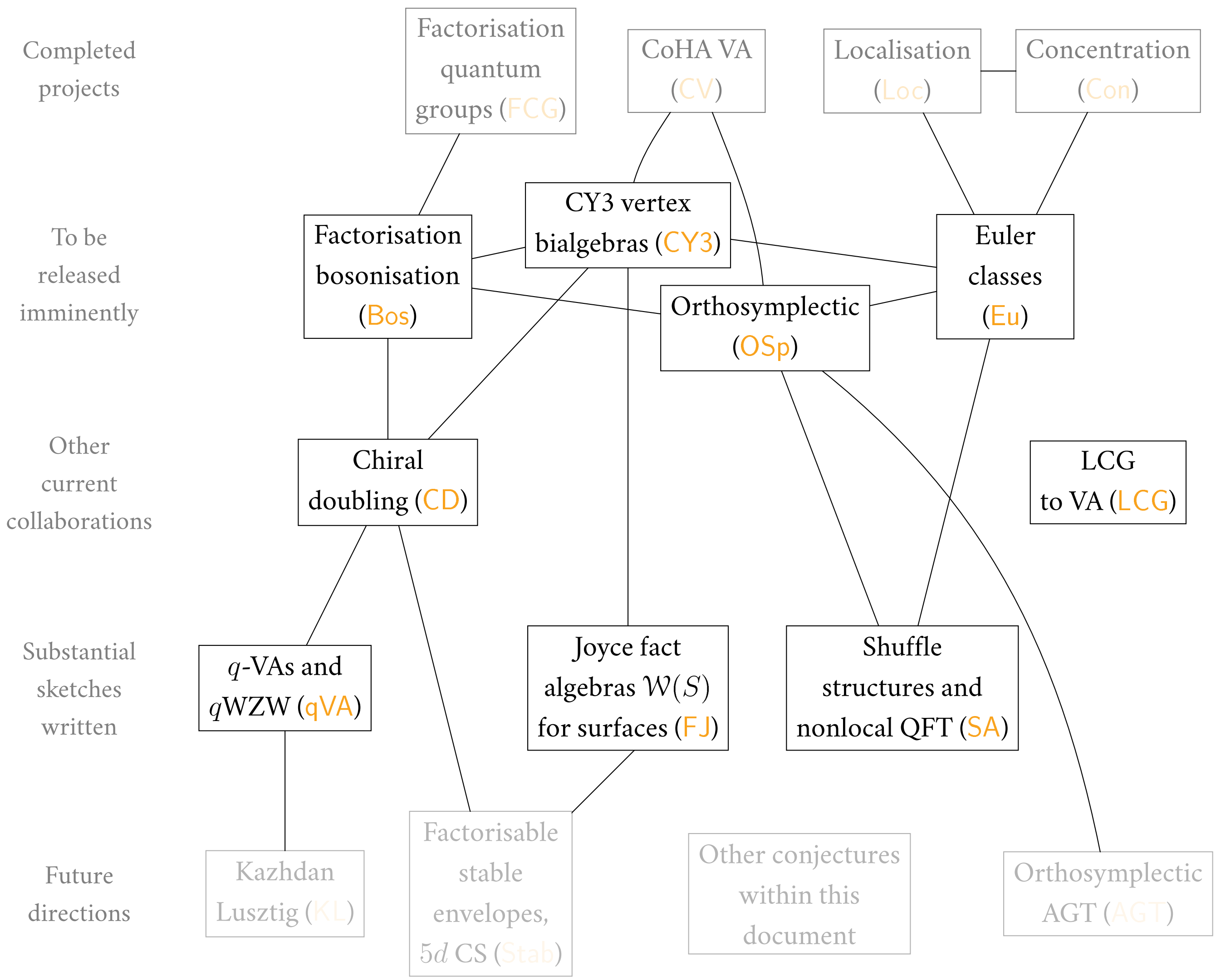
RESEARCH PLANS

ALEXEI LATYNTSEV

This is under construction!

See the following sections (with clickable links) for explanations of the projects and connections between them.



Contents

1. Summary of projects

The summary of projects (see §3 for a more detailed summary):

Project	Description	What existed before
CV	Dimension 1 CoHAs form vertex quantum groups (VQGs)	
CY3[1][2]	CoHAs for deformed CY3 categories form VQGs | Recover Drinfeld/Davison coproducts on Yangians | Configuration-to-Ran construction	
FJ [1]	factorisation stacks $\mathcal{M}_S^f$ over the canonical bundle K_S of more general algebraic surfaces; show its critical cohomology forms a S-*vertex algebra*; *configuration-to-Ran space* comparison, obtaining vertex algebra structures	The Jordan moduli stack $\mathcal{M}^f_{\mathbf{A}^2}$ insantiating Davison's localised coproduct;[3] generators-and-relations definition of W-algebra $\mathcal{W}(S)$ for algebraic surfaces S; [4] cohomological Hall algebras as factorisation algebras over the configuration space[5]

[1]Joint with S. Kaubrys.

[2]Joint with S. Jidnal.

[3] *B. Davison*, The integrality conjecture and the cohomology of preprojective stacks J. Reine Angew. Math. 804, 105–154 (2023)

[4]*A. Mellit, A. Minets, O. Schiffmann,* and *E. Vasserot*, "Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras," Preprint, arXiv:2311.13415 [math.AG] (2023).

[5]*Y. Yang* and *G. Zhao*, "Quiver varieties and elliptic quantum groups", Preprint, arXiv:1708.01418 [math.RT] (2017)

FCG Bos CD	Develop the theory of factorisation quantum groups Study factorisation algebras and quantum groups in the context of bosonisation Study the relation between chiral doubling and vertex algebras	
OSp[6] SA [6] AGT [6]	Generalise orthosymplectic CoHAs to arbitrary Dynkin-like spacetimes *Operadic* definition of ordinary shuffle algebras, extending to arbitrary '**Dykin**' systems of *Kac-Moody groups*; define *Dynkin vertex algebras* and give examples (type F, G, multiplicative, elliptic); *Dynkin* shuffle structure on loop spaces and Dynkin multiple zeta values; producing examples using deformation quantisation of on *orbifolds* Study orthosymplectic AGT and stable envelopes	Shuffle algebra formulas for CoHAs;[7] *orthosymplectic* $4d$ Chern-Simons and twisted Yangians[36]; orthosymplectic Joyce vertex bialgebras; boundary KZ equations
Eu Loc[8] Con [8]		
qVA KL	Develop the theory of q-vertex algebras and Study q-vertex algebras and qWZW models Uplift the *Zhu algebra* (**M6.1**) and *Huang-Lepowsky fusion product* (**M6.2**) to the level of factorisation and q-vertex algebras, recover [CF]'s proof of KL using q-WZW, and extend to *new* examples	factorisation proof [CF] of Kazhdan-Lusztig | New Kazhdan-Lusztig equivalences [BCDN]
LCG[9]	Construct a functor taking the LCG and GFF Segal CFT to Virasoro and Heisenberg VOA | Use VOA techniques to construct measures sent to affine and W vertex algebras	Rigorous construction of Liouville Quantum Gravity/Gaussian Free Field Feynman measures [DS] | DOZZ formula proof [KRV] | Proof of Segal axioms [GKRV]

[6]Joint with S. de Hority.

[7]*Y. Yang* and *G. Zhao*, Proc. Lond. Math. Soc. (3) 116, No. 5, 1029–1074 (2018)

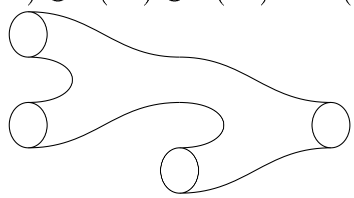
[8]Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.

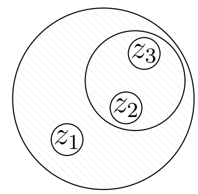
[9]Joint with V. Giri.

2. 40 Background

Formalising quantum field theory: factorisation algebras. The task of *axiomatising* topological QFTs was completed by Atiyah[10], as a functor from a *cobordism* category,

$$\mathcal{T}(S^1)\otimes\mathcal{T}(S^1)\otimes\mathcal{T}(S^1)\to\mathcal{T}(S^1)$$





Next, the theory of *vertex* and *chiral algebras* were developed by Borcherds[11] and Beilinson-Drinfeld[12] to axiomatise $2d$ *conformal* QFTs, where the spacetimes above have *holomorphic* structure, the former earning Borcherds a Fields medal and resolving the Moonshine Conjecture on modularity of monster group representations. In recent years, there has been a flurry[13][14] of activities related to *factorisation algebras* and *factorisation homology* as a way to understand *local operators* in a quantum field theory; formed by considering only cobordisms contained *within* a *fixed* manifold M; for instance, this was used to prove[15] a geometric analogue of *Weil's conjecture* for function fields.

