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Bayesian Methods for Estimating System Reliability Using Heterogeneous Multilevel Information

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Bayesian Methods for Estimating System Reliability Using Heterogeneous Multilevel Information

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We propose a Bayesian approach for assessing the reliability of multicomponent systems. Our models allow us to evaluate system, subsystem, and component reliability using multilevel information. Data are collected over time, and include binary, lifetime, and degradation data. We illustrate the methodology through two examples and discuss extensions. Supplementary materials are available online.

KEY WORDS: Degradation data; Hierarchical model; Lifetime data; Multicomponent system.

1. INTRODUCTION

This article proposes methodology to integrate system, subsystem, and component data to assess system reliability. The approach addresses two common problems in reliability: estimating how reliability changes over time and using information from multiple levels in the system to make inference. Generalizing previous work (Johnson et al. 2003; Reese et al. 2011), we discuss models for binary, lifetime, degradation, and expert opinion data at any level in the system.

Our original interest in this methodology arose from the assessment of weapons system reliability at Los Alamos National Laboratory and Sandia National Laboratories. The system models for these systems are similar to the one presented later in Figure 2. The data used to assess system reliability come from a variety of different tests on many different units, and the challenge is how to incorporate all of this information into an integrated assessment. Wilson, Anderson-Cook, and Huzurbazar (2011) presented a case study estimating the reliability of this system considering only component-level data; in Section 4, we generalize this analysis using system and subsystem data. Many parts of this problem have been addressed previously, and we provide a review of this literature while introducing new methodology to incorporate degradation data at the system or subsystem level and proposing a novel use of Bayesian melding to reconcile system-level priors with those at the component level.

Historically, much of the data collected for these assessments has been pass/fail (meet/do not meet a specification). There is considerable previous work considering Bayesian system reliability assessment using only binary data collected at a single time

point. Mastran (1976) and Mastran and Singpurwalla (1978) described a procedure to approximate the posterior mean reliability of a coherent system.¹ Martz, Waller, and Fickas (1988) and Martz and Waller (1990) proposed a bottom-up approach for approximating the posterior distribution of reliability of series and parallel systems of independent binomial subsystems and components. Tang, Tang, and Moskowitz (1997) proposed methods to obtain the exact posterior distributions in special cases. Johnson et al. (2003) developed full simultaneous Bayesian inference for multilevel binary data. Graves and Hamada (2005) extended the work by Johnson et al. (2003) to demonstrate simultaneous Bayesian inference for binary data collected over time at the system and component levels.

Generalizing beyond binary data, there are several papers that consider estimating system reliability with both time-to-failure and pass/fail data. Thompson and Chang (1975) and Chang and Thompson (1976) considered the reliability of subsystems with one or more components in series, where each component has an independent exponential distribution, and then compute Bayesian credible intervals for arbitrary series-parallel systems composed of these subsystems. Winterbottom (1984) surveyed classical and Bayesian results for estimating system reliability from binomial and exponential component data in coherent systems. Robinson and Dietrich (1988) considered a reliability growth model for component-level data with exponential lifetimes that have decreasing failure rates. Sharma and Bhutani (1994) estimated the availability of series and parallel systems where the components have exponential time to failure and repair. Bergman and Ringi (1997a) considered dependence between components induced by common operating environments; Bergman and Ringi (1997b) used data from non-identical environments.

¹Assume that a system has only two states: functioning or failed. A system is *coherent* when if all the components have failed, the system has failed; if all the components are functioning, the system is functioning; if the system has failed, an additional component failure does not cause the system to fail; if the system is functioning, replacing a failed component with a functioning component does not cause the system to fail (Rausand and Høyland 2004).

Hulting and Robinson (1994) were an exception to the above approaches, as they generalized the results by Martz, Waller, and Fickas (1988) to make approximate inferences about the reliability of a system using multilevel information. In particular, the article considered an automotive example with multilevel pass/fail, lifetime, and repair data. They approximated the reliability of a series system using nonhomogeneous Poisson processes to model the repair histories of repairable subsystems and time-to-failure data (modeled with a Weibull distribution) for nonrepairable subsystems. Reese et al. (2011) developed full simultaneous Bayesian inference for multilevel lifetime data.

Wilson et al. (2006) illustrated methodology for combining lifetime and degradation data at components with pass/fail data at the system. Anderson-Cook et al. (2008) assessed a missile system with pass/fail data for the system and degradation data developed from quality-assurance measurements at the components. In the context of weapons systems, we could also see degradation data at the subsystem or system levels. For example, one can collect health-monitoring data over time to watch for degradation in high-reliability systems. Historically, much of the data collected on weapons systems has been pass/fail. However, there is interest in changing the historical practices to collect more informative lifetime and degradation data, which requires the development of appropriate statistical methodology to analyze the (more costly) data if they are collected, and ultimately to assess whether the additional information provides for better inference and decision making.

Markov chain Monte Carlo (MCMC) has made fully Bayesian methods possible for addressing system reliability problems; for example, Johnson et al. (2003) and Hamada et al. (2004) proposed fully Bayesian approaches for simultaneously estimating the reliability for a system and its components described by a fault tree using binary data. Wilson, McNamara, and Wilson (2007) considered a system represented as a Bayesian network, also with binary data. Wilson et al. (2006) and Hamada et al. (2008) proposed approaches for assessing system reliability with binary data at the system, and binary, lifetime, or degradation data at the components.

One of the innovations in this article is to demonstrate how information can be combined when degradation data are collected at the system. Consider the following, adapted from an example by Wilson, McNamara, and Wilson (2007). In modern military usage, a *missile* is a self-propelled guided munition that travels through the air or space. Missiles have five major components: the structural system (frame), the payload system (warhead), the propulsion system, the targeting system, and the guidance system. The guidance system has two primary roles, which are to provide stability during flight and control during maneuvers. One key requirement for these roles is maintaining roll control (in an airplane, this would be equivalent to maintaining wings level).

Figure 1 shows the components that comprise a roll-control subsystem. (The representation is a Bayesian network, which is discussed in Section 5.) The actuators are motors; the engine control unit is an electronic control unit for the actuators; vehicle stability is a subsystem composed of structural elements (fins, frame, skin); the ailerons are a pair of hinged control surfaces used to maintain stability; the navigation sensor interprets

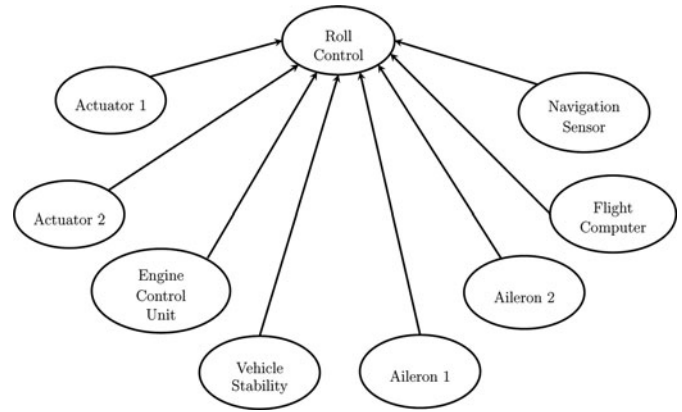


Figure 1. Missile roll control subsystem, adapted from the work by Wilson, McNamara, and Wilson (2007).

the movement of the missile, and the flight computer issues guidance commands.

In assessing the reliability, there are a variety of data that could be collected for each element. For example, roll control (the “system” in this example) could be assessed by making a pass/fail measurement about whether, given a particular set of conditions, the missile can reestablish acceptable roll control within a particular time period. More commonly, however, the time to reestablish control would be measured and monitored over time for increases, which would indicate degradation. In addition, changes in drag, lift, and rolling movement would be monitored, waiting for more drag, less lift, and higher rolling moment to indicate degradation.

This article presents a unified fully Bayesian approach for simultaneously estimating system, subsystem, and component reliability when there are binary, lifetime, degradation, or expert judgment data at any level of the system. In Section 2, we present a motivating example and introduce the modeling approach. In Section 3, we illustrate the approach with three simplified scenarios. In Section 4, we revisit the motivating example. Finally, in Section 5, we discuss extensions of the methodology.

2. MODEL SPECIFICATION

Figure 2 is the reliability block diagram for the example we consider in Section 4. A reliability block diagram provides a visual representation of how components and subsystems are configured to form a working system. There are a total of 17 distinct components or subsystems, with the notation K14(1) and K14(2) indicating two distinct units of the same kind of component (e.g., in a car, K14(u), $u = 1, \dots, 4$, might index the four spark plugs). K14(1) and K14(2) appear twice in the diagram, but represent the same physical part. While not evident from the diagram, JK20 is a subsystem, and K14-K15-K16 are physically configured so that they are often tested together.

