Comma category

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1 Diagonal functor

Definition 1.1 (Constant functor)**.** Consider a category $\mathcal{C}$. A constant functor

$$\Delta_c : \mathcal{D} \to \mathcal{C}$$

for a given $c : \mathrm{obj}(\mathcal{C})$ is defined as $\lambda f.\mathrm{id}_c$:

$$\begin{tikzcd} d \arrow[d, "f"'] & \Delta_c(d) := c \arrow[d, "\Delta_c(f) := \mathrm{id}_c"] \\ d' & \Delta_c(d') := c \end{tikzcd} .$$

Remark 1.2 (Natural transformations between constant functors)**.** Natural transformation $\alpha^{(f)} : \Delta_c \to \Delta_{c'}$ between constant functors corresponds to morphisms between their values:

$$\begin{tikzcd} d \arrow[d, "g"'] & \Delta_c(d) = c \arrow[r, "\alpha^{(f)}_c = f"] \arrow[d, "\mathrm{id}_c"'] & \Delta_{c'}(d) = c' \arrow[d, "\mathrm{id}_{c'}"] \\ d' & \Delta_c(d') = c \arrow[r, "\alpha^{(f)}_{c'} = f"'] & \Delta_{c'}(d') = c' \end{tikzcd} .$$

Definition 1.3 (Diagonal functor)**.** Consider categories $\mathcal{C}, \mathcal{D}$. Diagonal functor

$$\Delta : \mathcal{C} \to [\mathcal{D}, \mathcal{C}]$$

is defined as $\lambda f.\alpha^{(f)}$:

$$\begin{tikzcd} c \arrow[d, "f"'] & \Delta(c) := \Delta_c \arrow[d, "\Delta(f) := \alpha^{(f)}"] \\ c' & \Delta(c') := \Delta_{c'} \end{tikzcd} .$$

Remark 1.4 (Special case)**.** Let $\mathcal{D}$ be small and discrete. Then

$$[\mathcal{C}, \mathcal{D}] \simeq \prod_{d \in \mathrm{obj}(\mathcal{D})} \mathcal{C}$$

via $F \mapsto (Fd)_{d \in \mathrm{obj}(\mathcal{D})}$. Thus, diagonal functor takes the following form:

$$\begin{tikzcd} c \arrow[d, "f"'] & \Delta(c) := \prod_{d \in \mathrm{obj}(\mathcal{D})} c \arrow[d, "\Delta(f) := \prod_{d \in \mathrm{obj}(\mathcal{D})} f"] \\ c' & \Delta(c') := \prod_{d \in \mathrm{obj}(\mathcal{D})} c' \end{tikzcd} .$$

Remark 1.5 (Usage)**.** The diagonal functor is often used in precomposition with other functors to obtain constant sequences (e.g. in tensor algebra).

2 Comma category

Definition 2.1 (Comma category)**.** Given functors

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B},$$

the category $(S \downarrow T)$ is defined as follows:

- Objects of are the triples $(a : \mathrm{obj}(\mathcal{A}), h : \mathrm{arr}(\mathcal{C}), b : \mathrm{obj}(\mathcal{B}))$,
- Morphisms are pairs $(f : \mathrm{arr}(\mathcal{A}), g : \mathrm{arr}(\mathcal{B}))$ satisfying the diagram:

$$\begin{tikzcd} (a,h,b) \arrow[d, "{(f,g)}"] & Sa \arrow[r, "h"] \arrow[d, "Sf"'] & Tb \arrow[d, "Tg"] \\ (a',h',b') & Sa' \arrow[r, "h'"'] & Tb' \end{tikzcd},$$

- The composition is defined componentwise:

$$\begin{tikzcd} (a,h,b) \arrow[r, "{(f,g)}"] \arrow[dr, "{(g\circ f, g'\circ f')}"'] & (a',h',b') \arrow[d, "{(f',g')}"] & Sa \arrow[r, "Sf"] \arrow[d, "h"'] & Sa' \arrow[r, "Sf'"] \arrow[d, "h'"'] & Sa'' \arrow[d, "h''"'] \\ & (a'',h'',b'') & Tb \arrow[r, "Tg"'] & Tb' \arrow[r, "Tg'"'] & Tb'' \end{tikzcd}.$$

Remark 2.2.

– $(f,g) = (f',g')$ iff $f = f'$ and $g = g'$ – there's no quotienting by S or T;

– Thus, $(a,h,b) \simeq (a',h',b')$ iff f and g are isos.

Definition 2.3 ((Co)Slice category)**.**

– Slice category. Let $S = \mathrm{id}_{\mathcal{A}}$, $\mathcal{B} = 1$:

$$\mathcal{A} \xrightarrow{\mathrm{id}_{\mathcal{A}}} \mathcal{A} \xleftarrow{b} 1.$$

$$\begin{tikzcd} (a,h) := (a,h,*) \arrow[d, "{f := (f, \mathrm{id}_*)}"] & a \arrow[r, "h"] \arrow[d, "f"'] & b \\ (a',h) := (a',h',*) & a' \arrow[ur, "h'"'] & \end{tikzcd};$$

– Coslice category. Let $T = \mathrm{id}_{\mathcal{B}}$, $\mathcal{A} = 1$:

$$1 \xrightarrow{a} \mathcal{B} \xleftarrow{\mathrm{id}_{\mathcal{B}}} \mathcal{B},$$

$$\begin{tikzcd} (b,h) := (*,h,b) \arrow[d, "{g := (g, \mathrm{id}_*)}"] & a \arrow[r, "h"] \arrow[dr, "h'"'] & b \arrow[d, "g"] \\ (b',h) := (*,h',b') & & b' \end{tikzcd}.$$

Definition 2.4 (Arrow category)**.** Let $S = T = \mathrm{id}_{\mathcal{C}}$:

$$\mathcal{C} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C} \xleftarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C},$$

$$\begin{tikzcd}
(a, h, b) \arrow[d, "{(f,g)}"] \\
(a', h', b')
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "h"] \arrow[d, "f"'] & b \arrow[d, "g"] \\
a' \arrow[r, "h'"'] & b'
\end{tikzcd} .$$