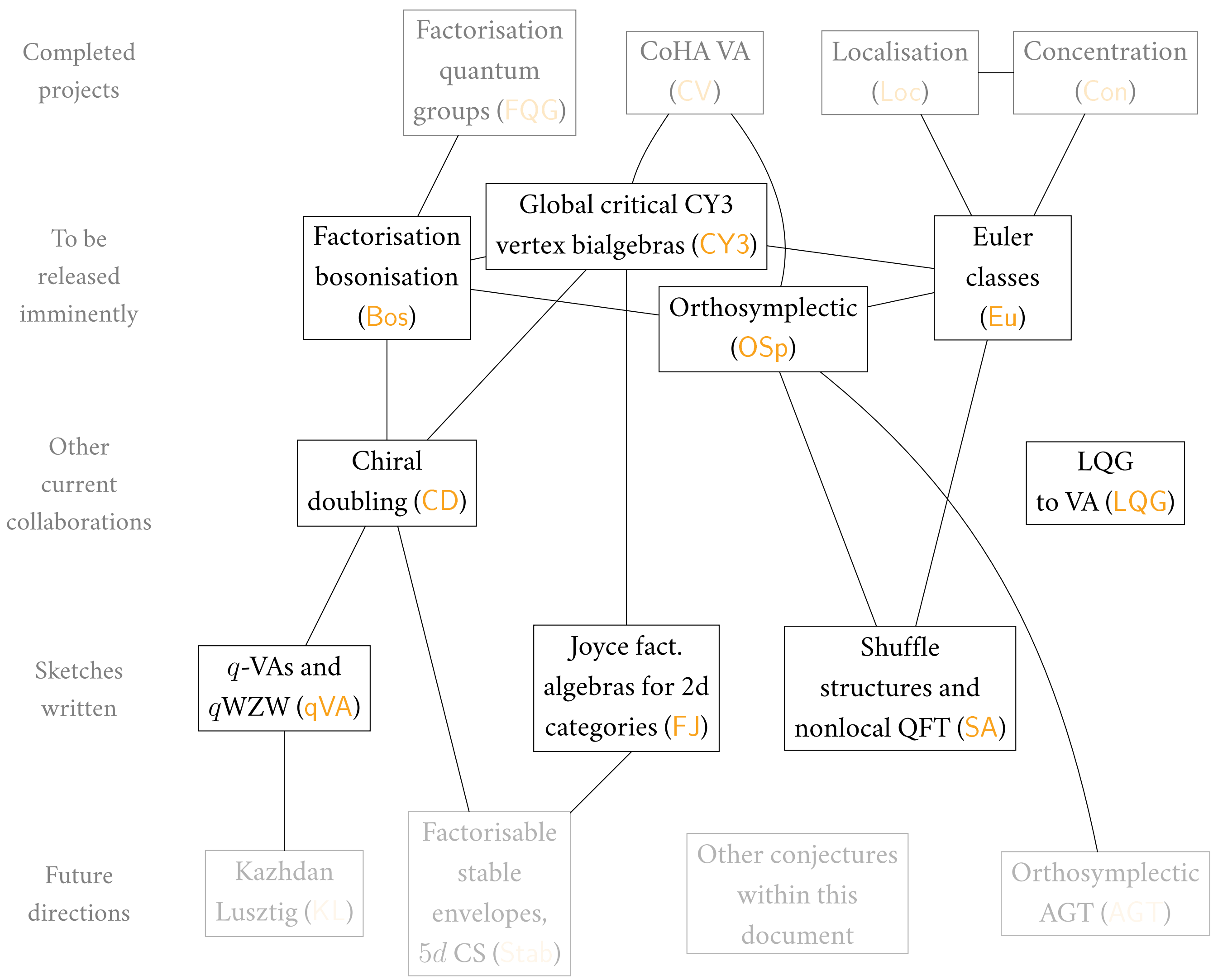
RESEARCH PLANS

ALEXEI LATYNTSEV

*For an **abridged** (4 pp.) version, see* https://alyoshalatyntsev.github.io/planabridged/planabridged.pdf.
*For a non-technical **summary** (2 pp.), see*
https://alyoshalatyntsev.github.io/plansummary/plansummary.pdf.



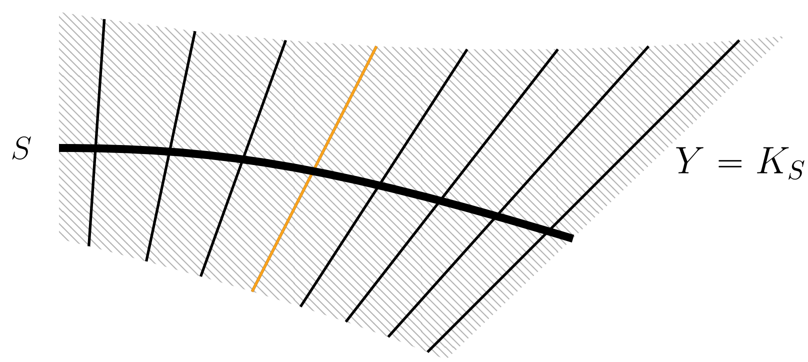
Note: CY3,FJ *is joint with S. Kaubrys,* CY3 *with S. Jindal,* CD *with W. Niu,* OSp, Bos, SA *with S. de Hority,* Loc, Con *with A. Khan, D. Aranha, H. Park, and C. Ravi, and* LQG *with V. Giri.*

Contents

1. Details of projects

1.1. Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ, Stab)

Global critical CY3 vertex quantum groups. The ultimate question this series of projects points towards is understanding how the assignment $Y \mapsto \mathrm{H}^\bullet(\mathcal{M}_Y, \mathcal{P}_Y)$ of a CoHA [KS; KPS; YZb] to a Calabi-Yau threefold or category Y behaves factorisably over Y.



If it only depended on the topological type of Y the answer would be "it is an $\mathbf{E}_6$-algebra". In actuality, in CY3 we show that, when the CY3 category $\mathcal{C}$ is a *global critical locus* inside a smooth ambient stack

$$\mathcal{M}_{\mathcal{C}} = \mathrm{Crit}(W) \subseteq \mathcal{M},$$

the CoHA has the structure of a "Joyce" *vertex coproduct* compatible with the CoHA: in the topological analogy this is analogous to the $\mathbf{E}_2 \subseteq \mathbf{E}_6$ structure. The main class of examples is *deformed CY3* completions, which include zero dimensional coherent sheaves on $Y = K_{T^*C}$ or local curves [KK] and the category of representations $\mathcal{C} = \mathrm{Rep}(Q, W)$ of a Jacobi algebra of a quiver with potential.

