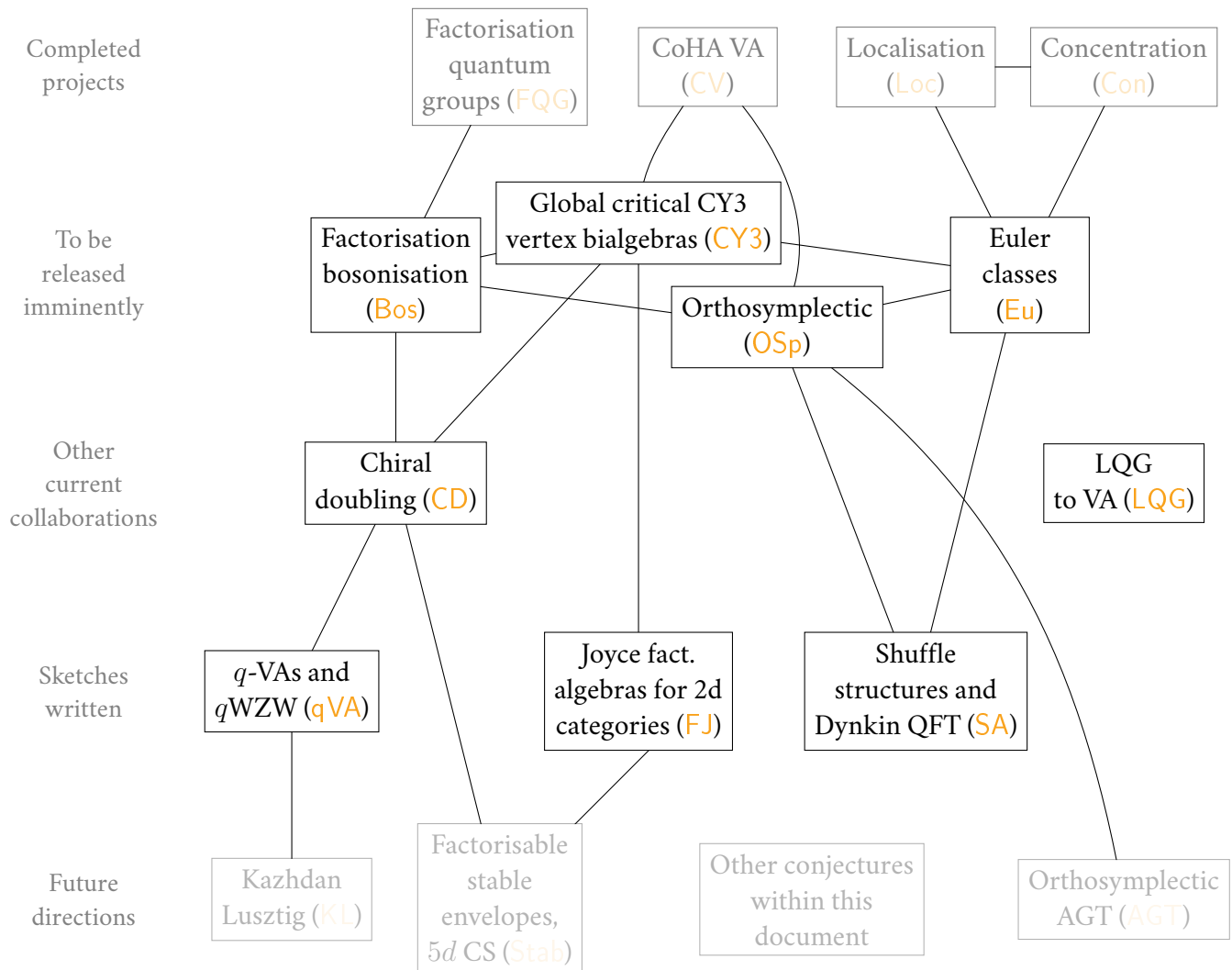


RESEARCH PLANS - ABRIDGED

ALEXEI LATYNTSEV

See <https://alyoshalatyntsev.github.io/plan/plan.pdf> for a more detailed research statement, including details/evidence on how we plan to prove the conjectures.



1. Research statement

I am a mathematician working in geometric representation theory.

1.1. Algebraic structures attached to CY3s (CV, CY3¹², FJ¹, Stab)

Cohomological Hall algebras are associative algebras attached to Calabi-Yau varieties or categories [Da; KS; KPS]. They relate to DT theory [KS; Sz], algebraic structures previously appearing in representation theory [BD; RSYZ; YZ] (e.g. quantum groups [MO]), hence are a key object in geometric representation theory/enumerative invariants.

