

# Random graphs : long problems

All of Mattekollo

# Domino tilings: long problems

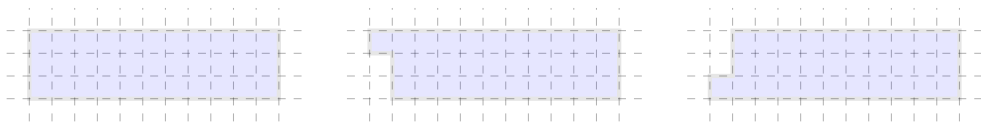
## All of Mattekollo

- Attempt problems in any order.
- Sometimes problems will use definitions introduced in later classes. Feel free to use one problem you've solved to solve another, but if it trivialises the problem, see if you can find another proof as well!
- More long problems, or if necessary, hints, will be added a bit each lesson.

### Formulas.

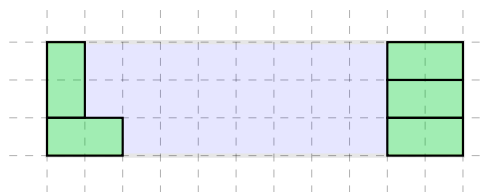
#### 1. Height three rectangles.

- Find a recurrence relation relating the numbers  $A_n, B_n, C_n$  of domino tilings of



where the base has length  $n$ .

- Use it to find a formula for  $A_n$  in terms of  $A_{n-1}, A_{n-2}, \dots, A_1$ .
  - Show that  $A_n, B_n, C_n = 0$  if  $n$  is odd.
  - Compute  $A_n$  for  $n = 1, 2, \dots, 10$ . Can you find a formula for the number of domino tilings of the  $3 \times n$  rectangle?
2. There are three possibilities for what a domino tiling looks of the  $3 \times n$  rectangle at the left (or right) boundary, e.g. one left and one right boundary is



Let  $B_L, B_R$  be boundary states on the left and right. For each choice of  $B_L, B_R$ , what is the probability

$$\mathbf{P}(B_L \rightarrow B_R)$$

that a random domino tiling with state  $B_L$  on the left has state  $B_R$  on the right?

Make sure you do the  $2 \times n$  case first!

3. Prove that the number of tilings of the  $n \times n$  Aztec diamond is  $2^{n(n-1)/2}$ .

One method (no guarantee it works!): can you solve this by finding a set  $S$  of size  $\binom{n}{2} = n(n-1)/2$  and a bijection

$$\{\text{domino tilings of the Aztec diamond}\} \simeq \{\text{finite subsets of } S\}?$$

4. Prove that the number of domino tilings of a dual  $A \times B \times C$  hexagonal graph is

$$\prod_{a=1}^A \prod_{b=1}^B \prod_{c=1}^C \frac{a+b+c-1}{a+b+c-2}.$$

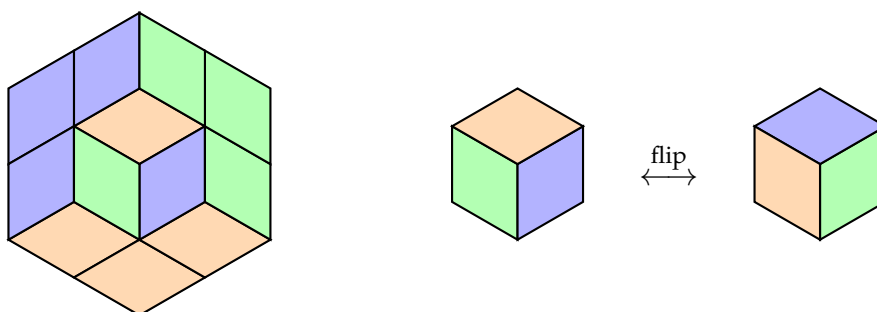
5. What is the expected number of 3d boxes in (the 3d interpretation of) a domino tiling of the dual  $A \times B \times C$  hexagonal graph?
6. What is the number of domino tilings of the  $m \times n$  rectangle?

#### Lattice paths.

7. Count the number of tilings of the  $3 \times n$  rectangle in terms of the number of paths on a certain graph.

a. Attempt if you know matrices. Define a matrix  $M_3$  such that the top left entry of  $M_3^n$  is the number of domino tilings of the  $3 \times n$  rectangle.

8. Show that any tiling of the hexagonal  $A \times B \times C$  graph by lozenges can be obtained from any other by a sequence of *flips*



- a. What does the hexagonal flip  $T \leftrightarrow T'$  do to the height function? i.e. how is  $h_{T'}$  related to  $h_T$ ?
- b. What about the  $m \times n$  rectangle? What about any square graph (subgraph of  $\mathbb{Z}^2$ )? Define a height function for the  $m \times n$  rectangle which has an analogous property to a.
- c. Can you define analogous paths in the  $m \times n$  rectangle attached to a tiling  $T$ ?