The intuition behind a reliability block diagram is that if there is a path on which all of the blocks that link the circle on the left to the circle on the right are working, then the system is working. In this block diagram, there are components in *series* and in *parallel*. When components are in series, all must work for the system to work: for example, components JE1, J5, J6, JK20, J7, J4, and J8. When components are in parallel, at

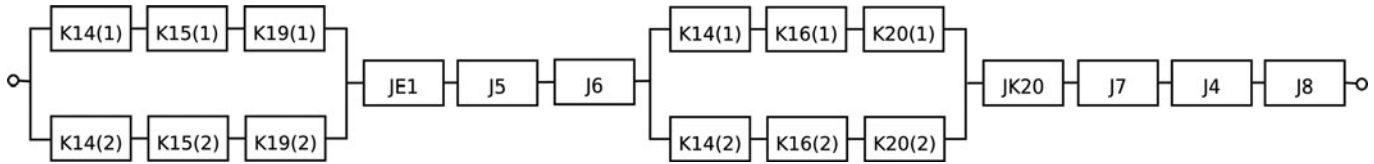


Figure 2. System reliability block diagram from Wilson, Anderson-Cook, and Huzurbazar (2011).

least one set must work for the system to work: for example, $\{K14(1), K15(1), K19(1)\}$ or $\{K14(2), K15(2), K19(2)\}$.

If we have point estimates for the reliability of each component, we could use the reliability equation implied by the block diagram to get a point estimate of the system reliability. In general, however, we are interested in estimating the joint distribution of the component and system reliabilities. This allows us to assess how well we know each component reliability and decide on how to allocate future test resources.

We return to this example in Section 4. For simplicity, we develop our methodology using a simple three-component series system, given in the fault tree in Figure 3. In this fault tree, each node represents the failure of a component (“basic events”) or the system (“top event”). Let C_0 denote the system, and C_1 , C_2 , C_3 denote the three components. Let $R_i(t | \Theta_i)$ denote the reliability of C_i at time t given parameters Θ_i . Let T_i be the random variable associated with the lifetime of C_i , with probability density function $f_i(t | \Theta_i)$ and cumulative distribution function $F_i(t | \Theta_i)$. We assume that $R_i(t | \Theta_i)$ is differentiable with respect to t and Θ_i . By definition, we have $R_i(t | \Theta_i) = \Pr\{T_i > t | \Theta_i\} = 1 - F_i(t | \Theta_i)$ and $f_i(t | \Theta_i) = -\frac{dR_i(t | \Theta_i)}{dt}$.

The first step of model development is specifying the reliability function $R_i(t | \Theta_i)$ for each component. The second step is to use the system structure to determine the reliability functions for the higher level (sub)systems. The system structure implies specific forms for the reliability functions. For example, in Figure 3, since the system works if and only if all three components work, the reliability function of system is the product of the reliability functions of the three components:

$$R_0(t | \Theta_0) = R_1(t | \Theta_1) \cdot R_2(t | \Theta_2) \cdot R_3(t | \Theta_3), \quad (1)$$

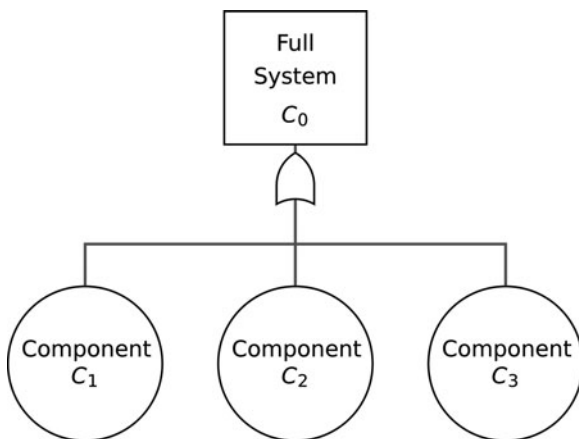


Figure 3. Three-component series system.

where Θ_0 includes Θ_1 , Θ_2 , and Θ_3 . The lifetime distributions follow immediately from the reliability functions. The next sections describe the development of likelihood functions. Once we have likelihoods, we specify prior distributions, and we can then use Bayes’ Theorem to perform inference. Note that to estimate the reliability of a single component, we may have data from hundreds of units.

2.1 Binary and Lifetime Data

Suppose at time s_{ik} ($k = 1, \dots, n_i$), N_{ik} tests have been conducted on C_i , with b_{ik} passing the test. We independently conduct n_i such tests, which results in $\sum_{k=1}^{n_i} N_{ik}$ total units tested. The likelihood function for C_i , using the binomial distribution, is given as (Johnson et al. 2003; Hamada et al. 2004):

$$L_i(\mathbf{b}_i | \Theta_i) = \prod_{k=1}^{n_i} \binom{N_{ik}}{b_{ik}} [R_i(s_{ik} | \Theta_i)]^{b_{ik}} [1 - R_i(s_{ik} | \Theta_i)]^{N_{ik} - b_{ik}}. \quad (2)$$

The reliability function in (2) can take many forms. Consider, for example, a logistic regression model (Wilson et al. 2006), where we specify the reliability function as

$$R_i(t | \Theta_i) = \text{logit}^{-1}(\theta_i + \eta_i t), \quad \theta_i > 0, \quad \eta_i < 0, \quad \Theta_i = (\theta_i, \eta_i),$$

where $\text{logit}^{-1}(y) \equiv [1 + \exp(-y)]^{-1}$.

For lifetime data, let $\mathbf{t}_i = (t_{i1}, t_{i2}, \dots, t_{im_i})$ be the failure times collected for C_i . In particular, we consider m_i units of type C_i and observe failure times t_{i1}, \dots, t_{im_i} . If the data are independent and identically distributed, the likelihood is straightforward (Reese et al. 2011),

$$L_i(\mathbf{t}_i | \Theta_i) = \prod_{k=1}^{m_i} f_i(t_{ik} | \Theta_i), \quad (3)$$

and can be easily generalized to censored data (Hamada et al. 2008).

2.2 Degradation Data

Degradation data measure some quantity about a component or system that is indirectly related to reliability or lifetime. In particular, degradation data are typically thought of as a continuous quantity that changes over time, with failure occurring when the degradation passes some threshold. Throughout the article, we assume that for two degradation measurements, d_1 and d_2 , if $d_1 < d_2$, then d_1 is more degraded (i.e., of poorer quality).

Consider degradation data for C_i , and suppose that we have measured a total of m_i different units. Denote the time of the measurements as q_{ijk} , where $j = 1, \dots, m_i$ and $k = 1, \dots, z_{ij}$. That is, for unit j , we make z_{ij} measurements. Let D_{ijk} denote the random variables associated with the degradation measurements d_{ijk} .

For components, the reliability function is derived from the degradation model: specifically, the reliability is the probability that the true degradation is above the threshold at time t . For example, consider the degradation process as specified by Wilson et al. (2006), where the measurements are assumed to be conditionally independent:

$$D_{ijk} | \alpha_i, \beta_{ij}, \sigma_i \sim \text{Normal}(\alpha_i - \beta_{ij}^{-1} q_{ijk}, \sigma_i^2), \\ \alpha_i > 0, \beta_{ij} > 0, \sigma_i > 0. \quad (4)$$

That is, all items are identical at $t = 0$, but they each degrade at their own rates. Let \mathbf{d}_i denote the degradation data for C_i . We can construct a likelihood function using (4):

$$L_i(\mathbf{d}_i | \beta_i, \alpha_i, \sigma_i) = \prod_{j=1}^{m_i} \prod_{k=1}^{z_{ij}} \frac{1}{\sigma_i} \phi\left(\frac{d_{ijk} - \alpha_i + \beta_{ij}^{-1} q_{ijk}}{\sigma_i}\right), \quad (5)$$

where $\phi(\cdot)$ is the probability density function of standard normal distribution $N(0, 1)$.

Extending the work by Wilson et al. (2006), to connect the degradation model with the lifetime distribution and reliability function, let τ_i be the threshold of the degradation process. This component fails when the true degradation is less than τ_i . Then we have

$$T_{ij} = \inf\{t \geq 0 : \alpha_i - \beta_{ij}^{-1} t \leq \tau_i\} = (\alpha_i - \tau_i)\beta_{ij}, \\ \alpha_i > \tau_i > 0. \quad (6)$$

Note that it is the unobserved “true” degradation, not the measured degradation, that determines whether the component fails.

