ST 740: Bayesian Reliability

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Definition of Reliability

Reliability is colloquially defined as the property that a thing works when we want to use it.

Formally, *reliability* is defined as the ability of an item to perform a required function under given environmental and operating conditions for a stated period of time.

Types of Data

- Binary data: testing to see if items work; often at a variety of ages under a variety of conditions
- Count data: observe items for a period of time to estimate a failure rate
- Lifetime data: observe items to determine distribution of "time to failure"
- Degradation data

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Interesting Functions of Parameters

The *reliability* function is the probability that an item's lifetime is greater than t

$$\begin{aligned} R(t) = \mathbf{P}(T > t) &= \int_t^\infty f(s \mid \theta) ds \\ &= 1 - \int_{-\infty}^t f(s \mid \theta) ds \\ &= 1 - F(t \mid \theta) \end{aligned}$$

The reliability function is also called the *survival function*.

Interestingly, in reliability, the c.d.f. is sometimes called the *unreliability function*.

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$$\mathbf{P}(t < T \leq t + \Delta t) = \frac{\mathbf{P}(t < T \leq t + \Delta t)}{\mathbf{P}(T > t)} = \frac{F(T + \Delta t) - F(t)}{R(t)}$$

We want to know the failure rate, so we divide by the length of the interval, Δt, and let $\Delta t \to 0$.

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{F(T + \Delta t) - F(t)}{\Delta t} \frac{1}{R(t)} \\ &= \left(\frac{d}{dt} F(t) \right) \frac{1}{R(t)} \\ &= \frac{f(t)}{R(t)} \end{aligned}$$

We could be in a regime where the item is becoming more likely to fail, less likely to fail, or where the number of failures is relative constant over time.

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LCD Data

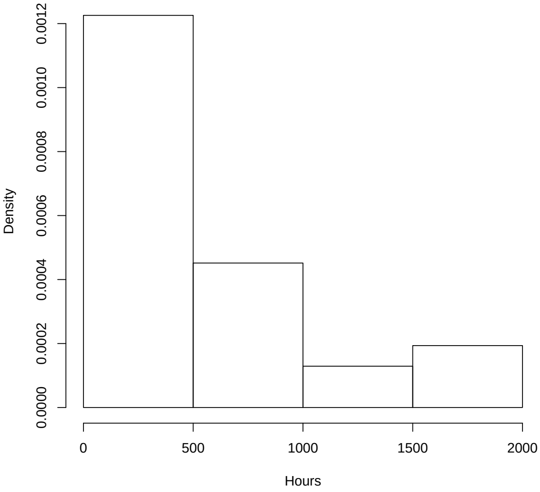
In business and educational settings, computer presentations use liquid crystal display (LCD) projectors. The most common failure mode of these projectors is the failure of the lamp. Many manufacturers include the "expected" lamp life in their technical specification documents, and one manufacturer claims that users can expect 1,500 hours of projection time from each lamp used under "normal operating conditions."

To test this claim, a large private university (BYU) placed identical lamps in three projector models for a total of 31 projectors. The university staff (grad students) recorded the number of projection hours (as measured by the projector) when each lamp burned out.

LCD Data

LCD Model	Projection Hours	LCD Model	Projection Hours
1	387	3	1895
1	182	2	158
1	244	1	974
1	600	2	345
1	627	1	1755
2	332	3	1752
2	418	1	473
2	300	2	81
1	798	1	954
2	584	2	1407
1	660	1	230
3	39	1	464
3	274	2	380
2	174	2	131
2	50	2	1205
3	34		

Histogram of LCD Data



Simple Model: Exponential Distribution with Conjugate Prior

Sampling Distribution: $f(t_i \mid \lambda) = \lambda \exp(-\lambda t_i)$
Prior Distribution: $\lambda \sim \text{Gamma}(2.5625, 2343.75)$
Posterior Distribution:

$$\begin{aligned} f(\lambda \mid \mathbf{t}) &\propto f(\theta \mid \lambda)\pi(\lambda) \\ &\propto \lambda^{31} \exp(-\lambda \sum_{i=1}^{31} t_i)\lambda^{2.5625-1} \exp(-2343.75\lambda) \\ &\propto \lambda^{2.5625+31-1} \exp(-\lambda(2343.75 + \sum_{i=1}^{31} t_i)) \\ &\sim \text{Gamma}(33.5625, 20250.75) \end{aligned}$$

Prior Choice

The manufacturer claims the mean is 1500 hours, but we are skeptical. We choose parameters for our gamma distribution so the expected value of the mean is 1500 and the standard deviation is 2000 hours.

