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Bayesian networks for multilevel system reliability

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Abstract

Bayesian networks have recently found many applications in systems reliability; however, the focus has been on binary outcomes. In this paper we extend their use to multilevel discrete data and discuss how to make joint inference about all of the nodes in the network. These methods are applicable when system structures are too complex to be represented by fault trees. The methods are illustrated through four examples that are structured to clarify the scope of the problem.

Keywords: Bayesian belief network; System reliability; Bayesian reliability

1. Introduction

Bayesian networks (BNs) have recently found many applications in reliability (cf. [3,5]). The focus of the applications has generally been on dichotomous variables representing binary outcomes, i.e., either components and systems are functioning or not functioning. A natural example of this is the BN representation of a fault tree (cf. [1,2,4,7]). However, many system structures are too complex to be represented by fault trees. For example, consider a system in the design phase. We may want to model the system with a BN to capture the uncertainty about whether two components that function individually will interoperate and produce a working system. This article extends complex systems modeling using BNs for multilevel discrete data.

Multilevel data are common in the surveillance of weapon stockpiles. Typically, it is too expensive to test large numbers of systems, and often, testing is destructive. Consequently, there are system tests with no component information. Component testing is often less expensive, and thus is also performed. Occasionally, there is data on the conditional distributions—perhaps enough data were collected to perform fault attribution when the system failed. Surveillance programs also have a need to make inference through the system. If we understand which components have large uncertainties that are reflected in the system uncertainty, we can allocate resources to additional testing.

Current applications of BNs focus primarily on cause/consequence relationships of the network, i.e., inference about an event (consequence) based on information about one or more events (causes) leading to that event. Our interest lies in possibly making joint inference about all of the events. For example, given varying amounts of data on the consequence, we would like to make statements about the likely causes or perhaps combine that data with existing data on the causes to make inference about the system. In other words, we would like to work the BN for inference in all possible directions. We also consider combining discrete data collected over time within a BN structure.

Section 2 reviews the basics of BNs, Section 3 provides several examples, and Section 4 contains discussion and conclusions.

2. BNs for system representation

Reliability block diagrams and fault trees are the most common representations in system reliability analysis. The

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most common use is to find the probability distribution for the state of the system. However, there are situations where these models do not offer enough flexibility to capture features of the system. BNs generalize fault trees and reliability block diagrams by allowing components and subsystems to be related by conditional probabilities instead of deterministic AND and OR relationships.

Formally, a BN is a pair $N = \langle (V,E),P \rangle$, where (V,E) are the nodes (vertices) and edges of a directed acyclic graph and P is a probability distribution on V. Each node contains a random variable, and the directed edges between them define conditional dependence or independence among the random variables. Fig. 1 summarizes the three probabilistic relationships that can be specified in a BN. The key feature of a BN is that it specifies the joint distribution P over the set of nodes V in terms of conditional distributions. In particular, the joint distribution of V is given by

$$\prod_{v \in V} P(v|\text{parents}[v]),$$

where the parents of a node are the set of nodes with an edge pointing to the node.

For example, in the serial structure in Fig. 1a, node C is called a root node. Node C is the parent (or predecessor) of node B which is also called the child (or descendant) of node C. The directed branch linking the two nodes is interpreted to mean that the parent node has a direct influence on the child node. Node A has no descendants and it is called a leaf.

Fig. 1b shows a converging BN while Fig. 1c shows a diverging BN. The primary applications of BNs have been

