KZ EQUATIONS

ALEXEI LATYNTSEV

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1. KZ equations

Let $V_1, ..., V_n$ be representations of a finite dimensional simple Lie algebra $\mathfrak{g}$. Pick extra data $\Omega \in \mathrm{Sym}^2\, \mathfrak{g}$ and $k - k_{crit} \in \mathbf{C}$. Then the **KZ equations** are the following n many differential operators

$$(k - k_{crit})\partial_{z_i} \;+\; \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

on the trivial vector bundle with fibre $V_1 \otimes \cdots \otimes V_n$ on the space $(\mathbf{C}^n)_\circ$.

1.1. **Warmup computation.** If we consider the differential operators

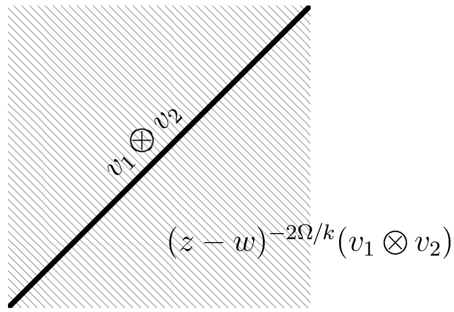
$$k\partial_z \;+\; \frac{\Omega_{12}}{z-w} \qquad\qquad k\partial_w \;+\; \frac{\Omega_{21}}{w-z}$$

then as Ω is symmetric solving these equations is equivalent to $\partial_{z+w} = 0$ and $\partial_{z-w} = \Omega/k(z-w)$. A solution to this is given by

$$v(z,w) \;=\; (z-w)^{-2\Omega/k}(v_1 \otimes v_2) \tag{1}$$

for any $v_i \in V_i$. In particular, the monodromy of this solution is given by $q^\Omega = e^{-\pi i \Omega/k}$.

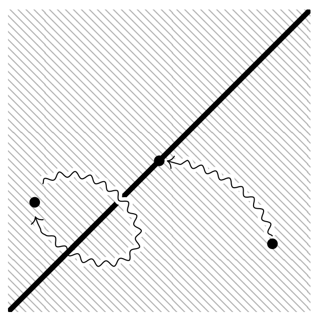
1.1.1. For the above solution (1), for (z, w) off the diagonal we would get the element $(z-w)^{-2\Omega/k}(v_1 \otimes v_2)$, and anywhere on the diagonal we would get $v_1 \otimes v_2$:



The monoidal structure

$$\otimes \;:\; \operatorname{Rep} U(\mathfrak{g}) \otimes \operatorname{Rep} U(\mathfrak{g}) \;\to\; \operatorname{Rep} U(\mathfrak{g})$$

looks like



Its braiding is given by monodromy around the diagonal; note that the braid group is

$$\mathfrak{B}_n \;=\; \pi_1((\mathbf{C}^n)_\circ).$$

Formal KZ equation. Note that the above only converges for $1/k$ small. Thus, we will consider the differential operators

$$\partial_{z_i} \;+\; \hbar \sum_{i \neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

on the trivial vector bundle with fibre $V_1 \otimes \cdots \otimes V_n[[\hbar]]$ on the space $(\mathbf{C}^n)_\circ$.

Theorem. (Kohno-Drinfeld) *The induced associator and braiding gives* $\operatorname{Rep} U(\mathfrak{g})$ *a different braided monoidal structure, equivalent to* $\operatorname{Rep} U_\hbar(\mathfrak{g})$*.*

Proof. Given a solution to the KZ equations, we can take:

- take its value away from the diagonals,
- take its residue along a diagonal $z_i = z_j$ avoiding the other diagonals, i.e. take the coefficient of $(z_i - z_j)^{-2\Omega_{ij}/(k-k_{crit})}$,

- take its residue along two diagonals $z_i = z_j$ and $z_k = z_\ell$, avoiding the other diagonals, i.e. take the coefficient of $(z_i - z_j)^{-2\Omega_{ij}/(k-k_{crit})}(z_k - z_\ell)^{-2\Omega_{k\ell}/(k-k_{crit})}$, (need the Ω_{ij}s to commute)
- and so on,

to get an element of $V_1 \otimes \cdots \otimes V_n[[\hbar]]$ attached to every point of $\mathbf{C}^n$. This will be an algebraic function on each locally closed stratum. We may parallel transport between these, since in a neighbourhood of a diagonal there is a unique function on $(\mathbf{C}^n)_\circ$ with that as residue. (check)

We now endow $\operatorname{Rep} U(\mathfrak{g})$ with the same monoidal structure, but choose a different associator $(V_1 \otimes V_2) \otimes V_3 \simeq V_1 \otimes (V_2 \otimes V_3)$, given by parallel transport from the $z_1 = z_2$ diagonal to the $z_2 = z_3$ diagonal.[1]

□

Remark. More generally, we consider

$$(k - k_{crit})\partial_{z_i} + \sum_{i>j} r_{ij}(z_i - z_j) - \sum_{j<i} r_{ji}(z_j - z_i).$$

$r(z)$ satisfies the classical Yang Baxter equation, i.e. $R(z) = e^{\hbar r(z)}$ satisfies the spectral Yang Baxter equation, if and only if these differential operators commute.

Remark. (check) Note that for any permutation $\sigma \in \mathfrak{S}_n$ acting on $\mathcal{D}_{(\mathbf{C}^n)_\circ}$ preserves the above set of differential operators. However, (might be possible actually, check [GL]) we cannot arrange the above to form a D module on $\operatorname{Ran} \mathbf{A}^1$.

1.2. **KZ equations on other curves.** Let us consider the sequence of maps

$$\mathbf{C} \overset{\exp}{\to} \mathbf{C}^\times \overset{\pi}{\to} E.$$

We construct analytic D-modules on each of these spaces, pulling back to each other, with the one on $\mathbf{C}$ being the KZ equations.

1.2.1. On $(\mathbf{C}^\times)^n_\circ$, the KZ equations are

$$z_i \partial_{z_i} + \sum_{i \neq j} \frac{\Omega_{ij}}{1 - z_i/z_j} + \lambda_i$$

on $(\mathbf{C}^\times)^n_\circ$, for some constant λ_i. Thus, the classical r-matrix is $r(z) = \frac{\Omega_{ij}}{1 - z_i/z_j}$, and $R(z) = e^{\hbar r(z)}$ satisfies the trigonometric Yang-Baxter equation.

Lemma 1.2.2. *This pulls back to the KZ D-module on* **C***. In other words, the pulled back differential equation is gauge equivalent to the KZ equation on* **C***.*

[1] Since we have $\partial_{z_1+z_2} = \partial_{z_2+z_3} = 0$, this does not depend on where on the diagonals we pick. (check)

Proof. Note that indeed under the exponential map we have $\exp_* \partial_z = z\partial_z$,[2] so this matches with our expectation in section 1.3. Next, we have as functions on $(\mathbf{C}^n)_\circ$ that

$$\exp^*(1 - z_i/z_j) \;=\; (1 - e^{z_i}/e^{z_j}) \;=\; (1 - e^{z_i - z_j}) \;=\; (z_i - z_j) \;+\; \mathcal{O}((z_i - z_j)^2).$$

Thus, the pullback of the KZ equation on $\mathbf{C}^\times$ is gauge equivalent to the KZ equation on $\mathbf{C}$ since the higher order terms of this expansion give holomorphic terms:

$$\frac{1}{1 - e^{z_i - z_j}} \;=\; \frac{1}{z_i - z_j} \;+\; \mathcal{O}(1),$$

thus this pullback can be gauged to the KZ equation on $\mathbf{C}$. □

1.2.3. On E, the equation is

$$\xi_i \;+\; \sum_{i \neq j} \frac{\Omega_{ij}}{w_i -_E w_j} \;+\; \mu_i$$

where ξ_i is the generating vector field on E, w_i are (what?) and μ_i are constants.

Lemma 1.2.4. *This pulls back to the KZ equations on* $\mathbf{C}^\times$ *and* $\mathbf{C}$*.*

Proof. (write) □

We need to replace

$$\frac{1}{z} \;\in\; \mathcal{O}_{\mathbf{C}}(1) \qquad \rightsquigarrow \qquad f \;\in\; \mathcal{O}_E(1).$$

There is no global section of $\mathcal{O}_E(1)$, only $\wp \in \Gamma(E, \mathcal{O}_E(2))$. If we choose a branch cut of E, i.e. remove planes so it becomes a square $U \subseteq E$, we may define $\sqrt{\wp} \in \Gamma(U, \mathcal{O}_E(1))$. (is this defined?)

Let $j : E \setminus 0 \to E$. A D-module structure on the pullback of coherent sheaf $\mathcal{V}_E = V \otimes \mathcal{O}_E$ to $E \setminus 0$ is called *regular singular* if the connection restricted to regular sections factors

$$\begin{array}{ccc} \mathcal{T}_E \otimes \mathcal{V}_E & \dashrightarrow & \mathcal{V}_E \otimes \mathcal{O}_E(1) \\ \downarrow & & \downarrow \\ \mathcal{T}_E \otimes j_* j^* \mathcal{V}_E & \xrightarrow{d} & j_* j^* \mathcal{V}_E \end{array}$$

In other words, if d has order one poles along 0 at worst. We are going to want to study D-modules on E^2 which are regular singular over E_Δ.

If we view

$$E \setminus 0 \;\hookrightarrow\; \mathbf{C}^2$$

given as the locus $w^2 = z^3 + az + b$, we may write the action of a generator of $\mathcal{T}_E$ as

$$\xi_i \;+\; \sum_{i \neq j} \frac{\Omega_{ij}}{(??)}$$

where $\xi_i = w\partial_z$.

[2]This formula follows from $\partial_z = e^z \partial_{e^z}$, which is an application of the chain rule.

To specify a vector bundle with connection on E, it is no longer enough to give an operator on $\Gamma(E, \mathcal{O}_E)$, because E is not affine. However, recall

Theorem 1.2.5. (Deligne) *Let $D \subseteq X$ be a normal crossings divisor and $\mathcal{V}$ vector bundle with connection on $X \setminus D$. Then after choosing residues, $\mathcal{V}$ extends uniquely to a D-module with with regular singular connection $\tilde{\mathcal{V}}$ on X. (details)*

For instance, we can apply this to an elliptic curve and any vector bundle with connection $\mathcal{V}$ on $E \setminus 0$ to

$$\nabla : \tilde{\mathcal{V}} \to \tilde{\mathcal{V}}(1) \otimes \Omega^1$$

(write explicitly)

We recall from [FGV] that the elliptic KZ equation are *not* valued in $\mathcal{O}_{E^n}$, but rather in the line bundle $\mathcal{L}$ on $E^n = \mathbf{C}^n/(\Lambda + \tau\Lambda)$ (where here Λ is the coroot lattice of the Lie algebra $\mathfrak{g}$ we are considering and $\mathbf{C}^n = \mathfrak{t}$) given by monodromy

$$\ell(z + \lambda_1 + \lambda_2\tau) = \exp\left(-\pi i\kappa(\lambda_2, \lambda_2) - 2\pi i\kappa(\lambda_2, z + \lambda_1)\right) \cdot \ell(z). \tag{2}$$

Note that in [FBZ, §I] they omit the z from this notation. We also assume that it is W_G-symmetric, and ℓ vanishes to a certain order along the coroot hyperplanes. (check)

Note that only degree *zero* line bundles can have connections. In particular, since $\mathcal{O}\cdot\theta \simeq \mathcal{O}\left(0 + \frac{1}{2} + \frac{\tau}{2} + \frac{\tau+1}{2}\right)$, the theta line bundle does not have a connection.

Note that if we consider

$$\mathbf{C}^\times \to E$$

then the pullback of the θ line bundle is trivial; since the monodromy of the θ line bundle in the Λ-direction was trivial:

$$\ell(z + \lambda_1) = \ell(z).$$

The KZ equations on E are now

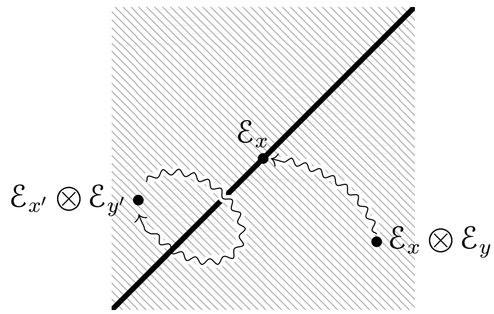
$$\xi_i + \sum_{i \neq j} \frac{\Omega_{ij}}{\theta_i - \theta_j}$$

where ξ_i acts on $\mathcal{L}$ as (write! does ∂_z descend to a vector field on $\mathcal{L} \hookrightarrow \mathcal{O}_{mer} = j_*\mathcal{O}$ where $j : \eta \to E$?) If we take the derivative of (2) then we get

$$\ell'(z+\lambda_1+\lambda_2\tau) = -2\pi i\kappa(\lambda_1, \lambda_2)e^{(-\pi i\kappa(\lambda_2,\lambda_2) - 2\pi i\kappa(\lambda_2,z+\lambda_1))}\cdot\ell(z) + e^{(-\pi i\kappa(\lambda_2,\lambda_2) - 2\pi i\kappa(\lambda_2,z+\lambda_1))}\cdot\ell'(z)$$

1.3. **Sheaves of categories.** How to think of all this structure? (maybe we need to consider $\mathrm{Conf}(\mathbf{C})$ instead of $\mathrm{Ran}\,\mathbf{C}$?)