Theorem A. [CY3] *For $\mathcal{C}$ as above, there is a vertex coproduct on the critical*[1] *CoHA*

$$\mathrm{H}^\bullet(\mathcal{M}, \varphi) \to \mathrm{H}^\bullet(\mathcal{M}, \varphi) \hat{\otimes} \mathrm{H}^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

[1] i.e. cohomology of the vanishing cycle sheaf $\varphi = \varphi_W$.

making it into a quasitriangular vertex bialgebra ("vertex quantum group") inside the braided factorisation category $\mathrm{Rep}(\mathrm{H}^\bullet(\mathcal{M}),\cup)$.[2]

To push the analogy with quantum groups, we consider the case $\mathcal{C}$ is preprojective representations of ADE quiver Q.

$\mathrm{Rep}_q T$	$U_q(\mathfrak{n})$	$U_q(\mathfrak{b})$	$U_q(\mathfrak{g})$
$\mathrm{Rep}\ \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M})$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)^{ext}$, c.f. Bos	c.f. CD
$\mathrm{Rep}Y_\hbar(\mathfrak{t})$	$Y_\hbar(\mathfrak{n})$	$Y_\hbar(\mathfrak{b})$	$Y_\hbar(\mathfrak{g})$

(1)

Just as $U_q(\mathfrak{n})$ is a braided commutative bialgebra in $\mathrm{Rep}_q T$ constructed as in [Ga] using the (co)free (co)algebra on $\mathfrak{n}$, Theorem A says that the CoHA forms a vertex bialgebra.

We sanity-check that this is an interesting structure, showing that the first three columns of the bottom part of (1) match up:

Theorem B. [CY3; CV for $W = 0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA* $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}})\simeq Y_\hbar(\mathfrak{n}_Q)$ *agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) Drinfeld's meromorphic coproduct.*

Just as $U_q(\mathfrak{b})$ is constructed by Tannakian reconstruction on $U_q(\mathfrak{b})\text{-Mod}(\mathrm{Rep}_q T)$, in Bos (work in progress) we develop a factorisable analogue of this. This results in a vertex bialgebra structure on the extended CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos} = \mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M})$,

$$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}\text{-Mod} \;=\; \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)\text{-Mod}(\mathrm{Rep}\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}))$$

which in the preprojective case recovers (Soibelman-Rapcak)-Yang-Zhao's construction on $Y_\hbar(\mathfrak{b}_Q)$.

Vertex coalgebras from configuration spaces. For Theorem B to be well-defined, we need to be able to compare localised and vertex coproducts. Recall from [Daa] that a *localised* bialgebra is a factorisation bialgebra over the configuration space, i.e. sheaf of algebras $\mathcal{A}$ with a map

$$\begin{tikzcd}[column sep=small] & (\mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1)_\circ \arrow[dl, "j"'] \arrow[dr, "\cup j"] & \\ \mathrm{Conf}\mathbf{A}^1\times\mathrm{Conf}\mathbf{A}^1 & & \mathrm{Conf}\mathbf{A}^1 \end{tikzcd} \qquad (\cup j)^*\mathcal{A}\xrightarrow{\Delta} j^*(\mathcal{A}\boxtimes\mathcal{A})$$

i.e. there are localised terms $1/(x_i-x_j)$ in the image of the coproduct. Now,

Theorem C. *There is a functor from localised bialgebras to vertex bialgebras.*

The functor is given by taking Talyor expansions $\frac{1}{x+z}=\frac{1}{z}\left(\frac{x}{z}-\left(\frac{x}{z}\right)^2+\cdots\right)$ where $z\in\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$ comes from the action of $\mathrm{B}\mathbf{G}_m$ on BGL, see SA. This construction suggests a resolution to the issue that so far there is no factorisable definition of vertex *co*algebras:

[2] The formalism of braided factorisation categories is developed in FQG.

Conjecture D. Configuration-to-Ran construction. *One may define a colimit* $\mathrm{Ran}_{\mathbf{P}}\mathbf{A}^1$ *of projective spaces* $\mathbf{P}^n$*, and factorisation coalgebras* $\mathcal{A}$ *on it induce vertex* coalgebras*. The construction above then lifts to a functor to these factorisation coalgebras* $\mathcal{A}$*.*

Lift to factorisation algebra. An interesting question is: does the above vertex coproduct factorise? In other words, is there a natural geometric structure over $\mathrm{Ran}Y$ which induces the vertex coproduct of Theorem A? Answering this is the content of FJ.

If we had an answer to this question,

- It may make it clearer how to generalise away from the case that $\mathcal{M}_{\mathcal{C}}$ is given as a global critical locus, to more general CY3s.
- It gives us new understanding of the [MMSV] of W-algebras attached to surfaces S.
- We could push the analogy with quantum groups further, by constructing a Yangian $Y_\hbar(\mathfrak{g})$ factorisably, mirroring [Ga].

In the case of quivers Q, we have an action of the torus $T_d = \prod T_{d_i}$ on the stack of representations, and

$$\mathcal{M}^f \;=\; \{(m,\lambda)\;:\;\lambda\in\mathfrak{t},\; m\in\mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d) \tag{2}$$

defines a factorisation space over the Q_0-coloured Ran space.

Conjecture E. [FJ] *The relative critical cohomology* $\mathcal{A} = \pi_*\varphi_W$ *defines a translation equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\mathrm{Ran}\mathbf{A}^1 \;\subseteq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1$$

recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{nilp})$ *of* [SV].

This should relate to Yang-Zhao's proof [YZa] that CoHAs form a localised factorisation bialgebra over $\mathrm{Conf}(\Sigma)$ by the Configuration-to-Ran construction D. Finally, this construction makes the role of the torus $\mathfrak{t}_d$ clear, and therefore in SA we may generalise it to arbitrary Kac-Moody groups.

Remark. Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work, see also [Dac]), and the above should be an algebraic analogue of their construction.

