DEFORMATION QUANTISATION

ALEXEI LATYNTSEV

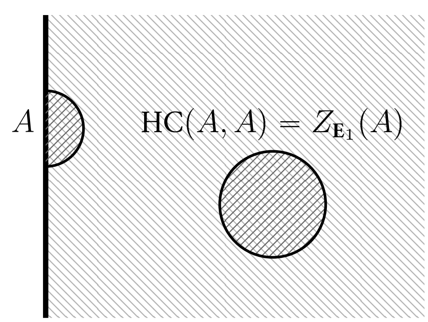
Contents

Questions:

(1) $\mathbf{T}_{A,\mathcal{O}}$, whose Maurer-Cartan element classify $\mathcal{O}$-deformations of A, has the structure of a $\mathcal{O}^!$-algebra. Does this lift to any extra structures on $\mathrm{Def}_{\mathcal{O}}(A)$? Why are the tangent space to $\mathrm{Def}_{\mathcal{O}}(A)$ computed just using the Lie algebra structure on $\mathbf{T}_{A,\mathcal{O}}$?

1. Deformation quantisation

Given an associative algebra A, its Hochschild cochains $\mathrm{HC}(A, A)$ has the structure of an $\mathbf{E}_2$-algebra acting on A.



This is Kontsevich-Thomas's Swiss cheese conjecture, see [Th], and this generalises to any $\mathbf{E}_n$-algebra A.

1.1. Sketch.

1.1.1. We have the following: for any commutative algebra A, for instance $\mathcal{O}(\mathfrak{g}^*)$, we have:

- $\mathrm{H}^\bullet(A,A)$ is an $\mathbf{H}^\bullet\mathbf{E}_2$-algebra, and $\mathrm{C}^\bullet(A,A)$ is an $\mathbf{C}^\bullet\mathbf{E}_2$-algebra. These structures are *boring*.
- We have a map
$$\varphi_{\mathbf{E}_2} \;:\; \mathbf{H}^\bullet\mathbf{E}_2 \;\to\; \mathbf{C}^\bullet\mathbf{E}_2$$
which gives
$$\mathbf{H}^\bullet\varphi_{\mathbf{E}_2} \;=\; \mathrm{id}$$
on $\mathbf{H}^\bullet\mathbf{H}^\bullet\mathbf{E}_2 = \mathbf{H}^\bullet\mathbf{E}_2 = \mathbf{H}^\bullet\mathrm{C}^\bullet\mathbf{E}_2$.
- *If* we have a map
$$\varphi_A \;:\; \mathrm{H}^\bullet(A,A) \;\to\; \mathrm{C}^\bullet(A,A)$$
which gives
$$\mathbf{H}^\bullet\varphi \;=\; \mathrm{id}$$
on $\mathbf{H}^\bullet\mathrm{H}^\bullet(A,A) = \mathrm{H}^\bullet(A,A) = \mathbf{H}^\bullet\mathrm{C}^\bullet(A,A)$, then we get an *interesting* structure of an $\mathbf{H}^\bullet\mathbf{E}_2 \simeq \mathbf{C}^\bullet\mathbf{E}_2$-algebra structure on $\mathrm{C}^\bullet(A,A)$ and $\mathrm{H}^\bullet(A,A)$, respectively.

Thus, the interesting data comes from φ_A.

1.1.2. However, the new algebra structure on $A[[\hbar]]$ will *not* be induced by the new $\mathbf{C}^\bullet\mathbf{E}_2$-algebra structure on $\mathrm{H}^\bullet(A,A)$.

Instead, we will consider an *element*
$$\omega \;\in\; \mathrm{H}^1(A,A) \;\subseteq\; \mathrm{H}^\bullet(A,A)\otimes\mathfrak{m}_{k[[\hbar]]}$$
satisfying the Maurer-Cartan equation, and take $\varphi_A\omega \in \mathrm{C}^\bullet(A,A)\otimes\mathfrak{m}_{k[[\hbar]]}$. We then use that φ_A is a map of L_∞-algebras to get
$$[\varphi_A\omega, \varphi_A\omega] \;=\; 0$$
and so $\varphi_A\omega$ defines an $\hbar$-adic deformation of A.

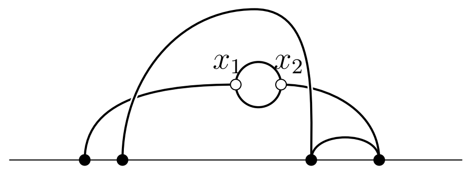
1.1.3. This uses that the $\mathbf{E}_1$-deformations of A are controlled by Maurer-Cartan elements of $\mathrm{C}^\bullet(A,A)$.

1.1.4. Note that L_∞ is an operad in chain complexes, and is a cofibrant relation of Lie. A map of L_∞ algebras is a map of chain complexes $f_0 : V \to W$ plus a homotopy making the following square commute:
$$\begin{array}{ccc} L_\infty(n)\otimes V^{\otimes n} & \longrightarrow & V \\ \downarrow{\scriptstyle \mathrm{id}\otimes f_0^{\otimes n}} & & \downarrow{\scriptstyle f_0} \\ L_\infty(n)\otimes W^{\otimes n} & \longrightarrow & W \end{array}$$
plus higher coherences. In other words, the homotopy is a map
$$f_n \;:\; L_\infty(n)\otimes W^{\otimes n} \;\to\; V$$
measuring the failure of this diagram to commute. (check)

1.1.5. Note that a dgla is an L_∞-algebra with vanishing higher brackets.

1.2. **Graphs.**

1.2.1. In the following section, we will be summing over *admissible graphs*, which loosely speaking will be the set of (oriented) graphs one can draw without loops or double edges



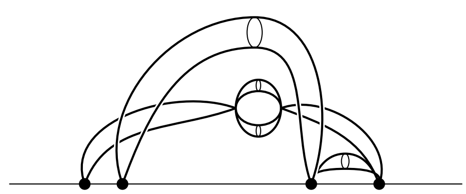
Given a picture as above, we form a graph by adding a vertex whenever the topology changes; these are marked in white in the above. The vertices and edges are ordered. We then quotient by the relation given by multiplying by $(-1)^d$ if we reverse an orientation, and by a sign $(-1)^d$ or $(-1)^{d-1}$ if we change the order of the vertices or edges (which will correspond to the Koszul sign rule).

A good reference is [LV].

1.2.2. Attached to each edge we can consider the sphere S^{d-1} given by only remembering the two end vertices. The volume form is then the class given by the two vertices rotating around each other; this is why edges are contribute degree $d-1$.

We then integrate the product of all of these over all possible x_i; this is why the internal vertices contribute degree d.

The way to view this is as the graph literally living inside $\mathbf{R}^d$, and draw a normal sphere around each edge, contracting around the vertices.



1.2.3. How should we view Kontsevich's map φ_A?

To begin with, it is not just a map f_n for every $n \geqslant 0$, it is a map f_Γ for every Feynman graph Γ. In other words, we have a homotopy

$$\mathrm{Graphs}(n) \otimes \mathrm{H}^\bullet(A, A)^{\otimes n} \;\to\; \mathrm{C}^\bullet(A, A)$$

which on restricting to $\Gamma \in \mathrm{Graphs}(|\Gamma|)$, where $|\Gamma|$ is the arity or number of external vertices, gives the map f_Γ.

