Natural transformations

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1 Basics

Definition 1.1 (Natural transformations)**.** Let $F, G : \mathcal{C} \to \mathcal{D}$. Natural transformation α between F and G is a collection of morphisms $(\alpha_c \mid c : \operatorname{obj} C)$ satisfying the diagrams

$$
\begin{tikzcd}
c \arrow[d, "f"'] \\
c'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc'
\end{tikzcd}.
$$

Notation:

$$
F \xrightarrow{\alpha} G
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}
\end{tikzcd}.
$$

Definition 1.2 (Vertical composition of natural transformations)**.** Let $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\alpha : F \to G$ and $\beta : G \to H$. The vertical composition $\beta \circ \alpha$ of natural transformations β and α is defined componentwise:

$$
\beta \circ \alpha : F \to H \qquad\qquad (\beta \circ \alpha)_c = \beta_c \circ \alpha_c,
$$

$$
\begin{tikzcd}
c \arrow[d, "f"'] \\
c'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[r, "\beta_c"] \arrow[d, dashed, "Gf" description] & Hc \arrow[d, "Hf"] \\
Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"'] & Hc'
\end{tikzcd}.
$$

Notation:

$$
\begin{tikzcd}
F \arrow[r, "\alpha"] \arrow[dr, "\beta\circ\alpha"'] & G \arrow[d, "\beta"] \\
& H
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
& \Downarrow\alpha & \\
\mathcal{C} \arrow[rr, bend left=60, "F"] \arrow[rr, "G" description] \arrow[rr, bend right=60, "H"'] & & \mathcal{D} \\
& \Downarrow\beta &
\end{tikzcd}.
$$

Definition 1.3 (Horizontal composition of natural transformations)**.** Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ with natural transformations $\alpha : F_1 \to F_2$ and $\beta : G_1 \to G_2$. The horizontal composition $\beta \bullet \alpha$ of natural transformations β and α is defined as the transformation of images of α_c under the action of F_1 and F_2 by β:

$$
\beta \bullet \alpha : G_1F_1 \to G_2F_2 \qquad\qquad (\beta \bullet \alpha)_c = G_2\alpha_c \circ \beta_{F_1c} = \beta_{F_2c} \circ G_1\alpha_c,
$$

$$
\begin{tikzcd}
& c' \\
c \arrow[ur, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& F_1c' \arrow[r, "\alpha_{c'}"] & F_2c' \\
F_1c \arrow[ur, "F_1f"] \arrow[r, "\alpha_c"'] & F_2c \arrow[ur, "F_2f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& G_1F_1c' \arrow[rr, "G_1\alpha_{c'}"] \arrow[dd, dotted, "\beta_{F_1c'}"' near start] & & G_1F_2c' \arrow[dd, "\beta_{F_1c'}"] \\
G_1F_1c \arrow[ur, "G_1F_1f"] \arrow[rr, "G_1\alpha_c" near end] \arrow[dd, "\beta_{F_1c}"'] & & G_1F_2c \arrow[ur, "G_1F_2f"] \arrow[dd, "\beta_{F_2c}" near end] & \\
& G_2F_1c' \arrow[rr, dotted, "G_2\alpha_{c'}" near end] & & G_2F_2c' \\
G_2F_1c \arrow[ur, dotted, "G_2F_1f"] \arrow[rr, "G_2\alpha_c"] & & G_2F_2c \arrow[ur, "G_2F_2f"] &
\end{tikzcd}
.
$$

Note that functors preserve commutative diagrams, so all of the faces of the cube commute. Notation:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
\qquad\qquad
G_1F_1 \xrightarrow{\beta \bullet \alpha} G_2F_2 \ .
$$

Definition 1.4 (Whiskering)**.** Wiskering is defined as a horizontal composition with identity natural transformation:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
\ ;
$$

$$
\begin{aligned}
G\alpha &:= 1_G \bullet \alpha : G \circ F_1 \to G \circ F_2 \qquad & (G\alpha)_c &= G\alpha_c, \\
\beta F &:= \beta \bullet 1_F : G_1 \circ F \to G_2 \circ F \qquad & (\beta F)_c &= \beta_{Fc}.
\end{aligned}
$$

Using this notation, we can present any horizontal transformation in the following form:

$$\beta \bullet \alpha = \beta F_2 \circ G_1\alpha = G_2\alpha \circ \beta F_1.$$

Proposition 1.5. Let $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$ with natural transformations $\alpha : F_1 \to F_2$, $\alpha' : F_2 \to F_3$ and $\beta : G_1 \to G_2$, $\beta' : G_2 \to G_3$:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E}
\end{tikzcd}
\ .
$$

Then there is an interchange law between the natural transformations:

$$(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = (\beta' \bullet \alpha') \circ (\beta \bullet \alpha).$$

Proof.

$$(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = \beta' F_3 \circ (\beta F_3 \circ G_1\alpha') \circ G_1\alpha = \beta' F_3 \circ G_2\alpha' \circ \beta F_2 \circ G_1\alpha,$$

$$(\beta' \bullet \alpha') \circ (\beta \bullet \alpha) = \beta' F_3 \circ G_2\alpha' \circ \beta F_2 \circ G_1\alpha.$$

2 Advanced whiskering

The naturality diagram from Def 1.1. basically means, that for given $f : \mathrm{arr}(\mathcal{C})$

$$\alpha_{\mathrm{cod}(f)} \circ \mathrm{dom}(\alpha)f = \mathrm{cod}(\alpha)f \circ \alpha_{\mathrm{dom}(\alpha)}.$$

The question arises: can we write it in the form of

$$\alpha\,\mathrm{dom}(\alpha) = \mathrm{cod}(\alpha)\alpha$$

using some polymorphic notation? We already have the notation for whiskering, but so far we only defined it on $\mathrm{obj}(\mathcal{C})$.