The answer is as a constructible sheaf of categories over $\mathrm{Ran}\,\mathbf{C}$, which factorises.



Here, the Ran space is endowed with the stratification by diagonals, and we have a constructible sheaf of categories on each $\mathbf{C}^n$, i.e. a functor

$$\mathcal{E}|_{\mathbf{C}^n} \ : \ \mathrm{Exit}_{\mathbf{C}^n} \ \to \ \mathrm{dgCat}$$

from the category of paths staying within the same strata except at the endpoints. Over $n = 1$ this just gives a category $\mathcal{E}_1$, and considering $n = 2$ gives it a braided monoidal structure, and then $n \geqslant 3$ corresponds to higher homotopy data.

To be precise, there is a functor

$$\mathrm{Fact} \ : \ \mathbf{E}_2\text{-}\mathrm{Alg}(\mathrm{dgCat}) \ \to \ \mathrm{FactCat}_c(\mathrm{Ran}\,\mathbf{C})$$

due to Lurie; see [CF, 6.3.3] or [Lu, §A.6] for an account. One expects an equivalence to $\mathrm{FactCat}(\mathrm{Ran}\,\mathbf{A}^1_{dR})$, but we cannot find this in the literature.

Lemma 1.3.1. *The **Drinfeld-Kohno** constructible sheaf of categories $\mathrm{Fact}\,\mathrm{Rep}\,U_\hbar(\mathfrak{g})$ has the following properties:*

- *Its fibre over $(z_1, \dots, z_n)$ is spanned by tuples $V_1 \boxtimes \cdots \boxtimes V_n$ of element of $(\mathrm{Rep}\,U_\hbar(\mathfrak{g}))^{\otimes n}$, where V_i are representations of $\mathfrak{g}$.*[3]
- *The exit path sending $z_i \to z_j$ is sent to the functor $\mathrm{Rep}\,U_\hbar(\mathfrak{g})_i \otimes \mathrm{Rep}\,U_\hbar(\mathfrak{g})_j \to \mathrm{Rep}\,U_\hbar(\mathfrak{g})$ given by $V_i \boxtimes V_j \mapsto V_i \otimes V_j$.*
- *The monodromy around the diagonal $z_i = z_j$ is given by the endomorphism of $(\mathrm{Rep}\,U_\hbar(\mathfrak{g}))^{\otimes n}$ given by swapping the two factors $V_i \boxtimes V_j \mapsto V_j \boxtimes V_i$.*
- *The contractible two-cell bounded by a loop around $z_i = z_j$ and two exit paths is the natural transformation*

[3]Note that $\mathrm{Rep}\,U(\mathfrak{g}) \simeq \mathrm{Rep}\,U_\hbar(\mathfrak{g})$ as categories if we forget the braided monoidal structure.

$$\begin{tikzcd} \operatorname{Rep} U_\hbar(\mathfrak{g}) \otimes \operatorname{Rep} U_\hbar(\mathfrak{g}) \arrow[dr, bend left, "\otimes"] \arrow[d, "\wr\downarrow\sigma"'] & \\ \operatorname{Rep} U_\hbar(\mathfrak{g}) \otimes \operatorname{Rep} U_\hbar(\mathfrak{g}) \arrow[r, bend right, "\otimes"'] & \operatorname{Rep} U_\hbar(\mathfrak{g}) \arrow[u, phantom, "\wr\Uparrow"] \end{tikzcd}$$

given on objects by the endomorphism $R = e^{\hbar\Omega} : V_i \otimes V_j \to V_j \otimes V_i$.

Likewise, it relates to the KZ equations as follows:

- *a flat section $v_1(z) \otimes \cdots \otimes v_n(z) \ : \ \mathrm{triv} \ \to \ \operatorname{Fact} \operatorname{Rep} U_\hbar(\mathfrak{g})$ over an open set $U \ \subseteq \ (\mathbf{C}^n)_\circ$ is precisely a solution to the KZ equations for $V_1 \otimes \cdots \otimes V_n$ on U.*

1.3.2. Note that if we were to consider other base curves, the restriction $\mathcal{E}_1$ becomes interesting. Whereas over $\mathbf{C}$ it only has the structure of a category, over $\mathbf{C}^\times$ and E it has one and two commuting automorphisms, which the structures we discuss above must respect. For instance, writing T for such an automorphism, we have

$$T(V \otimes V') \ = \ T(V) \otimes T(V')$$

respects the monoidal structure, and likewise the braiding.

If $\mathcal{E}_E$ is any such constructible factorisation category on an elliptic curve, we have functors

$$\mathcal{E}_1 \ = \ \Gamma(\mathbf{C}, \mathcal{E}_{\mathbf{C},1}) \ \stackrel{\exp^*}{\leftarrow} \ \Gamma(\mathbf{C}^\times, \mathcal{E}_{\mathbf{C}^\times,1}) \ \stackrel{\pi^*}{\leftarrow} \ \Gamma(E, \mathcal{E}_{E,1}).$$

Moreover, one expects a Galois correspondence between subcategories of $\mathcal{E}_1$ and subgroups of $\pi_1(E)$, and the above we expect is equal to

$$\mathcal{E}_1 \ \stackrel{\exp^*}{\leftarrow} \ \mathcal{E}_1^{\mathbf{Z}} \ \stackrel{\pi^*}{\leftarrow} \ \mathcal{E}_1^{\mathbf{Z}^2}.$$

For instance, the deck cover group of $\exp$ is generated by $\hbar \mapsto \hbar + 2\pi i$, so this conjecture is saying that

$$(\operatorname{Rep} U_\hbar(\mathfrak{g}))^{\mathbf{Z}} \ \stackrel{?}{\stackrel{\sim}{\leftarrow}} \ \operatorname{Rep} U_q(\mathfrak{g}).$$

This should extend to the entire constructible sheaves of categories, however we note that $\operatorname{Ran} \mathbf{C} \to \operatorname{Ran} \mathbf{C}^\times$ is not a $\mathbf{Z}$-covering map. We do not know the definition of $\operatorname{Rep} U_{q,t}(\mathfrak{g})$, but presumably if the above is correct it should be $\mathbf{Z}^2$-invariants inside $\operatorname{Rep} U_\hbar(\mathfrak{g})$.

The action of $\mathbf{Z}$ on the category $\mathcal{E}_{\mathbf{C},z} \simeq \mathcal{E}_{\mathbf{C}^\times, e^z}$ is given by the monodromy of the trigonometric KZ equation, computed in [EG, Thm. 3.2] to be

$$\tau \ = \ e^{\hbar(s+m(r))} m(R) \ = \ q^{s+m(r)} m(R)$$

where we have contracted $r = \Omega$ using the multiplication m in $U_\hbar(\mathfrak{g})$ and s is any even element with $[\Delta(s), \Omega] = 0$.

1.3.3. *Remark.* The inclusion[4] $U_q(\mathfrak{g}) \hookrightarrow U_\hbar(\mathfrak{g})$ allows us to form[5]

$$\mathrm{Ind} \;:\; \mathrm{Rep}\, U_q(\mathfrak{g}) \;\leftrightarrows\; \mathrm{Rep}\, U_\hbar(\mathfrak{g}) \;:\; \mathrm{Res} \;=\; \exp^* .$$

We do not know how to interpret Ind in terms of the constructible sheaf of categories.

1.3.4. *Partial inverses to* $\exp$ *and* π. Given a branch of the logarithm, i.e. a partially defined section $\log : \mathbf{C}^\times \to \mathbf{C}$ to the exponential map, we can consider the function on $(\mathbf{C}^\times)^n_\circ$

$$\log^*(z_i - z_j) \;=\; \log(z_i) - \log(z_j) \;=\; \log(z_i/z_j) \;=\; (1 - z_i/z_j) + \frac{1}{2}(1 - z_i/z_j)^2 + \cdots$$

thus $1/\log^*(z_i - z_j)$ is gauge-equivalent to $1/(1 - z_i/z_j)$. Likewise, $\log_*(z\partial_z) = \partial_z$.[6]

1.3.5. *Remark: affine analogue.* Why couldn't we have just applied the above section to $\mathfrak{g}$ an arbitrary Kac-Moody Lie algebra?

One answer is that we can of course define the equations, but since $\mathrm{Rep}\, U_\hbar(\mathfrak{g})$ is factorisation braided rather than braided, the Drinfeld-Kohno and Gaitsgory-Lysenko constructions cannot have applied in their usual forms.

1.4. **Comparison to [GL].** In [Ga], one considers the configuration space $\mathrm{Conf}_\Lambda(\mathbf{C})$ of ordered points labelled by nonnegative roots Λ.

One constructs a factorisable $\mathrm{B}\mathbf{G}_m$ gerbe $\mathcal{G}$ on $\mathrm{Conf}_\Lambda(\mathbf{C})$, and consider $\mathcal{G}$-twisted sheaves. Moreover (in [GL] somewhere) we have

$$\mathrm{Rep}_q(T) \;\simeq\; \mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C})).$$

The three integral forms of $U_q(\mathfrak{b})$ are constructed as pushforwards of constant sheaves from the open locus.

In [GL] one defines $u_q(N)$ inside $\mathrm{Rep}_q(T)$, then takes the relative Drinfeld double of $u_q(N)\text{-Mod}(\mathrm{Rep}_q(T))$ to get $u_q(\mathfrak{g})$-Mod. We get the (baby) renormalised version of this if we take the ind-completion with respect to finite dimensional modules (resp. before taking the Drinfeld double).

Then as in [GL, p202], we apply Lurie's construction of a factorisation algebra $\Omega_B \in \mathcal{D}\text{-Mod}(\mathrm{Ran}\,\mathbf{A}^1)$ attached to any $\mathbf{E}_2$-algebra B in braided monoidal category, with

$$\Omega_B\text{-FactMod}(\mathrm{Gr}_{T,\mathbf{A}^1}) \;\simeq\; B\text{-Mod}_{\mathbf{E}_2}.$$

We apply this to $B = \mathrm{Aug}(\mathrm{inv}_{u_q(\mathfrak{n})})$ being the augmentation ideal of the invariants functor

$$\mathrm{inv}_{u_q(\mathfrak{n})} \;:\; u_q(\mathfrak{n})\text{-Mod} \;\to\; \mathrm{Rep}_q(T)$$

[4]As $\mathbf{Z}[q, q^{-1}] \hookrightarrow \mathbf{Z}[[\hbar]]$-algebras.

[5]Here $\mathrm{Ind}\, V = V \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Z}[[\hbar]]$; in particular Ind might send non-isomorphic representations V, V' to isomorphic ones. We have an embedding $V \hookrightarrow V \otimes_{\mathbf{Z}[q,q^{-1}]} \mathbf{Z}[[\hbar]] = \mathrm{Res}\,\mathrm{Ind}\, V$.

[6]This again follows from the chain rule, $z\partial_z = \partial_{\log(z)}$.

which for general reasons has $B\text{-Mod}_{\mathbf{E}_2} \simeq Z_{\mathbf{E}_1}(u_q(\mathfrak{n})\text{-Mod}^{ren})$. On the other side, we have by a Riemann-Hilbert argument that factorisation modules over $\mathrm{Gr}_{T,\mathbf{A}^1}$ are equivalent to configuration factorisation modules over $\mathbf{C}$, and under this equivalence we have $\Omega_B\text{-FactMod}(\mathrm{Gr}_{T,\mathbf{A}^1}) \simeq \Omega_q^{sm}\text{-FactMod}(\mathrm{Conf}_\Lambda(\mathbf{C}))$.