The forgetful map Lie $\to \mathbf{E}_2$ corresponds to the map of operads

$$L_\infty \;\to\; \mathrm{Graphs}$$

which on degree n sends

$$[-,\cdots,-]_n \;\mapsto\; \sum_{|\Gamma|=n}\Gamma.$$

This explains why Kontsevich's f_n is given as a sum over graphs of degree n.

1.2.4. In any case, when $X = \mathbf{R}^d$ the map $f_\Gamma : \mathcal{T}_{poly}(X)^{|\Gamma|} \to \mathcal{D}_{poly}(X)$ for polyvector fields ξ_i is

$$f_\Gamma(\xi_1\otimes\cdots\otimes\xi_n) \;:\; f_1\otimes\cdots\otimes f_m \;\mapsto\; W_\Gamma \sum_{\psi:E_\Gamma\to\{1,\ldots,d\}} \prod\nolimits_{e:w\to v}\tfrac{\partial}{\partial x_{\psi(v)}}\xi_i(dx\otimes\cdots\otimes dx)$$

(check Kont p23 for the $dx\otimes\cdots\otimes dx$) where we take the sum over maps of partitions of the edge set into $d = \dim \mathbf{R}^d$ parts.

Here the weight W_Γ is (cont p23)

1.3. **Formality.**

1.3.1. If we have any operad $\mathcal{O}$ in chain complexes, we get a functor[1]

$$\mathcal{O}\text{-Alg} \;\to\; \mathrm{H}^\bullet(\mathcal{O})\text{-Alg}, \qquad\qquad A \;\mapsto\; \mathrm{H}^\bullet(A).$$

If in addition there is a quasiisomorphism $\mathcal{O} \simeq \mathrm{H}^\bullet(\mathcal{O})$ of operads in chain complexes, we can get an equivalence

$$\mathcal{O}\text{-Alg} \;\simeq\; \mathrm{H}^\bullet(\mathcal{O})\text{-Alg}, \qquad\qquad A \;\mapsto\; A.$$

In this case $\mathcal{O}$ is called *formal*.

1.3.2. The algebra A is called *formal* if there is an isomorphism $A \simeq \mathrm{H}^\bullet(A)$ of algebras over $\mathrm{H}^\bullet(\mathcal{O})$, or equivalently, of algebras over $\mathcal{O}$.

Theorem 1.3.3. [Ta, Ko2] *The operad $\mathbf{E}_n = \mathrm{C}^\bullet(\mathrm{Conf}(\mathbf{R}^n))$ is formal for $n \geqslant 2$.*

Proof. This proof is from [Ko2]: begin by taking the quotient

$$\overline{\mathrm{Conf}}_k(\mathbf{R}^n) \;=\; \mathrm{Conf}_k(\mathbf{R}^n)/(\mathbf{R}_{>0} \rtimes \mathbf{R}^n)$$

by scalings and translations. This is not an operad. We then form the operad $\mathrm{FM}(k)$ as the closure of the image of

$$\overline{\mathrm{Conf}}_k(\mathbf{R}^n) \;\hookrightarrow\; (S^{n-1})^{k(k-2)/2}, \qquad\qquad (x_1,\ldots,x_k) \;\mapsto\; \left(\frac{x_i-x_j}{|x_i-x_j|}\right)_{i<j}.$$

This a proper transform, i.e. the closure of Conf_k in the real oriented blowup of the diagonals in $(\mathbf{R}^n)^k$. (check) It has a natural stratification by how many points are infinitesimally close. We can form $\mathrm{FM}'(k)$

[1]Indeed, this is defined by

$$\left(\mathcal{O}(k)\otimes A^{\otimes k} \overset{a_A}{\to} A\right) \quad\rightsquigarrow\quad \left(\mathrm{H}^\bullet(\mathcal{O}(k))\otimes \mathrm{H}^\bullet(A)^{\otimes k} \overset{\mathrm{H}^\bullet(a_A)}{\to} \mathrm{H}^\bullet(A)\right)$$

where we have used the map $\mathrm{H}^\bullet(A)\otimes\mathrm{H}^\bullet(B) \to \mathrm{H}^\bullet(A\otimes B)$ for A, B chain complexes.

given by configurations of disks, but allowing the disks to be infinitely small; there are homotopy equivalences of operads

$$\mathrm{FM}(k) \to \mathrm{FM}'(k) \leftarrow \mathrm{Conf}_k(\mathbf{R}^n).$$

Note that $\mathrm{FM}'(k)$ is a manifold with corners, and we can consider the *exit path* operad[2] valued in chain complexes, with basis given by of stratified maps

$$\Delta^\bullet \to \mathrm{FM}'(k).$$

Formality will now follow from a chain of quasiisomorphisms

$$\mathrm{Graph}_n(k) \xrightarrow{\sim} \mathrm{C}^\bullet_{str}(\mathrm{FM}'(k)) \xrightarrow{\sim} \mathrm{C}^\bullet(\mathrm{FM}'(k)) \xleftarrow{\sim} \mathbf{E}_n(k) \tag{1}$$

and the fact that the admissible graph operad[3] is formal, by combinatorics.

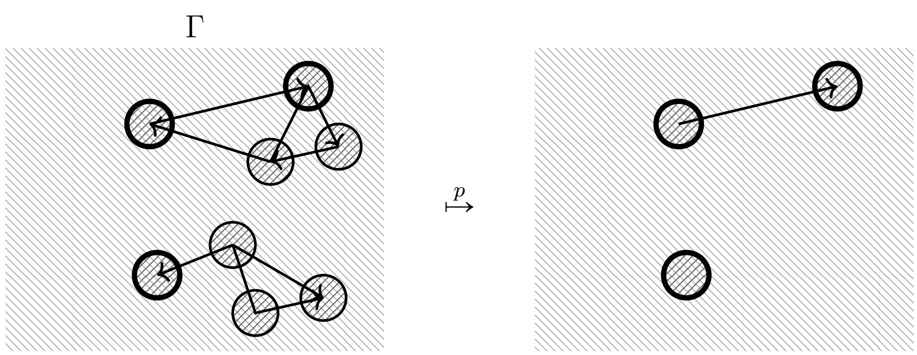
Given a admissible graph $\Gamma = \Gamma_{k,k',e}$, we get a differential form $\omega_\Gamma = p_* q^* \wedge dV_{S^{n-1}}$, defining a semialgebraic cochain (write Kont's proof of this), in terms of the forgetful maps

$$\mathrm{FM}'(2)^e \xleftarrow{q} \mathrm{FM}'(k+k') \xrightarrow{p} \mathrm{FM}'(k)$$

where p forgets the last k' circles, and q forgets all circles unattached to a particular edge. One can show ω_Γ form a basis for the semialgebraic cochains, so this defines the final quasiisomorphism in (1).