Global critical CY3 vertex quantum groups. Let \mathcal{C} be the CY3 category $\text{Rep}(Q, W)$, the representations of a Jacobi algebra of a quiver with potential, or $\text{Coh}_0(K_{T^*C})$, zero dimensional coherent sheaves on a local curve [KK], or more generally a deformed CY3 completion.

Theorem. [CY3] *The critical cohomology³ of the moduli stack of objects \mathcal{M} has a vertex coproduct*

$$H^\bullet(\mathcal{M}, \varphi) \rightarrow H^\bullet(\mathcal{M}, \varphi) \hat{\otimes} H^\bullet(\mathcal{M}, \varphi)((z^{-1}))$$

compatible with the CoHA: it forms a vertex quantum group (see FQG).

Theorem. [CY3] *There is a functor Φ from Q -localised bialgebras [Da] to vertex bialgebras. The former is equivalent to the category of factorisation bialgebras on the coloured configuration space $\text{Conf}_{Q_0}\mathbf{A}^1$.*

Theorem. [CY3; CV for $W = 0$] *For any quiver Q , the vertex coproduct on the preprojective CoHA $H_{\mathbf{G}_m}^\bullet(\mathcal{M}_{Q^{(3)}}, \varphi_{W^{(3)}})^{\text{ext}} \stackrel{[\text{BD}]}{\simeq} Y_h(\mathfrak{b}_Q)$ is identified by Φ with the Davison/Yang-Zhao localised coproduct [Da; YZ], and (when defined) Drinfeld's meromorphic coproduct [Dr; GLW].*

This generalises **Joyce-Liu's vertex coproduct** [Joa; Li], shows that all these well-known coproducts on the Yangian are **equal**, and gives a **geometric** definition using moduli stacks and configuration spaces. Finally, Theorem 1.1 points to a new Ran space definition of vertex-vertex bialgebras.

Lift to factorisation algebra and relation to stable envelopes. To move towards arbitrary CY3s, we first interpret the above in terms of factorisation algebras; second, we relate to the \mathcal{W} -algebras for surfaces of [MMSV], thus understanding the structure for K_S . Let Q be a quiver with torus $T = \prod T_d$.

Conjecture. [FJ] *Given $\mathcal{M}^f = \{(m, \lambda) : \lambda \in \mathfrak{t}, m \in \mathcal{M}^\lambda\} \xrightarrow{\pi} \text{colim}(\mathfrak{t}_d) = \text{Ran}_{Q_0}\mathbf{A}^1$, its relative BM-homology $\mathcal{A} = \pi_*\omega$ is a factorisation algebra over the coloured Ran space. Its restriction to the colour-diagonal recovers the vertex quantum group structure on the nilpotent CoHA [SV].*

Conjecture. [FJ] *For S a smooth algebraic surface, there is a braided factorisation category $\text{Rep } \mathcal{W}$ over $\text{Ran}_S K_S$ (c.f. FQG). Applying Bos/CD allows us to construct $\mathcal{W}(S)^{\geq 0}$ and $\mathcal{W}(S)$ from [MMSV]'s $\mathcal{W}(S)^{>0}$.*

The definition of \mathcal{M}^f is clearly reminiscent of [MO, §5.1.1]'s stable envelope construction, and suggests a resolution to their question: “we view this definition as provisional; perhaps a better set of axioms will emerge later”. Let \mathbf{w} be a multidimension vector of quiver Q and $M(\mathbf{w})$ the quiver variety.

Conjecture. [Stab] *There is a factorisation space $\pi_{\mathbf{w}} : M(\mathbf{w})^f \rightarrow \text{Ran}_Q\mathbf{A}^1$ and the factorisation category \mathcal{E} spanned by $\pi_{\mathbf{w},*}\omega$ is acted on by $\mathcal{A} = \pi_*\omega$. Applying chiral Tannakian reconstruction $\mathcal{E} \simeq \text{Rep } \mathbf{D}\mathcal{A}$ gives the double of \mathcal{A} with its (two) coproducts.*

The above is a Drinfeld-Kohno Theorem for [MO]'s Yangians $Y_h(\mathfrak{g}_Q)$ (see qVA for relations to qKZ).

1.2. The structure of factorisation quantum groups (FQG, Bos⁴, CD⁵)

Historically the definitions of (double) affine quantum groups $U_q(\hat{\mathfrak{g}})$, $Y_h(\mathfrak{g})$, $\mathcal{E}_{h,\tau}(\mathfrak{g})$, $Y_h(\hat{\mathfrak{g}})$, $\mathcal{W}_{1+\infty}(\mathfrak{g})$ were (ingeniously) made very explicitly [Dr; MO], and still much of the modern (e.g. CoHA) literature is based on explicit shuffle

¹Joint with S. Kaubrys.

²Joint with S. Jindal.

