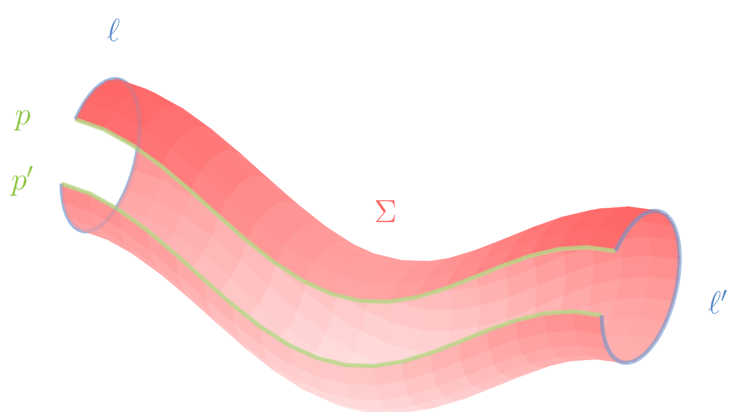
WHAT IS A VERTEX ALGEBRA?

ALEXEI LATYNTSEV

Vertex algebras, factorisation algebras, chiral algebras, ... are all (failed)[1] attempts to mathematically axiomatise *two dimensional conformal field theories.*

1. Two dimensional conformal field theories

1.1. Whatever its eventual definition, at very least a $2d$ *CFT* should assign a linear category $\mathcal{Z}(p) \in$ dgCat to every zero dimensional manifold p, a functor for every oriented one dimensional manifold ℓ and natural transformation for every Riemann surface Σ:



$$\begin{tikzcd} \mathcal{Z}(p) \arrow[r, bend left, "\mathcal{Z}(\ell)"] \arrow[r, bend right, "\mathcal{Z}(\ell')"'] \arrow[r, phantom, "\Downarrow \mathcal{Z}(\Sigma)"] & \mathcal{Z}(p') \end{tikzcd}$$

More precisely, it should at least be a symmetric monoidal functor between 2-categories[2,3]

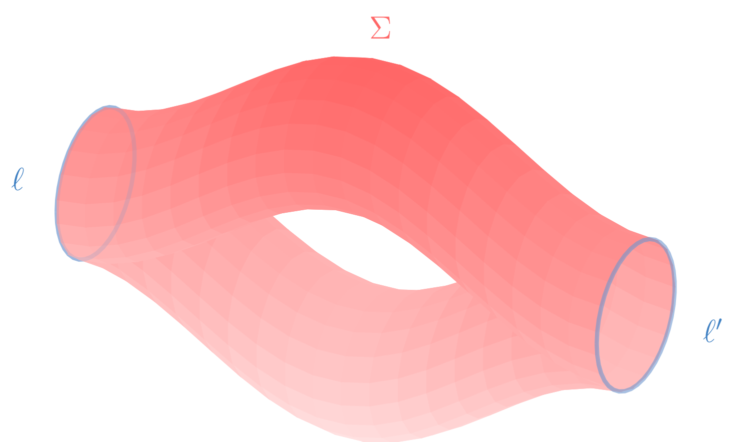
$$\mathcal{Z} \ : \ \mathrm{Cob}_2^{conf} \ \to \ \mathrm{dgCat}$$

[1]i.e. they only partially capture the data of a $2d$ CFT. They are all nevertheless extremely interesting.

[2]I do not know how to define Cob_2^{conf}.

[3]A minor technical point: we only consider Riemann surfaces Σ, bordisms between one manifolds ℓ and ℓ', if they are constant along the boundary of ℓ.

from the (undefined?) two dimensional conformal cobordism category.[4] For instance, the empty zero dimensional manifold is sent to the linear category $\mathcal{Z}(\varnothing) = \mathrm{Vect}$, so $\mathcal{Z}(\ell)$ is identified with (tensoring with a) vector space, and we get



$$\begin{array}{c}\mathcal{Z}(\ell)\\ \downarrow{\scriptstyle \mathcal{Z}(\Sigma)}\\ \mathcal{Z}(\ell')\end{array}$$

Moreover, this should vary reasonably with the complex structure on Σ, i.e. should somehow live over the moduli stack of Riemann surfaces with boundary.

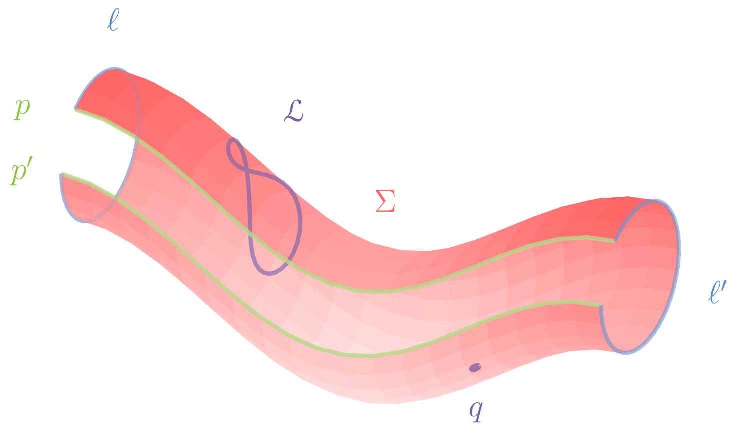
What other structure should there be? For any zero and one dimensional submanifolds of a Riemann surface $q \in \Sigma$ and $\mathcal{L} \subseteq \Sigma$, we have a vector space $\mathcal{V}_q$ and monoidal category $\mathcal{V}_{\mathcal{L}}$. For every element $\phi_q \in \mathcal{V}_q$ and object $\phi_{\mathcal{L}} \in \mathcal{V}_{\mathcal{L}}$, called *defects*, we get modified

$$\mathcal{Z}(\Sigma;\phi_q),\ \mathcal{Z}(\Sigma;\phi_{\mathcal{L}})\ :\ \mathcal{Z}(\ell)\ \Rightarrow\ \mathcal{Z}(\ell')$$

and similarly for any zero dimensional submanifolds $q \in \ell$, we get

$$\mathcal{Z}(\ell;\phi_q)\ :\ \mathcal{Z}(p)\ \rightarrow\ \mathcal{Z}(p').$$

[4]Note that zero dimensional manifolds, oriented one manifolds and Riemann surfaces are precisely the conformal manifolds in dimension up to two.



