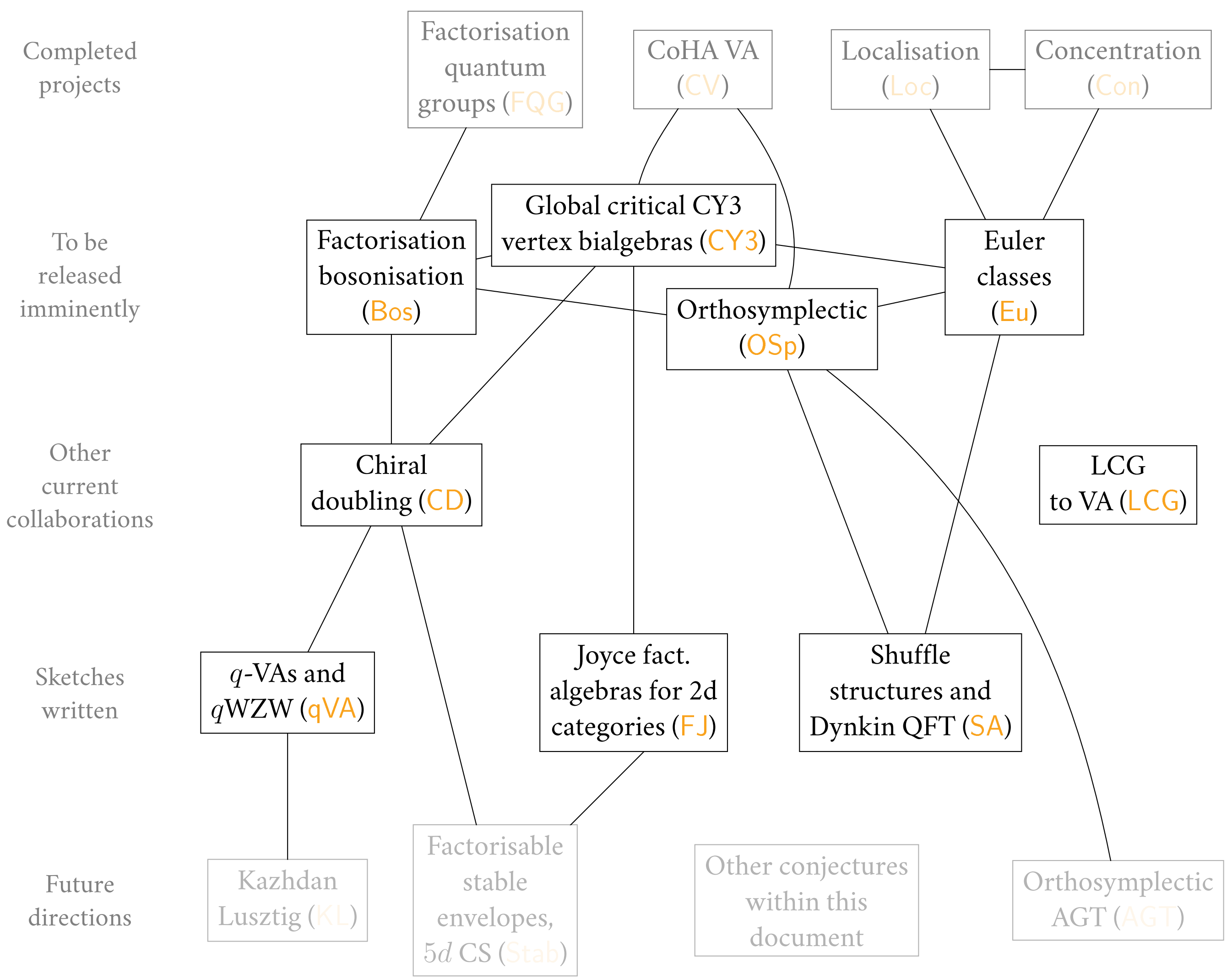
RESEARCH PLANS - ABRIDGED

ALEXEI LATYNTSEV

See https://alyoshalatyntsev.github.io/plan/plan.pdf *for a more detailed research statement.*



1. Research statement

I am a mathematician working in geometric representation theory.

1.1. Algebraic structures attached to CY3s (CV, CY3, FJ, Stab)

Cohomological Hall algebras are associative algebras attached to Calabi-Yau varieties or categories [Daa; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZ] (e.g. quantum groups [MO]), hence are a key object in geometric representation theory/enumerative invariants.

Global critical CY3 vertex quantum groups. Let $\mathcal{C}$ be the CY3 category $\mathrm{Rep}(Q, W)$, the representations of a Jacobi algebra of a quiver with potential, or $\mathrm{Coh}_0(K_{T^*C})$, zero dimensional coherent sheaves on a local curve [KK], or more generally a *deformed CY3* completion.

Theorem. [CY3] *The critical cohomology*[1] *of the moduli stack of objects $\mathcal{M}$ has a vertex coproduct*

$$\mathrm{H}^\bullet(\mathcal{M}, \varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M}, \varphi) \,\hat{\otimes}\, \mathrm{H}^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

compatible with the CoHA: it forms a vertex quantum group *(see FQG for a definition).*

Theorem. [CY3] *There is a functor Φ from Q-localised bialgebras [Daa] to vertex bialgebras. The former is equivalent to the category of* factorisation bialgebras *on the coloured configuration space* $\mathrm{Conf}_{Q_0}\mathbf{A}^1$.

Theorem. [CY3; CV for $W = 0$] *For any quiver Q, the vertex coproduct on the preprojective CoHA* $\mathrm{H}^\bullet_{\mathbf{G}_m}(\mathcal{M}_{Q^{(3)}}, \varphi_{W^{(3)}})^{ext} \overset{[\mathrm{BD}]}{\simeq} Y_\hbar(\mathfrak{b}_Q)$ *is identified by Φ with the Davison/Yang-Zhao localised coproduct* [Daa; YZ], *and (when defined) Drinfeld's meromorphic coproduct* [Dr; GLW].

This **generalises** Joyce-Liu's vertex coproduct [Joa; Li], shows that all these well-known coproducts on the Yangian are **equal**, and gives a **geometric** definition using moduli stacks and configuration spaces. Finally, Theorem 1.1 points to a new Ran space definition of vertex-vertex *bi*algebras.

