

Category theory basics

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1 Category

1.1 Basic definitions

Category theory extensively uses the language of commutative diagrams.

Definition 1.1 (Diagram). A diagram consists of a directed graph whose vertices are objects and whose edges are maps between these objects.

Definition 1.2 (Commutative diagram). A diagram is called commutative if for any two directed paths with the same starting object and the same ending object, the corresponding compositions of maps are equal.

Conventions:

- All diagrams shown are assumed to be commutative by default;
- By “satisfying the diagrams” we mean “such that the diagrams commute”.

Example 1.3. Commutativity of the diagram below means that $g \circ f = h \circ p$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ p \downarrow & & \downarrow g \\ C & \xrightarrow{h} & D \end{array} .$$

Definition 1.4 (Category). A category \mathcal{C} consists of:

- a collection $\text{obj}(\mathcal{C})$ of objects,
- a collection $\text{arr}(\mathcal{C})$ of morphisms (arrows) of the form $f : C \rightarrow D$, where $C, D : \text{obj}(\mathcal{C})$,

satisfying the following diagrams for any morphisms f, g, h of compatible types:

- Composition:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow g \circ f & \downarrow g \\ & C & \end{array} ,$$

- Identity:

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A & \quad & C & \xrightarrow{\text{id}_A \circ g = g} & A \\ & \searrow f \circ \text{id}_A = f & \downarrow f & \quad & \searrow \text{id}_A \circ g = g & & \\ & B & & \quad & A & \xrightarrow{\text{id}_A} & \end{array}$$

- Associativity:

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & & \\
 \searrow g \circ f & & \downarrow g & \nearrow h \circ g & \\
 & & C & \xrightarrow{h} & D
 \end{array} .$$

1.2 Additional definitions

Definition 1.5 (Mono, epi, iso). Consider the category \mathcal{C} .

- A morphism $m : \text{arr}(\mathcal{C})$ is called a monomorphism (mono) if it is left-cancellative:

$$A \xrightarrow[\substack{f_1 \\ f_2}]{} B \xrightarrow{m} C, \quad m \circ f_1 = m \circ f_2 \Rightarrow f_1 = f_2 ;$$

- A morphism $e : \text{arr}(\mathcal{C})$ is called an epimorphism (epi) if it is right-cancellative:

$$A \xrightarrow{e} B \xrightarrow[\substack{g_1 \\ g_2}]{} C, \quad g_1 \circ e = g_2 \circ e \Rightarrow g_1 = g_2 ;$$

- A morphism $i : \text{arr}(\mathcal{C})$ is called an isomorphism (iso) if has an inverse $i^{-1} : \text{arr}(\mathcal{C})$:

$$\begin{array}{ccc}
 A & \xrightarrow{i} & B \\
 \searrow \text{id}_A & & \downarrow i^{-1} \\
 & & A
 \end{array} \quad
 \begin{array}{ccc}
 B & & \\
 \downarrow i^{-1} & \searrow \text{id}_B & \\
 A & \xrightarrow{i} & B
 \end{array} .$$

Remark 1.6.

- Mono, epi and iso are the generalizations of injection, surjection and bijection respectively;
- Every isomorphism is both mono and epi, but the converse is not generally true.

Definition 1.7 (Initial, terminal and zero objects). Consider the category \mathcal{C} .

- $I : \text{obj}(\mathcal{C})$ is called initial object if for every $C : \text{obj}(\mathcal{C})$ there exists a unique morphism $I \rightarrow C$;
- $T : \text{obj}(\mathcal{C})$ is called terminal object if for every $C : \text{obj}(\mathcal{C})$ there exists a unique morphism $C \rightarrow T$;
- $0 : \text{obj}(\mathcal{C})$ is called zero object if it is both initial and terminal.

An object is called universal if it is initial or terminal.

Remark 1.8.

- Every two initial objects / every two terminal objects are isomorphic to each other;

- The concept of universal object combined with comma categories allows to gracefully define the variety of categorical structures.

Definition 1.9 (Small and locally small categories). The category is called

- Small, if both objects and morphisms form a set;
- Locally small, if morphisms between any two objects form a set.

1.3 List of examples

Example 1.10.