$$\begin{tikzcd} \mathcal{Z}(p) \arrow[r, bend left, "\mathcal{Z}(\ell)"] \arrow[r, bend right, "\mathcal{Z}(\ell')"'] \arrow[r, phantom, "\Downarrow \mathcal{Z}(\Sigma;\phi_q,\phi_{\mathcal{L}})"] & \mathcal{Z}(p') \end{tikzcd}$$

Moreover, and this is a special property of *conformal* field theories,

$$\mathcal{Z}(\mathrm{pt}) \;\simeq\; \mathcal{V}_{S^1}, \qquad \text{and} \qquad \mathcal{Z}(S^1) \;\simeq\; \mathcal{V}_{\mathrm{pt}}.$$

This is called the state-field correspondence, and you can see a map in one direction by putting a point insertion on a cap. Moreover, the set of all caps form a BS^1, which somehow relates to the expected Virasoro action.

There are probably other structures I am forgetting.

1.2. **Classical CFTs.** There is nothing special about Vect; we can replace it with any symmetric monoidal 2-category $\mathcal{C}$. For instance (see [CMR]) we can replace use the category of Poisson spaces with morphisms between them Lagrangian correspondences,

$$\begin{tikzcd} & \tilde{\mathcal{Z}}(\Sigma) \arrow[dl] \arrow[dr] & \\ \mathcal{Z}(\ell) & & \mathcal{Z}(\ell') \end{tikzcd}$$

We then expect to produce a vector valued $2d$ CFT after supplying a measure $d\mu$ on $\mathcal{O}(\tilde{\mathcal{Z}}(\Sigma))$, whose value on closed one manifold ℓ is $\mathcal{O}\left(\mathcal{Z}(\ell)\right)$ and value on Σ is pull-push along the correspondence:

$$\mathcal{O}\left(\mathcal{Z}(\ell)\right) \;\to\; \mathcal{O}\left(\mathcal{Z}(\ell')\right) \qquad\qquad f \;\mapsto\; \int_{\Psi\in\mathcal{O}\left(\tilde{\mathcal{Z}}(\Sigma)\right)\,:\,\Psi|_{\ell}=f} \Psi|_{\ell'}\, d\mu.$$

1.3. **Quantisation.** Sometimes the above space sits as a critical locus

$$\tilde{\mathcal{Z}}(\Sigma) \overset{dS=0}{\to} \tilde{\mathcal{Y}}(\Sigma)$$

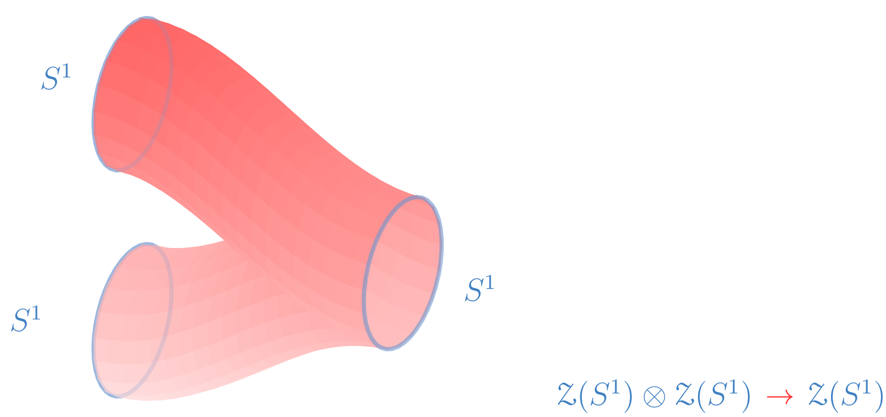
inside another space $\tilde{\mathcal{Y}}(\Sigma)$, and the entire diagram inside

$$\begin{array}{ccc} & \tilde{\mathcal{Y}}(\Sigma) & \\ \swarrow & & \searrow \\ \mathcal{Y}(\ell) & & \mathcal{Y}(\ell') \end{array}$$

If we can extend $d\mu$ to a measure on $\mathcal{O}(\tilde{\mathcal{Y}}(\Sigma))$, the associated *quantisation*[5] of the previous $2d$ CFT is

$$\mathcal{O}\left(\mathcal{Y}(\ell)\right) \ \to \ \mathcal{O}\left(\mathcal{Y}(\ell')\right) \qquad\qquad f \ \mapsto \ \int_{\Psi\in\mathcal{O}\left(\tilde{\mathcal{Y}}(\Sigma)\right)\,:\,\Psi|_\ell=f} \Psi|_{\ell'}\, e^{-S/\hbar} d\mu.$$

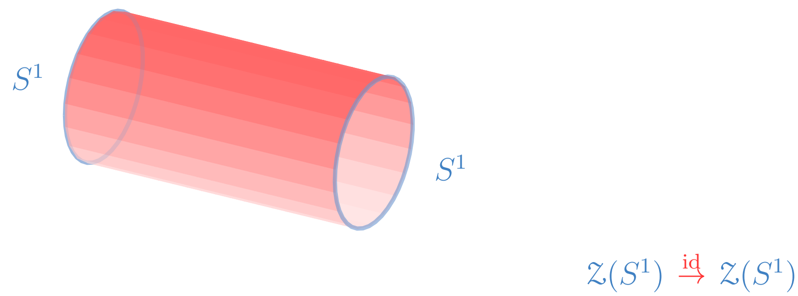
1.4. **Algebraic structures.** Let us explore consequences of the above approximate axioms. Firstly, the vector space $\mathcal{Z}(S^1) \simeq \mathcal{V}_{\mathrm{pt}}$ should have something like an algebra structure, induced by the pair of pants



Moreover, if we take into account that the cyclinder induces the identity[6]

[5]These sorts of theories are called *nonlinear sigma models*.