Theorem 1.4.1. *(prove this) The constructible factorisation category over* $\mathrm{Conf}_\Lambda(\mathbf{C})$

$$u_q(\mathfrak{g})\text{-Mod}^{baby\ ren} \simeq B\text{-Mod}_{\mathbf{E}_2} \simeq \Omega_B\text{-FactMod}(\mathrm{Gr}_{T,\mathbf{A}^1}) \simeq \Omega_q^{sm}\text{-FactMod}(\mathrm{Conf}_\Lambda(\mathbf{C}))$$

has sections being collections of $V_1, ..., V_n$ *together with their KZ equations over* $\mathbf{C}^n$.

Proof. The equivalences follow by the above discussion □

There is a factorisable version of (a completion of) $u_q(\mathfrak{g})$-Mod over $\mathrm{Conf}_\Lambda(\mathbf{C})$. It equivalent to factorisation modules over Ω_q.

1.4.2. *Relation to Riemann-Hilbert.* All the above is on the topological side; we now talk about how to pass to the algebraic side. As explained in [CF, Prop 6.3.3] there is a functor

$$\mathbf{E}_2\text{- Cat} \to \mathrm{FactCat}(\mathrm{Ran}\,\mathbf{A}^1)$$

compatible with the global sections functor (check). It sends (find reference)

$$U_q^{Lus}(\mathfrak{g})\text{-Mod} \mapsto \hat{\mathfrak{g}}\text{-Mod}^{G(\mathcal{O})}, \qquad \mathrm{Rep}_q(G)^{mxd} \mapsto \hat{\mathfrak{g}}\text{-Mod}^I.$$

1.5. **Relation to doubling.** Recall the following picture:

$$U_q(\mathfrak{n}) \in \mathrm{BiAlg}(\mathrm{Rep}_q\,\mathfrak{t}) \quad \overset{\text{Bosonisation}}{\rightsquigarrow} \quad U_q(\mathfrak{b}) \in \mathrm{BiAlg}(\mathrm{Vect})$$

$$U_q(\mathfrak{n})\text{-Mod}(\mathrm{Rep}_q\,\mathfrak{t}) = \underbrace{U_q(\mathfrak{b})\text{-Mod}(\mathrm{Vect})}_{\otimes} \quad \overset{\mathcal{Z}^{\mathbf{E}_1}}{\rightsquigarrow} \quad \underbrace{U_q(\mathfrak{g})\text{-Mod}(\mathrm{Vect})}_{\otimes_{\mathbf{E}_2}}$$

where the braiding on $\mathrm{Rep}_q\,\mathfrak{t}$ is given by $q^{\kappa(\lambda,\mu)} \in q^{\mathbf{R}} = \mathbf{C}[[\hbar]]$. Note that we need to use this instead of $\mathrm{Rep}_q\,T$ if we are to get an algebra $U_q(\mathfrak{b}) = U_q(\mathfrak{n}) \rtimes U_q(\mathfrak{t})$, since $\mathrm{Rep}_q\,\mathfrak{t}$ is a category of modules, for $U_q(\mathfrak{t})$.

The factorisaton story only works with the unbosonised $U_q(\mathfrak{n})$, rather than $U_q(\mathfrak{b})$.

1.5.1. For Yangians, we expect to have

$$Y_\hbar(\mathfrak{n}) \in \mathrm{BiAlg}_{ch,*}(\mathrm{Rep}\,Y_\hbar(\mathfrak{t})) \quad \overset{\text{Bosonisation}}{\rightsquigarrow} \quad Y_\hbar(\mathfrak{b}) \in \mathrm{BiAlg}_{ch,*}(\mathrm{Vect})$$

$$Y_\hbar(\mathfrak{n})\text{-Mod}(\mathrm{Rep}\,Y_\hbar(\mathfrak{t})) = \underbrace{Y_\hbar(\mathfrak{b})\text{-Mod}(\mathrm{Vect})}_{\otimes^{ch}} \quad \overset{\mathcal{Z}^{\mathbf{E}_1,\otimes}}{\rightsquigarrow} \quad \underbrace{Y_\hbar(\mathfrak{g})\text{-Mod}}_{\otimes,\otimes^{ch}}.$$

Note that $Y_\hbar(\mathfrak{t})$ has a chiral and standard coproduct, so its category of representations has $\otimes$ and $\otimes^{ch}$.

Thus, we expect that $Y_\hbar(\mathfrak{n})$ has a chiral coproduct inside $\mathrm{Rep}\,Y_\hbar(\mathfrak{t})$, and its double $Y_\hbar(\mathfrak{g})$ has a chiral and standard coproduct. Notice that the formula in [GT, §3.1] for the standard coproduct involves the Killing form (β, α_i), which is a smoking gun of it arising from a doubling construction.

1.5.2. In particular, we need to construct analogues to

$\mathrm{Conf}_\Lambda(\mathbf{C})$	$\mathcal{G}$	$\mathrm{Rep}_q\, T \simeq \mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C})_{x\cdot\infty})$
?	?	$\mathrm{Rep}\, Y_\hbar(\mathfrak{t})^{T(\mathcal{O})}$

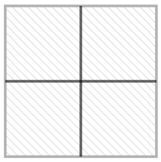
where we have taken the category of $Y_\hbar(\mathfrak{t})$-modules with integral eigenvalues for the action of t_i, where $t \in \mathfrak{t}$ and $i \geqslant 0$.

2. Other KZ equations

2.1. **Motivation.** We have the following picture: (not quite right, $\mathcal{W}_{1+\infty}$ is a vertex algebra not an algebra)

Δ, Δ'

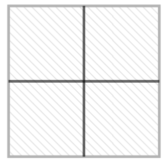
$U_\hbar(\mathfrak{g})$-Mod



$\mathbf{R}_1 \times \mathbf{R}_2$

Δ, Δ'

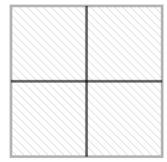
$Y_\hbar(\mathfrak{g})$-Mod



$\mathbf{C}_1 \times \mathbf{R}_2$

Δ, Δ'

$W_{1+\infty,\hbar}(\mathfrak{g})$-Mod



$\mathbf{C}_1 \times \mathbf{C}_2$

Each of the three algebras have two compatible coproducts Δ, Δ', hence their module categories are expected to factorise over the marked spaces. See [GRZ] for $W_{1+\infty}$.

2.1.1. To be precise, we expect sheaves of categories $\mathcal{C}$ over all three spaces, i.e. $\mathrm{Ran}(\mathbf{R}_1 \times \mathbf{R}_2)$ and so on, whose fibres are the three categories named above.

In addition, we need $\mathcal{C}$ to be endowed with a flat connection, loosely speaking because it comes from a TQFT or a holomorphic QFT and so has an action of Lie $\mathrm{Diff}(X) = \Gamma(X, \mathcal{T}_X)$ and Lie $\mathrm{Conf}(X) = \Gamma(X, \mathcal{T}_X^{hol})$. Flatness corresponds to it being a Lie algebra action.

Thus for instance, we expect a sheaf of categories on

$$(\mathbf{C}_1 \times \mathbf{R}_2)_{dR} \;=\; (\mathbf{C}_1 \times \mathbf{R}_2)/\exp(\mathcal{T}^{hol}_{\mathbf{C}_1} \boxplus \mathcal{T}^{sm}_{\mathbf{R}_2})$$

and likewise over $\mathrm{Ran}(\mathbf{C}_1 \times \mathbf{R}_2)$.

2.1.2. *Remark.* Let us consider the relation between these three. Identifying $\mathbf{C}/S^1 \simeq \mathbf{R}_{\geqslant 0}$, the above is presumably attached to

$$\mathbf{C}_{\theta_1,\theta_2} \longrightarrow \mathbf{C}^\times_{\theta_1,\theta_2} \longrightarrow E_{\theta_1,\theta_2}$$

where here $\mathbf{C}_{\theta_1,\theta_2} \simeq \mathbf{R}_{\theta_1} \times \mathbf{R}_{\theta_2}$ is the universal cover of the angle coordinate circles. Thus if we have analogues:

$$\mathbf{R}_{\theta_1} \times \mathbf{R}_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2} \leftsquigarrow S^1_{\theta_1} \times \mathbf{R}_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2} \leftsquigarrow S^1_{\theta_1} \times S^1_{\theta_2} \times \mathbf{R}_{r_1} \times \mathbf{R}_{r_2}$$

$$\mathbf{R}_1 \times \mathbf{R}_2 \longleftarrow \mathbf{C}_1 \times \mathbf{R}_2 \longleftarrow \mathbf{C}_1 \times \mathbf{C}_2$$

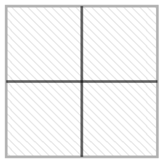
where this analogy matches collapsing an S^1 and taking its universal cover.

2.1.3. *Remark.* The KZ equations for $Y_\hbar(\mathfrak{g})$ are not expected to give the KZ (or qKZ) equations for $U_\hbar(\mathfrak{g})$. Instead, they are meant to be differential equations on valued in representations of $Y_\hbar(\mathfrak{g})$, with Ω replaced by the Casimir element of $Y_\hbar(\mathfrak{g})$. (check, seems dodgy, shouldn't we get [GT] stuff?)

2.1.4. *Remark.* There should also be multiplicative and elliptic versions of the above. The multiplicative version quotients the first (or second) space by $\mathbf{Z}$:

Δ, Δ'

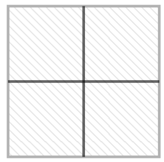
$U_q(\mathfrak{g})$-Mod



$$\mathbf{R}_1 \times \underbrace{\mathbf{R}_2/\mathbf{Z}}_{S^1_2}$$

Δ, Δ'

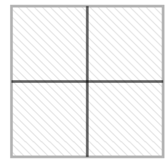
$U_q(\hat{\mathfrak{g}})$-Mod



$$\mathbf{C}_1 \times \underbrace{\mathbf{R}_2/\mathbf{Z}}_{S^1_2}$$

Δ, Δ'

$W_{1+\infty,q}(\mathfrak{g})$-Mod

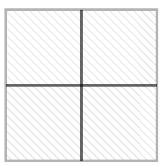


$$\mathbf{C}_1 \times \underbrace{\mathbf{C}_2/\mathbf{Z}}_{\mathbf{C}^\times_2}$$

and the elliptic analogue does it for both:

Δ, Δ'

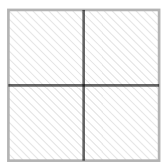
$U_{q,\tau}(\mathfrak{g})$-Mod



$$\underbrace{\mathbf{R}_1/\mathbf{Z}}_{S^\times_1} \times \underbrace{\mathbf{R}_2/\mathbf{Z}}_{S^1_2}$$

Δ, Δ'

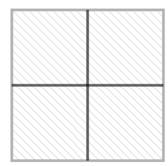
$\mathcal{E}_{q,\tau}(\mathfrak{g})$-Mod



$$\underbrace{\mathbf{C}_1/\mathbf{Z}}_{\mathbf{C}^\times_1} \times \underbrace{\mathbf{R}_2/\mathbf{Z}}_{S^1_2}$$

Δ, Δ'

$W_{1+\infty,q,\tau}(\mathfrak{g})$-Mod



$$\underbrace{\mathbf{C}_1/\mathbf{Z}}_{\mathbf{C}^\times_1} \times \underbrace{\mathbf{C}_2/\mathbf{Z}}_{\mathbf{C}^\times_2}$$

(maybe there should be other ways of quotienting which give you $E_1 \times \mathbf{C}_2$? This should be the story about taking limits)

2.2. **Conformal blocks.** If $\mathcal{V}$ is a factorisation algebra over Riemann surface Σ, its chiral homology $\mathrm{H}^\bullet(\mathrm{Ran}\,\Sigma, \mathcal{V}) = \mathrm{Conf}(\Sigma)$ is also called its *conformal blocks*.

