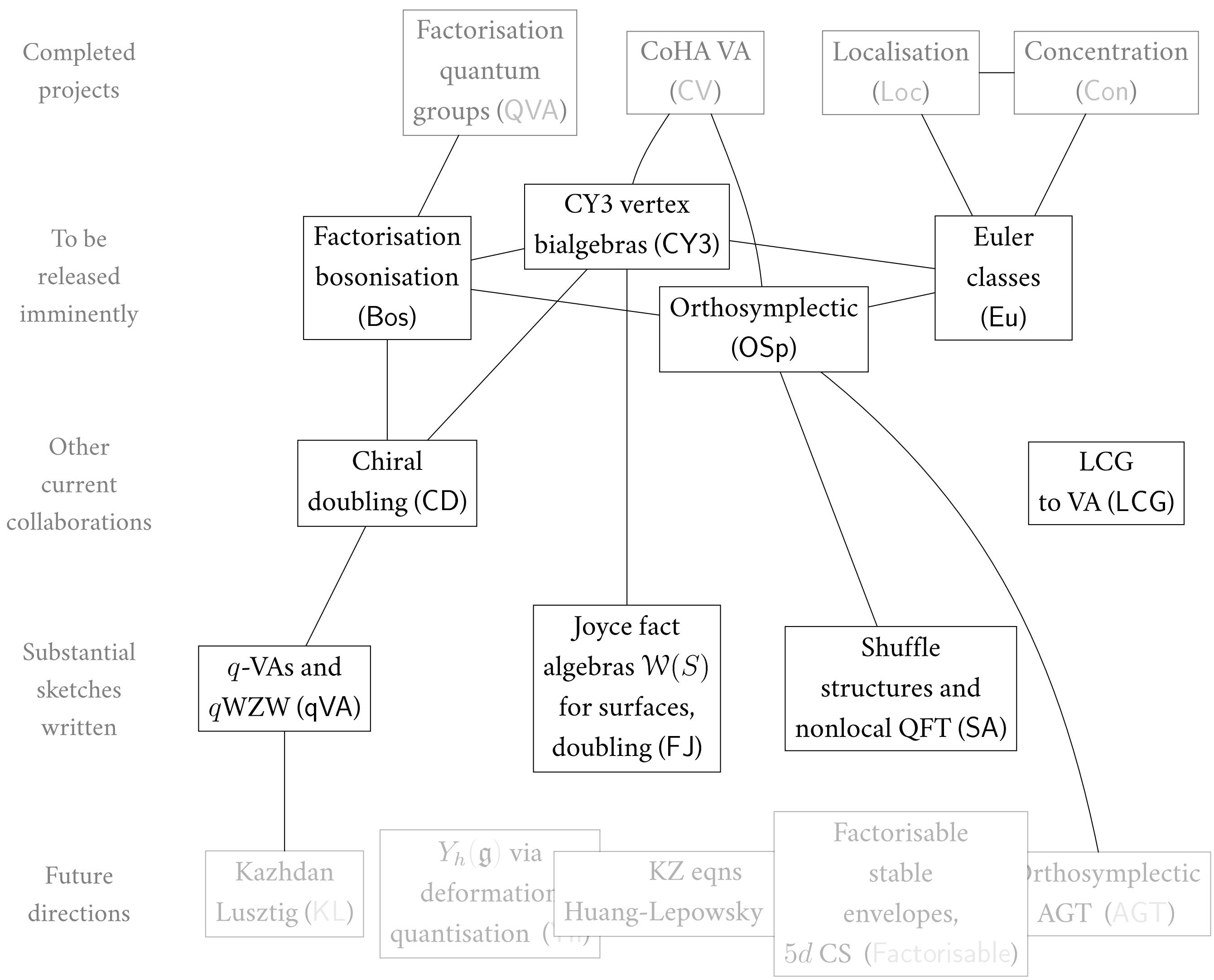
RESEARCH PLANS

ALEXEI LATYNTSEV

This is *extremely* under construction!

See the following sections (with clickable links*) for explanations of the projects and connections between them.*



Summary

0.1. Completed projects.

(1) **Quantum factorisation algebras**. Develop a theory of chiral $\mathbf{E}_n$-categories, which (should) recover previous notions of meromorphic tensor products. In this framework, build a theory of quantum factorisation (/vertex) algebras, spectral R-matrices.