Suppose that we further assume that $\log \beta_{ij} \sim \text{Normal}(\mu_i, \psi_i^2)$. Then the lifetime of C_i has a log-normal distribution. That is, $\log T_{ij} \sim \text{Normal}(\mu_i + \log(\alpha_i - \tau_i), \psi_i^2)$. As a result, the reliability function at time t is

$$R_i(t | \Theta_i) = 1 - \Phi\left(\frac{\log t - \mu_i - \log(\alpha_i - \tau_i)}{\psi_i}\right), \\ \Theta_i = (\mu_i, \psi_i, \alpha_i, \tau_i),$$

where $\Phi(\cdot)$ is the cumulative distribution function of $N(0, 1)$.

For (sub)systems, we first derive the induced lifetime distribution as described in Section 2. If we further assume the same degradation model as (4), the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ in (6) is determined for the (sub)systems. The choice of system degradation model is made here for convenience; in any given example, one can choose the appropriate degradation model subject to the derived constraint on the lifetime distribution.

In our Bayesian model, we must choose our prior distribution for $(\alpha_i, \tau_i, \beta_{ij})$ such that the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ is the same as the induced lifetime distribution for C_i . One simple way to achieve this is to specify the following conditional probability density distribution function for β_{ij} in terms of the induced lifetime distribution $f_i(t | \Theta_i)$:

$$g_i(\beta_{ij} | \Theta_i) = (\alpha_i - \tau_i) f_i[\beta_{ij}(\alpha_i - \tau_i) | \Theta_i]. \quad (7)$$

This specification for β_{ij} , along with any proper prior distributions for α_i and τ_i , makes the distribution of $(\alpha_i - \tau_i)\beta_{ij}$ coincide with the induced lifetime distribution. Consequently, the likelihood function for a (sub)system according to the above model specification still has the form of (5), but with constraints on β_{ij} from lower-level components and/or subsystems.

2.3 Prior Information

Specifying prior distributions in a Bayesian context is also part of the modeling process. An advantage of Bayesian methodology is that we can incorporate “nondata” information into our models; for example, information from expert opinions, historical data, and from similar systems.

Initial specification of prior distributions for all the model parameters described follows standard Bayesian practice. However, some thought must be given to the specification of prior distributions for the degradation data. Suppose that we are working with the model (4). Consider the specification of the priors for the degradation at time 0 (α_i) and the threshold τ_i , both of which are assumed to be positive. We consider two approaches.

A first approach, which is mentioned by Wilson et al. (2006), is to specify a gamma prior distribution on α_i and a beta distribution on τ_i/α_i given α_i . This approach is useful if we want to specify noninformative priors on τ_i or on both α_i and τ_i .

Our second approach allows us to incorporate detailed information about the threshold that implies an informative prior for τ_i . This can be difficult to specify using the preceding approach. However, consider the following. We specify an informative prior for τ_i and then a conditional prior distribution for α_i given τ_i . An example using a gamma distribution could be

$$\tau_i \sim \text{Gamma}(\nu_{\tau_i}, \xi_{\tau_i}); \quad (\alpha_i - \tau_i) | \tau_i \sim \text{Gamma}(\nu_{\alpha_i - \tau_i}, \xi_{\alpha_i - \tau_i}).$$

Prior information about the (sub)systems also requires additional thought. There are two potential issues: (1) the reliability and lifetime of (sub)systems are functions of the model parameters of components, and (2) if degradation data is observed at the (sub)system, the degradation model has constraints from lower levels. This implies that for (sub)systems, we need to specify prior distributions on functions of parameters. In addition, the prior distributions specified on the lower-level model parameters induce prior distributions on the reliability and lifetime of (sub)systems. Consequently, if we also have prior information about the reliability or lifetime of (sub) systems, we need a way to combine the information.

We use the *Bayesian melding* approach proposed by Poole and Raftery (2000), which has not been previously used in this context. Suppose that we have independent prior distributions on parameters θ and $\phi = M(\theta)$, specified by $q_1(\theta)$ and $q_2(\phi)$, respectively. $M(\cdot)$ is a deterministic function. The prior on θ induces an additional prior on $\phi = M(\theta)$, denoted by $q_1^*(\phi)$.

Poole and Raftery (2000) proposed pooling $q_1^*(\phi)$ and $q_2(\phi)$ and then inverting the pooled prior back to θ . Denote the inverted prior on θ by $\tilde{q}(\theta)$, with its formula given by

$$\tilde{q}(\theta) \propto q_1(\theta) \left(\frac{q_2(M(\theta))}{q_1^*(M(\theta))} \right)^{1-\alpha}, \quad (8)$$

where α is the pooling weight. The first issue is to reach an agreement on the prior of the vector (or scalar) parameter ϕ , as we have two priors for it: the induced prior and a direct prior. We use logarithmic pooling to combine the priors for ϕ ; more discussion of this choice can be found in the literature by Poole and Raftery (2000).

Table 1. Pass/fail data for component 1 (C_1) (25 new units are measured at each time), failure time data (years) data for 25 instances of component 2 (C_2), degradation data for component 3 (C_3) (one measurement per unit, with each unit measured once at the time given in parentheses)

C_1	Time	0	4	8	12	16	20	24	28	32	36	40
	Pass	25	25	25	24	22	23	20	14	9	7	3
	Total	25	25	25	25	25	25	25	25	25	25	25
C_2	23.8, 45.49, 64.61, 38.77, 11.22, 58.25, 29.93, 51.56, 75.42, 43.85, 44.01, 26.47, 26.9, 45.03, 21.11, 72.81, 64.04, 86.37, 56.67, 51.86, 69.88, 26.49, 71.24, 52.7, 67.84											
C_3	93.61 (2), 95.80 (4), 80.59 (6), 83.79 (8), 80.25 (10), 54.60 (12), 70.20 (14), 58.06 (16), 38.63 (18), 26.18 (20), 87.93 (2), 85.44 (4), 86.31 (6), 71.48 (8), 70.73 (10), 57.85 (12), 60.43 (14), 70.45 (16), 40.88 (18), 51.15 (20)											

In our system reliability setting, the reliability of a (sub)system at some time is a function of lower-level model parameters. Using the above notation, we have initial priors on θ for the components, denoted by $q_1(\theta)$, and initial priors on a (sub)system, which is some reliability function ϕ , denoted by $q_2(\phi)$. Then $q_1^*(\phi)$ is the prior induced by $q_1(\theta)$ on the reliability function and $\tilde{q}(\theta)$ is the final prior on the model parameters after melding. As a result, if we elicit the prior information on (sub)systems as prior on the reliability, we can use the Bayesian melding approach to combine multilevel prior information.

3. THREE-COMPONENT SERIES SYSTEM SCENARIOS

Before revisiting the complex example, we apply the proposed methodology to analyze the three-component series system. While the simpler system allows for easier presentation, it is also motivated by the example in Figure 2. The data cited by Wilson, Anderson-Cook, and Huzurbazar (2011) for K14-K15-K16 were collected as a subsystem, but estimates of individual component reliability are needed, and conceivably testing could have been done for each of the individual components.

We consider three scenarios for the system. All data in these scenarios are simulated. Each scenario has the same information for the components: binary data for C_1 , with 25 units evaluated at each of 11 time points (for a total of one measurement on each of 275 units); failure times for 25 units of type C_2 ; and degradation data for C_3 . The degradation data for C_3 are collected for 20 units that are measured one time each. These component data are given in Table 1.

Each scenario has different data collected for the entire system. Scenario 1 has binary data collected over time. Methodology for this scenario was developed by Wilson et al. (2006), and we will not consider it in much detail. Scenario 2 has lifetime data at the system, and Scenario 3 has degradation data at the system. The methodology for lifetime data at the system was introduced by Reese et al. (2011) considering only lifetime data at the components; the methodology for degradation data at the system is novel.

The system data are given in Table 2; they were simulated from the induced system distribution. To make the results more comparable, for Scenario 1, the binary data are directly derived from the lifetime data. For example, in Table 2, the two binary observations at age 10 correspond to the first two observations from the lifetime data (27.04 and 33.03). The degradation data are also simulated from the 10 systems with lifetime data as

given in Table 2. These systems are measured four times each. After giving the model (Section 3.1) and identifying prior parameters (Section 3.2), we first analyze the three scenarios when there is no prior information for the system (Section 3.3). We then introduce prior information for the system and use Bayesian melding to reanalyze Scenario 1 (Section 3.4).

3.1 Models

We use the logistic regression model for C_1 (Section 2.1), the Weibull lifetime distribution model (Section 2.1) for C_2 , and the degradation model (Section 2.2) for C_3 . For the system (C_0), the reliability function is determined from the specifications for the components, as discussed in Section 2.

