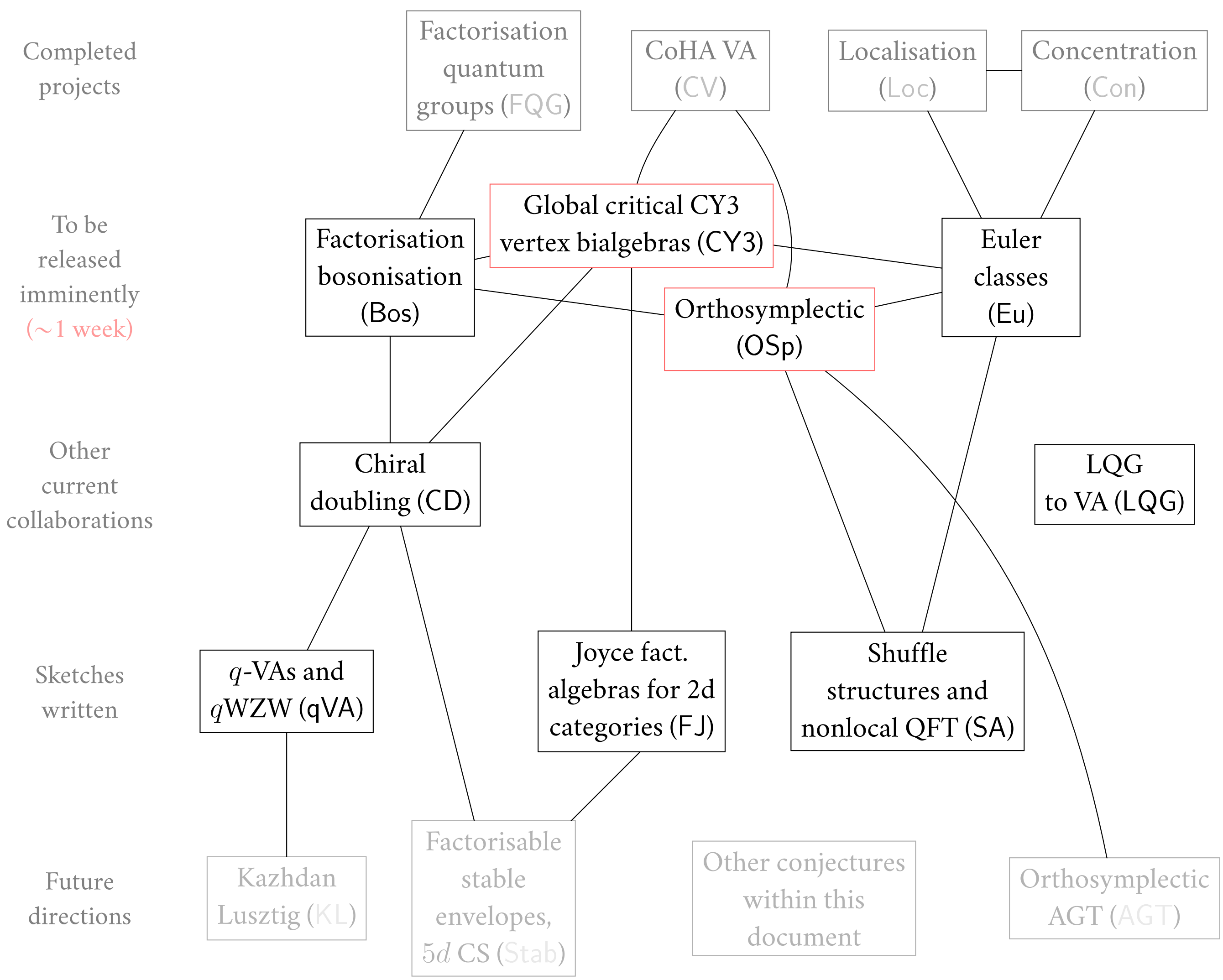
RESEARCH PLANS (7 PAGES)

ALEXEI LATYNTSEV

*For an **abridged** (4 pp.) version, see* https://alyoshalatyntsev.github.io/planabridged/planabridged.pdf.