Relation to $\mathcal{W}$-algebras. In the same way, for any surface S we may define the stack $\pi : \mathcal{M}^f_{\mathcal{C}} \to \mathrm{Ran}_S K_S$ parametrising a finite subset of points $x_i : S \to K_S$ and a dimension zero coherent sheaf supported on their joint image.

This gives the following new structure on the algebras $W^+(S)$ of [MMSV], which e.g. after completing CD may allow us to take its double.

Conjecture F. [FJ] *We have a global sheaf of categories over* $\mathrm{Ran}_S K_S$ *which have a braided factorisation structure along the fibres of* $K_S \to S$*.*

In general $\mathcal{M}^f_{\mathcal{C}}$ this will *not* be a global critical locus in a smooth factorisation space $\mathcal{M}^f$, only locally over an open cover $S = \bigcup U_i$ do we have such an $\mathcal{M}^f_i$. In particular, by applying the construction of CY3[3] we get a vertex quantum group $\mathcal{V}_i$ inside $\mathrm{H}^\bullet(\mathcal{M}_i)$-Mod, which will be Morita equivalent on the intersections $U_i \cap U_j$,[4] hence we expect $\mathcal{V}_i$-Mod($\mathrm{H}^\bullet(\mathcal{M}_i)$-Mod) to glue over $\mathrm{Ran}_S K_S$.

Relation to stable envelopes. The definition (2) is clearly reminiscent of the definition of stable envelopes and the structure [MO, §5.1.1] on Nakajima quiver varieties used to define $Y_\hbar(\mathfrak{g}_Q)$ via stable envelopes; we plan to study this once FJ is finished. This suggests the following answer to Maulik-Okounkov's question about how to interpret their data: for every multidimension vector w, define a factorisable version $\pi : M(w)^f \to \mathrm{Ran}\mathbf{A}^1$ of the quiver variety,[5] let $\mathcal{C}$ be the factorisation category generated by the relative Borel-Moore homology of $\pi_*\omega_{M(w)^f}$, then apply Tannakian reconstruction to get the definition [MO, Def. 5.2.6] of $Y_\hbar(\mathfrak{g})$.[6]

Thus, just as the Drinfeld-Kohno Theorem reconstructed $U_\hbar(\mathfrak{g})$ with its R-matrix from $U_q(\mathfrak{g})$-Mod (viewed [CF] as a perverse sheaf of categories over $\mathrm{Ran}\mathbf{R}^2$), we expect to do the same to $\mathcal{C} \simeq Y_\hbar(\mathfrak{g})$-Mod over $\mathrm{Ran}\mathbf{C} \times \mathbf{R}$.

1.2. The structure of factorisation quantum groups (FQG, Bos, CD)

History. A collection of structures all loosely called "quantum groups" have been some of the main objects in mathematical physics and geometric representation theory since the 80s:

(1) It is well-known that the representation categories of $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ should be controlled by "spectral" analogues $R(z)$ of R-matrices [CWY; GLW].

(2) The affine case has recently been understood much better: the algebras $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ in [GRZ; RSYZ]

(3) In [MO] Maulik-Okounkov define a bialgebra $Y_\hbar(\mathfrak{g}_Q)$ attached to *any* quiver Q.

Historically these definitions were (ingeniously) made very explicitly using formulas: generators and relations/RTT definitions, e.g. [Dr; MO], still much of the modern (e.g. CoHA) literature is based on explicit shuffle computations, e.g. [MMSV; SV; YZb].

The point of this series of projects is to first give a more conceptual definition of these structures, second recover the above formulas as a *consequence* of these definition, and third to generalise them to more general structures.

[3] We assume that K_S has a cover e.g. by local curves so that we can apply CY3, such as for instance if $S = \mathbf{P}^2$.

[4] Indeed, they are two different vertex algebra structures on the same vector space $\mathrm{H}^\bullet(\mathcal{M}_{U_i \cap U_j}, \varphi)$.

[5] To get this, one expects that one would have to interpret stable envelopes $\mathrm{Stab}_\lambda : \mathrm{H}^\bullet_T(M^\lambda) \to \mathrm{H}^\bullet_T(M) \simeq \mathrm{H}^\bullet_T(M^0)$ as a kind of factorisation structure.

[6] Notice how Maulik-Okounkov's definition essentially defines $Y_\hbar(\mathfrak{g})$ as something reminiscent of a subalgebra of "factorisation endomorphisms" of $\mathcal{C}$.

Factorisation quantum groups. In FQG we develop a theory of *braided factorisation categories*. The basic idea is to take Lurie's [Lu] result

$$\mathbf{E}_2\text{-dgCat} \simeq \mathbf{E}_2\text{-Alg}(\mathrm{Sh}(\mathrm{Ran}\mathbf{R}^2, \mathrm{dgCat}))$$

that a braided monoidal category $\mathcal{C}$ is equivalent to[7] a factorisable constructible sheaf of categories over the Ran space of $\mathbf{R}^2$, and replace $\mathrm{Ran}\mathbf{R}^2$ with an arbitrary *factorisation space* X.[8]

$$\begin{array}{ccc} & (\mathrm{Ran}\mathbf{R}^2 \times \mathrm{Ran}\mathbf{R}^2)_\circ & \\ \swarrow & & \searrow \\ \mathrm{Ran}\mathbf{R}^2 \times \mathrm{Ran}\mathbf{R}^2 & & \mathrm{Ran}\mathbf{R}^2 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & M_X & \\ \swarrow & & \searrow \\ X \times X & & X \end{array}$$

Examples include: ordinary groups G, configuration spaces $\mathrm{Conf}_\Lambda(\mathbf{A}^1)$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$, or any space X with the inclusion of the diagonal.

