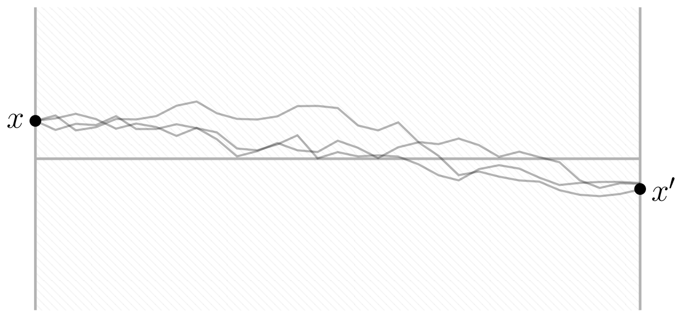
NOTES ON QFT

ALEXEI LATYNTSEV

2. Brownian motion as summing over paths



One can think of Brownian motion on X as a one-dimension quantum field theory. Given a point $x \in X$ (think of it as being a particle at x), the probability of it moving to point x' after time t is

$$\mathbf{P}(x \to x') \;=\; \frac{\#\{\text{Brownian walks } x \to x'\}}{\#\{\text{Brownian walks } x \to ?\}}.$$

In the discrete setting where X is a weighted graph, this formula is literally correct, and if X is a Riemannian manifold we need to replace each count by an integral over paths, and replace x' by an arbitrary measurable subset.

In the above case, the probability is a normal distribution with mean x and variance t.

2.1. The general structure of the above is we have a correspondence

$$\begin{tikzcd} & \{\text{paths } [0,t] \xrightarrow{\gamma} X\} \arrow[dl, "\mathrm{Res}_0"'] \arrow[dr, "\mathrm{Res}_t"] & \\ X & & X \end{tikzcd}$$

of measurable spaces, with relative measures on the restriction maps given in this case by the Brownian motion measure. Calling these correspondences C_{t_1,t_2}, we require them to be compatible in the obvious sense. This allows us to push-and-pull functions,

$$\begin{tikzcd} & \mathcal{O}\left(\mathrm{Maps}([0,1], X)\right) \arrow[dr, "\int_{\mathrm{Res}_t}"] & \\ \mathcal{O}(X) \arrow[ur, "\mathrm{Res}_0^*"] \arrow[rr, "\mathcal{Z}_t"] & & \mathcal{O}(X) \end{tikzcd}$$

and the compatibility condition thus gives us an action of $\mathbf{R}_{\geqslant 0}$ on $\mathcal{O}(X)$.[1] If we impose enough smoothness requirements, it must take the form $\mathcal{Z}_t = e^{Ht}$ for an endomorphism H of $\mathcal{O}(X)$.

[1] This includes the condition that $\mathcal{Z}_0 = \mathrm{id}$ and that $\mathcal{Z}_{t+t'} = \mathcal{Z}_t \cdot \mathcal{Z}_{t'}$.

2.1.1. In the above context, it might be useful to think of functions $f \in \mathcal{O}(X)$ also as "random points on X" if they have norm one, or as "the wavefunction of a particle on X". If we work with a function space that contains the Dirac delta δ_x, this should be thought of as a non-random point, although typically applying $\mathcal{Z}_t$ will not give back a delta function.

2.1.2. We can write the above definition in symbols as

$$\mathcal{Z}_t \ : \ f \ \mapsto \ \int_{\mathrm{Res}_0^{-1}(f)} \Psi \, d\mathcal{B}$$

where $\mathcal{B}$ is the Brownian measure on the set of paths and we have integrated along the fibres of Res_t. In other words, the coefficient of an element $\alpha \in \mathcal{O}(X)^*$ in the above is

$$\langle \alpha, \mathcal{Z}_t f \rangle \ = \ \int_{\Psi \,:\, \Psi|_0 = f} \langle \alpha, \Psi|_t \rangle \, d\mathcal{B}.$$

2.2. The compatibility condition between the correspondences is that we have a pullback

$$\begin{tikzcd}[column sep=small]
 & & (C_{t_1,t_3}, d\mathcal{B}_{t_1,t_3}) \arrow[dl] \arrow[dr] & & \\
 & (C_{t_1,t_2}, d\mathcal{B}_{t_1,t_3}) \arrow[dl] \arrow[dr] & & (C_{t_2,t_3}, d\mathcal{B}_{t_2,t_3}) \arrow[dl] \arrow[dr] & \\
X & & X & & X
\end{tikzcd}$$

of measure spaces. In other words, the Brownian measure is compatible under cutting up of the time interval; this is also called the *Markov* property of the measure.

