Name:

Midterm Examination
October 9, 2013

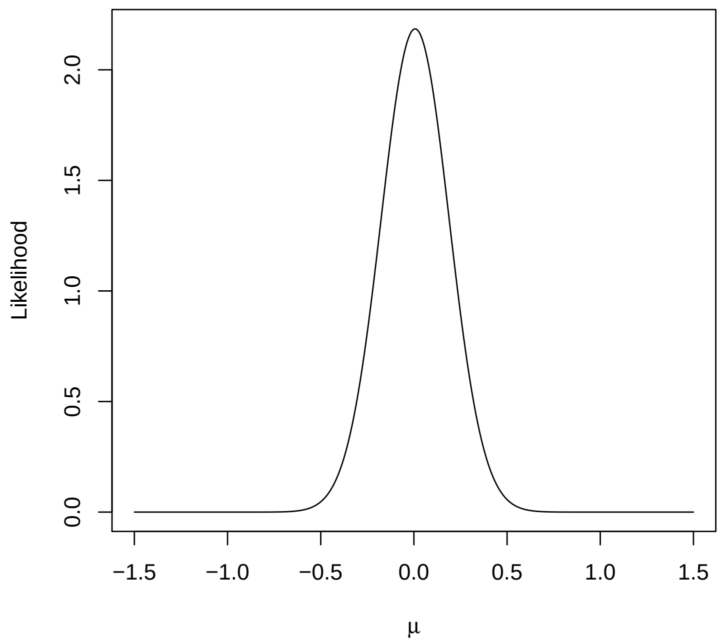
1. **Conjugate Priors.**

   (a) **(3 pts)** Define a *conjugate prior*.

   (b) **(4 pts)** What is the conjugate prior for the Exponential(λ) sampling distribution? Use the sampling distribution $f(x \mid \lambda) = \lambda \exp(-\lambda x)$. Both state the conjugate prior and show that it is conjugate.

2. **Jeffreys' Prior. (5 pts)** Derive the Jeffreys' prior for the normal mean assuming that the variance is known.

3. **Priors**. The plot is the likelihood function for the mean of a Normal($\mu, 1$) distribution evaluated using 30 observations.



(a) **(3 pts)** Sketch (and label) a *diffuse* prior on the plot. Write a short definition of a diffuse prior.

(b) **(3 pts)** Sketch (and label) an *informative* prior on the plot. Write a short definition of an informative prior.

(c) **(4 pts)** Sketch (and label) the Jeffreys' prior on the plot (see Question 2). Define an *improper* prior. Is the Jeffreys' prior proper or improper?

4. **Predictive Distribution.**

(a) **(3 pts)** Let $X_1, \ldots, X_n \mid p \sim \text{Bernoulli}(p)$ and assume that we observe x successes and $n - x$ failures. Let the prior distribution for p be $\text{Beta}(\alpha, \beta)$. What is the posterior distribution for p?

(b) **(3 pts)** Describe how to draw a sample from the predictive distribution of X_{n+1} assuming that you have a sample $p^{(i)}, i = 1, \ldots, m$ from the posterior distribution of p.

(c) **(4 pts)** Using the posterior distribution from (4a), derive the predictive distribution for X_{n+1} assuming that it is drawn from the same Bernoulli(p) distribution.

5. **Sampling.**

(a) **(5 pts)** Suppose that you have a posterior distribution $\pi(\theta \mid x_1, \ldots, x_n)$ with support on $(0, 1)$. Describe how you would use *brute force* sampling to draw a random sample from $\pi(\theta \mid x_1, \ldots, x_n)$.

(b) **(3 pts)** Suppose that we have a sample $\theta^{(i)}, i = 1, \ldots, m$ from $\pi(\theta \mid x_1, \ldots, x_n)$. How could we use this sample to evaluate the posterior probability that $\theta > 0.8$?

6. **Useful densities.**

- Bernoulli distribution
$$f(x \mid p) = p^x(1-p)^{1-x}$$
- Beta distribution
$$f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$
- Binomial distribution
$$f(x \mid p) = \binom{n}{x} p^x(1-p)^{n-x}$$
- Exponential distribution
$$f(x \mid \lambda) = \lambda \exp(-\lambda x)$$
- Gamma distribution
$$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \exp(-\beta x) x^{\alpha-1}$$
- Inverse gamma distribution
$$f(x \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \exp(-\beta/x) x^{-\alpha-1}$$
- Normal distribution
$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x-\mu)^2)$$
- Poisson distribution
$$f(x \mid \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$$