Note that as a vector bundle, $V|_{(\Sigma^n)_\circ} \simeq \mathcal{O} \otimes V^{\otimes n}$ is trivial. In particular, the restriction of a conformal block

$$\mathrm{H}^\bullet(\mathrm{Ran}\,\Sigma, \mathcal{V}) \;\to\; \mathrm{H}^\bullet((\Sigma^n)_\circ, \mathcal{V}), \qquad \Phi \;\mapsto\; \Phi|_{(\Sigma^n)_\circ}$$

is a $V^{\otimes n}$-valued function

$$\Phi|_{(\Sigma^n)_\circ} \;:\; (\Sigma^n)_\circ \;\to\; V^{\otimes n},$$

satisfying some differential equation given by the connection on $\mathcal{V}$. Moreover, as $z_i \to z_j$ this function satisfies

$$\Phi|_{(\Sigma^n)_\circ} \;\to\; Y_{ij}(z_i - z_j) \cdot \Phi|_{(\Sigma^{n-1})_\circ}, \tag{3}$$

where we have applied the vertex operator to the ijth entries, and $\Sigma^{n-1} \subseteq \Sigma^n$ is the diagonal $z_i = z_j$. We have shown

Proposition 2.2.1. *A conformal block is the same data as a collection of $V^{\otimes n}$-valued functions on $(\Sigma^n)_\circ$ satisfying:*

- *they satisfy the differential equations given by $\mathcal{V}$,*
- *they are $\mathfrak{S}_n$-invariant,*
- *they satisfy the operator product expansion (3) as $z_i \to z_j$.*

2.2.2. *Remark.* Often conformal blocks are presented after taking elements $\alpha_1, \dots, \alpha_n \in V^*$, and then using notation

$$\langle \alpha_1(z_1) \cdots \alpha_n(z_n) \rangle_\Phi \ := \ (\alpha_1 \otimes \cdots \otimes \alpha_n)\Phi|_{(\Sigma^n)_\circ}(z_1, \dots, z_n).$$

This is now a **C**-valued function on $(\Sigma^n)_\circ$ satisfying the same properties as above.

2.2.3. *Remark.* We have

$$\langle \alpha_1(z_1) \cdots (T\alpha_i)(z_i) \cdots \alpha_n(z_n) \rangle_\Phi \ = \ \partial_{z_i} \langle \alpha_1(z_1) \cdots \alpha_n(z_n) \rangle_\Phi,$$

and so it follows together with the structure of the $z_i \to z_j$ limit that a conformal block is determined by its values for $\{\alpha_i\}$ varying over (duals of) generating fields of V.

2.2.4. *Example.* For instance, when we take the Heisenberg vertex algebra a conformal block Φ consists of functions over $(\Sigma^n)_\circ$ denoted

$$\langle h^{(1)}(z_1) \cdots h^{(n)}(z_n) \rangle_\Phi \ \in \ \mathcal{O}((\Sigma^n)_\circ)$$

which as n vary are compatible according to the operator product expansion of the Heisenberg vertex algebra:

$$\langle h^{(1)}(z_1) \cdots h^{(n)}(z_n) \rangle_\Phi \ = \ \frac{1}{(z-w)^2} \langle h^{(1)}(z_1) \cdots \widehat{h^{(i)}(z_i)} \cdots \widehat{h^{(j)}(z_j)} \cdots h^{(n)}(z_n) \rangle_\Phi \ + \ \mathcal{O}(1).$$

as $z_i \to z_j$.

2.2.5. *Insertions.* Let us begin with the *wrong* definition of factorisation module $\mathcal{M}$ over $\mathcal{V}$. If we ask

$$j^*(\mathcal{V} \otimes \mathcal{M}) \ \xrightarrow{\sim} \ (\cup j)^* \mathcal{M}$$

then if we are working with unital Ran spaces, we get $\mathcal{V} \xrightarrow{\sim} \mathcal{M}$ by taking the restriction of the above map to

$$\begin{tikzcd} & (\operatorname{Ran} X \times \{\varnothing\})_\circ \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ \operatorname{Ran} X \times \{\varnothing\} & & \operatorname{Ran} X \end{tikzcd}$$

Instead, let us pull back along $f_x : \operatorname{Ran}_x X \to \operatorname{Ran} X$, the prestack of finite subsets containing $x \in X$, and form

$$\begin{tikzcd}
 & (\operatorname{Ran} X \times \operatorname{Ran}_x X)_\circ \arrow[dl, "j_x"'] \arrow[d] \arrow[dr, "\cup_x j_x"] & \\
\operatorname{Ran} X \times \operatorname{Ran}_x X \arrow[d] & (\operatorname{Ran} X \times \operatorname{Ran} X)_\circ \arrow[dl] \arrow[dr] & \operatorname{Ran}_x X \arrow[d] \\
\operatorname{Ran} X \times \operatorname{Ran} X & & \operatorname{Ran} X
\end{tikzcd} \tag{4}$$

where the left square is a pullback.

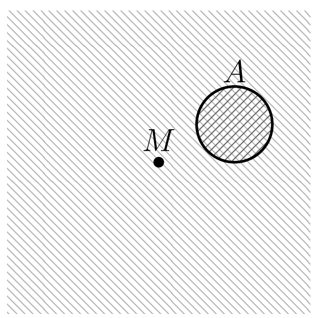
Definition 2.2.6. A $\mathcal{V}$-*module at* $x \in X$ is a factorisation $\mathcal{V}$-module $\mathcal{M}$ on $\operatorname{Ran}_x X$.

Said explicitly, it consists of a sheaf $\mathcal{M}$ along with structure map

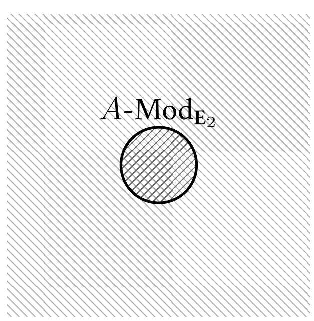
$$j_x^*(\mathcal{V} \boxtimes \mathcal{M}^x) \ \xrightarrow{\sim}\ (\cup_x j_x)^* \mathcal{M}^x$$

which is linear over $\mathcal{V}$. This should be viewed as the $2d$ CFT analogue of an associative algebra and a bimodule over it, i.e. a module over the two-sided Swiss cheese operad:

or rather the codimension two version of this, of a braided commutative algebra along with an $\mathbf{E}_2$-module for it:



We will now talk about the analogue of the fact that A-$\mathrm{Mod}_{\mathbf{E}_2}$ is itself braided monoidal, i.e. factorises over $\mathbf{R}^2$:



Note that if $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are modules at $x \in X$ then there is no obvious way that $\mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$ is also a $\mathcal{V}$-module at x. Instead, the category $\mathcal{V}$-Mod_x of such will itself form a factorisation category over X.

Definition 2.2.7. For any subset $S : T \to \operatorname{Ran} X$, the category $\mathcal{V}$-Mod(S) of **factorisation modules at** S is the category of $\mathcal{M} \in \mathcal{D}$-Mod$(\operatorname{Ran}_S X_T)$ along with structure map

$$j_S^*(\mathcal{V} \boxtimes \mathcal{M}) \xrightarrow{\sim} (\cup_S j_S)^* \mathcal{M} \tag{5}$$

linear over $\mathcal{V}$.

Here as before, we have correspondence of prestacks over T:

$$\begin{tikzcd} & (\operatorname{Ran} X_T \times \operatorname{Ran}_S X_T)_\circ \arrow[dl, "j_x"'] \arrow[dr, "\cup_S j_S"] & \\ \operatorname{Ran} X_T \times \operatorname{Ran}_S X_T & & \operatorname{Ran}_S X_T \end{tikzcd} \tag{6}$$

The structure map (5) is in the category of D-modules on $(\operatorname{Ran} X_T \times \operatorname{Ran}_S X_T)_\circ$. If $\mathcal{E}$ is any quasi-coherent sheaf on T, then extending (5) linearly gives $\mathcal{M} \otimes \mathcal{E}$ the structure of a factorisation module at S. Thus, (check)

Lemma 2.2.8. *The above forms a sheaf of categories* $\mathcal{V}$-Mod *over* $\operatorname{Ran} X$*, which factorises.*

Proof. To prove that $\mathcal{V}$-Mod factorises, we need to give an equivalence

$$\otimes_{\mathcal{V}} \; : \; j^*(\mathcal{V}\text{-Mod} \boxtimes \mathcal{V}\text{-Mod}) \xrightarrow{\sim} (\cup j)^*(\mathcal{V}\text{-Mod}). \tag{7}$$

Take two disjoint subsets $S_1, S_2 : T \to \operatorname{Ran} X$. The fibre of the left hand side of (7) over these consists of:

- $\mathcal{M}_i \in \mathcal{D}$-Mod$(\operatorname{Ran}_{S_i} X_T)$,
- structure maps $\varphi_i : j_{S_i}^*(\mathcal{V} \boxtimes \mathcal{M}_i) \to (\cup_{S_i} j_{S_i})^* \mathcal{M}_i$.

We use these to produce an object $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2 \in \mathcal{D}$-Mod$(\operatorname{Ran}_{S_1 \sqcup S_2} X_T)$ like so: take

$$\begin{tikzcd} & (\operatorname{Ran}_{S_1} X \times \operatorname{Ran}_{S_2} X)_\circ \arrow[dl, "j_{S_1,S_2}"'] \arrow[dr, "\cup_{S_1,S_2} j_{S_1,S_2}"] & \\ \operatorname{Ran}_{S_1} X_T \times \operatorname{Ran}_{S_2} X_T & & \operatorname{Ran}_{S_1 \sqcup S_2} X_T \end{tikzcd} \tag{8}$$

All three spaces are bimodules over $\operatorname{Ran} X$, and the maps in (8) are linear over $\operatorname{Ran} X$. We thus define the product using a chiral product like structure

- $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2 \;=\; (\cup_{S_1,S_1} j_{S_1,S_2})_* j_{S_1,S_2}^* (\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ as a D-module,
- we define the action map

$$\varphi \; : \; j_{S_1 \sqcup S_2}^*(\mathcal{V} \boxtimes (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)) \;\to\; (\cup_{S_1 \sqcup S_2} j_{S_1 \sqcup S_2})^* (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)$$

  as below, mimicking Huang and Lepowsky's [HL].

To finish, we consider

$$\begin{tikzcd}[column sep=tiny]
 & & (RX_T \times RX_T \times R_{S_1}X \times R_{S_2}X)_\circ \arrow[dl] \arrow[dr] & & \\
 & (RX_T \times RX_T)_\circ \times (R_{S_1}X \times R_{S_2}X)_\circ \arrow[dl] \arrow[dr] & & (RX_T \times R_{S_1 \sqcup S_2}X_T)_\circ \arrow[dl] \arrow[dr] & \\
(RX_T \times RX_T) \times (R_{S_1}X_T \times R_{S_2}X_T) & & RX_T \times R_{S_1 \sqcup S_2}X_T & & R_{S_1 \sqcup S_2}X_T
\end{tikzcd} \tag{9}$$

We then crucially use the *inverse* of the (invertible) factorisation structure on $\mathcal{V}$:

(if we try to do the obvious thing, we get):

$$\begin{aligned}
\mathcal{V}_w \otimes (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_{s_1,s_2,z} &= \bigoplus_{z=z_1 \sqcup z_2} \mathcal{V}_w \otimes \mathcal{M}_{1,s_1,z_1} \otimes \mathcal{M}_{2,s_2,z_2} \\
&\to \bigoplus_{z=z_1 \sqcup z_2,\, w=w_1 \sqcup w_2} \mathcal{V}_{w_1} \otimes \mathcal{M}_{1,s_1,z_1} \otimes \mathcal{V}_{w_2} \otimes \mathcal{M}_{2,s_2,z_2} \\
&\xrightarrow{\sim} \bigoplus_{z=z_1 \sqcup z_2,\, w=w_1 \sqcup w_2} \mathcal{M}_{1,s_1,z_1,w_1} \otimes \mathcal{M}_{2,s_2,z_2,w_2} \\
&= (\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2)_{s_1,s_2,z,w}
\end{aligned}$$

where the equalities are by the definition of $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2$, and the middle isomorphism is the action of $\mathcal{V}$ on $\mathcal{M}_i$. (However, note that crucially the second arrow is *not* an isomorphism. It is only an isomorphism on each factor, $\mathcal{V}_w \xrightarrow{\sim} \mathcal{V}_{w_1} \otimes \mathcal{V}_{w_2}$, not on the sum. We thus need to adjust $\mathcal{M}_1 \otimes_{\mathcal{V}} \mathcal{M}_2$ slightly to make this an isomorphism; this should be what is done in [HL].) □

We draw what the above proof is doing: (draw it)

(*this* is how to study intertwining operators, the braided monoidal structure on V-Mod, the fusion product, etc.)

2.2.9. *Remark.* Recall that if V is a vertex *operator* algebra with modules M_1, M_2. Then [HL] constructs a map (power series stuff not quite right)

$$\tau(z) \;:\; V \otimes \mathbf{C}[w, w^{-1}, (z^{-1} - w)^{-1}] \;\to\; \mathrm{End}\,((M_1 \otimes M_2)^*)$$

defined as in [HL, 13.2] by (approximately)

$$\tau(z)\delta((w-z)/u)Y(w) \;=\; \delta((w-u)/z)(Y(u)e^{wL_{-1}}w^{-2L_0} \otimes \mathrm{id}) \;+\; \delta((z-w)/u)(\mathrm{id} \otimes Y(w)).$$

Notice that we only use the first modes L_0, L_{-1} of the Virasoro. It involves:

- a translation by w: $\exp(wL_{-1})$,
- a scaling by $-2\log w$: w^{-2L_0}.