#### Determinants.

9. Colour the faces of the hexagonal graph in alternating white and black colours. Write  $N$  for the number of white or black squares (which we assume are equal, otherwise there are no domino tilings). Let  $K$  be the  $N \times N$  matrix of numbers

$$K_{i,j} = \begin{cases} 1 & \text{if the } i\text{th white vertex is adjacent to the } j\text{th black} \\ 0 & \text{otherwise.} \end{cases}.$$

Show that  $|\det K|$  is the number of domino tilings of the hexagonal graph by lozenges.

Can you find a matrix  $\tilde{K}$  that works for any planar bipartite graph?

**Random walks through the graph of all tilings.**

How do you in practice find a random lozenge tiling (there are *so many!*)? *Just start with any lozenge tiling, and randomly flip!*

**Theorem 1.** Let  $n$  be an integer and  $T_0$  any lozenge tiling. Pick

- $t \in \{0, 1, \dots, n\}$  randomly (uniformly),
- a random sequence of  $t$  many flips  $T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_t$ , at each step choosing all allowed flips equally likely.

Then if

$$P(T) : \{\text{lozenge tilings}\} \rightarrow [0, 1]$$

is the probability that  $T_t = T$ , we have

$$P(T) \rightarrow \frac{1}{|\{\text{lozenge tilings}\}|}$$

as  $n \rightarrow \infty$ .

**10.** *Good to practice linear algebra skills.* Let's prove Theorem 1.

a. Let  $G$  be an undirected graph, with set of vertices denoted by  $V(G)$ . Let

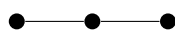
$$p_t(x) : V(G) \times \mathbf{N} \rightarrow [0, 1] \tag{1}$$

be the probability that, at time  $t = 0, 1, 2, \dots$ , a random walk starting at vertex  $i$  is at a given vertex  $x$ . Show that  $p_t$  satisfies

$$p_{t+1}(x) = \sum_{y \sim x} \frac{1}{\deg(y)} p_t(y) \tag{2}$$

and denote by  $A : \mathbf{C}^{V(G)} \rightarrow \mathbf{C}^{V(G)}$  the associated linear map.

b. Compute (and graph) for low values of  $t$  what  $p_t$  is for the graph



if we start at the middle, or edge. Conjecture what happens as  $t \rightarrow \infty$ .

c. Show that the function  $\lambda^t \cdot p_0$ , where  $p_0 : V(G) \rightarrow \mathbf{C}$  is a function independent of  $t$  and  $\lambda \in \mathbf{C}$ , satisfies (2) if and only if  $\lambda = 0$ , or

$$\lambda \cdot p_0(x) = A \cdot p_0(x).$$

*In the following, you may assume that  $p_t(x)$  is a sum of solutions of this form.*

- d. Assume from now on that  $G$  is connected. Pick  $N$  such that there is a path of length at most  $N$  between any two vertices, and let

$$B = \frac{1}{N}(I + A + \cdots + A^{N-1}).$$

Show that if  $Av = \lambda v$ , that

$$B \cdot v = \frac{1}{N}(1 + \lambda + \cdots + \lambda^{N-1})v.$$

- e. Show that if  $Bv = \mu v$  for some nonzero  $v$ ,  $|\mu| < 1$  unless for  $\mu = 1$ , in which there is a unique solution

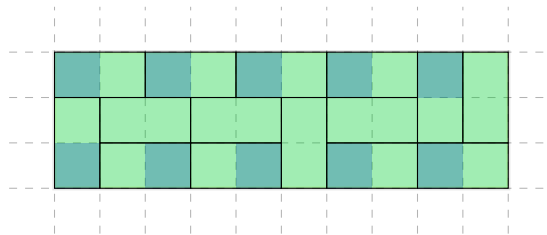
$$v = \frac{1}{|V(G)|}.$$

- f. Show that this implies Theorem 1.

**Spanning trees.**

### 11. Temperley bijection.

- a. Consider a certain square graph  $G$  within the  $n \times m$  rectangle.



When  $n = m$ , give a bijection between domino tilings of the rectangle minus a corner and spanning trees of the graph  $G$ .

- b. Can you find a similar result if you do not remove a corner?  
 c. What about if you remove an arbitrary subset of the boundary?  
 d. Can you get this to work for other  $n, m$ ?