Thus if we modify the measure to $e^S d\mathcal{B}$, it is consistent in the above sense if and only if the function

$$S_{0,t} \ \in \ \mathcal{O}(\mathrm{Maps}([0,t], X))$$

is memoryless, i.e. satisfies the cocycle condition

$$S_{t_1,t_3} \ = \ S_{t_1,t_2} \ + \ S_{t_2,t_3}$$

where all three are viewed as functions on $\mathrm{Maps}([t_1,t_3], X)$ by restriction. The set of such functions (modulo functions supported on the measure zero set measure zero $\{t_2\} \times X$) is closed under addition and multiplication.

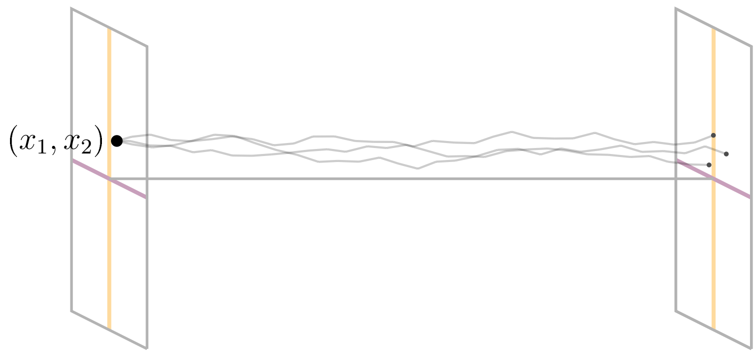
For instance, we may take a function $s \in \mathcal{O}(X)$ and integrate it over $[0,t]$ to get a function $S_{0,t}(\gamma) = \int_{[0,t]} \gamma^* s$ on the path space. Taking the constant function gives for instance $S_{0,t}(\gamma) = kt$. We could also take a covector field $\xi \in \mathcal{O}(T^*X)$ and evaluate it on the derivative of the path to get $S_{0,t}(\gamma) = \int_{[0,1]} \langle \xi, d\gamma \rangle$. Taking higher order differential forms gives more examples. A non-example is evaluating the path at a particular point.

A popular choice is

$$S_{0,t}(\gamma) \ = \ \int_{[0,t]} (\gamma', \gamma') dt$$

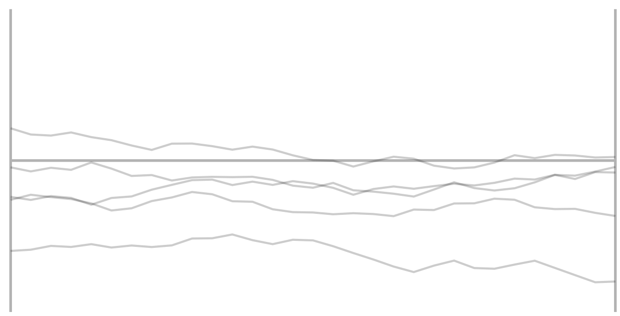
where we have used the Riemannian metric on X.

2.3. There are variants one could consider. For instance, one could consider *coloured* points on X, which is then equivalent to Brownian motion on $X^{\#\text{colours}}$,

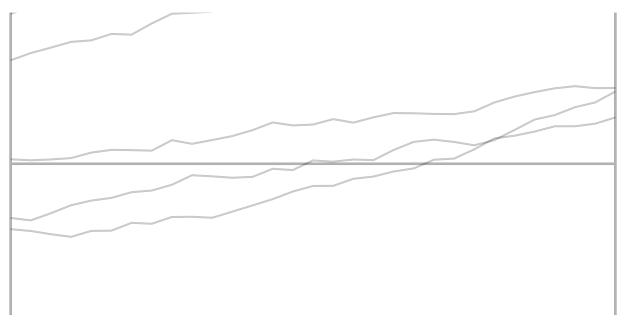


If we want to allow the colours to interact, then we need to change the metric on $X^{\#\text{colours}}$, adding off-diagonal terms. This will mean the Laplacian and hence Brownian motion will have off-diagonal terms.