However, despite recent progress on *axiomatising* quantum field theories, very few *examples* of (non-topological) quantum fields theories beyond dimension two have been constructed; mathematicians still must rely on (nonrigorous) QFT computations by physicists (e.g. based on *string theory*), which are turned into *provable conjectures*. Much of our proposed work concerns *extending* the range of rigorous mathematics further into physics; some of it concerns *proving* relations between mathematical structures (e.g. *CoHAs*, *vertex algebras*) conjectured by physics.

Cohomological Hall algebras and W-algebras. Cohomological Hall algebras (**CoHAs**) are a mathematical shadow of four-dimensional supersymmetric QFTs $\mathcal{T}$; as these QFTs are not yet rigorously defined, this is currently one of the best handles we have on their structure.

In *physics*, the study of CoHAs began with the space of *BPS states* of $\mathcal{T}$ was shown[16] to carry an *associative algebra* structure. Examples of $\mathcal{T}$ are given by compatifying an 11-dimensional string theory on

[10]Atiyah, M.F., 1988. Topological quantum field theory. Publications Mathématiques de l'IHÉS, 68, pp.175-186.

[11]Borcherds, R. (1986), "Vertex algebras, Kac-Moody algebras, and the Monster", Proceedings of the National Academy of Sciences of the United States of America.

[12]*A. Beilinson* and *V. Drinfeld*, Chiral algebras. Providence, RI: American Mathematical Society.

[13]Lurie, J., 2008. On the classification of topological field theories. Current developments in mathematics, 2008(1), pp.129-280.

[14]Costello, K. and Gwilliam, O., 2021. Factorization algebras in quantum field theory (Vol. 2). Cambridge University Press.

[15]Gaitsgory, D. and Lurie, J., 2014. Weil's conjecture for function fields. preprint.

[16]Harvey, J.A. and Moore, G., 1998. On the algebras of BPS states. Communications in Mathematical Physics, 197, pp.489-519.

a torically-compact Calabi-Yau threefold X, relating the subject to mirror symmetry and the *Geometric Langlands program*.[17]

Kontsevich-Soibelman[18] then discovered an algebra structure on the *critical cohomology*

$$\mathrm{H}^*(\mathcal{M}_{\mathcal{A}}, \varphi)$$

of certain moduli stacks $\mathcal{M}_{\mathcal{A}}$ of CY3 categories (specifically, Jacobi algebras of quivers with potential), which locally models coherent sheaves on CY3s[19], and related their graded dimensions to *Donaldson-Thomas* enumerative invariants. Recently, **Safronov** co-authored a breakthough paper[20] constructing cohomological Hall algebras for *arbitrary* CY3 categories, which will lead to a flurry of research activity in the near future.

Instantons and AGT. The breakthroughs of Grojnowski[21] and Nakajima[22] proved that the *Hilbert scheme* of points on a smooth surface S carries an action of the *Heisenberg vertex algebra* on its cohomology. Later generalisations were conjectured by Alday-Gaiotto-Tachikawa[23] and proved by Braverman-Finkelberg-Nakajima[24] [25] [26] to arbitrary surfaces and gauge groups with an action of $\mathcal{W}$-*vertex algbebras*, which were then realised as quotients of cohomological Hall algebras[27]. This begun the connection between cohomological Hall algebras, *vertex algebras* and *quantum groups*.

[17] Witten, E., 2009. Geometric Langlands from six dimensions. arXiv:0905.2720. (2009)

[18] *M. Kontsevich* and *Y. Soibelman*, Commun. Number Theory Phys. 5, No. 2, 231–352 (2011)

[19] Ben-Bassat, Oren; Brav, Christopher; Bussi, Vittoria; Joyce, Dominic A 'Darboux theorem' for shifted symplectic structures on derived Artin stacks, with applications. Geom. Topol. 19, No. 3, 1287-1359 (2015).

[20] injo, T., Park, H. and Safronov, P., 2024. Cohomological Hall algebras for 3-Calabi-Yau categories. arXiv preprint arXiv:2406.12838.

[21] Grojnowski, I., 1997. Instantons and affine algebras. I. The Hilbert scheme and vertex operators, Math. Res. Lett. 3 (1996)

[22] Nakajima, H., 1997. Heisenberg algebra and Hilbert schemes of points on projective surfaces. Annals of mathematics, 145(2), pp.379-388.

[23] Alday, L.F., Gaiotto, D. and Tachikawa, Y., 2010. Liouville correlation functions from four-dimensional gauge theories. Letters in Mathematical Physics,

[24] Nakajima, H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories,I, Adv. Theor. Math. Phys. (2016)

[25] Braverman, A., Finkelberg, M., and Nakajima, H., Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II, Adv. Theor. Math. Phys. (2018)

[26] Braverman, A., Finkelberg, M. and Nakajima, H., 2014. Instanton moduli spaces and $\mathcal{W}$-algebras. arXiv preprint arXiv:1406.2381.

[27] Rapcák, M., Soibelman, Y., Yang, Y. and Zhao, G., Cohomological Hall algebras, vertex algebras, and instantons, in Comm. Math. Phys.

Quantum groups and the Kazhdan-Lusztig equivalence. The theory of *quantum groups* (**QGs**) was preceded in the statistical physics literature by studies of *integrable systems*[28] and *spin chains*, e.g. studying formation of ice crystals.[29] In the 1986 ICM address Drinfeld[??] developed the mathematical theory of *quasitriangular Hopf algebras* to formalise this, and proved a fundamental result about *existence-uniqueness* of QGs $U_q(\mathfrak{g})$ deforming Lie bialgebras $\mathfrak{g}$.

Since then QGs have taken a central place in mathematics: they were connected to *Chern-Simons* and *knot theory* by Witten,[30] which predicted the famous *Kazhdan-Lusztig equivalence*[31]

$$(\mathrm{Rep}_k \hat{\mathfrak{g}})^{G(\mathcal{O})} \simeq \mathrm{Rep}\, U_q(\mathfrak{g})$$

relating representations of quantum groups to integrable representations of *vertex algebras* via the *KZ equations*,[68] more generally they relate to $3d$ *TQFTs* and *mirror symmetry*,[??][32] generalisations appear as *Yangians* and *affine/elliptic quantum groups* in Maulik-Okounkov's seminal work,[69] and more recently as *cohomological Hall algebras*.[33][34] The modern physics explanation is that QGs representations give *line operators* for certain QFTs;[37,38] thus the task of understanding/organising these different structures is crucial to understanding QFT and string theory.

There is a long *historical* connection between geometric representation theory and physics sketched in §**??**, two decades-long examples of the two-way exchange includes the Geometric Langlands programme[35] and Mirror Symmetry.

In **WP1**, we use work[36] on $4d$ Chern-Simons on orbifolds. Our results on quantum factorisation algebras for **WP3** are informed by work on $4d$ and $5d$ Chern-Simons theory[37][38] In **WP3** is related to physics-informed conjectures on the q-Langlands correspondence[39], and new *holomorphic-topological*

[28]Yang, C.N., 1967. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. Physical Review Letters, 19(23), p.1312.

[29]Lieb, E.H., 1967. Exact solution of the problem of the entropy of two-dimensional ice. Physical Review Letters, 18(17), p.692.

[30]Witten, E., 1989. Quantum field theory and the Jones polynomial. Communications in Mathematical Physics, 121(3), pp.351-399.

[31]David Kazhdan and George Lusztig. "Tensor structures arising from affine Lie algebras. I-IV". In: Journal of the American Mathematical Society 6.4 (1993-1994).

[32]Creutzig, T., Lentner, S. and Rupert, M., 2021. Characterizing braided tensor categories associated to logarithmic vertex operator algebras. arXiv preprint arXiv:2104.13262.

[33]Yang, Y. and Zhao, G., 2018. The cohomological Hall algebra of a preprojective algebra. Proceedings of the London Mathematical Society, 116(5), pp.1029-1074.

[34]Latyntsev, A., 2021. Cohomological Hall algebras and vertex algebras. arXiv preprint arXiv:2110.14356.

[35]Witten, E., 2009. Geometric Langlands from six dimensions. arXiv preprint arXiv:0905.2720.

[36]*R. Bittleston* and *D. Skinner*, J. High Energy Phys. 2019, No. 5, Paper No. 195, 53 p. (2019; Zbl 1416.81106)

[37] Costello, K., Witten, E. and Yamazaki, M., 2017. Gauge theory and integrability, I. arXiv preprint arXiv:1709.09993.

[38] Costello, K., Witten, E. and Yamazaki, M., 2018. Gauge theory and integrability, II. arXiv preprint arXiv:1802.01579.

[39]*Aganagic, M., Frenkel, E. and Okounkov, A.*, 2018. Quantum q-Langlands correspondence. Transactions of the Moscow Mathematical Society, 79, pp.1-83.

structures we wish to define will be informed by physics papers[40][41] on wide generalisations of Kontesevich's deformation quantisation. The deliverable on q-vertex algebras will be informed by Costello's[42] application of Nekrasov's Ω-background to $5d$ Chern-Simons theory.

