

Hom functor

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Conventions.

- We use λ -notation for functors and assume it implicitly tracks variance.
- All categories with set-valued Hom functors are assumed to be locally small.
- For brevity, when the context is clear, we assume

$$A, B := \text{Hom}(A, B) := \mathcal{C}(A, B).$$

1 Set-valued Hom functor

Definition 1.1 (Set-valued Hom functor). The set-valued Hom functor

$$\lambda AB. \mathcal{C}(A, B) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{Set},$$

is defined as follows:

- $\mathcal{C}(A, B)$ is the set of morphisms from A to B ,
- Its action on morphisms is given by

$$\mathcal{C}(f, g) := \lambda \varphi. g \circ \varphi \circ f,$$

i.e. by pre- and post-composition:

$$\begin{array}{ccc} \begin{array}{ccc} A & & B \\ f \uparrow & & g \downarrow \\ A' & & B' \end{array} & \begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ f \uparrow & & \downarrow g \\ A' & \xrightarrow{g \circ \varphi \circ f} & B' \end{array} & \begin{array}{ccc} \mathcal{C}(A, B) & & \\ \downarrow \mathcal{C}(f, g) = \lambda \varphi. g \circ \varphi \circ f & & \\ \mathcal{C}(A', B') & & \end{array} \end{array}$$

We will often write A, B or $\text{Hom}(A, B)$ instead of $\mathcal{C}(A, B)$ when the context is clear.

Definition 1.2 (Covariant and contravariant Hom functors). The functors

$$\mathcal{C}(A, -) := \lambda B. \mathcal{C}(A, B) : \mathcal{C} \rightarrow \mathcal{Set} \quad \text{and} \quad \mathcal{C}(-, B) := \lambda A. \mathcal{C}(A, B) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{Set}$$

are called the covariant and contravariant Hom functors, respectively. They are obtained from the bifunctor $\mathcal{C}(-, -)$ by fixing one argument. The induced action on morphisms is given by *post-composition* and *pre-composition*, respectively:

$$\begin{array}{ccc} \begin{array}{ccc} B & & \\ g \downarrow & & \\ B' & & \end{array} & \begin{array}{ccc} \mathcal{C}(A, -)(B) = \mathcal{C}(A, B) & & \\ \downarrow \mathcal{C}(A, -)(g) = g \circ - & & \\ \mathcal{C}(A, -)(B') = \mathcal{C}(A, B') & & \end{array} & \begin{array}{ccc} A & & \\ f \uparrow & & \\ A' & & \end{array} & \begin{array}{ccc} \mathcal{C}(-, B)(A') = \mathcal{C}(A, B) & & \\ \downarrow \mathcal{C}(-, B)(f) = - \circ f & & \\ \mathcal{C}(-, B)(A) = \mathcal{C}(A', B) & & \end{array} \end{array}$$

Natural transformations between covariant (resp. contravariant) Hom functors are induced by *pre-composition* (resp. *post-composition*):

$$\begin{array}{ccc}
\mathcal{C} & \begin{array}{c} \xrightarrow{A, -} \\ \Downarrow \alpha \\ \xrightarrow{B, -} \end{array} & \mathcal{S}et \\
\\
\begin{array}{ccc} X & A & A, X \xrightarrow{- \circ \alpha \bullet} B, X \\ f \downarrow & \alpha \bullet \uparrow & f \circ - \downarrow \quad \downarrow f \circ - \\ Y & B & A, Y \xrightarrow{- \circ \alpha \bullet} B, Y \end{array} & & \begin{array}{ccc} \varphi & \longmapsto & \varphi \circ \alpha \bullet \\ \downarrow & & \downarrow \\ f \circ \varphi & \longmapsto & f \circ \varphi \circ \alpha \bullet \end{array}
\end{array}$$

$$\begin{array}{ccc}
\begin{array}{ccc} X & A & X, A \xrightarrow{\beta \bullet \circ -} X, B \\ g \uparrow & \beta \bullet \downarrow & - \circ g \downarrow \quad \downarrow - \circ g \\ Y & B & Y, A \xrightarrow{\beta \bullet \circ -} Y, B \end{array} & & \begin{array}{ccc} \psi & \longmapsto & \beta \bullet \circ \psi \\ \downarrow & & \downarrow \\ \psi \circ g & \longmapsto & \beta \bullet \circ \psi \circ g \end{array}
\end{array}$$