Adjoint functors

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References

- GitHub:
  - Natural transformations;
  - Hom functor;
  - Comma category and Diagonal functor;
- Local: //not implemented yet

1 Introduction

Adjoint functors are a rather subtle topic, involving a specific categorical machinery. Many sources present them with an unclear flow, often leaving proofs to the reader and emphasizing intuition through examples rather than systematically exploring different conceptual perspectives.

Our primary goal is to provide both the necessary technical details and the underlying intuition. In this section, we present the high-level overview and the general flow of ideas without going into full proofs. The following section will be devoted to developing the full machinery and giving the detailed proofs.

Definition via Triangle identities. The most revealing high-level intuition for adjoint functors is probably the idea of *"the best thing after equivalence."* That is, we successively relax the notion of similarity between categories along the sequence:

$$\textit{isomorphism} \to \textit{equivalence} \to \textit{adjunction}.$$

Let's see how this looks in practice. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be:

– *Isomorphic*, if there exist $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $\mathrm{id}_{\mathcal{C}} = GF$, $FG = \mathrm{id}_{\mathcal{D}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\mathrm{id}_{\mathcal{C}} = G \circ F"'] & \mathcal{D} \arrow[d, dashed, "G"] \\ & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[d, "G"'] \arrow[dr, "F \circ G = \mathrm{id}_{\mathcal{D}}"] & \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{D} \end{tikzcd},$$

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "f"'] & GFc \arrow[d, "GFf = f"] \\ c' \arrow[r, equal] & GFc' \end{tikzcd} \quad \begin{tikzcd} Fc \arrow[d, "Ff"] \\ Fc' \end{tikzcd} \qquad \begin{tikzcd} Gd \arrow[d, "Gd"'] \\ Gd' \end{tikzcd} \quad \begin{tikzcd} FGd \arrow[r, equal] \arrow[d, "FGg = g"'] & d \arrow[d, "g"] \\ FGd' \arrow[r, equal] & d' \end{tikzcd};$$

– *Equivalent*, if there exist $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $\mathrm{id}_{\mathcal{C}} \simeq GF$, $FG \simeq \mathrm{id}_{\mathcal{D}}$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "G"] \arrow[dl, phantom, "\Nearrow \simeq"] \\ & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[d, "G"'] \arrow[dr, "\mathrm{id}_{\mathcal{D}}"] & \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow \simeq"] & \mathcal{D} \end{tikzcd},$$

$$\begin{tikzcd} c \arrow[r, "\cong"] \arrow[d, "f"'] & GFc \arrow[d, "GFf"] \\ c' \arrow[r, "\cong"'] & GFc' \end{tikzcd} \quad \begin{tikzcd} Fc \arrow[d, "Ff"] \\ Fc' \end{tikzcd} \qquad \begin{tikzcd} Gd \arrow[d, "Gd"'] \\ Gd' \end{tikzcd} \quad \begin{tikzcd} FGd \arrow[r, "\cong"] \arrow[d, "FGg"'] & d \arrow[d, "g"] \\ FGd' \arrow[r, "\cong"'] & d' \end{tikzcd};$$

– *Related by an adjunction* $F \dashv G$ given by $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, if there exist natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon : FG \to \mathrm{id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "\mathrm{id}_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "G"] \arrow[dl, phantom, "\Nearrow \eta"] \\ & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[d, "G"'] \arrow[dr, "\mathrm{id}_{\mathcal{D}}"] & \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Nearrow \varepsilon"] & \mathcal{D} \end{tikzcd},$$

$$
\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[d, "f"'] & GFc \arrow[d, "GFf"] & Fc \arrow[d, "Ff"] & & Gd \arrow[d, "Gd"'] & FGd \arrow[r, "\varepsilon_d"] \arrow[d, "FGg"'] & d \arrow[d, "g"] \\
c' \arrow[r, "\eta_{c'}"'] & GFc' & Fc' & & Gd' & FGd' \arrow[r, "\varepsilon_{d'}"'] & d'
\end{tikzcd},
$$

called *unit* and *counit* respectively, satisfying the *triangle identities* (coherence relations):

$$
\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[dr, "\mathrm{id}_G"'] & GFG \arrow[d, "G\varepsilon"] & & F \arrow[d, "F\eta"'] \arrow[dr, "\mathrm{id}_F"] & \\
& G & & FGF \arrow[r, "\varepsilon F"'] & F
\end{tikzcd}.
$$