### 12. Relation to entropy and Catalan's constant.

**Dual graphs.**

### 13. Let $G$ be a graph drawn on the sphere.

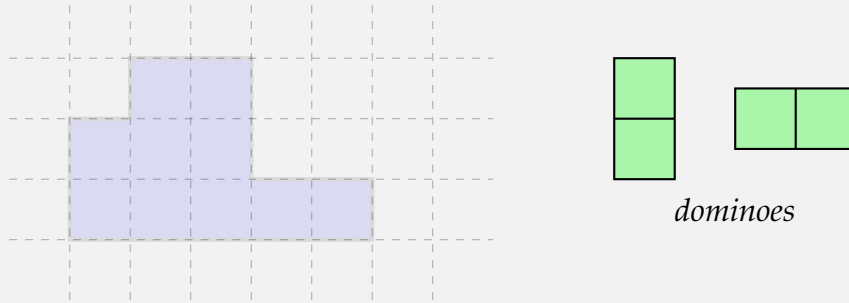
- a. Show that the *dual graph*  $\hat{G}$  is also a graph inside the sphere, and that  $\hat{\hat{G}} = G$ .  
 b. Show that  $G$  has no cycles if and only if  $\hat{G}$  is connected, and that  $G$  is connected if and only if  $\hat{G}$  has no cycles.  
 c. The *Euler characteristic* of a graph  $G$  with faces is defined as  $\chi(G) = |V(G)| - |E(G)| + |F(G)|$ , where  $V(G)$  is the set of vertices,  $E(G)$  is the set of edges, and  $F(G)$  is the set of faces. Show that  $\chi(G) = \chi(\hat{G})$ .

Now let  $G$  be a graph on the donut with  $g > 0$  many holes. Is a. - c. still true? Prove it or find a counterexample.

# 1. Domino tilings

1 July

**Definition.** A *domino tiling* of a region is a way of covering it completely using  $1 \times 2$  or  $2 \times 1$  dominoes, without overlaps or gaps.



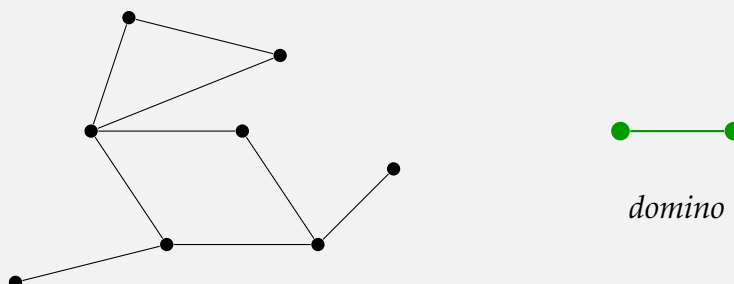
## 1. Counting domino tilings.

- How many domino tilings are there of the rectangles:  $2 \times 2$ ,  $2 \times 3$ ,  $2 \times 4$ ,  $2 \times 5$ ?
- Show that the number of domino tilings of a  $2 \times n$  rectangle is the  $n$ th Fibonacci number  $F_n$ . Conclude that

$$F_{n+m} = F_n F_m + F_{n-1} F_{m-1}.$$

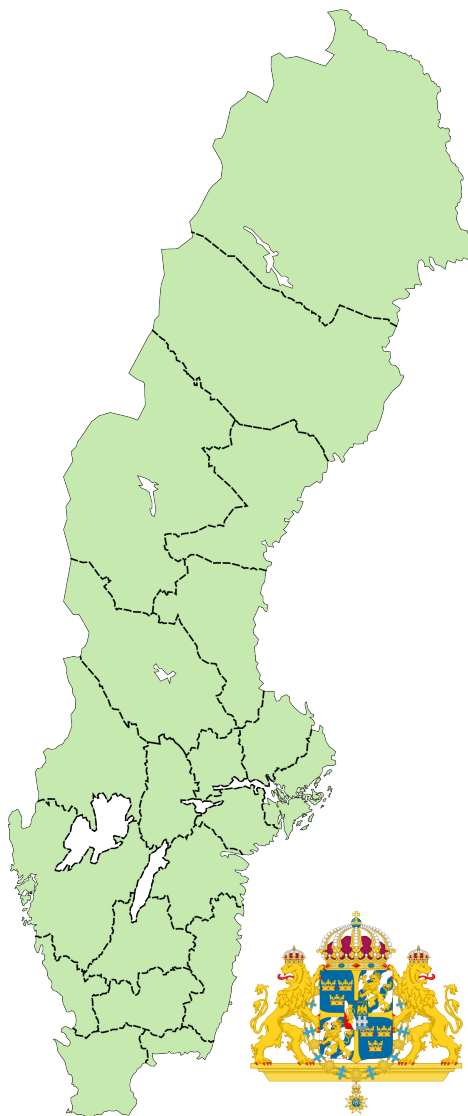
- How many domino tilings are there of a  $3 \times n$  rectangle ( $n = 2, 4, 7, 10$ )? Try to do the bigger numbers in a non-bashy way.
- How many tilings are there of the Aztec diamond with only one row of maximal length?
- How many tilings are there of the Aztec diamond with base length  $2, 4, 6$ ?