2.3.1. Another variant is Brownian motion with drift. If a random sample of paths with respect to the Brownian motion measure looks like



then Brownian motion with drift will look like



The Brownian motion with drift $\mathcal{D}$ satisfies the following stochastic differential equation:

$$d\mathcal{D} \;=\; d\mathcal{B} \;+\; k\,dt$$

where real number k is the drift term. Or, viewing $\mathcal{D}$ and $\mathcal{B}$ as random paths $[0,t] \to X$, we have

$$\mathcal{D} \;=\; \mathcal{B} \;+\; k\,t.$$

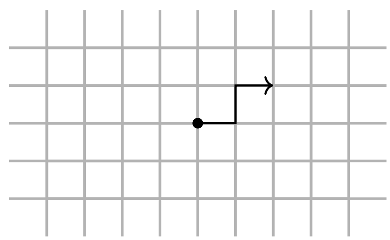
On a general space X with one parameter family of automorphisms $\varphi_t = e^{tv}$ where here v is a vector field on X, we can likewise define Brownian motion with drift as

$$\mathcal{D}_t = \varphi_t^* \mathcal{B}_t.$$

Taking the translation vector field on the real line gives back ordinary Brownian motion with drift. We can view the above as changing the projection map:

$$\begin{tikzcd} & \{\text{paths } [0,t] \xrightarrow{\gamma} X\} \arrow[dl, "\mathrm{Res}_0"'] \arrow[dr, "\varphi_t \cdot \mathrm{Res}_t"] & \\ X & & X \end{tikzcd} \tag{1}$$

2.3.2. Brownian motion on $\mathbf{R}^n$ are a limit of random walks on $r\mathbf{Z}^n$, taking the limit $r \to 0$.

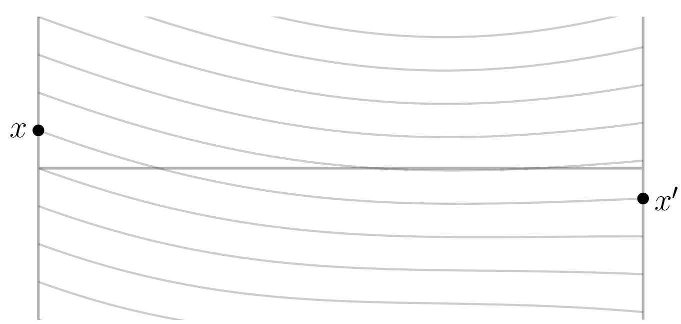


If we use discrete time, we can replace $X = \mathbf{Z}^n$ with any Markov chain, and define $\mathcal{Z}_t = P^t$ in terms of the Markov transition matrix P. We do not know how to pass to continuous time by taking limits in this general case.

2.3.3. In dynamical systems or ergodic theory, one often considers one-parameter families of automorphisms $\Phi_t : X \xrightarrow{\sim} X$. This gives a map

$$X \to \mathrm{Maps}([0,t], X)$$

and we can take the pushforward of the usual measure. In other words, given an initial starting point x the only point with nonzero probability it goes to is $\Phi_t(x)$.