The three reliability functions for C_1 , C_2 , and C_3 are given below.

$$\begin{aligned}
 R_1(t | \Theta_1) &= \text{logit}^{-1}(\theta_1 + \eta_1 t), \quad \Theta_1 = (\theta_1, \eta_1), \\
 R_2(t | \Theta_2) &= \exp \left[- \left(\frac{t}{\lambda_2} \right)^{\delta_2} \right], \quad \Theta_2 = (\delta_2, \lambda_2), \\
 R_3(t | \Theta_3) &= 1 - \Phi \left(\frac{\log t - \mu_3 - \log(\alpha_3 - \tau_3)}{\psi_3} \right), \\
 \Theta_3 &= (\mu_3, \psi_3, \alpha_3, \tau_3).
 \end{aligned}$$

Table 2. System data for the three scenarios: pass/fail data (10 units each measured once), lifetime data, and degradation data for 10 systems each measured four times

	Time	10	15	20	25	30
Pass/fail	Pass/total	2/2	2/2	2/2	1/2	0/2
Lifetime	27.04, 33.03, 33.73, 23.92, 27.49, 27.03, 26.31, 22.22, 27.46, 25.01					
Degradation	System	$t = 1$	$t = 6$	$t = 12$	$t = 19$	
	1	191.95	170.51	118.86	92.83	
	2	189.67	166.40	146.45	115.08	
	3	209.88	178.36	156.89	110.58	
	4	203.04	158.96	130.11	83.12	
	5	180.34	181.95	138.13	100.27	
	6	185.17	170.16	140.02	84.52	
	7	179.14	174.25	137.38	99.81	
	8	188.59	171.20	127.27	76.57	
	9	195.58	146.49	129.72	84.98	
	10	212.19	172.77	131.51	66.74	

Using (1), the reliability function for the system is

$$R_0(t | \Theta_0) = \text{logit}^{-1}(\theta_1 + \eta_1 t) \cdot \exp \left[- \left(\frac{t}{\lambda_2} \right)^{\delta_2} \right] \cdot \left[1 - \Phi \left(\frac{\log t - \mu_3 - \log(\alpha_3 - \tau_3)}{\psi_3} \right) \right]. \quad (9)$$

In Scenario 1, we model each system observation as a Bernoulli trial with the probability equal to the system reliability at the time of test. In Scenario 2, we model each system lifetime using the induced lifetime distribution. In Scenario 3, we model the degradation process with the restriction that the reliability function is determined by the system structure.

Let \mathbf{b}_1 denote the data for C_1 ; \mathbf{t}_2 for C_2 ; \mathbf{d}_3 for C_3 ; \mathbf{b}_0 , \mathbf{t}_0 , and \mathbf{d}_0 for C_0 . Additionally, let $\beta_3 = \{\beta_{3j} : j = 1, \dots, m_3\}$, $\beta_0 = \{\beta_{0j} : j = 1, \dots, m_0\}$. We can write down the likelihood functions for the three scenarios following (2), (3), and (5), respectively. In particular, the likelihood contribution from the system in Scenario 3 is:

$$L_0(\mathbf{d}_0 | \beta_0, \alpha_0, \sigma_0) = \prod_{j=1}^{m_0} \prod_{k=1}^{z_{0j}} \frac{1}{\sigma_0} \phi \left(\frac{d_{0jk} - \alpha_0 + \beta_{0j}^{-1} q_{0jk}}{\sigma_0} \right),$$

where $m_0 = 4$; $z_{0j} = 10$ for $j = 1, \dots, 4$. We specify the distribution for β_{0j} according to (7), with

$$g_0(\beta_0 | \Theta_0) = (\alpha_0 - \tau_0) f_0[\beta_0(\alpha_0 - \tau_0) | \Theta_0],$$

where $f_0(t | \Theta_0) = -\frac{dR_0(t | \Theta_0)}{dt}$ so that the reliability function for the system still satisfies (9).

3.2 Prior Distributions

When specifying prior distributions, we have the parameters from the basic events: θ_1 , η_1 , δ_2 , λ_2 , α_3 , τ_3 , ψ_3 , μ_3 , σ_3 . In Scenario 3, we also have α_0 , τ_0 , σ_0 . In real applications, these parameters are elicited; for illustration, we use fairly diffuse priors for many parameters. These priors are detailed in online supplementary Table 1. The priors for α_3 , τ_3 , α_0 , and τ_0 are developed following the discussion in Section 2.3.

3.3 Joint Posterior Distribution and Estimated Reliabilities

Let $L_1(\mathbf{b}_1 | \Theta_1)$ be the likelihood function for component 1 (from (2)); $L_2(\mathbf{t}_2 | \Theta_2)$ be the likelihood function for component 2 (from (3)); and $L_3(\mathbf{d}_3 | \Theta_3)$ be the likelihood function for component 3 (from (5)). For brevity, we present the posterior distribution for Scenario 3, which is the most complicated and includes all parameters involved in Scenarios 1 and 2.

The unnormalized joint posterior probability density function for Scenario 3 is given by

$$\begin{aligned} \pi(\Theta_0, \beta_3, \beta_0 | \mathbf{b}_1, \mathbf{t}_2, \mathbf{d}_3, \mathbf{d}_0) \\ \propto L_1(\mathbf{b}_1 | \Theta_1) L_2(\mathbf{t}_2 | \Theta_2) L_3(\mathbf{d}_3 | \beta_3, \Theta_3) L_0(\mathbf{d}_0 | \beta_0, \Theta_0) \\ \cdot \prod_{j=1}^{m_3} \beta_{3j}^{-1} \phi[(\log \beta_{3j} - \mu_3)/\psi_3] \cdot \prod_{j=1}^{m_0} g_0(\beta_{0j} | \Theta_0) \quad (10) \\ \cdot \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1} \phi(\log \lambda_2/100) \end{aligned}$$

$$\begin{aligned} & \cdot \phi(\mu_3/10) \\ & \cdot \alpha_3^3 \exp(-\alpha_3/30) \cdot (1/\alpha_3) \mathbb{1}_{[\alpha_3 > \tau_3 \geq 0]} \cdot \psi_3^3 \exp(-\psi_3/0.2) \\ & \cdot \sigma_3^3 \exp(-\sigma_3/2.5) \\ & \cdot (\alpha_0 - \tau_0)^{149} \exp(-(\alpha_0 - \tau_0)) \cdot \tau_0^{99} \exp(-\tau_0/0.5) \\ & \cdot \sigma_0^3 \exp(-\sigma_0/3). \end{aligned}$$

We can use MCMC to draw samples from the unnormalized joint posterior distributions. In particular, we used a one-variable-at-a-time random walk Metropolis algorithm to draw samples from the posterior distributions for the model parameters in these three scenarios. The marginal posterior distributions of the parameters of each scenario are summarized in online supplementary Tables 2, 4, and 5 for Scenarios 1, 2, and 3, respectively.

Perhaps more interesting, we can obtain the posterior distributions of the reliability functions for both the components and the system from the samples from posterior distributions. Plots for the functions with respect to time along with credible intervals are presented in Figures 4–6 for Scenarios 1–3, respectively. Note that the estimation of the reliability function of C_3 is not as accurate as those for C_1 and C_2 . The main reason is that we do not have much information about α_3 and τ_3 in the degradation model for C_3 . (This can also be seen from the posterior summary for τ_3 in online Supplementary Table 5.) Recall that a failure occurs when the true degradation passes the threshold. Here we have a noninformative prior for the threshold τ_3 , so the degradation data do not give much information about the reliability. Since we perform inference on the system as a whole, the information from the system contributes to the estimation of C_3 ; otherwise, we would not have information about the component reliability. Our methodology not only takes advantage of information at all levels to estimate the system reliability, but also it helps to estimate component reliability using data from the whole system.

An initial goal for considering three scenarios separately is to see which types of data improve estimation. This can be considered from an information perspective. Binary data can be considered censored lifetime data. In other words, the information we obtain from a pass/fail test at a time point is less than the exact lifetime information. For degradation data case, we are trying to infer the lifetime from the degradation process. Similarly, we have less information in the degradation data than in the lifetime data (although this relationship may change with heavy censoring of the lifetime data).

This is also illustrated in our three scenarios. Since the data are derived from the failure times, we can compare the credible intervals for the system reliability (as a function of time) in Figures 4–6. In particular, for the system's reliability function, we present the comparison results in Figure 7. Here we mainly focus on the estimation variability (the width of the credible intervals) instead of bias (compared with the actual reliability function). We can see from Figure 7 that the binary data is the worst case in terms of the estimation for the system reliability. The degradation data are comparable with the lifetime data possibly because each system is measured multiple times so that its lifetime can almost be inferred under the correct model specification.

