# $q$-DEFORMED D-MODULES

Contents



## 1. Introduction

1.1. **Vector fields.** Recall that a tangent vector is a map

$$\xi \; : \; \mathbf{D}_2 \; \to \; X$$

from the second order infinitesimal neighbourhood of the origin in the formal disk $\mathbf{D}$. Likewise we get the notion of $n$-jet for any $n = 1, 2, \cdots, \infty$, and stronger still we could ask for a map

$$\xi \; : \; \mathbf{G}_a \; \to \; X.$$

A vector field induces a map on functions

$$\mathcal{O}(X) \; \to \; \mathbf{C}[\epsilon]/\epsilon^2,$$

and the $\epsilon$ coeffient is the *derivative* of the function in the direction of the vector field.

1.1.1. *Multiplicative and elliptic jets.* We make the following redundant definition. If $G$ is a one-dimensional algebraic group, a *$G$-jet* is a map

$$\xi \; : \; \mathbf{D}^G \; \to \; X$$

from the formal neighbourhood of the identity in $G$. Of course, all of these are non-canonically isomorphic and so this is the same thing as an ordinary jet. Let $\chi_G$ be a left-invariant vector field on $G$, then

$$\mathbf{D}_2^G \; = \; \mathbf{D}_2 \cdot \chi_G.$$

However, when we pass to the quantum versions of the above definitions, the definitions for different $G$ will seperate.

1.1.2. *Vector fields.* A *vector field* is a map over $X$

$$\xi \;:\; X \times \mathbf{D}_2 \;\to\; X.$$

**Proposition 1.1.3.** *The sheaf* $\mathcal{T}_X$ *of vector fields is the Lie algebra of the group* $\mathrm{Aut}(X)$ *over* $X$.

*Proof.* A tangent vector inside $\mathrm{Aut}(X)$ is a map

$$\psi \;:\; \mathbf{D}_2 \;\to\; \mathrm{Aut}(X)$$

which by adjunction is the same as a map

$$\mathbf{D}_2 \times X \;\to\; X.$$

The condition that $\psi$ needs to be a tangent vector at the unit $\mathrm{id} \in \mathrm{Aut}(X)$ is equivalent to this map being over $X$. □

In exactly the same way, an $n$-jet field on $X$ is the same as an $n$-jet at the identity of $\mathrm{Aut}(X)$.

1.2. **Koszul dual picture.** If $X$ is a smooth scheme, we have a Koszul duality of sheaves of algebras over $X$,

$$\mathrm{KD}(\mathcal{D}_X) \;\simeq\; \Omega_X$$

where $\Omega_X$ is the de Rham complex. The equivalence is given by a bimodule, the de Rham complex $\mathcal{D}_X \otimes \Omega_X$ equipped with a differential which intertwines the factors.

Thus, if we define $q$-deformed D-modules on $X$ as D-modules on a noncommutative space $Y = Y_{\mathbf{q}}$, it is natural to expect that it be Koszul dual to the noncommutative de Rham complex $\Omega_Y$, if it is defined.

1.3. **Jets.** In the above, we considered jets, and moreover, the de Rham stack $X_{dR} \simeq X/\mathcal{G}$ is the quotient by

$$\mathcal{G} \;=\; \exp(\mathcal{T}_X) \;\simeq\; \mathcal{J}_\infty X$$

the formal group scheme over $X$ given by formal jets. In particular, below when we will want to define $q$-D modules on $X$ as D-modules on a certain noncommutative space $Y = Y_{\mathbf{q}}$, we will need to define the jet space $\mathcal{J}_\infty Y_{\mathbf{q}}$, and

$$Y_{\mathrm{dR}} \;=\; Y/\mathcal{J}_\infty Y.$$

For this we will use the machinery developed by Majid and Simao in [MS].

## 2. Quantum analogues

2.1. **$q$-vector fields.** Now let $\mathbf{G}_m$ act on our smooth scheme $X$. This makes $\mathcal{O}_X$ into a $\mathbf{Z}$-graded sheaf, so we can define the sheaf $\mathcal{T}_X^q \subseteq \mathcal{E}\mathrm{nd}(\mathcal{O}_X)$ of $q$-vector fields consisting of endomorphisms $\partial$ with

$$\partial(fg) \;=\; \partial(f)g \;+\; q^{|f|}f\partial(g)$$

for all pairs of homogenous functions $f, g \in \Gamma(\mathcal{O}_X)$.

