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Author(s): Joseph B. Kadane, James M. Dickey, Robert L. Winkler, Wayne S. Smith and Stephen C. Peters
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Interactive Elicitation of Opinion for a Normal Linear Model

JOSEPH B. KADANE, JAMES M. DICKEY, ROBERT L. WINKLER, WAYNE S. SMITH, and STEPHEN C. PETERS *

This article describes the mathematical theory underlying an interactive computer program for eliciting the hyperparameters of a subjective conjugate distribution for the multiple linear regression model with the usual normal error structure. Although the methods are heuristic, they are shown to produce hyperparameter estimates satisfying the constraints satisfied by the hyperparameters themselves. An application is given to the problem of predicting the time to fatigue failure of an asphalt-concrete road as a function of several design variables concerning the road.

KEY WORDS: Bayesian statistics; Conjugate families of opinion; Elicitation; Normal linear model; Interactive computing; Subjective probability.

1. INTRODUCTION

This article describes a practical method for the elicitation of subjective distributions useful for inference in the multiple linear regression model with the usual normal error structure. We choose this task in the belief that if subjective Bayesian methods are to be used, then technology for implementation has to be made available to empirical workers. We focus on the multiple linear regression model because it is widely used and because it offers a high dimensional parameter space. The latter is important because techniques for Bayesian inference that are limited to one or two parameters seem unlikely to have much influence on how statistics is actually practiced.

We also believe that a need exists for the quantification of expert knowledge. Subjective probability models permit experts to study their own uncertainties and to communicate them for interpersonal comparisons and for decision making. This need is especially acute in multidimensional contexts, where complexity often interferes with clear thinking. Formal statistical data may not even play a role in such applications.

To make the problem tractable, we impose a structure on the distributions assessed (the conjugate family), and this structure has parameters. To distinguish these latter parameters from those in the normal linear model, we call the parameters of the family of opinions *hyperparameters*, following Lindley (1971). The parameterized family of opinions has a status analogous to the usual conception of the sampling model. The former models the judgments of the experimenter, while the latter models the behavior of the experiment. The family can be a poor model of opinions, in which case discrepancies are likely to show up in the experimenter's answers to elicitation questions, much as the sampling model can succumb to traditional diagnostic checks.

Our attitude toward the prior distribution—that it represents the judgments of the experimenter—should be contrasted with the empirical Bayes attitude, which supposes a physical process generating the sampling parameters by an unknown probability distribution called a "prior" distribution. Rather than looking to the opinions of the experimenter for information about the hyperparameters, an empirical Bayes analysis looks to the sample data itself. Of course, empirical Bayes inference requires an additional symmetry assumption, such as the assumption that a vector of unobservable sampling parameters has the character of a repeated sample from a distribution. We do not require such a symmetry assumption, and we look to the expressed opinions of the experimenter as data to estimate the hyperparameters.

There are several possible strategies for asking the experimenter his opinions. One could ask for a joint distribution of the coefficients of the linear model, and various alternatives are available for eliciting such a distribution (e.g., see Zellner 1972). We have not pursued this approach because we believe that despite long familiarity with the linear model, its coefficients are hard to think about directly. Without touching the philosophical issue of the sense in which parameters may be said to "exist," we acknowledge that even experienced observers have

* Joseph B. Kadane is Professor and Head, Department of Statistics, Carnegie-Mellon University, Pittsburgh, PA 15213. Requests for reprints should be directed to him. James M. Dickey is Professor of Statistics, Department of Mathematics and Statistics, State University of New York at Albany, 1400 Washington Avenue, Albany, NY 12222. Robert L. Winkler is Professor of Quantitative Business Analysis, Graduate School of Business, Indiana University, Bloomington, IN 47401. Wayne S. Smith is in private business at 6961 Sunburst Way, Citrus Heights, CA 95610. Stephen C. Peters is a graduate student, Department of Statistics, Stanford University, Stanford, CA 94305. This research is supported in part by the National Science Foundation under Grant No. SOC-7727575 AOI, and in part by the National Cooperative Highway Research Program. The authors are grateful to James Chen, Morris DeGroot, Arthur Dempster, George Duncan, William Eddy, Prem Goel, Richard Mickelsen, Melvin Novick, and David Price for their helpful comments. We are particularly grateful for our interactions with James Chen, who commented on an early draft and discovered and helped correct an error in the constant in formula (2.3).

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trouble answering questions on their beliefs about them. These matters are discussed at greater length in Kadane (1980) and Winkler (1980).

Instead, we propose to ask questions about the predictive distribution of the dependent variable conditional on the independent variables, that is, the distribution after averaging out the sampling parameters according to their prior distribution. We make this choice because we believe that it accords with the way an experimenter might naturally think about a regression. While the experimenter may not be able to give us an opinion about a coefficient directly, by eliciting facts about his or her predictive distributions at various possible settings of the independent variables we can infer from the parameterized form of the predictive model what he or she must believe. This choice has the advantage that such a predictive distribution can be postulated without direct reference to the sampling model or prior model, and even posterior predictive inference from data can proceed as straightforward conditioning within the joint predictive distribution of data and future values. It also has the advantage of our being able to ask questions in the form of real bets (DeFinetti 1974). This latter feature is paramount in our thinking.

Considering what forms of questions to ask, one could elicit moments of the predictive distribution, but we believe that moments are particularly hard to assess. For this reason, we choose to elicit quantiles of the predictive distribution. Our hope is that quantiles are less subject to erratic behavior, easier for the experimenter to think about, and consequently easier for us to elicit. Of course, moments can be computed from the assessed prior distributions if they are desired.

At many stages our choices are dictated by what we think is the convenience of the experimenter. We want to ask convenient questions about the experimenter's predictive distribution at various values of the independent variables in order to estimate the hyperparameters. But even with convenient questions, we often find ourselves and others giving answers that are later regretted. Any reasonable program for elicitation, we think, must point out answers that appear "out of line," not to coerce the experimenter, but to allow reconsideration. Because of this and the need to take account of the answers to previous questions in formulating new questions, we find that interactive computing is very convenient to our task.

The methods of fitting used later to estimate the hyperparameters from the elicited responses are crude. To do a more sophisticated analysis, a model would have to be developed of how errors in elicitation occur. We hope that such a model can be found, but in order to do that we first need some experience with actual elicitation in a multivariate context (see Dickey 1979 for work along these lines). Consequently, we hope that although other statisticians may be disturbed by the nontheoretical way we have designed our methods, they may be in sufficient sympathy with our goals and philosophy to find them of interest nonetheless.

These ideas have been implemented in a computer program on the TROLL experimental system, operated on a computer at Massachusetts Institute of Technology and available nationwide on the Telenet system. The details of this implementation are described in Section 4. A second version of these ideas, with different choices made for implementation (not described here), is included in the 1980 CADA monitor. A concrete example of the use of the TROLL programs is given in the Appendix. A second area of application is described in Kadane and Sedransk (1979).