**Definition.** A *domino tiling* of a graph is a way of covering the vertices completely using dominoes, without overlaps or gaps.



Show that when the graph is a square grid graph, this is the same as the first definition of domino tiling.

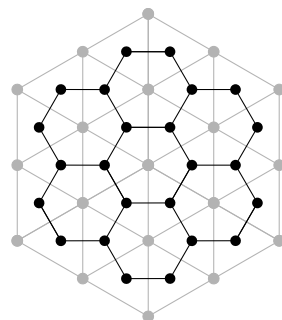
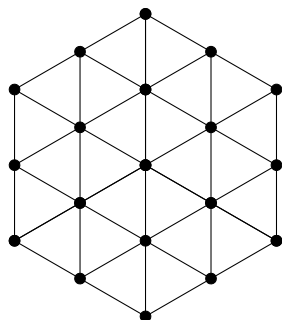
2. Consider the map of Sweden's Län



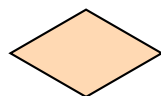
a. Draw its *dual graph*, which has one vertex for each region, and an edge whenever two regions share a 1d border (not just touch at a point).

3. Examples coming from 3d geometry.

a. Consider the hexagon graph and its dual

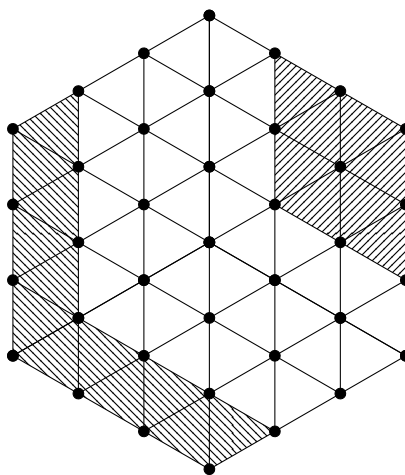


Draw three possible domino tilings of the dual graph, and show that they correspond to colouring in the regions of the hexagonal graph in by *lozenges*



*lozenge*

- b. How many domino tilings are there of the duals to the  $2 \times 2 \times 2$  and  $3 \times 3 \times 3$  hexagonal graphs?
- c. Give a 3D geometric description of such a tiling.
- d. If we remove the following region, how many domino tilings are there of the dual graph?





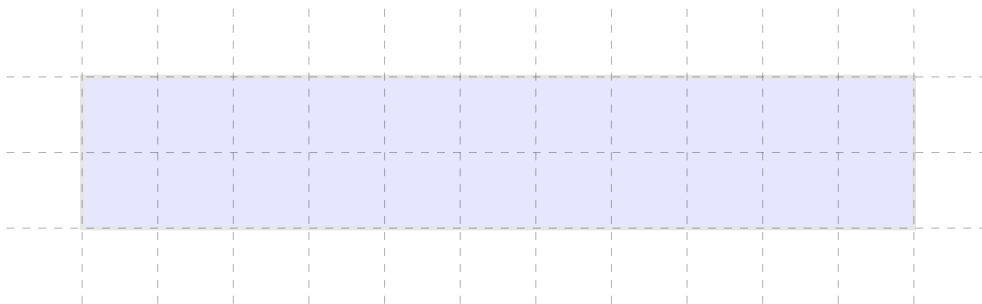
## 2. Random domino tilings

2 July

**Definition.** A *random domino tiling* of a finite region is a probability distribution (on the set of all tilings) where each domino tiling  $T$  is equally likely

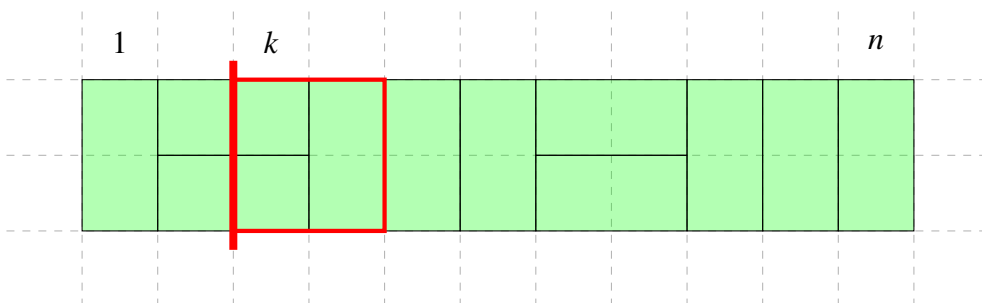
$$\mathbf{P}(\text{tiling } T) = \frac{1}{\#\{\text{all tilings } T'\}}.$$