*For a non-technical **summary** (2 pp.), see* https://alyoshalatyntsev.github.io/plansummary/plansummary.pdf.

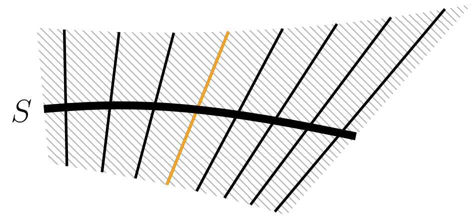


See section 2 for a the list of papers and papers in preparation corresponding to the above projects.

1. Details of projects

1.1. Algebraic structures attached to Calabi-Yau-threefolds (CV, CY3, FJ, Stab)

This series of projects asks: the assignment $Y \mapsto \mathrm{H}^\bullet(\mathcal{M}_Y, \mathcal{P}_Y)$ of a CoHA [KS; KPS; YZ] to a three-Calabi-Yau manifold or category Y, what are all the algebraic structures it has? What are the relations between them? Can we recover known structures on Yangians, $\mathcal{W}$-algebras, etc.?



The general expectation is that it should *factorise* over Y. We focus on cases where $Y = K_S$ is the canonical bundle of an algebraic surface, and the *vertex* structure on the fibres.

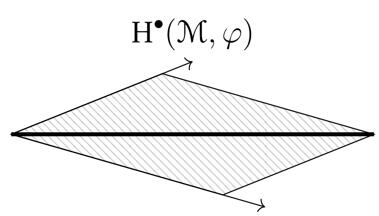
Global critical CY3 vertex quantum groups. Let $\mathcal{M}_Y$ be a *global critical locus* inside a smooth ambient stack $\mathcal{M}$. For instance Y is K_{T*C}, a local curve [KK], representations of quiver with potential; more generally or any *deformed CY3* completion.

Theorem A. [CY3] *There is a vertex coproduct on the critical CoHA*

$$\mathrm{H}^\bullet(\mathcal{M}, \varphi) \;\to\; \mathrm{H}^\bullet(\mathcal{M}, \varphi) \,\hat{\otimes}\, \mathrm{H}^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

which forms a quasitriangular vertex ***bialgebra*** *in meromorphic braided category* $\mathrm{H}^\bullet(\mathcal{M})$-Mod.[1]

Theorem B. [CY3] *View* $\mathrm{H}^\bullet(\mathcal{M}, \varphi)$ *as a quasicoherent sheaf on the coloured configuration space:*



$$\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1 \;=\; \mathrm{Spec}\,\mathrm{H}^\bullet(\mathcal{M})$$

There is a localised coproduct giving making this a quasitriangular ***factorisation bialgebra*** *over* $\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1$.

We obtain Theorem A from Theorem B by defining a **localised-to-vertex** functor between factorisation bialgebras on $\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1$ and $\mathrm{Ran}\mathbf{A}^1$. In the quiver case a factorisation bialgebra is the same a Davison localised bialgebra [Da], and

Theorem C. [CY3, CV] *In the quiver with potential case, the localised coproduct on the Yangian* $\mathrm{H}^\bullet(\mathcal{M}_Q, \varphi_W) \simeq Y_\hbar(\mathfrak{n}_Q)$ *agrees with the Davisons'/Yang-Zhao's. In the ADE case, the vertex product is* ***Drinfeld****'s meromorphic coproduct.*

[1]The formalism of braided factorisation categories is developed in FQG.