This article does not deal with the question of which circumstances favor formal analysis of decision problems and which do not. By making formal methods of analysis available for a canonical problem having many parameters, however, we hope we have made a contribution to that wider discussion.

Elicitation has been the subject of considerable speculation and some practice by statisticians and others; for example, see the bibliographies and papers of Savage (1971), Hogarth (1975), Lindley, Tversky, and Brown (1979), Spetzler and Stael Von Holstein (1975), Tversky (1974), Tversky and Kahneman (1974), and Winkler (1967).

2. NOTATION

Suppose that the dependent variable is denoted by y and the independent variable vector by $\mathbf{x}$, where $\mathbf{x}$ is r dimensional. Then the normal linear model specifies that y, conditional on $\mathbf{x}$, $\boldsymbol{\beta}$, and σ, has a normal distribution with mean $\mathbf{x}'\boldsymbol{\beta}$ and variance σ^2. The distribution of a set $(y_1, \ldots, y_m)$ conditional on $(X, \boldsymbol{\beta}, \sigma)$ is multivariate normal with mean $X\boldsymbol{\beta}$ and variance matrix $\sigma^2 I$, where X is the $m \times r$ matrix with rows $\mathbf{x}'_1, \ldots, \mathbf{x}'_m$. (Boldface symbols represent column vectors, and prime indicates matrix transpose.)

The family of prior distributions studied in this article is the family of natural conjugate priors, not because we think they necessarily represent the opinions of experimenters well—we know lamentably little about that yet—but because they are familiar and easy to integrate. We hope that the family of prior distributions will be extended in subsequent work. The conjugate prior on $(\boldsymbol{\beta}, \sigma)$ can be expressed as the product of a conditional and a marginal distribution. Let σ^2 be equal to $w\delta$ times the reciprocal of a chi-squared random variable with δ degrees of freedom. Define $n = \delta + r$. Given σ, $\boldsymbol{\beta}$ has a normal distribution with some mean $\mathbf{b}$ and some variance matrix $(\sigma^2/n)R^{-1}$.

Before discussing the predictive distribution, we introduce a language for the t distribution. When the degrees of freedom parameter of the t distribution is less than or equal to two, no variance exists, and when it is less than or equal to one, no mean exists. Since we do not wish to limit unnecessarily the possible opinions expressed by the experimenter, and since small degrees of freedom represent less information in a certain sense

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(which many investigators find appealing), we seek a parameterization for the t distribution that does not depend on the existence of moments. The following parameterization is also used by Box and Tiao (1973, p. 117), although our treatment of it differs from theirs.

The standard multivariate t vector with δ degrees of freedom $\mathbf{t}_\delta$ is distributed as a standard multivariate normal vector divided by the square root of the product of δ^{-1} and an independent chi-squared random variable with δ degrees of freedom. Then the general multivariate t family with δ degrees of freedom has generic random vector $\mathbf{z} = \mathbf{a} + B\mathbf{t}_\delta$ where $\mathbf{a}$ and B are constant and conformable (Press 1972, p. 127, (6.2.4)). We say that $C(\mathbf{z}) = \mathbf{a}$ is the *center* and that $S(\mathbf{z}) = BB'$ is the *spread*. When $\delta > 1$, so that the mean exists, $E(\mathbf{z}) = C(\mathbf{z})$, and when $\delta > 2$, so that the second moment exists,

$$\operatorname{var}(\mathbf{z}) = [\delta/(\delta - 2)]S(\mathbf{z}) \ . \tag{2.1}$$

Additionally, we speak of cospreads (CoS) to mean an off-diagonal element of a spread matrix.

Marginal and conditional distributions derived from the multivariate t also take the form of a multivariate t distribution. If $\mathbf{z}' = (\mathbf{z}'_1, \mathbf{z}'_2)$ where $\mathbf{z}$, $\mathbf{z}_1$, $\mathbf{z}_2$ have dimensions m, m_1, and $m_2 = m - m_1$, respectively, the marginal distribution of $\mathbf{z}_1$ is multivariate t with δ degrees of freedom, $C(\mathbf{z}_1) = \mathbf{a}_1$, and $S(\mathbf{z}_1) = S_{11}$, where $\mathbf{a}' = (\mathbf{a}'_1, \mathbf{a}'_2)$ and $S(\mathbf{z}) = (S_{ij})$, partitioned to conform to $\mathbf{z}$. The conditional distribution of $\mathbf{z}_2$ given $\mathbf{z}_1$ is multivariate t with $\delta + m_1$ degrees of freedom,

$$C(\mathbf{z}_2|\mathbf{z}_1) = \mathbf{a}_2 + S_{21}S_{11}^{-1}(\mathbf{z}_1 - \mathbf{a}_1) \ , \tag{2.2}$$

and

$$S(\mathbf{z}_2|\mathbf{z}_1) = (\delta/(\delta+m_1))\{1+\delta^{-1}(\mathbf{z}_1-\mathbf{a}_1)'S_{11}^{-1}(\mathbf{z}_1-\mathbf{a}_1)\} \cdot (S_{22}-S_{21}S_{11}^{-1}S_{12}) \ . \tag{2.3}$$

From the form of the conditional spread it is easy to observe that only in the normal case ($\delta = \infty$) does the conditional spread not depend on z_1.

Note further that $C(\mathbf{z}_2|\mathbf{z}_1)$ is linear in $\mathbf{z}_1$ and hence is itself a multivariate t-distributed random vector, with degrees of freedom δ, center

$$C[C(\mathbf{z}_2|\mathbf{z}_1)] = a_2 = C(\mathbf{z}_2) \ , \tag{2.4}$$

and spread $S_{21}S_{11}^{-1}S_{12}$.

The predictive distribution for a set $\mathbf{y} = (y_1, \ldots, y_m)$ conditional on X, $\mathbf{b}$, δ, w, and R is a multivariate t distribution with δ degrees of freedom and center and spread, respectively,

$$C(\mathbf{y}|X) = X\mathbf{b} \quad \text{and} \quad S(\mathbf{y}|X) = w(n^{-1}XR^{-1}X' + I) \ . \tag{2.5}$$

Thus the predictive distribution of some future observations conditional on some others is again a multivariate t distribution, with center and spread derived by applying (2.2) and (2.3) to the special form (2.5).

For a better understanding of the hyperparameters, we note that the distribution of $\mathbf{y}$ given X and $\boldsymbol{\beta}$, and of $\boldsymbol{\beta}$, are both t distributions with δ degrees of freedom for which

$$C(\mathbf{y}|X, \boldsymbol{\beta}) = X\boldsymbol{\beta}, \quad S(\mathbf{y}|X, \boldsymbol{\beta}) = wI \tag{2.6}$$

and

$$C(\boldsymbol{\beta}) = \mathbf{b}, \quad S(\boldsymbol{\beta}) = (w/n)R^{-1} \ . \tag{2.7}$$

After finding the experimenter's prior hyperparameters $\boldsymbol{\beta}$, R, w, and δ, we can combine them in the usual manner with the sufficient statistics from a data set to form the posterior distribution of $\boldsymbol{\beta}$ and σ and/or the posterior predictive distribution of future data. For details, see Raiffa and Schlaifer (1961). As stated earlier, however, it is sometimes useful to elicit a prior when no data will be available.