³i.e. $\mathcal{M} = \text{Crit}(W)$ is a critical locus inside a smooth moduli stack; we take the vanishing cycle sheaf $\varphi = \varphi_W$.

⁴Joint with S. de Hory.

⁵Joint with W. Niu.

computations, e.g. [MMSV; SV; YZ]. This series of projects **axiomatises** (operadically) these structures, allowing us to import techniques from the theory of ordinary **quantum groups**,⁶ to recover the above formulas as a **consequence** of these definition.

Factorisation quantum groups. In **FQG** we develop a theory of \mathbf{E}_n -factorisation categories over factorisation spaces X (including ordinary groups G , configuration spaces $\text{Conf}_{Q_0} \mathbf{A}^1$, and algebraic-topological Ran spaces $\text{Ran}(\mathbf{A}^n \times \mathbf{R}^m)$). We first give basic structure results for braided factorisation categories \mathcal{C} :

Theorem. [FQG] Let \mathcal{A} is a factorisation algebra in \mathcal{C} over X , a (braided) factorisation structure on $\mathcal{A}\text{-FactMod}(\mathcal{C})$ induces a factorisation bialgebra structure on \mathcal{A} (and a factorisation R -matrix $R : \mathcal{A} \otimes_{\mathcal{C}, X} \mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{C}, X} \mathcal{A}$).

Theorem. [FQG] When $X = \text{Ran} \mathbf{A}^1$ (resp. $\text{Conf} \mathbf{A}^1$), Theorem 1.2 recovers classical notions [EK; FR] of **quantum vertex algebras** (resp. localised algebras) and their R -matrices $R(z)$ satisfying the spectral YBE.

For instance, the structure [GLW] for **Yangians** (e.g. two (meromorphic) coproducts and R -matrices R^- , $R^{0, \epsilon}$, R^ϵ relating them) are equivalent to: $Y_h(\mathfrak{g})\text{-Mod}$ is a lax braided factorisation category. This makes rigorous the physics claim [CWY] that $Y_h(\mathfrak{g})\text{-Mod}$ from 4d Chern-Simons is a topological-holomorphic factorisations category over $\mathbf{R} \times \mathbf{C}$. The above may help understand **affine Yangians** (e.g. [GRZ]; **qVA** for relation to q -WZW) and the **new KL equivalences** [BCDN]. Finally, in principle we may get lots of examples from:

Theorem. [FQG] A generalisation of Borchers' twist construction [Bo] to arbitrary decomposition algebra.

Factorisation bosonisation. In the CoHA literature, a lot of algebraic effort needs to be expended each time [Da; RSYZ; YZ], [CY3; OSp] to add in the Cartan piece $H^\bullet(\mathcal{M}, \varphi) \rightsquigarrow H^\bullet(\mathcal{M}, \varphi)^{\text{ext}}$, e.g. to obtain Yangians of Borels $Y_h(\mathfrak{b}_Q)$. In the finite groups case, a clean construction is Tannakian reconstruction

$$U_q(\mathfrak{n}) \rightsquigarrow U_q(\mathfrak{n}) \quad \text{via} \quad U_q(\mathfrak{b})\text{-Mod} \xrightarrow{\sim} U_q(\mathfrak{n})\text{-Mod}(\text{Rep}_q T),$$

c.f. [Ga; Maa; Mab]; in **Bos** we apply the same to vertex and factorisation bialgebras to obtain:

Theorem. [Bos, in preparation] There is a Tannakian reconstruction functor from (braided) factorisation categories \mathcal{C} to (quasitriangular) factorisation quantum groups \mathcal{A} . In the preprojective case of Theorem 1.1,

$$Y_h(\mathfrak{b}_Q)\text{-Mod} \simeq Y_h(\mathfrak{n}_Q)\text{-Mod}(Y_h(\mathfrak{t}_Q)\text{-Mod})$$

we Tannakian reconstruct $Y_h(\mathfrak{b}_Q) \simeq H^\bullet(\mathcal{M}, \varphi)^{\text{ext}}$ with its localised or vertex bialgebra structure.

Applying this to $H^\bullet(\mathcal{M}, \varphi)\text{-Mod}(H^\bullet(\mathcal{M})\text{-Mod}^\cup)$ **automates** the process of extending CoHAs.