2.1.1. One way to axiomatise this is the following. Extend $\mathcal{O}(X)$ by adding the variable $\mathbf{q}$ with commutation relations

$$\mathbf{q}f \;=\; q^{|f|}f\mathbf{q}$$

for homogeneous elements, where $q \in k$ is central. Then

$$\mathbf{q}\partial(fg) \;=\; \mathbf{q}\partial(f)g \;+\; f\mathbf{q}\partial(g)$$

and so $\mathbf{q}\partial$ defines an honest vector field on $\langle \mathcal{O}(X), \mathbf{q}\rangle$. Thus a $q$-vector field induces an algebra map

$$\langle \mathcal{O}(X), \mathbf{q}\rangle \;\to\; \langle \mathbf{C}[\epsilon]/\epsilon^2, \mathbf{q}\rangle, \qquad\qquad f \;\mapsto\; f + \mathbf{q}\partial(f)\epsilon,$$

where $\mathbf{q}$ and $\epsilon$ commute. We now turn to the question of what this algebra $\langle \mathcal{O}(X), \mathbf{q}\rangle$ is.

**Proposition 2.1.2.** *$\langle \mathcal{O}(X), \mathbf{q}\rangle[q, q^{-1}]$ is a $\mathbf{Z}[q, q^{-1}]$-quantisation of $\mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})[q, q^{-1}]$ with the grading given by a $\mathbf{G}_m$-action on $X \times \mathbf{G}_{m,\mathbf{q}}$.*

For instance, if every function on $X$ has degree zero, then $\langle \mathcal{O}(X), \mathbf{q}\rangle = \mathcal{O}(X \times \mathbf{G}_{m,\mathbf{q}})$.

2.1.3. We are now in place to define $q$-vector field. To begin, we need to *choose* a quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \to G$ of $X \times \mathbf{G}_{m,\mathbf{q}}$ over $G$. Then,

**Definition 2.1.4.** A *$q$-vector field* on $X$ is a vector field

$$\xi \;:\; \mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) \;\to\; (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}})$$

on the noncommutative space $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$, i.e. a map as above, over $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$.

We have immediately

**Lemma 2.1.5.** *The restriction of a $q$-vector field to $X$ is a vector field.*

*Proof.* We take the pullback squares

$$\begin{array}{ccccc}
\mathbf{D}_2 \times X & \xrightarrow{\quad\xi_1\quad} & X & \xrightarrow{\qquad} & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{D}_2 \times (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) & \xrightarrow{\xi} & (X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}) & \longrightarrow & \mathbf{G}_m
\end{array}$$

which gives an ordinary vector field on $X$. □

Thus loosely speaking, a $q$-vector field is a quantised vector field on $X$.

2.1.6. *Example:* $X = \mathbf{A}^1$. The operator $\partial(x^n) = n_q x^{n-1}$, where $n_q$ is the $n$th $q$-integer,

$$n_q \;=\; 1 + q + \cdots + q^{n-1}, \qquad\qquad (-n)_q \;=\; q^{-1} + q^{-2} + \cdots + q^{-n}$$

which satisfies $(n+m)_q = n_q + q^n m_q$. In particular, $\partial(x^{n+m}) = n_q x^n \cdot x^m + q^n x^n \cdot m_q x^m$, and so this defines a $q$ vector field.

2.1.7. *Remark.* We could also just as well replace $\mathbf{G}_{m,\mathbf{q}}$ by $E_{\mathbf{q},\tau}$ or $G_{\mathbf{q}}$ any one-dimensional algebraic group.