Definitions via Universal property. Expanding the triangle identities, we obtain the following diagrams (underlining is just an annotation to show which elements we will be varying):

$$
\begin{tikzcd}
\underline{Gd} \arrow[r, "\eta_{Gd}"] \arrow[dr, "\mathrm{id}_{Gd}"'] & GF\underline{Gd} \arrow[d, "G\varepsilon_d"] & F\underline{Gd} \arrow[d, "\varepsilon_d"] & & c \arrow[d, "\eta_c"'] & Fc \arrow[d, "F\eta_c"'] \arrow[dr, "\mathrm{id}_{Fc}"] & \\
& Gd & d & & G\underline{Fc} & FG\underline{Fc} \arrow[r, "\varepsilon_{Fc}"'] & \underline{Fc}
\end{tikzcd}.
$$

It turns out that these identities taken together force each η_c for $c : \mathrm{obj}(\mathcal{C})$ to be an initial morphism, and each ε_d for $d : \mathrm{obj}(\mathcal{D})$ to be a terminal morphism. The diagrams above are therefore just special cases of the following ones:

$$
\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[dr, "\psi"'] & GFc \arrow[d, dashed, "G\psi^\flat"] & Fc \arrow[d, dashed, "\psi^\flat"] & & c \arrow[d, dashed, "\varphi^\sharp"'] & Fc \arrow[d, dashed, "F\varphi^\sharp"'] \arrow[dr, "\varphi"] & \\
& Gd & d & & Gd & FGd \arrow[r, "\varepsilon_d"'] & d
\end{tikzcd}.
$$

Formally, this means that for all $c : \mathrm{obj}(\mathcal{C})$ the object (η_c, Fc) is initial in $(c \downarrow G)$, and for all $d : \mathrm{obj}(\mathcal{D})$ the object (Gd, ε_d) is terminal in $(F \downarrow d)$:

$$1 \xrightarrow{c} \mathcal{C} \xleftarrow{G} \mathcal{D}, \qquad \mathcal{C} \xrightarrow{F} \mathcal{D} \xleftarrow{d} 1.$$

Moreover, these initial and terminal properties turn out to be equivalent, so it suffices to use either one of them to define the adjunction $F \dashv G$.

Definition via Hom isomorphism. Notice that the definitions via universal properties allow us, given an arrow $Fc \to d$, to obtain an arrow $c \to Gd$, as well as, given an arrow $c \to Gd$, to obtain an arrow $Fc \to d$, in a universal way. It turns out that this correspondence extends to a natural isomorphism.
Let $\mathcal{C}$ and $\mathcal{D}$ be locally small for simplicity, so the set-valued Hom functors are defined for both categories. Then the correspondence above provides us with the natural isomorphism $\lambda cd.c, Gd \simeq \lambda cd.Fc, d$:

$$\mathrm{Hom}_{\mathcal{C}}(c, Gd) \cong \mathrm{Hom}_{\mathcal{D}}(Fc, d).$$

2 Machinery and proofs

Definition 2.1 (Adjoint functors via set-valued Hom functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ and $\mathcal{D}$ are locally small. F is said to be left adjoint to G (resp. G is said to be right adjoint to F) if

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd).$$

Notation:

$$F \dashv G, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} .$$

In functorial notation this condition takes the following form:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\lambda cd.Fc{,}d"] \arrow[r, bend right, "\lambda cd.c{,}Gd"'] \arrow[r, phantom, "\Downarrow\simeq"] & \mathcal{S}et \end{tikzcd} .$$

Definition 2.2 (Φ; $\sharp$ and $\flat$; η_c and ε_d)**.** We denote the isomorphism between Hom functors as Φ:

$$\begin{tikzcd} c & d \arrow[d, "g"'] \\ c' \arrow[u, "f"] & d' \end{tikzcd} \qquad \begin{tikzcd} Fc, d \arrow[r, "\Phi_{c,d}"] \arrow[d, "g\circ - \circ Ff"'] & c, Gd \arrow[d, "Gg\circ - \circ f"] \\ Fc', d' \arrow[r, "\Phi_{c',d'}"'] & c', Gd' \end{tikzcd} .$$

The standard notation for action of Φ and its inverse on arrows is $\sharp$ and $\flat$ annotations respectively:

$$\begin{tikzcd} \varphi : Fc, d \arrow[r, mapsto] \arrow[d, mapsto] & \Phi_{c,d}(\varphi) := \varphi^\sharp \arrow[d, mapsto] \\ g \circ \varphi \circ Ff \arrow[r, mapsto] & (g \circ \varphi \circ Ff)^\sharp = Gg \circ \varphi^\sharp \circ f \end{tikzcd} \qquad \begin{tikzcd} \Phi^{-1}_{c,d}(\psi) := \psi^\flat \arrow[d, mapsto] & \psi : c, Rd \arrow[l, mapsto] \arrow[d, mapsto] \\ g \circ \psi^\flat \circ Ff = (Gg \circ \psi \circ f)^\flat & Gg \circ \psi \circ f \arrow[l, mapsto] \end{tikzcd} .$$

