Random graphs : long problems

All of Mattekollo

Domino tilings: long problems

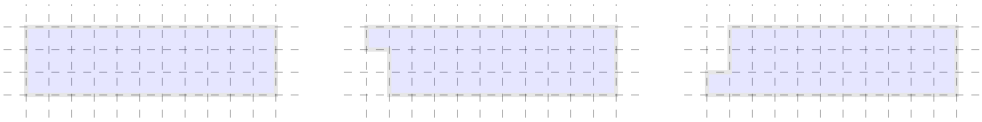
All of Mattekollo

- *Attempt problems in any order.*
- *Sometimes problems will use definitions intruduced in later classes. Feel free to use one problem you've solved to solve another, but if it trivialises the problem, see if you can find another proof as well!*
- *More long problems, or if necessary, hints, will be added a bit each lesson.*

Formulas.

1. Height three rectangles.

   a. Find a recurrence relation relating the numbers A_n, B_n, C_n of domino tilings of



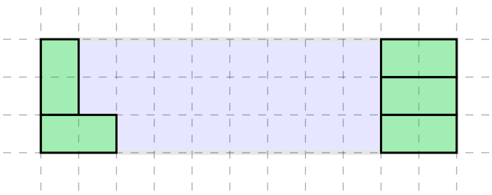
   where the base has length n.

   b. Use it to find a formula for A_n in terms of $A_{n-1}, A_{n-2}, \ldots, A_1$.

   c. Show that $A_n, B_n, C_n = 0$ if n is odd.

   d. Compute A_n for $n = 1, 2, \ldots, 10$. Can you find a formula for the number of domino tilings of the $3 \times n$ rectangle?

2. There are three possibilities for what a domino tiling looks of the $3 \times n$ rectangle at the left (or right) boundary, e.g. one left and one right boundary is



Let B_L, B_R be boundary states on the left and right. For each choice of B_L, B_R, what is the probability

$$\mathbf{P}(B_L \to B_R)$$

that a random domino tiling with state B_L on the left has state B_R on the right?

Make sure you do the $2 \times n$ case first!

3. Prove that the number of tilings of the $n \times n$ Aztec diamond is $2^{n(n-1)/2}$.
One method (no guarantee it works!): can you solve this by finding a set S of size $\binom{n}{2} = n(n-1)/2$ and a bijection

$$\{\text{domino tilings of the Aztec diamond}\} \simeq \{\text{finite subsets of } S\}?$$

4. Prove that the number of domino tilings of a dual $A \times B \times C$ hexagonal graph is

$$\prod_{a=1}^{A} \prod_{b=1}^{B} \prod_{c=1}^{C} \frac{a+b+c-1}{a+b+c-2}.$$

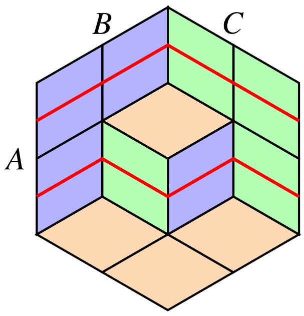
5. What is the expected number of 3d boxes in (the 3d interpretation of) a domino tiling of the dual $A \times B \times C$ hexagonal graph?

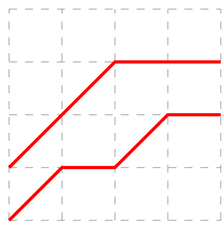
6. What is the number of domino tilings of the $m \times n$ rectangle?

Lattice paths.

7. Hexagonal paths.

   a. Give a bijection between tilings of the $A \times B \times C$ hexagon and certain a certain set of paths in $\mathbf{Z}^2$ (which you should precisely specify).





   b. What are the graphs corresponding to the lozenge tilings with the maximum and minimum number of 3d boxes?

   c. How can you count the number of 3d boxes from the paths?

   d. Can you find an analogous set of paths for the domino tilings of the $m \times n$ rectangle?

8. Count the number of tilings of the $3 \times n$ rectangle in terms of the number of paths on a certain graph.

   a. *Attempt if you know matrices.* Define a matrix M_3 such that the top left entry of M_3^n is the number of domino tilings of the $3 \times n$ rectangle.

Determinants.