Thus, let $X$ and $G_{\mathbf{q}}$ be viewed as constant schemes over $G$. Then we *choose* a quantisation $X \tilde{\times} G_{\mathbf{q}} \to G$ over $G$. In this case, a $G$*-jet* is a map

$$\xi \;:\; \mathbf{D}_n^G \times (X \tilde{\times} G_{\mathbf{q}}) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

over $X \tilde{\times} G_{\mathbf{q}}$. Over a point $x \in X$ we get

$$\xi_x \;:\; \mathbf{D}_n^G \times (G_{\mathbf{q}} \times G) \;\to\; (X \tilde{\times} G_{\mathbf{q}})$$

and so we get a map

$$\xi_x \;:\; \mathcal{O}(X \tilde{\times} G_{\mathbf{q}}) \;\to\; \mathcal{O}(\mathbf{D}_n^G \times G_{\mathbf{q}}) \otimes \mathcal{O}(G).$$

For instance, our ordinary notion of $q$-vector field corresponds to $G = \mathbf{G}_m$. We can define an $\hbar$-adic version by taking $G = \mathbf{G}_a$.

When dealing with elliptic curves, we may also require a compatible family of $\mathbf{G}_m$- and $E_\tau$-jets which glue over $\overline{\mathcal{M}}_{1,1}$.

2.2. $q$-**cotangent bundles.** The cotangent bundle over $X$ is given by taking the relative spectrum of the sheaf of vector fields.

2.2.1. Having chosen a quantisation $\tilde{X} = X \tilde{\times} G_{\mathbf{q}}$, the *quantum cotangent bundle* is

$$\tilde{\mathbf{T}}^*_{\tilde{X}} \;=\; \mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}} \times G}.$$

(define this, i.e. show that we get a quantisation)

**Lemma 2.2.2.** *This is a quantisation of the cotangent bundle of $X$ times $G_{\mathbf{q}} \times G$, i.e.*

$$\mathbf{T}^*_{\tilde{X}/G_{\mathbf{q}} \times G} \;=\; \mathbf{T}^*_X \tilde{\times} G_{\mathbf{q}}.$$

For instance, if $X = \mathbf{A}^1$ and $G = \mathbf{G}_m$, then we can take

$$\tilde{X} \;=\; \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}^{\pm}, q^{\pm}\rangle$$

where $q$ is central, and

$$\mathbf{T}^q_{\tilde{X}} \;=\; \operatorname{Spec} \mathbf{C}\langle x, p, \mathbf{q}^{\pm}, q^{\pm}\rangle$$

is a twisted product of $T^*\mathbf{A}^1$ and $\mathbf{G}_{m,\mathbf{q}} \times \mathbf{G}_m$, where $p = \partial_x$, and so we have that $\mathbf{q}p = q^{-1}p\mathbf{q}$. Notice that we get a closed subscheme

$$\mathbf{A}^2_q \;=\; \operatorname{Spec} \mathbf{C}\langle x, \mathbf{q}p\rangle$$

which is the quantum affine plane, since writing $y = \mathbf{q}p$, we get the defining relations $xy = qyx$.

2.3. **$q$-differential operators.** The $q$-differential operators $\mathcal{D}_q$ will be a filtered quantisation of

$$\operatorname{Spec} \operatorname{Sym}_{\tilde{X}} \tilde{\mathbf{T}}^*_{\tilde{X}}.$$

Notice that the role of $q$ and the $q$-quantisation is orthogonal to the role of the filtration and the filtered quantisation. We define it as usual: it is the sheaf of differential operators on $\tilde{X}$, i.e. it is the sheaf of subalgebras

$$\tilde{\mathcal{D}}_{\tilde{X}} \;\subseteq\; \mathcal{E}\mathrm{nd}_{\tilde{X}}(\mathcal{O}_{\tilde{X}})$$

generated by the $q$-vector fields and $\mathcal{O}_{\tilde{X}}$.

Notice that by the definition,

**Lemma 2.3.1.** *$\tilde{\mathcal{T}}_{\tilde{X}}$ forms a sheaf of Lie algebras over $\tilde{X}$.*

This allows us to give a Grothendieck definition of the sheaf of quantum differential operators:

**Lemma 2.3.2.** *$\tilde{\mathcal{D}}_{\tilde{X}} = \cup_{n\geqslant 0}\tilde{\mathcal{D}}_{\tilde{X},n}$, where the zeroeth term is $\tilde{\mathcal{O}}_{\tilde{X}}$, and above that*

$$\tilde{\mathcal{D}}_{\tilde{X},n} \;=\; \text{(recursive definition)}.$$

To summarise, we have the following

$$\begin{array}{cc} \operatorname{gr} \mathcal{D}_X & \operatorname{gr} \tilde{\mathcal{D}}_{\tilde{X}} \\ \\ \mathcal{D}_X & \tilde{\mathcal{D}}_{\tilde{X}} \end{array}$$

and the sheaves on the left are given by pulling back the sheaves on the right along $1 \to G$.