□

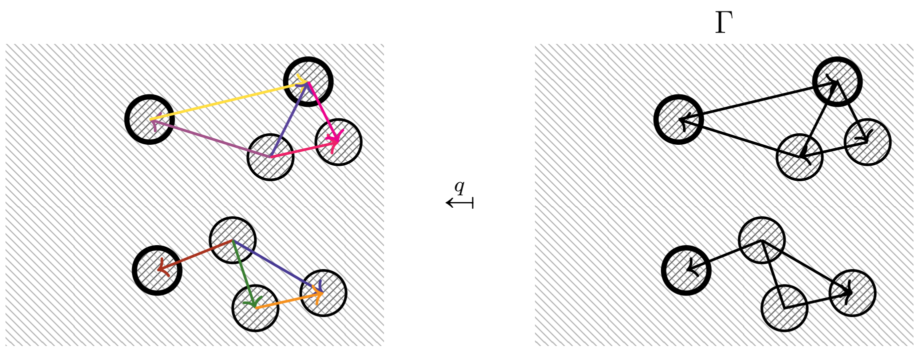
To understand the explicit basis ω_Γ of $\mathrm{C}^\bullet(\mathrm{FM}'(k))$ in the above, consider



where the vertices of the first kind are drawn in bold. Likewise,

[2] Also called *semialgebraic chains* in [Ko2].

[3] Here e is the number of edges of Γ, $k+k'$ is the number of vertices (split into two types, of which there are k and k' many respectively). The edges and vertices are ordered. A graph Γ is called *admissible* if every connected component contains a vertex of the first type, every vertex of the second type has degree $\geqslant 3$, there are no self-loops or multiple edges, and every edge comes with an orientation. The $\mathbf{Z}$-grading is $|\Gamma| = nk' - (n-1)k$. Finally, $\mathrm{Graphs}_n(k)$ is the the $\mathbf{Z}$-graded vector space of functions on the set of admissible graphs, behaving well (explain) as we change the labelling of the graph. The cochain map d is given by summing over admissible graphs $\Gamma' = \Gamma/e$ given by contracting an edge.



where each colour refers to a point in a single factor of a product of $\mathrm{FM}'(2) \simeq S^{n-1}$'s. For instance, ω_Γ is trivial if there are no edges. If there are no auxiliary thin circles of the second type, then it is just a product of $dV_{S^{n-1}}$'s.

The difference between $\mathbf{E}_n$ and $\mathrm{H}^\bullet(\mathbf{E}_n)$ in the above corresponds to taking the cohomology with respect to $d : \Gamma \mapsto \sum_e \Gamma/e$. Here $\omega_{\Gamma/e}$ is viewed as a form on a codimension one stratum of $\mathrm{FM}'(k)$. (check)

1.3.4. *Remark.* The only place where the ambient dimension $n = \dim \mathbf{R}^n$ shows up is in the definition of the $\mathbf{Z}$-grading on $\mathrm{Graphs}_n(k)$. (what is this $\mathbf{Z}$-grading in the Feynman sum point of view? In Feynman sums how do you see the ambient dimension?)

1.3.5. *Conjecture.* The above concerns factorisation algebras, i.e. in physics language, local operators of TQFTs. What about all the data of a TQFT?

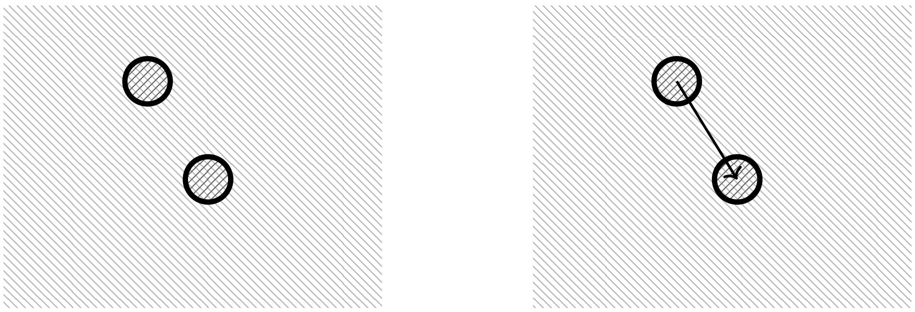
In the above, we considered abstract graphs, i.e. not equipped with an embedding into $\mathbf{R}^n$. However, for general d-manifolds X, embedded graphs with the same vertices can have different topological types, which we will need to keep track of in the data.

The equivalence $\mathrm{Graphs}_n \simeq \mathbf{E}_n$ for $n \geqslant 2$ is saying that instead of considering cobordisms between disks inside $\mathbf{R}^n$, we can consider combinatorial sums over graphs.

Conjecture 1.3.6. (Formality for TQFTs) *There is an equivalence between the data of a 1-functor* $\mathrm{Cob}_n \to \mathrm{Vect}$ *and* (combinatorial data) *This equivalence resticts to the previously defined* $\mathbf{E}_n$-Alg $\simeq \mathrm{Graphs}_n$-Alg.

1.3.7. *Example: $n = 2$ dimensions.* Note that $\mathrm{H}^\bullet(\mathbf{E}_2(2)) \simeq \mathrm{H}^\bullet(S^1)$ is generated by the multiplication and rotation, in degrees 0 and 1 respectively, corresponding to the graphs

Γ_0 Γ_1



(don't we have other graphs contributing also? Or are they not closed? Are they ones we drew closed?)

1.3.8. *Remark.* A Maurer-Cartan element of a Graphs_n-algebra A looks like (write).

1.3.9. (is there a Swiss Cheese version of this graph picture? And is there a graph version of Drinfeld doubling?)

1.3.10. (what is the analogue of the stratification and the compactification in the complex case? Just the proper transform of $(\mathbf{C})^n_\circ$ inside the blowup of $\mathbf{C}^n$ along the diagonals?)

1.4. **The HKR theorem.**

1.4.1. Note that by https://mathoverflow.net/questions/249114/multiplicativity-twisted-hochschild-kostant-rosen the Kontsevich map constructed below can be viewed as twisting by a square root of the Todd class.

1.4.2. *Twisted HKR theorem.* (how do you get $\mathcal{O}(\text{Crit}S)$ this way?) By [Ef], there is a notion of *twisted Hochschild homology*, and by [Ef, 3.14] there is a quasiisomorphism of mixed complexes

$$\text{HC}_\bullet(\mathcal{O}(X), W) \ \xrightarrow{\sim}\ (\Omega^\bullet(X), d, dW\wedge\ )$$

for X smooth of finite type with a function W, where $dW\wedge$ is corresponds to the Hochschild differential twisted by W, see [Ef, 3.1]:

$$b(f_0\otimes f_1) \ = \ (\pm f_0 f_1) \ + \ (df_0\otimes f_1 + f_0\otimes df_1) \ + \ (f_1\otimes W\otimes f_0 + f_1\otimes f_0\otimes W).$$

Notice that[4] we can read off functions on the critical locus from this:

$$\mathcal{O}(\text{Crit}W) \ = \ \mathcal{O}(X)/\ker(\mathcal{O}(X) \stackrel{dW}{\to} \Omega^1(X))$$

so in particular, $\mathcal{O}(\text{Crit}W) = \text{H}^0(\text{HC}_\bullet(\mathcal{O}(X), W), b)$ computes this.