3. ESTIMATION OF HYPERPARAMETERS

The hyperparameters to be elicited divide naturally into three groups: the central tendency hyperparameters $\mathbf{b}$, the degrees of freedom parameter δ, and the hyperparameters w and R. It is proper to consider these facts about the judgments of the experimenter as being "estimated" by our programs. First, we consider the central tendency hyperparameters.

3.1 Estimation of b

Suppose that for each of a sequence $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$ of $m \geq r$ values of the independent variable vector, the experimenter gives his or her median value of y, the dependent variable. The independent variable values are controlled so that $X'X$ is well conditioned. If the elicitation errors are independent of the value being elicited, of each other, and have zero mean, then the median values will tend to be centered at $(\mathbf{x}'_1\mathbf{b}, \ldots, \mathbf{x}'_m\mathbf{b})$, with a dispersion depending on how elicited quantiles vary relative to the quantiles of the true distribution. If the elicitation errors are additionally assumed to be normal with equal variances, this would justify unweighted least squares. Thus we form the estimate

$$\hat{\mathbf{b}} = (X'X)^{-1}X'\mathbf{y}_{.50} \ , \tag{3.1}$$

where $\mathbf{y}_{.50}$ is the vector of elicited medians. If the variances are not equal, some efficiency may be lost but consistency is preserved. It seems likely that the tails of the distribution of $\mathbf{y}_{.50}$ will be higher than that of a normal, suggesting the use of a more sophisticated estimator than (3.1), namely, some type of robust regression estimator.

Note that the vector of residuals from the least squares fit,

$$\mathbf{d} \equiv \mathbf{y}_{.50} - X\hat{\mathbf{b}} = [I - X(X'X)^{-1}X']\mathbf{y}_{.50} \ , \tag{3.2}$$

gives a measure of how "surprising" any given component answer $y_{.50}$ is. Components of $\mathbf{d}$ large in absolute value relative to the others can be used to trigger a notice that the experimenter's answer on that elicitation appears not to go as well with his or her other answers and the model as his or her other answers do. If $\mathbf{y}_{.50}$ had a multivariate normal or t distribution centered at $X\mathbf{b}$ with uncorrelated

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components, **d** would have a degenerate distribution with zero mean and variance-covariance matrix proportional to $[I - X(X'X)^{-1}X']$. Note that the off-diagonal elements of $I - X(X'X)^{-1}X'$ are not necessarily zero.

3.2 Estimation of δ

Next, we turn to the degrees of freedom hyperparameter δ. This parameter is likely to be hard to estimate well, but on the other hand, many inferences made from a posterior distribution do not radically depend on δ either.

At each point $\mathbf{x}_j (j = 1, \ldots, m)$ of the sequence of vectors of independent variables, the experimenter is asked first for a median, then for the median of the conditional distribution bounded below by the first median (the .75 quantile), next for the median of the conditional distribution bounded below by the second median (the .875 quantile), and finally for the similarly determined .9375 quantile. This has the advantage that at each state only medians are requested and the possible disadvantage that the experimenter must think conditionally, thus raising the possibility of an "anchoring" effect (Tversky and Kahneman 1974 and Spetzler and Stael Von Holstein 1975).

We compute for each point $\mathbf{x}$ for which we have assessments at $y_{.9375}$, $y_{.75}$, and $y_{.50}$ the ratio

$$a(\mathbf{x}) = (y_{.9375} - y_{.50})/(y_{.75} - y_{.50}) \ . \qquad (3.3)$$

By subtracting $y_{.50}$ from the numerator and denominator, both are independent of the center. By division, the quotient is independent of the spread. Consequently, $a(\mathbf{x})$ depends on δ only. From Table 1 it is easy to observe that $a(\mathbf{x})$ is a monotone function of δ, and, if the model were perfect and the experimenter infallible, we would have $a(\mathbf{x}) \geq 2.27$. To help the experimenter satisfy this criterion, we work with $a^*(\mathbf{x}) = \max(a(\mathbf{x}), 2.27)$. Under the same condition of a perfect model and an infallible experimenter, $a(\mathbf{x})$, and hence $a^*(\mathbf{x})$, would not depend on $\mathbf{x}$. To allow for human and model fallibility, we compute

$$\bar{a}^* = \sum a^*(\mathbf{x})/m \ , \qquad (3.4)$$

where the summation is over the m points $\mathbf{x}$ at which the assessments are available at all three quantiles. Now δ is found from Table 1. A similar method is now used for the degrees of freedom of the univariate t distribution in the CADA monitor (Novick, Isaacs, and DeKeyrel 1977).

Once again, peculiar values of $a^*(\mathbf{x})$ are easy to spot, as they are far from the average value $\bar{a}^*$. Thus the computer signals to the experimenter the existence of widely discrepant values and allows reassessment if the experimenter wishes. Of course, central statistics other than the average are of interest and can be used instead of $\bar{a}^*$.

Extreme values for δ are not uncommon in our meager experience to date. We are uncertain what to do about this problem. Part of the problem, at least, is in the difficulty of the assessment of $y_{.9375}$, which corresponds to a 1 in 16 bet. Yet the information about δ is carried in the tails of the predictive distribution (e.g., we have already seen that the median is invariant with respect to δ), and hence an assessment in the tail seems forced on us. This is certainly a topic for further research.

1. The .75 Fractile, the .9375 Fractile, and Their Ratio for the Standard t Distribution With Various Degrees of Freedom

Degrees of Freedom δ	t_δ (.75)	t_δ (.9375)	t_δ (.9375)/t_δ (.75)
.3	3.027	311.1	102.8
.4	1.975	65.1	32.95
.5	1.554	26.3	16.93
.6	1.364	14.7	11.04
.7	1.200	9.87	8.22
.8	1.111	7.39	6.65
.9	1.048	5.95	5.68
1.0	1.000	5.03	5.03
1.2	.934	3.95	4.23
1.4	.889	3.36	3.77
1.6	.858	2.99	3.48
1.8	.835	2.74	3.28
2.0	.816	2.56	3.13
2.5	.785	2.27	2.90
3.0	.765	2.11	2.76
4	.741	1.94	2.62
5	.727	1.84	2.53
6	.718	1.78	2.48
7	.711	1.74	2.45
8	.706	1.71	2.42
9	.703	1.69	2.40
10	.7	1.67	2.39
12	.695	1.65	2.37
14	.692	1.63	2.36
16	.69	1.62	2.35
18	.688	1.61	2.34
20	.687	1.60	2.33
30	.683	1.58	2.31
40	.681	1.57	2.31
60	.679	1.56	2.30
∞	.674	1.53	2.27

Source: Tables for the Distribution and Density Functions of t-Distribution, ed. N.V. Smirnov, trans. by P. Basu, New York: Pergamon Press (1961) and inverse t, for fractional degrees of freedom calculated using the inverse incomplete beta function of IMSL.