We first check that this is a reasonable definiton: it recovers R-matrices,

Theorem G. [FQG] *One may define factorisation categories, algebras, etc., over X. If $\mathcal{A}$ is a factorisation bialgebra in factorisation category $(\mathcal{C}, \otimes_{\mathcal{C},X})$ over X, each braided factorisation structure on $\mathcal{A}$-Mod$(\mathcal{C})$ induce a factorisation R-matrix $R : \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A}$ for which $\mathcal{A}$ is quasitriangular.*

In the vertex algebra case we recover classical notions [EK; FR] of quantum vertex algebras:

Theorem H. [FQG] *When $X = \mathrm{Ran}\mathbf{A}^1$ and $\mathcal{A}$ is a nonlocal vertex bialgebra V, a factorisation R-matrix is an endomorphism $R(z) : V \otimes V((z)) \xrightarrow{\sim} V \otimes V((z))$ satisfying the spectral hexagon relations, such that V is $R(z)$-twisted local (fields satisfy $R(z)$-twisted OPEs).*

For instance, this allows us to interpret result [GLW] on the strucutre of $\mathcal{A} = Y_\hbar(\mathfrak{g})$ as follows. It is a *lax* quasitriangular factorisation bialgebra with laxly compatible $\otimes^*$- and $\otimes^{ch}$-coproducts on the factorisation category of equivariant D-modules on $X = \mathrm{Ran}\mathbf{A}^1$, corresponding to the standard and meromorphic coproducts on $Y_\hbar(\mathfrak{g})$. The various R-matrices of [GLW]

$$\mathcal{M}_1 \otimes^* \mathcal{M}_2 \xrightarrow{R^-(z)} \mathcal{M}_2 \otimes^{ch} \mathcal{M}_2, \quad \mathcal{M}_1 \otimes^{ch} \mathcal{M}_2 \xrightarrow{R^{0,\epsilon}} \mathcal{M}_2 \otimes^{ch} \mathcal{M}_1, \quad \mathcal{M}_1 \otimes^* \mathcal{M}_2 \xrightarrow{R^\epsilon} \mathcal{M}_2 \otimes^* \mathcal{M}_1$$

then induce lax compatibility maps between the two coproducts on $\mathcal{A}$-Mod. So: we have lifted $Y_\hbar(\mathfrak{g})$-Mod to a topological-holomorphic factorisatsion algebra over $\mathbf{R} \times \mathbf{C}$ rigorously.

Theorem G specialises to the ordinary theory of R-matrices and quantum groups (e.g. for $U_q(\mathfrak{g})$) when $X = \mathrm{Ran}\mathbf{R}^2$. Likewise, when $X = \mathrm{Conf}\mathbf{A}^1$ a factorisation bialgebra $\mathcal{A}$ is equivalent to the *localised bialgebras* B of [Daa] (the main examples being CoHAs), a factorisation R-matrix is an endomorphism

$$R : (B \otimes B)_{\mathrm{loc}} \xrightarrow{\sim} (B \otimes B)_{\mathrm{loc}}$$

of the localised bialgebra B.

[7] To see a precise statement and of this, see [CF, Prop 7.0.2].

[8] Space here means derived prestack.

These techniques are fairly flexible and seem to be able to prove chiral analogues of many structure results in the theory of quantum groups, e.g. FQG ends by constructing a generalisation of the Borcherds twist construction [Bo].

Factorisation bosonisation. In Bos, we automate the construction of *adding in the Cartan*:

$$U_q(\mathfrak{n}) \overset{\text{[Maa; Mab]}}{\rightsquigarrow} U_q(\mathfrak{b}), \qquad Y_\hbar(\mathfrak{n}) \overset{\text{[Dr]}}{\rightsquigarrow} Y_\hbar(\mathfrak{b}), \qquad \mathrm{H}^\bullet(\mathcal{M},\varphi) \overset{\text{[Daa; RSYZ; YZb]}}{\rightsquigarrow} \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}.$$

Indeed, in both CY3 and OSp we had to do complicated generating series manipulations to extend the CoHA (i.e. add a Cartan part) as a vertex bialgebra. In the finite case [Maa; Mab], the bialgebra structure on $U_q(\mathfrak{b})$ can be written in terms of $U_q(\mathfrak{n})$'s using Tannakian reconstruction ("*bosonisation*"):

$$U_q(\mathfrak{b})\text{-Mod} \ \xrightarrow{\sim}\ U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

and we want to do the same for vertex algebras.

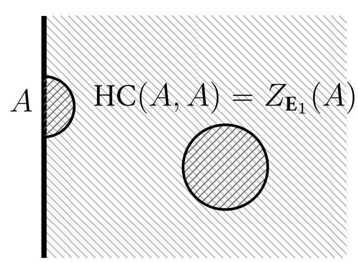
Theorem I. [Bos, in preparation] *A Tannakian reconstruction for factorisation categories, which when applied to*

$$Y_\hbar(\mathfrak{n})\text{-Mod}(Y_\hbar(\mathfrak{t})\text{-Mod}) \ \simeq\ \mathrm{H}^\bullet(\mathcal{M},\varphi)\text{-Mod}(\mathrm{H}^\bullet(\mathcal{M})\text{-Mod})$$

gives the vertex bialgebra $Y_\hbar(\mathfrak{b}) \simeq \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$.

The analogues of Majid's [Maa; Mab] formulas recovers e.g. the localised bialgebra structure on the extended CoHA from [YZb, §1.3] or [RSYZ].

Factorisation Drinfeld doubling. The *Drinfeld centre* (see [Lu]) of an $\mathbf{E}_n$-algebra by Kontsevich's conjecture [Th] gives an algebra over the Swiss cheese operad:



In CD the main aim is to replace $\mathbf{E}_n$-algebras with *chiral* $\mathbf{E}_n$-algebras:

Conjecture J. *There is a construction* $Z_{\mathbf{E}_1}(\mathcal{C})$ *of a chiral factorisation category over* $\mathrm{Ran}\mathbf{A}^1$*, which carries compatible chiral and ordinary monoidal structures. Applying this to the category of modules for the affine vertex algebra* $V^k(\mathfrak{g})$ *and the (dual) Yangian* $Y_\hbar(\mathfrak{b})^*$ *give category of modules over the quantisation [AN] of* $U(T^*\mathfrak{g}[t])$ *and* $Y_\hbar(\mathfrak{g})^*$*, respectively.*

The first part has followed easily from the results in FQG, and we are currently working on the second.

1.3. Orthosymplectic structures (OSp, SA, AGT)

Physics heuristic. In project OSp we make a mathematical theory of *boundary* $4d$ *Chern-Simons* [BS] on $\mathbf{R} \times (\mathbf{R} \times \mathbf{C})/\pm$, for instance our structures structures satisfy *boundary Yang-Baxter/Cherednik reflection equations*. More generally, we define boundary versions of compactifications of $4d$ SCFTs $\int_Y \mathcal{M}$ attached to a CY3 Y - at least, those for which non-boundary versions have been defined. It should relate to Finkelberg-Hanany-Nakajima's ongoing work on orthosymplectic Coulomb branches (see AGT).