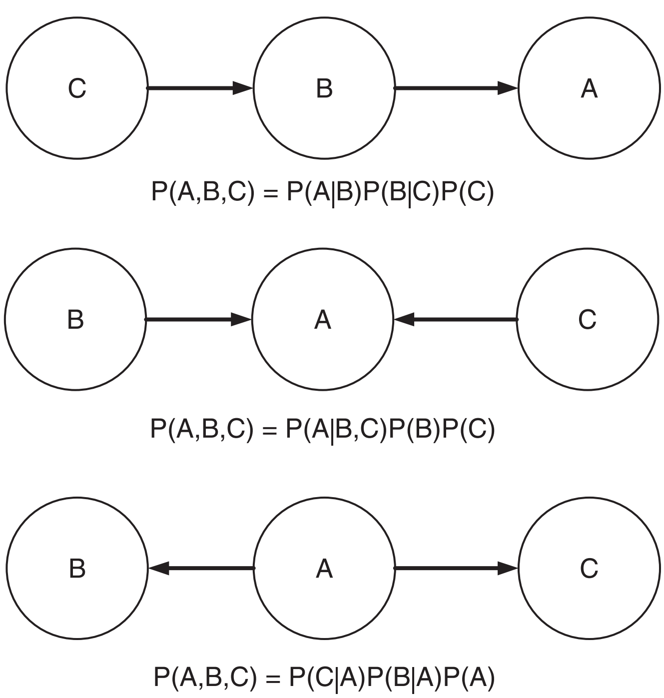

Fig. 1. Specifying joint probability distributions using a Bayesian network.

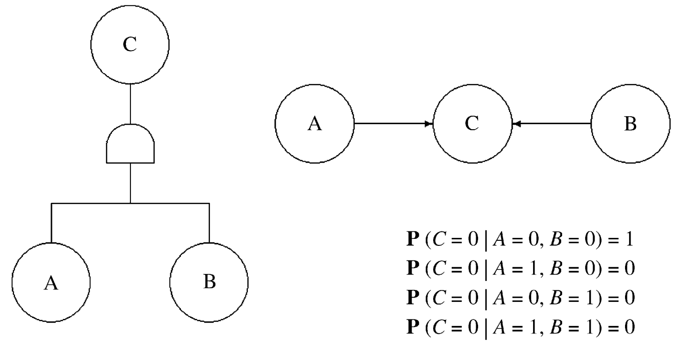

Fig. 2. Fault tree conversion to Bayesian network.

to make inferences about node A given information about nodes B and C. Our concern is to combine data available on node A along with data available on nodes B and C to make inference about all three nodes.

BNs can be used as a direct generalization of fault trees. The fault tree translation to a BN is straightforward, with the basic events that contribute to an intermediate event represented as parents and a child. Fig. 2 shows the correspondence between a fault tree AND gate and a BN converging structure.

Notice that a fault tree implies specific conditional probabilities. An OR gate in a fault tree can also be represented by the converging structure in Fig. 1b, with appropriate changes to the conditional probabilities.

3. Inference from multilevel data

Consider the BN from Fig. 3. Node S represents the full system and nodes C_1 and C_2 represent components. We illustrate the basic principles of inference with multilevel data in a BN using this particularly simple structure. Examples 1–3 consider the different cases with varying amounts of data. The simple structure illuminates the working of the BN in both directions.

The outcomes for C_1, C_2, and S are all binary, corresponding to nominal (1) and degraded (0) states of the components and system. Note that this is not a series system, nor does this system correspond to a fault tree. In fact, we may not have any details on the system architecture. It is possible that both C_1 and C_2 are in the degraded state and the system functions.

In Example 1, we calculate the unknown posterior marginal distributions for $P(S)$, $P(C_1)$, and $P(C_2)$ given that we know the conditional distributions from simulated data given in Table 1. In Example 2, we calculate the posterior distributions for the same marginal distributions and $P(S|C_1,C_2)$ when the both the marginal and conditional distributions are unknown. In Example 3, we calculate the posterior marginal and conditional distributions when we have additional data on some of the conditional distributions. In Example 4, we generalize the methods to the case when the data at each node are

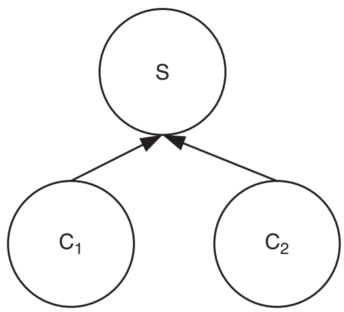

Fig. 3. Bayesian network for Examples 1–3.

Table 1
Conditional probabilities of S given C_1 and C_2 for Examples 1–3

C_1	C_2	$P(S=1 \mid C_1,C_2)$
1	1	0.95
1	0	0.70
0	1	0.40
0	0	0.10

collected over time. These scenarios allow us to study the system under the various types and levels of available data.