[6]This is baked into the definition of Cob_2^{conf}.

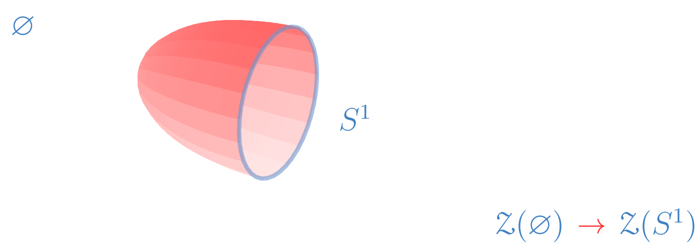


The moduli of pairs of pants, modulo adding cylinders, is (somehow??) isomorphic to $\mathbf{P}^1 \setminus \{0, \infty\}$. In particular, we will get a family of maps

$$Y(z) \; : \; \mathcal{Z}(S^1) \otimes \mathcal{Z}(S^1) \; \to \; \mathcal{Z}(S^1).$$

depending on $z \in \mathbf{P}^1 \setminus \{0, \infty\}$.[7] The cap should give an element

$$|0\rangle \; : \; \mathcal{Z}(\varnothing) \; = \; k \; \to \; \mathcal{Z}(S^1).$$



[7]A priori Y should depend on a point in the moduli space of pairs of pants, which depends on the lengths ℓ_1, ℓ_2, ℓ_3 of the three holes. I do not understand by what extra conditions we impose exactly to get that Y actually only depends on z, the centre of one of the circles (if the others' centres are at 0 and ∞).

1.5. The *(anti)chiral* part of the CFT is the subspace $\mathcal{Z}(S^1)_\pm \subseteq \mathcal{Z}(S^1)$ on which the operator $Y(z) : \mathcal{Z}(S^1) \to \mathcal{Z}(S^1)$ depends (anti)holomorphically on z. We have $\mathcal{Z}(S^1) = \mathcal{Z}(S^1)_+ \oplus \mathcal{Z}(S^1)_-$. Vertex, factorisation and chiral algebras study $\mathcal{Z}(S^1)_+$.

1.6. Likewise, the "looped circle" should give a pairing

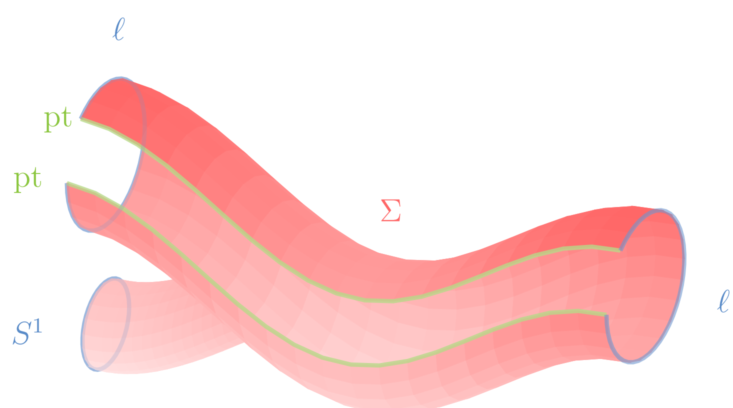
$$\mathcal{Z}(S^1) \otimes \mathcal{Z}(S^1) \ \to\ \mathcal{Z}(\varnothing) \ =\ k$$

and the reversed pants and cup give dual structures to the above. This is not to be confused with the folded cylinder, which gives

$$\mathcal{Z}(S^1) \otimes \mathcal{Z}(S^1)^* \ \to\ \mathcal{Z}(\varnothing) \ =\ k$$

which is just the evaluation map.

1.7. Next, the category $\mathcal{Z}(\mathrm{pt}) \simeq \mathcal{V}_{S^1}$ should have some sort of action of $\mathcal{Z}(S^1) \simeq \mathcal{V}_{\mathrm{pt}}$, by



$$\begin{tikzcd} \mathcal{Z}(\mathrm{pt}) \arrow[r, bend left, "\mathcal{Z}(S^1)\otimes \mathrm{id}"] \arrow[r, bend right, "\mathrm{id}"'] \arrow[r, phantom, "\Downarrow \mathcal{Z}(\Sigma)"] & \mathcal{Z}(\mathrm{pt}) \end{tikzcd}$$

where we have used that $\mathcal{Z}(\ell) = \mathrm{id}$ is trivial. Spelling out this natural transformation, for every object $\mathcal{M} \in \mathcal{Z}(\mathrm{pt})$ we have a map

$$\mathcal{Z}(S^1) \otimes \mathcal{M} \ \to\ \mathcal{M}$$

so $\mathcal{M}$ is something like a module for $\mathcal{Z}(S^1)$.

1.8. **Correlation functions and conformal blocks.** Here we just note that given a closed Riemann surface Σ and elements $\phi_i(x_i) \in \mathcal{V}_{x_i} \simeq \mathcal{Z}(S^1)$ where $x_i \in \Sigma$, we get a number

$$\mathcal{Z}\left(\Sigma; \phi_1(x_1), \cdots, \phi_n(x_n)\right) \ \in \ k.$$

Moreover, we expect that the vector spaces $\mathcal{V}_x$ should arrange to a vector bundle with connection $\mathcal{V}$, and if $\phi_1, ..., \phi_n$ are sections[8] then we get a function

$$\Sigma^n \setminus \Delta \ \to \ k \qquad\qquad (x_1, ..., x_n) \ \mapsto \ \mathcal{Z}\left(\Sigma; \phi_1(x_1), \cdots, \phi_n(x_n)\right)$$

on the open locus where the points are all distinct.

