ST 740: Multiparameter Inference

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Less Nice Example

Suppose that our sampling distribution is Gamma(α, β) and we choose independent marginal prior distributions $\pi(\alpha) \sim$ Gamma$(2, 1)$ and $\pi(\beta) \sim$ Gamma$(5, 1)$.

$$
\begin{aligned}
\pi(\alpha, \beta \mid \mathbf{y}) &\propto \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \exp(-\beta \sum y_i)(\prod y_i)^{\alpha-1} \exp(-(\alpha+\beta))\alpha\beta^4 \\
&\propto \frac{\beta^{n\alpha+4}}{\Gamma(\alpha)^n} \exp(-\beta(\sum y_i + 1))\alpha \exp(-\alpha)(\prod y_i)^{\alpha-1}
\end{aligned}
$$

We want to make posterior inferences about α, β, and the predictive distribution for the next observation.

Method 3: Rejection Sampling

We want a random sample from some distribution $\pi(\theta \mid \mathbf{y})$. We may not know this distribution's normalizing constant.

Step 1: Choose another probability density $p(\theta)$ such that

- It is easy to simulate draws from p
- The density of p resembles $\pi(\theta \mid \mathbf{y})$ in terms of location and spread
- For all θ and a constant c, $\pi(\theta \mid \mathbf{y}) \leq cp(\theta)$.

Rejection Sampling

If you can find such a $p(\theta)$, then the algorithm to follow is

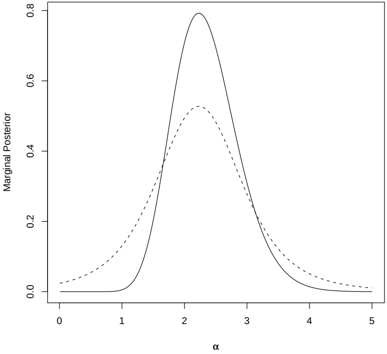
- Simulate independently $\theta^{(i)}$ from $p(\theta)$
- Simulate independently $u^{(i)}$ from a Uniform$(0, 1)$ distribution
- If $u^{(i)} \leq \frac{\pi(\theta^{(i)} \mid \mathbf{y})}{cp(\theta^{(i)})}$ then accept $\theta^{(i)}$ as a draw from the density $\pi(\theta \mid \mathbf{y})$. Otherwise reject $\theta^{(i)}$ (and throw it away).
- Keep going until you get a large enough sample of "accepted" θs

Notes:

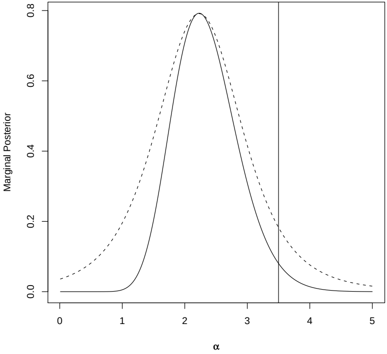
- For a proof that this works, see Devroye (1988) or Ripley (1987).
- The better the "envelope" ($cp(\theta)$) the more efficient the sampling.
- There are a variety of adaptive versions of this algorithm where you revise $p(\theta)$ as you go along, e.g., Gilks (1990).

Rejection Sampling

$\pi(\alpha \mid \mathbf{y})$ and t-distribution(df $= 4$, $\mu = 2.2304, \sigma^2 = 0.2555533$)



$\pi(\alpha \mid \mathbf{y})$ and 1.5*t-distribution(df $=$ 4, $\mu = 2.2304, \sigma^2 = 0.2555533$)



Method 4: SIR Sampling/Weighted Bootstrap

We want a random sample from some distribution $\pi(\theta \mid \mathbf{y})$. We may not know this distribution's normalizing constant.

Step 1: Choose another probability density $p(\theta)$ such that

- It is easy to simulate draws from p
- The density of p resembles $\pi(\theta \mid \mathbf{y})$ in terms of location and spread

But this time, we're having trouble figuring out what c should be.