Lift to factorisation algebra and relation to stable envelopes. To move towards *arbitrary* CY3s, we first interpret the above in terms of factorisation algebras; second, we relate to the $\mathcal{W}$-algebras for surfaces of [MMSV], thus understanding the structure for K_S. Let Q be a quiver with torus $T = \prod T_d$.

Conjecture. [FJ] *Given* $\mathcal{M}^f = \{(m, \lambda) \,:\, \lambda \in \mathfrak{t},\ m \in \mathcal{M}^\lambda\} \xrightarrow{\pi} \mathrm{colim}(\mathfrak{t}_d) = \mathrm{Ran}_{Q_0}\mathbf{A}^1$*, its relative BM-homology $\mathcal{A} = \pi_*\omega$ is a factorisation algebra over the coloured Ran space. Its restriction to the colour-diagonal recovers the vertex quantum group structure on the nilpotent CoHA* [SV].

Conjecture. [FJ] *For S a smooth algebraic surface, there is a braided factorisation category* $\mathrm{Rep}\mathcal{W}$ *over* $\mathrm{Ran}_S K_S$ *(c.f. FQG). Applying Bos/CD allows us to construct $\mathcal{W}(S)^{\geqslant 0}$ and $\mathcal{W}(S)$ from* [MMSV]*'s $\mathcal{W}(S)^{>0}$.*

The definition of $\mathcal{M}^f$ is clearly reminiscent of [MO, §5.1.1]'s stable envelope construction, and suggests a resolution to their question: *"we view this definition as provisional; perhaps a better set of axioms will emerge later"*. Let $\mathbf{w}$ be a multidimension vector of quiver Q and $M(\mathbf{w})$ the quiver variety.

Conjecture. [Stab] *There is a factorisation space $\pi_{\mathbf{w}} : M(\mathbf{w})^f \to \mathrm{Ran}_Q\mathbf{A}^1$ and the factorisation category $\mathcal{E}$ spanned by $\pi_{\mathbf{w},*}\omega$ is acted on by $\mathcal{A} = \pi_*\omega$. Applying chiral Tannakian reconstruction $\mathcal{E} \simeq \mathrm{Rep}\mathbf{D}\mathcal{A}$ gives the double of $\mathcal{A}$ with its (two) coproducts.*

The above is a Drinfeld-Kohno Theorem for [MO]'s Yangians $Y_\hbar(\mathfrak{g}_Q)$ (see qVA for relations to qKZ).

[1] i.e. $\mathcal{M} = \mathrm{Crit}(W)$ is a critical locus inside a *smooth* moduli stack; we take the vanishing cycle sheaf $\varphi = \varphi_W$.

1.2. The structure of factorisation quantum groups (FQG, Bos, CD)

Historically the definitions of (double) affine quantum groups $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ were (ingeniously) made very explicitly [Dr; MO], and still much of the modern (e.g. CoHA) literature is based on explicit shuffle computations, e.g. [MMSV; SV; YZ]. This series of projects **axiomatises** (operadically) these structures, allowing us to import techniques from the theory of ordinary **quantum groups**,[2] to recover the above formulas as a **consequence** of these definition.

Factorisation quantum groups. In FQG we develop a theory of $\mathbf{E}_n$*-factorisation categories* over *factorisation spaces* X (including ordinary groups G, configuration spaces $\mathrm{Conf}_{Q_0}\mathbf{A}^1$, and algebraic-topological Ran spaces $\mathrm{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$). We first give basic structure results for braided factorisation categories $\mathcal{C}$:

Theorem. [FQG] *Let $\mathcal{A}$ is a factorisation algebra in $\mathcal{C}$ over X, a (braided) factorisation structure on* $\mathcal{A}$-FactMod$(\mathcal{C})$ *induces a* factorisation bialgebra *structure on $\mathcal{A}$ (and a* factorisation R-matrix $R : \mathcal{A}\otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A}\otimes_{\mathcal{C},X} \mathcal{A}$*).*

Theorem. [FQG] *When $X = \mathrm{Ran}\mathbf{A}^1$ (resp. $\mathrm{Conf}\mathbf{A}^1$), Theorem 1.2 recovers classical notions* [EK; FR] *of* ***quantum vertex algebras*** *(resp. localised algebras) and their R-matrices $R(z)$ satisfying the spectral YBE.*

For instance, the structure [GLW] for **Yangians** (e.g. two (meromorphic) coproducts and R-matrices $R^-, R^{0,\epsilon}, R^\epsilon$ relating them) are equivalent to: $Y_\hbar(\mathfrak{g})$-Mod is a lax braided factorisation category. This makes rigorous the physics claim [CWY] that $Y_\hbar(\mathfrak{g})$-Mod from $4d$ Chern-Simons is a topological-holomorphic factorisatsion category over $\mathbf{R}\times\mathbf{C}$. The above may help understand **affine Yangians** (e.g. [GRZ]; qVA for relation to q-WZW) and the **new KL equivalances** [BCDN]. Finally, in principle we may get lots of examples from:

Theorem. [FQG] *A generalisation of Borcherds' twist construction* [Bo] *to arbitrary decomposition algebra.*

Factorisation bosonisation. In the CoHA literature, a lot of algebraic effort needs to be expended each time [Daa; RSYZ; YZ], [CY3; OSp] to *add in the Cartan piece* $\mathrm{H}^\bullet(\mathcal{M},\varphi) \rightsquigarrow \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$, e.g. to obtain Yangians of Borels $Y_\hbar(\mathfrak{b}_Q)$. In the finite groups case, a clean construction is Tannakian reconstruction

$$U_q(\mathfrak{n}) \rightsquigarrow U_q(\mathfrak{n}) \qquad \text{via} \qquad U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q T),$$

c.f. [Ga; Maa; Mab]; in Bos we apply the same to vertex and factorisation bialgebras:

Theorem. [Bos, in preparation] *There is a Tannakian reconstruction functor from (braided) factorisation categories $\mathcal{C}$ to (quasitriangular) factorisation quantum groups $\mathcal{A}$. In the preprojective case of Theorem 1.1,*

$$Y_\hbar(\mathfrak{b}_Q)\text{-Mod} \;\simeq\; Y_\hbar(\mathfrak{n}_Q)\text{-Mod}(Y_\hbar(\mathfrak{t}_Q)\text{-Mod})$$

we Tannakian reconstruct $Y_\hbar(\mathfrak{b}_Q) \simeq \mathrm{H}^\bullet(\mathcal{M},\varphi)^{ext}$ with its localised or vertex bialgebra structure.