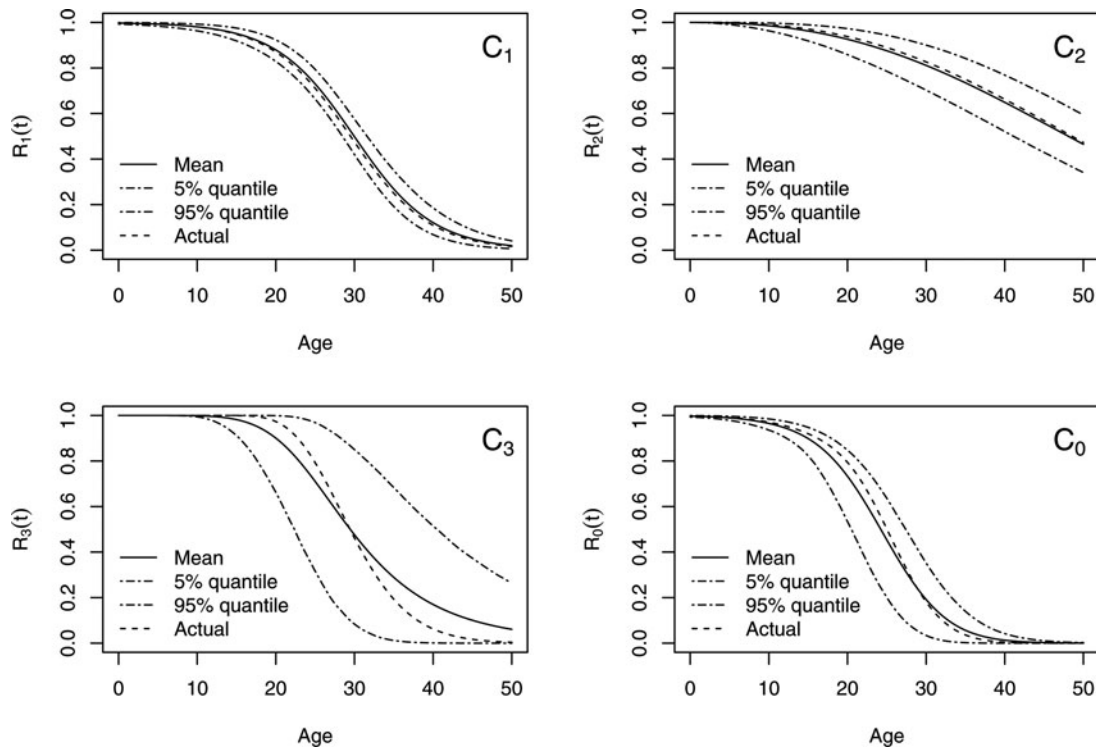


Figure 4. Reliability distributions as a function of time for Scenario 1.

3.4 Incorporating Prior Information About the System

In the above analyses, we have not incorporated any additional prior information about the system. Suppose we have additional independent prior information for the system, and

we believe the system reliability at 20 years, $R_0(t = 20 | \Theta_0)$, has a Beta(4, 2) distribution. From (9), the system reliability $R_0(t = 20 | \Theta_0)$ is a deterministic function (i.e., $M = R_0(t = 20 | \Theta_0)$ in the Bayesian melding context). Consequently, the prior on Θ_0 induces a prior on $R_0(t = 20 | \Theta_0)$.

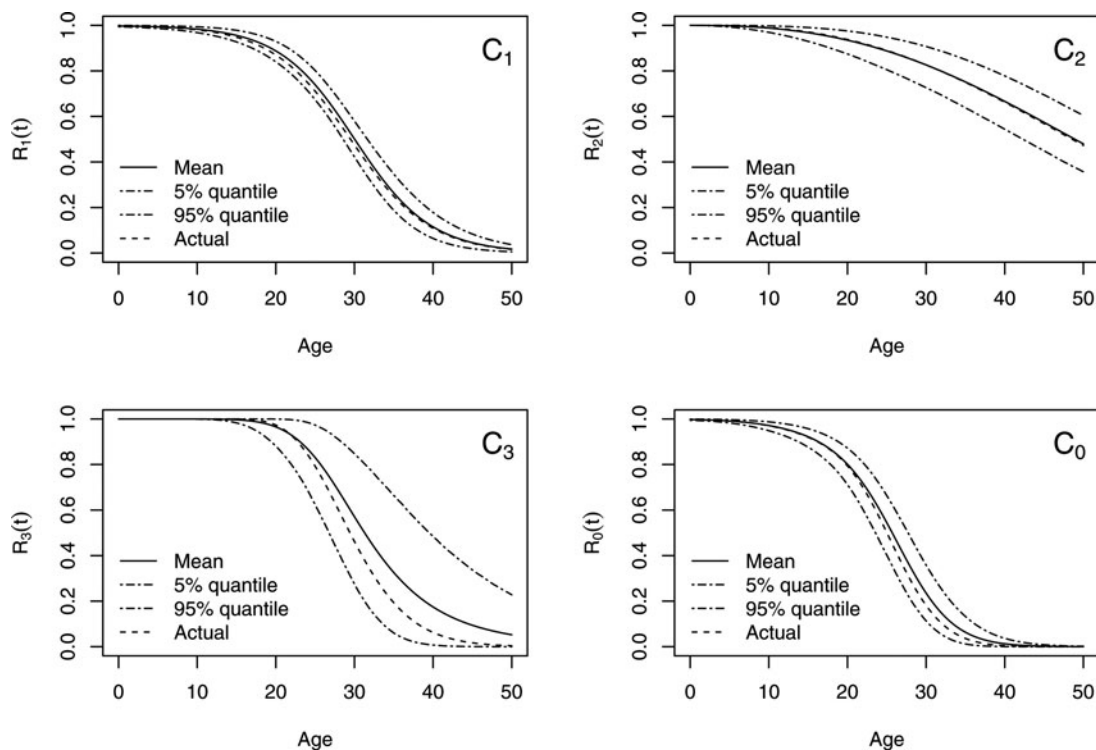


Figure 5. Reliability distributions as a function of time for Scenario 2.

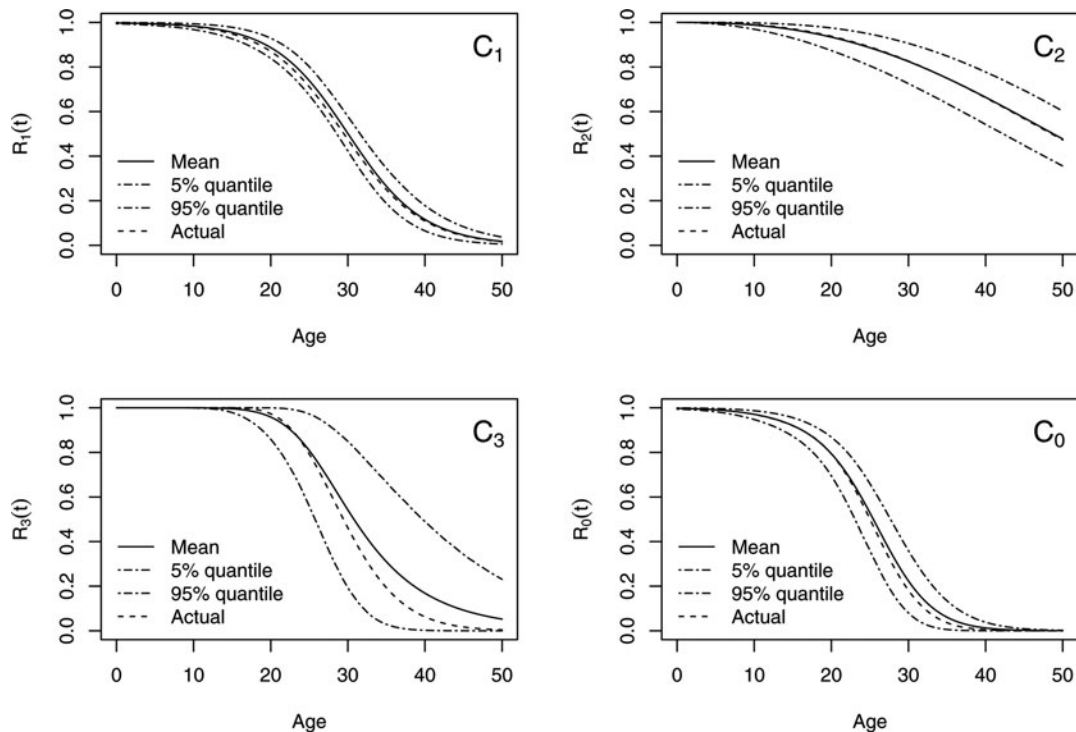


Figure 6. Reliability distributions as a function of time for Scenario 3.