Relevance. The above is a first step to giving a *factorisable, basis-free* definition of $Y_\hbar(\mathfrak{g})$-Mod with its meromorphic and ordinary tensor products, which *generalises* to $\mathcal{A}$-Mod for $\mathcal{A}$ any affine W-algebra, W-algebra attached to a surface, or double of a CoHA. The idea: take the factorisable definition [Ga] of $U_q(\mathfrak{g})$-Mod and replace

$$U_q(\mathfrak{n}) \;=\; \mathrm{co/free}(\mathfrak{n}) \;\in\; \mathrm{Perv}^{\mathrm{fact}}(\mathrm{Conf}_\Lambda \mathbf{R}^2) \qquad \text{by} \qquad \mathrm{H}^\bullet(\mathcal{M},\varphi) \;\in\; \mathrm{QCoh}^{\mathrm{fact}}(\mathrm{Conf}_{\mathcal{M}}\mathbf{A}^1).$$

Then apply the bosonisation Bos and doubling CD to get analogues of $U_q(\mathfrak{b})$ and $U_q(\mathfrak{g})$. This may also give a conceptual proof of the PBW generation of CoHAs by BPS Lie algebras $\mathfrak{g}_{\mathrm{BPS}}$, by analogy with the above definition of $U_q(\mathfrak{n})$.

Factorisation of moduli stacks. If Q is a quiver with torus T_d, we may define the *factorisable stack* parametrising Q-representations and a torus Lie algebra element fixing it:

$$\mathcal{M}^f \;=\; \{(V,\lambda)\;:\;\lambda\in\mathfrak{t}_\infty,\; V\in\mathcal{M}^\lambda\} \;\xrightarrow{p}\; \mathfrak{t}_\infty \;\simeq\; \mathrm{Ran}_{Q_0}\mathbf{A}^1 \tag{1}$$

where $\mathrm{T}_\infty = \cup \mathrm{T}_d$ is the union over diagonal maps and $\mathfrak{t}_\infty$ is its Lie algebra.

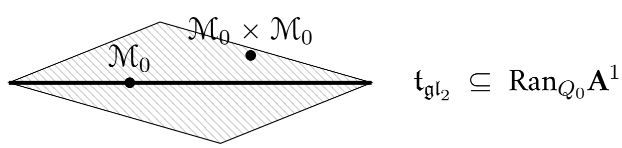
Conjecture D. [FJ] *This is a factorisation space over the Q_0-coloured Ran space. Thus, the relative critical cohomology $\mathcal{A} = \int_p \varphi_W$ defines a coloured vertex bialgebra. It agrees with the* ***nilpotent CoHA*** $\mathrm{H}^{\mathrm{BM}}_\bullet(\mathcal{M}_0)$ *of* [SV] *and its structures of Theorem A.*

Conjecture E. [FJ] *For S an appropriate algebraic surface, the moduli stack of* $\mathrm{Coh}_0(S)$ *is the total space of a factorisation stack $p:\mathcal{M}^f \to \mathrm{Ran}_S K_S$. The factorisation algebra $\mathcal{B} = \int_p \varphi_W$ recovers* ***Mellit*** *et al's W-algebra $W^+(S)$*-Mod *of* [MMSV].

Note that $\mathcal{B}$ is only a vertex algebra on opens with $K_U \simeq U\times\mathbf{A}^1$ and $\mathcal{M}|_U$ a critical locus. On the intersection $U_1\cap U_2$ the factorisation algebras $\mathcal{B}_{U_1}$ and $\mathcal{B}_{U_2}$ are only Morita-equivalent, so $\mathcal{B}$-$\mathrm{FactMod}_S$ is a sheaf of categories on S which is locally *meromorphic braided monoidal*.

Pinning down the above structure may allow one to generalise to other Calabi-Yau-threefolds Y.

A general theory of stable envelopes. Conjecture D's factorisation structure on the stack $\mathcal{M}^f$



looks a lot like Maulik-Okounkov's structures [MO] on quiver varieties $M_{\mathbf{w}}$. Second, [MO]'s contruction of the Yangian clearly looks like a Tannakian reconstruction. Thus,

Conjecture F. [Stab] $\mathrm{H}^{\mathrm{BM}}_\bullet(M_{\mathbf{w}})$ *is the global sections of a sheaf $\int_p \omega_{M^f_{\mathbf{w}}}$ on the coloured Ran space. These sheaves together form a meromorphic tensor category $\mathcal{C}_Q$ over $\mathrm{Ran}_{Q_0}\mathbf{A}^1$, with braiding $R_{\mathrm{MO}}(z)$. Applying factorisation Tannakian reconstruction of* Bos *recovers $Y_{\mathrm{MO}}(\mathfrak{g}_Q)$.*

1.2. The structure of factorisation quantum groups (FQG, Bos, CD)

There is an incongruity in the literature:

(1) There is by now a powerful general theory [EGNO; Lu; Maa] of all finite quantum groups A (quasitriangular bialgebras) and their representations.