Applying this to $\mathrm{H}^\bullet(\mathcal{M},\varphi)\text{-Mod}(\mathrm{H}^\bullet(\mathcal{M})\text{-Mod}^\cup)$ **automates** the process of extending CoHAs.

Factorisation Drinfeld doubling. An active problem is how the structures in CY3 relate to **wall crossing** and stability conditions, see [Br; Job]. As a first attempt, we use FQG to understand **doubling**, where the CoHA of heart $\mathcal{A}$ and its opposite $\mathcal{A}[1]$ are glued, in a similar way to Bos:

Conjecture. [CD] *There is a* Drinfeld centre *construction $Z_{\mathbf{E}_1}(\mathcal{C})$ of a chiral factorisation category $\mathcal{C}$, which carries compatible chiral and ordinary monoidal structures, and Tannakian reconstruction gives*

$$Z_{\mathbf{E}_1}^{Y_\hbar(\mathfrak{t}_Q)\text{-Mod}}(Y_\hbar(\mathfrak{b}_Q)\text{-Mod}) \;\simeq\; Y_\hbar(\mathfrak{g}_Q)\text{-Mod}$$

[2] Namely, when $X = \mathrm{Ran}\mathbf{R}^2$ in the below, via Lurie [Lu].

and likewise we recover the Takiff algebra double construction of [AN].

1.3. Orthosymplectic structures (OSp, SA, AGT)

Orthosymplectic CoHAs. We define *orthosymplectic moduli stacks* $\mathcal{M}^{\mathrm{OSp}}$, including perverse-coherent sheaves with a sympletic/orthogonal bilinear form, and quivers and potential with involution.

Theorem. [OSp] *For $\mathcal{M}$ as in $\mathcal{C}$ the vertex quantum group $\mathrm{H}^\bullet(\mathcal{M},\varphi)$ acts on $\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$, i.e.*

1. *there is a left module action a of the **CoHA** respecting the involution,*[3] *compatible with*
2. *a **symplectic vertex algebra** structure (factorisation coalgebra over symplectic Ran space $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\mathfrak{t}_{\mathfrak{sp}_{2n}}$, see SA). The defining R-/K-matrices satisfy the **Cherednik reflection equation**.*

Theorem. [OSp] *There is an action of $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M})$ of the CoHA of zero-dimensional coherent sheaves on surface S on $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}_{\sigma\text{-ss}})$ the BM homology of a compactification of the stack of $G\in\{\mathrm{Sp},\mathrm{O}\}$-bundles.*

This gives a rigorous definition of the objects coming from boundary $4d$ Chern-Simons theory [BS], should relate to work in ι**quantum groups** [LW] and Finkelberg-Hanany-Nakajima's ongoing work on **orthosymplectic Coulomb branches** (see also AGT). In the quiver with potential case, we have:

Theorem. [OSp] *An explicit shuffle formula for the CoHA action and vertex coaction on $\mathrm{H}^\bullet(\mathcal{M}^{\mathrm{OSp}}_{Q,W},\varphi^{\mathrm{OSp}})$.*

Conjecture. *When Q is type A_{2n+1} with the reflection involution, the preprojective orthosymplectic CoHA*[4] *is isomorphic to the **twisted Yangian** $Y_\hbar(\mathfrak{gl}_n)^{tw}$ of* [BR].

Dynkin QFT and shuffle structures. The structures (2) in OSp are defined over the *symplectic configuration space*[5] $\mathrm{Conf}_{\mathrm{Sp}}(\mathbf{A}^1) = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BSp})$, with singularities over the root hyperplanes; alternatively over $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1 = \mathrm{colim}\mathfrak{t}_{\mathfrak{sp}_{2n}}$.

Theorem. [OSp] *To give examples, we construct an **invariants** functor involving restricting along $\mathfrak{t}_{\mathfrak{sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$*

$$\iota \;:\; \mathrm{FactAlg}_{\mathrm{GL}}(\mathbf{A}^2) \;\to\; \mathrm{FactAlg}_{\mathrm{Sp}}(\mathbf{A}^1), \qquad\qquad (\mathcal{A},\tau)\;\mapsto\;(\mathcal{A},\mathcal{A}^\tau)$$

where $\mathcal{A}$ is a factorisation algebra with involution τ

In SA, we **generalise** the above to systems of **arbitrary Kac-Moody groups** G [Ku, §V]; this forms a category K_{KM} with morphisms the parabolics.

Theorem. [SA] *A shuffle algebra is equivalent to a monoidal functor $K_{\mathrm{GL}} \xrightarrow{A} \mathrm{Vect}$*

Namely, *all* the structures (moduli stacks, Hall algebras and its realisation as shuffle algebras, vertex coalgebra structure, (conjecturally, see AGT) action on Nakajima quiver varieties and the KZ equation, simultaneously generalise - this points towards this being a shadow of a more general theory.

The starting observation is this - the definition [KS; Gr] a shuffle algebra is equivalent to a monoidal functor $A : \mathrm{GL}\to\mathrm{Vect}$ from the category GL whose objects are finite products of the groups GL_n for $n\geqslant 0$, and the morphisms are parabolics between them. Indeed, the parabolics

$$\begin{tikzcd}[column sep=small] & P_{n,m}(\sigma) \arrow[dl] \arrow[dr] & \\ \mathrm{GL}_n\times\mathrm{GL}_m & & \mathrm{GL}_{n+m} \end{tikzcd} \quad \overset{A}{\mapsto} \quad A_n\otimes A_m \overset{m(\sigma)}{\to} A_{n+m}$$

[3]i.e. the left action a and the right action $a\cdot(\mathrm{id}\otimes\tau)$ commute, where τ is the involution.