Then by Theorem [HL, Cor. 13.11]:

Theorem 2.2.10. *If there is a V-module $M_1 \otimes_V M_2$ corepresenting intertwining operators, it takes the form of*

$$M_1 \otimes_V M_2 \;=\; (S)^* \;\twoheadleftarrow\; (M_1 \otimes_k M_2)^*$$

where S is the subspace of elements satisfying a dimension condition and $\tau(z)\delta(z-w)Y(w) = \delta(z-w)\tau(z)Y(w)$, see [HL, p.26]. Moreover, it exists if and only if τ makes S into a V-module.

2.2.11. *Boundary KZ and other singularities.* The boundary KZ equation looks like

$$\partial_{z_i} \;+\; \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j} \;+\; \frac{k_i}{z_i}.$$

Moreover, we have other KZ equations, with poles at:

- $z_i = z_j$, as usual,
- $z_i = z_j$ and $z_i = 0$, as above,
- $z_i = \pm z_j$,
- $z_i = \pm z_j$ and $z_i = 0$,

which depends on a choice of root system. These should arise from factorisation algebras living over:

- $\mathrm{Ran}\,\Sigma$, as usual,
- $\mathrm{Ran}(\Sigma \smallsetminus 0)$,
- $\mathrm{Ran}(\Sigma/(\mathbf{Z}/2))$,
- $\mathrm{Ran}((\Sigma \smallsetminus 0)/(\mathbf{Z}/2))$.

2.3. q-**conformal blocks.** Whatever our definition of q-vertex algebra and $V_q^k(\mathfrak{g})$ should recover the qKZ equations. In particular, we would like to have a $\mathcal{V}_q \in \mathcal{D}_q$-Mod$(\mathrm{Ran}\,\Sigma)$ such that $\mathrm{Conf}_q(\Sigma) = \Gamma(\mathrm{Ran}\,\Sigma, \mathcal{V}_q)$ is a q-conformal block.

Let us consider the restriction

$$\Gamma(\mathrm{Ran}\,\Sigma, \mathcal{V}_q) \;\to\; \Gamma((\Sigma^n)_\circ, \mathcal{V}_q) \qquad\qquad \Phi \;\mapsto\; \Phi|_{(\Sigma^n)_\circ}.$$

Assume for now that $\mathcal{V}_q$ is trivial as a vector bundle over $(\Sigma^n)_\circ$, so that we again get a function

$$\Phi|_{(\Sigma^n)_\circ} \;:\; (\Sigma^n)_\circ \;\to\; V^{\otimes n}$$

by the factorisation condition. Moreover,

- it is $\mathfrak{S}_n$-invariant,
- it satisfies a q-*difference* equation,

- it satisfies a q-operator product expansion as $z_i \to q^n z_j$ for any $n \in \mathbf{Z}$,
$$\Phi_{(\Sigma^n)_\circ} \;\to\; Y_{ij}^{q^n}(z_i - z_j)\cdot \Phi_{(\Sigma^{n-1})_\circ} \tag{10}$$
where $Y_{ij}^{q^n}$ is (bla) and $\Sigma^{n-1} \subseteq \Sigma^n$ is the q^n*-diagonal* $z_i = q^n z_j$.

Notice that in the above limit (10), only the $(z_i - q^n z_j)$ poles contribute.

(do we consider $\mathrm{Ran}(X_{dR})$ or $(\mathrm{Ran}\, X)_{dR}$ in the q-case? the above assumes the former)

2.3.1. *Remark.* We expect to have the following story.

$$\begin{tikzcd} & \mathcal{V}_q \arrow[dl, "\mathrm{Zhu}"', maps to] \arrow[dr, "q\to 1", maps to] & \\ A_q & & \mathcal{V} \end{tikzcd}$$

(and an associated projection functor on their conformal blocks, assuming that A_q has them. The fusion coproduct on $\mathcal{V}_q$, if it exists, should be sent to a braided monoidal product on A_q.)

2.4. **Affine analogue.** It is natural to ask whether there is a Gaitsgory Lysenko factorisation story when replacing
$$u_q(\mathfrak{n}) \;\rightsquigarrow\; Y(\mathfrak{g}_Q) = Y_\hbar(\mathfrak{n})?$$
To solve this question;

- we need to have a Riemann-Hilbert for difference equations, which we do; see [RSZ] or [KS],
- (partial evidence for this: BPS sheaf over $\mathfrak{X}$ or rather $\mathrm{Conf}_\Lambda(\mathbf{C})$ should be an analogue of $u_q(\mathfrak{n})$ over $\mathrm{Conf}_\Lambda(\mathbf{C})$)
- (the analogue of $\mathrm{Rep}_q\, T$ as a factorisation category $\mathrm{Sh}_{\mathcal{G}}(\mathrm{Conf}_\Lambda(\mathbf{C}))$ might be the limit $\lim \mathrm{H}^{\mathrm{BM}}(\mathcal{M}(v,w))$?)
- (unclear how the qKZ relates to the stable envelope, Nakajima quiver variety etc story)

2.4.1. Ignoring ellitpic, we have 2^4 choices,

- a, z are differential or difference (or elliptic?),
- whether a, z lie on $\mathbf{C}$ or $\mathbf{C}^\times$

We can have additive or multiplicative difference equation. We can have additive and multiplicative differential equations.

Ignore a for now (set it to be (??)), so we have 4 choices. The value of V_i are then:

- z differential equation on $\mathbf{C}$, $V_i \in \mathrm{Rep}\, U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}\, U(\mathfrak{g}[u])$,
- z differential equation on $\mathbf{C}^\times$, $V_i \in \mathrm{Rep}\, U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}\, U(\mathfrak{g}[u^{\pm 1}])$,
- z difference equation on $\mathbf{C}$, $V_i \in \mathrm{Rep}\, U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}\, Y_\hbar(\mathfrak{g})$,
- z difference equation on $\mathbf{C}^\times$, $V_i \in \mathrm{Rep}\, U(\mathfrak{g})$ or $\mathrm{Rep}^{ev}\, U_q(\hat{\mathfrak{g}})$,

2.4.2. In the affine case, you can replace $\mathfrak{g}$ with any Kac-Moody algebra. These KZ equations aren't well-studied.

2.4.3. We can also consider equivariant BM homology of X_Q, they satisfy differential equations (KZ equations) in the torus-equivariant parameters, a.

2.4.4. Note that for ζ a positive stability condition, there is an action of $\mathcal{M}$ on X_Q tautologically.

2.4.5. There is a completely different curve to the z, a curves; it's the quasimap curve, the curve over which you dimensionally reduce, the one where $\hbar$ is an equivariant parameter on that curve. And this has to do with the asymptotic R-matrices.

2.4.6. The Drinfeld coproduct comes from some the dimensional reduction curve, the $\mathbf{C}$ on

2.4.7. The KZ equations are the ward identity for the conformal transformations.

2.4.8. There are also notions of *twisted* and *coset* KZ equations.

2.4.9. Vanya's equation is a differential equation on $\mathbf{C}^\times/(\mathbf{Z}/2)$ or $\mathbf{C}/(\mathbf{Z}/2)$; this is not anywhere else in the literature. Call it DKZ. (Interesting question: what is the qKZ analogue of this?) The multiplicative DKZ equation look like

$$z_i\partial_{z_i} + \sum_{i\neq j}\frac{\Omega_{ij}}{1-z_i/z_j}, + \sum_{i\neq j}\frac{\Omega_{ij}^{long}}{1+z_i/z_j}$$

where $\Omega \in S^2\mathfrak{g}^{long}$ where $\mathfrak{g}^{long} \subseteq \mathfrak{g}$ are the long root Lie subalgebra of a simple Lie algebra $\mathfrak{g}$. For instance, $\mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \subseteq \mathfrak{sp}_4$.

If we want to understand orthosymplectic $Y_\hbar(\mathfrak{g})$, we then would have to consider the *difference* DKZ equations.

2.4.10. Read Agaganic Frenkel about quantum q-Langlands, (to get less confused about where all these curves come from; bottom of page 16 or picture on p17)

2.4.11. The KZB equations is the name for KZ equations over E. They are probably

$$\xi_i + \sum_{i\neq j}\frac{\Omega_{ij}}{z_i -_E z_j}$$

where ξ_i is the generating vector field on E.

3. Other versions of KZ

3.1. **As a Gauss-Manin connection.** The Coulomb branch fibres over $\mathbf{C}^n$, as

$$\begin{array}{c}\operatorname{Spec} \mathrm{H}^\bullet_T(\mathcal{M}_C)\\ \downarrow\pi \\ \operatorname{Spec} \mathrm{H}^\bullet(\mathrm{B}T) \;=\; \mathbf{C}^n\end{array}$$

where in examples $T = T_v$ is the torus attached to the non-framing vectors of the Nakajima quiver variety $\mathcal{M}_C = \mathcal{M}(v,w)$. Then

Proposition 3.1.1. *The KZ equation on $\mathbf{C}^{|v|}$ coincides with the Gauss-Manin connection on $\mathrm{H}^\bullet_T(\mathcal{M}(v,w))$.*

In particular, taking the sum over all v gives the KZ equation on all of $\operatorname{Ran}\mathbf{C}$. Moreover, we may take the T_w-equivariant version of the above. Note that π is $\mathfrak{S}_n$-equivariant.

3.1.2. Note that this allows us to generalise the KZ equations in the following way: take the *quantum cohomology* of $\mathcal{M}_C/T_v$, and the associated Dubrovin connection on that. This is called the *quantum KZ equation*. See e.g. [Ag].

3.1.3. We can likewise consider the multiplicative and (conjectural) elliptic Coulomb branches to get other KZ equations over $\operatorname{Spec} \mathrm{K}^\bullet(\mathrm{B}T)$ and $E^\bullet(\mathrm{B}T)$ respectively.

3.2. **Higher terms.** Whereas the KZ equations have to do with Lie algebra invariants, the higher terms of the KZ equation should correspond to higher Lie algebra cohomology, see [SV].

3.3. **The a, z variables.** In general, we expect a pair of differential or difference equations on

$$(\Sigma_a)^n_\circ \times (\Sigma'_z)^m_\circ$$

where $\Sigma, \Sigma' \in \{\mathbf{C}, \mathbf{C}^\times, E\}$. This is attached to a finite ADE quiver, i.e. is attached to the associated CY2 category; this gives the KZ equations.

The equation in the z variables will not contain any a terms, but the equation in the a variables will contain z terms. (see Kononov's thesis, or the Aganagic-Frenkel-Okounkov paper)

(how does this story relate to the story of KZ equations as coming from vertex algebras?)

3.3.1. In general, for X a local CY2 surface, we expect a pair of differential or difference equations on

$$(\Sigma_a)^n_\circ \times (\operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}^\times)_\circ$$

where $(\operatorname{Pic}(X) \otimes_{\mathbf{Z}} \mathbf{C}^\times)_\circ$ is the subspace of $e^{\omega + i\beta}$ where $\omega, \beta \in \operatorname{Pic}(X)$ have ω ample and β arbitrary. Here, $\Sigma_a = \mathfrak{a} \otimes_{\mathbf{Z}} \mathbf{C}^\times$ is given by the Lie algebra $\mathfrak{a}$ of the *framing torus* of "symmetries of the moduli problem preserving the holomorphic symplectic form on X", e.g. if $\mathcal{M}$ is the moduli stack of instantons, A scales the framing at infinity. n.b. when this is in fact GL_n, this is why we get Ran spacey behaviour.

For instance, when we consider framed representations $\mathcal{M}^{fr}(w)$ of a quiver with framing vector $w \in \mathbf{N}^{Q_0}$, we have $A = \prod A_i \simeq \prod \mathbf{G}_m^{w_i}$. Note that

$$\mathcal{M}^{fr}(w) \;=\; (\text{vector space})/\prod \mathrm{GL}_{v_i}$$

and $G = \prod \mathrm{GL}_{w_i}$ acts on this, and its good moduli space $\mathcal{X}^{fr}(w)$. The singularities of the KZ equations on $\mathfrak{a} \subseteq \mathfrak{g}$ will Lie along the locus where $\mathfrak{a}$ has higher than usual dimensonal fixed point locus when acting on $\mathcal{X}^{fr}(w)$.