9. Colour the faces of the hexagonal graph in alternating white and black colours. Write N for the number of white or black squares (which we assume are equal, otherwise there are no domino tilings). Let K be the $N \times N$ matrix of numbers

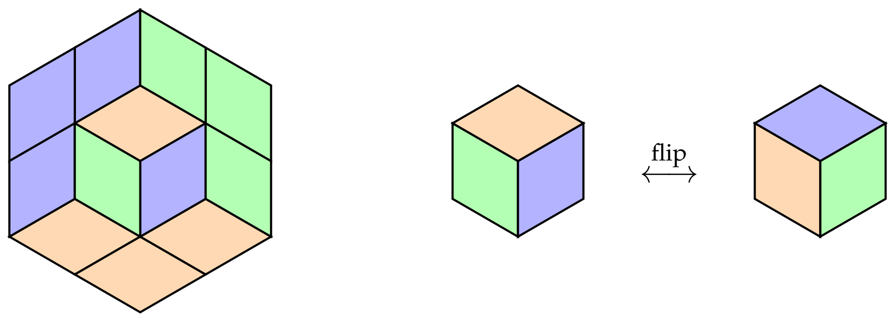
$$K_{i,j} = \begin{cases} 1 & \text{if the } i\text{th white vertex is adjacent to the } j\text{th black} \\ 0 & \text{otherwise.} \end{cases}.$$

Show that $|\det K|$ is the number of domino tilings of the hexagonal graph by lozenges.

Can you find a matrix $\tilde{K}$ that works for any planar bipartite graph?

Random walks through the graph of all tilings.

10. Show that any tiling of the hexagonal $A \times B \times C$ graph by lozenges can be obtained from any other by a sequence if *flips*



What about the $m \times n$ rectangle? What about any square graph (subgraph of $\mathbf{Z}^2$)?

How do you in practice find a random lozenge tiling (there are *so many!*)? *Just start with any lozenge tiling, and randomly flip!*

Theorem 1. Let n be an integer and T_0 any lozenge tiling. Pick

- $t \in \{0, 1, \ldots, n\}$ randomly (uniformly),
- a random sequence of t many flips $T_0 \to T_1 \to \cdots \to T_t$, at each step choosing all allowed flips equally likely.

Then if

$$P(T) \; : \; \{\text{lozenge tilings}\} \;\to\; [0,1]$$

is the probability that $T_t = T$, we have

$$P(T) \;\to\; \frac{1}{|\{\text{lozenge tilings}\}|}$$

as $n \to \infty$.

11. *Good to practice linear algebra skills.* Let's prove Theorem 1.

a. Let G be an undirected graph, with set of vertices denoted by $V(G)$. Let

$$p_t(x) \; : \; V(G) \times \mathbf{N} \;\to\; [0,1] \tag{1}$$

be the probability that, at time $t = 0, 1, 2, \ldots$, a random walk starting at vertex i is at a given vertex x. Show that p_t satisfies

$$p_{t+1}(x) = \sum_{y \sim x} \frac{1}{\deg(y)} p_t(y) \tag{2}$$

and denote by $A : \mathbf{C}^{V(G)} \to \mathbf{C}^{V(G)}$ the associated linear map.

b. Compute (and graph) for low values of t what p_t is for the graph

if we start at the middle, or edge. Conjecture what happens as $t \to \infty$.

c. Show that the function $\lambda^t \cdot p_0$, where $p_0 : V(G) \to \mathbf{C}$ is a function independent of t and $\lambda \in \mathbf{C}$, satisfies (2) if and only if $\lambda = 0$, or

$$\lambda \cdot p_0(x) = A \cdot p_0(x).$$

d. Show that $p_t(x)$ is a sum of solutions of this form. (give more hints)

e. Assume from now on that G is connected. Pick N such that there is a path of length at most N between any two vertices, and let

$$B = \frac{1}{N}(I + A + \cdots + A^{N-1}).$$

Show that if $Av = \lambda v$, that

$$B \cdot v = \frac{1}{N}(1 + \lambda + \cdots + \lambda^{N-1})v.$$

f. Show that if $Bv = \mu v$ for some nonzero v, $|\mu| < 1$ unless for $\mu = 1$, in which there is a unique solution

$$v = \frac{1}{|V(G)|}.$$

g. Show that this implies Theorem 1.