1.9. **Conformal symmetry.** All of the above should work in families somehow. For instance, given a family of Riemann surfaces

$$\Sigma \ \to \ B$$

and a family of points $q : B \to \Sigma$, then there should be a (flat?) vector bundle $\mathcal{V}_q$ on B, and similarly for all other $\mathcal{Z}_{(-)}$'s and $\mathcal{V}_{(-)}$'s.

In particular, we could take $\Sigma = \mathbf{C}$, on which the conformal group $\mathrm{Conf}(2)$ of angle preserving linear maps acts, fixing $0 \in \mathbf{C}$. This gives a family of Riemann surfaces (all isomorphic to $\mathbf{C}$)

$$\Sigma \ \to \ \mathrm{Conf}(2)$$

and a family of points (corresponding to $0 \in \mathbf{C}$). This probably implies that $\mathcal{V}_0$ is acted on by $\mathrm{Conf}(2)$.

In fact, we expect an action of the *local conformal groupoid* $\mathrm{Loc}\,\mathrm{Conf}(2)$ (more details??),[9] whose Lie algebra is two copies of the *Witt algebra* of holomorphic vector fields on the circle

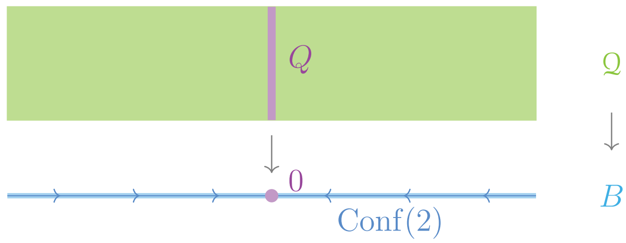
$$\mathfrak{locconf}(2) \ \simeq \ k\left\{ z^i \partial_z, \bar{z}^i \partial_{\bar{z}} \ : \ i \in \mathbf{Z} \right\}$$

and in general we expect an action of its central extension, the *Virasoro algebra*.

[8]When these sections presumably the resulting function will be nicer somehow.

[9]Its value over two open sets $U, V \subseteq \Sigma$ is the set of holomorphic maps $U \to V$.

1.10. **Renormalisation flow.** One might guess the following picture.



Picture a family of two dimensional QFTs (whatever that means) defined over some base

$$\mathcal{Q} \ \to \ B$$

and an action of the conformal group Conf(2) on B, for which this family $\mathcal{Q}$ is equivariant. Then over the fixed locus,

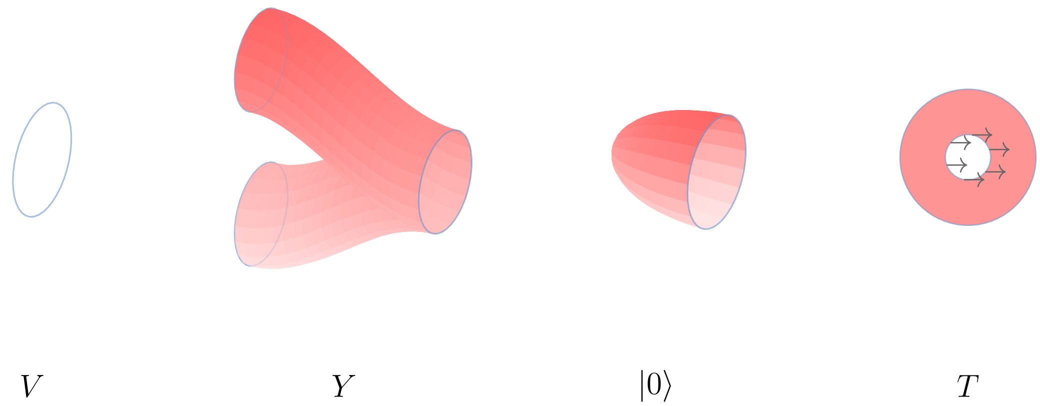
$$\mathcal{Q}|_{B^{\text{fix}}} \ \to \ B^{\text{fix}}$$

we expect an action (whatever that means) of Conf(2) on every QFT $Q \in \mathcal{Q}|_{B^{\text{fix}}}$, which in particular should give an action on most structures discussed above. Thus in examples, we could reasonably expect these to be $2d$ CFTs.

The famous case of the *renormalisation* (*semi*)*group* $\mathbf{R}_+ \subseteq \text{Conf}(2)$ was introduced by Wilson [Wi].

1.11. **Relation to** $3d$ **TQFTs.**

2. What is a vertex algebra?



Definition 2.1. A *vertex algebra* is a

1) vector space V with a map

$$Y(-,z)(-) \;:\; V \otimes V \;\to\; V((z))$$

such that the $Y(\alpha, z)$ for $\alpha \in V$ *weakly commute* (see below):

$$(z-w)^N [Y(\alpha,z), Y(\beta,w)] \;=\; 0 \qquad\qquad \text{for } N = N(\alpha,\beta) \gg 0,$$

2) a distiguished vector $|0\rangle$ with

$$Y(|0\rangle, z) \;=\; \mathrm{id}, \qquad\qquad Y(\alpha,z)|0\rangle \;=\; \alpha \quad \mod zV[[z]],$$

3) and an endomorphism T satisfying $T|0\rangle = 0$ and $[T, Y(\alpha,z)] = \partial_z Y(\alpha,z)$.