Factorisation Drinfeld doubling. An active problem is how the structures in **CY3** relate to **wall crossing** and stability conditions, see [Br; Job]. As a first attempt, we use **FQG** to understand **doubling**, where the CoHA of heart \mathcal{A} and its opposite $\mathcal{A}[1]$ are glued, in a similar way to **Bos**:

Conjecture. [CD] There is a Drinfeld centre construction $Z_{\mathbf{E}_1}(\mathcal{C})$ of a chiral factorisation category \mathcal{C} , which carries compatible chiral and ordinary monoidal structures, and Tannakian reconstruction gives

$$Z_{\mathbf{E}_1}^{Y_h(\mathfrak{t}_Q)\text{-Mod}}(Y_h(\mathfrak{b}_Q)\text{-Mod}) \simeq Y_h(\mathfrak{g}_Q)\text{-Mod}$$

and likewise we recover the Takiff algebra double construction of [AN].

1.3. Orthosymplectic structures (OSp⁴, SA⁴, AGT⁴)

Orthosymplectic CoHAs. We define orthosymplectic moduli stacks \mathcal{M}^{OSp} , including perverse-coherent sheaves with a symplectic/orthogonal bilinear form, and quivers and potential with involution.

Theorem. [OSp] For \mathcal{M} as in \mathcal{C} the vertex quantum group $H^\bullet(\mathcal{M}, \varphi)$ acts on $H^\bullet(\mathcal{M}^\tau, \varphi^\tau)$, i.e.

- (1) there is a left module action a of the **CoHA** respecting the involution,⁷ compatible with
- (2) a **symplectic vertex algebra** structure (factorisation coalgebra over symplectic Ran space $\text{Ran}_{\text{Sp}} \mathbf{A}^1 = \text{colim}_{\text{Sp}_{2n}} \text{t}_{\text{Sp}_{2n}}$, see **SA**). The defining R -/ K -matrices satisfy the **Cherednik reflection equation**.

⁶Namely, when $X = \text{Ran} \mathbf{R}^2$ in the below, via Lurie [Lu].

⁷i.e. the left action a and the right action $a \cdot (\text{id} \otimes \tau)$ commute, where τ is the involution.

Theorem. [OSp] *There is an action of $H_{\bullet}^{\text{BM}}(\mathcal{M})$ of the CoHA of zero-dimensional coherent sheaves on surface S on $H_{\bullet}^{\text{BM}}(\mathcal{M}_{\sigma\text{-ss}}^{\text{OSp}})$ the BM homology of a compactification of the stack of $G \in \{\text{Sp}, \text{O}\}$ -bundles.*

This gives a rigorous definition of the objects coming from boundary 4d Chern-Simons theory [BS], should relate to work in **quantum groups** [LW] and Finkelberg-Hanany-Nakajima's ongoing work on **orthosymplectic Coulomb branches** (see also AGT). In the quiver with potential case (resp. in the conjecture, type A_{2n+1} preprojective), we have:

Theorem. [OSp] *An explicit shuffle formula for the CoHA action and vertex coaction on $H^*(\mathcal{M}_{Q,W}^{\text{OSp}}, \varphi^{\text{OSp}})$.*

Conjecture. *The preprojective orthosymplectic CoHA⁸ is isomorphic to the **twisted Yangian** $Y_h(\mathfrak{gl}_n)^{tw}$ of [BR].*

Dynkin QFT and shuffle structures. The structures (2) in OSp are defined over the symplectic configuration space⁹ $\text{Conf}_{\text{Sp}}(\mathbf{A}^1) = \text{Spec} H^*(\text{BSp})$, with singularities over the root hyperplanes; alternatively over $\text{Ran}_{\text{Sp}} \mathbf{A}^1 = \text{colim}_{\text{Sp}_{2n}} \mathbf{A}^1$.

Theorem. [OSp] *To give examples: restricting along $\mathfrak{t}_{\text{Sp}_{2n}} \hookrightarrow \mathfrak{t}_{\mathfrak{gl}_{2n}}$ gives an **invariants** functor $\iota : \text{FactAlg}_{\text{GL}}(\mathbf{A}^2) \rightarrow \text{FactAlg}_{\text{Sp}}(\mathbf{A}^1)$. n.b. we expect the structures in OSp come from applying ι to \mathcal{M}^f (see FJ).*

In SA, we **generalise** the above to systems of **arbitrary Kac-Moody groups** G_i [Ku, §V]; this forms a category K_{KM} with morphisms the parabolics. Write $K_{\text{GL}}, K_{\text{Sp}}, \dots$ for the appropriate subcategories.