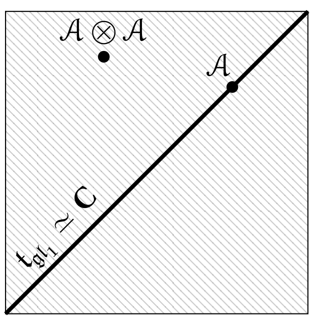
Details. Attached to an abelian category $\mathcal{A}$,[9] we construct the *orthosymplectic moduli stack* $\mathcal{M}_{\mathcal{A}}^{\mathrm{OSp}}$: a fixed point stack whose points are objects with a symmetric pairing $a \simeq a^*$.

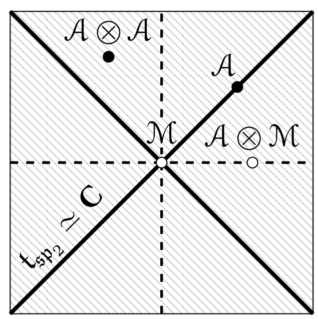
Theorem K. [OSp] *For $\mathcal{A}$ in* CY3 *or examples below, the vertex quantum group* $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ *"acts" on* $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}, \varphi^{\mathrm{OSp}})$*:*

1. *there is a left module action a of the CoHA respecting the involution,*[10] *compatible with*
2. *its* symplectic vertex algebra *structure: it is a factorisation coalgebra over symplectic Ran space* $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\,\mathfrak{t}_{\mathfrak{sp}_{2n}}$ *(coming from a localised structure over* $\mathrm{Conf}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$*).*

The data (1) and (2) is equivalent to a topological and holomorphic factorisation algebra over $\mathbf{R}/\pm$ and $\mathbf{C}/\pm$, respectively. We give an equivalent vertex algebra style definition of the latter in terms of fields $A \otimes M \to M((z))$. In pictures, a factorisation and symplectic factorisation algebra are:

$\mathfrak{t}_{\mathfrak{gl}_2} \simeq \mathbf{C}^2$





$\mathfrak{t}_{\mathfrak{sp}_4} \simeq \mathbf{C}^2$

To give examples, we construct an **invariants** functor involving restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$

$$\iota \;:\; \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \;\to\; \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1), \qquad\qquad (\mathcal{A}, \tau) \;\mapsto\; (\mathcal{A}, \mathcal{A}^\tau)$$

where $\mathcal{A}$ is a factorisation algebra with involution τ; we expect Theorem K may also be proved by applying ι to the factorisable moduli stack $\mathcal{M}^f$ from FJ. See also the link to stable envelopes Stab, and:

Conjecture L. *The* ***boundary KZ equations*** *may be derived by applying ι to the BD Grassmannian, taking distributions supported at the identity, and taking conformal blocks over* $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1$.

[9] More generally abelian category with involution $(\mathcal{A}, \tau)$, e.g. $\tau = (-)^*$.

[10] i.e. the left action a and the right action $a \cdot (\mathrm{id} \otimes \tau)$ commute, where τ is the involution.

Examples include $\mathbf{Z}/2$-equivariant quivers with potential,[11] or orthosymplectic perverse-coherent sheaves on surfaces, e.g. orthosymplectic ADHM quiver/perverse-coherent sheaves on $\mathbf{A}^2$:

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[loop above] \arrow[loop below] \arrow[r, shift left] & \mathrm{Sp}(2m) \arrow[l, shift left] \end{tikzcd} \qquad\qquad \mathcal{E} \simeq \mathrm{RHom}(\mathcal{E}, \mathcal{O})$$

Theorem M. [OSp] *In the quiver with potential case, an explicit shuffle formula for the CoHA action and vertex coaction on* $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}, \varphi^{\mathrm{OSp}})$.

Using techniques from Eu, we can give a geometric interpretation of this. We end with a conjecture:

Conjecture N. *The orthosymplectic CoHA for the "folded" linear quiver* A_{2n}[12] *is isomorphic to the twisted Yangian* $Y_\hbar(\mathfrak{gl}_n)^{tw}$ *of [BR].*

Nonlocal QFT and shuffle structures. Project SA begun by noticing the following interesting pattern in structures considered project OSp.

$$\mathrm{BGL} \rightsquigarrow \mathrm{BSp}, \qquad \mathrm{Conf}(\mathbf{A}^1) \rightsquigarrow \mathrm{Conf}(\mathbf{A}^1), \qquad \mathrm{VA} \rightsquigarrow \mathrm{OSpVA}, \qquad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties and the KZ equation, simultaneously generalise - this points towards this being a shadow of a more general theory.

The starting observation is this - the definition [KS; Gr] a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL} \to \mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n \geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

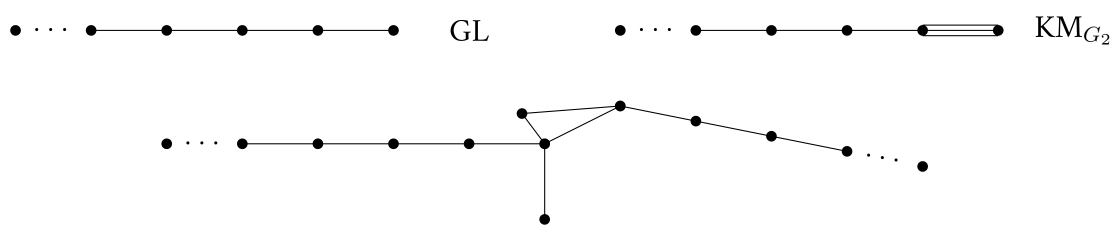
$$\begin{tikzcd} & P_{n,m}(\sigma) \arrow[dl] \arrow[dr] & \\ \mathrm{GL}_n \times \mathrm{GL}_m & & \mathrm{GL}_{n+m} \end{tikzcd} \quad \overset{A}{\mapsto} \quad A_n \otimes A_m \overset{m(\sigma)}{\to} A_{n+m}$$

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n, m)$.

The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.