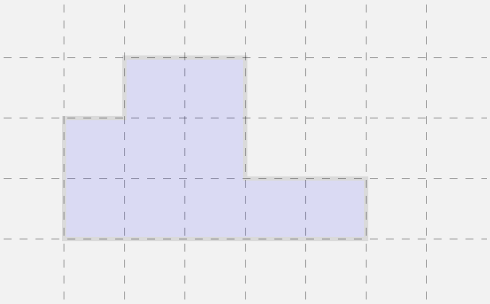
Spanning trees.

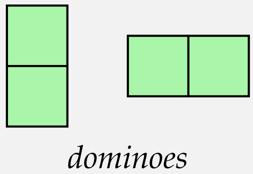
12. Temperley bijection.

1. Domino tilings

1 July

Definition. A *domino tiling* of a region is a way of covering it completely using 1×2 or 2×1 dominoes, without overlaps or gaps.





1. Counting domino tilings.

   a. How many domino tilings are there of the rectangles: 2×2, 2×3, 2×4, 2×5?

   b. Show that the number of domino tilings of a $2 \times n$ rectangle is the nth Fibonacci number F_n. Conclude that

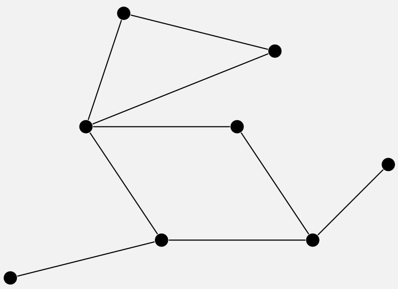
$$F_{n+m} = F_n F_m + F_{n-1} F_{m-1}.$$

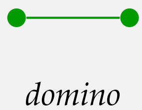
   c. How many domino tilings are there of a $3 \times n$ rectangle ($n = 2, 4, 7, 10$)? *Try to do the bigger numbers in a non-bashy way.*

   d. How many tilings are there of the Aztec diamond with only one row of maximal length?

   e. How many tilings are there of the Aztec diamond with base length $2, 4, 6$?

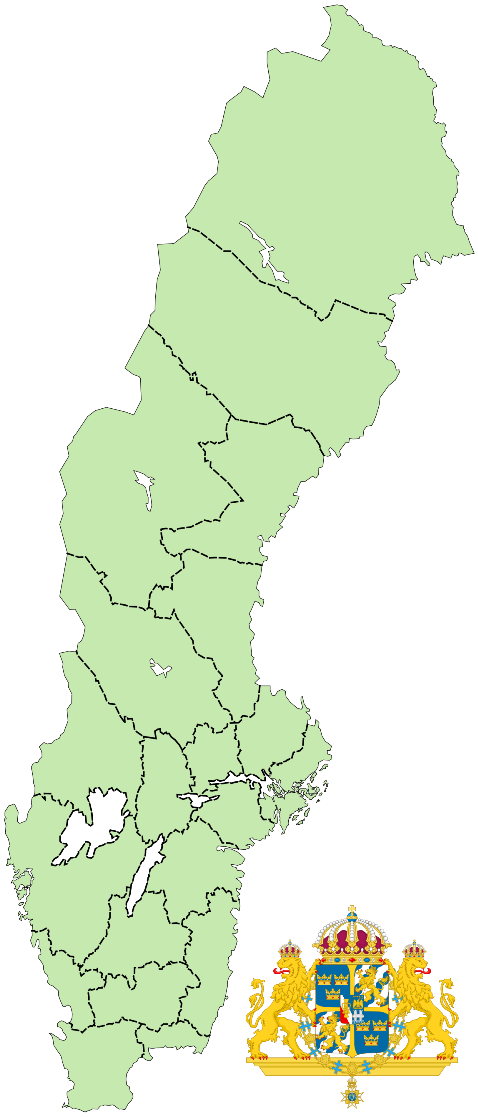
Definition. A *domino tiling* of a *graph* is a way of covering the vertices completely using dominoes, without overlaps or gaps.





Show that when the graph is a square grid graph, this is the same as the first definition of domino tiling.