Fact. [SA] *A shuffle algebra [KS; Gr] is equivalent to a monoidal functor $K_{\text{GL}} \xrightarrow{A} \text{Vect}$.*

There are KM analogues of *vertex algebra* and *localised bialgebra*, *quiver representation* with moduli stack $\mathcal{M}_Q = \sqcup \mathfrak{u}_e/G_i$, *quiver varieties*, and (factorisation) *braided monoidal categories* (generalising the G_2 -**reflection equation** [Ku]). We generalise¹⁰ **Chen's Theorem** [Ch] on the shuffle structure on cohomology $H^*(LX)$ of loop spaces; leading to the

Question. [SA] *Can we recover Dynkin/ q -analogues of **multiple zeta values** [KMT; Mi] using this KM-shuffle structure?*

We produce G_2, C_n, \dots structures by **folding** D_4, A_{2n+1}, \dots structures by outer automorphisms $\mathbf{Z}/3, \mathbf{Z}/2, \dots$

Conjecture. [SA] *We may recover $\{\emptyset, \text{boundary}, G_2, \dots\}$ spherical (or trigonometric) **KZ equations** [ES] by constructing a $K_{\text{GL}}, K_{\text{Sp}}, K_{G_2} \dots$ (or $K_{\widehat{\text{GL}}}, \dots$) **affine vertex algebra** via BD Grassmannians, and taking conformal blocks.*

Question. [SA] *Can we recover Dynkin/ q -analogues of **stable envelopes** (see [MO], Stab)?*

A twisted AGT correspondence. A foundational result in geometric representation theory is the **AGT correspondence** linking \mathcal{W} -vertex algebras and surfaces, first by Grojnowski and Nakajima [Gr; Na] then in [AGT; BFN] and [RSYZ].

Conjecture. [AGT] *The equivariant intersection homology of the invariant locus $\mathcal{U}_{\mathbf{p}^2, \text{GL}_n}^{\mathbf{Z}/2}$ in the Uhlenbeck space is a Verma module for an orthosymplectic analogue $\mathcal{W}^k(\mathfrak{gl}_n)^{\text{OSp}}$ of $\mathcal{W}^k(\mathfrak{gl}_n)$ (proof sketch: use SA-techniques on [BFN]).*

Conjecture. [AGT] *The dimension zero CoHA of \mathbf{A}^3 (which is $U_h(\mathcal{D}(\mathbf{C}/\pm))$ and admits $\mathcal{W}^{k_n}(\mathfrak{gl}_n)^{\text{OSp}}$ as quotients) acts on $\text{IH}_T^*(M^{\mathbf{Z}/2})$, the equivariant intersection cohomology of \mathbf{A}^3 -instantons. Likewise for any quiver with potential.*

1.4. q -vertex algebras (qVA, KL)

q -vertex algebras. There have been many attempts [FJW; FR; EK] to define q -affine vertex algebras; none are conceptual enough to apply factorisation techniques [CF]/physics heuristics [Co; GR; Wi] to relate to **q KZ connections** or the (conjecturally, affine) **Kazhdan-Lusztig equivalence**, or new KL equivalences [BCDN].

Conjecture. [qVA] *There is a factorisation category $\mathcal{D}\text{-Mod}$ over the noncommutative plane¹¹ \mathbf{A}_q^2 , a factorisation algebra in which induces a **q -vertex algebra** (i.e. ordinary vertex algebra with poles on the q -diagonals, e.g. [FR], [EK] for $\mathfrak{g} = \mathfrak{sl}_n$).*

Conjecture. [qVA] *There is an analogue of the BD Grassmannian $\text{Gr}_{G,q} \rightarrow \text{Ran} \mathbf{A}_q^2$ (this induces a q -affine VOA $V_q^k(\mathfrak{g})$).*

⁸i.e. the image of the CoHA in $\text{End}(H_{\bullet}^{\text{BM}}(\mathcal{M}^{\text{OSp}}))$.

⁹As opposed to the ordinary configuration space $\text{Conf} \mathbf{A}^1 = \sqcup \mathbf{A}^n // \mathfrak{S}_n = \text{Spec} H^*(\text{BGL})$.

¹⁰Replacing the n -simplex Δ^n with $\Delta_G = T_G/W_G$.

¹¹Its with ring of functions $\mathbf{C}\langle x, y, q \rangle / (yx - xyq)$ with q central.

The physics heuristic for this is: $V^k(\mathfrak{g})$ comes from 3d Chern-Simons with boundary \mathbf{C} [Wi]; $V_q^k(\mathfrak{g})$ comes from 5d **Chern-Simons** with noncommutative boundary \mathbf{C}_{nc}^2 [Co; GR; GRZ]. The category $\mathcal{D}\text{-Mod}$ will be related to q -**difference modules**¹² on \mathbf{A}^1 , so the above directly generalises the usual factorisation definition of vertex algebras.