- Trivial and finite categories:
 - The empty category 0 ;
 - The terminal category 1 : $\text{obj}(1) = \{\ast\}$, $\text{arr}(1) = \{\text{id}_\ast\}$;
 - A monoid: a category with a single object;
 - The category 2 with two objects and a single non-identity arrow;
 - The discrete category \mathcal{C}_X on a set X : $\text{obj}(\mathcal{C}_X) = X$, $\text{arr}(\mathcal{C}_X) = \{\text{id}_x \mid x \in X\}$;
 - The diagram category \mathcal{C}_G associated to a directed graph G : $\text{obj}(\mathcal{C}_G) = \text{vertices}(G)$, $\text{arr}(\mathcal{C}_G) = \text{edges}(G)$, with all compatible paths commuting;
- Order-theoretic categories:
 - Preorders as categories: objects are elements of preorder P , and there is a unique arrow $x \rightarrow y$ whenever $x \leq y$;
 - Preorders / Lattices / Boolean algebras: same as above, only the arrows satisfy additional axioms;
- Basic concrete categories:
 - $\mathcal{S}et$: objects are sets, morphisms are functions;
 - $\mathcal{F}in\mathcal{S}et$: objects are finite sets, morphisms are functions;
 - $\mathcal{R}el$: objects are sets, morphisms are binary relations;
 - $\mathcal{P}art$: objects are sets, morphisms are partial functions;
 - $\mathcal{P}oint$: objects are pointed sets (sets with a distinguished element), morphisms are functions preserving the base point;
- Logical / type-theoretic categories:
 - Boolean algebras / Heyting algebras: objects are Boolean or Heyting algebras, morphisms are structure-preserving functions;
 - Cartesian closed categories: categories with finite products and exponentials;

- Syntactic categories: constructed from logical theories or type theories; objects are formulas or types, morphisms are proofs or terms;
- Toposes: categories with certain logical and categorical structure supporting an internal logic;
- Algebraic categories:
 - $\mathcal{M}on$: monoids and monoid homomorphisms;
 - $\mathcal{G}rp$: groups and group homomorphisms;
 - $\mathcal{A}b$: abelian groups;
 - $(\mathcal{C})\mathcal{R}ing$: (commutative) rings and ring homomorphisms;
 - $\mathcal{V}ect_{\mathbf{k}}$: vector spaces over a field \mathbf{k} and linear maps;
 - $\mathcal{A}lg_{\mathbf{k}}$: algebras over a field or ring \mathbf{k} ;
 - $\mathcal{M}od_R$: modules over a ring R ;
 - $\mathcal{L}ie\mathcal{A}lg$ and $\mathcal{H}opf\mathcal{A}lg$: Lie and Hopf algebras and structure preserving maps;
- Topological / geometric categories:
 - $\mathcal{T}op$: topological spaces with continuous maps;
 - $\mathcal{T}op_*$: pointed spaces with basepoint-preserving maps;
 - $h\mathcal{T}op$: homotopy category; morphisms are homotopy classes of continuous maps;
 - $\mathcal{M}an$: smooth manifolds with smooth maps;
 - $\mathcal{D}iff$: diffeological spaces with smooth maps;
 - $\mathcal{S}ch$: schemes and morphisms of schemes;
 - $\mathcal{V}ar_{\mathbf{k}}$: algebraic varieties over a field \mathbf{k} ;
 - $\mathcal{C}ob$: cobordisms; objects are manifolds, morphisms are cobordism classes;
- Higher level constructions:
 - $\mathcal{C}at$: small categories and functors between them;
 - \mathcal{CAT} : (large) categories and functors — “set of all categories” is only valid if we allow a universe distinction to avoid size issues;
 - $\mathcal{G}rp_{\mathbf{d}}$: groupoids — categories in which all morphisms are invertible;
 - $\mathcal{M}on\mathcal{C}at$: monoidal categories and monoidal functors;
 - $\mathcal{A}b\mathcal{C}at$: abelian categories and additive functors;
 - $[\mathcal{C}, \mathcal{D}]$: functor category; objects are functors $\mathcal{C} \rightarrow \mathcal{D}$, morphisms are natural transformations;

- Comma categories ($F \downarrow G$): objects are triples $(C, D, f : F(C) \rightarrow G(D))$, morphisms are pairs making the corresponding squares commute.