- The Koszul complex is given by

$$\begin{array}{ccc} (s=0) & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ X & \xrightarrow{\ s\ } & E \end{array}$$

[4]For instance, if $X = \mathbf{A}^n$ we have

$$d(W)f \ = \ \sum \partial_i(W) f dx_i.$$

i.e. $\mathrm{K}_\bullet(X, E, s) = \mathcal{O}_X \otimes_{\mathrm{Sym}_{\mathcal{O}_X}\mathcal{E}^*} \mathcal{O}_X$.

- The critical locus of W is when $E = T^*X$

$$\begin{array}{ccc} \mathrm{Crit}(W) & \longrightarrow & X \\ \downarrow & & \downarrow 0 \\ X & \xrightarrow{dW} & T^*X \end{array}$$

- The Hochschild chain complex is given by

$$\begin{array}{ccc} LX & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

i.e. $\mathrm{HC}_\bullet(X) = \mathcal{O}(X) \otimes_{\mathcal{O}(X \times X)} \mathcal{O}(X)$.

- The HKR theorem says that $\mathrm{HH}_\bullet(X) = \Omega^*(X)$, i.e. Hochschild homology equals the de Rham complex.

1.4.3.

Theorem 1.4.4. [Ko2, Thm. 4] *If* $A = k[x_1, ..., x_n]$ *then* $\mathrm{HC}^\bullet(A, A)$ *formal as an* $\mathbf{E}_2$*-algebra.*

Proof. (reorganise) When $A = \mathcal{O}(X)$ its $\mathbf{E}_1$-algebra Hochschild homology is computed by the HKR Theorem

$$\mathrm{HH}(A, A) \simeq \mathrm{Sym}\mathcal{T}(X)[-1]$$

to be the algebra of polyvector fields on X, which is thus an $\mathrm{H}^\bullet(\mathbf{E}_2) \simeq \mathbf{E}_2$-algebra, or in other words a Gerstenhaber algebra. By [CRV, §7] the Hochschild cochains

$$\mathrm{HC}(A, A) \simeq (T(\mathcal{D}(X)), d)$$

are the polydifferential operators on X, which is an $\mathbf{E}_2$-algebra.

Theorem. [Ko, 4.6.2] *There is constructing a (canonical up to contractible choice) map of (homotopy)* Lie *algebras*

$$\mathcal{U} : \mathcal{T}_{poly}(X) \xrightarrow{\sim} \mathcal{D}_{poly}(X)$$

moreover, its first term is

$$\mathcal{U}_1^{(0)} : \xi_0 \wedge \cdots \wedge \xi_n \mapsto \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^\sigma \prod \xi_{\sigma(i)}$$

and is a quasiisomorphism of complexes.

Proof. When $X = \mathbf{R}^d$, the nth term

$$\mathcal{U}_n = \sum_\Gamma W_\Gamma \mathcal{U}_\Gamma : \otimes^n \mathcal{T}_{poly}(X) \to \mathcal{D}_{poly}(X)[1-n]$$

where we sum over all graphs Γ with n vertices of the first type, m of the second and $2n+m-2$ edges, and $W_\Gamma \in \mathbf{R}$ is its weight [Ko, §6.2]. Here, $\mathcal{U}_\Gamma$ is (write) $\square$

Note that by HKR, we have that $\mathcal{U}_1^{(0)} : \mathcal{T}_{poly}(X) \xrightarrow{\sim} \mathcal{D}_{poly}(X)$ is an isomorphism of dg vector spaces (not $\mathbf{E}_2$-algebras, unless we correct it with the higher homotopy terms as above), so in particular it gives a *new* $\mathbf{E}_2$-structure to $\mathcal{T}_{poly}(X)$ and on $\mathcal{D}_{poly}(X)$ written in terms of Feynman sums, given by $(\varphi^{-1}\mathcal{U}_1^{(0)})^{\pm 1}$. (one can presumably show this respects the Swiss cheese structure too:) $\square$

It follows from this that

Corollary 1.4.5. *If $A = k[x_1, ..., x_n]$ then there is an isomorphism of Lie algebras* $\mathrm{HC}^\bullet(A,A)[1] \simeq \mathrm{HH}^\bullet(A,A)[1]$.

Thus, taking the Maurer-Cartan spaces of these Lie algebras over Artin ring B:

$$\mathrm{Pois}_B(A) \;=\; \mathrm{MC}_{\mathrm{Lie}}(\mathrm{HH}^\bullet(A,A)\otimes \mathfrak{m}_B) \;\xrightarrow{\sim}\; \mathrm{MC}_{\mathrm{Lie}}(\mathrm{HC}^\bullet(A,A)\otimes\mathfrak{m}_B) \;=\; \mathrm{Def}_B(A).$$

Thus there is an equivalence between Poisson structures on $\mathbf{A}^n$ and classes of deformations on $\mathbf{A}^n$ over a base B.

1.4.6. *Remark.* If we had forgotten the Poisson structure on A, then its deformation theory is controlled by the *Harrison complex*. There is a map $\mathrm{Harr}^\bullet(A,A) \to \mathrm{HC}^\bullet(A,A)$, and the map on Maurer-Cartan elements

$$\mathrm{Def}_B^{\mathbf{E}_\infty}(A) \;=\; \mathrm{MC}_{\mathrm{Lie}}(\mathrm{Harr}^\bullet(A,A)\otimes\mathfrak{m}_B) \;\to\; \mathrm{MC}_{\mathrm{Lie}}(\mathrm{HC}^\bullet(A,A)\otimes\mathfrak{m}_B) \;=\; \mathrm{Def}_B^{\mathbf{E}_1}(A)$$

is not an isomorphism.

1.4.7. *Dimension $n = 1$ case.* Note that $\mathbf{E}_1$ is *not* formal, though in this section we will consider $\mathrm{C}^\bullet_{str}(\mathrm{FM}_1'(k))$ anyway. A point in the interior of $\mathrm{FM}_1'(3)$ looks like

We may scale and translate points so the endpoints are 0 and 1. It follows that

$$\mathrm{FM}_1'(k) \;=\; \Delta^k \times \mathfrak{S}_k$$

is the region inside $[0,1]^k$ defined by $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{k-2} \leqslant 1$. For instance, when $k=2$ this is just S^0.

1.4.8. *Remark.* If A is an associative algebra deformation over $\mathbf{C}[\hbar]$ of a commutative algebra $A_0 = A/\hbar$, then we have the following structures:

- $A/\hbar^2$ is an algebra over $k[\hbar]/\hbar^2$. The commutator of m gives a Poisson bracket on A_0.
- $A/\hbar^2$ has (what structure?)

Here we have written the product in A as $m = m_0 + \hbar m_1 + \hbar^2 m_2 + \cdots$. The above claims can be read off from the associativity conditions.[5]

1.4.9. *General spaces.* Now let X be a general smooth manifold.