SIR Sampling

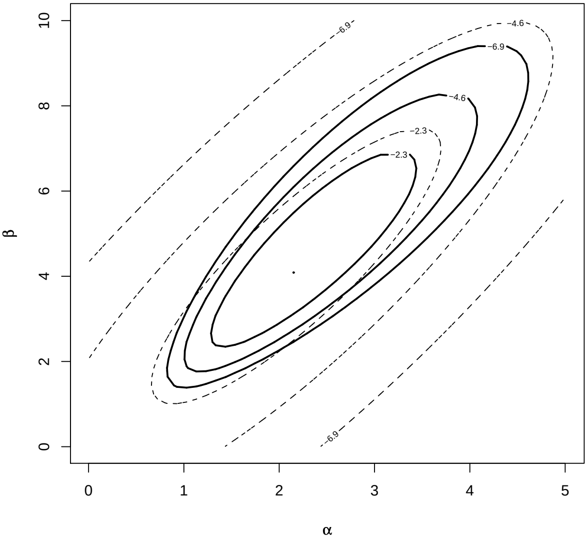
The algorithm to follow is

- Simulate independently $\theta^{(i)}$ from $p(\theta)$, $i = 1, \ldots, m$
- Calculate $w_i = \frac{\pi(\theta^{(i)} \mid \mathbf{y})}{p(\theta^{(i)})}$
- Normalize the weights, $q_i = \frac{w_i}{\sum_{i=1}^{m} w_i}$
- Resample (with replacement) the $\theta^{(i)}$ with weights q_i

Notes:

1. You want to make sure that the proposal distribution $p(\theta)$ has heavier tails than the posterior distribution $\pi(\theta \mid \mathbf{y})$ or you will have trouble getting samples from the tails.
2. t distributions are again a good choice here.
3. Check your weights and make sure that you don't have one or two large ones and the rest tiny. This depends on how well $p(\theta)$ approximates $\pi(\theta \mid \mathbf{y})$.

SIR Sampling



More Details

Works the same way for the multiparameter posterior distributions, but the proposal densities are usually multivariate t distributions.

Metropolis-Hastings Algorithm

The algorithm to generate a draw $\theta^{(i+1)}$:

1. Given a draw $\theta^{(i)}$ in iteration i, sample a *candidate* draw θ^* from a *proposal distribution* $J(\theta^* \mid \theta^{(i)})$
2. Calculate
$$r = \frac{\pi(\theta^* \mid \mathbf{y})/J(\theta^* \mid \theta^{(i)})}{\pi(\theta^{(i)} \mid \mathbf{y})/J(\theta^{(i)} \mid \theta^*)}$$
3. If $r \geq 1$, accept the draw θ^* and set $\theta^{(i+1)} = \theta^*$.
4. If $r < 1$, accept the draw and set $\theta^{(i+1)} = \theta^*$ with probability r.
5. Stay in place (do not accept the draw) with probability $1 - r$. Then $\theta^{(i+1)} = \theta^{(i)}$.

Random Walk Metropolis

- Most popular, easy to use.
- Proposal is a normal centered at the current draw.

$$J(\theta^{(*)} \mid \theta^{(i)}) = N(\theta^{(i)}, V)$$

- This proposal is symmetric in $\theta^{(*)}$ and $\theta^{(i)}$, so r simplifies. (When the proposal is symmetric in $\theta^{(*)}$ and $\theta^{(i)}$, the algorithm is called the *Metropolis* algorithm.)

Choosing V

1. V too small: takes long to explore parameter space
2. V too large: jumps to extremes are less likely to be accepted. Stay in the same place too long.

Do some experimentation to get V set right. The optimal acceptance rate (from some theory results) is between 25% and 50% for this type of proposal distribution. Gets lower with higher dimensional problem.

Independence Sampler

- Proposal does not depend on $\theta^{(i)}$.
- Just find a distribution $g(\theta)$ and sample from it.
- Can work well if $g(\theta)$ is a good approximation to $\pi(\theta \mid \mathbf{y})$ and has heavier tails.
- Think normal approximations to posterior (or t distributions) with inflated variances.

Metropolis-Hastings Algorithm

In general, you want to choose a $J(\theta^{(*)} \mid \theta^{(i)})$ so that starting from any $\theta^{(i)}$ you can move to any point in the support of $\pi(\theta \mid \mathbf{y})$. (The support of $J(\cdot \mid \theta)$ contains the support of $\pi(\theta \mid y)$ for every θ.)

Metropolis-Hastings Algorithm

The acceptance probability r is the product of the ratio of the target density evaluated at the candidate and current parameter values

$$\frac{\pi(\theta^* \mid \mathbf{y})}{\pi(\theta^{(i)} \mid \mathbf{y})}$$

and the ratio of the proposal distribution of the current and candidate point

$$\frac{J(\theta^{(i)} \mid \theta^*)}{J(\theta^* \mid \theta^{(i)})}$$

- The first ratio encourages the algorithm to move to parameter values that have high posterior probability.
- The second ratio accounts for the fact that the proposal distribution might favor some values of the parameter over others.