1. Random domino tilings of  $2 \times n$  rectangle.



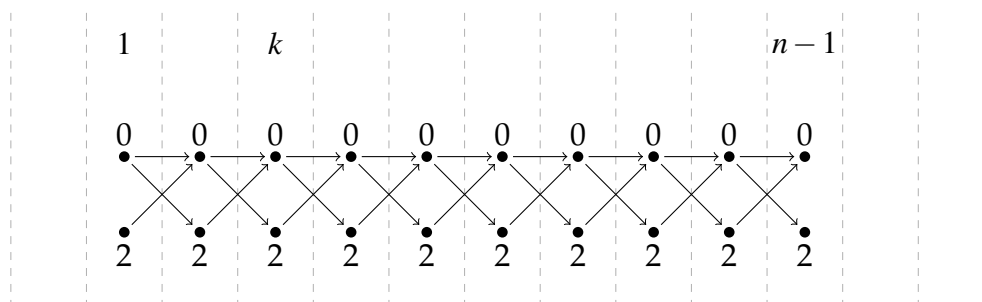
- Draw all domino tilings of the  $2 \times 5$  rectangle. Compute the probability that the leftmost domino is vertical.
- What is the answer for the  $2 \times n$  rectangles?
- If the leftmost domino of an  $2 \times n$  rectangle is vertical, what is the probability that the rightmost domino is vertical?
- What is the limit of b. and c. as  $n \rightarrow \infty$ ?

2. Take a domino tiling, and consider the sequence of boxes, labelled by  $k = 1, \dots, n-1$ .



- Let  $f : \{0, 1, \dots, n\} \rightarrow \{0, 2\}$  be the function counting the number of horizontal dominoes completely contained in the  $k$ th box. Show that  $f$  uniquely determines the domino tiling.

- b. Give a bijection between domino tilings of the  $2 \times n$  rectangle and maximal paths on the following graph:



- c. Give another proof that this is the  $n$ th Fibonacci number.  
d. *Attempt if you know matrices.* Define the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that the number of domino tilings of the  $2 \times n$  rectangle is equal to the top left entry  $a_n$  of  $M^n = M \times \cdots \times M$ .

- e. *Attempt if you know matrices.* Show that

$$M^2 = M + 1$$

hence show that  $a_{n+2} = a_{n+1} + a_n$ .

- f. *Attempt if you know matrices.* Consider the top-left entry  $b_n \in \mathbf{Z}[x, y]$  of  $N^n$ , for the matrix

$$N = \begin{pmatrix} x & y \\ y & 0 \end{pmatrix}.$$

Compute  $b_1, b_2, b_3, b_4$ . What is the interpretation of the  $x^i y^j$ -coefficient of  $b_n$ ? Show that  $b_{n+2} = x b_{n+1} + y^2 b_n$ . Find a formula for  $b_n$  of the form

$$b_n = c_1 \cdot \varphi_1^n + c_2 \cdot \varphi_2^n.$$

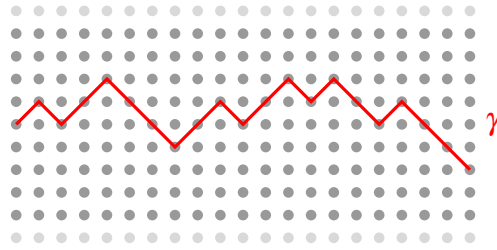
*Hint: find two solutions of the form  $\varphi_i^n$ , to the equation, then use  $b_1, b_2$  to find  $c_1, c_2$ .*

**Definition.** A *walk* on a graph  $G$  is a sequence of vertices  $v_0, v_1, \dots, v_t$ , where each vertex is adjacent to the next. A *random walk* is a probability distribution on the set of all walks, where each walk is equally likely,

$$\mathbf{P}(\text{walk } v_0, \dots, v_t) = \frac{1}{\#\{\text{all walks}\}}.$$

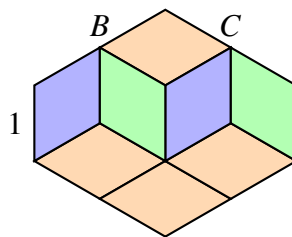
One can similarly define a *random walk starting at a fixed vertex  $v$* , or a random walk with some other set of properties.

3. Random walks. Consider a random walk on  $\mathbf{Z}$  of length  $t$ , which we draw as a path in  $\mathbf{Z} \times \{0, 1, \dots, t\}$ .



- How many walks of length  $t$  are there?
- What is the probability that a random walk  $\gamma$  has  $\gamma(t) \geq 0$ ?
- What is  $\mathbf{E}(\gamma(t))$ ?
- What is  $\mathbf{E}(\gamma(t)^2)$ ? In plain English, approximately how far away from the origin should we expect  $\gamma$  to be after  $t$  steps?

4. Show that the set of tilings of the  $1 \times B \times C$  hexagon by lozenges



is equivalent to a certain set of walks in  $\mathbf{Z}$  (which you should define precisely).