Specifically, let $q_1(\theta)$ denote the prior for the parameters used to model the three components in Scenario 1:

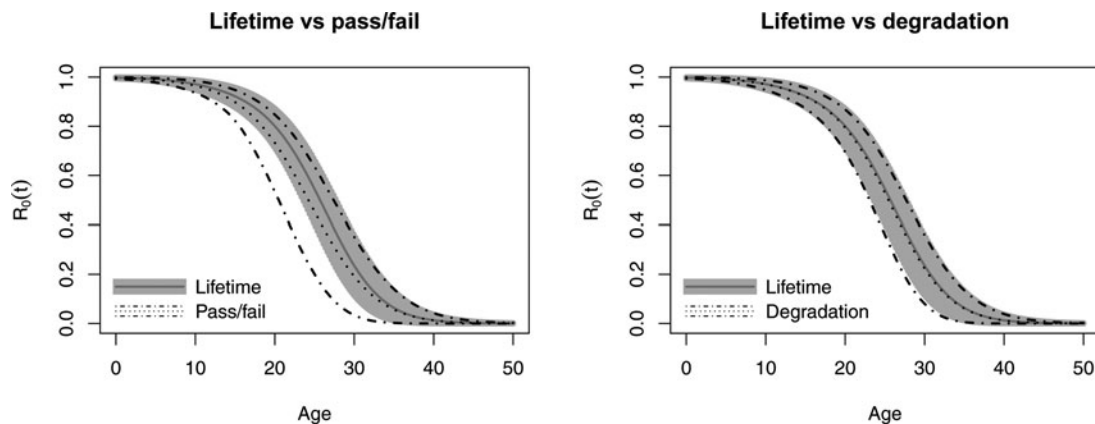
$$q_1(\theta) \propto \phi(\theta_1/100) \cdot \phi(\eta_1/100) \cdot \exp(-\delta_2) \cdot \lambda_2^{-1} \phi(\log \lambda_2/100) \\ \cdot \phi(\mu_3/10) \cdot \alpha_3^3 \exp(-\alpha_3/30) \cdot (1/\alpha_3) \mathbb{1}_{[\alpha_3 > \tau_3 \geq 0]} \\ \cdot \psi_3^3 \exp(-\psi_3/0.2) \cdot \sigma_3^3 \exp(-\sigma_3/2.5). \quad (11)$$

Then $q_1^*(M(\theta))$ is the prior distribution on $M(\theta)$ induced by the specification of (11), and $q_2[M(\theta) = R_0(t = 20 | \theta)]$ is the density function of the Beta(4, 2) distribution (i.e., $q_2[M(\theta)] \propto M(\theta)^{4-1} (1 - M(\theta))^{2-1}$).

In Figure 8, we plot the induced prior $q_1^*(M(\theta))$; the initial prior on $M(\theta)$, $q_2(M(\theta))$; and the pooling of $q_1^*(M(\theta))$ and $q_2(M(\theta))$. Inverting the pooled prior on $M(\theta)$ to a prior on θ gives the final Bayesian melding prior. We use the melded prior $\tilde{\theta}$ as in (8), with pooling weight $\alpha = 0.5$, to perform our posterior inference.

When executing the analyses, the induced prior often needs to be estimated numerically using, for example, kernel methods, since the deterministic function is complex. The MCMC can then be carried out with the updated posterior distribution. Notice that the induced prior is time-consuming to compute, and since its computation is required in every evaluation of the posterior distribution, the overall MCMC procedure can be quite slow.

We have employed two approximations to ease this computational burden. First, we can approximate the induced prior distribution using a parametric form. For example, we can find a beta distribution (or mixture of beta distributions) to approximate the induced prior on system reliability. A second approach is to first evaluate the induced prior at multiple points (say 10^7 points). We can then use a “table lookup,” which returns the density of the closest point to approximate the induced prior. This is the approach we used in our computations. The

Figure 7. System Reliability ($R_0(t)$) comparison for the three scenarios.

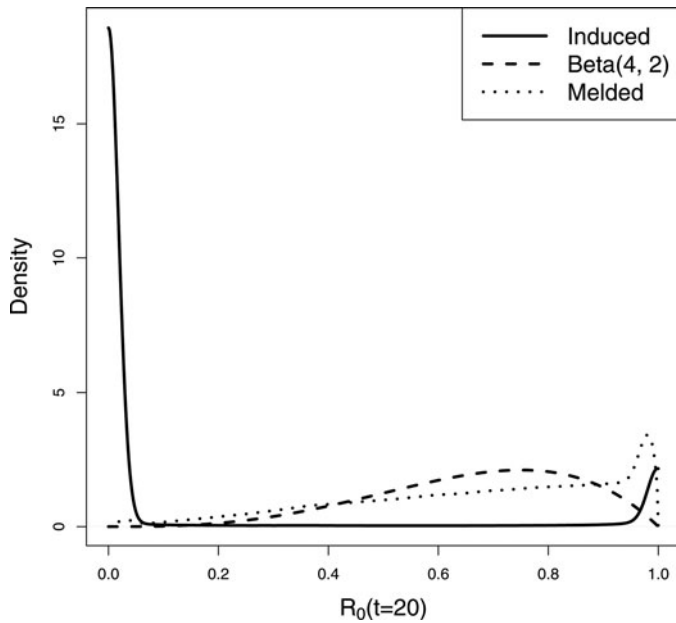


Figure 8. Bayesian melding priors (density functions) of priors on the system reliability when $t = 20$. The solid line represents the prior induced from the prior specified on the basic events; the dashed line Beta(4, 2); and the dotted line the pooled prior.

estimation results are presented in online Supplementary Table 3 and Figure 9.

Last, we also consider different values of pooling parameter α in (8). Considering $\alpha = (0.2, 0.4, 0.5, 0.6, 0.8, 0.9)$, we find that the estimation results are not sensitive to α . Most of our priors are noninformative; if we have stronger expert opinions, we expect that we would see more sensitivity.

4. A MORE COMPLEX EXAMPLE

To illustrate the scalability of the proposed methodology, we return to the example from Wilson, Anderson-Cook, and Huzurbazar (2011). The reliability block diagram (Figure 10) corresponds to an equation for system reliability. Details of this connection can be found in Hamada et al. (2008). The reliability block diagram can be described mathematically by the following equation, where R_A corresponds to the reliability of component A, $R_{B(1)}$ is the reliability of the first instance of part B, and R_{SYS} is system reliability.

$$\begin{aligned}
 R_{SYS} = & R_{JE1} R_{J5} R_{J6} R_{JK20} R_{J7} R_{J4} R_{J8} \\
 & (R_{K14(1)} R_{K14(2)} R_{K15(1)} R_{19(1)} R_{K15(2)} R_{19(2)} R_{K16(1)} R_{20(1)} \\
 & \times R_{K16(2)} R_{20(2)} \\
 & - R_{K14(1)} R_{K14(2)} R_{K15(1)} R_{19(1)} R_{K15(2)} R_{19(2)} R_{K16(2)} R_{20(2)} \\
 & - R_{K14(1)} R_{K14(2)} R_{K15(1)} R_{19(1)} R_{K16(1)} R_{20(1)} R_{K16(2)} R_{20(2)} \\
 & - R_{K14(1)} R_{K14(2)} R_{K15(1)} R_{19(1)} R_{K15(2)} R_{19(2)} R_{K16(1)} R_{20(1)} \\
 & - R_{K14(1)} R_{K14(2)} R_{K15(2)} R_{19(2)} R_{K16(1)} R_{20(1)} R_{K16(2)} R_{20(2)} \\
 & + R_{K14(1)} R_{K14(2)} R_{K15(1)} R_{19(1)} R_{K16(2)} R_{20(2)} \\
 & + R_{K14(1)} R_{K14(2)} R_{K15(2)} R_{19(2)} R_{K16(1)} R_{20(1)} \\
 & + R_{K14(1)} R_{K15(1)} R_{19(1)} R_{K16(1)} R_{20(1)} \\
 & + R_{K14(2)} R_{K15(2)} R_{19(2)} R_{K16(2)} R_{20(2)}). \quad (12)
 \end{aligned}$$

The equation for system reliability can be found directly for simple systems; for more complicated systems like this one, finding the reliability function of a nonbasic event in terms of component reliabilities requires the use of *structure functions* and *path* or *cut sets*. These algorithms are implemented in a variety of software packages; details of the methodology can be found in the literature by Rausand and Høyland (2004).