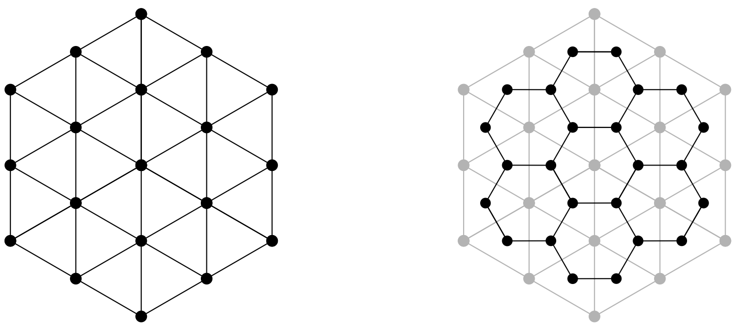
2. Consider the map of Sweden's Län



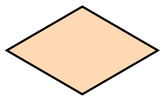
a. Draw its *dual graph*, which has one vertex for each region, and an edge whenever two regions share a 1d border (not just touch at a point).

3. Examples coming from 3d geometry.

a. Consider the hexagon graph and its dual



Draw three possible domino tilings of the dual graph, and show that they correspond to colouring in the regions of the hexagonal graph in by *lozenges*

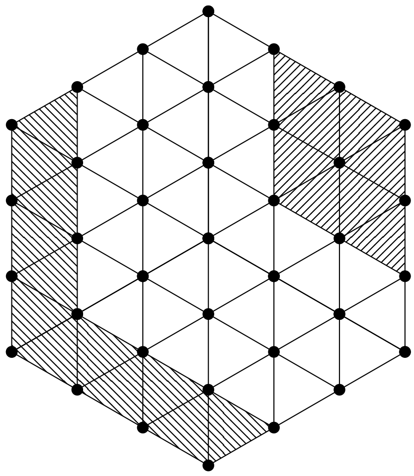


lozenge

b. How many domino tilings are there of the duals to the $2 \times 2 \times 2$ and $3 \times 3 \times 3$ hexagonal graphs?

c. Give a 3D geometric description of such a tiling.

d. If we remove the following region, how many domino tilings are there of the dual graph?



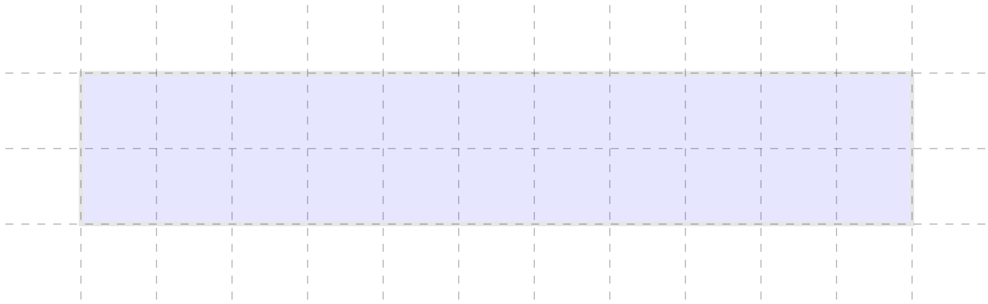
2. Random domino tilings

2 July

Definition. A *random domino tiling* of a finite region is a probability distribution (on the set of all tilings) where each domino tiling T is equally likely

$$\mathbf{P}(\text{tiling } T) \;=\; \frac{1}{\#\{\text{all tilings } T'\}}.$$

1. Random domino tilings of $2 \times n$ rectangle.



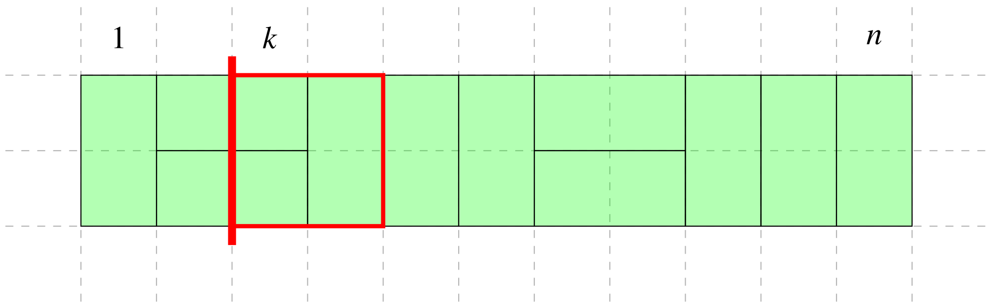
a. Draw all domino tilings of the 2×5 rectangle. Compute the probability that the leftmost domino is vertical.