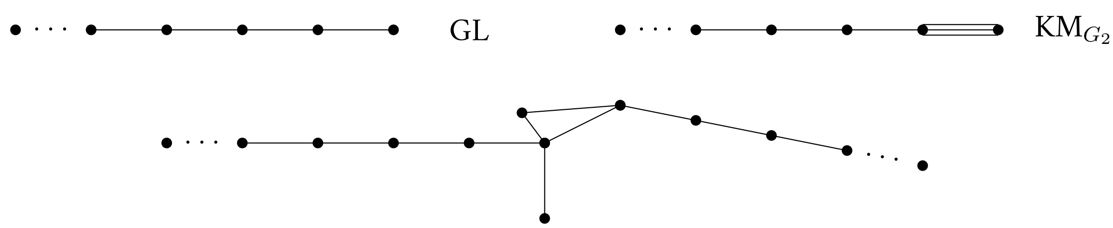
[4]i.e. the image of the CoHA in $\mathrm{End}(\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}^{\mathrm{OSp}}))$.

[5]As opposed to the ordinary configuration space $\mathrm{Conf}\mathbf{A}^1 = \sqcup\mathbf{A}^n/\!/\mathfrak{S}_n = \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{BGL})$.

are labelled by shuffles $\sigma \in \mathfrak{S}_{n+m}/\mathfrak{S}_n \times \mathfrak{S}_m = \mathrm{Sh}(n,m)$.

The motivating idea of SA is **replace** GL with the category KM of **arbitrary Kac-Moody groups**

. For convenience we often pass to full subcategories generated by a fixed set of generalised Cartan matrices/Dykin diagrams, e.g.



To summarise, we expect to define KM analogues of the following:

- *Shuffle algebras*, likewise analogues localised and vertex algebras living over new configuration and Ran spaces
$$\mathrm{Conf}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \coprod_G \mathrm{Spec}\, \mathrm{H}^\bullet(BG), \qquad\qquad \mathrm{Ran}_{\mathrm{KM}}(\mathbf{A}^1) \;=\; \mathrm{colim}_G \mathfrak{t}_G,$$
where $\mathfrak{t}_G$ is the Cartan of Kac-Moody group G. Topological sheaves on $\mathrm{Ran}_{\mathrm{KM}}\mathbf{C}$ gives analogues of $\mathbf{E}_2$-algebras/braided monoidal categories.
- Generalised quivers and quiver varieties. A quiver representation we can view as being attached to the groups
$$\underset{\bigcirc}{\mathrm{GL}_3} \xrightarrow{\;U_{3,5}\;} \underset{\bigcirc}{\mathrm{GL}_5} \xrightarrow{\;U_{5,4}\;} \underset{\bigcirc}{\mathrm{GL}_4}$$
where $P_{n,m} \to U_{n,m}$ is a unipotent. We can define the stack of KM-quiver representations as $\mathcal{M}_Q = \prod \mathfrak{u}_e/G_i$ the product over all maps $(G_i) : Q_0 \to \mathrm{KM}$ and U_e is a choice if unipotent for each edge e. Analogue of stable envelope construction Stab, e.g. giving [MO]-analogue construction of OSp CoHAs.
- Iterated integrals. Chen's [Ch] shuffle structure on cochains $\mathrm{C}^\bullet(LX)$ of the loop space may be deduced from a shuffle structure on the spaces $L_nX = \mathrm{Maps}(\Delta^n, X)$, where $\Delta^n = T^n/\mathfrak{S}_n$; in the general case we may replace this with the quotient $\Delta_G = T_G/\mathfrak{W}_G$ by the Weyl group of G. Understand the relation to Dynkin/q-analogues of multiple zeta values [KMT; Mi].

For instance, the structures in OSp (e.g. $\pm$-equivariant factorisation algebras on $\mathbf{C}$) are obtained from $\mathrm{KM}_{\mathrm{SO}(2n),\mathrm{Sp}(2n),\mathrm{SO}(2n+1)}$; so too let us consider K_{G_2} - factorisation algebras consist of ordinary factorisation algebras but where for any *triple* of points z_1, z_2, z_3 there is in addition equivariance with respect to

$$\begin{aligned}
\tau(z_1) &= z_3 + \sqrt{3}(z_1 + z_2 - 2z_3) \\
\tau(z_2) &= z_1 + \sqrt{3}(z_2 + z_3 - 2z_1) \\
\tau(z_3) &= z_2 + \sqrt{3}(z_3 + z_1 - 2z_2),
\end{aligned}$$

a square root of (231) generating $W_{G_2} \simeq D_{12} \supseteq \mathfrak{S}_3$. Just as in OSp we show $K_{\mathrm{Sp}_{2n}}$-analogues of (factorisation) braided monoidal categories give Cherednik's reflection equation, we expect to obtain the G_2-reflection equation [Ku]. For another example, considering the groups $\hat{\mathfrak{gl}}_n$ of affine type A gives $\mathrm{Ran}_{\widehat{\mathrm{GL}}}\mathbf{A}^1$, $\mathcal{D}$-modules on which are related to $\mathcal{D}$-modules on $\mathbf{A}^1/\mathbf{Z}$, so we expect this should relate to trigonometric KZ equations.

Just as in OSp we used that C_n is obtained by *folding* A_{2n},[6] we expect to be able to produce G_2 structures by taking $\mathbf{Z}/3$-invariants of type D structures.

Conjecture. *The **boundary KZ equations** may be derived by applying ι to the BD Grassmannian, taking distributions supported at the identity, and taking conformal blocks over* $\mathrm{Ran}_{\mathrm{Sp}}\mathbf{A}^1$.

A twisted AGT correspondence. After OSp, one natural next step (project AGT) is to construct a boundary version [AGT; BFN]:

Conjecture. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2,\mathrm{GL}_n}$ in the Uhlenbeck space is a Verma module for an orthosymplectic analogue of the vertex W-algebra $\mathcal{W}^k(\mathfrak{gl}_n)$.*

Likewise, we expect a version for an arbitrary smooth projective surface S. We expect the proof of the above should proceed in much the same way as in [BFN], but with the parabolic induction data replaced by

$$\begin{array}{ccc} & \mathrm{B}P & \\ \swarrow & & \searrow \\ \mathrm{BGL}_n \times \mathrm{BSp}_{2m} & & \mathrm{BSp}_{2n+2m} \end{array}$$

as in OSp; i.e. we expect a SA-type analogue of free field realisations. Likewise, we expect a generalisation of [RSYZ] for instantons on $\mathbf{A}^3$:

Conjecture. [AGT] *There is vertex algebra structure on the the orthosymplectic CoHA of the Jordan quiver, which acts on $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$, the equivariant intersection cohomology of the invariant locus in the quiver variety.*

Just as the CoHA $\mathcal{W}^+_{1+\infty}$ of the Jordan quiver is by [Dab] the universal enveloping algebra of positive half of differential operators on $\mathbf{C}^\times$ and admits the W-algebras of [BFN] as quotients, we expect the above to be a universal enveloping on differential operators on $\mathbf{C}^\times/\pm$, and admit the above W-algebras $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients.