[11] n.b. we can view any quiver with involution, as orbifold-valued quiver with vertices $Q_0/\mathbf{Z}/2$ and edges $Q_1/\mathbf{Z}/2$; it is natural to ask if we can generalise away from global quotients.

[12] i.e. with the involution being reflection in the linear direction.



To summarise, we expect to define KM analogues of the following:

- *Shuffle algebras*, likewise analogues localised and vertex algebras living over new configuration and Ran spaces
$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \textstyle\coprod_G \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{B}G), \qquad\qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G,$$
where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G. Topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras/braided monoidal categories.

- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups
$$\begin{array}{ccccc} \mathrm{GL}_3 & U_{3,5} & \mathrm{GL}_5 & U_{5,4} & \mathrm{GL}_4 \\ \bigcirc & \longrightarrow & \bigcirc & \longrightarrow & \bigcirc \end{array}$$
where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as $\mathcal{M}_Q = \coprod \mathfrak{u}_e/G_i$ the product over all maps $(G_i) : Q_0 \to \mathrm{KM}$ and U_e is a choice if unipotent for each edge e. Analogue of stable envelope construction Stab, e.g. giving [MO]-analogue construction of OSp CoHAs.

- Iterated integrals. Chen's [Ch] shuffle structure on cochains $\mathrm{C}^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX = \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n = T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G. Understand the relation to Dynkin/q-analogues of multiple zeta values [KMT; Mi].

For instance, the structures in OSp (e.g. $\pm$-equivariant factorisation algebras on $\mathbf{C}$) are obtained from $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$; so too let us consider K_{G_2} - factorisation algebras consist of ordinary factorisation algebras but where for any *triple* of points z_1, z_2, z_3 there is in addition equivariance with respect to
$$\begin{aligned} \tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3) \\ \tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1) \\ \tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2), \end{aligned}$$
a square root of (231) generating $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$. Just as in OSp we show $K_{\mathrm{Sp}_{2n}}$-analogues of (factorisation) braided monoidal categories give Cherednik's reflection equation, we expect to obtain the G_2-reflection equation [Ku]. For another example, considering the groups $\hat{\mathfrak{gl}}_n$ of affine type A gives

$\mathrm{Ran}_{\widehat{\mathrm{GL}}}\mathbf{A}^1$, $\mathcal{D}$-modules on which are related to $\mathcal{D}$-modules on $\mathbf{A}^1/\mathbf{Z}$, so we expect this should relate to trigonometric KZ equations.

Just as in OSp we used that C_n is obtained by *folding* A_{2n},[13] we expect to be able to produce G_2 structures by taking $\mathbf{Z}/3$-invariants of type D structures.

A twisted AGT correspondence. After OSp, one natural next step (project AGT) is to construct a boundary version [AGT; BFN]:

Conjecture O. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a Verma module for an orthosymplectic analogue of the vertex W-algebra $\mathcal{W}^k(\mathfrak{gl}_n)$.*

Likewise, we expect a version for an arbitrary smooth projective surface S. We expect the proof of the above should proceed in much the same way as in [BFN], but with the parabolic induction data replaced by

$$\begin{array}{ccc} & \mathrm{B}P & \\ \swarrow & & \searrow \\ \mathrm{BGL}_n \times \mathrm{BSp}_{2m} & & \mathrm{BSp}_{2n+2m} \end{array}$$

as in OSp; i.e. we expect a SA-type analogue of free field realisations. Likewise, we expect a generalisation of [RSYZ] for instantons on $\mathbf{A}^3$:

Conjecture P. [AGT] *There is vertex algebra structure on the the orthosymplectic CoHA of the Jordan quiver, which acts on $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$, the equivariant intersection cohomology of the invariant locus in the quiver variety.*

Just as the CoHA $\mathcal{W}^+_{1+\infty}$ of the Jordan quiver is by [Dab] the universal enveloping algebra of positive half of differential operators on $\mathbf{C}^\times$ and admits the W-algebras of [BFN] as quotients, we expect the above to be a universal enveloping on differential operators on $\mathbf{C}^\times/\pm$, and admit the above W-algebras $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients.

1.4. q-vertex algebras (qVA, KL)

q-vertex algebras. The main goal of project qVA is to develop the machinery of *q-vertex algebras* and define q-affine vertex algebras. In KL we hope to use it to produce the qKZ equations and relate to Kazhdan-Lusztig equivalences.

A natural first guess at a definition is to take the usual definition of vertex algebra but using $\mathcal{D}_q$-modules in place of $\mathcal{D}$-modules. The first observation is that a q-difference operator ∂_x on $\mathbf{A}^1$ induces a derivation $y\partial_x$ on the noncommutative plane[14] $\mathbf{A}^2_q$, and indeed the physics heuristic below points towards $\mathcal{D}$-modules on $\mathbf{A}^2_q$ (e.g. via [MS]) as the correct setting for q-vertex algebras:

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[13] i.e. the invariants construction discussed in OSp.

[14] Its with ring of functions $\mathbf{C}\langle x, y, q\rangle/(yx - xyq)$ with q central.

Conjecture Q. [qVA] *There is a factorisation category over the noncommutative space* $\mathbf{A}^2_q$*, such that any* $\mathcal{A} \in \mathrm{FactAlg}^{st}(\mathcal{D}\text{-}\mathrm{Mod}_{\mathrm{Ran}\mathbf{A}^2_q})$ *defines a* q***-vertex algebra*** *(generalising e.g. [FR]).*

To construct this category precisely, one needs to develop the theory of $\mathcal{D}$-modules (e.g. functoriality) over noncommutative spaces. We propose using work [FMW; MS] on jet spaces of noncommutative schemes to give a "q-crystal/de Rham" definition.

Physics heuristic. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V^k_\hbar(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

where $\mathbf{A}^2_q = (\mathbf{C} \times \mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x, y]/(xy - qyx)$, see [GRZ] or particularly Costello's [Co] work.

Examples. The above definition will have been correct if we may answer

Question R. *Is there an analogue of the Beilinson-Drinfeld Grassmannian* $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$*?*

Such a factorisation space would for free by Conjecture Q define for us a q-vertex algebra $V^k_q(\mathfrak{g})$, by the same construction as for the affine WZW vertex algebra: taking distributions supported at the identity. We expect that $V^k_q(\mathfrak{g})$ should be a q-deformation of the affine vertex algebra and should agree with Etingof-Kazhdan's RTT construction in [EK] when $\mathfrak{g} = \mathfrak{sl}_n$.