One possibility for permitting more flexible elicitations for δ is to ask questions of the type, What are the odds against observing a value of the dependent variable larger than y^* if the independent variables take the values $\mathbf{x}$? We would have to pay for this greater flexibility by generalizing Table 1, but this route may be worth exploring nonetheless.

An option for direct assessment for δ is also made available in our programs.

3.3 Estimation of *w* and *R* Using the Conditioning Method

To consider the elicitation of the hyperparameters w and R, we first give a method for finding the spread of a t-distributed variable y from its quantiles. Our method is to use the relation

$$S(y|\mathbf{x}) = (y_{.75} - y_{.50})^2/t_{75}^2 \ , \qquad (3.5)$$

where $t_{.75}$ is the 75th percentile of a standard t distribution with δ degrees of freedom.

In the conditional method, an "as if" data set is built

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up by the computer "declaring" certain data (y's) and asking the experimenter how these supposed data change his or her opinions. By asking for conditional medians (which we take as estimates of conditional centers), and occasionally for conditional upper quartiles (which we convert to conditional spreads by using (3.5)), we deduce the structure of the experimenter's prior.

An important design criterion for us in this effort is never to ask the experimenter to forget such "data" once asked to condition his or her inferences on it. While this imposes some analytical difficulties, we believe that the alternative, asking the experimenter to forget, would impose too great a psychological burden.

The organization of Section 3.3 is as follows. First, we introduce some necessary notation. Second, we describe how these elicitations can be used to find the matrix $U = w(n^{-1}XR^{-1}X' + I)$, which is the spread-cospread matrix of $(y_1, \ldots, y_m)$ from (2.5), and show that the resulting elicited matrix is positive definite. Then we describe how to use the elicitations to estimate w, review the elicitations required by the method, and finally show how to estimate R using our estimates for U and w.

3.3.1 The conditional elicitations. In this section, any variable y with an i subscript will stand for the value of y taken at $\mathbf{x}_i$, $i = 1, \ldots, m$ $(m \geq r)$. Additionally, the superscript 0 is taken to mean that this variable satisfies a condition, so that y_i indicates a random variable, y_i^0 a constant.

We plan to conduct certain elicitations unconditionally, then to condition on y_1^0 and conduct certain additional elicitations, then to condition on y_1^0 and y_2^0 and again conduct certain further elicitations, and so on. The values of $y_1^0, y_2^0, \ldots$ are general in our work, although they must satisfy $y_1^0 \neq C(y_1)$ and

$$y_i^0 \neq C(y_i \mid y_1^0, \ldots, y_{i-1}^0), \quad i = 2, \ldots, m \ , \tag{3.6}$$

so that the new data point is not "exactly what you would have thought."

3.3.2 Use of the elicited values to find the matrix U. Our approach is to exploit the formula for the conditional center (2.2) to successively find the values of the spread-cospread matrix U_i of $(y_1, \ldots, y_i)$. We do this in such a way that the resulting matrix U_m is sure to be positive definite.

We begin with

$$U_1 = S(y_1) > 0 \ , \tag{3.7}$$

as found from the elicitations done for Section 3.2 using (3.5). Suppose now that U_i is positive definite, and define the vector $\mathbf{g}_{i+1}$ of dimension i such that $U_i\mathbf{g}_{i+1}$ is the vector of cospreads of y_{i+1} with $(y_1, \ldots, y_i)$. We plan to form the matrix

$$U_{i+1} = \begin{bmatrix} U_i & U_i\mathbf{g}_{i+1} \\ \mathbf{g}'_{i+1}U_i & S(y_{i+1}) \end{bmatrix}. \tag{3.8}$$

Our tasks are to show how to use the elicited quantities to find $\mathbf{g}_{i+1}$ and $S(y_{i+1})$ and to show that U_{i+1} is positive definite if U_i is.

First, we have

$$C(y_{i+1} \mid y_1^0, \ldots, y_i^0) - C(y_{i+1}) = \begin{pmatrix} y_1^0 - C(y_1) \\ \vdots \\ y_i^0 - C(y_i) \end{pmatrix}' \mathbf{g}_{i+1} \ , \tag{3.9}$$

using (2.2). Furthermore, if we take the center of both sides of (3.9) conditional on $y_1^0, \ldots, y_k^0 (k \leq i)$, using (2.4), we have

$$C(y_{i+1} \mid y_1^0, \ldots, y_k^0) - C(y_{i+1}) = \begin{bmatrix} y_1^0 - C(y_1) \\ \vdots \\ y_k^0 - C(y_k) \\ C(y_{k+1} \mid y_1^0, \ldots, y_k^0) - C(y_{k+1}) \\ \vdots \\ C(y_i \mid y_1^0, \ldots, y_k^0) - C(y_i) \end{bmatrix}' \mathbf{g}_{i+1} \ , \tag{3.10}$$

for $k = 1, 2, \ldots, i$. This yields a system of i linear equations for $\mathbf{g}_{i+1}$. Let

$$\mathbf{h}_{i+1} = \begin{bmatrix} C(y_{i+1} \mid y_1^0) - C(y_{i+1}) \\ C(y_{i+1} \mid y_1^0, y_2^0) - C(y_{i+1}) \\ \vdots \\ C(y_{i+1} \mid y_1^0, y_2^0, \ldots, y_i^0) - C(y_{i+1}) \end{bmatrix}$$

be the vector of dimension i of the left-hand sides of these equations. Also, let

$$M_{i+1} = \begin{bmatrix} y_1^0 - C(y_1) & C(y_2 \mid y_1^0) - C(y_2) & \cdots & C(y_i \mid y_1^0) - C(y_i) \\ y_1^0 - C(y_1) & y_2^0 - C(y_2) & \cdots & C(y_i \mid y_1^0, y_2^0) - C(y_i) \\ \vdots & & & \\ y_1^0 - C(y_1) & y_2^0 - C(y_2) & \cdots & y_i^0 - C(y_i) \end{bmatrix}$$

Then the system of equations (3.10) can be written

$$\mathbf{h}_{i+1} = M_{i+1}\mathbf{g}_{i+1} \ . \tag{3.10a}$$

Multiplication on the left by the matrix

$$K_{i+1} = \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ -1 & 1 & 0 & \cdots & & 0 \\ 0 & -1 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & \cdots & & & -1 & 1 \end{bmatrix}$$

puts M_{i+1} into upper triangular form. The diagonal of this upper triangular matrix is $y_1^0 - C(y_1)$, $y_2^0 - C(y_2 \mid y_1^0)$, $\ldots$, $y_i^0 - C(y_i \mid y_1^0 \ldots y_{i-1}^0)$. The condition that none of these is zero, which is necessary and sufficient for nonsingularity of M_{i+1}, is precisely (3.6). This triangular expression for M_{i+1} also has computational advantages in solving the linear system (3.10a). Thus (3.10) can be solved for $\mathbf{g}_{i+1}$.