(2) Meromorphic quantum groups like CoHAs, $U_q(\hat{\mathfrak{g}})$, $Y_\hbar(\mathfrak{g})$, $\mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, $Y_\hbar(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g}), \ldots$ still rely solely on ingenious but laborious and hard-to-generalise explicit (shuffle) formulae [Dr; MO; MMSV; SV; YZ].

The purpose of this project is to take Lurie's definition of A-Mod as a topological factorisation category over $\mathbf{R}^2$, and generalise it to a theory of *factorisation categories* over arbitrary factorisation spaces X.

Factorisation quantum groups. In FQG we define $\mathbf{E}_n$- *factorisation spaces, categories,* and *algebras*.

Theorem G. [FQG] *A braided factorisation strucutre on $\mathcal{A}$-FactMod$_X(\mathcal{C})$ is equivalent to a factorisation bialgebra structure on $\mathcal{A}$ with a* ***factorisation*** ***R-matrix*** $R : \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C},X} \mathcal{A}$.

When $X = \operatorname{Ran} \mathbf{C}$ this gives $R(z)$ satisfying the *spectral Yang-Baxter equation* [Dr; CWY], and we expect that a factorisation category $\mathcal{C}$ generalises Soibelman's meromorphic tensor categories.

Theorem H. [FQG] *When $X = \operatorname{Ran}\mathbf{A}^1$ the data of Theorem G has the structure of an Etingof-Kazhdan and Frenkel-Reshetikhin* ***quantum vertex algebra****.*

An easy consequence[2] is for semisimple complex $\mathfrak{g}$,

Theorem I. *The standard and meromorphic coproduct on $Y_\hbar(\mathfrak{g})$ and the spectral R-matrices*

$$R^{\pm}(s),\ R^{0,\epsilon}(s),\ R^{\epsilon}(s)$$

relating them [GLW] *makes $Y_\hbar(\mathfrak{g})^\vee$ into a lax quasitriangular factorisation bialgebra over $X = \operatorname{Ran}\mathbf{A}^1$.*[3]

Finally, in [FQG] we generalise **Borcherd's twist construction** to arbitrary factorisation algebras. This twists a factorisation algebra by an "interaction term" to give more interesting examples.

Relevance. To understand *affine Yangians* [GRZ] or *q-vertex algebras* (see qVA) we need a good definition of factorisation algebra on the noncommutative affine plane, and we expect this definition is best made categorically. In turn this may help us understand the *new KL equivalances* [BCDN].

Factorisation bosonisation. A lot of algebraic effort [Da; RSYZ; YZ], [CY3; OSp] is spent each time one *extends* a CoHA by adding in a "Cartan piece", e.g. to obtain Yangians of Borels $Y_\hbar(\mathfrak{b}_Q)$.

But in the finite case this is easy: one just applies Tannakian reconstruction to $U_q(\mathfrak{n})$-Mod$(\operatorname{Rep}_q T)$ to recover $U_q(\mathfrak{b})$, and this can be done for any finite quantum group [Ga; Maa; Mab].

[2] Not written in the current version of [FQG].

[3] Here we use the equivalence between vertex bialgebras and ch-$*$ factorisation bialgebras on $\operatorname{Ran}\mathbf{A}^1$.