2.4. **Relation to automorphisms of $X$.** Recall that one may define a D-module on $X$ to be a quasicoherent sheaf which is equivariant for the action of the formal group $\exp(\mathcal{T}_X)$; this is the parallel transport map. Likewise, if $\Phi$ is an automorphism of $X$, one possible definition of quantum D-module is a $\Phi$-equivariant quasicoherent sheaf.

How does this relate to the above definition?

To begin with, what has this to do with the quantisation $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$? Let us consider the case when the quantisation and the automorphism both come from the same source: a single $\mathbf{G}_m$ action:

$$\begin{tikzcd} & \mathbf{G}_m \text{ action on } X \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\ \text{automorphism } \Phi_g \text{ for any } g \in \mathbf{G}_m & & \text{quantisation } X \tilde{\times} \mathbf{G}_{m,\mathbf{q}} \end{tikzcd}$$

A quasicoherent sheaf on $X \tilde{\times} \mathbf{G}_{m,\mathbf{q}}$ is the same as a quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(X)$ with a compatible action of $\mathbf{C}[\mathbf{q}^{\pm}]$, i.e. we have

$$\mathbf{q}_x \ : \ \mathcal{M}_x \ \xrightarrow{\sim} \ \mathcal{M}_x$$

for every point $x \in X$, and we have

$$\mathbf{q}_x f(x) \ = \ q^{|f|} f(x) \mathbf{q}_x$$

as automorphisms of $\mathcal{M}_x$. In particular, this has nothing to do with comparing $\mathcal{M}_x$ and $\mathcal{M}_{\Phi_g \cdot x}$, so *it is unlikely the definitions are related*.

The automorphism definition of quantum D-module is related to

$$\mathbf{Z} \ \xrightarrow{\Phi} \ \mathrm{Aut}(X) \ \leftarrow \ \exp(\mathcal{T}_X)$$

whereas the $q$-deformed D-module changes the underlying space,

$$\exp(\tilde{\mathcal{T}}_{\tilde{X}}) \ \to \ \exp(\mathcal{T}_X).$$

One expects that it might be possible to quantise both ways simultaneously.

2.5. **Relation to difference equations.** If instead we are to take $\tilde{X}_\hbar \ = \ X \tilde{\times} \mathbf{G}_a$, then we get (show how to get difference equations, might need to take $\mathbf{C}[[\hbar]]$)

2.6. **Relation to Beilinsorn-Bernstein.** Let $\lambda : \mathbf{G}_m \to G$ be a character with $\lambda B \lambda^{-1} \ = \ B$. Then we get an induced $\mathbf{G}_m$ action on the flag variety $G/B$, and can form the quantisation.

**Conjecture 2.6.1.** *We have a surjection* $\tilde{\mathcal{D}}_{\widetilde{G/B}} \twoheadrightarrow U_q(\mathfrak{g})$.

2.7. **Relation to quantum groups.** We are going to give a *different* relation to quantum groups, where

$$X \;=\; \operatorname{Spec} U_q(\mathfrak{g}), \qquad\qquad G \;=\; T.$$

Note that here we may be using a group of dimension greater than one. If $\mathbf{q}_\lambda$ corresponds to $\lambda \in \mathcal{O}(T) \subseteq \mathfrak{t}^*$, then we set

$$x\mathbf{q}_\lambda \;=\; q^{\lambda(x)}\mathbf{q}_\lambda x$$

for all $x \in \mathfrak{g} \subseteq U_q(\mathfrak{g})$.