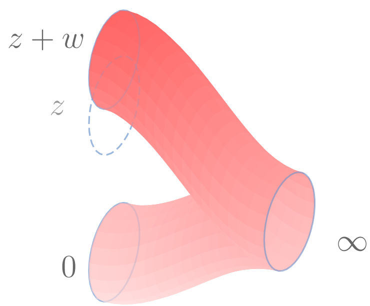
2.2. In the above, $[Y(\alpha,z), Y(\beta,w)]$ viewed as an element of the middle vector space:

$$\begin{array}{ccccc} (\mathrm{End}\, V)((z))((w)) & \to & (\mathrm{End}\, V)[[z^{\pm 1}, w^{\pm 1}]] & \leftarrow & (\mathrm{End}\, V)((w))((z)) \\ Y(\alpha,z)Y(\beta,w) & & & & Y(\beta,w)Y(\alpha,z) \end{array}$$

Noting the above are elements of $\mathrm{QCoh}\,(D_z \times D_w)$, weakly commuting means the commutator is supported on the diagonal.

2.3. **Sanity check.** Treat these as exercises, and see [FBZ] for the answers.

1) T is actually infinitesimal translation (so e^{wT} is translation):

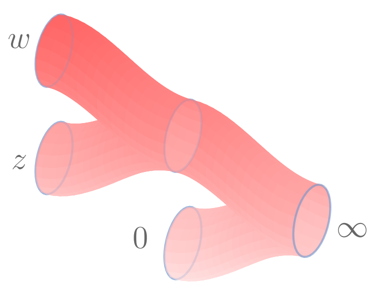


Lemma 2.4. $e^{wT}Y(\alpha,z)e^{-wT} = Y(\alpha, z+w)$.

Lemma 2.5. $Y(\alpha,z)|0\rangle = e^{zT}\alpha$.

This implies that $Y(\beta,z)\alpha = e^{zT}Y(\alpha,-z)\beta$, which is the analogue of commutativity.

2) Y is associative:



Lemma 2.6. $Y(\alpha,z)Y(\beta,w)\gamma = Y(Y(\alpha,z-w)\beta,w)\gamma$.

2.7. **First Examples.**

1) *Commutative algebras.* Any commutative algebra A is a vertex algebra, with

$$Y(a,z)b \;=\; a\cdot b,$$

as well as $T=0$ and $|0\rangle = 1$. These should be viewed as "topological" vertex algebras, since the structure does not depend on z, hence on the complex structure of the pair of pants. More precisely, A is the same as a partially defined $2d$ TQFT where we only allow (compositions of) cap and pants cobordisms.[10]

[10]Similarly, Frobenius algebras ($2d$ TQFTs) should be examples of $2d$ CFTs.

2) *Commutative algebras with derivation.* Any commutative algebra with a derivation (A, ∂) is a vertex algebra, with

$$Y(a,z)b \;=\; (e^{z\partial}a)\cdot b,$$

as well as $T = \partial$ and $|0\rangle = 1$. One can show these are all possible *holomorphic* examples: vertex algebras where $Y(-,z)$ includes no negative powers of z.[11]

$\star$) *Heuristic: loop spaces.* Given a manifold X, we can consider its *loop space* $\mathrm{Maps}(\mathbf{C}^\times, X)$. Its tangent space (however we define it) at the loop γ should be

$$T_\gamma \,\mathrm{Maps}(\mathbf{C}^\times, X) \;=\; \Gamma\left(\mathbf{C}^\times, \gamma^* TX\right).$$

In particular, considering

$$T\mathbf{C}^\times \overset{d\gamma}{\to} \gamma^* TX$$

it is plausible that any vector field on $\mathbf{C}^\times$ should induce one on $\mathrm{Maps}(S^1, X)$,[12] and using the translation vector field, that the function ring $\mathcal{O}\left(\mathrm{Maps}(\mathbf{C}^\times, X)\right)$ should be a (holomorphic) vertex algebra.

2)' *Jet spaces.* The *formal arc space*[13] $J_\infty X$ of a scheme X is defined as

$$J_\infty X(A) \;:=\; X\left(A[[t]]\right).$$

2)" *Canonical dequantisations.* Every vertex algebra has the *Li* filtration

$$V^{\leq n} \;=\; (\text{span of } \alpha_{-n_1-1}\beta_{-n_2-1}\cdots|0\rangle \;:\; n_1+n_2+\cdots \;\leq n)$$

where we take the span over all finite collections $\alpha, \beta, ... \in V$. This is a vertex ideal[14] and the associated graded $\mathrm{gr}\, V$ inherits a *holomorphic* vertex algebra structure. V is called a *chiral quantisation* of scheme X if

$$\mathrm{gr}\, V \;\simeq\; \mathcal{O}\left(J_\infty X\right).$$

This actually implies that X is Poisson (as $\mathrm{gr}\, V$ is always "vertex Poisson").

[11] Thus it makes sense to call vertex algebras without this property *meromorphic*.

[12] This works with any space replacing S^1.

[13] There is not a good definition yet of (formal) loop space of a scheme.The definition we want is different from the derived loop space $X \times_{X^2} X$, whose ring of functions is nothing like a vertex algebra.

[14] i.e. a vertex submodule of V.

2.8. **Operator product expansions.** The functions vanishing on the diagonal of

$$\hat{D}_z^\times \hat{\times} \hat{D}_w^\times \;=\; \operatorname{Spec} k[[z^{\pm 1}, w^{\pm 1}]]$$

form ideal generated by the *delta function*

$$\delta(z-w) \;:=\; \sum_{k\in\mathbf{Z}} z^k w^{-k-1}$$

and its derivatives $\partial_z^n \delta(z-w)$. Indeed, multiples of $\delta(z-w)$ are clearly precisely the functions killed by $(z-w)$, and similarly for its higher derivatives and $(z-w)^n$.