The mean of the exponential is $\frac{1}{\lambda}$, so if $\lambda \sim \text{Gamma}(\alpha, \beta)$, then $\frac{1}{\lambda} \sim \text{InverseGamma}(\alpha, \beta)$.

$$\begin{aligned} \frac{\beta}{\alpha - 1} &= 1500 \\ \sqrt{\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}} &= 2000 \end{aligned}$$

Then

$$\begin{aligned} \frac{\beta^2}{(\alpha-1)^2} &= 1500^2 \rightarrow \frac{1500^2}{\alpha - 2} = 2000^2 \rightarrow \alpha = 2.5625 \\ \frac{\beta}{1.5625} &= 1500 \rightarrow \beta = \frac{1500}{1.5625} = 2343.75 \end{aligned}$$

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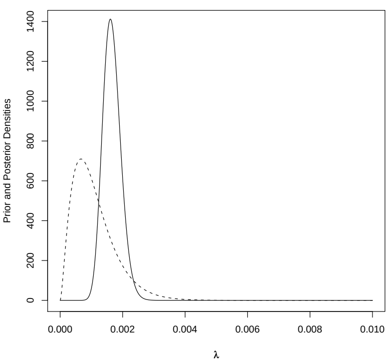
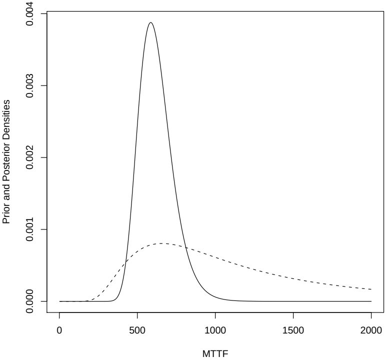
Prior and Posterior





Reliability Function

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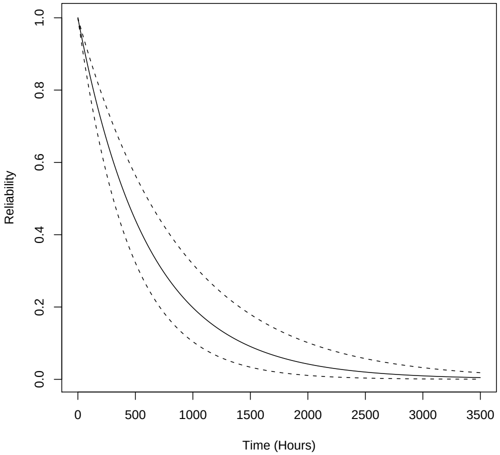
$$\begin{aligned}
R(t) = \mathbf{P}(T > t) &= \int_t^\infty f(s \mid \theta) ds \\
&= \int_t^\infty \lambda \exp(-\lambda s) ds \\
&= -\exp(-\lambda s)|_t^\infty \\
&= \exp(-\lambda t)
\end{aligned}$$

Reliability Function

Plot mean and central 95% credible interval for each time t



Goodness of Fit

Recall the Bayesian p-value:

$$\begin{aligned} p_B &= \mathbf{P}(T(\mathbf{y}^{rep},\theta) \geq T(\mathbf{y},\theta) \mid \mathbf{y}) \\ &= \int\int I_{T(\mathbf{y}^{rep},\theta)\geq T(\mathbf{y},\theta)} f(\mathbf{y}^{rep} \mid \theta)\pi(\theta \mid \mathbf{y}) d\mathbf{y}^{rep} d\theta \end{aligned}$$

Suppose

$$\begin{aligned} T(\mathbf{y},\theta) &= \sum_i \frac{(y_i - E(Y_i \mid \theta))^2}{\mathrm{Var}(Y_i \mid \theta)} \\ &= \sum_{i=1}^{33} \frac{t_i - \frac{1}{\lambda}}{\frac{1}{\lambda^2}} \end{aligned}$$

$p_B = 0.62$.