**Conjecture 2.7.1.** *We have*

$$\tilde{\mathcal{D}}_{\tilde{X}} \;=\; U_q(\mathfrak{g} \oplus_{\mathfrak{t}} \mathfrak{g}^*)$$

*is the Takiff algebra.*

## 3. Functoriality

3.1. In the above we defined the category of D-modules over $\operatorname{Spec} A$ for any (check Majid?) noncommutative algebras $A$ as an element of $\operatorname{QCoh}(\operatorname{Spec} A)$ which is equivariant for the action of the formal jet group $\mathcal{J}_\infty \operatorname{Spec} A$. (what about in the non-affine case)

3.1.1. Let $f : X \to Y$ be a map of noncommutative spaces. We then have functor

$$f^\dagger \; : \; \mathcal{D}\text{-Mod}(Y) \; \to \; \mathcal{D}\text{-Mod}(X)$$

induced by pullback of quasicoherent sheaves (i.e. restriction of modules) and functoriality of $\mathcal{J}_\infty$.

3.1.2. Now assume that $f$ is (noncommutative schematic and quasi-compact??). Then we have a pushforward functor

$$f_{dR,*} \; : \; \mathcal{D}\text{-Mod}(X) \; \to \; \mathcal{D}\text{-Mod}(Y)$$

defined by (pushforward on QCoh?). To be explicit, it acts on modules as

$$M \; \mapsto \; f_*(M \otimes \Omega_{X/Y})$$

where $\Omega_{X/Y}$ is the noncommutative de Rham complex of Majid and Simao [MS].

## 4. Quantum vertex algebras

If ordinary vertex algebras are meant to axiomatise two-dimensional chiral conformal field theory on a complex curve $\Sigma$, then $\mathbf{q}$-vertex algebras axiomatise the theory on *noncommutative* curves $\tilde{\Sigma}_{\mathbf{q}}$.

In physics terms, these should be two-dimensional CFTs on $\Sigma \times S^1$, which are compatified along a *nontrivial* $S^1$ action. (check)

One common way to get noncommutative curves is to quantise curves inside cotangent bundles

$$\Sigma \subseteq T^*C \qquad \rightsquigarrow \qquad \tilde{\Sigma} \subseteq \operatorname{Spec} \mathcal{D}_C$$

where if $\Sigma$ is the vanishing locus of the symbol $\sigma P$ of differential operator $P$, then the quantisation has ring of functions $\mathcal{D}_C/P$. If we want this to be an algebra over $k[[\hbar]]$, we may in the above take the $\hbar$-adically completed sheaf of D-modules $\mathcal{D}_{C,\hbar}$. There is a relation to opers, see for instance section 2 of [CPT].

### 4.1. Appearance of q-diagonals.

4.1.1. We now consider what the diagonal inside $X \tilde{\times} \mathbf{G}_{m_{\mathbf{q}}}$ looks like.

4.1.2. To begin, for a map $A \to B$ of algebras, note that the relative diagonal is given by the map

$$B \otimes_A B \twoheadrightarrow B, \qquad b \otimes b' \mapsto bb'.$$

4.1.3. For instance, let $X = \mathbf{A}^1 = \operatorname{Spec} k[x]$. Then the quantum diagonal is given by the ideal

$$\tilde{\Delta} : \tilde{X} \to \tilde{X} \times \tilde{X}$$

given by the ideal

$$I_\Delta \subseteq \langle k[x_1], \mathbf{q}_1^{\pm}, k[x_2], \mathbf{q}_2^{\pm} \rangle \twoheadrightarrow \langle k[x], \mathbf{q}^{\pm} \rangle, \qquad x_1, x_2 \mapsto x, \quad \mathbf{q}_i \mapsto \mathbf{q}$$

and where in the domain $x_1, x_2$ commute. For instance, the ideal of the diagonal contains the element

$$x_1 - x_2(\mathbf{q}_2\mathbf{q}_1^{-1})^n$$

for every integer $n \in \mathbf{Z}$.

### 4.2. q-additive group.

4.2.1. We consider the group structure,

$$m : \tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}} \to \tilde{\mathbf{A}}^1_{\mathbf{q}}$$

which is the unique map of noncommutative schemes so that

$$m^*\mathbf{q} = \mathbf{q} \otimes \mathbf{q}, \qquad m^*x = x \otimes 1 + \mathbf{q} \otimes x$$

for an integer $w \in \mathbf{Z}$ called the *weight*. This is well-defined, since

$$\begin{aligned} m^*(\mathbf{q}x) &= \mathbf{q}x \otimes \mathbf{q} \,+\, \mathbf{q}^2 \otimes \mathbf{q}x \\ &= q(x\mathbf{q} \otimes \mathbf{q} \,+\, \mathbf{q}^2 \otimes x\mathbf{q}) \\ &= q \cdot m^*(x\mathbf{q}). \end{aligned}$$

Denote this algebraic group $\mathbf{G}_{a\mathbf{q}}$.