2.9. In particular, it follows that

$$[Y(\alpha,z), Y(\beta,w)] \;=\; \gamma_0(z,w)\delta(z-w) + \cdots + \gamma_N(z,w)\partial_z^N \delta(z-w).$$

The functions γ_i should be viewed as the "structure constants" of the vertex algebra. They may be chosen which only depend on w (or only on z), because of the delta functions.

Theorem 2.10. (OPE) *The $\gamma_i(w)$ are themselves fields:*

$$\gamma_k(w) \;=\; \frac{1}{k!} Y\left(\alpha_{k-1}\beta, w\right)$$

where we have written $Y(\alpha,z) = \sum \alpha_k z^{-k-1}$. Moreover,

$$Y(\alpha,z)Y(\beta,w) \;=\; \sum_{k\in\mathbf{Z}} \frac{1}{(z-w)^k} Y\left(\alpha_{k-1}\beta, w\right).$$

2.11. **Important examples.**

3) *Lie algebras: Wess Zumino Witten models.* Let $(\mathfrak{g},\kappa)$ be a finite dimensional Lie algebra with an invariant bilinear form.

4) *Lattice vertex algebras.* Let (Λ,κ) be a lattice with an even bilinear form. We will construct a vertex algebra with associated graded

$$\mathcal{O}\left(J_\infty \mathfrak{h}\right) \otimes \mathbf{C}[\Lambda], \qquad\qquad \mathfrak{h} \;=\; \Lambda \otimes_{\mathbf{Z}} \mathbf{C}$$

which one should think of functions on the (undefined) loop space of the torus, $\mathrm{Maps}(\mathbf{C}^\times, \mathfrak{h}/\Lambda)$. We already know how to quantise the first piece,

$$\mathcal{O}(J_\infty \mathfrak{h}) \;\rightsquigarrow\; V_\kappa(\mathfrak{h})$$

where the vector space $\mathfrak{h}$ is viewed as a Lie algebra with trivial bracket. If we ask that $V_\kappa(\mathfrak{h}) \otimes e^\lambda$ is the Verma representation with weight $(\lambda, -)$, everything else follows. The OPE is

$$[\lambda(z), e^\mu(z)] \;=\; (\lambda,\mu) e^\mu(w)\delta(z-w) \qquad\qquad \text{for } \lambda,\mu \;\in\; \Lambda$$

which in turn implies that

$$e^\mu(z) \;=\; \pm e^\mu e^{-\sum_{k<0}\frac{1}{k}\mu_k z^{-k}} e^{-\sum_{k>0}\frac{1}{k}\mu_k z^{-k}}$$

where $\pm : \Lambda^2 \to \mathbf{Z}/2$ is an element of $\mathrm{H}^2(Q,\mathbf{Z}/2)$ trivial on $0\in\Lambda$.

5) *Affine W algebras.* Covered next time.

6) *Chiral differential operators.* A sheaf of vertex algebras on X whose associated graded is the pushforward of $\mathcal{O}_{J_\infty T^*X}$ along $J_\infty T^*X \to X$ is called a sheaf of *chiral differential operators.* The obstruction to its existence is

$$\frac{1}{2}\,\mathrm{ch}_2\left(\mathcal{T}_X\right) \;\in\; \mathrm{H}^2\left(X, \Omega^2_X \to \Omega^3_{X,cl}\right).$$

If this vanishes, the isomorphism classes of CDOs is H^1 and the infinitesimal automorphisms of each CDO is H^0.

For instance, when $X = \mathbf{A}^1$ the unique CDO is generated by two fields called $x(z), \partial(z)$, subject to

$$[\partial(z), x(w)] \;=\; \delta(z-w).$$

It has no automorphisms.

3. What is a factorisation algebra?

3.1. We first define an algebraic geometric analogue of the collection of finite subsets of X, which are allowed to "collide". Let X be any prestack, and take the functor defined on the category FSet^{surj} of nonempty finite sets with surjections

$$X^{(-)} \ : \ \mathrm{FSet}^{surj} \ \to \ \mathrm{PreStk}$$

. The *Ran space* of X is the colimit of this diagram

$$\mathrm{Ran}X \ = \ \mathrm{colim}_{I\in \mathrm{FSet}^{surj,op}} X^I.$$

Thus $\mathrm{Maps}(S, \mathrm{Ran}\, X)$ is the set of nonempty finite subsets of $\mathrm{Maps}(S, X)$, see e.g. sending I to X^I, and a surjection $I \to J$ to the associated diagonal map $\Delta_{I/J} : X^J \to X^I$

3.2. The Ran space is a (nonunital) commutative monoid in PreStk^{corr} in two different ways, meaning that it admits correspondences as below satisfying an associativity condition. The first comes from taking union of finite sets

$$\begin{tikzcd} & \mathrm{Ran}\, X \times \mathrm{Ran}\, X \arrow[dl, equal] \arrow[dr, "\pi"] & \\ \mathrm{Ran}\, X \times \mathrm{Ran}\, X & & \mathrm{Ran}\, X \end{tikzcd} \tag{1}$$

and the second from taking unions on the locus of *disjoint* finite subsets

$$\begin{tikzcd} & (\mathrm{Ran}\, X \times \mathrm{Ran}\, X)_{disj} \arrow[dl, "j"'] \arrow[dr, "\pi j"] & \\ \mathrm{Ran}\, X \times \mathrm{Ran}\, X & & \mathrm{Ran}\, X \end{tikzcd} \tag{2}$$

The fibre of π over a nonempty finite subset $I \subseteq X$ are the pairs of nonempty finite subsets I_1, I_2 with $I = I_1 \cup I_2$. Likewise for πj, except the subsets I_i are disjoint.

3.3. One can also define a *unital* Ran space $\mathrm{Ran}_{un}\, X$ (see [?]), a lax prestack which should be thought of as parametrising all finite subsets of X (including the empty one).

