Natural transformations

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1 Basics

Definition 1.1 (Natural transformations). Let $F, G : \mathcal{C} \to \mathcal{D}$. A natural transformation α between F and G is a collection of morphisms $(\alpha_c \mid c \in \mathrm{obj}(\mathcal{C}))$ satisfying the diagrams

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[d, "Gf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \end{tikzcd} .$$

Notation:

$$\begin{tikzcd} F \arrow[r, "\alpha"] & G \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} .$$

Definition 1.2 (Vertical composition of natural transformations). Let $F, G, H : \mathcal{C} \to \mathcal{D}$ with natural transformations $\alpha : F \to G$ and $\beta : G \to H$. The vertical composition $\beta \circ \alpha$ is defined componentwise:

$$\beta \circ \alpha : F \to H \qquad (\beta \circ \alpha)_c = \beta_c \circ \alpha_c,$$

$$\begin{tikzcd} c \arrow[d, "f"'] \\ c' \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "\alpha_c"] \arrow[d, "Ff"'] & Gc \arrow[r, "\beta_c"] \arrow[d, dashed, "Gf" description] & Hc \arrow[d, "Hf"] \\ Fc' \arrow[r, "\alpha_{c'}"'] & Gc' \arrow[r, "\beta_{c'}"'] & Hc' \end{tikzcd} .$$

Notation:

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta\circ\alpha"'] & G \arrow[d, "\beta"] \\ & H \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathcal{D} \end{tikzcd} .$$

Definition 1.3 (Horizontal composition of natural transformations). Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ with natural transformations $\alpha : F_1 \to F_2$ and $\beta : G_1 \to G_2$. The horizontal composition $\beta \bullet \alpha$ is defined as the transformation obtained by applying β to the images of α_c under F_1 and F_2:

$$\beta \bullet \alpha : G_1F_1 \to G_2F_2 \qquad (\beta \bullet \alpha)_c = G_2\alpha_c \circ \beta_{F_1c} = \beta_{F_2c} \circ G_1\alpha_c,$$

$$
\begin{tikzcd}
 & c' \\
c \arrow[ur, "f"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F_1c' \arrow[r, "\alpha_{c'}"] & F_2c' \\
F_1c \arrow[ur, "F_1f"] \arrow[r, "\alpha_c"'] & F_2c \arrow[ur, "F_2f"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=small, column sep=small]
 & G_1F_1c' \arrow[rr, "G_1\alpha_{c'}"] \arrow[dd, dotted, "\beta_{F_1c'}"] & & G_1F_2c' \arrow[dd, "\beta_{F_1c'}"] \\
G_1F_1c \arrow[ur, "G_1F_1f"] \arrow[rr, "G_1\alpha_c"] \arrow[dd, "\beta_{F_1c}"] & & G_1F_2c \arrow[ur, "G_1F_2f"] \arrow[dd, "\beta_{F_2c}"] & \\
 & G_2F_1c' \arrow[rr, dotted, "G_2\alpha_{c'}"] & & G_2F_2c' \\
G_2F_1c \arrow[ur, dotted, "G_2F_1f"] \arrow[rr, "G_2\alpha_c"] & & G_2F_2c \arrow[ur, "G_2F_2f"] &
\end{tikzcd} .
$$

Note that functors preserve commutative diagrams, so all of the faces of the cube commute. Notation:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd}
\qquad\qquad
G_1F_1 \xrightarrow{\beta\bullet\alpha} G_2F_2 \ .
$$

Definition 1.4 (Whiskering)**.** Whiskering is defined as a horizontal composition with an identity natural transformation:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "G"] & \mathcal{E}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{E}
\end{tikzcd} ;
$$

$$
\begin{aligned}
G\alpha &:= 1_G \bullet \alpha : G \circ F_1 \to G \circ F_2 & \qquad (G\alpha)_c &= G\alpha_c, \\
\beta F &:= \beta \bullet 1_F : G_1 \circ F \to G_2 \circ F & \qquad (\beta F)_c &= \beta_{Fc}.
\end{aligned}
$$

Using this notation, we can present any horizontal transformation in the following form:

$$
\beta \bullet \alpha = \beta F_2 \circ G_1\alpha = G_2\alpha \circ \beta F_1.
$$

Proposition 1.5. Let $F_1, F_2, F_3 : \mathcal{C} \to \mathcal{D}$, $G_1, G_2, G_3 : \mathcal{D} \to \mathcal{E}$ with natural transformations $\alpha : F_1 \to F_2$, $\alpha' : F_2 \to F_3$ and $\beta : G_1 \to G_2$, $\beta' : G_2 \to G_3$:

$$
\begin{tikzcd}[column sep=large]
\mathcal{C} \arrow[r, bend left=60, "F_1"] \arrow[r, "F_2" description] \arrow[r, bend right=60, "F_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\alpha'"] & \mathcal{D} \arrow[r, bend left=60, "G_1"] \arrow[r, "G_2" description] \arrow[r, bend right=60, "G_3"'] \arrow[r, phantom, shift left=3, "\Downarrow\beta"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathcal{E}
\end{tikzcd} .
$$

Then the following interchange law holds:

$$
(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = (\beta' \bullet \alpha') \circ (\beta \bullet \alpha).
$$

Proof.

$$
(\beta' \circ \beta) \bullet (\alpha' \circ \alpha) = \beta' F_3 \circ (\beta F_3 \circ G_1\alpha') \circ G_1\alpha = \beta' F_3 \circ G_2\alpha' \circ \beta F_2 \circ G_1\alpha,
$$

$$
(\beta' \bullet \alpha') \circ (\beta \bullet \alpha) = \beta' F_3 \circ G_2\alpha' \circ \beta F_2 \circ G_1\alpha.
$$