4.2.2. Likewise, we have an action for every integer $w$

$$m_w^*\mathbf{q} \;=\; \mathbf{q} \otimes \mathbf{q}, \qquad\qquad m_w^* x \;=\; x \otimes 1 \,+\, \mathbf{q}^w \otimes x$$

giving a group law as above.

4.2.3. If we write points of $\mathbf{G}_{a\mathbf{q}}$ as $z$, then the above group law we will write as $(z_1, z_2) \mapsto z_1 + \mathbf{q}_1 z_2$.

4.2.4. Given a representation of $\mathbf{G}_{a\mathbf{q}}$, i.e.

$$V \;\to\; V \otimes \langle k[x], \mathbf{q}^{\pm} \rangle,$$

then the invariants are the elements $v$ sent to

$$v \;\mapsto\; v \otimes 1.$$

4.2.5. What are the $\mathbf{G}_{a\mathbf{q}}{}^w$-invariants of $\mathcal{O}(\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}})$? Note that the coaction is given by

$$m^*\mathbf{q}_i \;=\; \mathbf{q}_i \otimes \mathbf{q}, \qquad\qquad m^* x_i \;=\; x_i \otimes 1 \,+\, \mathbf{q}_i^w \otimes x,$$

where the right hand side tensor multiplicand lies in $\mathcal{O}(\mathbf{G}_{a\mathbf{q}})$, and so

$$\begin{aligned} m^*(x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n) &= (x_1 \otimes 1 \,+\, \mathbf{q}_1^w \otimes x) \,-\, (x_2 \otimes 1 + \mathbf{q}_2^w \otimes x)\,((\mathbf{q}_2/\mathbf{q}_1)^n \otimes 1) \\ &= (x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n) \otimes 1 \,+\, (\mathbf{q}_1^w - \mathbf{q}_2^w(\mathbf{q}_2/\mathbf{q}_1)^n) \otimes x. \end{aligned}$$

In particular, $(x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^n)$ is invariant with respect to the $\mathbf{G}_{a\mathbf{q}}$-action of weight $w = -n$. Thus we get

**Proposition 4.2.6.** *For any integer $w \in \mathbf{Z}$, the functions on the complement of the main quantum diagonal which are invariant with respect to the weight $w$ action are*

$$\mathcal{O}((\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}})_{\mathbf{q},\circ})^{\mathbf{G}_{a\mathbf{q}}{}^w} \;=\; \langle (x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^w) \rangle_{k[\mathbf{q}_1^{\pm},\mathbf{q}_2^{\pm}]},$$

*which is spanned as a vector space by $\mathbf{q}_1^a\mathbf{q}_2^b(x_1 - x_2(\mathbf{q}_2/\mathbf{q}_1)^w)^c\mathbf{q}_1^d\mathbf{q}_2^e$.*

4.2.7. We now ask the question: what is the category of D-modules on $\tilde{\mathbf{A}}^1_{\mathbf{q}}$ which are weakly equivariant with respect to the weight $w$ action of $\mathbf{G}_{a\mathbf{q}}$? Recall that without the $\mathbf{q}$ the answer was it is the category of a vector space (the invariant sections) with endomorphism (the action of $\partial_z$).

(write)

4.2.8. Notice that the Ran space of $\tilde{\mathbf{A}}^1_{\mathbf{q}}$ is still a symmetric factorisation space,

$$\begin{tikzcd}
 & (\mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}})_\circ \arrow[d, "\sigma"', "\wr"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & (\mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}})_\circ \arrow[dl] \arrow[dr] & \\
\mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}} & & \mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}}
\end{tikzcd}$$

because for instance in $\tilde{\mathbf{A}}^1_{\mathbf{q}} \times \tilde{\mathbf{A}}^1_{\mathbf{q}}$ functors on the left and right factors commute, so the swap map is indeed a map of noncommutative schemes; considering higher powers of the quantum affine plane induces the symmetric factorisation structure $\sigma$ considered above.