3.4. From now on, assume that X is a seperated scheme of finite type over a field k. It follows from the definition of the Ran space as a colimit that its category of $\mathcal{D}$ modules (see section **??**) is

$$\mathcal{D}(\mathrm{Ran}\, X) \ = \ \lim_{I\in \mathrm{FSet}^{surj}} \mathcal{D}(X^I),$$

meaning a $V \in \mathcal{D}(\operatorname{Ran} X)$ corresponds to a collection of $V_I \in \mathcal{D}(X^I)$ with compatible isomorphisms $V_J \simeq \Delta^!_{I/J} V_I$ for all surjections of (nonempty) finite sets $I \twoheadrightarrow J$. To give a $\mathcal{D}$ module on the *unital* Ran space is to in addition supply compatible maps $\Delta^!_{I/J}\mathcal{F}_I \to \mathcal{F}_J$ for all maps of finite sets $I \to J$. For instance, this gives a map $V_\emptyset \otimes \omega_{X^I} \to V_I$ for all I.

3.5. By smooth base change, each (nonunital) commutative monoid structure on $\operatorname{Ran} X$ as an object in PreStk^{corr} where the rightwards map to $\operatorname{Ran} X$ is an open immersion induces a (nonunital) symmetric monoidal structure on $\mathrm{Sh}(\operatorname{Ran} X)$. Applying this to the above monoidal structures, we get the $*$ and *chiral* tensor products

$$\mathcal{A} \otimes^* \mathcal{B} \;=\; \pi_*(\mathcal{A} \boxtimes \mathcal{B}), \qquad\qquad \mathcal{A} \otimes^{ch} \mathcal{B} \;=\; \pi_* j_* j^!(\mathcal{A} \boxtimes \mathcal{B}).$$

3.6. It is easy to describe these tensor products explicitly [**?**, §2.3], first

$$(\mathcal{A} \otimes^* \mathcal{B})_I \;=\; \bigoplus_{I = I_1 \cup I_2} \Delta^!_{I_1 \amalg I_2 / I}(\mathcal{A}_{I_1} \boxtimes \mathcal{B}_{I_2}),$$

where direct sum is over all two nonempty subsets I_1, I_2 with $I = I_1 \cup I_2$, not necessarily disjoint. To describe the chiral tensor product, we write $j : (X^{I_1} \times X^{I_2})_{disj} \hookrightarrow X^I$ for the open locus where the first I_1 and last I_2 points are disjoint. Since $j^! = j^*$, we have

$$(\mathcal{A} \otimes^{ch} \mathcal{B})_I \;=\; (\pi_* j_* j^* \mathcal{A} \boxtimes \mathcal{B})_I \;=\; \bigoplus_{I = I_1 \amalg I_2} j_{I*} j_I^*(\mathcal{A}_{I_1} \boxtimes \mathcal{B}_{I_2}),$$

where direct sum is over partitions $I = I_1 \amalg I_2$ into disjoint nonempty subsets.

3.7. We now define a factorisation algebra over a scheme X of finite type over a field of characteristic 0.

Definition 3.8. [**?**, **?**] A *factorisation algebra* is a (chiral) cocommutative coalgebra

$$\mathcal{B} \;\in\; \mathrm{commCoAlg}\left(\mathcal{D}(\operatorname{Ran} X), \otimes^{ch}\right)$$

which *factorises*: considering the coproduct $\mathcal{B} \to \mathcal{B} \otimes^{ch} \mathcal{B}$, each component

$$\mathcal{B}_I \;\to\; j_{I*} j_I^* \mathcal{B}_{I_1} \boxtimes \mathcal{B}_{I_2} \qquad\qquad I \;=\; I_1 \amalg I_2$$

becomes an equivalence when restricted to the open locus (i.e. after applying j_I^*).

Definition 3.9. [**?**, **?**] A *chiral algebra* is a (chiral) Lie algebra

$$\mathcal{A} \ \in \ \mathrm{Lie}\left(\mathcal{D}(\mathrm{Ran}\, X), \otimes^{ch}\right)$$

lying in the image of $\Delta_* : \mathcal{D}(X) \to \mathcal{D}(\mathrm{Ran}\, X)$.

3.10. **Koszul duality.**

3.11. **Equivalence with vertex algebras.**

Theorem 3.12. *Weakly* $\mathbf{G}_a$ *equivariant chiral algebras...*

3.13. **Examples.**

1) There is a map $\otimes^* \to \otimes^{ch}$, so any $*$ factorisation coalgebra

$$\mathcal{A} \ \to \ \mathcal{A} \otimes^* \mathcal{A}$$

is in particular a chiral factorisation coalgebra too. In the above equivalence, these give *holomorphic* vertex algebras.

2) *Lie algebra: Wess Zumino Witten model.*

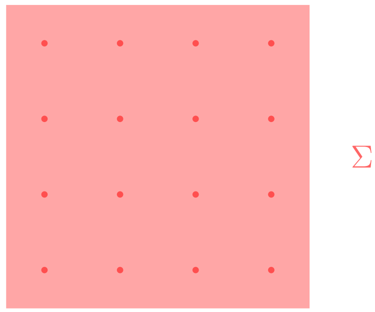
3) *Lattice models.* Let T be a torus. Apply the above, but noting that $\pi_0\left(\mathrm{Gr}_T\right) = \Lambda$ is the cocharacter lattice of T, consider

$$\Lambda \times \mathrm{Ran}\, X \ \to \ \mathrm{Gr}_{T,X}$$

and run the above.

4. Examples of $2d$CFTs, vertex, factorisation and chiral algebras

4.1. **Ising model.** Consider a finite graph Λ of particles on a Riemann surface, each can be in two states $\{\pm 1\}$, called *spin up* or *spin down*. Pick a positive real number T called the *temperature*.



