(Co)Limits

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1 (Co)Limits

1.1 Main definitions

Definition 1.1 (Comma category)**.** Given functors

$$\mathcal{A} \xrightarrow{S} \mathcal{C} \xleftarrow{T} \mathcal{B},$$

the category $(S \downarrow T)$ is defined as follows:

- Objects of are the triples $(a : \mathrm{obj}(\mathcal{A}), h : \mathrm{arr}(\mathcal{C}), b : \mathrm{obj}(\mathcal{B}))$,
- Morphisms are pairs $(f : \mathrm{arr}(\mathcal{A}), g : \mathrm{arr}(\mathcal{B}))$ satisfying the condition:

$$\begin{tikzcd} (a,h,b) \arrow[d, "{(f,g)}"] \\ (a',h',b') \end{tikzcd} \qquad \begin{tikzcd} Sa \arrow[r, "h"] \arrow[d, "Sf"'] & Tb \arrow[d, "Tf"] \\ Sa' \arrow[r, "h'"'] & Tb' \end{tikzcd} .$$

The concept of a comma category allows us to define (co)limits in a uniform way.

Definition 1.2 ((Co)limit)**.** The limit of a diagram D is defined as a terminal object of the category $(\Delta \downarrow D)$. Dually, the colimit of a diagram E is defined as an initial object of the category $(E \downarrow \Delta)$.

$$\mathcal{D} \xrightarrow{\Delta} \mathcal{D}^{\mathcal{J}} \xleftarrow{D} 1; \qquad 1 \xrightarrow{E} \mathcal{C}^{\mathcal{J}} \xleftarrow{\Delta} \mathcal{C}.$$

Notation:

$$\lim_{\mathcal{J}} D := \mathrm{ter}(\Delta \downarrow D); \qquad \operatorname*{colim}_{\mathcal{J}} E := \mathrm{ini}(E \downarrow \Delta).$$

Visual intuition:

$$\begin{tikzcd} \Delta_d \arrow[r, dashed] \arrow[dr, "h'"'] & \Delta_{\lim D} \arrow[d, "h"] \\ & D \end{tikzcd} \qquad \begin{tikzcd} d \arrow[rr, dashed] \arrow[d] \arrow[drr, dotted] \arrow[ddr] & & \lim D \arrow[d, "h_2"] \arrow[dll, dotted, "h_1"'] \arrow[dl, "h_n"'] \\ Dj_1 \arrow[rr, dotted, "Dg"] \arrow[dr, "Df \circ Dg"'] & & Dj_2 \arrow[dl, "Df"] \\ & Dj_n & \end{tikzcd} .$$

Remark 1.3. From here on we use $(\mathrm{co})\lim D$ also to denote colimit object.

Definition 1.4 ((Co)limit funtor)**.** If $\mathcal{C}$ has all (co)limits of shape $\mathcal{J}$, then there exists a (co)limit functor

$$(\mathrm{co})\lim_{\mathcal{J}} \colon \mathcal{C}^{\mathcal{J}} \to \mathcal{C}.$$

The action on morphisms is defined via the universal property:

$$
\begin{tikzcd}
\Delta \lim D \arrow[r, "h"] \arrow[d, dashed, "\Delta(\phi(\alpha\circ h))"'] \arrow[dr, "\alpha\circ h"] & D \arrow[d, "\alpha"] & & \lim D \arrow[d, dashed, "\phi(\alpha\circ h)"] \\
\Delta \lim D' \arrow[r, "h'"'] & D' & & \lim D'
\end{tikzcd}
.
$$

Remark 1.5. Thus, limit functor is a right adjoint to diagonal functor, colimit functor is a left adjoint to diagonal functor.

1.2 Examples

Remark 1.6 (Terminal and initial objects)**.** Terminal and initial objects in a category are special cases of limits and colimits, respectively.

Definition 1.7 ((Co)products)**.** (Co)Product is defined as the (co)limit of a discrete diagram:

$$
\prod_{\mathcal{J}} D := \lim_{\mathcal{J}} D; \qquad \coprod_{\mathcal{J}} E := \operatorname*{colim}_{\mathcal{J}} E.
$$

Setting $J := \mathrm{obj}(\mathcal{J})$, one can write the standard low-level universal property diagrams for (co)product:

$$
\begin{tikzcd}
c_k \arrow[r, "i_k"] \arrow[dr, "f_k"'] & \coprod_{j\in J} c_j \\
& c \arrow[u, dashed, "\coprod_{j\in J} f_k"']
\end{tikzcd}
\qquad
\begin{tikzcd}
d \arrow[d, dashed, "\prod_{j\in J} g_k"'] \arrow[dr, "g_k"] & \\
\prod_{j\in J} d_j \arrow[r, "\pi_j"'] & d_i
\end{tikzcd}
.
$$

Definition 1.8 ((Co)Equalizers)**.** A (co)equalizer is defined as the (co)limit of a two-arrow diagram:

$$
\mathcal{J}: \qquad
\begin{tikzcd}
j_1 \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & j_2
\end{tikzcd}
,
$$

$$
\begin{tikzcd}
E \arrow[r, "\mathrm{eq}"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\
O \arrow[u, dashed] \arrow[ur, "h'"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\mathrm{coeq}"] \arrow[dr, "h'"'] & C \arrow[d, dashed] \\
& & P
\end{tikzcd}
.
$$

Remark 1.9 ((Co)Kernels)**.** A (co)kernel is a special case of a (co)equalizer where one of the morphisms is a zero morphism.

Definition 1.10 (Pullbacks and pushouts)**.** Pullback and pushout are defined as limit and colimit of the following diagrams:

$$
\begin{tikzcd}
 & j_3 \arrow[d, "f_2"] \\
j_2 \arrow[r, "f_1"'] & j_1
\end{tikzcd}
\qquad
\begin{tikzcd}
j_1 \arrow[r, "f_2"] \arrow[d, "f_1"'] & j_3 \\
j_2 &
\end{tikzcd}
,
$$

$$
\begin{tikzcd}
D \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
 & A \times_C B \arrow[r] \arrow[d] & A \arrow[d] \\
 & B \arrow[r] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & A +_C B \arrow[dr, dashed] & \\
 & & D
\end{tikzcd}
.
$$

1.3 (Co)Continuous functors

Definition 1.11 ((Co)Continuous functors)**.** A functor $F : \mathcal{C} \to \mathcal{D}$ is called (co)continuous if it preserves (co)limits:

$$F(\text{co})\lim D \cong (\text{co})\lim F \circ D.$$