1.5. **Deformation theory and Drinfeld centres.**

1.5.1. There are two different ways Hochschild cochains appear. The first is the notion of *Drinfeld centre* of an algebra over an operad:

$$\mathcal{Z}_{\mathcal{O}} \;:\; \mathcal{O}\text{-Alg} \;\to\; \mathbf{E}_1 \otimes \mathcal{O}\text{-Alg}, \qquad\qquad A \;\mapsto\; \mathrm{End}_{A\text{-Mod}_{\mathcal{O}}}(A),$$

and the second is the *tangent complex* of a $\mathcal{P}$-algebra formal moduli problem:

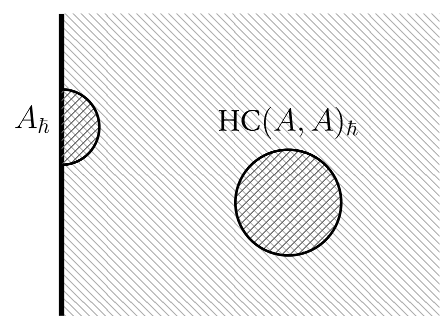
$$T_{\mathcal{P}}[-1] \;:\; \mathrm{FMP}_{\mathcal{P}} \;\xrightarrow{\sim}\; \mathcal{P}^![-1]\text{-Alg}.$$

In very special cases like $\mathcal{O} = \mathcal{P} = \mathbf{E}_1$, then we have for associative algebra A that these two notions agree:

$$\mathcal{Z}_{\mathbf{E}_1}(A) \;=\; \mathrm{HC}^\bullet(A, A) \;=\; T_{\mathbf{E}_1, \mathrm{Def}(A)}[-1]$$

where we have taken the formal moduli problem deforming A as an associative algebra. In other words, Hochschild cochains are both the appropriate derived notion of the centre of A, and also Maurer-Cartan elements inside it classify deformations of A.

1.6. $2d$ **TQFT picture.**



[5]Associativity is

$$m(m(a,b),c) - m(a,m(b,c)) \;=\; \sum_{n\geqslant 0} \hbar^n \sum_{i+j=n} (m_i(m_j(a,b),c) - m_i(a,m_j(b,c))) \;=\; 0.$$

The first few terms of this are $m_0(m_0(a,b),c) = m_0(a, m_0(b,c))$, or $(ab)c = a(bc)$ if we suppress m_0 from the notation, then

$$m_1(ab,c) + m_1(a,b)c - m_1(a,bc) - am_1(b,c) \;=\; 0$$

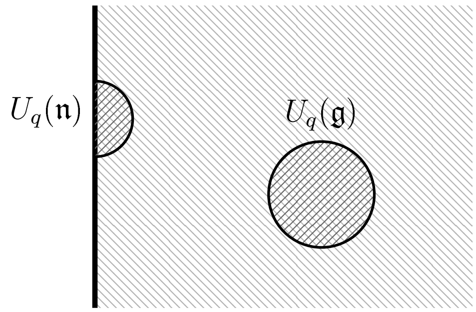
(actually we haven't used the fact that A is Poisson anywhere, maybe we need this data to go to the boundary in the above)

1.6.1. *Relation to the tree operad.* (there is a relation between the exit path stuff in Lurie/Gaitsgory/KZ and Kontsevich's formulas?)

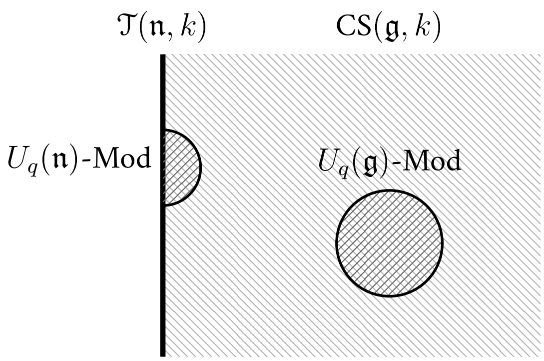
Remark. Note that the Swiss cheese operad is *not* formal, by [IV].

2. Quantum groups

2.0.1. The example of relative Drinfeld doubling coming from quantum groups is:



Of course, by Drinfeld doubling we in fact mean taking the Drinfeld *centre* of the appropriate category of representations:



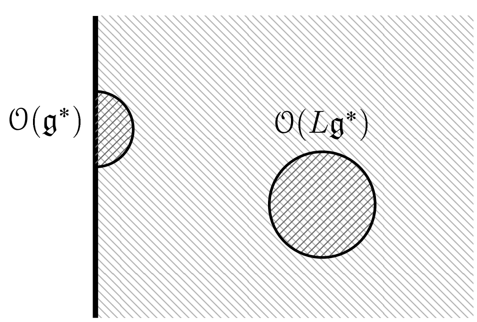
Here we have $3d$ Chern-Simons with a (relative?) topological boundary condition, and the above are the associated category of line operators.

2.0.2. Let $\mathcal{C}$ be an n-category defining an n dimensional TQFT, and c : triv $\to \mathcal{C}$ be a boundary condition.

2.0.3. *Example: quantum groups.* We can apply deformation quantisation to the above picture *again*, following [Ta2].

If $\mathfrak{g}$ has a Lie bialgebra structure, then $\mathcal{O}(\mathfrak{g}^*)$

$\mathcal{O}(L\mathfrak{g}^*)$ is a P_2-algebra in a *different* way;



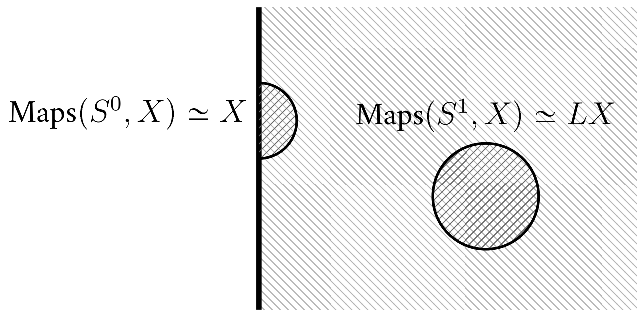
(how is this related to Drinfeld doubling?)

Note that the Drinfeld doubling procedure

$$\begin{tikzcd} \mathrm{BiAlg} \arrow[r, "Z"] \arrow[d, "\mathrm{KD}"] & \mathrm{QuasiTriangBiAlg} \arrow[d, "\mathrm{KD}"] \\ \mathbf{E}_2\text{-Alg} \arrow[r, "Z_{\mathbf{E}_2}"] & \mathbf{E}_3\text{-Alg} \end{tikzcd} \qquad U_\hbar(\mathfrak{g}) \longmapsto U_\hbar(\mathfrak{g} \oplus \mathfrak{g}^*)$$

3. Physics point of view

We can consider the classical sigma model with target $X = (X, x)$ a pointed space. The local operators are:

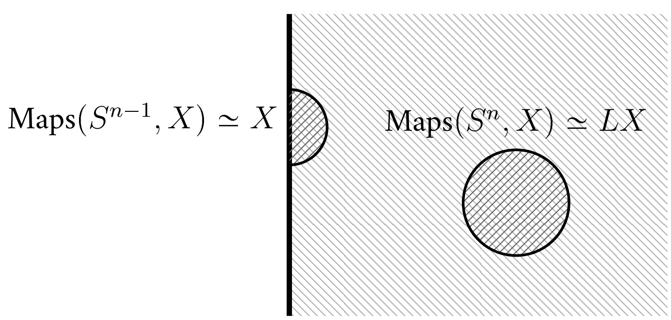


and the Swiss cheese map is given by the shifted Lagrangian

$$\begin{tikzcd} & \mathrm{Maps}(\Sigma, X) \arrow[dl] \arrow[dr] & \\ \mathrm{Maps}(S^0, X) \times \mathrm{Maps}(S^1, X) & & \mathrm{Maps}(S^0, X) \end{tikzcd}$$

i.e. we get a $2d$ TQFT with boundary valued in the category of (-1)-shifted Poisson manifolds. Forgetting some data then gives an algebra in the same category for the Swiss cheese operad.