Something that is close to (but not) a classical CFT is:

$$\tilde{\mathcal{Z}}(\Sigma) \;=\; \mathrm{Fun}\,(\Lambda, \{\pm 1\})$$

with the probability measure given by

$$\mu(\sigma) \;\propto\; \exp\left(-\frac{1}{T}\sum_{\lambda\sim\lambda'}\sigma(\lambda)\sigma(\lambda')\right)$$

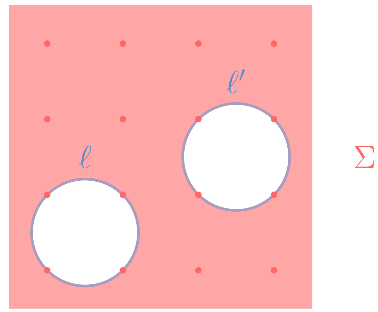
where the sum is called the *energy*. One can compute that (for square lattices in $\mathbf{C}$),

$$\mathbf{E}\left(\sigma(\lambda_1)\sigma(\lambda_2)\right) \;\approx\; \begin{cases}\log|\lambda_1-\lambda_2| & |\lambda_1-\lambda_2|\ll L\\ e^{-|\lambda_1-\lambda_2|/L}\cdot|\lambda_1-\lambda_2|^{1/2} & |\lambda_1-\lambda_2|\gg L\end{cases}$$

Where the so called *length scale* L is a function of T that has a single pole at T_c, the *critical temperature* (see [To]). Thus,

- away from critical temperature, a generic σ will have blobs of the same spin, with most blobs of radius approximately L,
- at critical temperature there are blobs of all sizes, and the correlation between the value of $\sigma(\lambda_1)$ and $\sigma(\lambda_2)$ is to leading order $\log|\lambda_1-\lambda_2|$.

Now given one-manifolds



then setting $\mathcal{Z}(\ell) = \mathrm{Fun}\,(\Lambda \cap \ell, \{\pm 1\})$, we have restriction maps

$$\begin{tikzcd} & \tilde{\mathcal{Z}}(\Sigma) \arrow[dl] \arrow[dr] & \\ \mathcal{Z}(\ell) & & \mathcal{Z}(\ell') \end{tikzcd}$$

which we can pull-push along using the measure:

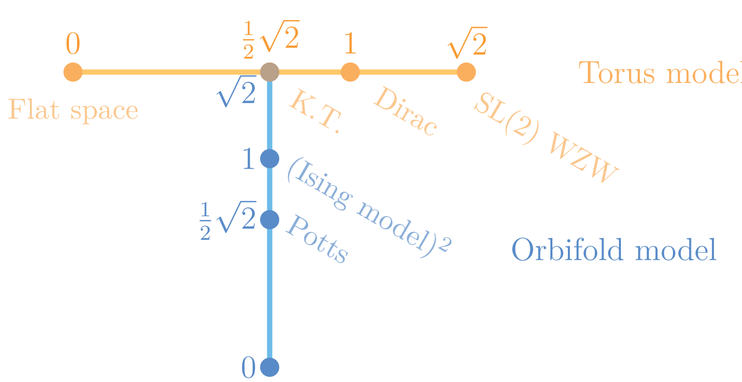
$$\mathcal{Z}(\ell) \;\to\; \mathcal{Z}(\ell') \qquad\qquad f \;\mapsto\; \sum_{\sigma\,:\,\sigma|_\ell = f} \mu(\sigma)\cdot \sigma|_{\ell'}.$$

Thus the functions $\sigma \in \tilde{\mathcal{Z}}(\Sigma)$ that contribute the most to this map are the low energy ones.

Conjecture 4.2. *If we take some appropriate limit over finer and finer graphs $\Lambda \subseteq \Sigma$, the Ising model at critical temperature gives a $2d$ conformal field theory.*

This is related to *conformal nets* in mathematics, see [He].

4.3. **All $2d$ CFTs with charge one.** Physicists expect that the only conformal field theories on which the Virasoro acts with central charge $c = 1$ are *torus models* and *orbifold models*, see [DVV]:



The *torus* (or *lattice*) *model* is meant to be a quantisation of maps

$$\mathcal{Z}(S^1) \;=\; \mathcal{O}\left(\mathrm{Maps}(S^1,T)\right) \qquad \text{where} \quad T \;=\; \mathbf{R}/2\pi\frac{R}{2}\mathbf{Z} \times \mathbf{R}/2\pi\frac{1}{R}\mathbf{Z},$$

with R a positive real number and Maps means homotopy classes of maps. The elements of the mapping space are

$$r \;\mapsto\; \left(\frac{1}{2}nRr, \frac{m}{R}r\right), \quad n,m \in \mathbf{Z}.$$

4.4. **Loop spaces and their quantisations.** If X is a Poisson space, then setting

$$\mathcal{Z}(S^1) \;=\; \mathrm{Maps}(S^1,X), \qquad \tilde{\mathcal{Z}}(\Sigma) \;=\; \mathrm{Maps}(\Sigma,X)$$

should probably give a classical $2d$ conformal field theory, e.g. the Poisson form on X induces a shifted Poisson form of degrees -1 and -2 on the above spaces, respectively.

Examples:

1) *Lie algebras* $\mathfrak{g}$. The function on $\mathrm{Maps}(\Sigma, G)$ here is an integral
$$S(\gamma) \;=\; k\int_\Sigma (\cdots) + \int_B \pm(\cdots,[\cdots,\cdots])$$
where B is a three manifold with $\partial B = \Sigma$. The latter is the pullback of the tautological three form on G.

2) *Tori* $T = \mathbf{G}_m^n$.

3) *Slodowy slices.*

4) *ALE spaces* $\mathrm{Hilb}^n\, \mathbf{C}^2/\Gamma$.

5) *Cotangent bundles* T^*X.

6) *Nakajima quiver varieties.*

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