1.4. q-vertex algebras (qVA, KL)

q-vertex algebras. The main goal of project qVA is to develop the machinery of *q-vertex algebras* and define q-affine vertex algebras. In KL we hope to use it to produce the qKZ equations and relate to Kazhdan-Lusztig equivalences.

A natural first guess at a definition is to take the usual definition of vertex algebra but using $\mathcal{D}_q$-modules in place of $\mathcal{D}$-modules. The first observation is that a q-difference operator ∂_x on $\mathbf{A}^1$ induces a derivation $y\partial_x$ on the noncommutative plane[7] $\mathbf{A}^2_q$, and indeed the physics heuristic below points towards $\mathcal{D}$-modules on $\mathbf{A}^2_q$ (e.g. via [MS]) as the correct setting for q-vertex algebras:

Conjecture. [qVA] *There is a factorisation category over the noncommutative space $\mathbf{A}^2_q$, such that any $\mathcal{A} \in$* $\mathrm{FactAlg}^{st}(\mathcal{D}\text{-}\mathrm{Mod}_{\mathrm{Ran}\mathbf{A}^2_q})$ *defines a q-**vertex algebra** (generalising e.g. [FR]).*

[6] i.e. the invariants construction discussed in OSp.

[7] Its with ring of functions $\mathbf{C}\langle x, y, q\rangle/(yx - xyq)$ with q central.

To construct this category precisely, one needs to develop the theory of $\mathcal{D}$-modules (e.g. functoriality) over noncommutative spaces. We propose using work [FMW; MS] on jet spaces of noncommutative schemes to give a "q-crystal/de Rham" definition.

Physics heuristic. Our guiding heuristic from physics is the following: much as $V^k(\mathfrak{g})$ and $U_\hbar(\mathfrak{g})$ have module categories giving line operators for "$3d$ Chern-Simons with boundary" on

$$\mathbf{C} \times \mathbf{R}_{\geqslant 0}$$

or more cleanly, on the suspension $S(\mathbf{CP}^1)$, so then module categories for $V_\hbar^k(\mathfrak{g})$ and $Y_\hbar(\hat{\mathfrak{g}})$ should define line operators for "$5d$ Chern-Simons theory with boundary" on

$$(\mathbf{C} \times \mathbf{C})_{nc} \times \mathbf{R}_{\geqslant 0}$$

where $\mathbf{A}_q^2 = (\mathbf{C} \times \mathbf{C})_{nc}$ is the noncommutative plane with ring of functions $\mathbf{C}[x,y]/(xy - qyx)$, see [GRZ] or particularly Costello's [Co] work.

Examples. The above definition will have been correct if we may answer

Question. *Is there an analogue of the Beilinson-Drinfeld Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}_q^2$?*

Such a factorisation space would for free by Conjecture 1.4 define for us a q-vertex algebra $V_q^k(\mathfrak{g})$, by the same construction as for the affine WZW vertex algebra: taking distributions supported at the identity. We expect that $V_q^k(\mathfrak{g})$ should be a q-deformation of the affine vertex algebra and should agree with Etingof-Kazhdan's RTT construction in [EK] when $\mathfrak{g} = \mathfrak{sl}_n$.

Kazhdan-Lusztig. Conditional on having defined a factorisation algebra $V_q^k(\mathfrak{g})$ as in Question 1.4, many interesting questions follow. To begin with, for formal reasons just as for ordinary vertex algebras, given $V_q^k(\mathfrak{g})$-modules $M_1, \dots, M_n$ we expect to obtain a $\mathcal{D}_q$-module of *conformal blocks* $\mathrm{C}^\bullet(M_1, \dots, M_n)$ on $(\mathbf{A}_q^2)^n$.

Question. [KL] *Is its restriction to $(\mathbf{A}^1)^n_\circ$ equal to the qKZ connection?*

It has been long expected that one may define an affine analogue of the Kazhdan-Lusztig equivalence, and answering the above question would be a first step in understanding whether the geometric proof [CF] might be generalised to the affine setting.[8]

Orthogonally to this, we can try to understand ordinary Kazhdan-Lusztig better. First, we ask whether there is a lift of the *Zhu algebra* functor to the q-setting, fitting into a commuting square

$$\begin{array}{ccc} V_q^k(\mathfrak{g}) & \xmapsto{q\to 1} & V^k(\mathfrak{g}) \\ \downarrow{\scriptstyle \mathrm{Zhu}} & & \downarrow{\scriptstyle \mathrm{Zhu}} \\ U_q(\mathfrak{g}) & \xmapsto{q\to 1} & U(\mathfrak{g}) \end{array}$$

Noting the appearance of both objects $U_q(\mathfrak{g})$ and $V^k(\mathfrak{g})$ appearing in the Kazhdan-Lusztig equivalence, having done this we then ask whether this these are the special and general fibres of a structure on $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$:

Question. [KL] *Does $V_q^k(\mathfrak{g})$ induce a topological-holomorphic factorisation algebra $\mathcal{A}$ on $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$, whose restriction to $\mathbf{C}$ is $V^k(\mathfrak{g})$ and whose restriction to $\mathbf{C} \times \mathbf{R}_{>0}$ is $U_q(\mathfrak{g})$?*

[8] Specifically, one wants a Riemann-Hilbert type functor "$\mathrm{RH} : \mathrm{FactCat}(\mathbf{A}_q^2) \to \mathrm{FactCat}^{\mathrm{QCoh}}(\mathbf{C}_q^2)$", which sends the category $V_q^k(\mathfrak{g})$-Mod to $Y_\hbar(\mathfrak{g})$-Mod^{fd}.

One would then hope to interpret the fact that [CF]'s RH functor sends $V^k(\mathfrak{g})$-FactMod to $\mathrm{KD}(U_q(\mathfrak{g}))$-FactMod as some sort of flatness statement for $\mathcal{A}$-FactMod over $\mathbf{R}_{\geqslant 0}$.[9] This may give a new way to understand the recent Kazhdan-Lusztig equivalences [BCDN] coming from $3d$ mirror symmetry.