3.0.1. Likewise we have for higher dimensional TQFTs:

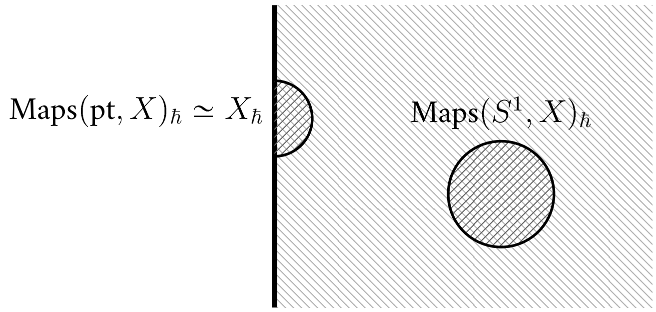


3.0.2. *Remark.* LX is the Drinfeld centre of X in this category. (check)

3.0.3. There is a quantisation of this, via Hochschild homology.

(write the $P_2 \simeq \mathbf{E}_2 \simeq \mathrm{Graphs}_2$ structure on Hochschild cochains)

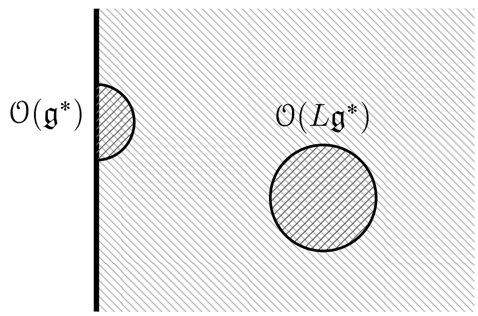
3.0.4. A Poisson bracket on X deforms this to first order. Physicists compute that "for $2d$ TQFTs there all contributions to the Feynman sum above 3 vertices are trivial", which corresponds to their being no higher Maurer-Cartan equations, i.e. $\mathcal{M}(\mathbf{C}[\hbar]/\hbar^2) \simeq \mathcal{M}(\mathbf{C}[[\hbar]])$ and so the first-order deformation determines a whole $\hbar$-adic deformation:



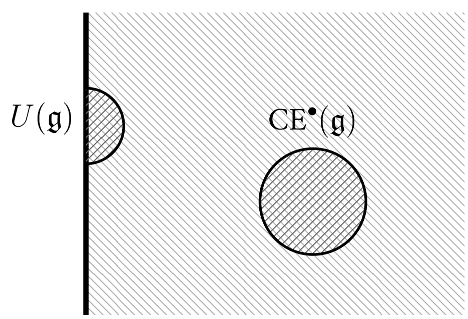
The boundary of the resulting $2d$ TQFT, whose local operators we denote $\mathrm{Maps}(S^1, X)_\hbar$, gives the deformation $X_\hbar$ of $\mathcal{O}(X)$.

3.0.5. *Remark.* It is apparently not easy to check the triviality of the contributions of the Feynman sums in degree above 3. It is false for $1d$ TQFTs.

3.0.6. *Example: Lie algebras.* For any vector space $\mathfrak{g}^*$ with basepoint 0, we have



For any Lie algebra structure on $\mathfrak{g}$, we get a quantisation of this:



In both cases we have taken Hochschild cochains. Note that $\mathrm{CE}^\bullet(\mathfrak{g})$ is equal to $\mathcal{O}(L\mathfrak{g}^*)$ if the Lie bracket vanishes. The operadic structure corresponds to the map

$$\mathrm{CE}^\bullet(\mathfrak{g}) \otimes U(\mathfrak{g}) \ \to \ U(\mathfrak{g})$$

of an $\mathbf{E}_2$-algebra on a $\mathbf{E}_1$-algebra. (how does this story relate to KZ equations?)

3.0.7. *Dumb coproduct on this.* Recall that the symmetric algbera $\mathrm{CE}^\bullet(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[-1])$ is given a differential by a Lie bracket on $\mathfrak{g}$, viewed as a map

$$d \ : \ \mathfrak{g}[-1] \otimes \mathfrak{g}[-1] \ \to \ \mathfrak{g}[-1], \qquad\qquad \mathfrak{g}[-1] \xrightarrow{0} k$$

in Vect. It is a derivation, and also a coderivation with respect to the standard coproduct on $\mathrm{CE}^\bullet(\mathfrak{g}[-1])$, e.g.

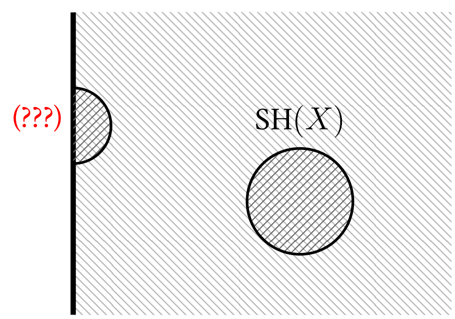
$$\Delta(dx) \;=\; \Delta(0) \;=\; 0 \;=\; (d\otimes \mathrm{id} + \mathrm{id}\otimes d)(x\otimes 1 + 1\otimes x) \;=\; (d\otimes \mathrm{id} + \mathrm{id}\otimes d)\Delta(x)$$

as $dx = d1 = 0$. (check)

(how does the commutative, cocommutative bialgebra structure on $\mathcal{O}(\mathfrak{g}^*)$ relate to this?)

Note that the coproduct on $\mathrm{CE}^\bullet(\mathfrak{g})$ should *not* be confused with the shifted Lie bracket induced by a cobracket on $\mathfrak{g}$.

3.0.8. *A-model example: quantum cohomology.* If X is a symplectic manifold, we have a quantisation

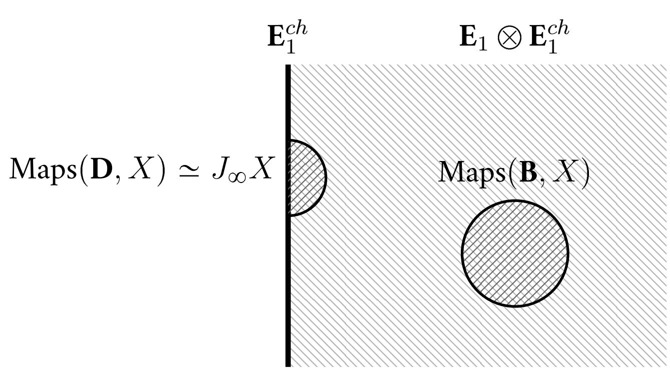


where $\mathrm{SH}(X)$ is the symplectic cohomology of X, see [Ri]. (what does it quantise?)