- Picking a random tiling (or equivalently, a random walk), what is the expected number of 3d cubes in a random  $1 \times B \times C$  lozenge tiling?
- Given a tiling  $T$  on the hexagonal graph  $G$ , we define the *height function*

$$h_T : V(G) \rightarrow \mathbf{N}$$

whose value is the number of 3d boxes under ( $\downarrow$ ) that vertex. Describe the walk attached to  $T$  in terms of  $h_T$ .

- If you're interested in continuing, try the *lattice paths long problem*.

### 3. Random dominoes 3

3 July

Probability.

**Definition.** A *probability space* is a countable (i.e. finite or has as many elements as  $\mathbf{N}$ ) set  $\Omega$  together with a function

$$\mathbf{P} : \{\text{subsets of } \Omega\} \rightarrow [0, 1]$$

such that

1.  $\mathbf{P}(\Omega) = 1$ ,
2. if  $A_1, A_2, \dots$  are disjoint subsets of  $\Omega$ , then  $\mathbf{P}(A_1 \cup A_2 \cup \dots) = \mathbf{P}(A_1) + \mathbf{P}(A_2) + \dots$ .

If  $X : \Omega \rightarrow \mathbf{R}$  is a function, its *expectation* is defined to be

$$\mathbf{E}(X) = \sum_{a \in \Omega} \mathbf{P}(\{a\}) \cdot X(a).$$

A function  $X$  is called a *random variable*, or a *random real number*.

1. Consider one white and one black 6-sided dice, both with faces numbered 1, 2, 3, 4, 5, 6.



- a. Draw the set  $\Omega$  of possible outcomes of rolling the two dice (in a  $6 \times 6$  table).
- b. Let  $W, B : \Omega \rightarrow \mathbf{R}$  denote the values of the white and black dice, respectively. What are the expectations

$$\mathbf{E}(W), \mathbf{E}(B), \mathbf{E}(10W + 2B)?$$

What are *variances*

$$\mathbf{E}((W - \mathbf{E}(W))^2), \mathbf{E}((B - \mathbf{E}(B))^2), \mathbf{E}((10W + 2B - \mathbf{E}(10W + 2B))^2)?$$

Write down on paper a plain English/Swedish description of what this means.

- c. The *correlator*

$$\mathbf{E}((W - \mathbf{E}(W))(B - \mathbf{E}(B)))$$

measures how much information you get about the value  $W(a)$  if you know the value  $B(a)$ , and vice versa. Draw the values of the function

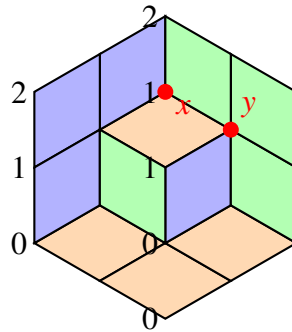
$$(W - \mathbf{E}(W))(B - \mathbf{E}(B)) : \Omega \rightarrow \mathbf{R}$$

on the  $6 \times 6$  table. What is the value of this correlator?

- d. The dice are now enchanted, so that the rolls are never more than 4 apart. Other than that all pairs of rolls are equally likely. What is the new answer to a., b. and c.?

**Back to random domino tilings.**

## 2. Correlators.



- Show that there are 20 lozenge tilings of the  $2 \times 2 \times 2$  hexagon.
- Write down the *height function*  $h_T : V(G) \rightarrow \mathbf{N}$  of two tilings  $T$ . Here  $V(G)$  is the vertex set of the hexagonal graph, some heights are given above for the above tiling.
- What are the expected values  $\mathbf{E}(h_T(x))$  and  $\mathbf{E}(h_T(y))$  for a random tiling  $T$ ?
- What is the value of the *correlator*

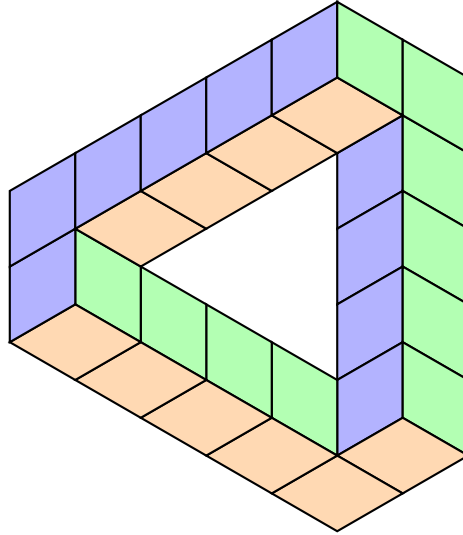
$$C(x, y) = \mathbf{E}(h_T(x)h_T(y)) - \mathbf{E}(h_T(x))\mathbf{E}(h_T(y))$$

for the two points  $x, y$ ?

