(Co)Limits

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1 (Co)Limits

1.1 Main definitions

Definition 1.1 (Comma category)**.** Given functors

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B},$$

the category $(S \downarrow T)$ is defined as follows:

- Objects of are the triples $(a : \mathrm{obj}(\mathcal{A}), h : \mathrm{arr}(\mathcal{C}), b : \mathrm{obj}(\mathcal{B}))$,
- Morphisms are pairs $(f : \mathrm{arr}(\mathcal{A}), g : \mathrm{arr}(\mathcal{B}))$ satisfying the condition:

$$\begin{tikzcd}
(a,h,b) \arrow[d, "{(f,g)}"] & Sa \arrow[r, "h"] \arrow[d, "Sf"'] & Tb \arrow[d, "Tf"] \\
(a',h',b') & Sa' \arrow[r, "h'"'] & Tb'
\end{tikzcd}.$$

The concept of a comma category allows us to define (co)limits in a uniform way.

Definition 1.2 ((Co)Limit)**.** The limit of a diagram D is defined as a terminal object of the category $(\Delta \downarrow D)$. Dually, the colimit of a diagram E is defined as an initial object of the category $(E \downarrow \Delta)$.

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{D}^{\mathcal{J}} \xleftarrow{D} 1; \qquad 1 \xrightarrow{E} \mathcal{C}^{\mathcal{J}} \xleftarrow{\Delta} \mathcal{C}.$$

Notation:

$$\lim_{\mathcal{J}} D := \mathrm{ter}(\Delta \downarrow D); \qquad \operatorname*{colim}_{\mathcal{J}} E := \mathrm{ini}(E \downarrow \Delta).$$

Visual intuition (limits):

$$\begin{tikzcd}
\Delta d \arrow[rr, dashed] \arrow[drr, "h'"'] & & \Delta \lim D \arrow[d, "h"] \\
& & D
\end{tikzcd}
\qquad
\begin{tikzcd}
d \arrow[rr, dashed] \arrow[d] \arrow[ddr] \arrow[drr, dotted] & & \lim D \arrow[dll, dotted, "h_1"'] \arrow[d, "h_2"] \arrow[ddl, "h_n"] \\
Dj_1 \arrow[rr, dotted, "Dg"] \arrow[dr, "Df \circ Dg"'] & & Dj_2 \arrow[dl, "Df"] \\
& Dj_n &
\end{tikzcd}.$$

Remark 1.3.

- From here on we use (co)lim D also to denote colimit object;
- (Co)Limits are defined up to isomorphism.

Definition 1.4 ((Co)limit funtor)**.** If $\mathcal{C}$ has all (co)limits of shape $\mathcal{J}$, then there exists a (co)limit functor

$$\underset{\mathcal{J}}{\text{(co)lim}}\colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}.$$

The action on morphisms is defined via the universal property:

$$\begin{tikzcd}
\Delta \lim D \arrow[r, "h"] \arrow[d, dashed, "\Delta(\phi(\alpha\circ h))"'] \arrow[dr, "\alpha\circ h"] & D \arrow[d, "\alpha"] & & \lim D \arrow[d, dashed, "\phi(\alpha\circ h)"] \\
\Delta \lim D' \arrow[r, "h'"'] & D' & & \lim D'
\end{tikzcd} .$$

Remark 1.5. Thus, limit functor is a right adjoint to diagonal functor, colimit functor is a left adjoint to diagonal functor.

1.2 Examples

Remark 1.6 (Terminal and initial objects)**.** Terminal and initial objects in a category are special cases of limits and colimits, respectively.

Definition 1.7 ((Co)products)**.** (Co)Product is defined as the (co)limit of a discrete diagram:

$$\prod_{\mathcal{J}} D := \lim_{\mathcal{J}} D; \qquad \coprod_{\mathcal{J}} E := \underset{\mathcal{J}}{\text{colim}}\, E.$$

Setting $J := \text{obj}(\mathcal{J})$, one can write the standard low-level universal property diagrams for (co)product:

$$\begin{tikzcd}
c_k \arrow[r, "i_k"] \arrow[dr, "f_k"'] & \coprod_{j\in J} c_j \arrow[d, dashed, leftarrow, "\coprod_{j\in J} f_k"] & & d \arrow[d, dashed, "\prod_{j\in J} g_k"'] \arrow[dr, "g_k"] & \\
& c & & \prod_{j\in J} d_j \arrow[r, "\pi_j"'] & d_i
\end{tikzcd} .$$

Definition 1.8 ((Co)Equalizers)**.** A (co)equalizer is defined as the (co)limit of a two-arrow diagram:

$$\mathcal{J}: \qquad \begin{tikzcd} j_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & j_2 \end{tikzcd} ,$$

$$\begin{tikzcd}
e \arrow[r, "\text{eq}"] & x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y & & x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & y \arrow[r, "\text{coeq}"] \arrow[dr, "h'"'] & c \arrow[d, dashed] \\
o \arrow[u, dashed] \arrow[ur, "h'"'] & & & & & & p
\end{tikzcd} .$$

Remark 1.9 ((Co)Kernels)**.** A (co)kernel is a special case of a (co)equalizer where one of the morphisms is a zero morphism.