Theorem J. [Bos, in preparation] *There is a factorisation Tannakian reconstruction functor*

$$F \ : \ \text{(braided) factorisation categories } \mathcal{C} \ \to \ \text{(quasitriangular) factorisation quantum groups } \mathcal{A}.$$

Applying this to $\mathcal{C} = Y_\hbar(\mathfrak{n}_Q)\text{-Mod}(Y_\hbar(\mathfrak{t}_Q)\text{-Mod})$ *or* $\mathrm{H}^\bullet(\mathcal{M},\varphi)\text{-Mod}(\mathrm{H}^\bullet(\mathcal{M})\text{-Mod})$ *gives the Yangian of the Borel and the extended CoHAs with its vertex bialgebra structure.*

Factorisation Drinfeld doubling. The purpose of the project [CD] is to define a factorisation analogue of the **Drinfeld centre** construction, and as a result, a functor from vertex bialegbras to vertex quantum groups. We aim to show this sends $Y_\hbar(\mathfrak{b}_Q)$ to $Y_\hbar(\mathfrak{g}_Q)$, the affine Grassmannian to the bubble affine Grassmannian, and recovers Takiff algebra double construction of [AN].

This may help understand the *doubling* of CoHAs [RSYZ; YZ] and the relation of this double to the derived category.

1.3. Orthosymplectic structures (OSp, SA, AGT)

When $\mathrm{G} = \mathrm{SO}_k, \mathrm{Sp}_{2n}$, an *(ortho)symplectic* object of $\mathcal{M}$ a moduli stack as in CY3 is an object $c \in \mathcal{C}$ with

$$\kappa \ : \ c \otimes_{\mathcal{C}} c \ \to \ 1$$

nondegenerate and (anti)symmetric.

Theorem K. [OSP] *The moduli of orthosymplectic objects is a fixed point stack* $\mathcal{M}^\tau$ *for an involution* τ. *There is a* τ*-equivariant CoHA action*

$$\mathrm{H}^\bullet(\mathcal{M},\varphi) \otimes \mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau) \ \to \ \mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau) \tag{2}$$

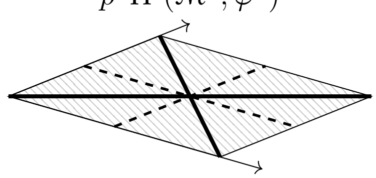
and a τ*-equivariant vertex coaction*

$$\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau) \ \to \ \mathrm{H}^\bullet(\mathcal{M},\varphi) \otimes \mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)((z^{-1})). \tag{3}$$

These together form a τ***-twisted vertex Yetter-Drinfeld module****.*

More than this, $\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$ forms an *orthosymplectic localised coalgebra*: it factorises over

$p^*\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)$



$$\mathrm{Conf}\mathbf{A}^1 \ \xrightarrow{p} \ \mathrm{Conf}_{\mathrm{G}}\mathbf{A}^1 \ = \ \mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{B}G)$$

We prove Theorem K by defining an *orthosymplectic localised to vertex* functor. Thus

$$\mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau) \ \to \ \mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau) \otimes \mathrm{H}^\bullet(\mathcal{M}^\tau,\varphi^\tau)(((w_1 \pm w_2)^{-1}))$$

in addition has the structure of a **orthogonal vertex algebra** (which we develop the theory of), e.g. OPEs have poles along the SO_k root hyperplanes.

Theorem L. [OSP] *In the quiver with potential case,* (2) *and* (3) *are given by* W_{G}*-shuffle (co)products for the Weyl group* W_{G}*.*

Another example where the maps (2) and (3) are defined is **orthosymplectic perverse-coherent** sheaves on a surface. In general, the Borcherds twist defining (3) satisfies the **Cherednik reflection equation** [BS], and so $\mathrm{H}^\bullet(\mathcal{M})$-Mod and $\mathrm{H}^\bullet(\mathcal{M}^\tau)$-Mod define a $\pm$-equivariant factorisation braided monoidal and module category.

Conjecture M. *When Q is a ADE, the subalgebra of* $\mathrm{End}(\mathrm{H}^\bullet(\mathcal{M}^\tau, \varphi^\tau))$ *generated by the CoHA is the positive part of the* ***twisted Yangian*** [BR].