At this point, the theory of quantum groups $U_q(\mathfrak{g})$ is well-developed:

(1) There are basis-free constructions [Ga] of $U_q(\mathfrak{g})$-Mod,

 (a) by working in the category $\mathrm{Perv}(\mathrm{Conf}_\Lambda(\mathbf{A}^1))$ of perverse sheaves on the configuration spaces,

 (b) by double-bosonisation [Maa].

(2) There is a "geometric" proof [CF] of the Kazhdan-Lusztig equivalence $U_q(\mathfrak{g})\text{-Mod}^{ren} \simeq \hat{\mathfrak{g}}\text{-Mod}_k^I$

Background. A main theme of geometric representation theory/enumerative geometry is: attached to certain Calabi-Yau-threefolds Y or categories $\mathcal{C}$, it has long been conjectured [KS] (now proven [**KPS**]) a "cohomological Hall" algebra structure on

$$\mathrm{H}^\bullet(\mathcal{M}_\mathcal{C}, \mathcal{P}) \tag{1}$$

where $\mathcal{P}$ is Joyce's DT sheaf (reference), and

- structure thing one
- two

From the physics perspective, the algebra structure is explained by (1) arising from an 11-dimensional "M" theory compactified on Y, which gives a $5d$ theory, then taking its algebra of BPS states [**Mo**] gives a q-deformed algebra structure. The other structures then arise from varying Y, to get an Alg-valued factorisation algebra over it; the analogy in the trivial toy model where Y is a $6d$ topological manifold is

$$\begin{array}{ccc} \mathrm{TQFT}_{11d} & \overset{\int_{\mathbf{R}^4\times S^1}}{\dashrightarrow} & \mathrm{TQFT}_{6d}(\mathrm{Alg}) \\ \downarrow\int_Y & & \downarrow\int_Y \\ \mathrm{TQFT}_{5d} & \xrightarrow{\int_{\mathbf{R}^4\times S^1}} & \mathrm{Alg} \end{array}$$

The motivating example is when $Y = K_S$ for a smooth algebraic surface S; then in FJ we expect a vertex algebra structures in the fibres of $K_S \to S$; this is proven in some 2CY cases in CY3.

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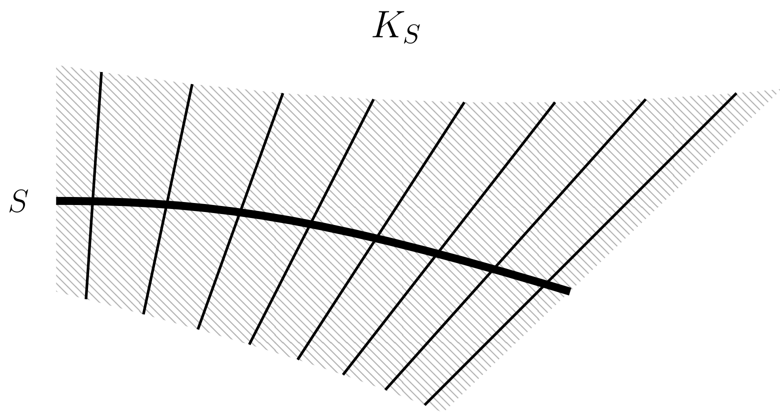
[40] *Gaiotto, D., Kulp, J. and Wu, J.*, 2024. Higher Operations in Perturbation Theory. arXiv preprint arXiv:2403.13049.

[41] *Balduf, P.H. and Gaiotto, D.*, 2024. Combinatorial proof of a Non-Renormalization Theorem. arXiv preprint arXiv:2408.03192.

[42] *Costello, K.*, 2016. M-theory in the Omega-background and 5-dimensional non-commutative gauge theory. arXiv preprint arXiv:1610.04144.



. Explanation: standard and nonstandard coproduct on $Y_\hbar(\mathfrak{g}_Q)$.

. Define CoHAs

3. Details of projects

3.1. 70 Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ)

CoHAs as vertex quantum groups. One aim of project CY3[43] and project Bos[44] is to push the analogy between CoHAs and finite quantum groups:

$\mathrm{Rep}_q T$	$U_q(\mathfrak{n})$	$U_q(\mathfrak{b})$	$U_q(\mathfrak{g})$
$\mathrm{Rep}\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M})$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)$	$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}$	c.f. CD

To begin with, whereas $U_q(\mathfrak{n})$ a braided cocommutative bialgebra inside the braided monoidal category $\mathrm{Rep}_q T$,

Theorem A. [CY3] *For any deformed CY3 category (e.g. coherent sheaves on local curve $K_{T^*\Sigma}$, quiver with potential) there is a vertex coproduct on the CoHA*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \ \to\ \mathrm{H}^\bullet(\mathcal{M},\varphi)\,\hat{\otimes}\,\mathrm{H}^\bullet(\mathcal{M},\varphi)((z^{-1}))$$

making it into a braided colocal vertex bialgebra inside the braided factorisation category $\mathrm{Rep}(\mathrm{H}^\bullet(\mathcal{M}),\cup)$.[45]

We sanity-check that this is an interesting structure:

Theorem B. [CY3; CV for $W = 0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}},\varphi_{W^{(3)}}) \simeq Y_\hbar(\mathfrak{n}_Q)$ agrees with the Davison/Yang-Zhao localised coproduct, and (when defined) Drinfeld's meromorphic coproduct.*

Next, $U_q(\mathfrak{b})$ is constructed by Tannakian reconstruction on $U_q(\mathfrak{b})\text{-Mod}(\mathrm{Rep}_q T)$, and in Bos we develop a factorisable analogue of this. This results in a vertex bialgebra structure on the extended CoHA $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos} = \mathrm{H}^\bullet(\mathcal{M},\varphi)\otimes\mathrm{H}^\bullet(\mathcal{M})$,

$$\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)_{bos}\text{-Mod} \ =\ \mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M},\varphi)\text{-Mod}(\mathrm{Rep}\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}))$$

which in the preprojective case recovers (Soibelman-Rapcak)-Yang-Zhao's construction on $Y_\hbar(\mathfrak{b}_Q)$. This "automates" difficult generating-series definitions of CoHA extensions: they follow as a consequence of factorisable Tannakian reconstruction. (give more evidence/details)

Vertex coalgebras from configuration spaces. (recall what localised (bi)algebras are!) To compare localised and vertex coproducts in CY3, we introduce a *Ran-to-Conf* construction: taking localised terms $1/x$, pulling back by a $\mathrm{H}^\bullet(\mathrm{B}\mathbf{G}_m)$-coaction and taking a power seires expansion in z^{-1}

$$\frac{1}{x+nz} \ =\ \frac{1}{nz}\left(\frac{x}{nz} - \left(\frac{x}{nz}\right)^2 + \cdots\right)$$

defines a functor from localised coalgebras to vertex coalgebras.

[43] Joint with S. Jidnal and S. Kaubrys.

[44] Joint with S. de Hority.

[45] The formalism of braided factorisation categories is developed in FQG.

Conjecture C. *The Ran-to-Conf construction lifts to a functor* $\mathrm{FactCoAlg}(\mathrm{Conf}\mathbf{A}^1) \to \mathrm{VertexCoAlg}$.

We notice as an aside that the

Conjecture D. (Properadic vertex algebra-coalgebras) *Vertex coalgebras from factorisation algebras*

3.1.1. *FJ.* we will define *factorisable* versions of Joyce's vertex algebras for *dimension zero coherent sheaves* over canonical bundles of arbitrary algebraic *surfaces* S[46] to give a sheaf over S of S-*vertex algebras* which are Morita equivalent on intersections, and relate this to existing presentations of *cohomological Hall W-algebras*;[47] the more *conceptual* (i.e. operadic) nature of this *novel* approach to constructing vertex structures for *non-Calabi-Yau* surfaces will allow for easier generalisation, e.g. to *multiplicative/elliptic* cases, or to more general CY *threefolds*, as it makes visible structure not accessible to the explicit generators-and-relations approach.

TM: *virtual torus localisation*[48][49] for cohomological computations, the *stable envelope* construction to produce factorisation quantum groups. **C**: $\mathcal{M}_S^f$ is *not a global critical locus* over $\mathrm{Ran}K_S$, and so the results of[50] will not apply. **S**: It will only be a vertex algebra *relative* to S: we will get a *sheaf* of factorisation categories; alternatively, use techniques in[20].

Lift to factorisation algebra. To finish the analogy with [Ga], it remains to construct the Yangian factorisably, which we plan to do in FJ[51]

Conjecture E. *Construction of* $Y_\hbar(\mathfrak{g})$ *factorisably. (finish)*

In the case of quivers Q, we have an action of the torus $T_d = \prod T_{d_i}$ on the stack of representations, and

$$\mathcal{M}^f \;=\; \{(m,\lambda)\;:\;\lambda\in\mathfrak{t}^*,\; m\in\mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d)$$

defines a factorisation space over the Q_0-coloured Ran space.

Conjecture F. *The relative critical cohomology* $\mathcal{A} = \pi_*\varphi_W$ *defines a* $\mathbf{G}_a^{Q_0}$*-equivariant factorisation algebra over the coloured Ran space. Moreover, restricting to the colour-diagonal*

$$\mathrm{Ran}\mathbf{A}^1 \;\subseteq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1$$

recovers the Joyce-CoHA vertex bialgebra structure on the nilpotent CoHA $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_{nilp})$ *of* [SV].

