Notes on Dependent Type Theory

Alyson Mei

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These notes are based on video recordings of R. Harper's lectures.

Conventions.

- Bullet lists are used for strongly logically connected items (primarily in definitions), while dashed lists are used for (somewhat) distinct items;
- Remarks marked with * are additions by the present author.

1 Introduction

Boolean algebra is associated with classical logic, Heyting algebra – with intuitionistic logic.

Definition 1.1 (Boolean algebra)**.** A *Boolean algebra* can be defined as a complemented distributive lattice:

- Pre-order:
$$x \leq x, \qquad x \leq y \,\&\, y \leq z \to x \leq z;$$
- Has finite meets and joins:
$$x \leq 1, \qquad z \leq x \,\&\, z \leq y \to z \leq (x \wedge y), \qquad x \wedge y \leq x, \qquad x \wedge y \leq y;$$
$$0 \leq x, \qquad x \leq z \,\&\, y \leq z \to x \vee y \leq z, \qquad x \leq x \vee y, \qquad y \leq x \vee y;$$
- Has complements:
$$1 \leq \bar{x} \vee x, \qquad \bar{x} \wedge x \leq 0;$$
- Distributive:
$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z).$$

Additionally, the exponential is defined as

$$y^x := \bar{x} \vee y.$$

Definition 1.2 (Heyting algebra)**.** A *Heyting algebra* is defined as a lattice with exponentials:

- Pre-order;
- Has finite meets and joins;

- Has exponentials:
$$y^x \wedge x \leq y, \qquad z \wedge x \leq y \to z \leq y^x.$$

Exercise 1.3.

1. "Yoneda lemma": $x \leq y$ iff $\forall z \ (z \leq x \to z \leq y)$;
2. Every Heyting algebra is distributive.

Quotes:

- "Boolean algebra (closed world) = Heyting algebra (open world) with complements";
- "Classical logic is a logic with complete information".

The definitions provide us with standard rules:

- Weakening:
$$x \leq x \vee y \qquad (x \wedge y \leq x);$$
- Contraction:
$$x \leq x \wedge x;$$
- Exchange:
$$x \wedge y \equiv y \wedge x.$$

Relation to classical logic:

- Sequent $\Gamma \vdash A$, where $\Gamma = A_1, \ldots, A_n$, corresponds to:
$$A_1 \wedge \cdots \wedge A_n \leq A;$$
- Rules for $\wedge$:
$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \frac{}{\Gamma, A \wedge B \vdash A} \quad \frac{}{\Gamma, A \wedge B \vdash B}\ ;$$
- ...and so on.

Definition 1.4 (Lindenbaum algebra)**.** A *Lindenbaum algebra* is defined as the algebra of equivalence classes of a given theory:

- $[A] = \{B \mid B \equiv A\}$;
- $[A] \wedge [B] := [A \wedge B]$;
- ... and so on.

Theorem 1.5 (Soundness and Completeness for Intuitionistic Propositional Logic)**.** Let Γ be a context and A a formula. Then
$$\Gamma \vdash A \quad \text{iff} \quad \forall H \ \big(\llbracket \Gamma \rrbracket_H \leq \llbracket A \rrbracket_H\big),$$
where H ranges over all Heyting algebras, and $\llbracket - \rrbracket_H$ denotes the interpretation of formulas as elements of H under a valuation of propositional variables.

2 Simple Type Theory

Definition 2.1 (Simple type theory)**.** We define the *simple type theory* as follows:

- Unit 1:

$$\frac{}{\Gamma \vdash 1 \ \text{type}}\ 1\text{-}F \qquad \frac{}{\Gamma \vdash \langle\rangle : 1}\ 1\text{-}I \qquad \text{(no 1-E)}\ ;$$

- Product $\times$:

$$\frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \times B \ \text{type}}\ \times\text{-}F \qquad \frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M, N\rangle : A \times B}\ \times\text{-}I \qquad \frac{\Gamma \vdash M : A \times B}{\begin{array}{c}\Gamma \vdash \mathrm{fst}(M) : A \\ \Gamma \vdash \mathrm{snd}(M) : B\end{array}}\ \times\text{-}E \quad ;$$

- Exponential $\to$:

$$\frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A \to B \ \text{type}} \to\text{-}F \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \to\text{-}I$$

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M(N) : B} \to\text{-}E \quad ;$$

- Void 0:

$$\frac{}{\Gamma \vdash 0 \ \text{type}}\ 0\text{-}F \qquad \text{(no 0-I)} \qquad \frac{\Gamma \vdash M : 0}{\Gamma \vdash \mathrm{abort}(M) : A}\ 0\text{-}E \quad ;$$

- Coproduct $+$:

$$\frac{\Gamma \vdash A \ \text{type} \quad \Gamma \vdash B \ \text{type}}{\Gamma \vdash A + B \ \text{type}}\ +\text{-}F \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{inl}(M) : A + B}\ +\text{-}I_1 \qquad \frac{\Gamma \vdash N : B}{\Gamma \vdash \mathrm{inr}(N) : A + B}\ +\text{-}I_2 \quad ,$$

$$\frac{\Gamma, x : A \vdash N : C \quad \Gamma, y : B \vdash P : C}{\Gamma, z : A + B \vdash \mathrm{case}(x.N, y.P)(z) : C}\ +\text{-}E \quad .$$

Remark 2.2 (Categorical interpretation*)**.** The type theory given above corresponds to a category $\mathcal{C}$ that is both cartesian closed and cocartesian. We also assume that $\mathcal{C}$ has all morphisms from the terminal object 1 to the objects corresponding to the types of context variables (this is somewhat experimental).