(2) **CoHA and vertex algebras**. Show that dimension one cohomological Hall product and Joyce vertex algebra structures agree. Give new methods for computing CoHAs, giving new formulas.

(3) **Localisation**. Proved a localisation formula for arbitrary quasismooth derived schemes, relating the pushforward and pullback to a closed substack to the virtual Euler class.

(4) **Concentration**. Gave a sufficient condition for the Chow homology to be concentrated on a closed substack.

0.2. Imminently completing projects.

(1) **Orthosymplectic**. Work out how to take group-invariants of constructions in geometric representation theory. Define orthosymplectic moduli stacks, CoHAs, and orbifold vertex algebras (which we develop the theory of). Gives shuffle formulas. Show that we get twisted Yangians.

(2) **Euler classes**. Define Gysin maps, localisation formulas and virtual Euler classes for non-quasismooth closed embeddings between quasismooth spaces, by using a version of the exponential map to relate to the normal complex. Give localisation formulas for arbirary sheaf cohomologies. Develops a general theory of shuffle algebras.

(3) **CY3 vertex bialgebras**. For a large class of CY3 categories (local curves, quiver with potential), we define a Joyce-Liu vertex bialgebra structure on their critical cohomology. Show that Drinfeld's meromorphic and Davison's localised coproducts are examples of Joyce vertex coproducts. Develops a theory of "vertex bosonisation" and applies them to Yangians.

(4) **Chiral centres**. Work out how to take Drinfeld centres of chiral categories. Recovers notions of doubling chiral bialgebras, bubble Grassmannians (when applied to Rep$\mathfrak{g}(\mathcal{O})$), Yangians. Generalises BZFN's derivd loop spaces and centres construction.

0.3. Next stage projects.

(1) **Orthosymplectic AGT**. Proves AGT for the orthosymplectic CoHAs, i.e. that the action of the CoHA on intersection homology of good moduli spaces is the action on orthosymplectic affine $\mathcal{W}$-algebras.

(2) **q-WZW and q-vertex algebras via q-$\mathcal{D}$ modules**. Develop a theory of $\mathcal{D}_q$-modules, and define q-vertex algebras as factorisation algebras over "noncommutative spacetime". Now the ordinary definitions of Virasoro and affine vertex algebras via factorisation algebras carry over

immediately to the noncommutative setting. The q-affine vertex algebra lives on $\mathbf{C}^2$ with a Nekrasov's Ω-background.

(3) **Shuffle algebras**. Develops an operadic theory of shuffle algebras, giving new algebraic structures attached to any sequence of reductive groups, e.g. G_2 vertex algebras and their KZ equations, G_2 iterated loop spaces, and G_2 multiple zeta values. This is more general than the theory of ordinary (chiral) operads.

(4) **Factorisation algebras attached to a surface**. Shows that $\mathcal{W}(S)$ locally in (certain) S forms a sheaf of factorisation algebras over K_S, i.e. "S-vertex algebras", which are Morita equivalent on intersections. Gives an example of the M2-M5 brane construction.

0.4. **Prospective projects.**

(1) **Kazhdan-Lusztig**. Give a new proof of the Kazhdan-Lusztig equivalence which generalises to the new structures defined by Creutzig, Dimofte et al. Use the qWZW affine vertex algebra.

(2) **Stable envelopes factorisably**. Give a "Ran space" version of Maulik-Okounkov construction that includes all generalisations, e.g. the dynamical R-matrices.

(3) **Chiral deformation quantisation**. Make Gaiotto's recent paper rigorous, proving a formality result for chiral $\mathbf{E}_n$-algebras. Use it to recover the construction of Yangians and affine quantum groups.

1. Summary of projects

Contents

1.1. Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ).

1.1.1. *Background.* A main theme of geometric representation theory/enumerative geometry is: attached to certain Calabi-Yau-threefolds Y or categories $\mathcal{C}$, it has long been conjectured [KS] (now proven [**KPS**]) a "cohomological Hall" algebra structure on

$$\mathrm{H}^\bullet(\mathcal{M}_\mathcal{C}, \mathcal{P}) \tag{1}$$

where $\mathcal{P}$ is Joyce's DT sheaf (reference), and

- structure thing one
- two

From the physics perspective, the algebra structure is explained by (1) arising from an 11-dimensional "M" theory compactified on Y, which gives a $5d$ theory, then taking its algebra of BPS states [**Mo**] gives a q-deformed algebra structure. The other structures then arise from varying Y, to get an Alg-valued factorisation algebra over it; the analogy in the trivial toy model where Y is a $6d$ topological manifold is

$$\begin{array}{ccc} \mathrm{TQFT}_{11d} & \overset{\int_{\mathbf{R}^4\times S^1}}{\dashrightarrow} & \mathrm{TQFT}_{6d}(\mathrm{Alg}) \\ \downarrow\int_Y & & \downarrow\int_Y \\ \mathrm{TQFT}_{5d} & \xrightarrow{\int_{\mathbf{R}^4\times S^1}} & \mathrm{Alg} \end{array}$$