Conjecture. [KL] *The restriction of **conformal blocks** of $V_q^k(\mathfrak{g})$ -modules to $(\mathbf{A}^1)_{\circ}^n \subseteq (\mathbf{A}_q^2)_{\circ}^n$ equal to the qKZ connection.*

Conjecture. [KL] *There is a **Zhu algebra** functor, $\text{Zhu} : V_q^k(\mathfrak{g}) \mapsto U_q(\mathfrak{g})$, and the functor $\text{RH} : V^k(\mathfrak{g})\text{-Mod} \mapsto U_q(\mathfrak{g})\text{-Mod}$ in the proof [CF] of KL is “parallel transport” along a factorisation category $V_q^k(\mathfrak{g})\text{-Mod}$ on $\mathbf{C} \times \mathbf{R}_{\geq 0}$.*

The above would help understand **affine KL** and the **new KL equivalences** [BCDN] from 3d mirror symmetry.

1.5. Sheaf methods (Con¹³, Loc⁴, Eu)

Localisation methods. We generalise *torus localisation* and *Graber-Pandharipande formulas* [GP] (one of the main technique in enumerative geometry) to the Artin moduli stacks appearing in modern enumerative/algebraic geometry.

Theorem. [Conc; Loc] *We give conditions for the cohomology of an Artin stack \mathcal{X} to be **concentrated** on a closed substack \mathcal{Z} ; when $\mathcal{Z} = \mathcal{X}^T$ is fixed points of a quasismooth dg scheme we give **Atiyah-Bott** $\text{id} = i_* (i^!(-)/e(N_{\text{vir}}))$ and **Graber-Pandharipande localisation** formulas $[\mathcal{X}]^{\text{vir}} = i_* ([\mathcal{X}^T]^{\text{vir}}/e(N_{\text{vir}}))$.*

Theorem. [Eu] *We strengthen both parts of the theorem to **critical cohomology** of **arbitrary** closed embeddings $\mathcal{Z} \hookrightarrow \mathcal{X}$ quasismooth over a common base. As a result, for \mathcal{M} as in CY3, we have the following Theorem:*

Theorem. [Eu] *For any “split locus” map $\pi : \mathcal{M}^s \rightarrow \mathcal{M}$, we have $\text{CoHA} = \text{CoHA}^s/e(N_{i,\text{vir}})$.*

Taking \mathcal{M}^s a *shuffle space*¹⁴, this gives a **universal, geometric** way of producing/interpreting shuffle products for CoHAs [Da; SV; YZ]. Taking $\mathcal{M}^s = \mathcal{M} \times \mathcal{M} \xrightarrow{\oplus} \mathcal{M}$ proves **compatibility** [CV,CY3,Li] between **coproducts** and the CoHA (c.f. CY3). This easily **generalises** to the **OSp**/Dynkin **SA** cases.

1.6. Liouville quantum gravity to vertex algebras (LQG¹⁵)

Probabilists are beginning construct Feynman measures (GFF [BPR], LQG [KRV]) **rigorously** to understand 2d CFTs [CRV; DS; Sh]. There is almost no intereaction with the algebraic geometry/functorial school to QFT, and they understand/have access to techniques (e.g. SLE curves [MS; SS], combinatorial approximations) that the latter school does not. We want to build a bridge:

Conjecture. [LQG] *There is a **chiral part** functor $F^{\text{ch}} : \text{CFT} \rightarrow \text{VertexAlg}$, from Segal-style full 2d CFTs to vertex algebras. It sends [GKRV]’s LQG (resp. GFF) to the Virasoro (resp. Heisenberg) VOA, and [KRV]’s DOZZ formula to the Virasoro OPE.*

We will then use **free field embeddings** to produce a probabilistic CFT analogue of the **g-affine vertex algebra**.

¹²e.g. a q -difference operator ∂_x on \mathbf{A}^1 induces a derivation $y\partial_x$ on \mathbf{A}_q^2 .

¹³Joint with A. Khan, D. Aranha, H. Park, and C. Ravi.

¹⁴i.e. shuffle algebra in the category of spaces, see SA.

¹⁵Joint with V. Giri.

