Hom functor

Alyson Mei

December 24, 2025

1 Intro

Conventions.

- We use λ-notation for functors and assume it implicitly tracks variance.
- All categories with set-valued Hom functors are assumed to be locally small.
- For brevity, when the context is clear, we write

$$A, B := \mathrm{Hom}(A, B) := \mathcal{C}(A, B).$$

TODO.

- Hom functors in monoidal categories.
- Enriched categories.

2 Set-valued Hom functor

Definition 2.1 (Hom functor)**.** Set-valued Hom functor

$$\lambda AB.\mathcal{C}(A,B) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}et,$$

is defined as follows:

- $\mathcal{C}(A,B)$ is the set of morphisms from A to B,

– its action on morphisms is given by

$$\mathcal{C}(f,g) := \lambda\varphi.\, g \circ \varphi \circ f,$$

i.e. by pre- and post-composition:

$$\begin{tikzcd} A & B & A \arrow[r, "\varphi"] & B & \mathcal{C}(A,B) \arrow[d, "\mathcal{C}(f,g)=\lambda\varphi.g\circ\varphi\circ f"] \\ A' \arrow[u, "f"] & B' \arrow[from=u, "g"'] & A' \arrow[u, "f"] \arrow[r, "g\circ\varphi\circ f"'] & B' \arrow[from=u, "g"] & \mathcal{C}(A',B') \end{tikzcd} .$$

We will often write A, B or $\mathrm{Hom}(A,B)$ instead of $\mathcal{C}(A,B)$ when the context is clear.

Definition 2.2 (Covariant and contravariant Hom functors)**.** The functors

$$\lambda A.\, \mathcal{C}(A,-) : \mathcal{C} \to \mathcal{S}et \quad \text{and} \quad \lambda B.\, \mathcal{C}(-,B) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$$

are called the covariant and contravariant Hom functors, respectively. They are obtained from the bifunctor $\mathcal{C}(-,-)$ by fixing one argument. The induced action on morphisms is given by *post-composition* and *pre-composition*, respectively:

$$\begin{tikzcd} B \arrow[d, "g"'] & \mathcal{C}(A,-)(B) = \mathcal{C}(A,B) \arrow[d, "\mathcal{C}(A,-)(g)=g\circ-"] & A & \mathcal{C}(-,B)(A') = \mathcal{C}(A,B) \arrow[d, "\mathcal{C}(-,B)(f)=-\circ f"] \\ B' & \mathcal{C}(A,-)(B') = \mathcal{C}(A,B') & A' \arrow[u, "f"] & \mathcal{C}(-,B)(A) = \mathcal{C}(A',B) \end{tikzcd} .$$

Natural transformations between covariant (resp. contravariant) Hom functors are induced by *pre-composition* (resp. *post-composition*):

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "{A,-}"] \arrow[r, bend right, "{B,-}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{S}et \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, bend left, "{-,A}"] \arrow[r, bend right, "{-,B}"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{S}et \end{tikzcd}$$

$$
\begin{tikzcd}
X \arrow[d, "f"'] & A \\
Y & B \arrow[u, "\alpha_\bullet"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A, X \arrow[r, "\_\circ\alpha_\bullet"] \arrow[d, "f\circ\_"'] & B, X \arrow[d, "f\circ\_"] \\
A, Y \arrow[r, "\_\circ\alpha_\bullet"'] & B, Y
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi \arrow[r, mapsto] \arrow[d, mapsto] & \varphi \circ \alpha_\bullet \arrow[d, mapsto] \\
f \circ \varphi \arrow[r, mapsto] & f \circ \varphi \circ \alpha_\bullet
\end{tikzcd}
.
$$

$$
\begin{tikzcd}
X & A \arrow[d, "\beta_\bullet"'] \\
Y \arrow[u, "g"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X, A \arrow[r, "\beta_\bullet\circ\_"] \arrow[d, "\_\circ g"'] & X, B \arrow[d, "\_\circ g"] \\
Y, A \arrow[r, "\beta_\bullet\circ\_"'] & Y, B
\end{tikzcd}
\qquad
\begin{tikzcd}
\psi \arrow[r, mapsto] \arrow[d, mapsto] & \beta_\bullet \circ \psi \arrow[d, mapsto] \\
\psi \circ g \arrow[r, mapsto] & \beta_\bullet \circ \psi \circ g
\end{tikzcd}
$$