Furthermore, the spread $S(y_{i+1} \mid y_1^0, \ldots, y_i^0)$ is now used as follows to find the spread of y_{i+1}, using (2.3):

$$\begin{aligned} &S(y_{i+1} \mid y_1^0, \ldots, y_i^0)(1 + i/\delta) \\ &\quad \cdot \{1 + \delta^{-1}(y_1^0 - C(y_1), \ldots, y_i^0 - C(y_i))' \\ &\quad \cdot U_i^{-1}(y_1^0 - C(y_1), \ldots, y_i^0 - C(y_i))\}^{-1} \\ &\quad\quad + \mathbf{g}'_{i+1}U_i\mathbf{g}_{i+1} = S(y_{i+1}) \ . \end{aligned} \tag{3.11}$$

Now we have found ways to elicit the two quantities $\mathbf{g}_{i+1}$ and $S(y_{i+1})$ needed for form (3.8). Notice that while we already have an estimate for $S(y_{i+1})$ from the unconditional elicitations found previously, we choose, in

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(3.11), not to use it, but instead to use something more complicated. The reason for this is to ensure positive definiteness of U_{i+1}, as will now become apparent.

The following elementary result is used in relation to (3.8): Let

$$A_{p+1} = \begin{bmatrix} A_p & \mathbf{b} \\ \mathbf{b}' & c \end{bmatrix},$$

where A_{p+1} is $(p + 1) \times (p + 1)$ and A_p is $p \times p$ for $p = 1, 2, 3, \ldots$. Then A_{p+1} is positive definite iff A_p is positive definite and $c - \mathbf{b}'A_p^{-1}\mathbf{b} > 0$. (For proof, see Bellman 1970, p. 75 and Anderson 1958, p. 103.)

Thus we can check the positive definiteness of U_{i+1} provided

$$S(y_{i+1}) - \mathbf{g}'_{i+1}U_i\mathbf{g}_{i+1} > 0 \ , \tag{3.12}$$

which is assured by the construction (3.11). How well conditioned U_{i+1} is depends on the magnitude of first summand on the right-hand side of (3.11).

Had we used the alternate unconditional method of assessing $S(y_{i+1})$ instead of (3.11), the inequality in (3.12), and hence the positive definiteness of U_{i+1}, would not be assured. However, any estimate satisfying (3.12) would suffice: for example, one could use the larger of the two estimates for $S(y_{i+1})$, or one could use (3.11) if (3.11) is the larger, and otherwise the average, and so on. For simplicity, we use (3.11) alone. However, a comparison of the two estimates of $S(y_{i+1})$ provides another, and very interesting, check on the model or the assessor.

In this way U_i is built up to U_{i+1}, and the process is repeated until U_m is reached. By induction, then, $U_m = U$ is positive definite.

Note that condition (3.6) can be checked before y_i^0 is chosen, by observing that for $i = 1$,

$$C(y_1) = x'_1b \ , \tag{3.13}$$

and for $i > 1$,

$$C(y_i | y_1^0, \ldots, y_{i-1}^0) = C(y_i) + \begin{pmatrix} y_1^0 - C(y_1) \\ y_{i-1}^0 - C(y_{i-1}) \end{pmatrix}' \mathbf{g}_i \ . \tag{3.14}$$

3.3.3 Estimation of w. To estimate w, note the conditional on $y_1^0, \ldots, y_{i-1}^0$, the variables y_i and y_i^*, two "independent" realizations at $\mathbf{x} = \mathbf{x}_i$, have a joint bivariate t distribution, with degrees of freedom $\delta + (i - 1)$, equal centers, with

$$C(y_i | y_1^0, \ldots, y_{i-1}^0) = \mathbf{x}'_i\mathbf{b} + (y_1^0 - \mathbf{x}'_1\mathbf{b}, \ldots, y_{i-1}^0 - \mathbf{x}'_{i-1}\mathbf{b})\mathbf{g}_i \ , \tag{3.15}$$

and spread matrix

$$\begin{aligned} S(y_i, y_i^* | y_1^0, \ldots, y_{i-1}^0) &= (\delta/(\delta + i - 1)) \\ &\cdot \{1 + \delta^{-1}(y_1^0 - \mathbf{x}_1\mathbf{b}, \ldots, y_{i-1}^0 - \mathbf{x}'_{i-1}\mathbf{b}) \\ &\cdot U_{i-1}^{-1}(y_1^0 - \mathbf{x}'_1\mathbf{b}, \ldots, y_{i-1}^0 - \mathbf{x}'_{i-1}\mathbf{b})'\} \\ &\cdot [wI + (wn^{-1}\mathbf{x}'_iR^{-1}\mathbf{x}_i - \mathbf{g}'_iU_{i-1}\mathbf{g}_i)\mathbf{1}\,\mathbf{1}'] \ , \end{aligned} \tag{3.16}$$

where $\mathbf{1}' = (1, 1)$.

Consequently,

$$\begin{aligned} &S(y_i^* | y_1^0, \ldots, y_{i-1}^0) - CoS(y_i^*, y_i | y_1^0, \ldots, y_{i-1}^0) \\ &= w(\delta/(\delta + i - 1))\{1 + \delta^{-1}(y_1^0 - \mathbf{x}'_1\mathbf{b}, \ldots, y_{i-1}^0 - \mathbf{x}'_{i-1}\mathbf{b}) \\ &\cdot U_{i-1}^{-1}(y_i^0 - \mathbf{x}'_1\mathbf{b}, \ldots, y_{i-1}^0 - \mathbf{x}'_{i-1}\mathbf{b})'\} \ . \end{aligned} \tag{3.17}$$

Under our model

$$\begin{aligned} &C(y_i^* | y_1^0, \ldots, y_{i-1}^0, y_i^0) - C(y_i | y_1^0, \ldots, y_{i-1}^0) \\ &= [CoS(y_i^*, y_i | y_1^0, \ldots, y_{i-1}^0)/S(y_i | y_1^0, \ldots, y_{i-1}^0)] \\ &\cdot [y_i^0 - C(y_i | y_1^0, \ldots, y_{i-1}^0)] \ . \end{aligned} \tag{3.18}$$

Now (3.18) can be solved for the conditional cospread, and then (3.17) can be solved for w. This yields m estimates of w, denoted by $\hat{w}_1, \ldots, \hat{w}_m$.

Additionally, from the constraint that $U - wI$ must be positive semidefinite, w must be less than or equal to the smallest eigenvalue of U. Consequently, we propose to estimate w by the average of those $\hat{w}_i$'s that are smaller than the minimum root of U. If there are none, we propose to allow the experimenter to reassess U or to choose a w directly constrained to be positive and less than the smallest root of U. Estimated in this way, $U - wI$ is positive definite.

An alternative, suggested to one of us by A.P. Dempster in private conversation and in the spirit of Lindley, Tversky, and Brown (1979), is to have a Wishart prior distribution for U, which would have the effect of raising the smallest eigenvalues of elicited U to allow realistic assessment of w. Implementation of this suggestion would require a prior positive-definite matrix for each problem and hence would be somewhat more complex to administer than the present programs. But this may be a fruitful direction for further work.