[46] *B. Davison*, The integrality conjecture and the cohomology of preprojective stacks J. Reine Angew. Math. 804, 105–154 (2023)

[47] *A. Mellit, A. Minets, O. Schiffmann,* and *E. Vasserot*, "Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras," Preprint, arXiv:2311.13415 [math.AG] (2023).

[48] Atiyah, M.F. and Bott, R., 1984. The moment map and equivariant cohomology. Topology, 23(1), pp.1-28.

[49] *Aranha, D., Khan, A.A., Latyntsev, A., Park, H. and Charanya, R.*, 2022. Virtual localization revisited. arXiv preprint arXiv:2207.01652.

[50] *Kaubrys, S., Jidnal, S., Latyntsev, A.*, Vertex bialgebras for Calabi-Yau-three categories. *In preparation*

[51] Joint with S. Kaubrys.

Only the last part is nontrivial. This would be interesting for the following reasons:

- This should relate to Yang-Zhao's proof [YZ] that CoHAs form a localised factorisation bialgebra over $\mathrm{Conf}_\Lambda(E)$. We expect that the relation should be a factorisation space version of the Conf-to-Ran construction in CY3.
- This should relate to Maulik-Okounkov's stable envelope construction [MO] of Yangians.
- This construction makes the role of the torus $\mathfrak{t}_d$ clear, and therefore in (ref) we may generalise it to arbitrary Kac-Moody groups.

Crucially, having repackaged the vertex bialgebra structure as a factorisation algebra, we can consider applying them to more general CY3 categories.

Davison-Kinjo have defined similar structures on analytic moduli stacks (upcoming work), and the above should be an algebraic analogue of their construction.

Relation to $\mathcal{W}$-algebras.

Conjecture G. *When $\mathcal{A}$ is the category of zero-dimensional coherent sheaves on a surface S, $\mathcal{A}$ is equivalent to the factorisation bialgebra $\mathcal{W}^+(S)$ of [MMSV].*

This could give a coceptual explanation for the "off-local" terms in [MMSV]

(write),

i.e. $\mathcal{A}$ will be braided colocal for the factorisation category $(\mathcal{B}, \cup)$-Mod, where $\mathcal{B} = \pi_* k$. Moreover, one might expect that the techniques in CD may explain how to form doubles of these algebras.

Shows that $\mathcal{W}(S)$ locally in (certain) S forms a sheaf of factorisation algebras over K_S, i.e. "S-vertex algebras", which are Morita equivalent on intersections. Gives an example of the M2-M5 brane construction.

Joyce factorisable $\mathcal{W}(S)$-algebras. Define *factorisable* moduli stacks of *coherent sheaves* over canonical bundles of algebraic curves and a sheaf of critical charts[52] (**M1.1**), glue Joyce-Liu's vertex algebras[53][54] *factorisably* over *canonical bundles* of algebraic surfaces (**M1.2**), give a *new* construction of W-algebras[55] $\mathcal{W}(S)$ (**M1.3**);

[52] *B. Davison*, The integrality conjecture and the cohomology of preprojective stacks J. Reine Angew. Math. 804, 105–154 (2023)

[53] Joyce, D., 2018. Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces. Incomplete work.

[54] Liu, H., 2022. Multiplicative vertex algebras and quantum loop algebras. arXiv preprint arXiv:2210.04773.

[55] *A. Mellit, A. Minets, O. Schiffmann,* and *E. Vasserot*, "Coherent sheaves on surfaces, COHAs and deformed $W_{1+\infty}$-algebras," Preprint, arXiv:2311.13415 [math.AG] (2023).

Factorisable stable envelopes. Give a *Tannakian* (*factorisation category*) reformulation of the stable envelope construction over the Ran space (**M2.1**), obtain give a *vertex bialgebra* action of $\mathcal{W}(S)$ and *factorisation bialgebra* structure on the *nilpotent CoHA*[56][57] (**M2.2**), generalise to the *elliptic/multiplicative* case (**M2.3**).

Relation to stable envelopes. (write)

3.2. 80 The structure of factorisation quantum groups (FCG, Bos, CD, Stab)

History. A collection of structures all loosely called "quantum groups" have been some of the main objects in mathematical physics and geometric representation theory since the 80s:

(1) It is well-known that the representation categories of $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$ should be controlled by "spectral" analogues $R(z)$ of R-matrices [CWY; GLW].

(2) The affine case has recently been understood much better: the algebras $Y_\hbar(\hat{\mathfrak{g}}), \mathcal{W}_{1+\infty}(\mathfrak{g})$ in [GRZ]

(3) In [MO] Maulik-Okounkov define a bialgebra $Y_\hbar(\mathfrak{g}_Q)$ attached to *any* quiver Q.

(4) Understanding stability conditions/derived CoHA

Historically these definitions were (ingeniously) made very explicitly using generators and relations/RTT definitions, e.g. [Dr; MO], still much of the modern (e.g. CoHA) literature is based on explicit shuffle computations, e.g. [MMSV; SV; YZ].

The point of this series of projects is to first give a more conceptual definition of these structures, second to recover the above formulas as a consequence of this definition, and third to generalise them to more general structures.

Understanding the precise factorisation algebra structure we expect will ultimately help us understand the SCFTs $\int_Y \mathcal{M}$. We use these techniques in FJ.

Factorisation quantum groups. In FCG we develop a theory of *braided factorisation categories*. The basic idea is to take Lurie's [Lu] result

$$\mathbf{E}_2\text{-dgCat} \;\simeq\; \mathbf{E}_2\text{-Alg}(\mathrm{Sh}(\mathrm{Ran}\mathbf{R}^2, \mathrm{dgCat}))$$

that a braided monoidal category $\mathcal{C}$ is equivalent to[58] a factorisable constructible sheaf of categories over the Ran space of $\mathbf{R}^2$, and replace $\mathrm{Ran}\mathbf{R}^2$ with an arbitrary *factorisation space* X.[59]

[56] *O. Schiffmann* and *E. Vasserot*, J. Reine Angew. Math. 760, 59–132 (2020; Zbl 1452.16017)

[57] *Y. Yang* and *G. Zhao*, "Quiver varieties and elliptic quantum groups", Preprint, arXiv:1708.01418 [math.RT] (2017)

[58] To see a precise statement and of this, see [CF, Prop 7.0.2].

[59] Space here means derived prestack.

$$\begin{array}{ccc} & (\mathrm{Ran}\mathbf{R}^2\times\mathrm{Ran}\mathbf{R}^2)_\circ & \\ \swarrow & & \searrow \\ \mathrm{Ran}\mathbf{R}^2\times\mathrm{Ran}\mathbf{R}^2 & & \mathrm{Ran}\mathbf{R}^2 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} & M_X & \\ \swarrow & & \searrow \\ X\times X & & X \end{array}$$

Examples include: ordinary groups G, configuration spaces $\mathrm{Conf}_\Lambda(\mathbf{A}^1)$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n\times\mathbf{R}^m)$, or any space X with the inclusion of the diagonal.

We first check that this is a reasonable definiton: it recovers R-matrices,

Theorem H. [FCG] *One may define factorisation categories, algebras, etc., over X. If $\mathcal{A}$ is a factorisation bialgebra in factorisation category $(\mathcal{C},\otimes_{\mathcal{C},X})$ over X, each braided factorisation structure on $\mathcal{A}$-Mod$(\mathcal{C})$ induce a factorisation R-matrix $R:\mathcal{A}\otimes_{\mathcal{C},X}\mathcal{A}\xrightarrow{\sim}\mathcal{A}\otimes_{\mathcal{C},X}\mathcal{A}$ for which $\mathcal{A}$ is quasitriangular.*

In the vertex algebra case we recover classical notions [EK; FR] of quantum vertex algebras:

Theorem I. [FCG] *When $X=\mathrm{Ran}\mathbf{A}^1$ and $\mathcal{A}$ is a nonlocal vertex bialgebra V, a factorisation R-matrix is an endomorphism $R(z):V\otimes V((z))\xrightarrow{\sim}V\otimes V((z))$ satisfying the spectral hexagon relations, such that V is $R(z)$-twisted local (fields satisfy $R(z)$-twisted OPEs).*

For instance, this allows us to interpret result [GLW] on the strucutre of $\mathcal{A}=Y_\hbar(\mathfrak{g})$ as follows. It is a *lax* quasitriangular factorisation bialgebra with laxly compatible $\otimes^*$- and $\otimes^{ch}$-coproducts on the factorisation category of equivariant D-modules on $X=\mathrm{Ran}\mathbf{A}^1$, corresponding to the standard and meromorphic coproducts on $Y_\hbar(\mathfrak{g})$. The various R-matrices of [GLW]

$$\mathcal{M}_1\otimes^*\mathcal{M}_2\xrightarrow{R^-(z)}\mathcal{M}_2\otimes^{ch}\mathcal{M}_2,\qquad \mathcal{M}_1\otimes^{ch}\mathcal{M}_2\xrightarrow{R^{0,\epsilon}}\mathcal{M}_2\otimes^{ch}\mathcal{M}_1,\qquad \mathcal{M}_1\otimes^*\mathcal{M}_2\xrightarrow{R^\epsilon}\mathcal{M}_2\otimes^*\mathcal{M}_1$$

then induce lax compatibility maps between the two coproducts on $\mathcal{A}$-Mod. So: we have lifted $Y_\hbar(\mathfrak{g})$-Mod to a topological-holomorphic factorisatsion algebra over $\mathbf{R}\times\mathbf{C}$ rigorously.