Note that viewing $\Sigma_a^n = \mathfrak{a}$, the singularities of the KZ equations will lie along the root hyperplanes of the full framing group $\mathfrak{g}$. For instance, for $\mathfrak{sp}_{2n}$ (type C) these are $a_i = \pm a_j$ and $a_i = 0$, for $\mathfrak{so}_{2n}$ (type D) we have $a_i = \pm a_j$ for $i \neq j$, and for $\mathfrak{so}_{2n+1}$ (type B) they are $a_i = \pm a_j$ and $a_i = 0$.[7]

3.3.2. *Remark.* We have that $\pi_1((\mathbf{C}^\times)_\circ)$ is the *affine* braid group, so we get an affine braid group action on $V_1 \otimes \cdots \otimes V_n$. See [EG, Lem. 5.5], where the monodromy around $\mathbf{C}^\times$ is given in terms of $q = e^{\hbar}$.

Likewise, $\pi_1((E^n)_\circ)$ is the elliptic braid group, see [Jo].

3.4. Passing to a quantisation of the KZ equation corresponds to Etingof-Kazhdan quantising $r(z) \rightsquigarrow R(z)$.

3.5. **Other KZ equations.** The multiplicative KZ equation are the differential operators

$$(k - k_{crit}) z_i \partial_{z_i} \;+\; \sum_{i \neq j} r(z_i/z_j) + \pi_i(\lambda) \tag{11}$$

see [FR, p5], where λ is a weight of $\mathfrak{g}$ and $\pi_i(\lambda)$ denotes action of this weight on the ith representation. Likewise for the elliptic KZ equation,

$$(k - k_{crit})\xi_i \;+\; \sum_{i \neq j} r(z_i -_E z_j) + (corrections?) \tag{12}$$

where ξ_i is the generating vector field on elliptic curve E.

The multiplicative qKZ equations (attached to $V_i \in \operatorname{Rep} U_q(\mathfrak{g})$, [GT2, §8.9]) are the *difference* operators

$$q_i^{-2(k-k_{crit})} \;+\; \prod_{i>j} R_{ij}(q^{2(k-k_{crit})} z_i/z_j) \cdot (\overline{R}_{i0}\pi_i(q^{2\rho})\overline{R}_{iN}^{-1}) \cdot \prod_{i<j} R_{ij}(z_i/z_j)$$

as in [FR, 1.12] and [FR, p33], where $q_i : (z_1, ..., z_n) \mapsto (z_1, ..., qz_i, ..., z_n)$, and both products are taken over j decreasing. Here $\overline{R}_{ij}$ are the R-matrices for $U_q(\mathfrak{g})$, ρ is the sum of the positive roots

[7] The root hyperplanes are (from Fulton and Harris):

- $D/\mathfrak{so}_{2n}$ are $\pm a_i \pm a_j$ for $i \neq j$,
- $B/\mathfrak{so}_{2n+1}$ are $a_i \pm a_j$ for all $i \neq j$ and $a_i = 0$,
- $C/\mathfrak{sp}_{2n}$ are $\pm a_i \pm a_j$ for $i \neq j$ and $2a_i = 0$.

in $\mathfrak{g}$ and π_i is the action of $\mathfrak{g}$ on the ith factor. (Presumably) the additive qKZ equation (attached to $V_i \in \operatorname{Rep} Y_\hbar(\mathfrak{g})$, [GT2, §2.11]) is of the form

$$q_i^{-2(k-k_{crit})} + \prod_{i>j} R_{ij}(q^{2(k-k_{crit})}z_i - z_j) \cdot (\overline{R}_{i0}\pi_i(q^{2\rho})\overline{R}_{iN}^{-1}) \cdot \prod_{i<j} R_{ij}(z_i - z_j).$$

The elliptic analogue of the qKZ equation by [FTV, §2], are differential operators valued on the vector bundle with value $\mathrm{Fun}_{mer}(\mathbf{A}^1_\lambda, V_1 \otimes \cdots \otimes V_n)$ (is that right? why no periodicity in λ? What is the actual data the elliptic qKZ is attached to?) given by

$$p_i + \prod_{i>j} R_{ij}(z_i - z_j + p, \lambda - 2\hbar \sum_{r=1, r\neq i}^{j-1} h^{(r)}) \cdot \Gamma_i \cdot \prod_{i<j} R_{ij}(z_i - z_j)$$

where $p_i : (z_1, ..., z_n) \mapsto (z_1, ..., z_i +_E p, ..., z_n)$, $h^{(i)}$ is a basis of the Cartan, Γ_i translates $\lambda \mapsto \lambda - 2\hbar\mu$ if μ is the eigenvalue of $h^{(i)}$. (finish this definition)

The R matrices $R_{ij}(z, \lambda)$ depend on two complex numbers (z, λ), unlike the additive or multiplicative case (compare [TV]).

3.5.1. Compare the multiplicative qKZ equations to [GT2, §8.9],

$$\overline{\mathcal{R}}_{V_1,V_2}(q^{2\ell}\zeta) = \mathcal{A}_{V_1,V_2}(\zeta)\overline{\mathcal{R}}_{V_1,V_2}(\zeta).$$

Here $\mathcal{A}_{V_1,V_2}(\zeta)$ is the monodromy of the difference equation.

3.6. **Affinised analogue.** We can do the above for an arbitrary quiver Q, or replace $\mathfrak{g}$ with an arbitrary Kac-Moody Lie algebra in the above. We should have which are valued on tensor products $V_1(a_1) \otimes \cdots \otimes V_n(a_n)$ of evaluation representations of $Y_\hbar(\mathfrak{g}_Q)$, $U_q(\mathfrak{g}_Q)$ or $\mathcal{E}_{\hbar,\tau}(\mathfrak{g}_Q)$.

3.6.1. There is also a *boundary KZ equation* ∂KZ, which looks like

$$\partial_{z_i} + \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j} + \frac{k_i}{z_i}$$

where $k \in \mathfrak{g}$ is a classical K-matrix.

4. Physics explanation

4.1. Recall that Nakajima quiver varieties are Higgs branches, $X = \operatorname{Spec}\mathcal{Z}(S^2)$, of three dimensional theories. Recall from [BFN] that the theory are $3d\ \mathcal{N}=4$ quiver gauge theories, attached to a quiver Q and v, w dimension and framing vectors, with Higgs and Coulomb branches

$$\mathcal{M}_H = X_Q(v,w), \qquad \mathcal{M}_C = \operatorname{Spec} \mathrm{H}^{\mathrm{BM}}_{G(\mathcal{O}),\bullet}(\mathcal{R}).$$

These both have quantisations (does $X_Q(v,w)$?).

4.1.1. *Relation to Chern-Simons.* Consider Chern Simons on $\Sigma\times\mathbf{R}_{\geqslant 0}$ with line operators $V_i \in \operatorname{Rep} U_\hbar(\mathfrak{g})$ living on $\{z_i\}\times\mathbf{R}_{\geqslant 0}$. Its value is

$$\operatorname{LocSys}_G^{(V_i,z_i)}\Sigma$$

where we consider local systems on $\Sigma\smallsetminus\{z_i\}$ valued in $V_1\otimes\cdots\otimes V_n$ whose monodromy around z_i is given by the action of the representation V_i. There is a quantisation of this

$$\mathcal{O}(\operatorname{LocSys}_G^{(V_i,z_i)}\Sigma) \quad\rightsquigarrow\quad \mathcal{O}_\hbar(\operatorname{LocSys}_G^{(V_i,z_i)}\Sigma) = C^0((V_i,z_i)),$$

is the space of conformal blocks. Note that varying z_i makes $\operatorname{LocSys}_G^{(V_i,z_i)}\Sigma$ into a family of spaces. This gives the structure of a vector bundle with connection on conformal blocks,

$$C^0((V_i,-)) \to (\mathbf{C}^n)_\circ.$$

4.1.2. (what is the analogue of this for an arbitrary CY2 surface as in the previous section?)

4.1.3. Recall that an example of a quiver gauge theory is (a circle reduction of) $4d$ super Yang-Mills theory.

4.2. **Questions.**

(1) $Y(\mathfrak{g}_Q)$ (or its double) is Koszul dual to local operators in what theory (of what dimension)? What does doubling correspond to physically? (Sam's not sure; see Costello and Yagi "unification of integrability"-chapter 6 or something)

(2) X_Q is the Higgs branch of which theory? ($3d\ \mathcal{N}=4$ dimensionally reduced $4d\ \mathcal{N}=2$ quiver gauge theory)

(3) Why do we expect asymptotic Higgs branches to have a factorisation structure? (it's probably some $5d$ Chern-Simons $W_{1+\infty}$ or $5d$ SYM thing)

(4) What is the relation between this Coulomb branch stuff and $4d$ Chern Simons (i.e. Yangians)?

(5) Is the trichotomy in a and z orthogonal to the issue of taking double loops? i.e. is the quiver fixed as we vary a, z? If so, what is different when we take double loops, e.g. affine ADE?

(6) Is Kazhdan Lusztig to KZ what double affine Kazhdan Lusztig is to qKZ?

(7) (see Stable envelopes CoHA section) (is there a sense in which Ω_q is over $\mathrm{Conf}_\Lambda(\mathbf{C})$ in the finite ADE case, but there is something over $\mathrm{Conf}_\Lambda(\mathbf{C}\times\mathbf{R})$ in the affine case?) (is this to do with the rational sections stuff in YZ's elliptic quantum groups?)

(8) In the tri$\times$trichotomy, what is the fibre of the vector bundle? I assume something like $\mathrm{Maps}(G, V_1\otimes\cdots\otimes V_n)$ (evaluation reps) for V_i representations of $Y_\hbar(\mathfrak{g}), U_q(\hat{\mathfrak{g}}), \mathcal{E}_{\hbar,\tau}(\mathfrak{g})$, but if so, why are conformal blocks expected to be this?

(9) Continue: KZ, qKZ, ?

(10) What do differential equations, difference equations and elliptic difference equations have to do with $\mathbf{G}_a, \mathbf{G}_m, E$?

(11) In just the KZ case, we get a braided monoidal structure $\mathrm{Rep}\, U_\hbar(\mathfrak{g})$ when the base is $\mathbf{C}$. What structure do we get when the base is $\mathbf{G}_m$ or E? Is the factorisable category on $\mathrm{Conf}_\Lambda(\mathbf{G}_m)$ and $\mathrm{Conf}_\Lambda(E)$ still $\mathrm{Rep}\, U_\hbar(\mathfrak{g})$? Or it is $\mathrm{Rep}\, U_q(\mathfrak{g})$? Or is the fibre $\mathrm{Rep}\, U_\hbar(\mathfrak{g})$, but the global sections are $\mathrm{Rep}\, U_q(\mathfrak{g})$? (c.f. Vanya's work about monodromy around $\mathbf{C}^\times$ and the trigonometric (i.e. $\mathbf{C}^\times$) KZ equation)

4.2.1. There is a pair of commuting differential equations, one in the a-variables, one in the z-variables.

5. Kazhdan-Lusztig equivalence

5.1. **Relation to conformal blocks.** Let $V_{\lambda_i,k}$ be representations of $V^k(\mathfrak{g})$ induced by highest weights λ_i of $\mathfrak{g}$. Then we can by [FBZ, 13.3.5] define a vector bundle of conformal blocks

$$C^0(\mathbf{P}^1, \infty, z_1, \ldots, z_n)$$

over $(\mathbf{C}^n)_\circ$, with fibres $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$ the tensor product of finite dimensional representations of $\mathfrak{g}$. It is a subbundle

$$C^0(\mathbf{P}^1, \infty, z_1, \ldots, z_n) \ \subseteq \ (\boxtimes \mathcal{V}_{\lambda_i,k})^*$$

where $\mathcal{V}_{\lambda_i,k}$ is the vector bundle attached to the $V(\mathfrak{g})$-module

Proposition 5.1.1. [FBZ, Lem. 13.3.7] *The differential operator $\partial_{z_i} + T_i$ on $(\otimes \mathcal{V}_{\lambda_i,k})^*$ preserves the conformal blocks.*

It is (expected?) that there are differential operators for all VOAs and modules. For instance the Virasoro and the *BPZ equations*.

5.1.2. *Remark.* Conformal blocks (three points at $0, z, \infty$) are called *intertwining operators*.

5.1.3. *Other vertex algebras.* According to Sujay, if V_i is an arbitrary representation of a VOA V, then if

$$V_i \ = \ \mathrm{Ind}_{\mathrm{Zhu}\, V}^{V} W_i$$

is induced from the Zhu algebra of V_i, then $C^0(\mathbf{P}^1, \infty, z_1, \ldots, z_n, V_i)$ is a subquotient of $\otimes W_i$, or something similar. In particular, we get braiding data on these subquotients.