- Types are the objects of $\mathcal{C}$;
- A context $\Gamma = x_1 : A_1, \ldots, x_n : A_n$ corresponds to the product of its types, together with a morphism from the terminal object naming the variables:

$$\Gamma := A_1 \times \cdots \times A_n, \qquad \langle x_1, \ldots, x_n\rangle : 1 \to \Gamma;$$

- Terms are morphisms in $\mathcal{C}$:

$$\Gamma \vdash M : A \quad \mapsto \quad [\![M]\!] : \Gamma \to A;$$

– Unit 1 is a terminal object; the 1-I rule corresponds to arrows *to* 1. Void 0 is an initial object; the 0-E rule corresponds to arrows *from* 0:

$$
\begin{tikzcd}
\Gamma \arrow[d, dashed, "\langle\rangle"] \\
1
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \arrow[r, "M"] \arrow[dr, "\mathrm{abort}(M)"'] & 0 \arrow[d, dashed, "\mathrm{abort}"] \\
& A
\end{tikzcd};
$$

– Product $\times$ and coproduct $+$ are the usual categorical limits and colimits (finite in our case):

$$
\begin{tikzcd}
& \Gamma \arrow[dl, "M"'] \arrow[dr, "N"] \arrow[d, dashed, "\langle M{,}N\rangle"] & \\
A & A \times B \arrow[l, "\mathrm{fst}"] \arrow[r, "\mathrm{snd}"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Gamma \arrow[dl, "\mathrm{fst}(M)"'] \arrow[dr, "\mathrm{snd}(M)"] \arrow[d, dashed, "M"] & \\
A & A \times B \arrow[l, "\mathrm{fst}"] \arrow[r, "\mathrm{snd}"'] & B
\end{tikzcd},
$$

$$
\begin{tikzcd}
& A + B \\
A \arrow[ur, "\mathrm{inl}"] & \Gamma \arrow[l, "M"] \arrow[u, "\mathrm{inl}(M)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A + B & \\
B \arrow[u, "\mathrm{inr}(N)"] & \Gamma \arrow[l, "N"] \arrow[ul, "\mathrm{inr}"']
\end{tikzcd},
$$

$$
\begin{tikzcd}
& \Gamma \times 1 \arrow[dl, "\Gamma\times x"'] \arrow[dr, "\Gamma\times y"] \arrow[d, dashed, "\Gamma\times z"] & \\
\Gamma \times A \arrow[dr, "N"'] & \Gamma \times (A + B) \arrow[d, dashed, "\mathrm{case}(x.N{,}y.P)(z)"] & \Gamma \times B \arrow[dl, "P"] \\
& C &
\end{tikzcd};
$$

– Exponential $\to$ is the right adjoint to product $\times$:

$$
\begin{tikzcd}
\Gamma \arrow[d, "\lambda x.M"'] \\
A \to B
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \times A \arrow[d, "M"'] & \Gamma \times 1 \arrow[l, "\mathrm{id}_\Gamma\times x"'] \arrow[dl] \\
B &
\end{tikzcd};
$$

$$
\begin{tikzcd}
\Gamma \arrow[d, "M"'] \\
A \to B
\end{tikzcd}
\quad
\begin{tikzcd}
\Gamma \arrow[d, "N"'] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma \arrow[d, "M(N)"] \\
B
\end{tikzcd}.
$$

Definition 2.3 (β and η equivalences)**.**

– Unit 1:

$$(\beta)\ \text{none} \qquad (\eta)\ \Gamma \vdash \langle\rangle \equiv M : 1;$$

– Product $\times$:

$$(\beta)\ \begin{array}{l} \Gamma \vdash \mathrm{fst}(\langle M, N\rangle) \equiv M : A \\ \Gamma \vdash \mathrm{snd}(\langle M, N\rangle) \equiv N : B \end{array} \qquad (\eta)\ \Gamma \vdash \langle \mathrm{fst}\, M, \mathrm{snd}\, M\rangle \equiv M : A \times B;$$

– Exponential $\to$:

$$(\beta)\ \Gamma \vdash (\lambda x.M)(N) \equiv [N/x]M : B \qquad (\eta)\ \Gamma \vdash (\lambda x.M(x)) \equiv M : A \to B;$$

– Zero 0:

$$(\beta)\ \text{none} \qquad (\eta)\ \Gamma, z : 0 \vdash R \equiv \text{abort}(M) : C;$$

– Coproduct $+$:

$$(\beta)\ \begin{array}{l} \text{case}(x.M; y.N)(\text{inl}(P)) \equiv [P/x]M : C \\ \text{case}(x.M; y.N)(\text{inr}(Q)) \equiv [Q/y]N : C \end{array},$$

$$\frac{\Gamma \vdash [\text{inl}(P)/z]R \equiv [P/x]M \quad \Gamma \vdash [\text{inr}(Q)/z]R \equiv [Q/y]N}{\Gamma, z : A + B \vdash R \equiv \text{case}(x.M; y.N)(z)}\ (\eta).$$