3.3.4 Summary of required elicitations. Suppose that the conditioning values to date are $y_1^0, \ldots, y_i^0$ $(i < m)$, chosen to satisfy (3.6). Then we suggest the following elicitations in the order given:

1. The median of $y_i^* | y_1^0, \ldots, y_i^0$, which we use to estimate $C(y_i^* | y_1^0, \ldots, y_i^0)$ for use in (3.18);
2. The median of $y_{i+1} | y_1^0, \ldots, y_i^0$, to be treated as an estimate of $C(y_{i+1} | y_1^0, \ldots, y_i^0)$ and used in (3.18) in the next analysis and also in (3.10);
3. The 75th percentile of $y_{i+1} | y_1^0, \ldots, y_i^0$, to be used with (3.5) to estimate $S(y_{i+1} | y_1^0, \ldots, y_i^0)$ and used in the next analysis of (3.18);
4. The median of each $y_{i+j} | y_1^0, \ldots, y_i^0$, for $j = 2, \ldots, m - i$ used to estimate $C(y_{i+j} | y_i^0, \ldots, y_i^0)$, for use in (3.10).

When $i = m$, only the median of $y_m^* | y_1^0, \ldots, y_m^0$ is elicited.

3.3.5 Finding an estimate for R using U and w. According to our model of the assessor's uncertainty, the predictive spread $S(\mathbf{y} | X)$ should take the form $w((1/n)XR^{-1}X' + I)$. To fit to this the elicited matrix U, define the matrix

$$A = U - wI \ . \tag{3.19}$$

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The matrix A is an elicited version of

$$S(X\beta \mid X) = (w/n)XR^{-1}X' .$$

The true matrix $S(\mathbf{y} \mid X)$ depends on R only through $S(X\beta \mid X)$, and our estimate $\hat{R}$ will be a function of the assessed U through A. By our construction of w, A is positive definite. This will guarantee the same property for our estimate of R.

Having assessed at least an $r \times r$ matrix U, and hence A, the equations (3.19) are "solved" for R^{-1} as

$$\hat{R}^{-1} = (X'X)^{-1}X'AX(X'X)^{-1}/(w/n) . \quad (3.20)$$

Since the elicitation has assured that A is positive definite, so is $\hat{R}$.

Thus, by substitution, the "fitted values" for A are

$$\hat{A} = (w/n)X\hat{R}^{-1}X' = P_X A P_X , \quad (3.21)$$

where P_X is the orthogonal projection operator onto the subspace spanned by the columns of X,

$$P_X = X(X'X)^{-1}X' . \quad (3.22)$$

These fitted values $\hat{A}$ resemble the elicited matrix A in the sense that $X'\hat{A}X = X'AX$ and that the two associated quadratic forms are related in the following way:

$$\mathbf{z} = P_X\mathbf{z} \quad \text{implies} \quad \mathbf{z}'\hat{A}\mathbf{z} = \mathbf{z}'A\mathbf{z} ,$$
$$\mathbf{z} = (I - P_X)\mathbf{z} \quad \text{implies} \quad \mathbf{z}'\hat{A}\mathbf{z} = 0 . \quad (3.23)$$

Note that $\mathbf{z}'A\mathbf{z}$ is an elicited version of $S(\mathbf{z}'X\beta \mid X, \mathbf{z})$, using (3.19). Thus the spreads of certain linear combinations of β coordinates are preserved by the fit, namely, those combinations that are linear functions of the combinations whose scales were assessed and for which the coefficient vector of the linear function belongs to the subspace spanned by the columns of X.

3.4 Averaging in Obtaining Elicited Hyperparameters

Common sense suggests that to have a one-to-one relationship between answers to questions and hyperparameters is a mistake; some kind of averaging seems critical for reliable answers. This matter has recently been reemphasized in Lindley, Tversky, and Brown (1979). In our case, averaging to obtain an estimate of $\mathbf{b}$ occurs in (3.1), for δ in (3.4), for w in the last paragraph of (3.3.3), and for R in (3.20). If the experimenter's opinion were exactly represented by the predictive t distribution of Section 3.2 (i.e., if the experimenter's model were exactly the normal linear model and the prior a member of the conjugate family), and if no elicitation errors were made, then the averages would be averages of equal quantities. The extent to which these quantities are not equal (or differ from their average) is a good measure of the extent to which either the model does not fit, or the experimenter makes elicitation errors, or both. At present we report the largest deviations from the averages to the experimenter, so that he or she can judge the extent to which some of his or her answers may be in error and require changing, and the extent to which this model does not represent his or her opinions.

3.5 Concluding Remarks

With all the hyperparameters elicited, a computer program can easily update to yield a posterior distribution and posterior predictive distributions from regression data using Bayes Theorem. At present, we have only algebraic ways of reporting the results, although we can imagine that the experimenter might want to ask quantile questions of the computer similar to those the computer has been asking the experimenter.

There are important cases in which the exogenous variables are themselves functions of some underlying space, as in the case of polynomial regression. The methods outlined in this article can be used without change, since all that is required is judgments on quantiles at certain values of the exogenous variables and judgments on quantiles conditional on certain pseudodata. Similarly, the dependent variable can have an arbitrary monotone transformation without disturbing the method. That is, the experimenter's elicitations can be made in terms of the original variables and each elicitation transformed before use. In particular, lognormal models can be treated in this fashion.

We have found that the technique introduced in this section is a practical, usable method of elicitation.

4. IMPLEMENTATION

In the TROLL implementation, vectors of independent variables at which to elicit the experimenter's predictive quantiles are constructed as follows: If the experimenter has given the computer data on the independent variables, those data are used to compute a midpoint and a step size equal to 30 percent of the range. For each independent variable the experimenter is asked to accept the computed midpoint and step size or to supply them, and is required to supply them when no data are available. The computer then asks the experimenter about each of the 3^r points generated, by allowing each independent variable the values of the midpoint, and the midpoint $\pm$ step size. The experimenter has the option of accepting or rejecting each point when the computer offers it. After the experimenter has given sufficient information to allow the regression in (3.1) to be computed, the computer offers, at each subsequent vector of independent variables, the option to stop or to continue.

Values of y_i^0 are chosen by our program to be $C(y_i \mid y_1^0, \ldots, y_{i-1}^0) + \frac{1}{2}\{S(y_i \mid y_1^0, \ldots, y_{i-1}^0)\}^{\frac{1}{2}}$. This choice, together with our choice to use the upper quartile to estimate scales, implies that we concentrate on the upper tail of the predictive distribution in our elicitation and rely solely on symmetry to fit the lower tail. Possibly, at the cost of additional elicitations, this reliance on symmetry could be reduced.

An interesting alternative that might be considered, if

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real data the experimenter has not seen are available, is to elicit at the data values of the independent variables. Then real bets could be made about the data outcomes or scoring rules could be used to reward the experimenter (Savage 1971).

