Functions of several variables

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This notes are based on Mathematical Analysis by V.A.Zorich.

Contents

1 $\mathbb{R}^m$ space and the most important classes of its subsets

1) $\mathbb{R}^m$, Distance

Definition 1.1 ($\mathbb{R}^m$)**.**

$$\mathbb{R}^m := \{(x^1, ..., x^m) \mid x^i \in \mathbb{R}\}.$$

Definition 1.2 (Distance)**.**

$$d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} :$$

- $d(x_1, x_2) \geq 0$;
- $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$;
- $d(x_1, x_2) = d(x_2, x_1)$;
- $d(x_1, x_3) \leq d(x_1, x_2) + d(x_1, x_3)$.

We define d as:

$$d(x_1, x_2) = \sqrt{\sum_{i=1}^{m} (x_1^i - x_2^i)^2}.$$

Property:

$$|x_1^i - x_2^i| \leq d(x_1, x_2) \leq \sqrt{m} \max_{1 \leq i \leq m} |x_1^i - x_2^i|.$$

2) Open and closed sets in $\mathbb{R}^m$

Definition 1.3 (Open ball)**.**

$$B(a, \delta) = \{x \in \mathbb{R}^m \mid d(a, x) < \delta\}.$$

(also δ-heighborhood)

Definition 1.4 (Open set, closed set)**.**

- G is open if every its point has a neighborhood $\subseteq G$;
- F is closed, if its complement is open.

Proposition 1.5.

- Finite $\bigcap$ of open sets is open, finite $\bigcup$ of closed sets is closed;
- Any $\bigcup$ of open sets is open, any $\bigcap$ of closed set is closed.

Definition 1.6 (Neighborhood; interior, exterior and boundary point; limit point; closure)**.** (standard definitions)

Proposition 1.7. *F is closed $\Leftrightarrow F = \bar{F}$.*

2 Vector structure in $\mathbb{R}^m$

Definition 2.1 ($\mathbb{R}^m$ as a vector space)**.**

- $x = (x^1, ..., x^m) \in \mathbb{R}^m$;
- λx, $x_1 + x_2$ work componentwise;
- Basis $e_i := (0_{(1)}, ..., 0_{(i-1)}, 1_{(i)}, 0_{(i+1)}, ..., 0_{(m)})$, thus

$$x = \sum_{i \le m} x^i e_i =: x^i e_i.$$

Definition 2.2 (Linear maps $L : \mathbb{R}^m \to \mathbb{R}^n$)**.**

- Linear map L: $L(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 L(x_1) + \lambda_2 L(x_2)$. Thus, on the elements of basis:

$$L(e_i) = a_i^j \tilde{e}_j \qquad (i = 1, ..., m),$$

$$L(h) = L(h^i e_i) = h^i a_i^j \tilde{e}_j, \text{ in coordinates } L(h) = (a_i^1 h^i, ..., a_i^n h^i).$$

- Matrix form: $L(h) = Ah$,

$$L(h) = \begin{pmatrix} L_1(h) \\ L_2(h) \\ \vdots \\ L_m(h) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix};$$

- Let $L_1 : \mathbb{R}^m \to \mathbb{R}^n$, $L_2 : \mathbb{R}^n \to \mathbb{R}^k$, then $(L_2 \circ L_1)(h) = A_2 A_1 h$.

Definition 2.3 (Norm in $\mathbb{R}^m$)**.** Norm is a function $\|\| : \mathbb{R}^m \to R$ defined as:

$$\|x\| = \sqrt{(x^1)^2 + ... + (x^m)^2}.$$

Norm properties:

- $\|x\| \ge 0$,
- $\|x\| = 0 \Leftrightarrow x = 0$,
- $\|\lambda x\| = |\lambda| \cdot \|x\|$,
- $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$ (and $\|\sum_i x_i\| \le \sum_i \|x_i\|$).

Definition 2.4 (Distance in $\mathbb{R}^m$)**.** Distance is a function $d : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined as:

$$d(x_1, x_2) = \|x_2 - x_1\|.$$

Definition 2.5 ($\mathbb{R}^m$ as a Euclidean space)**.** Scalar product on vector space V is a positively defined bilinear function

$$\langle x, y\rangle : V \times V \to \mathbb{R}.$$

This leads to its presentation in the following form:

$$\langle x, y\rangle = \langle e_i, e_j\rangle x^i y^j =: g_{ij} x^i y^j.$$

$\mathbb{R}^m$ is just a special case here.

Definition 2.6 (Orthogonality)**.**

- $x \perp y :\Leftrightarrow \langle x, y\rangle = 0$;
- Basis is called orthonormal if $\langle e_i, e_j\rangle = \delta_{ij}$. In this case:

$$\langle x, y\rangle = \delta_{ij} x^i y^j = \sum_{i,j} x^i \cdot y^j.$$

Definition 2.7 (Angle between vectors)**.** It is known that

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle = \|x\|^2\|y\|^2.$$

Thus, for all $x, y \in \mathbb{R}^m$ there exists an angle $\varphi \in [0, \pi]$ such that:

$$\langle x, y\rangle = \|x\|\|y\| \cos\varphi.$$

Fact: Any $L : \mathbb{R}^m \to \mathbb{R}$ has the form

$$L(x) = \langle \xi, x\rangle,$$

where ξ is unique for each L.