Note that $\mathrm{SH}(X)$ is the Drinfeld centre of an A if a ring A exists with A-Mod $\simeq$ $\mathrm{Fuk}(X)$; this does not always exist (when it does this is the affine case; in the B-model case we could also consider $\mathrm{QCoh}(X)$ for a general X).

3.1. **Conjectural $3d$ holomorphic-topological generalisation.**

3.1.1. The natural analogue of the above for the conjectural $\mathbf{E}_1 \otimes \mathbf{E}_1^{ch}$ operad is



where $\mathbf{B}$ is the bubble and $\mathbf{D}$ is the disk. Here $\mathrm{Maps}(\mathbf{B}, X) = Z_{\mathbf{E}_1^{ch}}(\mathrm{Maps}(\mathbf{D}, X))$ is the chiral $\mathbf{E}_1^{ch}$-centre. An analogue of Kontsevich's Theorem would then be

Conjecture. *The (chiral?) operad $\mathbf{E}_1 \otimes \mathbf{E}_1^{ch}$ is formal, and there is an equivalence of $\mathbf{E}_1 \otimes \mathbf{E}_1^{ch}$-algebras*

$$\mathrm{H}^\bullet(B) \xrightarrow{\sim} B$$

where $B = \mathcal{O}(\mathrm{Maps}(\mathbf{B}, X))$.

Warning. In the $3d$ holomorphic-topological situation, Davide thinks the 4 vertex terms might contribute, so it doesn't work. Davide expects that $4d$ holomorphic-topological is OK though.

3.1.2. We have

$2d$ TQFT	$3d$ HTQFT
$A = \mathcal{O}(X)$	$A = \mathcal{O}(J_\infty X)$
$\mathfrak{g} = \mathrm{HH}^\bullet(A, A)[1]$ is a Lie algebra	is $\mathfrak{g} = \mathrm{HH}^\bullet(A, A)$ a Lie* algebra?
?	Maurer Cartan equations
Does $\mathcal{M}_\mathfrak{g}$ control vertex deformations of A?	$\mathcal{M}_\mathfrak{g}$ controls deformations of A
Is $\mathcal{O}(\mathcal{M}_\mathfrak{g}) = \mathrm{CE}^{ch}(\mathfrak{g}[-1])$	$\mathcal{O}(\mathcal{M}_\mathfrak{g}) = \mathrm{CE}(\mathfrak{g}[-1])$

where we expect that CE^{ch} comes from a conjectural duality of chiral operads.[6]

Example. Take $A = \mathcal{O}(J_\infty \mathbf{A}^2)$, a Poisson vertex algebra. Then

$$\mathrm{HH}(A, A) := \mathrm{End}_{A\text{-Mod},\star_A}(A) \simeq \mathrm{End}_{U(A)^*}(A) \simeq \mathcal{O}(J_\infty T^*[-1]\mathbf{A}^2)$$

which is a commutative algebra and (± 1)-shifted vertex Lie algebra.

A-model version. There is also an A-model version of this story, for if we take *Poisson cohomology* of a Poisson manifold X. We can also take Poisson cohomology shifted with respect to a function $W \in \mathcal{O}(X)$; this corresponds to the *Landau-Ginzburg* two dimensional TQFT.

3.2. Categorification.

3.2.1. Note that we also have $\mathrm{QCoh}(X)$ doubling to $\mathrm{QCoh}(LX)$. Now if X has a symplectic form, we can consider a deformation $\mathrm{QCoh}(X)_\hbar$ and $\mathrm{QCoh}(LX)_\hbar$. (write this)

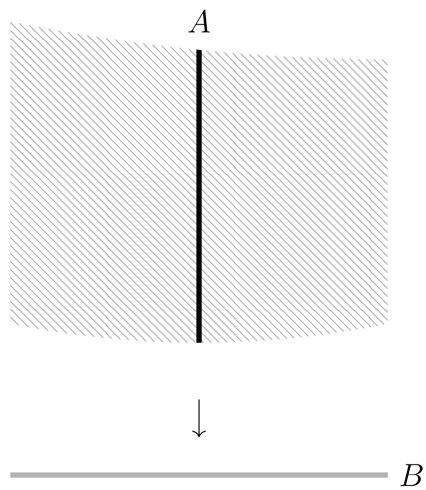
[6]This should be distinct from Francis-Gaitsgory's chiral Koszul duality, which is about the redundancy of the definition of $\mathbf{E}_1^{ch}$-algebra in terms of topological operads.

4. Twisted version

Let now let X be endowed with a function W.

Appendix A. Reminder on deformation theory

If A is a commutative, associative, Lie, ... algebra, we may consider the groupoid $\mathrm{Def}_A(B)$ of *deformations* over an Artin commutative, assocoative, Lie, ... algebra B.



This defines a *formal moduli problem* for the operad $\mathcal{P}$ we are considering, a functor

$$F \;:\; \mathcal{P}\text{-}\mathrm{Alg}_{\mathrm{Art}} \;\to\; \mathrm{Set}.$$

But by [CG], any such is uniquely determined by a $\mathcal{P}^!$-algebra T_F, and

$$F(B) \;=\; \mathrm{MC}(T_F \otimes B).$$

In the formal moduli problem Def_A where we're studying deformations of A, if the operad is sufficiently nice $T_F = A^!$ is just the Koszul dual.

Some examples of tangent complexes T_F are:

- If $\mathfrak{g}$ is a Lie algebra, then $T_{\mathrm{Def}_\mathfrak{g}} \;=\; \mathrm{CE}^\bullet(\mathfrak{g}) \;\simeq\; U(\mathfrak{g}) \otimes \wedge^\bullet \mathfrak{g}^*$ (check) is the Chevalley Eilenberg complex. For instance, an element $[\,,\,]_1 \in \mathrm{CE}^2(\mathfrak{g}) \subseteq \wedge^2\mathfrak{g}^* \otimes \mathfrak{g}$ measures a first order deformation

  $$[\,,\,] \;=\; [\,,\,]_\mathfrak{g} \;+\; \epsilon[\,,\,]_1$$

  and the Maurer-Cartan equation is equivalent to this being antisymmetric and satisfying the Jacobi equation. (check)

- If A is an associative algebra, first order deformations are measured by $m_1 \in \mathrm{HH}^2(A, A) \;=\; \mathrm{MC}(\mathrm{HC}^\bullet(A, A) \otimes \mathbf{C})$, i.e. in

  $$m \;=\; m_A \;+\; \epsilon m_1.$$

  Note that $\mathrm{HC}^\bullet(A, A) \simeq \mathrm{Hom}(\mathrm{B}A, A)$.

