Adjoint functors

Alyson Mei

January 11, 2026

References

- GitHub:
  - Natural transformations;
  - Hom functor;
  - Comma category and Diagonal functor;
- Local: //not implemented yet

1 Adjoint functors

Definition 1.1 (Adjoint functors via set-valued Hom functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, where $\mathcal{C}$ and $\mathcal{D}$ are locally small. F is said to be left adjoint to G (resp. G is said to be right adjoint to F) if

$$\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cong \mathrm{Hom}_{\mathcal{C}}(c, Gd).$$

Notation:

$$F \dashv G, \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} .$$

In functorial notation this condition takes the following form:

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \arrow[r, bend left, "\lambda cd.Fc{,}d"] \arrow[r, bend right, "\lambda cd.c{,}Gd"'] \arrow[r, phantom, "\Downarrow\simeq"] & \mathcal{S}et \end{tikzcd} .$$

Definition 1.2 (Φ; $\sharp$ and $\flat$; η_c and ε_d)**.** We denote the isomorphism between Hom functors as Φ:

$$\begin{tikzcd} c & d & Fc, d \arrow[r, "\Phi_{c,d}"] \arrow[d, "g\circ-\circ Ff"'] & c, Gd \arrow[d, "Gg\circ-\circ f"] \\ c' \arrow[u, "f"] & d' \arrow[from=u, "g"'] & Fc', d' \arrow[r, "\Phi_{c',d'}"'] & c', Gd' \end{tikzcd} .$$

The standard notation for action of Φ and its inverse on arrows is $\sharp$ and $\flat$ annotations respectively:

$$
\begin{tikzcd}
\varphi : Fc, d \arrow[r, mapsto] \arrow[d, mapsto] & \Phi_{c,d}(\varphi) := \varphi^\sharp \arrow[d, mapsto] & \Phi^{-1}_{c,d}(\psi) := \psi^\flat \arrow[d, mapsto] & \psi : c, Rd \arrow[l, mapsto] \arrow[d, mapsto] \\
g \circ \varphi \circ Ff \arrow[r, mapsto] & (g \circ \varphi \circ Ff)^\sharp = Gg \circ \varphi^\sharp \circ f & g \circ \psi^\flat \circ Ff = (Gg \circ \psi \circ f)^\flat & Gg \circ \psi \circ f \arrow[l, mapsto]
\end{tikzcd}.
$$

The action of Φ on id_{Fc} and id_{Gd} is denoted as follows:

$$\eta_c := \mathrm{id}^\sharp_{Fc}, \qquad \varepsilon_d := \mathrm{id}^\flat_{Gd}.$$

Special cases:

- $f = \mathrm{id}_c$:
$$(g \circ \varphi)^\sharp = Gg \circ \varphi^\sharp, \qquad g \circ \psi^\flat = (Gg \circ \psi)^\flat;$$
- $g = \mathrm{id}_d$:
$$(\varphi \circ Ff)^\sharp = \varphi^\sharp \circ f, \qquad \psi^\flat \circ Ff = (\psi \circ f)^\flat;$$
- $d = Fc,\ \varphi = \mathrm{id}_c$:
$$(g \circ Ff)^\sharp = Gg \circ \eta_c \circ f;$$
- $c = Gd,\ \psi = \mathrm{id}_d$:
$$g \circ \varepsilon_d \circ Ff = (Gg \circ f)^\flat.$$

Definition 1.3 (Adjoint functors via Universal property)**.** We can define adjunctions via the universal property using the data from the Hom-functor definition.

- $F \dashv G :\Leftrightarrow$ for all $\psi : c \to Gd$ there exists unique morphism $\psi^\flat : Fc \to d$ satisfying the diagram
$$
\begin{tikzcd}
c \arrow[r, "\eta_c"] \arrow[dr, "\psi"'] & GFc \arrow[d, dashed, "G\psi^\flat"] & Fc \arrow[d, dashed, "\psi^\flat"] \\
& Gd & d
\end{tikzcd};
$$
- $F \dashv G :\Leftrightarrow$ for all $\varphi : Fc \to d$ there exists unique morphism $\varphi^\sharp : c \to Gd$ satisfying the diagram
$$
\begin{tikzcd}
c \arrow[d, dashed, "\varphi^\sharp"'] & Fc \arrow[d, dashed, "F\varphi^\sharp"'] \arrow[dr, "\varphi"] & \\
Gd & FGd \arrow[r, "\varepsilon_d"'] & d
\end{tikzcd}.
$$

Proposition 1.4 (Equivalence of definitions via Hom functors and via Universal property)**.** The Hom-functor and universal property definitions of adjoint functors (Defs 1.1 and 1.3) are equivalent.

Proof. $(\Rightarrow)$:

1. *Existence.* The morphisms $\psi^\flat$ and $\varphi^\sharp$ are obtained directly from the natural transformation Φ in Def 1.2. Commutativity of the diagrams follows from the corresponding special cases in Def 1.2:
   - Set $f := \mathrm{id}_c$, $g := \psi^\flat$ in special case (3) to obtain the first diagram.
   - Set $g := \mathrm{id}_d$, $f := \varphi^\sharp$ in special case (4) to obtain the second diagram.
2. *Uniqueness.* Suppose $\tilde{\varphi}$ and $\tilde{\psi}$ are alternative arrows satisfying the diagrams:

$$\psi = G\tilde{\psi} \circ \eta_c, \qquad \varphi = \varepsilon_d \circ F\tilde{\varphi}.$$

   Applying special cases (3) and (4) and using the invertibility of $\sharp$ and $\flat$, we get

$$\psi = \tilde{\psi}^\sharp \Leftrightarrow \psi^\flat = \tilde{\psi}, \qquad \varphi = \tilde{\varphi}^\flat \Leftrightarrow \varphi^\sharp = \tilde{\varphi}.$$

($\Leftarrow$) // not implemented yet

$\square$