5.2. **Gautam and Toledano Laredo's [GT].** We have an inclusion (of meromorphic tensor categories [GT2])

$$\mathrm{Rep}^{fd}\, Y_\hbar(\mathfrak{g}) \ \to \ \mathrm{Rep}^{fd}\, U_q(\hat{\mathfrak{g}})$$

over Vect, whose definition involves choosing a branch of $\log(z)$. It exponentiates the roots of the Drinfeld polynomials $P_i(u)$ of representations, which are defined for $Y_\hbar(\mathfrak{g})$ by

$$\xi_i(u)v \ = \ \frac{P_i(u + d_i\hbar)}{P_i(u)} v$$

where v is a generating vector, $\xi_i(u) \ = \ 1+\hbar \sum_{r \geqslant 0} \xi_{i,r} u^{-r-1}$ and if $\mathfrak{h}$ acts as λ, then $d_i v = \xi_{i,0}/\lambda(\alpha_i^\vee)v$. For $U_q(\hat{\mathfrak{g}})$ they have a similar definition.

This is in some sense a pullback along

$$\log \ : \ \mathbf{C} \setminus \ell \ \to \ \mathbf{C}^\times$$

a section of $\exp : \mathbf{C} \to \mathbf{C}^\times$. The meromorphic tensor structures are given by the Drinfeld coproducts on $Y_\hbar(\mathfrak{g})$ and $U_q(\hat{\mathfrak{g}})$, see [GT2, §2].

5.3. **Relation to Chern-Simons.** Recall the physics story of Chern-Simons theory: given a Riemannian three-manifold M with boundary and $P = G \times M \to M$ the trivial G-bundle, we take the sheaf

$$\mathrm{Conn}'(P) \;\to\; M$$

of smooth $\mathfrak{g}$-connections

$$\nabla \;:\; \mathcal{T}_M \to \mathrm{End}(\mathrm{ad}\, P),$$

such that for a normal vector ξ along $\partial M \subseteq M$, we have that the *boundary condition* that $\nabla(\xi) = 0$ as an element of $\mathrm{End}(\mathrm{ad}\, P)|_{\partial M}$ (moreover, we need to ask that it vanishes to which order?); see the discussion around [Wi, Eqn. 3.1].

One can define a function on $\mathrm{Conn}'(P)$ by

$$\nabla \;\mapsto\; \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

which physics-defines a $3d$ QFT, which Witten physics-showed does not depend on the metric (i.e. is topological) if $\partial M = 0$, and is independent under rescaling of the metric (i.e. is conformal) in a neighbourhood of ∂M. Note that ∂M has measure zero, so we do not need to worry about whether the integrand is well-defined on the boundary.

5.3.1. *Classical version.* There is a classical version of this, instead taking the sheaf of sections of P itself:

$$P \;\to\; M.$$

Given a section $\gamma : U \to P$, we can take the differential 2- and 3-forms $\alpha_2, \alpha_3 \in \Omega^\bullet(G)$, and define the function

$$\gamma \;\mapsto\; \int_U \gamma^* \alpha_3 \;+\; 3k \int_{\partial U} \gamma^* \alpha_2.$$

Note that α_2 is given by the Killing form κ, and α_3 is given by $\kappa(-, [-, -])$.

Note that a function $\gamma : U \to G$ induces a map $\mathcal{T}_U \to \mathcal{T}_G \twoheadrightarrow \mathfrak{g}$. (is this how we get the connection above?)

5.3.2. *Line defects.* To add in line operators, mathematically one considers instead *parabolic* G-bundles, i.e. those equipped with a flag plus weights. Given any complex structure on ∂M, one can geometrically quantise this moduli stack using the level line bundle $\mathcal{L}$, giving

$$\mathrm{Bun}_G^{\mathrm{Par}}(\mathcal{E}_{\partial M}) \qquad \rightsquigarrow \qquad V_{\partial M, G} \;\to\; \mathcal{M}_{\partial M, n}$$

a vector bundle over the moduli stack of complex structures on ∂M. Here $\mathcal{E}$ is the universal curve over $\mathcal{M}_{\partial M, n}$.

One can show that this is the bundle of conformal blocks for $V(\mathfrak{g})$,[8] and has a KZ connection ∇_{KZ}.

[8]This is called the *Pauly isomorphism*.

One can *also* construct a so-called "Hitchin connection" ∇_{Hitch}, which *projectively* flat, is different from the KZ connection (but is projectively equivalent to KZ).

One can show that the vector space *Chern-Simons theory* attaches to a surface is

$$\partial M \qquad \rightsquigarrow \qquad \Gamma_{\nabla_{\text{Hitch}}\text{-flat}}(\mathcal{M}_{\partial M}, V_{\partial M,G}) \ (\overset{?}{=})\ \Gamma_{\nabla_{\text{KZ}}\text{-flat}}(\mathcal{M}_{\partial M}, V_{\partial M,G}) \ (\overset{?}{=})\ \text{Conf}(V(\mathfrak{g}), \partial M_{\sigma})$$

where σ is here a complex structure.

6. Differential and difference equations

Let $\mathcal{G}$ be a group or formal group scheme which acts on X. For instance, we could consider

$$\mathcal{G} = \exp(\mathcal{T}_X)$$

the exponential of the sheaf of Lie algebras over X given by the tangent bundle, or any subgroup generated by some vector fields. We have an action (is that right?)

$$\mathcal{G} \times_X X \to X.$$

The *de Rham stack* of this action is the quotient stack $X_{dR,\mathcal{G}} = X/\mathcal{G}$. For instance,

Lemma 6.0.1. *When $\mathcal{G} = \exp(\mathcal{T}_X)$, we recover the usual notion of the de Rham stack $X_{dR,\mathcal{G}} = X^{\wedge}_{X^2}$, usually denoted just X_{dR}.*

Proof. (write) □

6.0.2. *Motivation.* We should view the de Rham stack as being the pushout

$$\begin{array}{ccc} \mathcal{G} \times_B X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X/\mathcal{G} \end{array} \qquad \begin{array}{ccc} \{(x, g \cdot x)\} & \longmapsto & g \cdot x \\ \downarrow & & \\ x & & \end{array}$$

For instance, let $v \in \Gamma(X, \mathcal{T}_X)$ be a nonvanishing vector field and $\mathcal{G}$ be the formal group over X it generates. Assume the flow of v is complete, so it exponentiates to an action of $\mathbf{G}_a$. Then we can consider

$$\begin{array}{ccc} X \times \mathbf{A}^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ X & \longrightarrow & X_{\mathrm{dR},v} \end{array} \qquad \begin{array}{ccc} \{(x, e^{tv} \cdot x)\} & \longmapsto & e^{tv} \cdot x \\ \downarrow & & \\ x & & \end{array}$$

Likewise, if all vector fields' flows are complete, we have that $\mathcal{G} \times X = X^2$ (check this) and so we get (probably wrong)

$$\begin{array}{ccc} X \times X & \longrightarrow & X \\ \downarrow & & \\ X & & \end{array} \qquad \begin{array}{ccc} \{(x, e^{tv} \cdot x)\} & \longmapsto & e^{tv} \cdot x \\ \downarrow & & \\ x & & \end{array}$$

Taking the completion gives back X_{dR}.

6.1. We thus make the definition that a $\mathcal{G}$-**differential equation** is a quasicoherent sheaf $M \in \mathrm{QCoh}(X_{dR,\mathcal{G}})$. Explicitly this consists of *parallel transport* isomorphisms

$$\varphi_{g,x} : M_x \xrightarrow{\sim} M_{g\cdot x}$$

for every pair of points $g \in \mathcal{G}$ and $x \in X$.

The *solutions* to a $\mathcal{G}$-differential equation are its image under the pushforward

$$\mathrm{QCoh}(X/\mathcal{G}) \;\to\; \mathrm{QCoh}(\mathrm{pt}) \;\simeq\; \mathrm{Vect},$$

which generalises the notion of flat sections (or de Rham cohomology) of a vector bundle with connection.

6.1.1. *Example.* For the ordinary de Rham space this is equivalent to a $\mathcal{D}$-module structure.

For instance, if the vector bundle $\mathcal{V}_X = V \otimes \mathcal{O}_X$ is trivial then we get an isomorphism

$$\varphi \;:\; \mathrm{act}^* \mathcal{V} \;\xrightarrow{\sim}\; \pi_2^* \mathcal{V}$$

where $\mathrm{act}, \pi_2 \;:\; \mathcal{G} \times_X X \;\rightrightarrows\; X$. In other words, this gives an automorphism of $\mathcal{V}_{\mathcal{G}\times X}$, with the condition that it pull back along X to the trivial automorpism of $\mathcal{V}_X$, plus the cocycle condition. On global sections for $X = \mathbf{A}^1$, this gives (check)

$$\Gamma(X, \mathcal{V}_X) \otimes \mathbf{C}[[t]] \;\xrightarrow{\sim}\; \Gamma(X, \mathcal{V}_X) \otimes \mathbf{C}[[t]].$$

The conditions imply that

Lemma 6.1.2. *This map is of the form $e^{\partial\otimes t}$, where ∂ is an $\mathrm{End}(V)$-valued derivation on $\Gamma(X, \mathcal{O}_X)$.*

Proof. (write) □

In other words, we get a derivation ∂. (how do we get higher order ODEs?) The flat sections consist of functions $f(x)$ with

$$e^{t\otimes\partial} f(x) \;=\; f(x) + t\partial f(x) + \cdots \;=\; f(x)$$

which is equivalent to $\partial f(x) = 0$.

6.1.3. *Example.* We can consider $\mathcal{G} = \mathbf{Z}$ acting on X generated by automorphism q, in which case a $\mathcal{G}$-differential equation is just a quasicoherent sheaf M along with compatible automorphisms

$$q^* M \;\simeq\; M.$$

Examples of this are when X it itself a group and the automorphism is action by a point $q \in X$.

For instance, if $\mathcal{V} = V \otimes \mathcal{O}_X$ is the trivial vector bundle, then the sections consist of functions consist of functions $f(x)$ with

$$q \cdot f(x) \;=\; f(qx) \;=\; f(x).$$

6.1.4. *Example.* We can contruct "mixed" examples as follows. Say a two dimensional torus $T \simeq \mathbf{G}_m \times \mathbf{G}_m$ acts on X, and $v = (1, 0) \in \mathfrak{t}$ and $q = (1, t) \in T$. Then we can take

$$\mathcal{G} \;=\; \exp(\mathcal{O} \cdot v) \times \mathbf{Z} \cdot q.$$

Loosely speaking, a $\mathcal{G}$-differential equation is a connection along the flowlines of action of the first $\mathbf{G}_m$, and a difference equation along the second.

6.1.5. *Example.* For instance, we may take $X = \mathbf{C}$ and $v = \partial_z$, then renaming $t = \hbar$ the $\mathcal{G}$-differential equation becomes

$$e^{\hbar\partial_z} f(x) \;=\; f(x).$$

Note that we have $e^{\hbar\partial_z} f(x) = f(x + \hbar z)$ by Taylor's Theorem, which under the exponential map $\exp : \mathbf{C} \to \mathbf{C}^\times$ corresponds to multiplication by $q = e^{\hbar z}$,

$$\begin{tikzcd} \mathcal{O}(\mathbf{C}) \arrow[r, "+\hbar z"] & \mathcal{O}(\mathbf{C}) & f(x) \arrow[r, mapsto] & f(x+\hbar z) \\ \mathcal{O}(\mathbf{C}^\times) \arrow[u, "\exp^*"] \arrow[r, "q"] & \mathcal{O}(\mathbf{C}^\times) \arrow[u, "\exp^*"] & f(x) \arrow[r, mapsto] & f(qx) \end{tikzcd}$$

where $X = \mathbf{C}^\times$ and $\mathcal{G} = \mathbf{Z}$. (write in a more canonical way)

6.1.6. *Example.* An action of a group G on X gives a map of groups in PreStk

$$G \;\simeq\; \mathrm{Maps}(\mathrm{pt}, G) \xrightarrow{\mathrm{id}} \mathrm{Maps}(\mathrm{pt}, G) \times \mathrm{Maps}(X, X) \;\to\; \mathrm{Maps}(X, G \times X) \xrightarrow{\mathrm{act}} \mathrm{Maps}(X, X).$$

Taking the associated map on Lie algebras (i.e. applying $\mathrm{Maps}_*(\mathrm{Spec}\, k[\epsilon]/\epsilon^2, -)$, where we take pointed maps) gives

$$\mathfrak{g} \;\to\; \Gamma(X, \mathcal{T}_X).$$

In particular, we get $\exp(\mathfrak{g})_X \to \exp(\mathcal{T}_X)$

6.1.7. Note that $\mathrm{Lie}\,\mathrm{Aut}(X) = \Gamma(X, \mathcal{T}_X)$, thus we can consider

$$X/\exp(\mathcal{T}_X) \qquad \rightsquigarrow \qquad X/\mathrm{Aut}(X).$$

Or likewise,

$$X/\exp(\mathcal{O}_X \cdot v) \qquad \rightsquigarrow \qquad X/e^{\mathbf{C}\cdot v}$$

or take a subgroup $q^{\mathbf{Z}} \subseteq e^{\mathbf{C}\cdot v}$.

