Yoneda lemma

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Conventions

- $\mathcal{C}$ – locally small category;
- $\lambda AB.\mathcal{C}(A,B) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ – Hom functor;
  - For brevity, if the context is clear, $A, B := \mathcal{C}(A,B)$;
  - $A, \_ := \mathcal{C}(A, \_) := \lambda B.\mathcal{C}(A,B) : \mathcal{C} \to \mathcal{S}et$; similarly for $\_, B : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$;
- $F, G := [F, G]$ – class of natural transformations between F and G;
- $\simeq$ denotes isomorphisms of objects, $\cong$ emphasizes that isomorphism between objects extends to isomorphism between functors.

1 Yoneda lemma

Theorem 1.1 (Yoneda lemma)**.** For a given $F : \mathcal{C} \to \mathcal{S}et$ (resp. $F : \mathcal{C}^{\text{op}} \to \mathcal{S}et$)

$$\lambda AF.\mathcal{C}(A,-), F \simeq \lambda AF.FA \text{ (resp. } \lambda AF.\mathcal{C}(-,A), F \simeq \lambda AF.FA\text{)} :$$

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{S}et^{\mathcal{C}} \arrow[rr, bend left, "\lambda AF.\mathcal{C}(A,_),F"] \arrow[rr, bend right, "\lambda AF.FA"'] \arrow[rr, phantom, "\Downarrow \simeq"] & & \mathcal{S}et
\end{tikzcd}
\quad \Bigg| \quad
\begin{tikzcd}
\mathcal{C}^{\text{op}} \times \mathcal{S}et^{\mathcal{C}^{\text{op}}} \arrow[rr, bend left, "\lambda AF.\mathcal{C}(_,A),F"] \arrow[rr, bend right, "\lambda AF.FA"'] \arrow[rr, phantom, "\Downarrow \simeq"] & & \mathcal{S}et
\end{tikzcd} .$$

Proof (Covariant option)**.**

1. $\eta : \mathcal{C}(A,-) \to F$ is uniquely determined by the choice of $\eta_A(\text{id}_A) \in FA$:

$$\begin{tikzcd}
A \arrow[d, "f"'] & A,A \arrow[r, "\eta_A"] \arrow[d, "f\circ -"'] & FA \arrow[d, "Ff"] & \text{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & \eta_A(\text{id}_A) \arrow[d, mapsto] \\
B & A,B \arrow[r, "\eta_B"'] & FB & f \arrow[r, mapsto] & \eta_B(f) = Ff(\eta_A(\text{id}_A))
\end{tikzcd} .$$

Note, that this is possible, because for all $B : \text{obj}(\mathcal{C})$ $\bigcup_{f \in A,B} \text{im}(f \circ -) = A,B$. So, we established a family of isomorphisms:

$$\Phi_{A,F} : \mathcal{C}(A,-), F \xrightarrow{\simeq} FA;\ \Phi^{-1} : x \mapsto \eta^x, \text{ where } \eta^x_A(\text{id}_A) = x.$$

2. Φ extends to a natural transformation between $\lambda AF.\mathcal{C}(A,-), F$ and $\lambda AF.FA$:

$$\begin{tikzcd}
A \arrow[d, "\alpha_\bullet"] & F \arrow[d, "\beta"] & \mathcal{C}(A,-), F \arrow[r, "\Phi_{A,F}"] \arrow[d, "\beta \circ - \circ (\lambda\varphi.\varphi \circ \alpha_\bullet)"'] & F(A) \arrow[d, "F'\alpha_\bullet \circ \beta_A"] \\
A' & F' & \mathcal{C}(A',-), F' \arrow[r, "\Phi_{A',F'}"'] & F'(A')
\end{tikzcd} ;$$

$$\begin{tikzcd}
\eta^x \arrow[r, mapsto] \arrow[d, mapsto] & x \arrow[d, mapsto] \\
\eta^{x'} := \beta \circ \eta^x \circ (\lambda\varphi.\alpha_\bullet \circ \varphi) \arrow[r, mapsto] & \Phi(\eta^{x'}) = F'\alpha_\bullet \circ \beta_A(x)
\end{tikzcd} .$$

To prove the equality, we need to compare the $x' = \eta^{x'}(\text{id}_{A'})$ and $F'\alpha_\bullet \circ \beta_A(x)$. It follows from the naturality conditions for β and η^x:

$$\begin{tikzcd}
A \arrow[d, "\alpha_\bullet"'] & FA \arrow[r, "\beta_A"] \arrow[d, "F\alpha_\bullet"'] & F'A \arrow[d, "F'\alpha_\bullet"] & \text{id}_A : A,A \arrow[r, mapsto, "\eta^x_A"] \arrow[d, mapsto, "\alpha_\bullet \circ -"'] & \eta^x_{A'}(\text{id}_{A'}) = x : FA \arrow[d, mapsto, "F\alpha_\bullet"] \\
A' & FA' \arrow[r, "\beta_{A'}"'] & F'A' & \alpha_\bullet : A,A' \arrow[r, mapsto, "\eta^x_{A'}"'] & \eta^x_{A'}(\alpha_\bullet) = F\alpha_\bullet(x) : FA'
\end{tikzcd}$$

$$\beta_{A'} \circ \eta^x_{A'} \circ (\lambda\varphi.\varphi \circ \alpha_\bullet)(\text{id}_{A'}) = \beta_{A'} \circ \eta^x_{A'}(\alpha_\bullet) = \beta_{A'} \circ F\alpha_\bullet(x) = F'\alpha_\bullet \circ \beta_A(x).$$

2 Corollaries

Definition 2.1 (Yoneda embedding)**.**

$$Y := \lambda A.\mathcal{C}(A,-) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et^{\mathcal{C}} \qquad \begin{tikzcd} A \arrow[d, "f"'] & B,- \arrow[d, "-\circ f"'] \\ B & A,- \end{tikzcd} .$$

Proposition 2.2 (Fully faithfulness)**.** Yoneda embedding is a fully faithful functor.

Proof. For an arbitrary $B : \mathrm{obj}(\mathcal{C})$, in the covariant Yoneda lemma set $F := \mathcal{C}(B,-)$. This gives us a natural isomorphism:

$$\mathcal{C}(A,-), \mathcal{C}(B,-) \cong \mathcal{C}(B,A)$$

So, the mapping of morphisms under Y is bijective.

Corollary 2.3 (Small rephrasings)**.**

1. Every natural transformation between Hom functors from $\mathcal{S}et^{\mathcal{C}}$ is given by a unique corresponding arrow between objects in $\mathcal{C}$;
2. $A \simeq B \Leftrightarrow \mathcal{C}(A,-) \simeq \mathcal{C}(B,-)$ (note that functors preserve isomorpisms).