1.5. Sheaf methods (Con, Loc, Eu)

Localisation methods. *Torus localisation* is one of the main methods in enumerative geometry, and projects Con and Loc were concerned with extending these techniques to the Artin moduli stacks appearing in enumerative geometry. Given a closed Artin substack

$$\mathcal{Z} \hookrightarrow \mathcal{X}$$

not necessarily quasicompact,

Theorem. [Conc] *If $\mathcal{L}_i$ are a collection of line bundles such that at least one of them vanishes on each geometric point $x \in \mathcal{X} \smallsetminus \mathcal{Z}$, then*

$$\mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X} \smallsetminus \mathcal{Z})_{\mathrm{loc}} \;=\; 0,$$

so then the cohomology of $\mathcal{X}$ is "concentrated" on $\mathcal{Z}$: we have $i_* : \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{Z}) \xrightarrow{\sim} \mathrm{C}^{\mathrm{BM}}_{\bullet}(\mathcal{X})$.

Here we have localised with respect to $c_1(\mathcal{L}_i)$, for instance we show the condition holds if $\mathcal{Z}_0/T \hookrightarrow \mathcal{X}_0/T$ is an inclusion of quotient stacks with $\dim \mathrm{Stab}_x(T)$ non-maximal for all $x \in \mathcal{X}_0 \smallsetminus \mathcal{Z}_0$, and we take for $\bigoplus \mathcal{L}_i$ the tautological T-bundle.

Theorem. [Loc] *If $i : \mathcal{X}^T \hookrightarrow \mathcal{X}$ is the inclusion of the homotopy fixed points of a torus action on quasismooth dg scheme $\mathcal{X}$, there is a* ***Gysin pullback*** *map $i^! : \mathrm{C}^{\mathrm{BM}}_{T,\bullet}(\mathcal{X})_{\mathrm{loc}} \to \mathrm{C}^{\mathrm{BM}}_{T,\bullet}(\mathcal{X}^T)_{\mathrm{loc}}$ satisfying Atiyah-Bott and Graber-Pandharipande formulas:*

$$\mathrm{id} \;=\; i_* \frac{i^!(-)}{e(N_{vir})}, \qquad\qquad [\mathcal{X}]^{vir} \;=\; i_* \frac{[\mathcal{X}^T]^{vir}}{e(N_{vir})}, \tag{1}$$

relating to pushforward and fundamental classes.

This recovers the usual torus localisation results when $\mathcal{Z} = X^T/T$ and $\mathcal{X} = X/T$ are quotients of smooth finite-type schemes by tori.

Virtual Euler classes and shuffle structures. In Eu, we strengthen the above results in Con and Loc until:

- they give a general geometric method to output *shuffle products* for CoHAs,
- and show CoHAs are compatible with Davison/Yang-Zhao localised/Joyce vertex coproducts.

Specifically, we prove analogues of Theorems 1.5 and 1.5 for the *vanishing cycle* (or any sheaf) cohomology of arbitrary closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ which is quasismooth other a common base, and concentrated with respect to a multiplicative subset $\mathcal{S} \subseteq \mathrm{H}^\bullet(\mathcal{X})$. As a result,

Theorem. [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we get a diagram*

$$\begin{tikzcd}
\mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, dashed, "1/e(\mathrm{N}_{i,vir})"] & \mathrm{C}^\bullet(\mathcal{M}^s \times \mathcal{M}^s, \varphi^s \boxtimes \varphi^s) \arrow[r, "p^s_* q^{s,*}"] & \mathrm{C}^\bullet(\mathcal{M}^s, \varphi^s) \\
\mathrm{C}^\bullet(\mathcal{M} \times \mathcal{M}, \varphi \boxtimes \varphi) \arrow[u, "(\pi\times\pi)^*"] \arrow[rr, "p_* q^*"] & & \mathrm{C}^\bullet(\mathcal{M}, \varphi) \arrow[u, "\pi^*"']
\end{tikzcd} \tag{2}$$

[9]By means of extra evidence, it seems plausible that ordinary Riemann-Hilbert $\mathcal{D}\text{-Mod}^{rh} \xrightarrow{\sim} \mathrm{Perv}$ may be interpreted this way, where we consider $\mathcal{A}$ a sheaf of algebras generated by $\mathcal{O}_{\Sigma\times\mathbf{R}_{\geqslant 0}}$ and the Lie algebra $\mathcal{T}_{\Sigma\times\mathbf{R}_{\geqslant 0}}$ of infinitesimal automorphisms of $\Sigma \times \mathbf{R}_{\geqslant 0}$ whose restriction to the boundary is antiholomorphic.

saying that up to an Euler class term, the pullback map intertwines the CoHA and the split locus CoHA.

Here p, p^s (proper) and q, q^s (quasismooth) are

$$\begin{tikzcd} & & \mathrm{SES}^s \arrow[dl, "q^s"'] \arrow[d] \arrow[dr, "p^s"] & & \\ & \mathcal{M}^s \times \mathcal{M}^s \arrow[d] & \mathrm{SES} \arrow[dl, "q"] \arrow[dr, "p"'] & \mathcal{M}^s \arrow[d] & \\ & \mathcal{M} \times \mathcal{M} & & \mathcal{M} & \end{tikzcd}$$

for instance $\mathcal{M}$ is *smooth* moduli stack containing as a critical locus $\mathrm{Crit}W$ the deformed CY3 moduli stacks considered in CY3, and $\varphi = \varphi_W$, and we apply localisation to $i : \mathrm{SES}^s \to \mathrm{SES} \times_{\mathcal{M}^s} \mathcal{M}$.

Two consequences of this are:

- If we take $\mathcal{M}^s$ to be a *shuffle space*[10] given by products of "simple" moduli stacks, e.g. parametrising tuples of rank one quiver representations, then (2) recovers shuffle formulas [Daa; SV; YZ] for CoHAs and localised/vertex coproducts.
- If we take $\mathcal{M}^s = \mathcal{M} \times \mathcal{M}$ together with its direct sum map to $\mathcal{M}$, (2) recovers the compatibility [CV,CY3,Li] between Davison/Yang-Zhao/Joyce localised/vertex coproducts and the CoHA.

Thus, this turns algebraic properties of stacks (shuffle/bialgebra-type structures) into algebraic properties on their critical cohomology. In OSp this explains the OSp-shuffle module structure on $\mathrm{H}^\bullet(\mathcal{M}^\tau, \varphi^\tau)$, and plan to generalise this in SA.

