

# Hom functor

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## Conventions.

- We use  $\lambda$ -notation for functors and assume it implicitly tracks variance.
- All categories with set-valued Hom functors are assumed to be locally small.
- For brevity, when the context is clear, we assume

$$A, B := \text{Hom}(A, B) := \mathcal{C}(A, B).$$

## 1 Set-valued Hom functor

**Definition 1.1** (Set-valued Hom functor). The set-valued Hom functor

$$\lambda AB.\mathcal{C}(A, B) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{S}\text{et},$$

is defined as follows:

- $\mathcal{C}(A, B)$  is the set of morphisms from  $A$  to  $B$ ,
- Its action on morphisms is given by

$$\mathcal{C}(f, g) := \lambda \varphi. g \circ \varphi \circ f,$$

i.e. by pre- and post-composition:

$$\begin{array}{ccccccc} A & & B & & A & \xrightarrow{\varphi} & B \\ \uparrow f & & \downarrow g & & \uparrow f & & \downarrow g \\ A' & & B' & & A' & \xrightarrow{g \circ \varphi \circ f} & B' \\ & & & & & & \\ & & & & & & \mathcal{C}(A, B) \\ & & & & & & \downarrow \mathcal{C}(f, g) = \lambda \varphi. g \circ \varphi \circ f \\ & & & & & & \mathcal{C}(A', B') \end{array}$$

We will often write  $A, B$  or  $\text{Hom}(A, B)$  instead of  $\mathcal{C}(A, B)$  when the context is clear.

**Definition 1.2** (Covariant and contravariant Hom functors). The functors

$$\mathcal{C}(A, -) := \lambda B. \mathcal{C}(A, B) : \mathcal{C} \rightarrow \mathcal{S}\text{et} \quad \text{and} \quad \mathcal{C}(-, B) := \lambda A. \mathcal{C}(A, B) : \mathcal{C}^{\text{op}} \rightarrow \mathcal{S}\text{et}$$

are called the covariant and contravariant Hom functors, respectively. They are obtained from the bifunctor  $\mathcal{C}(-, -)$  by fixing one argument. The induced action on morphisms is given by *post-composition* and *pre-composition*, respectively:

$$\begin{array}{ccccc} B & & \mathcal{C}(A, -)(B) = \mathcal{C}(A, B) & & A \\ \downarrow g & & \downarrow \mathcal{C}(A, -)(g) = g \circ - & & \uparrow f \\ B' & & \mathcal{C}(A, -)(B') = \mathcal{C}(A, B') & & A' \\ & & & & \\ & & & & \mathcal{C}(-, B)(A') = \mathcal{C}(A, B) \\ & & & & \downarrow \mathcal{C}(-, B)(f) = - \circ f \\ & & & & \mathcal{C}(-, B)(A) = \mathcal{C}(A', B) \end{array}$$

Natural transformations between covariant (resp. contravariant) Hom functors are induced by *pre-composition* (resp. *post-composition*):

$$\begin{array}{ccc}
\mathcal{C} & \begin{array}{c} \xrightarrow{A,-} \\ \Downarrow \alpha \\ \xrightarrow{B,-} \end{array} & \mathcal{Set} \\
& & \\
\begin{array}{ccccc} X & & A & & A, X \xrightarrow{- \circ \alpha_\bullet} B, X \\ \downarrow f & & \alpha_\bullet \uparrow & & \downarrow f \circ_- \\ Y & & B & & A, Y \xrightarrow{- \circ \alpha_\bullet} B, Y \end{array} & & \begin{array}{ccc} \varphi & \longmapsto & \varphi \circ \alpha_\bullet \\ \downarrow & & \downarrow \\ f \circ \varphi & \longmapsto & f \circ \varphi \circ \alpha_\bullet \end{array} \\
& & & & \\
\begin{array}{ccccc} X & & A & & X, A \xrightarrow{\beta_\bullet \circ -} X, B \\ \uparrow g & & \beta_\bullet \downarrow & & \downarrow \circ g \\ Y & & B & & Y, A \xrightarrow[\beta_\bullet \circ -]{} Y, B \end{array} & & & \begin{array}{ccc} \psi & \longmapsto & \beta_\bullet \circ \psi \\ \downarrow & & \downarrow \\ \psi \circ g & \longmapsto & \beta_\bullet \circ \psi \circ g \end{array}
\end{array}$$