Censored Data

Type of Observation	Failure Time	Contribution
Uncensored	$T = t$	$f(t)$
Left censored	$T \leq t_L$	$F(t_L)$
Interval censored	$t_L < T \leq t_R$	$F(t_R) - F(t_L)$
Right censored	$T > t_R$	$1 - F(t_R)$

Censored Data

Suppose that it is university policy that bulbs are replaced when they fail or at 1700 hours, whichever comes first. In our data set, we would have three *censored* observations. Instead of observing that a bulb failed at 1895 hours, we would observe that it had not failed before 1700 hours.

The likelihood is the product of the terms from the uncensored observations and the censored observations. For the uncensored observations we have:

$$\prod_{i=1}^{28} \lambda \exp(-\lambda t_i)$$

For the censored observations we have:

$$\prod_{i=1}^{3} \int_{1700}^{\infty} \lambda \exp(-\lambda s) ds = \exp(-1700\lambda))$$

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$$
h(t) = \frac{f(t)}{R(t)}
$$

Model Checking

Bayes Factor

$$
\begin{aligned}
f(\mathbf{t}) &= \frac{f(\mathbf{t} \mid \lambda)\pi(\lambda)}{\pi(\lambda \mid \mathbf{t})} = \int f(\mathbf{t} \mid \lambda)\pi(\lambda) d\lambda \\
&= \frac{2343.75^{2.5625}}{\Gamma(2.5625)} \frac{\Gamma(31 + 2.5625)}{(2343.75 + \sum t_i)^{31+2.5625}} \\
&= \exp(-229.7288)
\end{aligned}
$$

Model Checking

DIC

$$
\begin{aligned}
D(\mathbf{t}, \lambda) &= -2 \log f(\mathbf{t} \mid \lambda) = -2(n \log(\lambda) - \lambda \sum t_i) \\
D_{avg}(\mathbf{t}) &= \int -2 \log f(\mathbf{t} \mid \lambda) \pi(\lambda \mid \mathbf{t}) d\lambda \\
&\approx \frac{1}{M} \sum -2 \log f(\mathbf{t} \mid \lambda^{(j)}) = 457.2537 \\
p_D &= D_{avg}(\mathbf{t}) - D(\mathbf{t}, \hat{\lambda}) \\
&= D_{avg}(\mathbf{t}) - 2 \log(\hat{\lambda}^n \exp(-\hat{\lambda} \sum t_i)) \\
&= 457.2547 - 456.2555 = 0.9982 \\
\text{DIC} &= D_{avg} + p_D = 457.2547 + 0.9982 = 458.2519
\end{aligned}
$$

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Censored Data

- Suppose that we observe that a bulb fails at 532 hours. What term do we use in the likelihood?
- Suppose that we observe that a bulb fails between 300 and 500 hours. What term do we use in the likelihood?
- Suppose that we observe that a bulb fails before 20 hours. What term do we use in the likelihood?

Hierarchical Model

Suppose that we want to use the information that there are three different kinds of projectors. We could propose the following hierarchical model.

$$
\begin{aligned}
T_{ij} &\sim \text{Weibull}(\lambda_i, \beta) \\
\lambda_i &\sim \text{Gamma}(0.6, 0.0004) \\
\beta &\sim \text{Gamma}(10, 10)
\end{aligned}
$$