Relevance. This should relate to Finkelberg-Hanany-Nakajima's ongoing work on orthosymplectic Coulomb branches (see AGT). This mathematicises *boundary 4d Chern-Simons* of [BS] on $\mathbf{R} \times (\mathbf{R} \times \mathbf{C})/\pm$. We suspect orthosymplectic analogues of affine vertex algebras, whose conformal blocks give **boundary KZ equations**.

Kac-Moody shuffle structures and "Dynkin spacetimes". What happens if we let G in OSp vary over any Kac-Moody group?

We have analogues of the **configuration space** $\sqcup_{\mathrm{G}} \mathrm{Spec}\, \mathrm{H}^\bullet(\mathrm{B}G)$, **Ran space** $\cup_{\mathrm{G}} \mathfrak{t}_{\mathrm{G}}$, hence of braided monoidal category and vertex algebra, **iterated loop space** $L_{\mathrm{G}}X = \mathrm{Maps}(T^{\mathrm{rkG}}/W_{\mathrm{G}}, X)$, quiver representation and (we expect) quiver variety.

Conjecture N. [SA] *When we restrict to the category of groups with Coxeter diagram*

 $\mathrm{KM}_{\mathrm{G}_2}$

a vertex/localised algebra is equivalent to a vertex/localised algebra equivariant for the action of

$$\tau : (z_1, z_2, z_3) \mapsto \Big(z_3 + \sqrt{3}(z_1 + z_2 - 2z_3),\ z_1 + \sqrt{3}(z_2 + z_3 - 2z_1),\ z_2 + \sqrt{3}(z_3 + z_1 - 2z_2)\Big),$$

a square root of (231) *generating* W_{G_2} *acting on* $\mathbf{C}^3$.

Likewise, restricting to $\mathrm{KM}_{\mathrm{SO}}$ we expect recovers the *orthosymplectic* vertex algebras of OSp, restricting to $\mathrm{KM}_{\widehat{\mathrm{GL}}}$ gives factorisable D-modules on $\mathbf{A}^1/\mathbf{Z}$; we expect this to relate to **trigonometric KZ equations**.

We define a *shuffle algebra* to be a monoidal functor $\mathcal{A} : \mathrm{KM}_{\mathrm{Par}} \to \mathrm{Vect}$ from the category of Kac-Moody groups [Ku] with morphisms parabolics induction diagrams. Restricting to products of GL_n's gives the usual definition [KS; Gr]. Generalising Chen's Theorem [Ch],

Conjecture O. [SA] *The cohomologies* $\{\mathrm{H}^\bullet(L_{\mathrm{G}}X)\}_{\mathrm{G}}$ *form a shuffle algebra.*

We expect this relates to Dynkin and q-analogues of multiple zeta values [KMT; Mi].

A twisted AGT correspondence. Finishing OSp, we want to generalise the *AGT corrrespondence* linking $\mathcal{W}$-vertex algebras and surfaces [Gr; Na; AGT; BFN; RSYZ].

Conjecture P. [AGT] *The equivariant intersection homology of the invariant locus* $\mathcal{U}^{\mathbf{Z}/2}_{\mathbf{P}^2, \mathrm{GL}_n}$ *in the Uhlenbeck space is a* ***Verma module*** *for an orthosymplectic analogue* $\mathcal{W}^k(\mathfrak{gl}_n)^{\mathrm{OSp}}$ *of a* $\mathfrak{gl}_n$ *W-algebra.*

We expect the proof to proceed in much the same way as in [BFN], but with with free field realisations and parabolic induction replaced by their OSp-analogue; see the shuffle discussion in SA.

Conjecture Q. [AGT] *The dimension zero CoHA $U_\hbar(\mathcal{D}(\mathbf{C}/\pm))$ of $\mathbf{A}^3$, admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\mathrm{OSp}}$ as quotients, and acts on the equivariant intersection cohomology $\mathrm{IH}^\bullet_T(M^{\mathbf{Z}/2})$ of $\mathbf{A}^3$-**instantons** in* [RSYZ].