The motivating example is when $Y = K_S$ for a smooth algebraic surface S; then in FJ we expect a vertex algebra structures in the fibres of $K_S \to S$; this is proven in some 2CY cases in CY3.

1.1.2. Explanation: standard and nonstandard coproduct on $Y_\hbar(\mathfrak{g}_Q)$.

1.1.3. In the project CY3 (joint with S. Jidnal and S. Kaubrys), we prove

Theorem. (CY3, or CV for $W=0$) *For any quiver Q, there is a Joyce vertex coproduct on*

$$\mathrm{H}^\bullet_T(\mathcal{M}_{Q^{(3)}}, \varphi_{W^{(3)}}) \;\simeq\; Y_\hbar(\mathfrak{n}_Q)$$

compatible with the algebra structure: they form a vertex quantum group. It agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) the Drinfeld meromorphic coproduct.

Theorem. (CY3) *For any deformed CY3 category (e.g. coherent sheaves on local curve K_{T^*C}) there is a Joyce vertex coproduct on the cohomological Hall algebra $\mathrm{H}^\bullet(\mathcal{M},\varphi)$, which forms a vertex quantum group.*

One should think of this as a generalisation of the construction

$$U_q(\mathfrak{t})\text{-Mod} \;\rightsquigarrow\; \mathrm{H}^\bullet_T(\mathcal{M})\text{-Mod}^\cup, \qquad\qquad U_q(\mathfrak{n}) \;\rightsquigarrow\; \mathrm{H}^\bullet_T(\mathcal{M},\varphi)$$

and identify them with $Y_\hbar(\mathfrak{t})$-modules and $Y_\hbar(\mathfrak{n})$, respectively.

1.1.4. To compare localised and vertex coproducts in CY3, we introduce a *Ran-to-Conf* construction: taking localised terms $1/x$, pulling back by a $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$-coaction and taking a power seires expansion in z^{-1}

$$\frac{1}{x+nz} \;=\; \frac{1}{nz}\left(\frac{x}{nz} - \left(\frac{x}{nz}\right)^2 + \cdots\right)$$

defines a functor from localised coalgebras to vertex coalgebras.

1.1.5. Note that the map $\mathcal{M}\to \mathrm{BGL}$ makes $\mathrm{H}^\bullet(\mathcal{M},\varphi)\in \mathrm{QCoh}(\mathrm{Conf}(\mathbf{A}^1))$, moreover it is equivariant for the action (actually the $\mathbf{A}^1$ is acting on Conf $\times$ Conf)

$$\mathbf{A}^1\times \mathrm{Conf}(\mathbf{A}^1) \;\xrightarrow{a}\; \mathrm{Conf}(\mathbf{A}^1) \qquad \text{induced by applying cohomology to} \qquad \mathrm{B}\mathbf{G}_m\times \mathrm{BGL} \;\to\; \mathrm{BGL}.$$

The map a carries a flat connection, (we also need π_1 to, no?) so we may take

$$\pi_{1,*}a^*\mathrm{H}^\bullet(\mathcal{M},\varphi) \;\in\; \mathcal{D}\text{-Mod}(\mathbf{A}^1).$$

(just say I want to understand this structure operadically, be honest)

Thus, if $\mathcal{A}$ is a localised coalgebra,

$$\begin{tikzcd}[column sep=small] & (\mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1)_\circ \arrow[dl] \arrow[dr] & \\ \mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1 & & \mathrm{Conf}\mathbf{A}^1 \end{tikzcd} \qquad \Delta \;:\; \cup^*\mathcal{A} \;\to\; j^*(\mathcal{A}\boxtimes\mathcal{A})$$

Then the fibre over the antidiagonal (doesn't preserve disjointness)

$$D_\infty^\times \;\to\; \mathbf{A}^1 \;\xleftarrow{p}\; \mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1 \;\xrightarrow{(a\otimes a)(-\Delta\otimes\mathrm{id})}\; \mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1$$

has that $\Delta_{z^{-1}}$ defines a vertex coalgebra structure.