Theorem H specialises to the ordinary theory of R-matrices and quantum groups (e.g. for $U_q(\mathfrak{g})$) when $X=\mathrm{Ran}\mathbf{R}^2$. Likewise, when $X=\mathrm{Conf}\mathbf{A}^1$ a factorisation bialgebra $\mathcal{A}$ is equivalent to the *localised bialgebras* B of [Da] (the main examples being CoHAs), a factorisation R-matrix is an endomorphism

$$R\ :\ (B\otimes B)_{\mathrm{loc}}\ \xrightarrow{\sim}\ (B\otimes B)_{\mathrm{loc}}$$

of the localised bialgebra V.

These techniques are fairly flexible and seem to be able to prove chiral analogues of many structure results in the theory of quantum groups, e.g. FCG ends by constructing a generalisation of the Borchers twist construction [Bo].

Factorisation bosonisation. In Bos, we automate the construction of *adding in the Cartan*:

$$U_q(\mathfrak{n})\overset{\text{[Maa; Mab]}}{\rightsquigarrow}U_q(\mathfrak{b}),\qquad Y_\hbar(\mathfrak{n})\overset{\text{[Dr]}}{\rightsquigarrow}Y_\hbar(\mathfrak{b}),\qquad \mathrm{H}^\bullet(\mathcal{M},\varphi)\overset{\text{[Da; RSYZ; YZ]}}{\rightsquigarrow}\mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}.$$

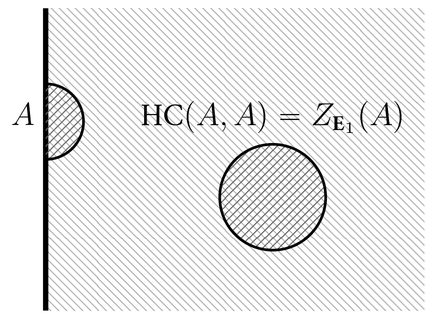
Indeed, in both CY3 and OSp we had to do complicated generating series manipulations to extend the CoHA (i.e. add a Cartan part) as a vertex bialgebra. In the finite case [Maa; Mab], the bialgebra structure on $U_q(\mathfrak{b})$ can be written in terms of $U_q(\mathfrak{n})$'s using Tannakian reconstruction ("*bosonisation*"):

$$U_q(\mathfrak{b})\text{-Mod} \ \tilde{\to}\ U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

and we want to do the same for vertex algebras.

Theorem 3.2.1. [Bos, in preparation]

Factorisation Drinfeld doubling. In project CD[60] Work out how to take Drinfeld centres of chiral categories. Recovers notions of doubling chiral bialgebras, bubble Grassmannians (when applied to $\mathrm{Rep}\,\mathfrak{g}(\mathcal{O})$), Yangians. Generalises BZFN's derivd loop spaces and centres construction.



Stable envelopes. Give a "Ran space" version of Maulik-Okounkov construction that includes all generalisations, e.g. the dynamical R-matrices.

3.3. 80 Orthosymplectic structures (OSp, SA, AGT)

Physics heuristic. In project OSp[61] we make a mathematical theory of *boundary* $4d$ *Chern-Simons* [BS] on $\mathbf{R} \times (\mathbf{R} \times \mathbf{C})/\pm$, for instance our structures structures satisfy *boundary Yang-Baxter/Cherednik reflection equations*. More generally, we define boundary versions of compactifications of $4d$ SCFTs $\int_Y \mathcal{M}$ attached to a CY3 Y - at least, those for which non-boundary versions have been defined. It should relate to Finkelberg-Hanany-Nakajima's ongoing work on orthosymplectic Coulomb branches (see AGT).

Details. Attached to an abelian category $\mathcal{A}$,[62] we construct the *orthosymplectic moduli stack* $\mathcal{M}_{\mathcal{A}}^{\mathrm{OSp}}$: a fixed point stack whose points are objects with a symmetric pairing $a \simeq a^*$.

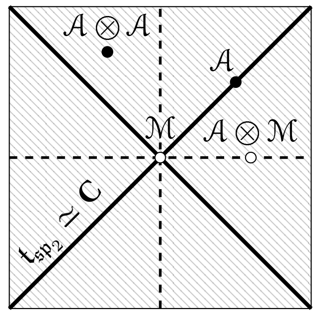
Theorem J. [OSp] *For $\mathcal{A}$ in CY3 or examples below, the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ "acts" on $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}, \varphi^{\mathrm{OSp}})$:*

(1) there is a left module action a of the CoHA respecting the involution,[63] compatible with

[60]Joint with W. Niu.
[61]Joint with S. de Hority.
[62]More generally abelian category with involution $(\mathcal{A}, \tau)$, e.g. $\tau = (-)^*$.
[63]i.e. the left action a and the right action $a \cdot (\mathrm{id} \otimes \tau)$ commute, where τ is the involution.

(2) *its* ***symplectic vertex algebra*** *structure: it is a factorisation coalgebra over symplectic Ran space* $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\,\mathfrak{t}_{\mathfrak{sp}_{2n}}$ *(coming from a localised structure over* $\mathrm{Conf}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$*).*

Data (1) and (2) are equivalent to a topological and holomorphic factorisation algebra over $\mathbf{R}/\pm$ and $\mathbf{C}/\pm$, respectively. We give an equivalent definition of the latter in terms of fields $A \otimes M \to M((z))$, etc.



$\mathfrak{t}_{\mathfrak{sp}_4} \simeq \mathbf{C}^2$

To give examples, we construct an **invariants** functor involving restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$

$$\iota \;:\; \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \;\to\; \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1), \qquad\qquad (\mathcal{A}, \tau) \;\mapsto\; (\mathcal{A}, \mathcal{A}^\tau)$$

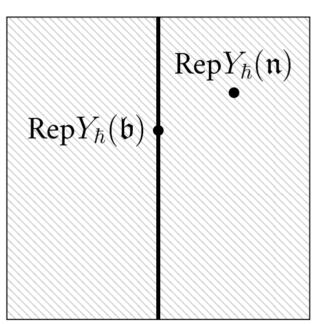
where $\mathcal{A}$ is a factorisation algebra with involution τ; we expect Theorem J may also be proved by applying ι to the factorisable moduli stack $\mathcal{M}^f$ from FJ. See also the link to stable envelopes (ref), and:

Conjecture K. *The* ***boundary KZ equations*** *may be derived by applying* ι *to the BD Grassmannian, taking distributions supported at the identity, and taking conformal blocks over* $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1$*.*

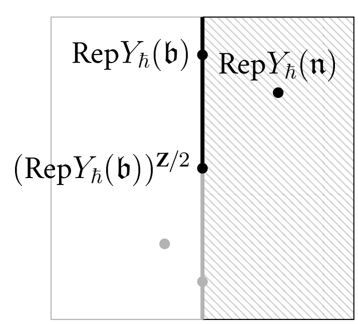
Examples include $\mathrm{B}\mathbf{Z}/2$ orbifold quivers with potential,[64] or orthosymplectic perverse-coherent sheaves on surfaces, e.g. orthosymplectic ADHM quiver/perverse-coherent sheaves on $\mathbf{A}^2$:

$$\begin{tikzcd} \mathrm{SO}(n) \arrow[loop above] \arrow[loop below] \arrow[r, shift left] & \mathrm{Sp}(2m) \arrow[l, shift left] \end{tikzcd} \qquad\qquad \mathcal{E} \simeq \mathrm{RHom}(\mathcal{E}, \mathcal{O})$$

$\mathbf{C} \times \mathbf{R}$



$(\mathbf{C} \times \mathbf{R})/\pm$



[64]i.e. either an ordinary quiver with involution, or an orbifold-valued quiver.

Theorem L. [OSp] *In the quiver with potential case, an explicit shuffle formula for the CoHA action.*

We end with a conjecture:

Conjecture M. *The orthosymplectic CoHA for the "folded" linear quiver A_{2n}*[65] *is isomorphic to the twisted Yangian $Y_\hbar(\mathfrak{gl}_n)^{tw}$ of [BR].*

3.3.1. *Dynkin QFTs*. Develop the theory of analogues of *Ran space, loop spaces, quiver varieties, MZVs, vertex algebras* and *KZ equations* attached to Coxeter/Kac-Moody data (**M3.1**), give *affine* examples of associated *vertex algebras, quantum groups* (using a variant of *Kontsevich formality*[??]) and *Yangians* (**M3.2**); compute the *generalised KZ equations* on their conformal blocks, formulate analogue of *Drinfeld's conjecture* (**M3.3**).