1.4. q-vertex algebras (qVA, KL)

q-vertex algebras. Our understanding of double affine quantum groups or Kazhdan Lusztig equivalences, qKZ equations, . . . is hampered by the abscence of a good definition of q-affine vertex algebra. Note that many attempts at defining q-affine vertex algebras used q-difference modules, and:

(1) From physics, Costello [Co] suggests these should factorise over the q-affine plane $\mathbf{A}^2_q$.

(2) The derivation $y\partial_x$ defines a q-difference operator on $\mathbf{A}^1_{y=1} \subseteq \mathbf{A}^2_q$.

Conjecture R. [qVA] *Using the theory* [MS] *of jet spaces on noncommutative schemes X, one may define $\mathcal{D}$-Mod(X) and functors $f^!$, f_* for any map f satisfying the usual properties. There is a functor from $\mathcal{D}$-Mod$(\mathbf{A}^2_q)$ to the category of q-difference modules on $\mathbf{A}^1$.*

We then may define $\mathcal{D}$-Mod$(\mathrm{Ran}\mathbf{A}^2_q)$ as a (co)limit of $\mathcal{D}$-Mod$((\mathbf{A}^2_q)^n)$. We expect this carries a q-**chiral** monoidal product, allowing us to define q-vertex algebras as usual, and prove they are equivalent to concrete definitions [EK; FR] in terms of field maps with q-OPEs.

Conjecture S. [qVA] *For* G *semisimple, there is a q-deformation of the BD Grassmannian $\mathrm{Gr}_{G,q} \to \mathrm{Ran}\mathbf{A}^2_q$, defining the q-affine vertex algebra $V^k_q(\mathfrak{g})$. Their conformal blocks*[4] *give the qKZ connection.*

Kazhdan-Lusztig equivalences. The goal of this project is to relate q-vertex algebras to the geometric proof [CF] of Iwahori Kazhdan-Lusztig.

Conjecture T. [KL] *There is a **Zhu algebra** functor from q-affine VOAs to q-deformed associative algebras, which sends $V^k_q(\mathfrak{g}) \mapsto U_q(\mathfrak{g})$.*

Having this would imply $V^k_q(\mathfrak{g})$ is thus an object which interpoles between $U_q(\mathfrak{g})$ and $V^k(\mathfrak{g})$. Then, to make this precise, we aim to construct a factorisation category $\mathcal{C} = V^k_q(\mathfrak{g})$-Mod on $\mathbf{C} \times \mathbf{R}_{\geqslant 0}$ with a connection ∇, whose fibres over give $V^k(\mathfrak{g})$-Mod and $U_q(\mathfrak{g})$-Mod, and ∇-parallel transport gives $\mathrm{RH} : V^k(\mathfrak{g})$-Mod $\mapsto U_q(\mathfrak{g})$-Mod in the proof [CF].

Relevance. Constructing analogues of $\mathcal{C}$ may give proofs of new Kazhdan-Lusztig equivalences [BCDN] from $3d$ mirror symmetry, or affine analogues of Kazhdan-Lusztig.

1.5. Sheaf methods (Con, Loc, Eu)

Localisation methods. One of the main techniques in enumerative geometry are the *torus localisation* and *Graber-Panharipande formulas* [GP]. We generalise these to the Artin moduli stacks appearing in modern enumerative/algebraic geometry.

[4] i.e. homology of the factorisation algebra over $(\mathbf{A}^2_q)^n_\circ$.

Theorem U. [Conc; Loc] *We give conditions for the cohomology of an Artin stack $\mathcal{X}$ to be **concentrated** on a closed substack $\mathcal{Z}$; when $\mathcal{Z} = \mathcal{X}^T$ is fixed points of a quasismooth dg scheme we give **Atiyah-Bott*** $\mathrm{id} = i_* \left(i^!(-)/e(N_{vir})\right)$ *and **Graber-Pandharipande localisation** formulas* $[\mathcal{X}]^{vir} = i_* \left([\mathcal{X}^T]^{vir}/e(N_{vir})\right)$.