Conjecture. (Properadic vertex algebra-coalgebras) *Vertex coalgebras from factorisation algebras*

Conjecture. *The Ran-to-Conf construction lifts to a functor*

$$\mathrm{FactCoAlg}(\mathbf{A}^1) \;\to\; \mathrm{VertexCoAlg}.$$

1.1.6. Project FJ (joint with S. Kaubrys) aims to define factorisable lift of these structures. In the case of quivers Q, we have an action of the torus $T_d = \prod T_{d_i}$ on the stack of representations, and

$$\mathcal{M}^f \;=\; \{(m,\lambda)\;:\;\lambda\in\mathfrak{t}^*,\; m\in\mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d)$$

defines a factorisation space over the Q_0-coloured Ran space.

Conjecture. *The relative Borel-Moore homology of π defines a $\mathbf{G}_a^{Q_0}$-equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\mathrm{Ran}\mathbf{A}^1 \;\subseteq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1$$

recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{nilp})$ of [SV].

This would be interesting for two reasons:

- This should relate to Yang-Zhao's proof [YZ] that CoHAs form a localised factorisation bialgebra over $\mathrm{Conf}_\Lambda(E)$. We expect that the relation should be a factorisation space version of the Conf-to-Ran construction in CY3.
- This should relate to Maulik-Okounkov's stable envelope construction [MO] of Yangians.

Crucially, having repackaged the vertex bialgebra structure as a factorisation algebra, we can consider applying them to more general CY3 categories.

Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work), and the above should be an algebraic analogue of their construction.

1.2. **The structure of factorisation quantum groups (QVA, Bos, CD).**

1.2.1. At this point, the theory of quantum groups $U_q(\mathfrak{g})$ is well-developed:

(1) There are basis-free constructions [Ga] of $U_q(\mathfrak{g})$-Mod,
   (a) by working in the category $\mathrm{Perv}(\mathrm{Conf}_\Lambda(\mathbf{A}^1))$ of perverse sheaves on the configuration spaces,
   (b) by double-bosonisation [Ma].

(2) There is a "geometric" proof [Fu] of the Kazhdan-Lusztig equivalence $U_q(\mathfrak{g})\text{-Mod}^{ren} \simeq \hat{\mathfrak{g}}\text{-Mod}_k^I$

1.2.2. Explain what structures the (affine) Yangian is meant to have.

1.3. **Orthosymplectic vertex algebras (OSp, SA).**

1.3.1. In the project OSp (joint with S. de Hority), we

Work out how to take group-invariants of constructions in geometric representation theory. Define orthosymplectic moduli stacks, CoHAs, and orbifold vertex algebras (which we develop the theory of). Gives shuffle formulas. Show that we get twisted Yangians.

Theorem.

1.3.2. In the project OSp, one notices the following interesting thing.

$$\mathrm{BGL} \rightsquigarrow \mathrm{BSp}, \qquad \mathrm{Conf}(\mathbf{A}^1) \rightsquigarrow \mathrm{Conf}(\mathbf{A}^1), \qquad \mathrm{VA} \rightsquigarrow \mathrm{OSpVA}, \qquad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties, (KZ equations)) simultaneously generalise - this points towards this being a shadow of a more general theory.

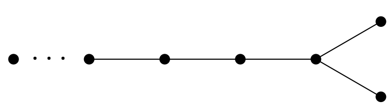
The starting observation is this - the definition (ref) a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL} \to \mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n \geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

$$\begin{tikzcd} & P_{n,m}(\sigma) \arrow[dl] \arrow[dr] & & & \\ \mathrm{GL}_n \times \mathrm{GL}_m & & \mathrm{GL}_{n+m} & \overset{A}{\mapsto} & A_n \otimes A_m \xrightarrow{m(\sigma)} A_{n+m} \end{tikzcd}$$

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n,m)$.

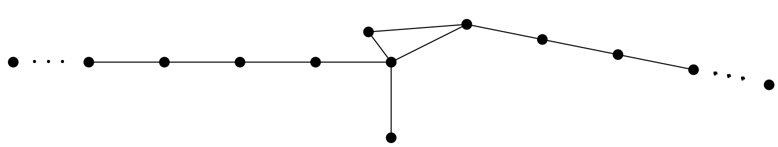
The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.