Or in the previous notation, the only path γ restricting at 0 to x is the path $\gamma(-) = \Phi_{-}(x)$,

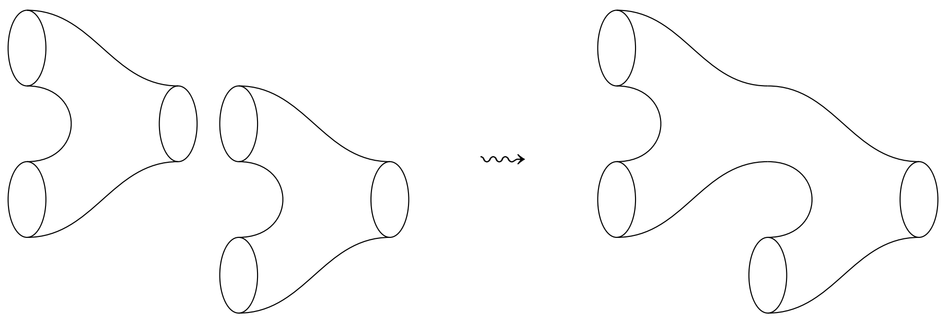
$$\begin{tikzcd} & \gamma \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & \Phi_t(x) \end{tikzcd}$$

Thus this is a “classical” example. To get non-classical examples, one needs to consider *random* dynamical systems, see for instance [Ar].

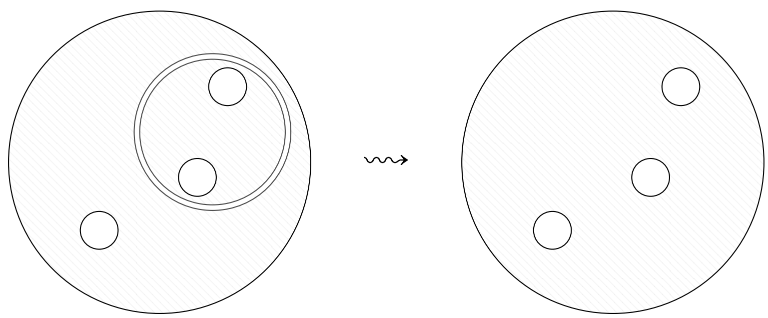
3. Two dimensions

We now replace $[0, t]$ with a two-dimensional surface Σ, i.e. consider two-dimensional quantum field theories. Thus we consider “particles on two-dimensional spacetime”. The Markov compatibility condition, which previously had to do with gluing intervals:

will now be replaced with the Markov *domain* property, which has to do with gluing surfaces:



or in other words, it is a Markov property for splitting up a region using codimension one walls:



The basic example is the Gaussian free field. (explain)

3.1. The general structure is (cobordism stuff)

3.2. We now consider variants.

3.2.1. What is the two-dimensional analogue of Brownian motion with drift? To begin with we need to understand the role of $\mathbf{R}_{\geqslant 0}$ in the one-dimensional case: we identify it as

$$\mathbf{R}_{\geqslant 0} \simeq \mathrm{Hom}_{\mathrm{Cob}_1}(\mathrm{pt}, \mathrm{pt}) = \mathrm{Cob}_1(\mathrm{pt}, \mathrm{pt}).$$

Thus, in line with (1) given a functor $F : \mathrm{Cob}_1 \to \mathrm{Sp}^\mu$ out from Cob_1 to the category of spaces equipped with a measure, for each correspondence

$$\begin{tikzcd} & C \simeq [t_1, t_2] & \\ \{t_1\} \arrow[ur] & & \{t_2\} \arrow[ul] \end{tikzcd}$$

we can ask for an action $\varphi_C : F(\{t_2\}) \to F(\{t_2\})$. In the case of Brownian motion with drift this will be just be $\varphi_{t_2-t_1}$. We want this to be compatible in the the sense that the composition (pullback) of

$$\begin{tikzcd} & F(C) \arrow[dl, "\mathrm{Res}_{t_1}"'] \arrow[dr, "\varphi_C \cdot \mathrm{Res}_{t_2}"] & & F(C') \arrow[dl, "\mathrm{Res}_{t_2}"'] \arrow[dr, "\varphi_{C'} \cdot \mathrm{Res}_{t_3}"] & \\ \{t_1\} & & \{t_2\} & & \{t_3\} \end{tikzcd}$$

is equal to

$$\begin{tikzcd} & F(C \sqcup_{\{t_2\}} C') \arrow[dl, "\mathrm{Res}_{t_1}"'] \arrow[dr, "\varphi_{C \sqcup_{\{t_2\}} C'} \cdot \mathrm{Res}_{t_3}"] & \\ \{t_1\} & & \{t_3\} \end{tikzcd}$$

