

Notes on Dependent Type Theory

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These notes are based on [video recordings](#) of R. Harper's lectures.

1 Introduction

Boolean algebra is associated with classical logic, Heyting algebra – with intuitionistic logic.

Definition 1.1 (Boolean algebra). A *Boolean algebra* can be defined as a complemented distributive lattice:

- Pre-order:

$$x \leq x, \quad x \leq y \ \& \ y \leq z \rightarrow x \leq z;$$

- Has finite meets and joins:

$$x \leq 1, \quad z \leq x \ \& \ z \leq y \rightarrow z \leq (x \wedge y), \quad x \wedge y \leq x, \quad x \wedge y \leq y;$$

$$0 \leq x, \quad x \leq z \ \& \ y \leq z \rightarrow x \vee y \leq z, \quad x \leq x \vee y, \quad y \leq x \vee y;$$

- Has complements:

$$1 \leq \bar{x} \vee x, \quad \bar{x} \wedge x \leq 0;$$

- Distributive:

$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z).$$

Additionally, the exponential is defined as

$$y^x := \bar{x} \vee y.$$

Definition 1.2 (Heyting algebra). A *Heyting algebra* is defined as a lattice with exponentials:

- Pre-order;
- Has finite meets and joins;
- Has exponentials:

$$y^x \wedge x \leq y, \quad z \wedge x \leq y \rightarrow z \leq y^x.$$

Exercise 1.3.

1. “Yoneda lemma”: $x \leq y$ iff $\forall z (z \leq x \rightarrow z \leq y)$;

- 2. Every Heyting algebra is distributive.

Quotes:

- “Boolean algebra (closed world) = Heyting algebra (open world) with complements”;
- “Classical logic is a logic with complete information”.

The definitions provide us with standard rules:

- Weakening:

$$x \leq x \vee y \quad (x \wedge y \leq x);$$

- Contraction:

$$x \leq x \wedge x;$$

- Exchange:

$$x \wedge y \equiv y \wedge x.$$

Relation to classical logic:

- Sequent $\Gamma \vdash A$, where $\Gamma = A_1, \dots, A_n$, corresponds to:

$$A_1 \wedge \dots \wedge A_n \leq A;$$

- Rules for \wedge :

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \frac{}{\Gamma, A \wedge B \vdash A} \quad \frac{}{\Gamma, A \wedge B \vdash B};$$

- ...and so on.

Definition 1.4 (Lindenbaum algebra). A *Lindenbaum algebra* is defined as the algebra of equivalence classes of a given theory:

- $[A] = \{B \mid B \equiv A\};$
- $[A] \wedge [B] := [A \wedge B];$
- ... and so on.

Theorem 1.5 (Soundness and Completeness for Intuitionistic Propositional Logic). Let Γ be a context and A a formula. Then

$$\Gamma \vdash A \quad \text{iff} \quad \forall H \left(\llbracket \Gamma \rrbracket_H \leq \llbracket A \rrbracket_H \right),$$

where H ranges over all Heyting algebras, and $\llbracket - \rrbracket_H$ denotes the interpretation of formulas as elements of H under a valuation of propositional variables.

2 Simple Type Theory

Definition 2.1 (Simple type theory). We define the *simple type theory* as follows:

- Unit 1:

$$\frac{}{\Gamma \vdash 1 \text{ type}} \text{1-}F \quad \frac{}{\Gamma \vdash \langle \rangle : 1} \text{1-}I \quad (\text{no 1-E}) ;$$

- Product \times :

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \times B \text{ type}} \times\text{-}F \quad \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N \rangle : A \times B} \times\text{-}I \quad \frac{\Gamma \vdash M : A \times B}{\Gamma \vdash \text{fst}(M) : A \quad \Gamma \vdash \text{snd}(M) : B} \times\text{-}E ;$$

- Exponential \rightarrow :

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A \rightarrow B \text{ type}} \rightarrow\text{-}F \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x. M : A \rightarrow B} \rightarrow\text{-}I$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M(N) : B} \rightarrow\text{-}E ;$$

- Void 0:

$$\frac{}{\Gamma \vdash 0 \text{ type}} \text{0-}F \quad (\text{no 0-I}) \quad \frac{\Gamma \vdash M : 0}{\Gamma \vdash \text{abort}(M) : A} \text{0-}E ;$$

- Coproduct $+$:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash A + B \text{ type}} \text{+}\text{-}F \quad \frac{\Gamma \vdash M : A}{\Gamma \vdash \text{inl}(M) : A + B} \text{+}\text{-}I_1 \quad \frac{\Gamma \vdash N : B}{\Gamma \vdash \text{inr}(N) : A + B} \text{+}\text{-}I_2 ,$$

$$\frac{\Gamma, x : A \vdash N : C \quad \Gamma, y : B \vdash P : C}{\Gamma, z : A + B \vdash \text{case}(x.N, y.P)(z) : C} \text{+}\text{-}E .$$

Definition 2.2 (β and η equivalences).

- Unit 1:

$$(\beta) \text{ none} \quad (\eta) \Gamma \vdash \langle \rangle \equiv M : 1;$$

- Product \times :

$$(\beta) \frac{\Gamma \vdash \text{fst}(\langle M, N \rangle) \equiv M : A \quad \Gamma \vdash \text{snd}(\langle M, N \rangle) \equiv N : B}{\Gamma \vdash \langle M, N \rangle \equiv M : A \times B} \quad (\eta) \Gamma \vdash \langle \text{fst } M, \text{snd } M \rangle \equiv M : A \times B;$$

- Exponential \rightarrow :

$$(\beta) \Gamma \vdash (\lambda x. M)(N) \equiv [N/x]M : B \quad (\eta) \Gamma \vdash (\lambda x. M(x)) \equiv M : A \rightarrow B;$$

- Zero 0:

$$(\beta) \text{ none} \quad (\eta) \Gamma, z : 0 \vdash R \equiv \text{abort}(M) : C;$$

– Coproduct $+$:

$$\begin{array}{l}
(\beta) \quad \text{case}(x.M; y.N)(\text{inl}(P)) \equiv [P/x]M : C \\
\quad \text{case}(x.M; y.N)(\text{inr}(Q)) \equiv [Q/y]N : C \text{ ,} \\
\frac{\Gamma \vdash [\text{inl}(P)/z]R \equiv [P/x]M \quad \Gamma \vdash [\text{inr}(Q)/z]R \equiv [Q/y]N}{\Gamma, z : A + B \vdash R \equiv \text{case}(x.M; y.N)(z)} (\eta) .
\end{array}$$