References

- [AGT] Luis F Alday, Davide Gaiotto, and Yuji Tachikawa. “Liouville correlation functions from four-dimensional gauge theories”. In: *Letters in Mathematical Physics* 91.2 (2010), pp. 167–197.
- [AN] Raschid Abedin and Wenjun Niu. “Yangian for cotangent Lie algebras and spectral R -matrices”. In: *arXiv preprint arXiv:2405.19906* (2024).
- [BCDN] Andrew Ballin, Thomas Creutzig, Tudor Dimofte, and Wenjun Niu. “3d mirror symmetry of braided tensor categories”. In: *arXiv preprint arXiv:2304.11001* (2023).
- [BD] Tommaso Maria Botta and Ben Davison. “Okounkov’s conjecture via BPS Lie algebras”. In: *arXiv preprint arXiv:2312.14008* (2023).
- [BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima. “Instanton moduli spaces and \mathcal{W} -algebras”. In: *arXiv preprint arXiv:1406.2381* (2014).
- [Bo] R. E. Borcherds. “Quantum Vertex Algebras”. In: *Advanced Studies in Pure Mathematics*. Vol. 31. Mathematical Society of Japan, 1999, pp. 51–74.
- [BPR] Roland Bauerschmidt, Jiwoon Park, and Pierre-François Rodriguez. “The Discrete Gaussian model, II. Infinite-volume scaling limit at high temperature”. In: *The Annals of Probability* 52.4 (2024), pp. 1360–1398.
- [BR] Samuel Belliard and Vidas Regelskis. “Drinfeld J presentation of twisted Yangians”. In: *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications* 13 (2017), p. 011.
- [Br] Tom Bridgeland. “Geometry from donaldson-thomas invariants”. In: *arXiv preprint arXiv:1912.06504* (2019).
- [BS] Roland Bittleston and David Skinner. “Gauge theory and boundary integrability”. In: *Journal of High Energy Physics* 2019.5 (2019), pp. 1–53.
- [CF] Li Chen and Yuchen Fu. “An Extension of the Kazhdan-Lusztig Equivalence”. PhD thesis. Harvard University, 2022.
- [Ch] Kuo-tsai Chen. “Iterated integrals of differential forms and loop space homology”. In: *Annals of Mathematics* 97.2 (1973), pp. 217–246.
- [Co] Kevin Costello. “M-theory in the Omega-background and 5-dimensional non-commutative gauge theory”. In: *arXiv preprint arXiv:1610.04144* (2016).
- [CRV] Baptiste Cerclé, Rémi Rhodes, and Vincent Vargas. “Probabilistic construction of Toda conformal field theories”. In: *arXiv preprint arXiv:2102.11219* (2021).
- [CWY] Kevin Costello, Edward Witten, and Masahito Yamazaki. “Gauge theory and integrability, I”. In: *arXiv preprint arXiv:1709.09993* (2017).
- [Da] Ben Davison. “The critical CoHA of a quiver with potential”. In: *Quarterly Journal of Mathematics* 68.2 (2017), pp. 635–703. arXiv: arXiv:1311.7172 [math . AG].
- [Dr] Vladimir Gershonovich Drinfeld. “Hopf algebras and the quantum Yang–Baxter equation”. In: *Doklady Akademii Nauk*. Vol. 283. 5. Russian Academy of Sciences. 1985, pp. 1060–1064.
- [DS] Bertrand Duplantier and Scott Sheffield. “Liouville quantum gravity and KPZ”. In: *Inventiones mathematicae* 185.2 (2011), pp. 333–393.
- [EK] P. Etingof and D. Kazhdan. “Sel. Math., New Ser. 6, No. 1, 105–130”. In: *Selecta Mathematica, New Series* 6.1 (2000), pp. 105–130.
- [ES] Pavel I Etingof and Olivier Schiffmann. “Lectures on quantum groups”. In: *(No Title)* (1998).
- [FJW] Igor B Frenkel, Naihuan Jing, and Weiqiang Wang. “Quantum Vertex Representations via Finite Groups and the McKay Correspondence”. In: *Communications in Mathematical Physics* 211 (2000), pp. 365–393.
- [FR] Edward Frenkel and Nicolai Reshetikhin. “Towards deformed chiral algebras”. In: *arXiv preprint* (1997). arXiv: q-alg/9706023 [q-a.lg].
- [Ga] Dennis Gaiatsgory. “On factorization algebras arising in the quantum geometric Langlands theory”. In: *Advances in Mathematics* 391 (2021), p. 107962.