b. What is the answer for the $2 \times n$ rectangles?

c. If the leftmost domino of an $2 \times n$ rectangle is vertical, what is the probability that the rightmost domino is vertical?

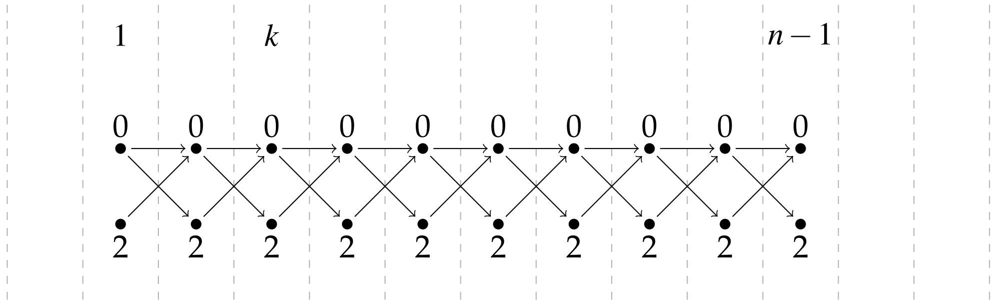
d. What is the limit of b. and c. as $n \to \infty$?

2. Take a domino tiling, and consider the sequence of boxes, labelled by $k = 1, \ldots, n-1$.



a. Let $f : \{0, 1, \ldots, n\} \to \{0, 2\}$ be the function counting the number of horizontal dominoes completely contained in the kth box. Show that f uniquely determines the domino tiling.

b. Give a bijection between domino tilings of the $2 \times n$ rectangle and maximal paths on the following graph:



c. Give another proof that this is the nth Fibonacci number.

d. *Attempt if you know matrices.* Define the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that the number of domino tilings of the $2 \times n$ rectangle is equal to the top left entry a_n of $M^n = M \times \cdots \times M$.

e. *Attempt if you know matrices.* Show that

$$M^2 = M + 1$$

hence show that $a_{n+2} = a_{n+1} + a_n$.

f. *Attempt if you know matrices.* Consider the top-left entry $b_n \in \mathbf{Z}[x, y]$ of N^n, for the matrix

$$N = \begin{pmatrix} x & y \\ y & 0 \end{pmatrix}.$$

Compute b_1, b_2, b_3, b_4. What is the interpretation of the $x^i y^j$-coefficient of b_n? Show that $b_{n+2} = x b_{n+1} + y^2 b_n$. Find a formula for b_n of the form

$$b_n = c_1 \cdot \varphi_1^n + c_2 \cdot \varphi_2^n.$$

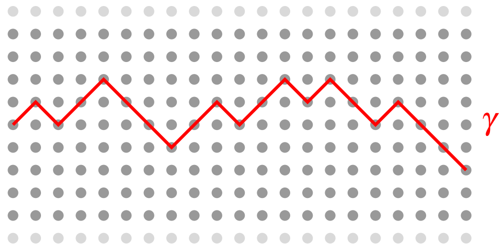
Hint: find two solutions of the form φ_i^n, to the equation, then use b_1, b_2 to find c_1, c_2.

Definition. A *walk* on a graph G is a sequence of vertices $v_0, v_1, \ldots, v_t$, where each vertex is adjacent to the next. A *random walk* is a probability distribution on the set of all walks, where each walk is equally likely,

$$\mathbf{P}(\text{walk } v_0, \ldots, v_t) = \frac{1}{\#\{\text{all walks }\}}.$$

One can similarly define a *random walk starting at a fixed vertex v*, or a random walk with some other set of properties.

3. Random walks. Consider a random walk on $\mathbf{Z}$ of length t, which we draw as a path in $\mathbf{Z} \times \{0, 1, \ldots, t\}$.