3.3.2. *SA*. is to generalise key objects in geometric representation theory to live on *Dynkin* spacetimes, then use this as a method to prove new relations between these objects. I will extend my previous work on orthosymplectic CoHAs[66] to *arbitrary* Dynkin-like spacetimes, and prove/contruct analogues of *Chen's Theorem*[67] on the cohomology of loop spaces and *multiple zeta values* (**MZVs**), variants of *vertex algebras* and *boundary KZ equations* (generalising/help understanding *Drinfeld's conjecture*[68] on the relation to MZVs), and *Nakajima quiver varieties*, and Maulik-Okounkov's[69] *stable envelopes* and *Yangians*; simultaneously generalising these topics will make new *connections* betewen them more apparent. Finally, we will prove an analogue of *Kontsevich's formality* theorem, with $\mathbf{E}_n$-algebras replaced by factorisation algebras over Dynkin spacetimes.

Nonlocal QFT and shuffle structures. project SA begun by noticing the following interesting pattern in structures considered project OSp.

$$\mathrm{BGL} \rightsquigarrow \mathrm{BSp}, \qquad \mathrm{Conf}(\mathbf{A}^1) \rightsquigarrow \mathrm{Conf}(\mathbf{A}^1), \qquad \mathrm{VA} \rightsquigarrow \mathrm{OSpVA}, \qquad \text{etc.}$$

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties, (KZ equations)) simultaneously generalise - this points towards this being a shadow of a more general theory.

The starting observation is this - the definition (ref) a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL} \to \mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n \geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

[65] i.e. with the involution being reflection in the linear direction.

[66] *deHority, S.* and *Latyntsev, A.*, *Orthosymplectic boundary cohomological Hall algebras*, in preparation.

[67] Chen, K.T., 1973. Iterated integrals of differential forms and loop space homology. Annals of Mathematics, 97(2), pp.217-246.

[68] *Etingof, P.I.* and *Schiffmann, O.*, 1998. Lectures on quantum groups.

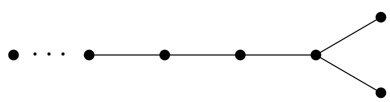
[69] *D. Maulik* and *A. Okounkov*, Quantum groups and quantum cohomology. Paris: Société Mathématique de France (SMF) (2019)

$$\begin{tikzcd} & P_{n,m}(\sigma) \arrow[dl] \arrow[dr] & \\ \mathrm{GL}_n \times \mathrm{GL}_m & & \mathrm{GL}_{n+m} \end{tikzcd} \quad \overset{A}{\mapsto} \quad A_n \otimes A_m \overset{m(\sigma)}{\to} A_{n+m}$$

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n,m)$.

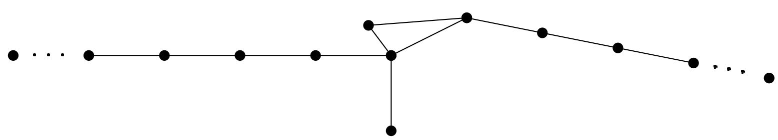
The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups** [Ku, §V]. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.

GL



$\mathrm{KM}_{\mathrm{SO}_{2n}}$

KM_{G_2}



To summarise:

- We get analogues of *shuffle algebras*.
- We get new configuration and Ran spaces

$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \textstyle\coprod_G \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{B}G), \qquad\qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G^*,$$

  where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G, so can define generalised *localised* and *vertex* algebras (and as in CY3 a Conf-to-Ran construction relating them). We expect to recover *boundary KZ* equations by taking conformal blocks (i.e. cohomology over $\mathrm{Ran}_{\mathrm{KM}}\mathbf{A}^1$).
- Topological case - topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras, then by considering $\mathrm{FactAlg}^{top}(\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}, \mathrm{Cat})$ we get analogues of the notion of *braided monoidal categories*.
- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups

$$\begin{tikzcd} \underset{\bigcirc}{\mathrm{GL}_3} \arrow[r, "U_{3,5}"] & \underset{\bigcirc}{\mathrm{GL}_5} \arrow[r, "U_{5,4}"] & \underset{\bigcirc}{\mathrm{GL}_4} \end{tikzcd}$$

  where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as

$$\mathcal{M}_Q \;=\; \textstyle\coprod \mathfrak{u}_e/G_i$$

  the product over all maps $(G_i) : Q_0 \to \mathrm{KM}$ and U_e is a choice if unipotent for each edge e.

Relation to orbifolding.

- Stable envelope construction.
- Chen's [Ch] shuffle structure on cochains $C^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX = \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n = T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G.
- Iterated integrals.

For the orthosymplectic example $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$, many of these structures are considered in OSp. Let us consider K_{G_2}

Example: G_2. For K_{G_2}, factorisation algebras consist of ordinary factorisation algebras but for any *triple* of points there is in addition equivariance with respect to the group $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$ acting on $\mathbf{C}^3$, in which the element

$$\begin{aligned}\tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3)\\ \tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1)\\ \tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2)\end{aligned}$$

squares to $\tau^2 = (231)$. Thus for instance a G_2 vertex algebra is a vertex algebra V with an (copy-paste from notes), and a topological G_2 factorisation category is a braided monoidal category $\mathcal{C}$ along with (copy-paste from notes; G_2 reflection equations)

Relation to folding. We expect there to be a folding construction of G_2 structures. (reference conjecture on twisted Yangians)

3.3.3. *free field realisations*[70] for producing actions of W-algebras in proving a Dynkin AGT Theorem, the theory of *Coxeter groups* to organise our combinatorial definitions,[71] the theory of *qKZ* and *KZB*[72] equations which we hope to generalise in the multiplicatie/elliptic case, and *quiver varieties*[73]. **C**: The good moduli spaces are *no longer smooth*. **S**: Use *intersection homology*,[74] adapt the *fixed point* techniques in my upcoming collaboration[66] which resolves these issues in the orthosymplectic case.

A twisted AGT correspondence. In the *finite type* case, define an action of *twisted CoHA* on the quiver varieties (**M4.1**), prove an *AGT* result: that this is a *Verma module* for a *twisted affine W-algebra*, which we define (**M4.2**).[24,25]

After OSp, one natural next step (project AGT) is to construct a boundary version [BFN]:

[70] Frenkel, E. and Ben-Zvi, D., 2004. Vertex algebras and algebraic curves (No. 88). American Mathematical Soc..

[71] Björner, A. and Brenti, F., 2005. Combinatorics of Coxeter groups (Vol. 231, pp. xiv+-363). New York: Springer.

[72] *G. Felder*, in: Quantum symmetries/ Symétries quantiques. Proceedings of the Les Houches summer school (1995)

[73] Hiraku Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. Duke Math. J. 76 (1994)

[74] Goresky, M. and MacPherson, R., 1983. Intersection homology 11. Inc. Mat, 71, pp.77-129.

Conjecture N. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a Verma module for an orthosymplectic analogue of the vertex W-algebra $\mathcal{W}^k(\mathfrak{gl}_n)$.*

We expect the proof should proceed in much the same way as in [BFN], but with the parabolic induction data replaced by

(write OSp correspondence)

Likewise, we expect a generalisation of [RSYZ] for instantons on $\mathbf{A}^3$:

Conjecture O. [AGT] *There is vertex algebra structure on the the orthosymplectic CoHA of the Jordan quiver, which acts on the equivariant critical cohomology of $\mathcal{M}^{\mathbf{Z}/2}$, the invariant locus in the quiver variety.*

and likewise for arbitrary quivers with potential. We expect this CoHA should be equal to (W algebra thing), which admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients

(nonabelian stable envelopes)

3.4. 70 q-vertex algebras (qVA, KL)

is focussed on developing the machinery of *q-vertex algebras*, then applying it to prove *Kazhdan-Lusztig equivalences*. I will give a definition of q-vertex algebras, generalising factorisation algebras to live on *noncommtutative* spacetimes; note that such factorisation algebras are *new*; this requires giving a sufficiently functorial modern definition of *q-D-modules*.[75] I will then use it to give a new proof of the Kazhdan-Lusztig equivalence and recent generalisations,[??] giving an *uniform* explanation.

The q-WZW vertex algebra. Build a theory of *q-D modules*/D-modules on noncommutative schemes and prestacks, then apply it to define/prove structural results on *q-vertex algebras* (**M5.1**), use *q-affine Grassmannians* and *q-coordinate bundles* to define q-WZW and q-Virasoro vertex algebras (**M5.2**).

It has been long expected that one may define a q-analogue of the Kazhdan-Lusztig equivalence, but this has been hampered by the lack of a good definition of q-WZW algebras: currently, the available definition is an RTT-style definition from [EK].

q-vertex algebras. The main goal of project qVA is:

Conjecture P. *There is a factorisation category over the noncommutative space $\mathbf{A}^2_q$, such that any*

$$\mathcal{A} \in \mathrm{FactAlg}^{st}(\mathcal{D}\text{-}\mathrm{Mod}_{\mathrm{Ran}\mathbf{A}^2_q})$$

*defines a q-**vertex algebra**.*

Moreover, for any complex finite-dimensional simple Lie algebra $\mathfrak{g}$, we may ask

Question Q. *Is there an analogue of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$?*

[75] Majid, S. and Simão, F., 2023. Quantum jet bundles. Letters in Mathematical Physics, 113(6), p.120.