5. CRITIQUE OF ASSUMPTIONS

The techniques of this article rest on certain assumptions:

1. The mean of the dependent variable y is, within the relevant range, a linear function of the independent variables x. In many applications this can be justified, within a sufficiently small x-domain, by Taylor's theorem.
2. The errors in this relationship are normal, having vector zero mean and covariance matrix proportional to I.
3. The experimenter's opinion is a member of the conjugate family of prior distributions.
4. The experimenter's errors of elicitation are unbiased, uncorrelated with each other, have equal variances, and are normal.
5. The experimenter is coherent in the sense that his or her underlying opinions can be modeled by subjective probability.

While the first two assumptions are familiar to most readers, the latter three require some comment. Conjugate prior distributions are assumed here only for their analytic convenience. They have some well-known failings, especially in the assumed dependence between β and σ that they require. Properties of errors of elicitation are a current topic of investigation in psychology. We have made some use of this literature in our work and expect that many improvements in our techniques will come from a close partnership between statisticians and psychologists studying these problems. Dickey (1979) discusses further the probabilistic structure of elicitation errors. The last assumption, that the experimenter is coherent is to some an article of faith and to others an assumption so rigid as to be absurd. Our own attitude toward coherence is that this assumption vastly simplifies our undertaking, and, like any model, it need not be justified by the claim of being exactly true to be very useful in practice.

6. PERSPECTIVES ON FURTHER WORK

A natural enhancement of our programs would be the design question: the selection of vectors of independent variables for which predictive distributions are to be elicited. The objective here is to maximize the information received from the experimenter. Such an enhancement would be helped by having a parametric model of the elicitation process, we believe, but even without a parametric model of the elicitation process, some reasonable guidelines might be given. For example, the computer might inquire whether some area of the space of independent variables is especially important. If so, points from that area could be chosen for elicitation, and the answers could be given extra weight in the analysis.

Symmetries that may exist in the experimenter's uncertainty can be exploited to reduce the number of elicitations needed. Thus if the experimenter views the effects of two variables as exchangeable, answers to some elicitation questions will imply answers to others. Again the trade-off between relying on structure (here symmetry) and asking more questions (to see if the claimed symmetry is reflected in behavior) must be faced. We expect that future versions of our programs will incorporate appropriate options for declaring symmetry.

Extension of the ideas of this article to a multivariate setting are given in Dawid, Dickey, and Kadane (1979).

Finally, we note an interesting kind of experiment that might be done with the programs at hand. An experimenter with a model he or she is convinced is a homoscedastic linear regression could elicit a prior, study some new data, and then elicit a new opinion. This could be compared with the posterior distribution computed from Bayesian theory. Of course, there might be any number of explanations for a discrepancy, but it will be interesting to have such results and to find out when the computed and elicited posteriors tend to be close or distant (a relevant reference is Peterson and Beach 1967).

APPENDIX: AN APPLICATION TO A PROBLEM IN THE DESIGN OF HIGHWAY PAVEMENTS

For flexible (asphalt-concrete) pavements, cracking of the surface layer is generally considered to be the most significant manifestation of pavement distress. Premature cracking (i.e., cracking occurring at an early life or after less accumulated traffic than anticipated during design) is particularly troublesome. A study was conducted to prepare a series of recommendations that could be used by pavement designers to reduce the possibility of premature cracking. A portion of the study is discussed briefly in this Appendix; for further details, see Smith et al. (1978) and Winkler, Smith, and Kulkarni (1978).

The generally accepted approach to verification of pavement design recommendations (materials, construction, proportions) has been to plan, construct, test, and analyze results from full-scale field trials or test sections. In considering the test-section approach for the verification of designer-controlled options to minimize fatigue-cracking distress, it was evident that a large number of sections would be required because of the large number of variables involved. It was concluded that field trials would be impractical, as they would require too great a commitment of funds and other resources. Also, the time required to obtain results would be objectionable. Fortunately, highway departments contain a number of experienced individuals who have observed over long periods of time the field behavior of existing pavements constructed with a wide variety of design options. Thus the

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experience of practicing highway engineers was viewed as a very valuable resource suitable for quantification using the theory described in this article.

It was recognized that many variables influence the occurrence of fatigue cracking. Even when the list of variables considered is restricted to those a designer can control, the list is extensive. Therefore, a questionnaire was used in six different states to identify the variables having a dominant influence on fatigue cracking. In addition to the results of the questionnaire, the following criteria were used to select the designer-controlled variables that would constitute the regression model: (a) consideration of the recommendations to be verified, (b) preference for variables directly related to assessor experience, (c) practical limitations on the number of variables, and (d) a set of variables not strongly correlated to one another. On the basis of these criteria and the results of the questionnaire, the following four designer-controlled variables were selected as the independent variables in the regression model: (a) asphalt consistency, (b) asphalt content, (c) percentage of asphalt concrete, and (d) base density. The fatigue life (FL) of a pavement was defined to be the number of years from construction until 10 percent of the wheel path area of the pavement exhibited fatigue cracking. All pavements were considered to be properly designed (using standard design procedures) for the traffic level that they actually carried. The objective of the analysis is to define the mathematical relationship between the dependent variable (fatigue life) and the independent variables (asphalt consistency, asphalt content, proportion of asphalt concrete, and base density).

The asphalt consistency (PEN) was measured in terms of the standard penetration of the recovered asphalt at 77°F. To account for any aging effects, the asphalt was considered to be that recovered from the pavement after three to five years of service. Asphalt content (AC) was measured as the percent asphalt by weight of mix. The proportion of asphalt concrete in the structural section (TAC) was defined as the percent thickness of pavement materials above subgrade consisting of asphalt concrete. The density of base material (DEN) was measured in terms of relative density as determined from the AASHTO T-180 compaction test.

The multiple regression model for this application is

$$\mathrm{FL} = \beta_1 + \beta_2(\mathrm{PEN}) + \beta_3(\mathrm{AC}) + \beta_4(\mathrm{TAC}) + \beta_5(\mathrm{DEN}) + \text{error}.$$

Asking a highway engineer directly for a probability distribution of the model parameters seemed unreasonable, but asking about predictive distributions of FL corresponding to fixed values of the independent variable seemed reasonable. The results of the interview process with one experienced individual will be summarized here to illustrate the procedures discussed in this article. As shown in Table 2, assessments concerning FL were made for nine different sets of values of the independent variables. The least squares estimate of b from the medians is $b = (-14.125, .167, 1.75, .075, .1)$, and the predicted FL values from this estimate of b are given in Table 2.

The values of $a(\mathbf{x})$ corresponding to $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_5$, and $\mathbf{x}_8$ are 2.75, 3.00, 2.89, and 3.14, yielding $\bar{a}^* = 2.945$. From Table 1, this corresponds to $\delta = 2.4$, which is our estimate of δ.