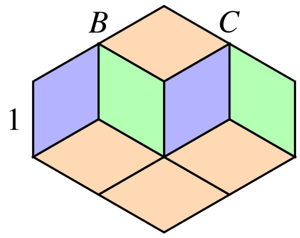
a. How many walks of length t are there?

b. What is the probability that a random walk γ has $\gamma(t) \geq 0$?

c. What is $\mathbf{E}(\gamma(t))$?

d. What is $\mathbf{E}(\gamma(t)^2)$? In plain English, approximately how far away from the origin should we expect γ to be after t steps?

4. Show that the set of tilings of the $1 \times B \times C$ hexagon by lozenges



is equivalent to a certain set of walks in $\mathbf{Z}$ (which you should define precisely).

a. Picking a random tiling (or equivalently, a random walk), what is the expected number of 3d cubes in a random $1 \times B \times C$ lozenge tiling?

b. Given a tiling T on the hexagonal graph G, we define the *height function*

$$h_T : V(G) \to \mathbf{N}$$

whose value is the number of 3d boxes under ($\downarrow$) that vertex. Describe the walk attached to T in terms of h_T.

c. *If you're interested in continuing, try the lattice paths long problem.*

3. Random dominoes 3

3 July

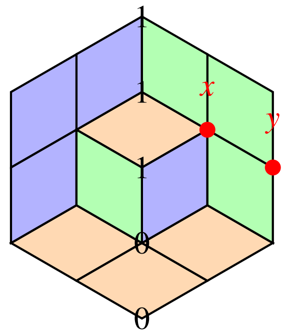
Probability.

Definition. A *probability space* is a countable (i.e. finite or has as many elements as $\mathbf{N}$) set Ω

1.

Back to random domino tilings.

2. Correlators.



a. Show that there are 20 lozenge tilings of the $2 \times 2 \times 2$ hexagon.

b. Write down the *height function* $h_T : V(G) \to \mathbf{N}$ of two tilings T. Here $V(G)$ is the vertex set of the hexagonal graph, some heights are given above for the above tiling.

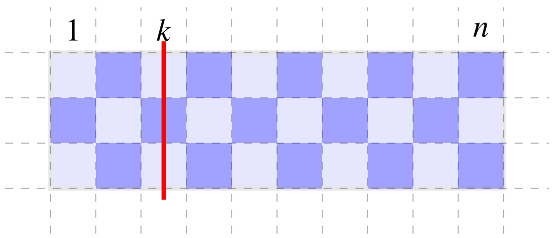
c. What are the expected values $\mathbf{E}(h_T(x))$ and $\mathbf{E}(h_T(y))$ for a random tiling T?

d. What is the value of the *correlator*

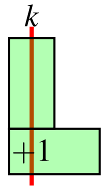
$$C(x,y) \;=\; \mathbf{E}(h_T(x)h_T(y)) - \mathbf{E}(h_T(x))\mathbf{E}(h_T(y))$$

for the two points x, y?

3. Flux.



Colour the squares of an $n \times m$ rectangle in a chessboard pattern. The *flux* $\Phi(k)$ *through the kth slice* of a domino tiling is the sum of ± 1 for each domino perpendicular to the slice



where it is $+1$ or -1 depending on the colour of the square. (*The word "flux" means "flow", like the amount of water flowing through a pipe.*)

a. Show that for every domino tiling, $\Phi(k)$ is a constant, i.e. independent of k.

b. For a random domino tiling of the $2 \times n$ and $3 \times n$ rectangle, what is the expected value of the flux Φ?

4. Random dominoes 4

5. Random graphs : Electrical networks

2 July

Definition. Reminder on expected value, etc.

4. Exericse on expected value, etc.

Definition. The *resistance* between two points i and j in a graph is

$$R_{i,j} = \mathbf{E}[\#\text{steps } \gamma \text{ takes before first hitting } j]$$

where we take expectation over the set of all paths γ starting at i. `https://math.dartmouth.edu/~pw/math100w13/qin.pdf`

5. In the graph

what is the resistance between 1 and 2? Between any pair of vertices i and j?

6. Random graphs 2

7. Random graphs 3

8. Random graphs 4