Kazhdan-Lusztig. Conditional on having defined a factorisation algebra $V^k_q(\mathfrak{g})$ as in Question R, many interesting questions follow. To begin with, for formal reasons just as for ordinary vertex algebras, given $V^k_q(\mathfrak{g})$-modules $M_1, \dots, M_n$ we expect to obtain a $\mathcal{D}_q$-module of *conformal blocks* $\mathrm{C}^\bullet(M_1, \dots, M_n)$ on $(\mathbf{A}^2_q)^n$.

Question S. [KL] *Is its restriction to* $(\mathbf{A}^1)^n_\circ$ *equal to the qKZ connection?*

It has been long expected that one may define an affine analogue of the Kazhdan-Lusztig equivalence, and answering the above question would be a first step in understanding whether the geometric proof [CF] might be generalised to the affine setting.[15]

Orthogonally to this, we can try to understand ordinary Kazhdan-Lusztig better. First, we ask whether there is a lift of the *Zhu algebra* functor to the q-setting, fitting into a commuting square

[15]Specifically, one wants a Riemann-Hilbert type functor "RH : $\mathrm{FactCat}(\mathbf{A}^2_q) \to \mathrm{FactCat}^{\mathrm{QCoh}}(\mathbf{C}^2_q)$", which sends the category $V^k_q(\mathfrak{g})$-Mod to $Y_\hbar(\mathfrak{g})$-Mod^{fd}.

$$\begin{tikzcd} V_q^k(\mathfrak{g}) \arrow[r, mapsto, "q\to 1"] \arrow[d, "\mathrm{Zhu}"] & V^k(\mathfrak{g}) \arrow[d, "\mathrm{Zhu}"] \\ U_q(\mathfrak{g}) \arrow[r, mapsto, "q\to 1"] & U(\mathfrak{g}) \end{tikzcd}$$

Noting the appearance of both objects $U_q(\mathfrak{g})$ and $V^k(\mathfrak{g})$ appearing in the Kazhdan-Lusztig equivalence, having done this we then ask whether this these are the special and general fibres of a structure on $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$:

Question T. [KL] *Does $V_q^k(\mathfrak{g})$ induce a topological-holomorphic factorisation algebra $\mathcal{A}$ on $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$, whose restriction to $\mathbf{C}$ is $V^k(\mathfrak{g})$ and whose restriction to $\mathbf{C} \times \mathbf{R}_{>0}$ is $U_q(\mathfrak{g})$?*

One would then hope to interpret the fact that [CF]'s RH functor sends $V^k(\mathfrak{g})$-FactMod to $\mathrm{KD}(U_q(\mathfrak{g}))$-FactMod as some sort of flatness statement for $\mathcal{A}$-FactMod over $\mathbf{R}_{\geqslant 0}$.[16] This may give a new way to understand the recent Kazhdan-Lusztig equivalences [BCDN] coming from $3d$ mirror symmetry.

1.5. Sheaf methods (Con, Loc, Eu)

Localisation methods. *Torus localisation* is one of the main methods in enumerative geometry, and projects Con and Loc were concerned with extending these techniques to the Artin moduli stacks appearing in enumerative geometry. Given a closed Artin substack

$$\mathcal{Z} \hookrightarrow \mathcal{X}$$

not necessarily quasicompact,

Theorem U. [Conc] *If $\mathcal{L}_i$ are a collection of line bundles such that at least one of them vanishes on each geometric point $x \in \mathcal{X} \smallsetminus \mathcal{Z}$, then*

$$\mathrm{C}_\bullet^{\mathrm{BM}}(\mathcal{X} \smallsetminus \mathcal{Z})_{\mathrm{loc}} \;=\; 0,$$

so then the cohomology of $\mathcal{X}$ is "concentrated" on $\mathcal{Z}$: we have $i_* : \mathrm{C}_\bullet^{\mathrm{BM}}(\mathcal{Z}) \xrightarrow{\sim} \mathrm{C}_\bullet^{\mathrm{BM}}(\mathcal{X})$.

Here we have localised with respect to $c_1(\mathcal{L}_i)$, for instance we show the condition holds if $\mathcal{Z}_0/T \hookrightarrow \mathcal{X}_0/T$ is an inclusion of quotient stacks with $\dim \mathrm{Stab}_x(T)$ non-maximal for all $x \in \mathcal{X}_0 \smallsetminus \mathcal{Z}_0$, and we take for $\bigoplus \mathcal{L}_i$ the tautological T-bundle.

Theorem V. [Loc] *If $i : \mathcal{X}^T \hookrightarrow \mathcal{X}$ is the inclusion of the homotopy fixed points of a torus action on quasismooth dg scheme $\mathcal{X}$, there is a **Gysin pullback** map $i^! : \mathrm{C}_{T,\bullet}^{\mathrm{BM}}(\mathcal{X})_{\mathrm{loc}} \to \mathrm{C}_{T,\bullet}^{\mathrm{BM}}(\mathcal{X}^T)_{\mathrm{loc}}$ satisfying Atiyah-Bott and Graber-Pandharipande formulas:*

$$\mathrm{id} \;=\; i_* \frac{i^!(-)}{e(N_{vir})}, \qquad\qquad [\mathcal{X}]^{vir} \;=\; i_* \frac{[\mathcal{X}^T]^{vir}}{e(N_{vir})}, \tag{3}$$

relating to pushforward and fundamental classes.

[16] By means of extra evidence, it seems plausible that ordinary Riemann-Hilbert $\mathcal{D}\text{-Mod}^{rh} \xrightarrow{\sim} \mathrm{Perv}$ may be interpreted this way, where we consider $\mathcal{A}$ a sheaf of algebras generated by $\mathcal{O}_{\Sigma \times \mathbf{R}_{\geqslant 0}}$ and the Lie algebra $\mathcal{T}_{\Sigma \times \mathbf{R}_{\geqslant 0}}$ of infinitesimal automorphisms of $\Sigma \times \mathbf{R}_{\geqslant 0}$ whose restriction to the boundary is antiholomorphic.

This recovers the usual torus localisation results when $\mathcal{Z} = X^T/T$ and $\mathcal{X} = X/T$ are quotients of smooth finite-type schemes by tori.