The action of Φ on id_{Fc} and id_{Gd} is denoted as follows:

$$\eta_c := \mathrm{id}^\sharp_{Fc}, \qquad \varepsilon_d := \mathrm{id}^\flat_{Gd}.$$

Special cases:

- $f = \mathrm{id}_c$:

$$(g \circ \varphi)^\sharp = Gg \circ \varphi^\sharp, \qquad g \circ \psi^\flat = (Gg \circ \psi)^\flat;$$

- $g = \mathrm{id}_d$:

$$(\varphi \circ Ff)^\sharp = \varphi^\sharp \circ f, \qquad \psi^\flat \circ Ff = (\psi \circ f)^\flat;$$

– $d = Fc,\ \varphi = \mathrm{id}_c$:

$$(g \circ Ff)^\sharp = Gg \circ \eta_c \circ f;$$

– $c = Gd,\ \psi = \mathrm{id}_d$:

$$g \circ \varepsilon_d \circ Ff = (Gg \circ f)^\flat.$$

Definition 2.3 (Adjoint functors via Universal property)**.** We can define adjunctions via the universal property using the data from the Hom-functor definition.

– $F \dashv G :\Leftrightarrow$ for all $c : \mathrm{obj}(\mathcal{C})$ there exists an initial morphism $\eta_c : c \to GFc$, meaning that for this fixed morphism η_c and for all $\psi : c \to Gd$ there exists a unique morphism $\psi^\flat : Fc \to d$ satisfying the diagram

$$\begin{tikzcd} c \arrow[r, "\eta_c"] \arrow[dr, "\psi"'] & GFc \arrow[d, dashed, "G\psi^\flat"] \\ & Gd \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[d, dashed, "\psi^\flat"] \\ d \end{tikzcd};$$

– $F \dashv G :\Leftrightarrow$ for all $d : \mathrm{obj}(\mathcal{D})$ there exists a terminal morphism $\varepsilon_d : FGd \to d$, meaning that for this fixed morphism ε_d and for all $\varphi : Fc \to d$ there exists a unique morphism $\varphi^\sharp : c \to Gd$ satisfying the diagram

$$\begin{tikzcd} c \arrow[d, dashed, "\varphi^\sharp"'] \\ Gd \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[d, dashed, "F\varphi^\sharp"'] \arrow[dr, "\varphi"] & \\ FGd \arrow[r, "\varepsilon_d"'] & d \end{tikzcd}.$$

Proposition 2.4 (Equivalence of definitions via initial and via terminal morphisms)**.** The definitions using initial and terminal morphisms are equivalent.

Proof.

$(\Rightarrow)$: We are given an initial morphism η_c for all $c : \mathrm{obj}(\mathcal{C})$, our goal is to provide a terminal morphism ε_d for all $d : \mathrm{obj}(\mathcal{D})$. Consider the special case

$$c := G_d, \qquad \psi := \mathrm{id}_{Gd}.$$

ε_d is then defined using the universality of η_{Gd} :

$$\begin{tikzcd} Gd \arrow[r, "\eta_{Gd}"] \arrow[dr, "\mathrm{id}_{Gd}"'] & GFGd \arrow[d, dashed, "G\varepsilon_d"] \\ & Gd \end{tikzcd} \qquad \begin{tikzcd} FGd \arrow[d, dashed, "\varepsilon_d := \mathrm{id}_{Gd}^\flat"] \\ d \end{tikzcd}.$$

... (unfinished)

Proposition 2.5 (Equivalence of definitions via Hom functors and via Universal property)**.** The Hom-functor and universal property definitions of adjoint functors (Defs 1.1 and 1.3) are equivalent.

Proof.

($\Rightarrow$) :

1. *Existence.* The morphisms $\psi^\flat$ and $\varphi^\sharp$ are obtained directly from the natural transformation Φ in Def 1.2. Commutativity of the diagrams follows from the corresponding special cases in Def 1.2:
    - Set $f := \mathrm{id}_c$, $g := \psi^\flat$ in special case (3) to obtain the first diagram.
    - Set $g := \mathrm{id}_d$, $f := \varphi^\sharp$ in special case (4) to obtain the second diagram.
2. *Uniqueness.* Suppose $\tilde{\varphi}$ and $\tilde{\psi}$ are alternative arrows satisfying the diagrams:

$$\psi = G\tilde{\psi} \circ \eta_c, \qquad \varphi = \varepsilon_d \circ F\tilde{\varphi}.$$

Applying special cases (3) and (4) and using the invertibility of $\sharp$ and $\flat$, we get

$$\psi = \tilde{\psi}^\sharp \Leftrightarrow \psi^\flat = \tilde{\psi}, \qquad \varphi = \tilde{\varphi}^\flat \Leftrightarrow \varphi^\sharp = \tilde{\varphi}.$$

($\Leftarrow$) // not implemented yet

$\square$