- [GKRV] Colin Guillarmou, Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. “Segal’s axioms and bootstrap for Liouville Theory”. In: *arXiv preprint arXiv:2112.14859* (2021).
- [GLW] Sachin Gautam, Valerio Toledano Laredo, and Curtis Wendlandt. “The meromorphic R-matrix of the Yangian”. In: *Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin*. Springer, 2021, pp. 201–269.
- [GP] Tom Graber and Rahul Pandharipande. “Localization of virtual classes”. In: *arXiv preprint alg-geom/9708001* (1997).
- [GR] Davide Gaiotto and Miroslav Rapčák. “Miura operators, degenerate fields and the M2-M5 intersection”. In: *Journal of High Energy Physics* 2022.1 (2022), pp. 1–80.
- [Gr] James Alexander Green. *Shuffle algebras, Lie algebras and quantum groups*. Vol. 9. Departamento de Matemática da Universidade de Coimbra, 1995.
- [GRZ] Davide Gaiotto, Miroslav Rapčák, and Yehao Zhou. “Deformed Double Current Algebras, Matrix Extended $\mathcal{W}_{1+\infty}$ Algebras, Coproducts, and Intertwiners from the M2-M5 Intersection”. In: *arXiv preprint arXiv:2309.16929* (2023).
- [Joa] Dominic Joyce. “Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces”. In: *Incomplete work* (2018).
- [Job] Dominic Joyce. “Enumerative invariants and wall-crossing formulae in abelian categories”. In: *arXiv preprint arXiv:2111.04694* (2021).
- [KK] Tasuki Kinjo and Naoki Koseki. *Cohomological χ -independence for Higgs bundles and Gopakumar-Vafa invariants*. 2023. arXiv: 2112.10053 [math . AG] .
- [KMT] Yasushi Komori, Kohji Matsumoto, and Hirofumi Tsumura. “A study on multiple zeta values from the viewpoint of zeta-functions of root systems”. In: *Functiones et Approximatio Commentarii Mathematici* 51.1 (2014), pp. 43–46.
- [KPS] Tasuki Kinjo, Hyeonjun Park, and Pavel Safronov. “Cohomological Hall algebras for 3-Calabi-Yau categories”. In: *arXiv preprint arXiv:2406.12838* (2024).
- [KRV] Antti Kupiainen, Rémi Rhodes, and Vincent Vargas. “Integrability of Liouville theory: proof of the DOZZ formula”. In: *Annals of Mathematics* 191.1 (2020), pp. 81–166.
- [KS] Maxim Kontsevich and Yan Soibelman. “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants”. In: *arXiv preprint arXiv:1006.2706* (2010).
- [Ku] Atsuo Kuniba. “Matrix product solutions to the G 2 reflection equation”. In: *Journal of Integrable Systems* 3.1 (2018), xyy008.
- [Li] Henry Liu. “Multiplicative vertex algebras and quantum loop algebras”. In: *arXiv preprint arXiv:2210.04773* (2022).
- [Lu] Jacob Lurie. *Higher Algebra*. Preprint, available at <http://www.math.harvard.edu/~lurie>. 2016.
- [LW] Ming Lu and Weiqiang Wang. “ ι Hall algebras and ι quantum groups”. In: *arXiv preprint arXiv:2209.12416* (2022).
- [Maa] Shahn Majid. “Transmutation theory and rank for quantum braided groups”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 113. 1. Cambridge University Press. 1993, pp. 45–70.
- [Mab] Shahn Majid. “Double-bosonization of braided groups and the construction of $U_q(\mathfrak{g})$ ”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 125. 1. Cambridge University Press. 1999, pp. 151–192.
- [Mi] Antun Milas. “Generalized multiple q-zeta values and characters of vertex algebras”. In: *arXiv preprint arXiv:2203.156* (2022).
- [MMSV] Anton Mellit, Alexandre Minets, Olivier Schiffmann, and Eric Vasserot. “Coherent sheaves on surfaces, COHAs and deformed $\mathcal{W}_{1+\infty}$ -algebras”. In: *arXiv preprint arXiv:2311.13415* (2023).

- [MO] Daveshe Maulik and Andrei Okounkov. “Quantum groups and quantum cohomology”. In: *arXiv preprint arXiv:1211.1287* (2012).
- [MS] Shahn Majid and Francisco Simão. “Quantum jet bundles”. In: *Letters in Mathematical Physics* 113.6 (2023), p. 120.
- [Na] Hiraku Nakajima. “Instantons and affine Lie algebras”. In: *arXiv preprint alg-geom/9510003* (1995).
- [RSYZ] Miroslav Rapčák, Yan Soibelman, Yaping Yang, and Gufang Zhao. “Cohomological Hall algebras, vertex algebras and instantons”. In: *Communications in Mathematical Physics* 376.3 (2020), pp. 1803–1873.
- [Sh] S Sheffield. “Gaussian free fields for mathematicians. preprint”. In: *arXiv preprint math.PR/0312099* (2003).
- [SS] O. Schramm and S. Sheffield. “Contour lines of the two-dimensional discrete Gaussian free field. preprint”. In: *Annals of Probability* 33.6 (2005), pp. 2127–2148.
- [SV] Olivier Schiffmann and Eric Vasserot. “On cohomological Hall algebras of quivers: generators”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2020.760 (2020), pp. 59–132.
- [Sz] Balázs Szendroi. “Cohomological Donaldson–Thomas theory”. In: *Proceedings of String-Math 2015* (2014).
- [Wi] Edward Witten. “Quantum field theory and the Jones polynomial”. In: *Communications in Mathematical Physics* 121.3 (1989), pp. 351–399.
- [YZ] Yaping Yang and Gufang Zhao. “Cohomological Hall algebras and affine quantum groups”. In: *Selecta Mathematica* 24.2 (2018), pp. 1093–1119.