Such a factorisation space would for free by Conjecture P define for us a q-vertex algebra $V_q^k(\mathfrak{g})$, by the same construction as for the affine WZW vertex algebra (and which we expect it would be is a q-deformation of) and we expect should agree with [EK] when $\mathfrak{g} = \mathfrak{sl}_n$. We expect there to be an algebra of modes fucntor $A(-)$, and we propose to finish with a sanity-check of our definitions by showing $A(V^k(\mathfrak{g})) \simeq U_q(\hat{\mathfrak{g}})$.

We spell out evidence for Conjecture P, first from physics, then give explicit mathematical details.

Physics: $5d$ Chern-Simons. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V_\hbar^k(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

where $\mathbf{A}_q^2 = (\mathbf{C} \times \mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x, y]/(xy - qyx)$. Thus by analogy with the $3d$ case, to search for $V_q^k(\mathfrak{g})$ we need to understand factorisation algebras over $\mathbf{A}_q^2$.

Mathematical details. (copy-paste from the notes)

One can interpret the ordinary KZ equations as (factorisation stuff), much like (drinfeld kohno). Thus, a natural question is:

Question R. *Can we recover the qKZ equations by taking (i.e. q-confomal blocks)?*

quantum jet spaces**??** and *de Rham* definition of D-modules via *crystals*[76]; vertex algebras as factorisation/chiral algebras; non-operadic definition of deformed vertex algebras[77] | KZ equations and fusion product of vertex modules[78] | Chen-Fu's proof of Kazhdan-Lusztig equivalence;**??** new Kazhdan-Lusztig equivalences from $3d$ *mirror symmetry* and new *quantum groups*/vertex algebras[79]

de Rham definition of q-D-modules and their functoriality; q-vertex algebras as *factorisation algebras* on *noncommutative schemes*; *q-affine* and *q-Virasoro* factorisation algebras | factorisation category explanation of KZ equations, *Zhu algebra* and *fusion product*

[76]*D. Gaitsgory* and *N. Rozenblyum*, Pure Appl. Math. Q. 10, No. 1, 57–154 (2014; Zbl 1327.14013)

[77]*E. Frenkel* and *N. Reshetikhin*, "Towards Deformed Chiral Algebras", Preprint, arXiv:q-alg/9706023 (1997)

[78]*Y.-Z. Huang*, J. Pure Appl. Algebra 100, No. 1–3, 173–216 (1995)

[79]The following paper, and upcoming work by the authors along with *C. Beem*: *A. Ballin*, *T. Creutzig*, *T. Dimofte*, *W. Niu*, "3d mirror symmetry of braided tensor categories", Preprint, arXiv:2304.11001 [hep-th] (2023)

Kazhdan-Lusztig. One ultimate goal of projects FJ and qVA is to understand Kazhdan-Lusztig equivalences: potentially, our above techniques may be used to either:

- give an affine analogue of the factorisable proof [CF] of the Kazhdan-Lusztig equivalence, or
- proving a Zhu/$q \to 1$ correspondence to obtain Chen-Fu's proof from q-affine vertex algebras; generalising to give a blanket proof of the new Kazhdan-Lusztig equivalences

Question S. *Is there a Riemann-Hilbert functor* $\mathrm{RH} : \mathrm{FactCat}(\mathbf{A}^2_q) \to \mathrm{FactCat}^{QCoh}(\mathbf{C}^2_q)$*, which sends the category* $V^k_q(\mathfrak{g})\text{-Mod}$ *to* $Y_\hbar(\mathfrak{g})\text{-Mod}$*? (too vague)*

(need to write down what topological factorisation algebras on $\mathbf{C}^2_q$ are)

3.5. 90 Sheaf methods (Con, Loc, Eu)

Localisation methods. *Torus localisation* is one of the main methods in enumerative geometry, and projects Con and Loc were concerned with extending these techniques to the Artin moduli stacks appearing in enumerative geometry. Given a closed Artin substack

$$\mathcal{Z} \hookrightarrow \mathcal{X}$$

not necessarily quasicompact,

Theorem T. [Conc] *If* $\mathcal{L}_i$ *are a collection of line bundles such that at least one of them vanishes on each geometric point* $x \in \mathcal{X} \smallsetminus \mathcal{Z}$*, then*

$$\mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X} \smallsetminus \mathcal{Z})_{\mathrm{loc}} \;=\; 0,$$

so then the cohomology of $\mathcal{X}$ *is "concentrated" on* $\mathcal{Z}$*: we have* $i_* : \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{Z}) \xrightarrow{\sim} \mathrm{C}^{\mathrm{BM}}_\bullet(\mathcal{X})$.

Here we have localised with respect to $c_1(\mathcal{L}_i)$, for instance we show the condition holds if $\mathcal{Z}_0/T \hookrightarrow \mathcal{X}_0/T$ is an inclusion of quotient stacks with $\dim \mathrm{Stab}_x(T)$ non-maximal for all $x \in \mathcal{X}_0 \smallsetminus \mathcal{Z}_0$, and we take for $\bigoplus \mathcal{L}_i$ the tautological T-bundle.

Theorem U. [Loc] *If* $i : \mathcal{X}^T \hookrightarrow \mathcal{X}$ *is the inclusion of the homotopy fixed points of a torus action on quasismooth dg scheme* $\mathcal{X}$*, there is a* ***Gysin pullback*** *map* $i^! : \mathrm{C}^{\mathrm{BM}}_{T,\bullet}(\mathcal{X})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_{T,\bullet}(\mathcal{X}^T)_{\mathrm{loc}}$ *satisfying Atiyah-Bott and Graber-Pandharipande formulas:*

$$\mathrm{id} \;=\; i_* \frac{i^!(-)}{e(N_{vir})}, \qquad [\mathcal{X}]^{vir} \;=\; i_* \frac{[\mathcal{X}^T]^{vir}}{e(N_{vir})}, \tag{2}$$

relating to pushforward and fundamental classes.

This recovers the usual torus localisation results when $\mathcal{Z} = X^T/T$ and $\mathcal{X} = X/T$ are quotients of smooth finite-type schemes by tori.

Virtual Euler classes and shuffle structures. In Eu, we strengthen the above results until:

- they give a general geometric method to ourput *shuffle products* for CoHAs,
- and show CoHAs are compatible with Davison/Yang-Zhao localised/Joyce vertex coproducts.

Specifically, we prove analogues of Theorems T and U for the *vanishing cycle* (or any sheaf) cohomology of arbitrary closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ which is quasismooth other a common base, and concentrated with respect to a multiplicative subset $\mathcal{S} \subseteq \mathrm{H}^\bullet(\mathcal{X})$. As a result,

Theorem V. [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we get a diagram*

$$\begin{tikzcd} \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, dashed, "1/e(\mathrm{N}_{i,vir})"] & \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, "p^s_* q^{s,*}"] & \mathrm{C}^\bullet(\mathcal{M}^s, \varphi^s) \\ \mathrm{C}^\bullet(\mathcal{M} \times \mathcal{M}, \varphi \boxtimes \varphi) \arrow[u, "(\pi\times\pi)^*"] \arrow[rr, "p_* q^*"] & & \mathrm{C}^\bullet(\mathcal{M}, \varphi) \arrow[u, "\pi^*"'] \end{tikzcd} \tag{3}$$

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Here p and q are

$$\begin{tikzcd} & \mathrm{SES} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{M} \times \mathcal{M} & & \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{SES}^s \arrow[dl, "q^s"'] \arrow[dr, "p^s"] & \\ \mathcal{M}^s \times \mathcal{M}^s & & \mathcal{M}^s \end{tikzcd}$$

Two consequences of this are:

- If we take $\mathcal{M}^s$ to be a *shuffle space*[80] given by products of "simple" moduli stacks, e.g. rank one quiver representations, then (3) recovers shuffle formulas for CoHAs.
- If we take $\mathcal{M}^s = \mathcal{M} \times \mathcal{M}$ together with its direct sum map to $\mathcal{M}$, (3) recovers the compatibility between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

The first we plan to use in SA to give more general shuffle-style products.

This gives a unified torus localisation way to compute cohomological Hall type products.

As a result, we recover shuffle descriptions of CoHAs and a new proof of compatibility between them and Davison's localised coproduct

3.6. 100 Liouville quantum gravity to vertex algebras (LCG)

History. In recent years, probabilists have increasingly understood quantum field theory, giving rigorous definitions of Feynman measures for $2d$ CFTs, e.g. [CRV; DS; Sh] whose "holomorphic part" are expected to be W-algebras, Virasoro, and Heisenberg vertex algebras.

