# Notes on Dependent Type Theory

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These notes are based on video recordings of R. Harper's lectures.

## 1 Introduction

Boolean algebra is associated with classical logic, Heyting algebra – with intuitionistic logic.

**Definition 1.1** (Boolean algebra)**.** A Boolean algebra can be defined as a complemented distributive lattice:

- Pre-order:
$$x \leq x, \qquad x \leq y \,\&\, y \leq z \to x \leq z;$$
- Has finite meets and joins:
$$x \leq 1, \qquad z \leq x \,\&\, z \leq y \to z \leq (x \wedge y), \qquad x \wedge y \leq x, \qquad x \wedge y \leq y;$$
$$0 \leq x, \qquad x \leq z \,\&\, y \leq z \to x \vee y \leq z, \qquad x \leq x \vee y, \qquad y \leq x \vee y;$$
- Has complements:
$$1 \leq \bar{x} \vee x, \qquad \bar{x} \wedge x \leq 0;$$
- Distributive:
$$x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z), \qquad x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z).$$

Additionally, the exponential is defined as

$$y^x := \bar{x} \vee y.$$

**Definition 1.2** (Heyting algebra)**.** A Heyting algebra is defined as a lattice with exponentials:

- Pre-order;
- Has finite meets and joins;
- Has exponentials:
$$y^x \wedge x \leq y, \qquad z \wedge x \leq y \to z \leq y^x.$$

**Exercise 1.3.**

1. "Yoneda lemma": $x \leq y$ iff $\forall z\ (z \leq x \to z \leq y)$;

2. Every Heyting algebra is distributive.

Quotes:

– "Boolean algebra (closed world) = Heyting algebra (open world) with complements";

– "Classical logic is a logic with complete information".

The definitions provide us with standard rules:

– Weakening:

$$x \leq x \vee y \qquad (x \wedge y \leq x);$$

– Contraction:

$$x \leq x \wedge x;$$

– Exchange:

$$x \wedge y \equiv y \wedge x.$$

**Relation to classical logic**:

– Sequent $\Gamma \vdash A$, where $\Gamma = A_1, \ldots, A_n$, corresponds to:

$$A_1 \wedge \cdots \wedge A_n \leq A;$$

– Rules for $\wedge$:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B} \quad \frac{}{\Gamma, A \wedge B \vdash A} \quad \frac{}{\Gamma, A \wedge B \vdash B}\ ;$$

– ...and so on.

**Definition 1.4** (Lindenbaum algebra)**.** A Lindenbaum algebra is defined as the algebra of equivalence classes of a given theory:

- $[A] = \{B \mid B \equiv A\}$;
- $[A] \wedge [B] := [A \wedge B]$;

– ... and so on.

**Theorem 1.5** (Soundness and Completeness for Intuitionistic Propositional Logic)**.** Let $\Gamma$ be a context and $A$ a formula. Then

$$\Gamma \vdash A \quad \text{iff} \quad \forall H \left( [\![\Gamma]\!]_H \leq [\![A]\!]_H \right),$$

where $H$ ranges over all Heyting algebras, and $[\![-]\!]_H$ denotes the interpretation of formulas as elements of $H$ under a valuation of propositional variables.