Pitzer MATH194 Alyssa Sawyer, HMC'26
Experimental Mathematics
Fall 2025

Homework 1

Due Friday, September 12 on Canvas

Solutions/Proofs.

1. We will exhibit and verify a homeomorphism between $S^1 - \{(-1,0)\}$ and $\mathbb{R}$.

   Our homeomorphism is as follows: $f(x) = \frac{y}{x+1}$, where we are mapping from S^1 to $\mathbb{R}$.

   Now, we will prove that this is a valid homeomorphism by first showing it is continuous and then proving that it has a continuous inverse.

   It is clearly continuous since $x \neq -1$ is in $S^1 - \{(-1,0)\}$.

   Next, we can solve for the inverse by parameterizing our equations in order to write it in terms of one variable, t: $t = \frac{y}{x+1}$. We can solve for y and then substitute that into the unit circle equation, $x^2 + y^2 = 1$, which gives us:

   $$x^2(t(x+1))^2 = 1$$

   We can expand to get the following equation:

   $$x^2(1+t^2) + x(2t^2) + (t^2 - 1) = 0$$

   Now, we can solve using the quadratic equation, which tells us $x = \frac{1-t^2}{1+t^2}$, considering the domain excludes -1. That corresponds to $y = t\frac{2}{1+t^2}$.

   Thus, $f^{-1}(t) = (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$

   This function is continuous as there is no possible point for it to be undefined. Since we were able to find a continuous inverse with the same dimension as our original space, that implies surjectivity and injectivity. Thus, we have found a homeomorphism, as desired.

2. Now, we will prove that $\{(x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}-\{(0, 0, -1)\}$ is homeomorphic to R^2.

   The homeomorphism is as follows: $F : (x, y, z) \mapsto (\frac{x}{z+1}, \frac{y}{z+1})$, where we are going from $S^2 - \{(0, 0, -1)\}$ to $\mathbb{R}^2$.

   To prove it is a homeomorphism, we will show that it is continuous, has an inverse (implies injectivity), and there is a bijection between itself and it's inverse (implies surjectivity).

   The function is clearly continuous as $z \neq 0$ in $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} - \{(0, 0, -1)\}$.

   We can find the inverse of this function as follows. We will use the same parameterizing approach as the previous problem. First, we will write our equation in terms of two new variables, $\bar{x}, \bar{y}$: $\bar{x} = \frac{x}{z+1}$ and $\bar{y} = \frac{y}{z+1}$. We solve for x and y, then substitute that into the equation for the unit sphere, $x^2 + y^2 + z^2 = 1$:

$$((\bar{x})(z + 1))^2 + ((\bar{y})(z + 1))^2 + z^2 = 1$$

   Then, we simplify the algebra in order to find a quadratic equation in terms of z:

$$(\bar{x}^2 + \bar{y}^2 + 1)z^2 + (2\bar{x}^2 + 2\bar{y}^2)z + (\bar{x}^2 + \bar{y}^2 - 1) = 0$$

   Solving yields us

$$z = \frac{-\bar{x}^2 - \bar{y}^2 + 1}{\bar{x}^2 + \bar{y}^2 + 1}$$

   Now, we can plug that back into our initial equations to get the inverse function. We have that $z + 1 = \frac{2}{1+\bar{x}^2+\bar{y}^2}$ and recall we defined $\bar{x}$ and $\bar{y}$ such that $x = \bar{x}z + 1$ and $y = \bar{y}z + 1$. Thus, we can construct the inverse equation as follows:

$$f^{-1}(\bar{x}, \bar{y}) = (\bar{x}(\frac{2}{1 + \bar{x}^2 + \bar{y}^2}), \bar{y}(\frac{2}{1 + \bar{x}^2 + \bar{y}^2}), \frac{-\bar{x}^2 - \bar{y}^2 + 1}{\bar{x}^2 + \bar{y}^2 + 1})$$

   This function is also clearly continuous, since there is no point at which it would become undefined.

   Since we were able to find an inverse, injectivity follows, as desired.

   Finally, because the dimension of S_2 is the same as R_2, we know they are surjective, as we were able to find an inverse that maps between them so surjectivity follows naturally.

3. Next, I will define a topological invariant, provide examples, and justify those examples.

   First, a topological invariant refers to a specific property of any given topological space that does not change under homeomorphism.

   Now, I will go through 3 examples and rigorous justifications for those examples.

   1. First, the genus is a topological invariant. In other words, the genus represents the number of times you can make circular cuts in the space without making the space disconnected.

   From class, we are given that $\chi(s) = 2 - 2g - n$, where g represents the genus. If two objects are homeomorphic, they must have the same χ value, since that is Euler's characteristic, which is a known invariant. n represents the boundary condition, which is also a known invariant. Thus, in order for that equation to hold, if two objects have the same genus they must also have the same χ value, which implies they are homeomorphic, as desired. In converse, if they have different χ values they must have different genus values. Thus, two objects are homeomorphic if and only if they have the same genus, which is the definition of a topological invariant, as desired.

   2. Second, the number of boundary components is a topological component.

   Suppose for the purpose of contradiction that we have two manifolds that are homeomorphic where one manifold has a boundary and one of them doesn't. By definition, they cannot be locally homeomorphic due to the difference in boundary. Supposed a extremal boundary neighborhood in the bounded manifold. There cannot be a corresponding neighborhood in the open manifold, as that would only consists of interior points. That is illogical, which means that we have generated a contradiction, as desired. Thus, the number of boundary components must be preserved under homeomorphism.

   3. Finally, the last topological invarient is Euler's characteristic.

   Suppose for the purposes of contradiction that there are two spaces that do not have the same Euler's characteristic but are homeomorphic. That means one of the spaces has to have a different number of verticies, edges, or faces. From what we know about homeomorphisms, it is not possible to construct a homeomorphism where one of those values is inconsistent, as it would make the relationship not bijective. Thus, Euler's characteristic must be a topological invarient, as desired.

4. Finally, we will show that a 2-dimensional unit sphere is a topological manifold.

   We will do this by showing that it is locally homeomorphic to $\mathbb{R}^2$, or in other words, that $N_{p \in S^2} \cong \mathbb{R}^2$.

   We define our homeomorphism as follows, where we are going from any arbitrary neighborhood in S^2 to $\mathbb{R}^{\nvDash}$: $f : (x, y) \to \frac{(x,y)}{||(x,y)||+1}$.

   Now, we will show that this is a valid homeomorphism by showing that it is one to one, onto, and continuous.

   1. First, we will prove that this function is injective. Begin by assuming two sets of inputs into this function x_1, y_1 and x_2, y_2. We begin by assuming that $f(x_1, y_1) = f(x_2, y_2)$. Plugging this into the equation yields:

   $$\frac{(x_1, y_1)}{||(x_1, y_1)|| + 1} = \frac{(x_2, y_2)}{||(x_1, y_1)|| + 1}$$

   Evaluating the x and y terms separately, it is clear that in order for that to hold, it must be the case that $x_1 = x_2$ and $y_1 = y_2$. Therefore, this function is one to one, as desired. This also makes sense if we think about it topologically speaking – every arbitrary neighborhood of points on the sphere should correspond to an open plane.

   2. This is onto, suppose we have some point in the output, we know it has to map back into the input because of the way that we have defined our homeomorphism. Any point in $\mathbb{R}$ will map back to it's corresponding spot within any arbitrary neighborhood of points.

   3. Finally, this function is continuous. There is no value that could make it non continuous.