1.6. Liouville quantum gravity to vertex algebras (LCG)

History. In recent years, probabilists have increasingly understood quantum field theory, giving rigorous definitions of Feynman measures for $2d$ CFTs, e.g. [CRV; DS; Sh] whose "holomorphic part" are expected to be W-algebras, Virasoro, and Heisenberg vertex algebras.

This approach is very different to the factorisation/vertex algebra/functorial QFT approach in the above projects, e.g. it can directly study level sets of fields as SLE curves [MS; SS], there is a rigorous connection to combinatorial toy models like the discrete Gasussian Free Field [BPR], and it is able to access the *full* CFT, not just the chiral part as we are in geometric representation theory, e.g. [KRV] proves the *DOZZ* formula for full OPEs in the Liouville CFT.

However, there is currently not much interaction between the two approaches, and this project aims to build a bridge between the two so that techniques/results/heuristics can move between subjects more easily (then give a simple example of this).

Goal. In LCG we aim to define a functor from Segal-style $2d$ conformal field theories to vertex algebras

$$\mathrm{CFT} \overset{(-)^{ch}}{\to} \mathrm{CFT}^{hol} \overset{\mathrm{Res}}{\to} \mathrm{FactAlg}(\mathbf{C})^{hol}_{\mathbf{C}\rtimes\mathbf{C}^\times} \overset{[\mathrm{CG}]}{\to} \mathrm{VertexAlg}, \tag{3}$$

then show that the Gaussian Free Field and Liouville Quantum Gravity Segal CFTs are sent to the Heisenberg and Virasoro vertex algebras, repectively.

[10] i.e. shuffle algebra in the category of spaces, see SA.

Details. We will need to upgrade $\mathcal{Z} \in$ CFT to a definition that remembers the geometric structure on the category Cob_2 of conformal cobordisms. Namely, consider a complex vector bundle $\mathcal{V}$ with connection over the Teichmuller space $\mathcal{T}_{g,n}$ satisfying a factorisation condition, and with a section ψ. The fibre of this data over Σ is the vector space $\mathcal{Z}(\partial\Sigma)$ and $\mathcal{Z}(\Sigma) : \mathbf{C} \to \mathcal{Z}(\partial\Sigma)$.

The induced factorisation algebra over $\mathbf{C}$ is automatically smoothly translation and rotation equivariant, so if it is *holomorphic* (i.e. $\partial_{\bar{z}}\psi = 0$) then it is by [CG] a vertex algebra; these are the last two maps in (3). The equivariance comes from a G-action on $\mathcal{T}_{0,n}$, since then the Lie algebra $\mathfrak{g}$ acts on $\mathcal{V}$ by the connection, e.g. the vertex algebras in the image of (3) will automatically have an action by vector fields on $\mathbf{P}^1$, so we expect they are VOAs.

The main task is to define a chiralisation functor $(-)^{ch}$ to holomorphic CFTs, and prove that [GKRV]'s LQG Segal CFT (upgraded appropriately in the above sense) is sent by (3) to the Virasoro vertex algebra, and relate the DOZZ formula [KRV, (1.12)] to the Virasoro OPE. Having done this, we plan to do the same for the GFF, and finally to give a new example of these methods, construct a probability measure in the domain of (3) recovering the affine vertex algebra, e.g. by using the free field embedding [FB, $\S$11] to a direct sum of Heisenberg algebras.

References

[AGT] Luis F Alday, Davide Gaiotto, and Yuji Tachikawa. "Liouville correlation functions from four-dimensional gauge theories". In: *Letters in Mathematical Physics* 91.2 (2010), pp. 167–197.

[AN] Raschid Abedin and Wenjun Niu. "Yangian for cotangent Lie algebras and spectral R-matrices". In: *arXiv preprint arXiv:2405.19906* (2024).

[BCDN] Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu. "3d mirror symmetry of braided tensor categories". In: *arXiv preprint arXiv:2304.11001* (2023).

[BD] Tommaso Maria Botta and Ben Davison. "Okounkov's conjecture via BPS Lie algebras". In: *arXiv preprint arXiv:2312.14008* (2023).

[BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. "Instanton moduli spaces and $\mathcal{W}$-algebras". In: *arXiv preprint arXiv:1406.2381* (2014).

[Bo] R. E. Borcherds. "Quantum Vertex Algebras". In: *Advanced Studies in Pure Mathematics*. Vol. 31. Mathematical Society of Japan, 1999, pp. 51–74.

[BPR] Roland Bauerschmidt, Jiwoon Park, and Pierre-François Rodriguez. "The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature". In: *The Annals of Probability* 52.4 (2024), pp. 1360–1398.

[BR] Samuel Belliard and Vidas Regelskis. "Drinfeld J presentation of twisted Yangians". In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 13 (2017), p. 011.

[Br] Tom Bridgeland. "Geometry from donaldson-thomas invariants". In: *arXiv preprint arXiv:1912.06504* (2019).

[BS] Roland Bittleston and David Skinner. "Gauge theory and boundary integrability". In: *Journal of High Energy Physics* 2019.5 (2019), pp. 1–53.

[CF] Li Chen and Yuchen Fu. "An Extension of the Kazhdan-Lusztig Equivalence". PhD thesis. Harvard University, 2022.

[CG] Kevin Costello and Owen Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*. Vol. 31. New Mathematical Monographs. Cambridge University Press, Cambridge, 2017, pp. ix+387.

[Ch] Kuo-tsai Chen. "Iterated integrals of differential forms and loop space homology". In: *Annals of Mathematics* 97.2 (1973), pp. 217–246.

[Co] Kevin Costello. "M-theory in the Omega-background and 5-dimensional non-commutative gauge theory". In: *arXiv preprint arXiv:1610.04144* (2016).

[CRV] Baptiste Cerclé, Rémi Rhodes, and Vincent Vargas. "Probabilistic construction of Toda conformal field theories". In: *arXiv preprint arXiv:2102.11219* (2021).

[CWY] Kevin Costello, Edward Witten, and Masahito Yamazaki. "Gauge theory and integrability, I". In: *arXiv preprint arXiv:1709.09993* (2017).