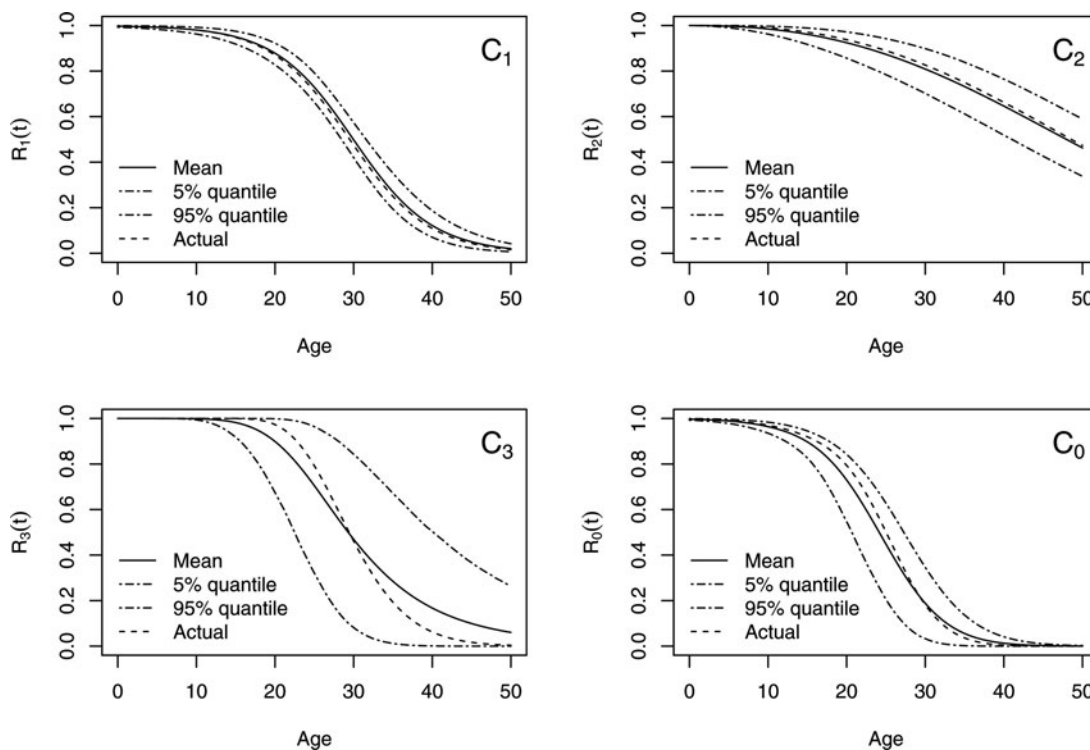


Figure 9. Reliability distributions as a function of time for Scenario 1 with Bayesian melding.

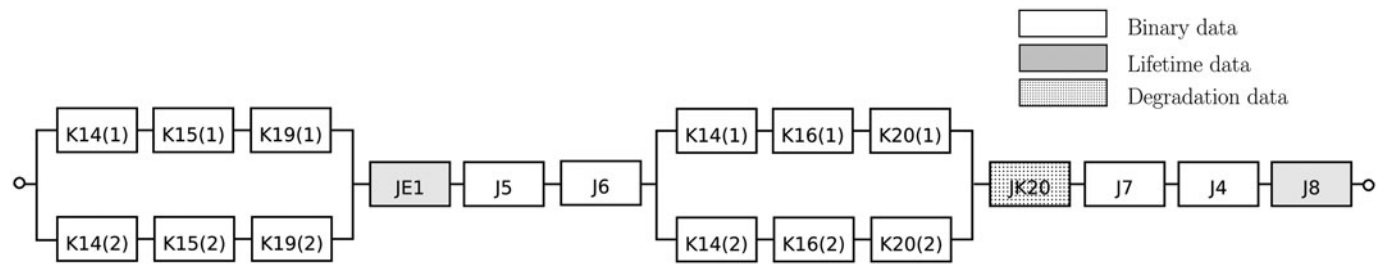


Figure 10. System reliability block diagram from Wilson, Anderson-Cook, and Huzurbazar (2011) with data sources indicated.

This system was analyzed by Wilson, Anderson-Cook, and Huzurbazar (2011). To illustrate our methodology, we have made some changes to the data used there. The simulated data are provided in online Supplementary Tables 6–8. Figure 11 shows plots of the data for K14, JE1, and JK20. In particular:

- We used the degradation data as given by Wilson, Anderson-Cook, and Huzurbazar (2011) for subsystem JK20, although we did not use the binary data related to catastrophic failures. Figure 10 shows that the subsystem analysis used degradation data with dotted shading.
- Components J4, J5, J6, J7, K19, and K20 have pass/fail data collected over time. In the literature by Wilson, Anderson-Cook, and Huzurbazar (2011), these data were aggregated because the times when the data were collected were not recorded. Here, we assume that we have these times; in particular, we assume that we have observed data at $t = 5, 50, 100, 150$, and 180 .
- Components K14, K15, and K16 were only tested as a subsystem by Wilson, Anderson-Cook, and Huzurbazar (2011). The data were binary, and no times of testing were recorded. Here we have added times for the subsystem testing ($t = 5, 50, 100, 150$, and 180) and have simulated additional independent testing of the individual components.
- Components JE1 and J8 had only expert opinion available in the literature by Wilson, Anderson-Cook, and Huzurbazar (2011). To illustrate our methodology, we have sim-

ulated failure times for these components (indicated by shading in Figure 10).

- No full-system data were analyzed by Wilson, Anderson-Cook, and Huzurbazar (2011). We simulated degradation data where the degradation rate is modeled as in (7) (and the induced lifetime distribution density is derived from (12)).

As in the three-components case, we model the binary data for components using the logistic regression model; the reliability function for the binary data for K14-K15-K16 can be derived simply. To model the failure times for components, we use a Weibull model.

JK20 is actually composed of two components in parallel. A failure occurs when the sum of the logarithm of amplitudes ($\log(A_1) + \log(A_2)$) is less than 13.12. Figure 11 shows the amplitude data (natural logarithm scale). As in the literature by Wilson, Anderson-Cook, and Huzurbazar (2011), we assume that $\log(A_1)$ and $\log(A_2)$ are independent and identically distributed $\text{Normal}(a + bt, \sigma^2)$ so that the reliability function of JK20 is

$$R_{JK20}(t) = \int_{13.12}^{\infty} \frac{1}{\sqrt{2\pi(2\sigma^2)}} \exp\left(-\frac{(x - 2a - 2bt)^2}{2(2\sigma^2)}\right) dx.$$

The data for the amplitudes are given in online supplementary Table 8 and represent one measurement per unit.

Posterior distributions for system reliability and for selected components are given in Figure 12. The estimation for the

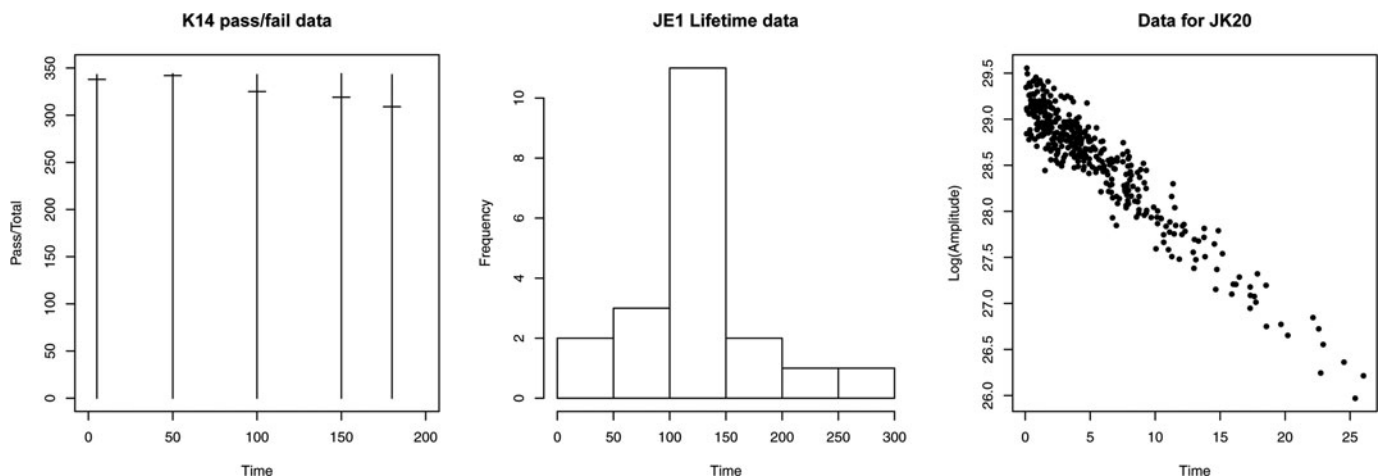


Figure 11. Examples of component-level data for the complex system. The first panel shows binary data for K14. Each unit is measured only once, and for each time point, the line heights indicate the total number of units tested, with the short horizontal lines indicating the number of successful trials. The degradation data for JK20 are the natural log of the amplitude data.