Virtual Euler classes and shuffle structures. In Eu, we strengthen the above results in Con and Loc until:

- they give a general geometric method to output *shuffle products* for CoHAs,
- and show CoHAs are compatible with Davison/Yang-Zhao localised/Joyce vertex coproducts.

Specifically, we prove analogues of Theorems U and V for the *vanishing cycle* (or any sheaf) cohomology of arbitrary closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ which is quasismooth other a common base, and concentrated with respect to a multiplicative subset $\mathcal{S} \subseteq \mathrm{H}^\bullet(\mathcal{X})$. As a result,

Theorem W. [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we get a diagram*

$$\begin{tikzcd} \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, dashed, "1/e(\mathrm{N}_{i,vir})"] & \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, "p^s_* q^{s,*}"] & \mathrm{C}^\bullet(\mathcal{M}^s, \varphi^s) \\ \mathrm{C}^\bullet(\mathcal{M} \times \mathcal{M}, \varphi \boxtimes \varphi) \arrow[u, "(\pi\times\pi)^*"] \arrow[rr, "p_* q^*"] & & \mathrm{C}^\bullet(\mathcal{M}, \varphi) \arrow[u, "\pi^*"'] \end{tikzcd} \tag{4}$$

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Here p, p^s (proper) and q, q^s (quasismooth) are

$$\begin{tikzcd} & \mathrm{SES}^s \arrow[dl, "q^s"'] \arrow[d] \arrow[dr, "p^s"] & \\ \mathcal{M}^s \times \mathcal{M}^s \arrow[d] & \mathrm{SES} \arrow[dl, "q"'] \arrow[dr, "p"] & \mathcal{M}^s \arrow[d] \\ \mathcal{M} \times \mathcal{M} & & \mathcal{M} \end{tikzcd}$$

for instance $\mathcal{M}$ is *smooth* moduli stack containing as a critical locus $\mathrm{Crit} W$ the deformed CY3 moduli stacks considered in CY3, and $\varphi = \varphi_W$, and we apply localisation to $i : \mathrm{SES}^s \to \mathrm{SES} \times_{\mathcal{M}^s} \mathcal{M}$.

Two consequences of this are:

- If we take $\mathcal{M}^s$ to be a *shuffle space*[17] given by products of "simple" moduli stacks, e.g. parametrising tuples of rank one quiver representations, then (4) recovers shuffle formulas [Daa; SV; YZb] for CoHAs and localised/vertex coproducts.
- If we take $\mathcal{M}^s = \mathcal{M} \times \mathcal{M}$ together with its direct sum map to $\mathcal{M}$, (4) recovers the compatibility [CV,CY3,Li] between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

Thus, this turns algebraic properties of stacks (shuffle/bialgebra-type structures) into algebraic properties on their critical cohomology. In OSp this explains the OSp-shuffle module structure on $\mathrm{H}^\bullet(\mathcal{M}^\tau, \varphi^\tau)$, and plan to generalise this in SA.

[17] i.e. shuffle algebra in the category of spaces, see SA.

1.6. Liouville quantum gravity to vertex algebras (LQG)

History. In recent years, probabilists have increasingly understood quantum field theory, giving rigorous definitions of Feynman measures for $2d$ CFTs, e.g. [CRV; DS; Sh] whose "holomorphic part" are expected to be W-algebras, Virasoro, and Heisenberg vertex algebras.

This approach is very different to the factorisation/vertex algebra/functorial QFT approach in the above projects, e.g. it can directly study level sets of fields as SLE curves [MS; SS], there is a rigorous connection to combinatorial toy models like the discrete Gasussian Free Field [BPR], and it is able to access the *full* CFT, not just the chiral part as we are in geometric representation theory, e.g. [KRV] proves the *DOZZ* formula for full OPEs in the Liouville CFT.

However, there is currently not much interaction between the two approaches, and this project aims to build a bridge between the two so that techniques/results/heuristics can move between subjects more easily (then give a simple example of this).

Goal. In LQG we aim to define a functor from Segal-style $2d$ conformal field theories to vertex algebras

$$\mathrm{CFT} \overset{(-)^{ch}}{\to} \mathrm{CFT}^{hol} \overset{\mathrm{Res}}{\to} \mathrm{FactAlg}(\mathbf{C})^{hol}_{\mathbf{C}\rtimes\mathbf{C}^\times} \overset{[\mathrm{CG}]}{\to} \mathrm{VertexAlg}, \tag{5}$$

then show that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs are sent to the Heisenberg and Virasoro vertex algebras, repectively.

Details. We will need to upgrade $\mathcal{Z} \in \mathrm{CFT}$ to a definition that remembers the geometric structure on the category Cob_2 of conformal cobordisms. Namely, consider a complex vector bundle $\mathcal{V}$ with connection over the Teichmuller space $\mathcal{T}_{g,n}$ satisfying a factorisation condition, and with a section ψ. The fibre of this data over Σ is the vector space $\mathcal{Z}(\partial\Sigma)$ and $\mathcal{Z}(\Sigma) : \mathbf{C} \to \mathcal{Z}(\partial\Sigma)$.

The induced factorisation algebra over $\mathbf{C}$ is automatically smoothly translation and rotation equivariant, so if it is *holomorphic* (i.e. $\partial_{\bar{z}}\psi = 0$) then it is by [CG] a vertex algebra; these are the last two maps in (5). The equivariance comes from a G-action on $\mathcal{T}_{0,n}$, since then the Lie algebra $\mathfrak{g}$ acts on $\mathcal{V}$ by the connection, e.g. the vertex algebras in the image of (5) will automatically have an action by vector fields on $\mathbf{P}^1$, so we expect they are VOAs.

The main task is to define a chiralisation functor $(-)^{ch}$ to holomorphic CFTs, and prove that [GKRV]'s LQG Segal CFT (upgraded appropriately in the above sense) is sent by (5) to the Virasoro vertex algebra, and relate the DOZZ formula [KRV, (1.12)] to the Virasoro OPE. Having done this, we plan to do the same for the GFF, and finally to give a new example of these methods, construct a probability measure in the domain of (5) recovering the affine vertex algebra, e.g. by using the free field embedding [FB, §11] to a direct sum of Heisenberg algebras.

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