GL



$\mathrm{KM}_{\mathrm{SO}_{2n}}$

KM_{G_2}



To summarise:

- We get analogues of *shuffle algebras*.
- We get new configuration and Ran spaces

$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \textstyle\coprod_G \mathrm{Spec}\, \mathrm{H}^\bullet(BG), \qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G^*,$$

where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G, so can define generalised *localised* and *vertex* algebras (and as in CY3 a Conf-to-Ran construction relating them). We expect to recover *boundary KZ* equations by taking conformal blocks (i.e. cohomology over $\mathrm{Ran}_{\mathrm{KM}}\mathbf{A}^1$).

- Topological case - topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras, then by considering $\mathrm{FactAlg}^{top}(\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}, \mathrm{Cat})$ we get analogues of the notion of *braided monoidal categories*.
- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups

$$\begin{array}{ccccc} \mathrm{GL}_3 & U_{3,5} & \mathrm{GL}_5 & U_{5,4} & \mathrm{GL}_4 \\ \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc \end{array}$$

  where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as

$$\mathcal{M}_Q \;=\; \coprod \mathfrak{u}_e/G_i$$

  the product over all maps $(G_i) : Q_0 \to$ KM and U_e is a choice if unipotent for each edge e.

  Relation to orbifolding.
- Stable envelope construction.
- Chen's [Ch] shuffle structure on cochains $\mathrm{C}^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX = \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n = T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G.
- Iterated integrals.

For the orthosymplectic example $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$, many of these structures are considered in OSp. Let us consider K_{G_2}

For instance, for K_{G_2},

For instance, we have an action of $W_{G_2} \simeq D_{12}$ on $\mathbf{C}^3$. It has a normal subgroup $\mathfrak{S}_3$ whose quotient is generated by the rotation τ by one-sixth, in coordinates it sends

$$\begin{aligned} \tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3) \\ \tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1) \\ \tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2) \end{aligned}$$

which one can check squares to (231).

Thus, a factorisation algebra over $\mathrm{Ran}_{G_2}\mathbf{C}$ restricts to $\mathrm{Ran}\mathbf{C}$, giving a braided monoidal category. However, in addition we have an isomorphism

$$\tau^*\mathcal{C}|_{\mathbf{C}^3} \;\simeq\; \mathcal{C}|_{\mathbf{C}^3}.$$

Fibrewise, this means we have an isomorphism

$$\mathcal{C}_{\tau(z_1)} \otimes \mathcal{C}_{\tau(z_2)} \otimes \mathcal{C}_{\tau(z_3)} \;\simeq\; \mathcal{C}_{z_1} \otimes \mathcal{C}_{z_2} \otimes \mathcal{C}_{z_3}$$

squaring to the identity, for disjoint triples $z_i \in \mathbf{C}$.

Lemma 1.3.3. *A G_2 factorisation category over $\mathbf{C}$, i.e. over $\mathrm{Ran}_{G_2}\mathbf{C}$, is equivalent to a braided monoidal category together with an involution*

$$\tau \;:\; \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{C} \;\xrightarrow{\sim}\; \mathcal{C} \boxtimes \mathcal{C} \boxtimes \mathcal{C}$$

(maybe actually a natural transformation?)

For instance, we

1.4. q-vertex algebras (qVA, KL).

1.4.1. It has been long expected that one may define a q-analogue of the Kazhdan-Lusztig equivalence, but this has been hampered by the lack of a good definition of q-WZW algebras: currently, the available definition is an RTT-style definition from [**EK**].

1.4.2. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V^k_\hbar(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

where $(\mathbf{C} \times \mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x,y]/(xy - qyx)$.

The goal of qVA is then to define a structure $V^k_q(\mathfrak{g})$ that "lives on $(\mathbf{C} \times \mathbf{C})_{nc}$". The hope would be then that a subcategory of its modules should be relate to $Y_\hbar(\hat{\mathfrak{g}})$-Mod.

1.4.3.

1.5. Localisation methods (Con, Loc, Eu).

1.5.1. In Con and Loc (joint with A. Khan, D. Aranha, H. Park, and C. Ravi)

1.5.2. In Eu

1.6. Louiville quantum gravity to vertex algebras (LCG).

1.6.1. In LCG (joint with V. Giri)

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