be a few different integral forms of the CoHAs we are considering)

First, it is defined in [Ga, p28] as the image of $U_q^{free}(\mathfrak{n}) \to U_q^{cofree}(\mathfrak{n})$, i.e. the quotient of the free thing by the Serre relations. (there should be a version of this but for the BPS Lie algebra)

Second, it is defined in [Ga, p30] as the factorisation algebra Ω_H on $\mathrm{Conf}_\Lambda(\mathbf{C})$ arising from an explicit construction; and under Lurie's $H \mapsto \Omega_H$, these constructions agree.

(In the ADE case we will begin with the vertex Lie algebra $\mathfrak{n}[t]$; we will need to gerbe-twist to pick up the level κ when we take universal enveloping algebra)

(We would have to come up with a version of [GL, 29.5] but for vertex Hopf algebras H, such that if we take Kozul dual B we get a map $B \mapsto \Omega_B$ with Ω_B-FactMod $\simeq$ B-$\mathrm{Mod}_{\mathbf{E}_1^{ch}\mathbf{E}_1}$. In our case, $H = U(\mathrm{BPS}[u])$ is a vertex Hopf algebra with m_{CoHA} and $\Delta(z) = \mathrm{act}_1^* \Delta_{\mathrm{loc}}$; thus Ω_B should live over $\mathrm{Conf}_\Lambda(\mathbf{C} \times \mathbf{R})$ somehow) (is this what's happening with qKZ equations, where we take the algebra of rational sections as in [YZ]?)

5.6. **CoHA action on quiver varieties.** (maybe it's already in the literature that the CoHA acts as a localised bialgebra on quiver varieties)

(the cohomology of X should define an element in Ω_H-FactMod; thus X and $\mathcal{M}$ should factorise over the same space probably)

6. Sujay remarks

6.1. To summarise: for the Nakajima quiver variety $M(v,w) = \mu^{-1}(m)//^{\theta}\,\mathrm{GL}$, we have

- a universal Poisson deformation
$$M(v,w)^{def} \ \to \ \mathfrak{t}^*_m.$$
- a map
$$M(v,w)^f \ \to \ \mathfrak{t}^*_w$$
- $\varnothing$,
- θ gives stable envelope data.

6.1.1. Likewise, for the *stack* $\mathcal{M}(v) = \mu^{-1}(0)/\,\mathrm{GL}$, we have

- a map
$$\mathcal{M}(v)^{def} = \mathrm{Rep}(Q,v)/\,\mathrm{GL} \ \to \ \mathfrak{t}^*_m.$$
- a map
$$\mathcal{M}(v)^f \ \to \ \mathfrak{gl}_v//\,\mathrm{GL}_v \ = \ \mathfrak{t}^*_v//Sk_v$$
- $\mathrm{H}^{\mathrm{BM}}_{\bullet}(\mathcal{M}(v)) \in \mathrm{QCoh}(\mathrm{Spec}\,\mathfrak{t}^*_v//\mathfrak{S}_v)$
- $\varnothing$

6.1.2. m is called the mass parameter (and in the Higgs branch has to do with GL_v), and θ the Kahler parameter (and has to do with GL_w).

There is a dual Coulomb branch version of this, in which the Kahler parameter θ of BFN has to do with GL_v, and GL_w has to do with the base of $\mathcal{M}_C$, i.e. the base of the universal deformation.

7. Henry notes

7.1. His asymptotic R-matrices has to do with

$$\oplus \; : \; \mathcal{M}(v,w) \times \mathcal{M}(v,w) \;\to\; \mathcal{M}(2v,2w)$$

where $\mathcal{M}$ is the quiver variety.

We also have a $\mathbf{G}_m$ action on $\mathcal{M}(2v,2w)$ acting on the framing vectors, given by $\exp\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. There is a locus inside $\mathcal{M}(2v,2w)$ where the $\mathbf{G}_m$ gives a map of quiver representations $A : V_1 \to V_2$ of weights v, w. Then A has weight z.

Alternatively, consider the moduli space of maps of quiver reps $V_1 \to V_2$ of sizes (v,w). One can consider chains of length n, $\mathcal{M}_n(v,w)$, which has $\mathrm{H}^\bullet(\mathcal{M}_n(v,w)) = \mathrm{H}^\bullet(\mathcal{M}(v,w))^{\otimes n}$ or something similar.

Given a chain $V_1 \to \cdots \to V_n$ and $\lambda \in \mathbf{G}_m$, we get a new chain on the same V_i by acting by λ (weight one) on the maps between the V_i.

In particular, the $\mathbf{G}_m$ weights of the tensor factors of $\mathrm{H}^\bullet(\mathcal{M}_n(v,w))$ have weights $\alpha, \lambda\alpha, ..., \lambda^n\alpha$. Letting $\hbar$ be the weight of the $\mathbf{G}_m$ scaling the cotangent fibres: iff $\lambda = \hbar$ then this is irreducible.

Henry gave a GRT construction of the irreducible factors: the simple factors correspond to chains of quiver representations where the maps are injective, or something like that: it depends subtly on the stability condition and the quiver.

There is an R-matrix for this story, where we take the torus T which is independent of n (i.e. it's not $T^{\times n}$).

We also consider the category $Y_\hbar(\mathfrak{g})$-Mod, or whatever for a general quiver, independently of n.

When $n \to \infty$, when considering R-matrices, we need to be able to swap

$$V \otimes (W_1 \otimes W_2 \otimes \cdots) \;\simeq\; (W_1 \otimes W_2 \otimes \cdots) \otimes V$$

which will be given by an infinite product of $n \to \infty$ terms.

If you apply RTT to this, you get the shifted Yangian.

7.2. If you get the factorisation story to work for CY2 setting, the above allows you to bootstrap this to CY3.

In examples, the shifted Yangian is CY3 CoHAs. For instance, the Hilbert scheme of $\mathbf{C}^3$.

7.3. If you look at the positive part of this algebras, you don't see the shift - Y and Y shifted have the same positive part (perhaps noncanonically), and the $\mathbf{C}^3$-Yangian after doubling should be the shifted affine $\mathfrak{gl}_1$ Yangian. The shifted Yangian should probably have the same coproducts as the Yangian, because it's contructed in an RTT way the same way.

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