In other words, whenever C, C' are composable cobordisms (this is always true in the one dimensional case), we have the cocycle condition

$$\varphi_{C \cup C'} \;=\; \varphi_C \cdot \varphi_{C'}.$$

Equivalently, note that $\mathrm{Mor}\,\mathrm{Cob}_1$ is a groupoid over Cob_1, i.e. we have head and tail maps

$$\begin{tikzcd} & \mathrm{Mor}\,\mathrm{Cob}_1 \arrow[dl] \arrow[dr] & \\ \mathrm{Cob}_1 & & \mathrm{Cob}_1 \end{tikzcd}$$

and φ may be viewed as an action of this groupoid on F. (write details)

3.2.2. In particular, in the two-dimensional case we will need an action of (write explicitly)

As a consequence, we can ask for an action of the semigroup $\mathcal{A}$ of a parametrised annuli (or, monoid of thin annuli) on $F(S^1)$. To be explicit, it is

$$\mathcal{A} \;=\; \{A \subseteq \mathbf{C} \text{ an annulus},\; S^1 \sqcup S^1 \xrightarrow{\sim} \partial A\}/\Delta S^1,$$

see [Se], which as a topological space is homeomorphic to

$$\mathcal{A} \;\simeq\; (0,1) \times (\mathrm{Aut}^+(S^1) \times \mathrm{Aut}^+(S^1))/\Delta S^1,$$

given by the ratio of the two annulus radii, and automorphisms of the parametrisations. For thin annuli, we (presumably) use $\mathbf{R}_{\geqslant 0}$ instead of $(0,1)$. One step up, for each pair of Riemannian pants we have an action on $F(S^1)$, and this action is compatible with the semigroup of annuli action. Likewise we have compatible data for other surfaces.

(maybe we want to act on $F(S^1 \sqcup S^1)$ also?)

3.2.3. What is the two-dimensional analogue of Markov chains?

For ordinary Markov chains, we use that $r\mathbf{N}$ is a discrete analogue of $\mathbf{R}_{\geqslant 0}$, which in some sense converges to $\mathbf{R}_{\geqslant 0}$ as $r \to 0$. Thus, we need to construct a discrete analogue of the category of cobordisms. (or something like that?)

To begin with, we find a discrete analogue of $\mathcal{A}$. We have $r \cdot \mathbf{N}_{\geqslant 0}$ a discrete analogue of $\mathbf{R}_{\geqslant 0}$, and a discrete analogue of $\mathrm{Aut}^+(S^1)$ is (what? use the root lattice of $\mathfrak{aut}^+(S^1)$ and exponentiate it) This is

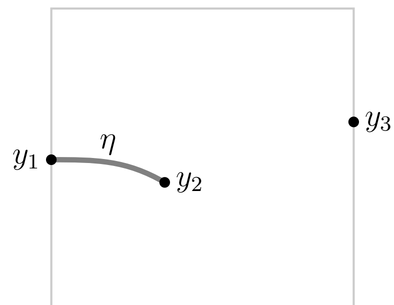
$$\mathbf{N}_{\geqslant 0} \times \Lambda$$

where $\Lambda \subseteq \mathfrak{witt}$ is a lattice inside the Witt Lie algebra of vector fields on S^1, closed under the bracket. For instance, in the holomorphic case we can take

$$\mathbf{N}_{\geqslant 0} \times \mathbf{Z}[z^n \partial_z].$$

3.3. The loop-erased random walk does *not* give an example, however it in some sense lies between the dimension one and two cases.

It satisfies the domain Markov property in the sense that if we have a loop-erased random η walk on Y,



then the loop-erased random walk conditional on starting at η is equivalent to the loop-erased random walk from y_2 to y_3. In other words, it has to do with gluing

$$Y_1 = \eta, \ Y_2 = (Y \setminus \eta) \qquad \rightsquigarrow \qquad Y.$$

We thus define a category whose objects are dimension zero and one manifolds with boundary, and morphisms are *cobordisms*, i.e. manifolds with boundary N with submanifolds

$$Y_1, Y_2 \hookrightarrow N$$

such that the complement has no boundary and (what?)