## 4. Random dominoes 4

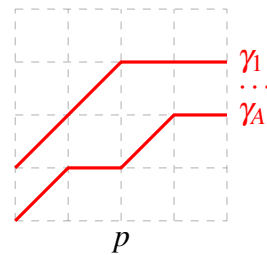
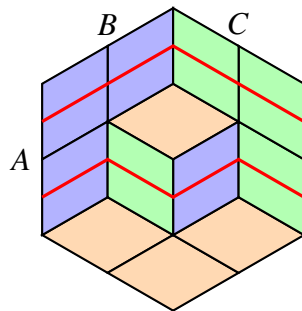
4 July 2025

We'll begin with a proof that lozenge tilings of the  $A \times B \times C$  hexagonal graph biject with 3d cube stackings of the  $A \times B \times C$  3d box. Be careful - it's not true if for certain subgraphs of the hexagonal graph!



### 1. Hexagonal paths.

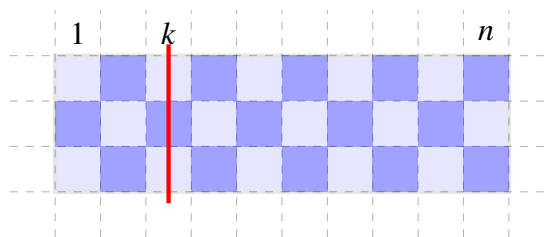
- Give a bijection between tilings of the  $A \times B \times C$  hexagon and certain a certain set of paths in  $\mathbb{Z}^2$  (which you should precisely specify).



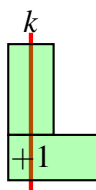
- What are the graphs corresponding to the lozenge tilings with the maximum and minimum number of 3d boxes?
- How can you see the number of 3d boxes from the paths?
- What is the expected value of  $\gamma_1(p), \dots, \gamma_A(p)$  for the  $2 \times 2 \times 2$  hexagon?

In the next problem we will find an analogous set of paths for the domino tilings of the  $m \times n$  rectangle.

## 2. Flux.



Colour the squares of an  $n \times m$  rectangle in a chessboard pattern. The *flux*  $\Phi(k)$  through the  $k$ th slice of a domino tiling is the sum of  $\pm 1$  for each domino perpendicular to the slice



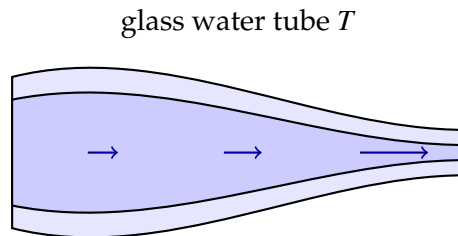
where it is  $+1$  or  $-1$  depending on the colour of the square. (The word “flux” means “flow”, like the amount of water flowing through a pipe.)

- Pick an arbitrary domino tiling of a  $3 \times n$  and a  $4 \times n$  square, and draw  $\Phi(k)$  for each.
- What do you notice about the values of  $\Phi(k)$ ? Form a conjecture, and prove it.
- For a random domino tiling of the  $2 \times n$  and  $3 \times n$  rectangle, what is the expected value of the flux  $\Phi$ ?

## 5. Random graphs : networks

6 July

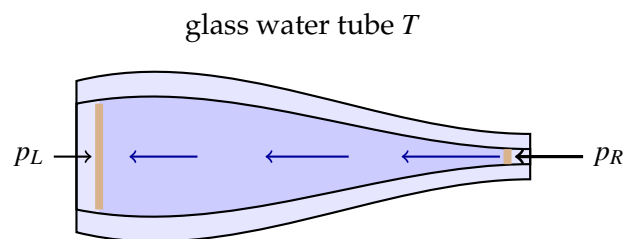
Imagine a water flowing through a glass tube



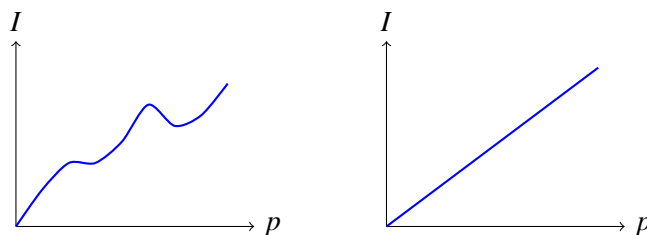
In the above, the speed of individual water molecules is marked.

**Axiom. (Water is incompressible)** The volume per second  $I$  of water flowing through each vertical slice in the tube is the same.

Now place two frictionless wooden sheets at each end, and push different amounts on each end



The flow of water  $I$  through the tube in response to the difference  $p = p_L - p_R$  in pressure forms some graph, e.g.