For the procedure discussed in Section 3.3, m was chosen to be eight, and the hypothetical values of FL provided sequentially for $x_1, \ldots, x_8$ were, respectively, 17.03, 19.44, 17.90, 21.53, 23.66, 17.54, 19.45, and 20.16. Given the assessed conditional medians and .75 quantiles (which are omitted here to save space), U was found to be

$$U = \begin{bmatrix} 25.63 & 5.06 & 5.06 & 5.06 & 5.06 & 5.06 & 5.06 & 5.06 \\ & 38.12 & 7.90 & 7.90 & 1.00 & 7.90 & 7.90 & 7.90 \\ & & 24.58 & 14.03 & 18.62 & 14.03 & 14.03 & 14.03 \\ & & & 54.50 & 28.47 & 26.65 & 26.65 & 26.65 \\ & & & & 103.57 & 43.21 & 43.21 & 43.21 \\ & & & & & 29.46 & 27.37 & 27.37 \\ & & & & & & 35.02 & 30.78 \\ & & & & & & & 136.72 \end{bmatrix}.$$

The smallest eigenvalue of U is 3.72, and the estimate of w is 2.084. Finally, the estimate of R is

$$R = \begin{bmatrix} 3330.13 & -16.128 & -107.404 & 4.098 & -28.293 \\ & .149 & .012 & -.002 & .014 \\ & & 11.902 & -.429 & .667 \\ & & & .042 & -.043 \\ & & & & .274 \end{bmatrix}.$$

The preceding predictive model can be used to assist in developing pavement design criteria as well as material and construction specifications. The model predictions can also be used to plan pavement maintenance

2. *Assessments for Pavement Example*

x	AC	DEN	PEN	TAC	$FL_{.50}$	$FL_{.75}$	$FL_{.875}$	$FL_{.9375}$	Least Squares Prediction of $FL_{.50}$
$\mathbf{x}_1$	5.5	95	30	60	14	18	22	25	14.50
$\mathbf{x}_2$	6.5	95	30	60	16	20	25	28	16.25
$\mathbf{x}_3$	5.5	100	30	60	14	18	—	—	15.00
$\mathbf{x}_4$	5.5	95	45	60	17	21	—	—	17.00
$\mathbf{x}_5$	5.5	95	30	100	19	23.5	29	32	17.50
$\mathbf{x}_6$	4.5	95	30	60	12.5	16	—	—	12.75
$\mathbf{x}_7$	5.5	90	30	60	13	17	—	—	14.00
$\mathbf{x}_8$	5.5	95	15	60	12	15.5	20	23	12.00
$\mathbf{x}_9$	5.5	95	30	20	13	16.5	—	—	11.50

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scheduling and budgeting. In addition, the Bayesian techniques developed provide the means by which expert judgment can be qualified in a meaningful format for use by others. The interview process used to quantify judgments makes the individuals think hard about what they really believe and why. This process alone is very beneficial to the assessor as well as the interviewer. If a number of persons are interviewed, the degree of agreement can be quantified and areas of disagreement identified. This can provide a basis for meaningful discussions of differences in opinion and provide guidance for developing research needs and field data acquisition programs.

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REFERENCES

Anderson, T.W. (1958), *An Introduction to Multivariate Statistical Analysis*, New York: John Wiley & Sons.

Bellman, R. (1970), *Introduction to Matrix Analysis* (2nd ed.), New York: McGraw-Hill.

Box, G.E.P., and Taio, G.C. (1973), *Bayesian Inference in Statistical Analysis*, Reading, Mass.: Addison-Wesley.

Dawid, A.P., Dickey, J.M., and Kadane, J.B. (1979), "Distribution Theory and Assessment Methods for Matrix t and Multivariate t Models," Technical Report, University College of Wales, Statistics Dept.

DeFinetti, B. (1974), *Theory of Probability* (2 vols.), New York: John Wiley & Sons.

Dickey, J.M. (1979), "Beliefs About Beliefs, A Theory for Stochastic Assessments of Subjective Probabilities," Invited Paper given at the International Meeting on Bayesian Statistics, Valencia, Spain.

Hogarth, R.M. (1975), "Cognitive Processes and the Assessment of Subjective Probability Distribution" (with discussion), *Journal of the American Statistical Association*, 70, 271–289.

Kadane, J.B. (1980), "Predictive and Structural Methods for Eliciting Prior Distributions," in *Studies in Bayesian Econometrics and Statistics in Honor of Harold Jeffreys*, ed. A. Zellner, Amsterdam, North-Holland Publishing Co.

Kadane, J.B., and Sedransk, N. (1979), "Toward a More Ethical Clinical Trial," Invited Paper given at the International Meeting on Bayesian Statistics, Valencia, Spain.

Lindley, D.V. (1971), *Bayesian Statistics, A Review*, Philadelphia: Society for Industrial and Applied Mathematics.

Lindley, D.V., Tversky, A., and Brown, R.V. (1979), "On the Reconciliation of Probability Assessments" (with discussion), *Journal of the Royal Statistical Society*, Ser. A, 146–180.

Novick, M.R., Isaacs, G.L., and DeKeyrel, D.F. (1977), *Manual for the Computer-Assisted Data Analysis Monitor*, Iowa City, Iowa: University of Iowa.

Peterson, C.R., and Beach, L.R. (1967), "Man As an Intuitive Statistician," *Psychological Bulletin*, 68, 29–46.

Press, S. James (1972), *Applied Multivariate Analysis*, New York: Holt, Rinehart & Winston.

Raiffa, H., and Schlaifer, R. (1961), *Applied Statistical Decision Theory*, Cambridge, Mass.: Harvard University, Graduate School of Business Administration.

Savage, L.J. (1971), "Elicitation of Personal Probabilities and Expectations," *Journal of the American Statistical Association*, 66, 783–801.

Smirnov, N.V. (ed.) (1961), *Tables for the Distribution and Density Functions of t-Distribution*, trans. by P. Basu, New York: Pergamon Press.

Smith, W., Finn, F., Saraf, C., and Kulkarni, R. (1978), "Bayesian Analysis Methodology for Verifying Recommendations to Minimize Asphalt Pavement Distress," Final Report 9-4A, National Cooperative Highway Research Program.

Spetzler, C.S., and Stael Von Holstein, C.A. (1975), "Probability Encoding in Decision Analysis," *Management Science*, 22, 340–358.

Tversky, A. (1974), "Assessing Uncertainty" (with discussion), *Journal of the Royal Statistical Society*, Ser. B, 36, 148–159.

Tversky, A., and Kahneman, D. (1974), "Judgment Under Uncertainty: Heuristics and Biases," *Science*, 185, 1124–1131.

Winkler, R.L. (1967), "The Assessment of Prior Distributions in Bayesian Analysis," *Journal of the American Statistical Association*, 62, 776–800.

——— (1980), "Prior Information, Predictive Distribution and Bayesian Model-Building," in *Studies in Bayesian Econometrics and Statistics in Honor of Harold Jeffreys*, ed. A. Zellmer, Amsterdam: North-Holland Publishing Co., 95–109.

Winkler, R.L., Smith, W., and Kulkarni, R. (1978), "Adaptive Forecasting Models Based on Predictive Distributions," *Management Science*, 24, 977–986.

Zellner, A. (1972), "On Assessing Informative Prior Distributions for Regression Coefficients," unpublished mimeo.

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