For instance, the above picture represents two cobordisms

$$\{y_1\} \xrightarrow{\eta} \{y_2\} \xrightarrow{Y \setminus \eta} \{y_3\}.$$

(maybe instead we should consider manfiolds with defect?)

maybe instead we need to consider

$$\eta \xrightarrow{Y} \{y_3\}$$

3.3.1. One can likewise consider loop-erased Markov chains, see [La].

Brownian motion is a certain random real-valued function on the interval $[0, t]$. In particular, it is a measurable map

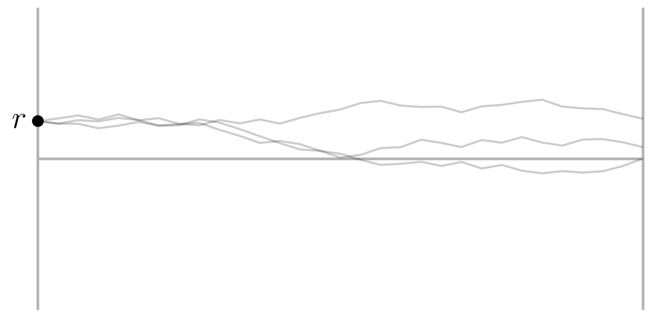
$$B : \Omega \to \mathrm{Fun}([0, t])$$

and so this induces a probability measure on $\mathrm{Fun}([0, t])$. See above for a few functions picked randomly according to this distribution.

For any real number r we can also define a random function on the interval that always begins at r,

$$B_r : \Omega \to \mathrm{Fun}_r([0, t]) \subseteq \mathrm{Fun}([0, t]).$$

Some samples from the induced measure on $\mathrm{Fun}_r([0, t])$:



Taking average endpoint of one of these random functions gives us a linear map:

$$H_t : \mathbf{R} \xrightarrow{\quad r \,\mapsto\, \int_{\mathrm{Fun}_r([0,t])} B_r(t) \quad} \mathbf{R}$$

Thus H_t is defined as "summing over all paths" to get a transformation. Note that

$$H_t \cdot H_{t'} = H_{t+t'}$$

by the Markov property of Brownian motion. In physics terminology, this gives us a $1d$ quantum field theory. In fact in this case $H_t = \mathrm{id}$, but we will now follow these ideas to get more interesting examples.

3.4. **General picture.** We can restrict functions on an interval to either endpoint:

$$\begin{tikzcd} & \mathrm{Fun}([0,t]) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbf{R} \simeq \mathrm{Fun}(\{0\}) & & \mathrm{Fun}(\{t\}) \simeq \mathbf{R} \end{tikzcd}$$

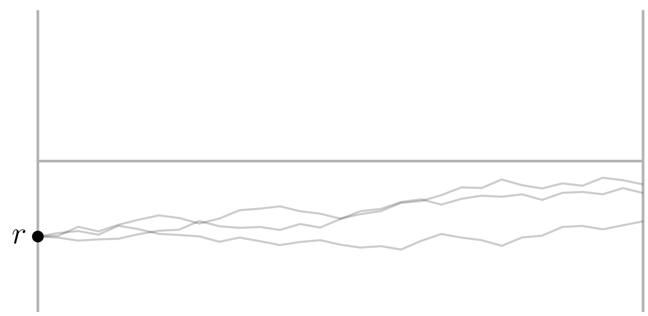
Whenever we have a measure μ_t on $\mathrm{Fun}([0,t])$ plus conditional probability data along p, q, then we get a linear map

$$H_t \,:\, \mathrm{Fun}(\{0\}) \,\to\, \mathrm{Fun}(\{t\}) \qquad\qquad r \,\mapsto\, p_* q^* r \,:=\, \int_{F \in q^{-1}(r)} F(t)$$

as before. We need compatibility data to ensure that $H_t \cdot H_{t'} \;=\; H_{t+t'}$.