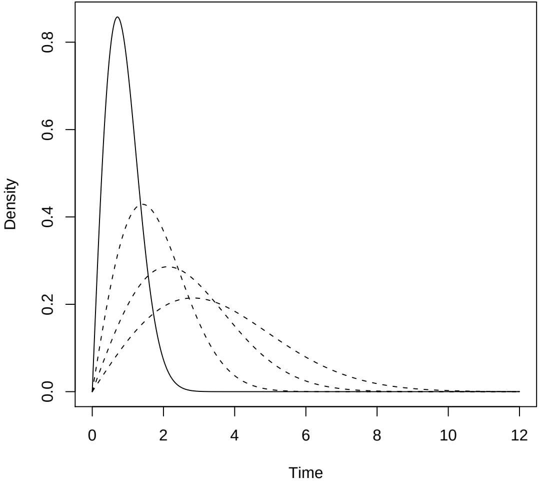
The Weibull pdf is

$$
f(t \,|\, \lambda, \beta, \theta) = \frac{\beta}{\lambda}\left(\frac{t-\theta}{\lambda}\right)^{\beta-1} \exp\left(-\left[\frac{t-\theta}{\lambda}\right]^{\beta}\right)
$$

$$
0 \leq \theta < t, \quad \lambda > 0, \quad \beta > 0.
$$

λ is the scale parameter, β is the shape parameter, and θ is the location parameter.

Weibull Density, Same Shape, Different Scale

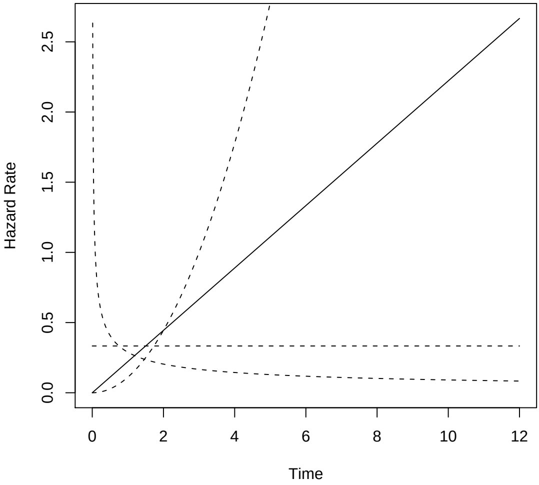


Interesting Functions, Weibull Density

$$R(t) = \exp\left(-\frac{t}{\lambda}^{\beta}\right)$$

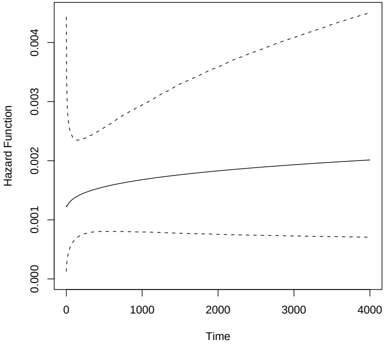
$$h(t) = \frac{\beta}{\lambda}\left(\frac{t}{\lambda}\right)^{\beta-1}$$

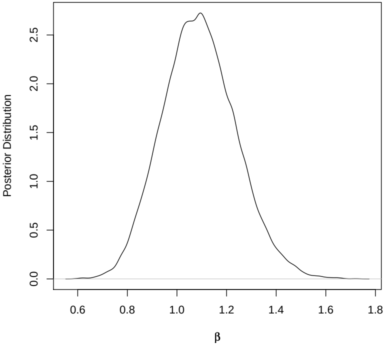
Weibull Hazard, Different Shape, Same Scale



Posterior Inference

What if we wanted to calculate the probability that the mean lifetime of a bulb in an LCD of type 1 was greater than 1500 hours?





Model Checking

- Bayes Factor computation of $f(\mathbf{t})$: -233.2527 (from Laplace approximation)
- Compating the first model to the second $\exp(-229.7288 + 233.2527) \approx 34$.
- DIC: Model 1 458.2519, Model 2 461.9344

More Bayes

- What models have you learned? (e.g., linear regression)
  - Model prior information
  - Difference techniques for inference
  - Some different models (e.g., hierarchical)
- Axiomatics and foundations
- Bayesian time series
- Bayesian nonparametrics: Let G be the set of all distributions. We've been studying models of the form $\{F_\theta : \theta \in \Theta\}$ with a prior distribution on θ. This assigns mass to just a small subset of G. We can develop priors on G that assign mass to much larger subsets.