- If A is a commutative algebra, derivations are measured by a subcomplex $\mathrm{Harr}^\bullet(A,A) \subseteq \mathrm{HC}^\bullet(A,A)$ called the *Harrison complex*. By [Lo], if A is flat then we have
$$\mathrm{Harr}^\bullet(A,A) \ \simeq \ \mathbf{T}_A[1]$$
so that $\mathrm{Harr}^n(A,A) = \mathrm{H}^{n-1}(\mathbf{T}_A)$. In particular, deformations of flat X over $\mathrm{Spec} B$ are given by
$$\mathrm{MC}^\bullet(\mathbf{T}_X \otimes \mathfrak{m}_B)$$
where $\mathfrak{m}_B$ is the augmentation ideal, e.g. $\mathfrak{m}_{\mathbf{C}[\epsilon]/\epsilon^2} \simeq \mathbf{C}$. If X is smooth then $\mathbf{T}_X = T_X$ has no differential, but
$$\mathrm{C}^\bullet(X, T_X)$$
does, and
$$\mathrm{MC}^\bullet(\mathrm{C}^\bullet(X,T_X) \otimes \mathfrak{m}_B)$$
is what measures deformations of X over $\mathrm{Spec} B$. When $B = \mathbf{C}[\epsilon]/\epsilon^2$, this is identified with $\mathrm{H}^1(X,T_X)$ i.e. the Maurer-Cartan equation becomes $dv = 0$ because every element will have self-bracket $[v,v] = 0$, and we have implicitly modded out by the image of $\mathrm{C}^0(X,T_X)$. Note that $dv \in \mathrm{C}^2(X,T_X)$ is the obstruction to v defining a deformation.

We have maps
$$\mathrm{Harr}^\bullet(A,A) \ \to \ \mathrm{HC}^\bullet(A) \ \to \ \mathrm{CE}^\bullet(A)$$
where A is a commutative algebra; the latter is also defined when A is merely associative. When $A = \mathcal{O}(V)$ the latter map is a quantisation of the projection (check)

$$\begin{array}{ccc} V_\Delta \times V_{-\Delta}[1] & \longrightarrow & V \\ \downarrow & & \downarrow\Delta \\ V & \xrightarrow{\Delta} & V_\Delta \times V_{-\Delta} \end{array} \qquad \mapsto \qquad \begin{array}{ccc} V[1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & V \end{array}$$

A.0.1. *Remark.* There should be a module version of the above story.[7]

A.1. **Miscellaneous.**

[7]Compare the Lie algebra casr to the fact that maps $\mathrm{Hom}^2_{U(\mathfrak{g})}(\mathrm{CE}^\bullet(\mathfrak{g}), V)$ measure the set of extensions
$$0 \ \to \ V \ \to \ \mathfrak{e} \ \to \ \mathfrak{g} \ \to \ 0.$$
It is classified by a chain $\wedge^2\mathfrak{g} \to V$. Thus, the Lie algebra cohomology $\mathrm{Hom}^2_{U(\mathfrak{g})}(\mathrm{CE}^\bullet(\mathfrak{g}), \mathfrak{g})$ classifies first-order deformations of $\mathfrak{g}$ as a Lie algebra,

Reminder on deformation theory. W have the story of *formal deformation theory* giving as in [CCN, Thm. 1] an equivalence[8] [CG, Thm 3.64]

$$\mathrm{FMP}_{\mathcal{P}} \overset{\sim}{\leftrightarrows} \mathcal{P}^!\text{-Alg} \qquad F \mapsto \mathrm{KD}(\mathbf{T}_{F,\mathcal{P}})$$

between the category of formal moduli problems and algebras over the Koszul dual $\mathcal{P}^!$. For A a $\mathcal{P}$-algebra, an example of a formal moduli problem is $\mathrm{Def}_{\mathcal{P}}(A) \in \mathrm{FMP}_{\mathcal{P}}$, measuring the $\mathcal{P}$-deformations of A. Here $\mathbf{T}_{F,\mathcal{P}}$ is the $\mathcal{P}$-*tangent complex* of [CG, Def. 3.17] endowed with a $\mathcal{P}^!$-structure, see [CG, Rem 3.54].

If $V \in \mathrm{Vect}$, then by [CG, 2.29] there is a Lie algebra $\mathfrak{g}_{\mathcal{P}^!,V} = \mathrm{Tot}(\mathrm{Conv}(\mathcal{P}^!, \mathrm{End}_V))$, such that

$$\{\mathcal{P}^!\text{-algebra structures on } V\} \simeq \mathrm{MC}(\mathfrak{g}_{\mathcal{P}^!,V}).$$

If $A \in \mathcal{P}^!$-Alg, defining an element $\phi \in \mathfrak{g}_{\mathcal{P}^!,A}$, we can define a Lie algebra by changing the differential $d_{\mathfrak{g}^\phi} = d_{\mathfrak{g}} + [\phi, -]$, giving by [CG, 2.30] $\mathfrak{g}^!_{\mathcal{P}^!,A}$. Note that

$$\mathrm{MC}(\mathfrak{g}_{\mathcal{P}^!,A}) - \phi = \mathrm{MC}(\mathfrak{g}^\phi_{\mathcal{P}^!,A}) \hookrightarrow \mathrm{Def}_{\mathcal{P}^!}(A).$$

where the left hand equality is taken inside the vector space $\mathfrak{g}_{\mathcal{P}^!,A} = \mathfrak{g}^\phi_{\mathcal{P}^!,A}$ and the right hand inclusion is [CG, Prop 3.14]. We have taken Maurer-Cartan elements at ϕ.

The above is functorial in $\mathcal{P}, \phi$, i.e. in $\mathcal{P}, A \in \mathcal{P}^!$-Alg. For instance, we have for A a commutative algebra

$$\mathfrak{g}^\phi_{\mathbf{E}^!_\infty,A} \to \mathfrak{g}^\phi_{\mathbf{E}^!_1,A} \to \mathfrak{g}^\phi_{\mathrm{Lie}^!,A}$$

and applying Maurer-Cartan elements gives

$$\mathrm{Harr}^\bullet(A) \to \mathrm{HC}^\bullet(A,A) \to \mathrm{CE}^\bullet(A)$$

the complexes which measure the deformations of A as a commutative, associative, and Lie algebra, respectively. If A is just an associative algebra, the second map still exists. Note that these are just twisted bar complexes of A, see [CG, §1.6].

Note that we should view A as an element $\mathrm{Def}_{\mathcal{P}}(A) \in \mathrm{FMP}_{\mathcal{P}}$, and the above three are just elements of $\mathcal{P}^!$-Alg, i.e. as in [CG, 2.39]

$$\mathrm{Harr}^\bullet(A) \in \mathrm{Lie}\text{-Alg} \qquad \mathrm{HC}^\bullet(A,A) \in \mathbf{E}_1\text{-Alg} \qquad \mathrm{CE}^\bullet(A) \in \mathbf{E}_\infty\text{-Alg}.$$

(how do we explain the $\mathbf{E}_2$ structure on the middle?)

Note also that

[8] Note, what we have written $\mathcal{P}^!$ is actually $\mathcal{P}_\infty$ in [CG].

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