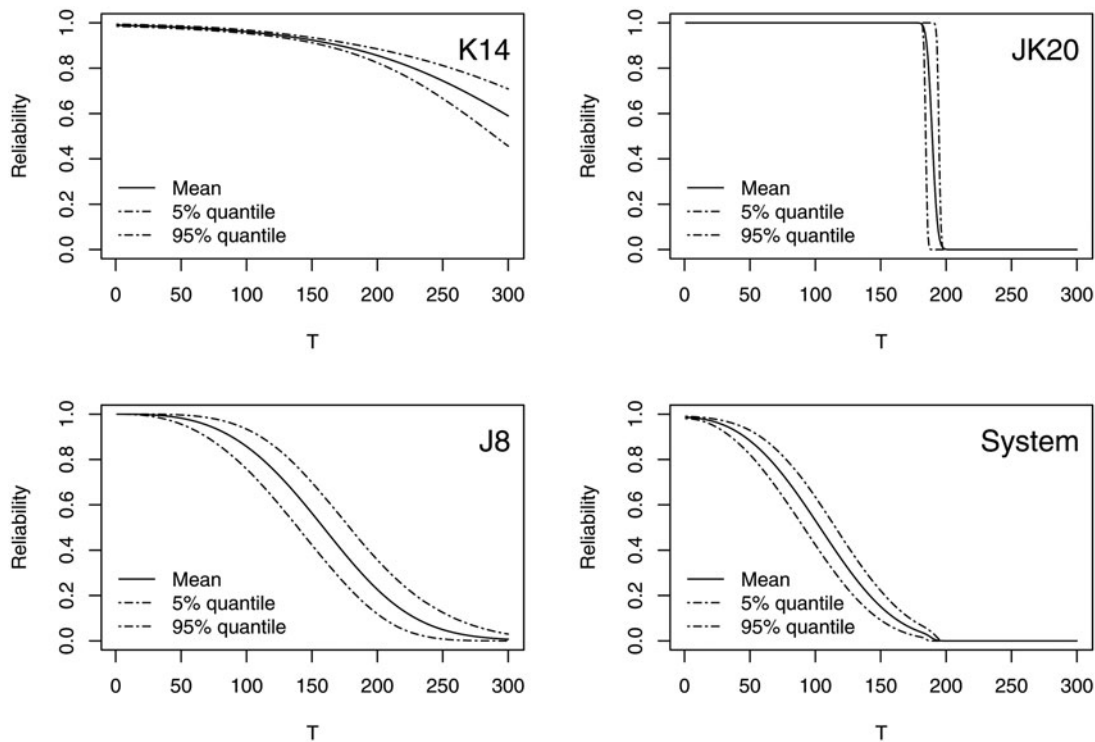


Figure 12. Posterior distributions for system reliability and for components K14, JK20, and J8.

more complex system is very similar to the three-component series system except that the reliability function and its derivative, used to determine the induced lifetime distribution, and the MCMC implementation require considerably more detailed bookkeeping.

5. EXTENSION AND DISCUSSION

In this article, we propose a unified methodology to estimate system reliability for a multicomponent complex system with different types of information. This methodology uses the relationships among reliability functions between a system and its components to combine models at different levels into one model. The model for the system is developed in a consistent and compatible way so that it naturally eliminates the aggregation errors. As a result, all the data and information are used to assess the system and component reliabilities.

This methodology can clearly be extended to more complex models for the data. For example, we might have dependence between basic events, which we could model using bivariate lifetime distributions, and different forms of degradation models. We can easily extend the approach to deal with censored lifetime data. In this case, we just need to replace $f_i(t_{ik} | \Theta_i)$ in the likelihood function of (3) by the corresponding forms depending on censoring type (e.g., see table 4.3 by Hamada et al. 2008). In fact, the binary data for the system are essentially right censored (pass) or left censored (fail). Another example requiring a more complex model is the situation where we have field data for systems and lab data for components. There can be difficulties in reconciling the differences between the two types of data; this is certainly a limitation of the simple models presented in this article.

Another extension is the application of the methodology to systems represented as generalizations of the fault tree. For example, consider the Bayesian network in Figure 13. Using $C_i = 0$ (1) to denote that component i is working (not working), the relationships given in (13) describing the dependence among the components are used to fully specify the Bayesian network.

$$\begin{aligned}
 \Pr(C_0 = 1 | C_1 = 1, C_2 = 1, C_3 = 1) &= 0.9, \\
 \Pr(C_0 = 1 | C_1 = 0, C_2 = 1, C_3 = 1) &= 0.4, \\
 \Pr(C_0 = 1 | C_1 = 1, C_2 = 0, C_3 = 1) &= 0.3, \\
 \Pr(C_0 = 1 | C_1 = 1, C_2 = 1, C_3 = 0) &= 0.5, \\
 \Pr(C_0 = 1 | C_1 = 0, C_2 = 0, C_3 = 1) &= 0.1, \\
 \Pr(C_0 = 1 | C_1 = 1, C_2 = 0, C_3 = 0) &= 0.05, \\
 \Pr(C_0 = 1 | C_1 = 0, C_2 = 1, C_3 = 0) &= 0.25, \\
 \Pr(C_0 = 1 | C_1 = 0, C_2 = 0, C_3 = 0) &= 0.
 \end{aligned} \tag{13}$$

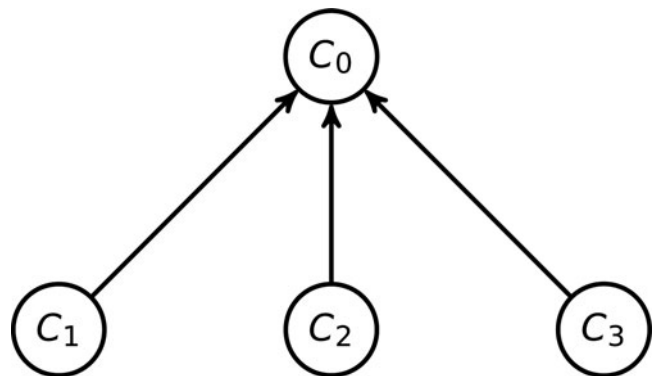


Figure 13. Bayesian network generalization of the example system.

For this generalized system, the relationships between reliability functions become more complicated. With the parameters suppressed, $R_0(t)$ is expressed as

$$\begin{aligned} R_0(t) = & 0.9R_1(t)R_2(t)R_3(t) + 0.4(1 - R_1(t))R_2(t)R_3(t) \\ & + 0.3R_1(t)(1 - R_2(t))R_3(t) + 0.5R_1(t)R_2(t) \\ & \times (1 - R_3(t)) + 0.1(1 - R_1(t))(1 - R_2(t))R_3(t) \\ & + 0.05R_1(t)(1 - R_2(t))(1 - R_3(t)) \\ & + 0.25(1 - R_1(t))R_2(t)(1 - R_3(t)). \end{aligned}$$

Then all the procedures for estimating the reliabilities for the system in fault tree quantification can be applied to this system with dependent components by updating the reliability function for the system. Of course, a more plausible model would be one in which the conditional probabilities are estimated from data.

In general, we can apply our methodology to any data structure as long as we can build up models for nonbasic events from the relationships among the reliability functions of the basic events. As the systems become more complicated, it may be difficult to explicitly perform the differentiation required to determine the probability density function for, say, the lifetime data. We can then employ numerical differentiation instead of writing down the explicit analytical form of the probability density function or use automatic differentiation to help obtain the analytical form. In addition, model checking becomes much more difficult; appropriate diagnostics are an open research question.

SUPPLEMENTARY MATERIALS

Code: C and R code for implementing MCMC algorithms for the proposed models (zip file).

Supplementary text: Tables containing the data for the example of complex system, prior distributions for the three-component example, and results for each scenario in the three-component example (including Scenario 1 with the Bayesian melding prior) (pdf).

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