Theorem V. [Eu] *We strengthen both parts of the theorem to **critical cohomology** of **arbitrary** closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ quasismooth over a common base. As a result, for $\mathcal{M}$ as in* CY3*, we have the following Theorem:*

Theorem W. [Eu] *For any "split locus" map $\pi : \mathcal{M}^s \to \mathcal{M}$, we have* $\mathrm{CoHA} = \mathrm{CoHA}^s/e(\mathrm{N}_{i,vir})$.

Taking $\mathcal{M}^s$ a *shuffle space*[5], this gives a **universal, geometric** way of producing/interpreting shuffle products for CoHAs [Da; SV; YZ]. Taking $\mathcal{M}^s = \mathcal{M} \times \mathcal{M} \overset{\oplus}{\to} \mathcal{M}$ proves **compatibility** [CV,CY3,Li] between **coproducts** and the CoHA (c.f. CY3). This easily **generalises** to the OSp/Dynkin SA cases.

1.6. Liouville quantum gravity to vertex algebras (LQG)

Probabilists are beginning construct Feynman measures (GFF [BPR], LQG [KRV]) **rigorously** to understand $2d$ CFTs [CRV; DS; Sh]. There is almost no intereaction with the algebraic geometry/functorial school to QFT, and they understand/have access to techniques (e.g. SLE curves [MS; SS], combinatorial approximations) that the latter school does not. We want to build a bridge:

Conjecture X. [LQG] *Let $\mathcal{Z} : \mathrm{Cob}_2^{\mathrm{hol}} \to \mathrm{Vect}$ be a Segal CFT. If its unstraightening*

$$\mathcal{C}_{\mathcal{Z}} \to \mathrm{Cob}_2^{\mathrm{hol}}$$

respects the holomorphic structure on (Teichmuller) Hom spaces, we may take its chiral part $\mathcal{Z}^{ch}$, which gives a translation-equivariant holomorphic factorisation algebra, hence [CG] *vertex algebra.*

Conjecture Y. [LQG] *The Liouville Quantum Gravity CFT* [GKRV] *satisfies the condition of Conjecture X, and its chiral part is the Virasoro vertex al gebra. Likewise the Gaussian Free Field CFT* [BPR] *gives the Heisenberg vertex algebra.*

Conjecture Y would give an explicit relation between [KRV]'s *DOZZ* formula to the Virasoro OPE.

[5] i.e. shuffle algebra in the category of spaces, see SA.

2. Paper list

Conc Aranha, D., Khan, A.A., Latyntsev, A., Park, H. and Ravi, Charanya, 2024, *The stacky concentration theorem*. arxiv:2407.08747 2024

FQG Latyntsev, A., 2023. *Factorisation quantum groups*. arXiv:2312.07274 2023

Loc Aranha, D., Khan, A.A., Latyntsev, A., Park, H. and Ravi, Charanya, 2022. *Virtual localization revisited.* arXiv preprint arXiv:2207.01652. 2022

CV Latyntsev, A., 2021. *Cohomological Hall algebras and vertex algebras*. arXiv:2110.14356 2021

To appear imminently:[6]

CY3 Jindal, S., Kaubrys, S., Latyntsev, A. *Vertex quantum groups for deformed CY3 completions and the Drinfeld coproduct on Yangians*

Bos de Hority, S., Latyntsev, A. *Factorisation bosonisation*.

OSp de Hority, S., Latyntsev, A. *Orthosymplectic instantons and cohomological Hall algebras*.

Eu Latyntsev, A. *Virtual Euler classes for Artin stacks*.

In preparation:

CD Latyntsev, A. and Niu, W. *Chiral doubling*.

LQG Giri, V. and Latyntsev, A. *Louiville quantum gravity and vertex algebras*.

[6]See e.g. https://arxiv.org/search/math?searchtype=author&query=Latyntsev,+A

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[BCDN] Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu. "3d mirror symmetry of braided tensor categories". In: *arXiv preprint arXiv:2304.11001* (2023).

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[Ch] Kuo-tsai Chen. "Iterated integrals of differential forms and loop space homology". In: *Annals of Mathematics* 97.2 (1973), pp. 217–246.

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