4.2.9. In particular, this means we should consider the categories

$$\bigoplus_{w \in \mathbf{Z}} \mathcal{D}\text{-Mod}(\mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}})^{\mathbf{G}_{a\,\mathbf{q}},w}$$

of D-modules which are weakly equivariant respect to some weight $w$. (how to combine these together more naturally?) Notice that

**Proposition 4.2.10.** *For each weight $w$, the $w$ summand upgrades to a symmetric factorisation category $\mathcal{D}\text{-Mod}^{\mathbf{G}_{a\,\mathbf{q}},w}$ over $\mathrm{Ran}\,\tilde{\mathbf{A}}^1_{\mathbf{q}}$.*

4.2.11. We can finally define a **q**-*vertex algebra* to be a strong factorisation algebra in this category.

**Theorem 4.2.12.** *A* **q**-*vertex algebra is equivalent to a direct sum of vector spaces (or $k[\mathbf{q}^{\pm}]$-comodules?)*

$$V \;=\; \bigoplus_{w \in \mathbf{Z}} V_w$$

*along with a map of $\mathcal{D}(\tilde{\mathbf{A}}^1_{\mathbf{q}})$-modules (how should this interact with the weight $w$?)*

$$Y \;:\; V \otimes V \;\to\; V((\{z_1 - \mathbf{q}^n z_2\}))$$

*satisfying (a commutativity and associativity condition), and equipped with a vector $|0\rangle \in V_0$ and (whatever data is equivalent to a $\mathcal{D}(\tilde{\mathbf{A}}^1_{\mathbf{q}})$-module)*

# 5. Misc

5.1. **Ordinary $\mathcal{D}$ modules.** Consider the category $\mathrm{Sh}_X$ of sheaves of abelian groups on smooth scheme $X$. We have a functor

$$\mathcal{O}_X\text{-Mod} \ \to \ \mathrm{Sh}_X$$

which is lax monoidal, i.e. we have a map $\mathcal{M}\otimes\mathcal{M}' \to \mathcal{M}\otimes_{\mathcal{O}}\mathcal{M}'$ for any $\mathcal{O}$-modules $\mathcal{M}, \mathcal{M}'$. If in addition $\mathcal{O}_X$ forms a bialgebra in $\mathrm{Sh}_X$, then we may ask that $\otimes, \otimes_{\mathcal{O}}$ form a lax braided monoidal structure on $\mathcal{O}$-Mod,

$$(\mathcal{M}_1 \otimes_{\mathcal{O}} \mathcal{M}_2) \otimes (\mathcal{M}_3 \otimes_{\mathcal{O}} \mathcal{M}_4) \xrightarrow{\beta} (\mathcal{M}_1 \otimes \mathcal{M}_3) \otimes_{\mathcal{O}} (\mathcal{M}_2 \otimes \mathcal{M}_4)$$

for all $\mathcal{O}$-modules $\mathcal{M}_i$.

5.1.1. *Example: $X = \mathbf{A}^n$.* The sheaf $\mathcal{O}_{\mathbf{A}^n}$ has a natural coalgebra structure in which the coordinates $x_i$ are primitive. Moreover, this bialgebra structure is graded with respect to (any) linear action of $\mathbf{G}_m$ on $\mathbf{A}^n$.

5.1.2. In particular, we can define $\otimes$-$\otimes_{\mathcal{O}}$ *bilalgebras* $\mathcal{A}$, which are $\mathcal{O}$-modules equipped with maps

$$\mathcal{A} \otimes \mathcal{A} \ \to \ \mathcal{A}, \qquad\qquad \mathcal{A} \ \to \ \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}$$

which are compatible as such:

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \arrow[r] \arrow[d] & (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \otimes (\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}) \arrow[r, "\beta"] & (\mathcal{A} \otimes \mathcal{A}) \otimes_{\mathcal{O}} (\mathcal{A} \otimes \mathcal{A}) \arrow[d] \\ \mathcal{A} \arrow[rr] & & \mathcal{A} \otimes_{\mathcal{O}} \mathcal{A} \end{tikzcd}$$

commute as a diagram in $\mathrm{Sh}_X$, and finally $\mathcal{A}$ has a unit and counit which are compatible with each other and the above data.