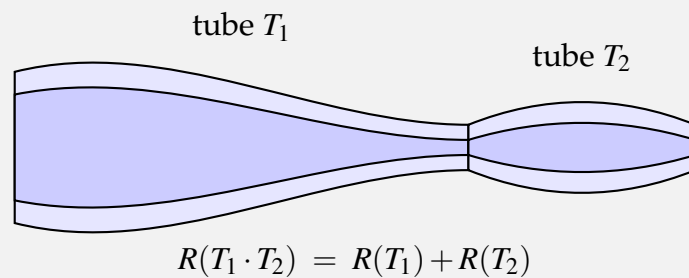
The *resistance* of a tube  $T$  is  $R(T) = \frac{d(p_L - p_R)}{dI}$ . We will make the simplifying **assumption** that all above graphs are linear (which is approximately true for small  $p, I$ ), so

**Definition.** The *resistance* of a tube  $T$  is the ratio

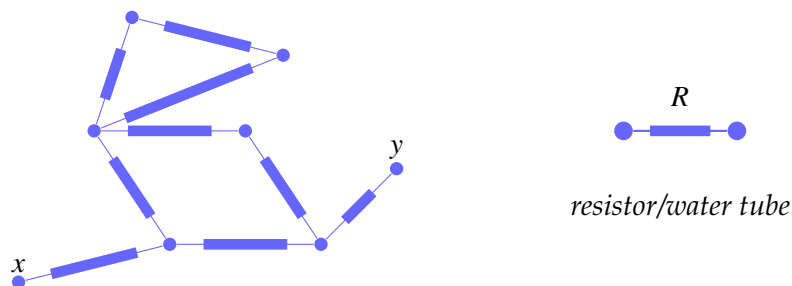
$$R(T) = \frac{p}{I}.$$



**Axiom. (Addition of resistances)** Putting two tubes  $T_1$  and  $T_2$  after one another adds their resistances:



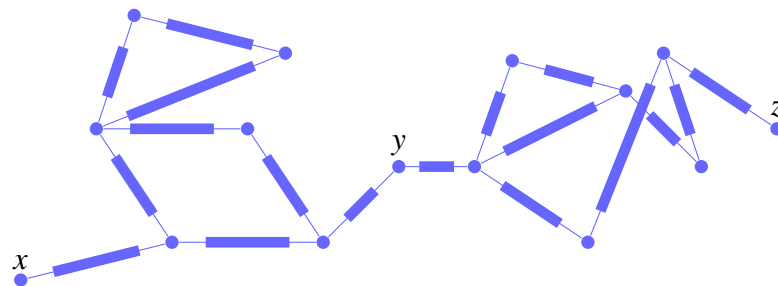
We represent a collection of tubes joined together as a graph



where the ends of the graph are considered closed off.

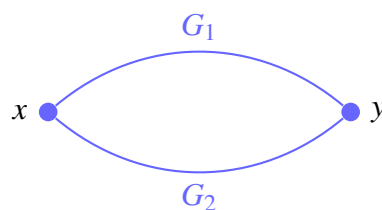
1. We open up the tubes at  $x$  and  $y$ , and add small wooden sheets to keep in the water. We will compute the resistance between  $x$  and  $y$ .

- a. Show that joining two graphs  $G_1$  and  $G_2$  together along a single vertex to form  $G_1 \cdot_y G_2$



gives  $R_{G_1 \cdot G_2}(x, z) = R_{G_1}(x, y) + R_{G_2}(y, z)$ .

- b. What happens if we join two graphs  $G_1$  and  $G_2$  together along multiple points  $y_1, \dots, y_k$ ?
- c. Show that adding two graphs in parallel to form  $G_1 + G_2$



gives  $\frac{1}{R_{G_1+G_2}(x,y)} = \frac{1}{R_{G_1}(x,y)} + \frac{1}{R_{G_2}(x,y)}$ .

d. What is the resistance of the above graph between  $x$  and  $y$ ?

2. In this problem we will show that

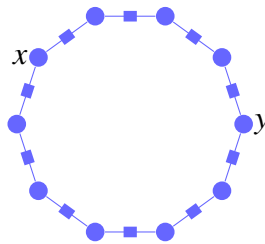
$R(x,y) = \mathbf{E}[\text{\#steps a random walk } \gamma \text{ starting at } x \text{ takes before first hitting } y]$ .

a. <https://math.dartmouth.edu/~pw/math100w13/qin.pdf>

3. Assuming problem 2, give a new proof that

- a. Parts a.-c. of Problem 1.
- b. The axiom of addition of resistances.
- c. Removing an edge increases or leaves unchanged the resistance between any two points on the network.

4. Consider the circular graph



What is the resistance between a pair of points  $x$  and  $y$ ?

## 6. Random graphs 2

6 July

5. Resistance of trees.

## 7. Random graphs 3

## 8. Random graphs 4