6.2. **Elliptic differential equations.** Take the universal curve

$$\pi \;:\; \mathcal{E} \;\to\; \mathcal{M}_{1,1}$$

and consider both:

- an automorphism $+p$ on $\mathcal{E}_\tau$ given by adding a point, (we need to specify a point, i.e. work with $\mathcal{M}_{1,2}$, or quotient by all of $\mathrm{Aut}(\mathcal{E}_\tau)$),
- vector fields on the base, $\mathcal{T}_{\mathcal{M}_{1,1}}$.

In particular, we have that

$$\mathcal{E}_{dR} \;=\; \mathcal{E}/(p^{\mathbf{Z}} \times \exp(\pi^* \mathcal{T}_{\mathcal{M}_{1,1}}))$$

and so $M \in \mathrm{QCoh}(\mathcal{E}_{dR})$ corresponds to a quasicoherent sheaf with an action of differential operators on the base and an automorphism of the fibre. In an important example of M in [FTV2], these are called the *heat equation* and the *qKZB* equation, respectively.

6.2.1. If one considers
$$\overline{\pi} : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$$
then we can consider:

- the automorphism $+p$ extends to $\overline{\mathcal{E}}$, and above $\mathcal{E}_\infty$ it becomes multiplication by $q = f(p)$ (which function?)
- an action of differential operators on $\overline{\mathcal{M}}_{1,1}$, which is still smooth.

One can thus define as before
$$\overline{\mathcal{E}}_{dR} = \overline{\mathcal{E}}/(\overline{p}^{\mathbf{Z}} \times \exp(\overline{\pi}^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}})).$$
Note that (check)
$$\overline{\mathcal{E}}_{dR,\infty} = E_\infty/q^{\mathbf{Z}}$$
which contains $\mathbf{C}^\times/q^{\mathbf{Z}}$ as an open subset, and its normalisation is $\mathbf{P}^1/q^{\mathbf{Z}}$,
$$\mathbf{C}^\times/q^{\mathbf{Z}} \overset{j}{\hookrightarrow} E_\infty/q^{\mathbf{Z}} \twoheadleftarrow \mathbf{P}^1/q^{\mathbf{Z}}.$$
In particular, an element $M \in \mathrm{QCoh}(E_\infty/q^{\mathbf{Z}})$ is equivalent to $M \in \mathrm{QCoh}(\mathbf{P}^1/q^{\mathbf{Z}})$ with a $q^{\mathbf{Z}}$-equivariant identification of $M_0 \simeq M_\infty$, which is (check!) equivalent to an element $M \in \mathrm{QCoh}(\mathbf{A}^1/q^{\mathbf{Z}})$ with (what other data?).

6.2.2. One should probably actually consider
$$\overline{\mathcal{E}}_{dR} = \overline{\mathcal{E}}/(\overline{\mathcal{E}} \times \exp(\pi^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}}))$$
where $\mathcal{E}$ via the group law on the universal elliptic curve. In particular, this allows us to *both*:

- pass to the formal completion of the identity in $\mathcal{E}$, *and*
- pass to the boundary of $\mathcal{M}_{1,1}$.

These give group maps
$$\exp(\mathcal{T}_{\overline{\mathcal{E}}}) \simeq \hat{\bar{\mathfrak{e}}} \times \exp(\pi^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}}) \to \overline{\mathcal{E}} \times \exp(\pi^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}}) \leftarrow (\overline{\mathcal{E}} \times \exp(\pi^* \mathcal{T}_{\overline{\mathcal{M}}_{1,1}}))_\infty$$
interpolating between D-modules, elliptic differential modules, and difference modules.

6.3. **Riemann-Hilbert.** We have defined parallel transport, by definition.

This should be related to ongoing work by Kontsevich and Soibelman [KS].

6.3.1. For q-difference modules, Riemann Hilbert was developed in [RSZ]

7. Quantum vertex algebras

We expect that we have

$$\begin{array}{ccc} \mathcal{D}\text{-Mod}(\mathbf{C}) & \xrightarrow{\exp_*} & \mathcal{D}\text{-Mod}(\mathbf{C}^*) \\ \downarrow & & \downarrow \\ \mathcal{D}_\hbar\text{-Mod}(\mathbf{C}) & \xrightarrow{\exp_*} & \mathcal{D}_q\text{-Mod}(\mathbf{C}^*) \end{array}$$

and we have likewise the notions of vertex algebras for these, where on the bottom the OPEs have singularities of the form $(z - w - n\hbar)$ and $(z - q^n w)$.

7.1. Let G act on a space X. The factorisation space to consider is then $(\operatorname{Ran} X)/G$, with factorisation structure

$$\begin{tikzcd} & (\operatorname{Ran} X \times \operatorname{Ran} X)_{G,\circ}/G \arrow[dl] \arrow[dr] & \\ (\operatorname{Ran} X)/G \times (\operatorname{Ran} X)/G & & (\operatorname{Ran} X)/G \end{tikzcd}$$

where $(\operatorname{Ran} X \times \operatorname{Ran} X)_{G,\circ}$ is the open subset of (S, S') with $gS \cap S' = \varnothing$ for all $g \in G$. The left map is the composition

$$(\operatorname{Ran} X \times \operatorname{Ran} X)_{G,\circ}/G \;\to\; (\operatorname{Ran} X \times \operatorname{Ran} X)_{G,\circ}/G \times G \;\to\; (\operatorname{Ran} X \times \operatorname{Ran} X)/G \times G.$$

7.1.1. *Remark.* Note that for the above to work the open subset $(\operatorname{Ran} X \times \operatorname{Ran} X)_{G,\circ}$ must be a $G \times G$-invariant, which is why we made this definition.

7.1.2. *Remark.* The above is a colimit of

$$\begin{tikzcd} & (X^n \times X^m)_{G,\circ}/G \arrow[dl] \arrow[dr] & \\ X^n/G \times X^m/G & & X^{n+m}/G \end{tikzcd} \tag{13}$$

7.2. Let $\mathcal{M}$ be a factorisation algebra on $(\operatorname{Ran} X)/G$, in the category:

(1) $\operatorname{QCoh}(\operatorname{Ran} X/G)$, i.e. $\mathcal{D}_q$-Mod(X) when $G = \mathbf{Z} \cdot q$, or otherwise
(2) $\mathcal{D}$-Mod$(\operatorname{Ran} X/G)$.

We consider the restriction of the factorisation map to (13) when $n, m = 1$, in which case we have open and closed complements

$$\Delta_G X/G \overset{i_G}{\to} (X^n \times X^m)/G \overset{j_G}{\leftarrow} (X^n \times X^m)_{G,\circ}/G$$

where $\Delta_G X \subseteq X \times X$ consists of points of the form (x, gx) for $g \in G$, and G acts diagonally.

We now consider the cofibre sequence

$$i_G^*\mathcal{M} \;\to\; i_G^* j_* j^* \mathcal{M} \;\to\; \text{cofib}\,.$$

7.2.1. In the D-module case $\operatorname{cofib} = i_G^! \mathcal{M}[1]$. Assume that $i_G^* \mathcal{M} = V \otimes \mathcal{O}$ and z, w are local coordinates on X, then taking global sections of the cofibre sequence gives

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G}[\{(z-gw)^{\pm 1}\}_{g\in G}] \to V \otimes \mathcal{O}_{X/G}[\{\delta_{z-gw}\}_{g\in G}].$$

Crucially, because we have only quotiented by a single, diagonal, G-action throughout, in the above we have *not* taken G-invariants with respect to the antidiagonal action, which would have killed the $z - gw$ terms.

Lemma 7.2.2. *The data of the above is equivalent to a map*

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G} \to V \otimes \mathcal{O}_{X/G} \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z-gw)).$$

Assuming that X is itself an algebraic group, we take X-invariant sections of the above to get a map

$$Y_G : V^{\otimes 2} \to V((z-g_1 w, z-g_2 w, \cdots)) = V \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z-gw)).$$

7.2.3. In the quasicoherent sheaf case, since the pullback/forgetful functor $\mathcal{D}\text{-Mod}(Z) \to \operatorname{QCoh}(Z)$ is exact, we have that cofib is the same as above, and given the above assumptions, taking global sections of the cofibre sequence gives

$$V^{\otimes 2} \otimes \mathcal{O}_{X/G}[\{(z-gw)^{\pm 1}\}_{g\in G}] \to V \otimes \mathcal{O}_{X/G}[\{\delta_{z-gw}\}_{g\in G}].$$

What is different is that we have only remembered that this is a map inside $\operatorname{QCoh}(X/G)$. However,

Lemma 7.2.4. *(check) When $X = \mathbf{C}$ and $G = \mathbf{Z} \cdot q \simeq \mathbf{Z}$, this is equivalent to a map*

$$Y_G : V^{\otimes 2} \to V((z-g_1 w, z-g_2 w, \cdots)) = V \hat{\otimes} \prod_{\mathbf{C}[[z,w]]} \mathbf{C}((z-gw))$$

(with some T action, or rather $q = \exp(\hbar T)$.)

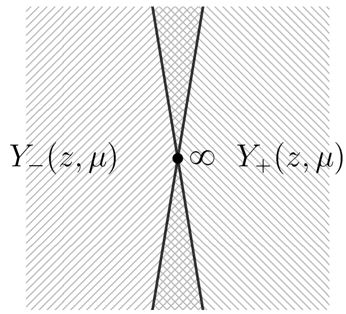
Proof. (The same proof as for the D-module case should work, except that instead of asking that the map commutes with ∂, we ask that it be $\mathbf{Z}$-graded, a $\mathbf{Z}$ acts on $k[x]$ as $x^n \mapsto (qx)^n$.) □

8. Stokes phenomena and dynamical KZ

8.1. One can consider the *dynamical* KZ equation

$$\mu_i \;+\; (k - k_{crit})\partial_{z_i} \;+\; \sum_{i\neq j} \frac{\Omega_{ij}}{z_i - z_j}$$

where $\mu \in \mathfrak{t}^{reg} \subseteq \mathfrak{g}$ acts on the ith factor, see [Xu]. This picks up an *irregular* singularity at $z_i = \infty$, around which there is a unique formal solution $Y(z,\mu)$ but on different sectors in the z_i-plane around $z_i = \infty$ there are *different* holomorphic solutions:



which are unique if we prescribe behaviour $Y(z,\mu) \to z^{\hbar\Omega} e^{z\mu_1}\mathcal{O}(1)$ as $z \to \infty$ along any sector. The *Stokes matrix* is

$$S_+ \;=\; Y_+(z,\mu)/Y_-(z,\mu) \;\in\; U(\mathfrak{g}) \otimes U(\mathfrak{g})[[\hbar]]$$

where the z independence is due to [TLX, §4].

The main Theorem [TLX, Thm. 4.2] says that

$$R \;=\; e^{i\pi\hbar\Omega} S_+$$

defines the R-matrix for $U_\hbar(\mathfrak{g})$-Mod.

8.1.1. *Why care?* From [Xu, §3], if we play the same game around $z = 0$, we can define $Y^0_\pm(z,\mu)$, and set

$$J_+ \;=\; Y^\infty_+(z,\mu)/Y^0_+(z,\mu)$$

by [TLX, Thm. 3.12] kills the associator of $U_\hbar(\mathfrak{g})$-Mod, and so it follows that *all* information of $U_\hbar(\mathfrak{g})$ as a braided monoidal 1-category is contained in the $n = 2$ case, unlike when $\mu = 0$, where we need $n \leqslant 3$ to also get the associator. (write/think more precisely)

8.1.2. The above seems to give a factorisable perverse sheaf of categories over (or $\mathrm{Ran}\,\mathbf{P}^1$)

$$\mathrm{Conf}(\mathbf{P}^1).$$

In the elliptic case, we can consider the dynamical KZ equation

$$\mu^i \;+\; \xi_i \;+\; \sum_{i\neq j} \frac{\Omega_{ij}}{(???)}$$

over the universal curve $\overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$, where ξ is a generating vector field. (is that defined over all $\overline{\mathcal{E}}_{1,1}$?) This does not add any more singularities to the KZ equation. (maybe in the $\mathbf{G}_m$ case though?)

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