3.5. **Examples.**

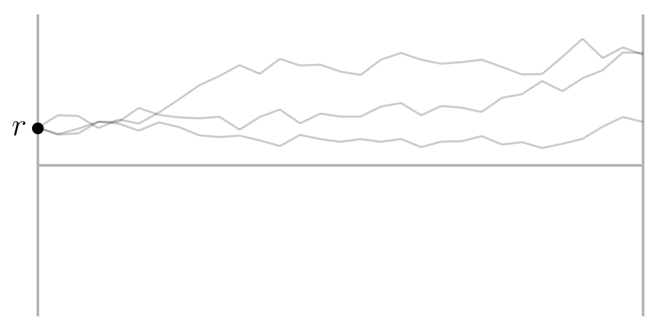
3.5.1. *Brownian motion with drift.* We get that $H_t(r) = r + t$.



3.5.2. *Polynomials.* We can also take polynomials in B, for instance,

$$B^2 + B \,:\, \Omega \,\to\, \mathrm{Fun}([0,t]).$$

All such random functions are bounded below by $-1/2$, i.e. the induced measure on $\mathrm{Fun}([0,t])$ gives measure zero to any measurable set of functions not of this form.



The resulting $H_t \,:\, \mathbf{R} \to \mathbf{R}$ will clearly be non-linear. It is easy to compute as $H_t = t + r^2 + r$ since we know the expectation of $B_0(t)^2$ is t since it is a Gaussian distribution.[2] Thus it does not satisfy the Markov property so cannot come from a quantum field theory.

[2] Indeed, $\mathbf{E}(B_r(t)^2 + B_r(t)) = \mathbf{E}((B_0(t)+r)^2 + (B_0(t)+r)) = t + r^2 + r$.

3.5.3. *Remark.* The Markov or *memoryless* property of a random function is related to the fact that physics theories are *local*.

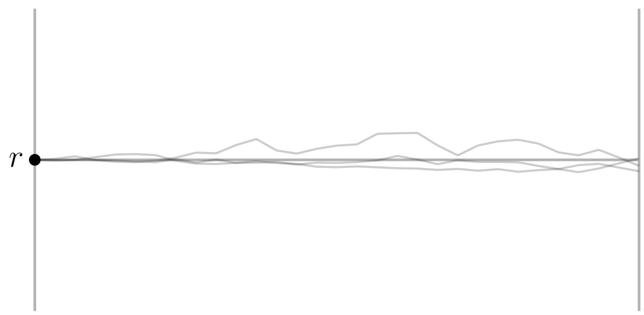
3.5.4. *Ito processes.* To get more examples with the Markov property, note that $B_r(t) = \int_0^r dB_r$, where dB_r is a random one-form. (check) Ito showed that

$$X(t) = \int_0^t f(B)dB$$

is a Markov process for f any L^2 function, and more generally (write). For instance,

$$\int_0^t BdB = \frac{1}{2}(B^2 - t)$$

which still gives $H_t = \mathrm{id}$ since its expectation is zero.



(is this Markov?)

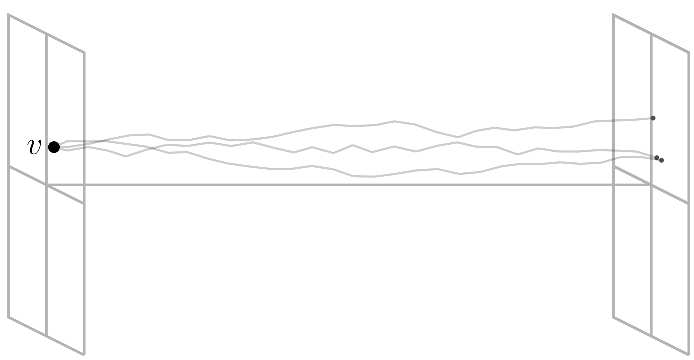
3.5.5. *Brownian motion in* $\mathbf{R}^d$. We can consider Brownian motion valued in a vector field V, which is a random function as before

$$B : \Omega \to \mathrm{Fun}([0,t], V)$$

where V is a vector space.[3] For a vector $v \in V$, we get a random function

$$B_v : \Omega \to \mathrm{Fun}_v([0,t], V) = q^{-1}(v) \subseteq \mathrm{Fun}([0,t], V)$$

as before, some samples of which are:

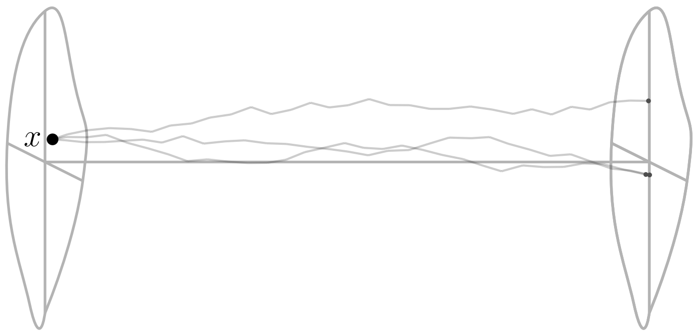


As before, $H_t : V \to V$ is the identity map, but we can take e.g. coordinatewise polynomials in B to get other maps.

[3]To specify B we also need to give a symmetric bilinear form on V giving the covariance of B.

3.5.6. *Brownian motion on general spaces, i.e. sigma models.* For a Riemannian manifold X, we can consider again Brownian motion on X,

$$B \,:\, \Omega \;\to\; \mathrm{Fun}([0,1], X)$$



Because X does not have a group structure, we are not able to take the average value of $B_x(t)$ like before. As before we can restrict

$$\begin{tikzcd} & \mathrm{Fun}([0,t], X) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ X \simeq \mathrm{Fun}(\{0\}, X) & & \mathrm{Fun}(\{t\}, X) \simeq X \end{tikzcd}$$

But even if we have a measure on $\mathrm{Fun}([0,t], X)$ with appropriate conditionals defined push-pull only gives a map on *functions*, which if we normalise to have integral one we can think of as a map on *random points*

$$p_* q^* \,:\, \mathcal{R}X \;\to\; \mathcal{R}X.$$

Here, if Y is a measurable space $\mathcal{R}Y = \mathrm{Maps}(\Omega, Y)$ is the space of measurable maps from a fixed probability space Ω to Y, in other words the random points of Y.

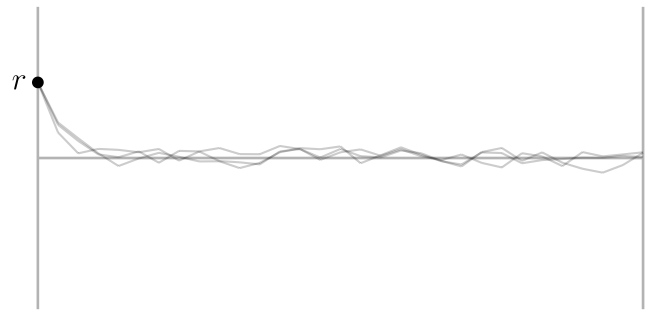
3.5.7. *Remark.* All Markov maps are of the form $\exp(tv) \,:\, X \to X$ where v is a vector field on X whose flow is complete.[4] Pushing forward by this map induces $\exp(tv) : \mathcal{R}X \to \mathcal{R}X$, which inherits the Markov property.

3.5.8. *Ornstein-Uhlenbeck process.* We consider an equation

$$dX(t) \;=\; -2X(t)dt \,+\, dB(t)$$

which is Markov. Some samples of it are

[4]Indeed, if we have a homomorphism $\varphi : \mathbf{G}_a \to \mathrm{Aut}(X)$ then the map on Lie algebras is $\mathbf{C} \to \Gamma(X, \mathcal{T}_X)$, the image of 1 gives a vector field v which exponentiates to φ.



Note that this is Markov in the sense that $H_t \cdot H_{t'} = H_{t+t'}$ as functions $\mathcal{R}X \to \mathcal{R}X$.

3.5.9. *Remark.* If we restrict to random functions $\mathcal{R}X$ which are smooth, this is preserved under $p_* q^*$, and the solution H_t satisfies the *Fokker-Planck* equation.

References

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