[Daa] Ben Davison. "The critical CoHA of a quiver with potential". In: *Quarterly Journal of Mathematics* 68.2 (2017), pp. 635–703. arXiv: arXiv:1311.7172 `[math.AG]`.

[Dab] Ben Davison. "Affine BPS algebras, W algebras, and the cohomological Hall algebra of $\mathbf{A}^2$". In: *arXiv preprint arXiv:2209.05971* (2022).

[Dr] Vladimir Gershonovich Drinfeld. "Hopf algebras and the quantum Yang–Baxter equation". In: *Doklady Akademii Nauk*. Vol. 283. 5. Russian Academy of Sciences. 1985, pp. 1060–1064.

[DS] Bertrand Duplantier and Scott Sheffield. "Liouville quantum gravity and KPZ". In: *Inventiones mathematicae* 185.2 (2011), pp. 333–393.

[EK] P. Etingof and D. Kazhdan. "Sel. Math., New Ser. 6, No. 1, 105–130". In: *Selecta Mathematica, New Series* 6.1 (2000), pp. 105–130.

[FB] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*. 88. American Mathematical Soc., 2004.

[FMW] Keegan J Flood, Mauro Mantegazza, and Henrik Winther. "Jet functors in noncommutative geometry". In: *arXiv preprint arXiv:2204.12401* (2022).

[FR] Edward Frenkel and Nicolai Reshetikhin. "Towards deformed chiral algebras". In: *arXiv preprint* (1997). arXiv: q-alg/9706023 [q-alg].

[Ga] Dennis Gaitsgory. "On factorization algebras arising in the quantum geometric Langlands theory". In: *Advances in Mathematics* 391 (2021), p. 107962.

[GKRV] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. "Segal's axioms and bootstrap for Liouville Theory". In: *arXiv preprint arXiv:2112.14859* (2021).

[GLW] Sachin Gautam, Valerio Toledano Laredo, and Curtis Wendlandt. "The meromorphic R-matrix of the Yangian". In: *Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin*. Springer, 2021, pp. 201–269.

[Gr] James Alexander Green. *Shuffle algebras, Lie algebras and quantum groups*. Vol. 9. Departamento de Matemática da Universidade de Coimbra, 1995.

[GRZ] Davide Gaiotto, Miroslav Rapčák, and Yehao Zhou. "Deformed Double Current Algebras, Matrix Extended $\mathcal{W}_{1+\infty}$ Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection". In: *arXiv preprint arXiv:2309.16929* (2023).

[Joa] Dominic Joyce. "Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces". In: *Incomplete work* (2018).

[Job] Dominic Joyce. "Enumerative invariants and wall-crossing formulae in abelian categories". In: *arXiv preprint arXiv:2111.04694* (2021).

[KK] Tasuki Kinjo and Naoki Koseki. *Cohomological χ-independence for Higgs bundles and Gopakumar-Vafa invariants*. 2023. arXiv: 2112.10053 [math.AG].

[KMT] Yasushi Komori, Kohji Matsumoto, and Hirofumi Tsumura. "A study on multiple zeta values from the viewpoint of zeta-functions of root systems". In: *Functiones et Approximatio Commentarii Mathematici* 51.1 (2014), pp. 43–46.

[KPS] Tasuki Kinjo, Hyeonjun Park, and Pavel Safronov. "Cohomological Hall algebras for 3-Calabi-Yau categories". In: *arXiv preprint arXiv:2406.12838* (2024).

[KRV] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. "Integrability of Liouville theory: proof of the DOZZ formula". In: *Annals of Mathematics* 191.1 (2020), pp. 81–166.

[KS] Maxim Kontsevich and Yan Soibelman. "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants". In: *arXiv preprint arXiv:1006.2706* (2010).

[Ku] Atsuo Kuniba. "Matrix product solutions to the G 2 reflection equation". In: *Journal of Integrable Systems* 3.1 (2018), xyy008.

[Li] Henry Liu. "Multiplicative vertex algebras and quantum loop algebras". In: *arXiv preprint arXiv:2210.04773* (2022).

[Lu] Jacob Lurie. *Higher Algebra*. Preprint, available at http://www.math.harvard.edu/~lurie. 2016.

[LW] Ming Lu and Weiqiang Wang. "$\imath$Hall algebras and $\imath$ quantum groups". In: *arXiv preprint arXiv:2209.12416* (2022).

[Maa] Shahn Majid. "Transmutation theory and rank for quantum braided groups". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 113. 1. Cambridge University Press. 1993, pp. 45–70.

[Mab] Shahn Majid. "Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$". In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 125. 1. Cambridge University Press. 1999, pp. 151–192.

[Mi] Antun Milas. "Generalized multiple q-zeta values and characters of vertex algebras". In: *arXiv preprint arXiv:2203.15642* (2022).

[MMSV] Anton Mellit, Alexandre Minets, Olivier Schiffmann, and Eric Vasserot. "Coherent sheaves on surfaces, COHAs and deformed $\mathcal{W}_{1+\infty}$-algebras". In: *arXiv preprint arXiv:2311.13415* (2023).

[MO] Davesh Maulik and Andrei Okounkov. "Quantum groups and quantum cohomology". In: *arXiv preprint arXiv:1211.1287* (2012).

[MS] Shahn Majid and Francisco Simão. "Quantum jet bundles". In: *Letters in Mathematical Physics* 113.6 (2023), p. 120.

[RSYZ] Miroslav Rapčák, Yan Soibelman, Yaping Yang, and Gufang Zhao. "Cohomological Hall algebras, vertex algebras and instantons". In: *Communications in Mathematical Physics* 376.3 (2020), pp. 1803–1873.

[Sh] S Sheffield. "Gaussian free fields for mathematicians. preprint". In: *arXiv preprint math.PR/0312099* (2003).

[SS] O. Schramm and S. Sheffield. "SS". In: *Annals of Probability* 33.6 (2005), pp. 2127–2148.

[SV] Olivier Schiffmann and Eric Vasserot. "On cohomological Hall algebras of quivers: generators". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2020.760 (2020), pp. 59–132.

[Sz] Balázs Szendroi. "Cohomological Donaldson–Thomas theory". In: *Proceedings of String-Math 2015* (2014).

[YZ] Yaping Yang and Gufang Zhao. "Cohomological Hall algebras and affine quantum groups". In: *Selecta Mathematica* 24.2 (2018), pp. 1093–1119.