This approach is very different to the factorisation/vertex algebra/functorial QFT approach in the above projects, e.g. it can directly study level sets of fields as SLE curves [MS; SS], there is a rigorous

[80] i.e. shuffle algebra in the category of spaces, see SA.

connection to combinatorial toy models like the discrete Gasussian Free Field [BPR], and it is able to access the *full* CFT, not just the chiral part as we are in geometric representation theory, e.g. [KRV] proves the *DOZZ* formula for full OPEs in the Liouville CFT.

However, there is currently not much interaction between the two approaches, and this project aims to build a bridge between the two so that techniques/results/heuristics can move between subjects more easily (then give a simple example of this).

Goal. In LCG[81] we aim to define a functor from Segal-style $2d$ conformal field theories to vertex algebras

$$\mathrm{CFT} \overset{(-)^{ch}}{\to} \mathrm{CFT}^{hol} \overset{\mathrm{Res}}{\to} \mathrm{FactAlg}(\mathbf{C})^{hol}_{\mathbf{C}\rtimes\mathbf{C}^\times} \overset{[\mathrm{CG}]}{\to} \mathrm{VertexAlg}, \tag{4}$$

then show that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs are sent to the Heisenberg and Virasoro vertex algebras, repectively.

Details. We will need to upgrade $\mathcal{Z} \in \mathrm{CFT}$ to a definition that remembers the geometric structure on the category Cob_2 of conformal cobordisms. Namely, consider a complex vector bundle $\mathcal{V}$ with connection over the Teichmuller space $\mathcal{T}_{g,n}$ satisfying a factorisation condition, and with a section ψ. The fibre of this data over Σ is the vector space $\mathcal{Z}(\partial\Sigma)$ and $\mathcal{Z}(\Sigma) : \mathbf{C} \to \mathcal{Z}(\partial\Sigma)$.

The induced factorisation algebra over $\mathbf{C}$ is automatically smoothly translation and rotation equivariant, so if it is *holomorphic* (i.e. $\partial_{\bar{z}}\psi = 0$) then it is by [CG] a vertex algebra; these are the last two maps in (4). The equivariance comes from a G-action on $\mathcal{T}_{0,n}$, since then the Lie algebra $\mathfrak{g}$ acts on $\mathcal{V}$ by the connection, e.g. the vertex algebras in the image of (4) will automatically have an action by vector fields on $\mathbf{P}^1$, so we expect they are VOAs.

The main task is to define a chiralisation functor $(-)^{ch}$ to holomorphic CFTs, and prove that [GKRV]'s LQG Segal CFT (upgraded appropriately in the above sense) is sent by (4) to the Virasoro vertex algebra, and relate the DOZZ formula [KRV, (1.12)] to the Virasoro OPE. Having done this, we plan to do the same for the GFF, and finally to give a new example of these methods, construct a probability measure in the domain of (4) recovering the affine vertex algebra, e.g. by using the free field embedding [FB, §11] to a direct sum of Heisenberg algebras.

[81] Joint with V. Giri.

References

[BCDN] Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu. "3d mirror symmetry of braided tensor categories". In: *arXiv preprint arXiv:2304.11001* (2023).

[BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. "Instanton moduli spaces and $\mathcal{W}$-algebras". In: *arXiv preprint arXiv:1406.2381* (2014).

[Bo] R. E. Borcherds. "Quantum Vertex Algebras". In: *Advanced Studies in Pure Mathematics*. Vol. 31. Mathematical Society of Japan, 1999, pp. 51–74.

[BPR] Roland Bauerschmidt, Jiwoon Park, and Pierre-François Rodriguez. "The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature". In: *The Annals of Probability* 52.4 (2024), pp. 1360–1398.

[BR] Samuel Belliard and Vidas Regelskis. "Drinfeld J presentation of twisted Yangians". In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 13 (2017), p. 011.

[BS] Roland Bittleston and David Skinner. "Gauge theory and boundary integrability". In: *Journal of High Energy Physics* 2019.5 (2019), pp. 1–53.

[CF] Li Chen and Yuchen Fu. "An Extension of the Kazhdan-Lusztig Equivalence". PhD thesis. Harvard University, 2022.

[CG] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*. Vol. 31. New Mathematical Monographs. Cambridge University Press, Cambridge, 2017, pp. ix+387.

[Ch] Kuo-tsai Chen. "Iterated integrals of differential forms and loop space homology". In: *Annals of Mathematics* 97.2 (1973), pp. 217–246.

[CRV] Baptiste Cerclé, Rémi Rhodes, and Vincent Vargas. "Probabilistic construction of Toda conformal field theories". In: *arXiv preprint arXiv:2102.11219* (2021).

[CWY] Kevin Costello, Edward Witten, and Masahito Yamazaki. "Gauge theory and integrability, I". In: *arXiv preprint arXiv:1709.09993* (2017).

[Da] Ben Davison. "Quantum field theory and the cohomology of the moduli space of G-bundles on a Riemann surface". In: *Quarterly Journal of Mathematics* 68.2 (2017), pp. 635–703. arXiv: arXiv:1311.7172 `[math.AG]`.

[Dr] Vladimir Gershonovich Drinfeld. "Hopf algebras and the quantum Yang–Baxter equation". In: *Doklady Akademii Nauk*. Vol. 283. 5. Russian Academy of Sciences. 1985, pp. 1060–1064.

[DS] Bertrand Duplantier and Scott Sheffield. "Liouville quantum gravity and KPZ". In: *Inventiones mathematicae* 185.2 (2011), pp. 333–393.

[EK] P. Etingof and D. Kazhdan. "Sel. Math., New Ser. 6, No. 1, 105–130". In: *Selecta Mathematica, New Series* 6.1 (2000), pp. 105–130.

[FB] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*. 88. American Mathematical Soc., 2004.

[FR] Edward Frenkel and Nicolai Reshetikhin. "Towards deformed chiral algebras". In: *arXiv preprint* (1997). arXiv: q-alg/9706023 [q-alg].

[Ga] Dennis Gaitsgory. "On factorization algebras arising in the quantum geometric Langlands theory". In: *Advances in Mathematics* 391 (2021), p. 107962.

[GKRV] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. "Segal's axioms and bootstrap for Liouville Theory". In: *arXiv preprint arXiv:2112.14859* (2021).

[GLW] Sachin Gautam, Valerio Toledano Laredo, and Curtis Wendlandt. "The meromorphic R-matrix of the Yangian". In: *Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin*. Springer, 2021, pp. 201–269.

[GRZ] Davide Gaiotto, Miroslav Rapčák, and Yehao Zhou. "Deformed Double Current Algebras, Matrix Extended $\mathcal{W}_{1+\infty}$ Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection". In: *arXiv preprint arXiv:2309.16929* (2023).

[KRV] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. "Integrability of Liouville theory: proof of the DOZZ formula". In: *Annals of Mathematics* 191.1 (2020), pp. 81–166.

[KS] Maxim Kontsevich and Yan Soibelman. "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants". In: *arXiv preprint arXiv:1006.2706* (2010).

[Ku] Shrawan Kumar. *Kac-Moody groups, their flag varieties and representation theory*. Vol. 204. Springer Science & Business Media, 2012.

[Lu] Jacob Lurie. *Higher Algebra*. Preprint, available at http://www.math.harvard.edu/~lurie. 2016.

[Maa] Shahn Majid. "Transmutation theory and rank for quantum braided groups". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 113. 1. Cambridge University Press. 1993, pp. 45–70.

[Mab] Shahn Majid. "Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 125. 1. Cambridge University Press. 1999, pp. 151–192.

[MMSV] Anton Mellit, Alexandre Minets, Olivier Schiffmann, and Eric Vasserot. "Coherent sheaves on surfaces, COHAs and deformed $\mathcal{W}_{1+\infty}$-algebras". In: *arXiv preprint arXiv:2311.13415* (2023).

[MO] Davesh Maulik and Andrei Okounkov. "Quantum groups and quantum cohomology". In: *arXiv preprint arXiv:1211.1287* (2012).

[MS] J Miller and S Sheffield. "Gaussian free field light cones and $\mathrm{SLE}_\kappa(\rho)$". In: *Ann. Probab., Preprint, June* (2016).

[RSYZ] Miroslav Rapčák, Yan Soibelman, Yaping Yang, and Gufang Zhao. “Cohomological Hall algebras, vertex algebras and instantons”. In: *Communications in Mathematical Physics* 376.3 (2020), pp. 1803–1873.

[Sh] S Sheffield. “Gaussian free fields for mathematicians. preprint”. In: *arXiv preprint math.PR/0312099* (2003).

[SS] O. Schramm and S. Sheffield. “SS”. In: *Annals of Probability* 33.6 (2005), pp. 2127–2148.

[SV] Olivier Schiffmann and Eric Vasserot. “On cohomological Hall algebras of quivers: generators”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2020.760 (2020), pp. 59–132.

[YZ] Yaping Yang and Gufang Zhao. “Cohomological Hall algebras and affine quantum groups”. In: *Selecta Mathematica* 24.2 (2018), pp. 1093–1119.