We have then, assuming throughout that $\mathcal{O}_X$ is a bialgebra,

**Proposition 5.1.3.** *For any Lie algebroid $\mathcal{L}$, its universal enveloping algebra $U(\mathcal{L})$ is a bialgebra.*

*Proof.* (write, should be abstract nonsense) □

Examples of Lie algebroids include tangent bundles and relative tangent bundles. Thus,

**Corollary 5.1.4.** *The sheaf $\mathcal{D}_X$ forms a bialgebra.*

As a consequence,

**Corollary 5.1.5.** *The symmetric monoidal structure $\otimes_{\mathcal{O}}$ has a canonical lift along $\mathcal{D}_X\text{-Mod}^{\otimes} \to \mathcal{O}_X\text{-Mod}$.*

5.1.6. *Example:* $X = \mathbf{A}^1$. In this case, the coalgebra structure on $\mathcal{D}_{\mathbf{A}^1}$, which we identify with $k\langle x, \partial_x\rangle$, is

$$\Delta(\partial_x) \;=\; \partial_x \otimes 1 \,+\, 1 \otimes \partial_x.$$

Note that by the coalgebra axioms and $\mathcal{O}$-linearity, $\Delta(1) = 1 \otimes 1$ and

$$\Delta(x^n \partial_x^m) \;=\; x^n(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^m$$

are forced, likewise if we are to ask that it be a bialgebra (how to define bialgebra?) this forces

$$\Delta(x^{n_1}\partial_x^{m_1} \cdots x^{n_k}\partial_x^{m_k}) \;=\; x^{n_1}(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^{m_1} \cdots x^{n_k}(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)^{m_k}.$$

Note that

$$\Delta([x, \partial_x]) \;=\; x(\partial_x \otimes 1 \,+\, 1 \otimes \partial_x) - (\partial_x \otimes 1 \,+\, 1 \otimes \partial_x)x \;=\; 1 \otimes 1 \;=\; \Delta(1).$$

In particular, we have $\mathrm{Prim}(\mathcal{D}_{\mathbf{A}^1}) = \mathcal{T}_{\mathbf{A}^1}$.

## 5.2. The tangent bundle as a Lie bialgebroid.

The tangent sheaf $\mathcal{T}$ is naturally a Lie algebroid.

5.2.1. *Lie bialgebroids.* We can now define *Lie bialgebroids* over $X$ as a sheaf $\mathcal{L} \in \mathcal{O}$-Mod with a Lie algebra structure in $\mathrm{Sh}_X$

$$[\,,\,] \;:\; \mathcal{L} \otimes \mathcal{L} \;\to\; \mathcal{L}$$

which is $\mathcal{O}$-linear, in the sense that there is a map $\rho : \mathcal{L} \to \mathcal{T}_X$ with $[\ell, f\ell'] = (\rho(\ell)f)\ell' + f[\ell, \ell']$, and a Lie coalgebra structure in $\mathcal{O}_X$-Mod

$$\delta \;:\; \mathcal{L} \;\to\; \mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}$$

such that the usual axiom of a Lie bialgebra holds:

$$\delta([\ell, \ell']) \;=\; (\mathrm{ad}_\ell \otimes_{\mathcal{O}} \mathrm{id} \,+\, \mathrm{id} \otimes_{\mathcal{O}} \mathrm{ad}_{\ell'})\delta(\ell) \;-\; (\mathrm{ad}_{\ell'} \otimes_{\mathcal{O}} \mathrm{id} \,+\, \mathrm{id} \otimes_{\mathcal{O}} \mathrm{ad}_\ell)\delta(\ell'),$$

the relation viewed as a map $\mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}} \mathcal{L}$.

5.2.2. *Example:* $X = \mathbf{A}^1$. In this case, we identify $\mathcal{T}_{\mathbf{A}^1}$ with the free $\mathcal{O}(\mathbf{A}^1)$-module $k[x]\partial_x$. Then as for any Lie algebroid, $\delta = 0$ defines a Lie bialgebroid structure on $\mathcal{T}_{\mathbf{A}^1}$.

## References


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