Definition 1.10 (Pullbacks and pushouts)**.** Pullback and pushout are defined as limit and colimit of the following diagrams:

$$
\begin{tikzcd}
 & j_3 \arrow[d, "f_2"] \\
j_2 \arrow[r, "f_1"'] & j_1
\end{tikzcd}
\qquad
\begin{tikzcd}
j_1 \arrow[r, "f_2"] \arrow[d, "f_1"'] & j_3 \\
j_2 &
\end{tikzcd},
$$

$$
\begin{tikzcd}
d \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
 & a \times_c b \arrow[r] \arrow[d] & a \arrow[d] \\
 & b \arrow[r] & c
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[r] \arrow[d] & b \arrow[d] \arrow[ddr] & \\
a \arrow[r] \arrow[drr] & a +_c b \arrow[dr, dashed] & \\
 & & d
\end{tikzcd}.
$$

1.3 (Co)Continuous functors

Definition 1.11 ((Co)Continuous functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called (co)continuous if it preserves (co)limits:

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, "\Delta"] \arrow[d, "G"'] & \mathcal{D}^{\mathcal{J}} \arrow[d, "G^{\mathcal{J}}"'] & 1 \arrow[l, "D"'] \arrow[d, "\mathrm{id}"'] \\
\mathcal{C} \arrow[r, "\Delta"'] & \mathcal{C}^{\mathcal{J}} & 1 \arrow[l, "G \circ D"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "E"] \arrow[d, "\mathrm{id}"'] & \mathcal{C}^{\mathcal{J}} \arrow[d, "F^{\mathcal{J}}"'] & \mathcal{C} \arrow[l, "\Delta"'] \arrow[d, "F"'] \\
1 \arrow[r, "F \circ E"'] & \mathcal{D}^{\mathcal{J}} & \mathcal{D} \arrow[l, "\Delta"]
\end{tikzcd};
$$

$$
G\,\mathrm{ter}(\Delta \downarrow D) \simeq \mathrm{ter}(\Delta \downarrow G \circ D) \Leftrightarrow G \lim D \cong \lim G \circ D,
$$

$$
F\,\mathrm{ini}(E \downarrow \Delta) \simeq \mathrm{ini}(\Delta \downarrow F \circ E) \Leftrightarrow F \operatorname{colim} E \cong \operatorname{colim} F \circ E.
$$

Visual intuition (limits):

$$
\begin{tikzcd}
d \arrow[rr, dashed] \arrow[d] \arrow[dr, bend left=0] \arrow[drr, dotted] \arrow[ddr] & & \lim D \arrow[dll, dotted] \arrow[d] \arrow[ddl] \\
Dj_1 \arrow[rr, dotted] \arrow[dr] & & Dj_2 \arrow[dl] \\
 & Dj_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[rr, dashed] \arrow[d] \arrow[drr, dotted] \arrow[ddr] & & G \lim D \arrow[dll, dotted] \arrow[d] \arrow[ddl] \\
GDj_1 \arrow[rr, dotted] \arrow[dr] & & GDj_2 \arrow[dl] \\
 & GDj_n &
\end{tikzcd}.
$$

Remark 1.12.

– $\simeq$ for iso of objects in category, $\cong$ is to show that the iso is potentially can be extended to some natural isomorphism (depends on the context);

– Limit preserving property implies that there exists a unique iso f and g satisfying the diagrams:

$$
\begin{tikzcd}
\Delta G \lim D \arrow[r, dashed, shift left, "f"] \arrow[dr, "h'"'] & \Delta \lim G \circ D \arrow[l, dashed, shift left, "f^{-1}"] \arrow[d, "h"] \\
& G \circ D
\end{tikzcd}
\qquad
\begin{tikzcd}
F \circ E \arrow[d, "p"'] \arrow[dr, "p'"] & \\
\Delta \operatorname{colim} F \circ E \arrow[r, dashed, shift right, "g^{-1}"'] & \Delta F \operatorname{colim} E \arrow[l, dashed, shift right, "g"']
\end{tikzcd}
\quad ;
$$

Actually, there are always arrows

$$f : G \lim D \to \lim GD \text{ and } g : F \operatorname{colim} E \to \operatorname{colim} FE,$$

and they become isomorphisms if G is continuous (resp. F is cocontinuous).

– Functor names align with RAPL (right adjoint preserve limits) and LAPC (left adjoint preserve colimits).