Example 1 (*Conditional distributions known*). This example considers the network of Fig. 3. C_1, C_2, and S each have binary outcomes corresponding to nominal (1) and degraded (0) states of the components and system. The system probability is represented by

$$P(S) = \sum_{i=0}^{1} \sum_{j=0}^{1} P(S=1 \mid C_1=i, C_2=j) P(C_1=i, C_2=j).$$

In this example, we assume that $P(S=1 \mid C_1=\mathrm{i}, C_2=j)$ is known and given in Table 1 for a given (i,j), but $P(C_1=i, C_2=j)$ is unknown for any given (i,j).

Table 3 gives data on the full system S and components C_1 and C_2. Given these data, the likelihood has the form

$$L(p_1,p_2) = p_1^{11}(1-p_1)^3 p_2^{37}(1-p_2)^4 p_S^8 (1-p_S)^3, \quad (1)$$

where

$$p_S = 0.95p_1p_2 + 0.70p_1(1-p_2) + 0.40(1-p_1)p_2 + 0.10(1-p_1)(1-p_2). \quad (2)$$

We use a Bayesian approach to calculate the unknown marginal probabilities. We specify a prior distribution for the unknown joint distribution $P(C_1=i, C_2=j)$ as the product of independent Uniform(0,1) distributions. The posterior distributions are calculated using Markov Chain Monte Carlo (MCMC).

Fig. 4 shows the marginal posterior distribution for $P(C_1=1)$ using all of the observed data (solid) and only the data collected directly about C_1 (dashed). Figs. 5 and 6 show the results for $P(C_2=1)$ and $P(S=1)$. The solid vertical lines in the figures are the values of $P(C_1=1)$,

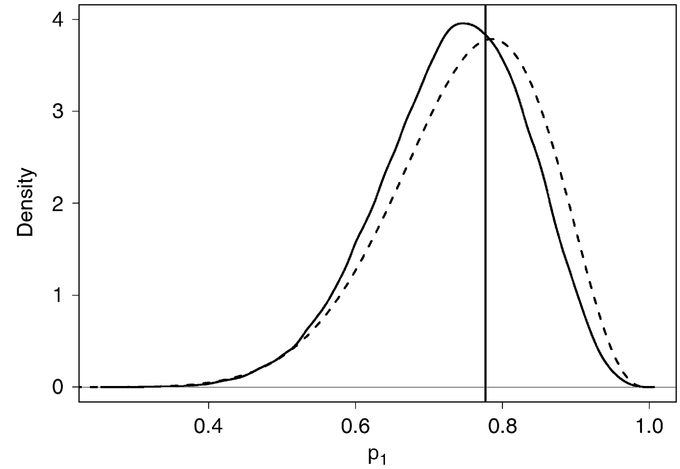

Fig. 4. Posterior distribution for $P(C_1=1)$: (i) using all data (solid); (ii) using only data observed at C_1 (dashed).

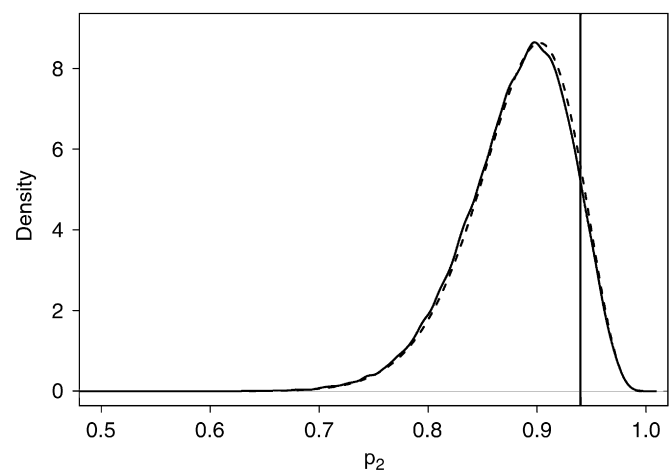

Fig. 5. Posterior distribution for $P(C_2=1)$: (i) using all data (solid); (ii) using only data observed at C_2 (dashed).

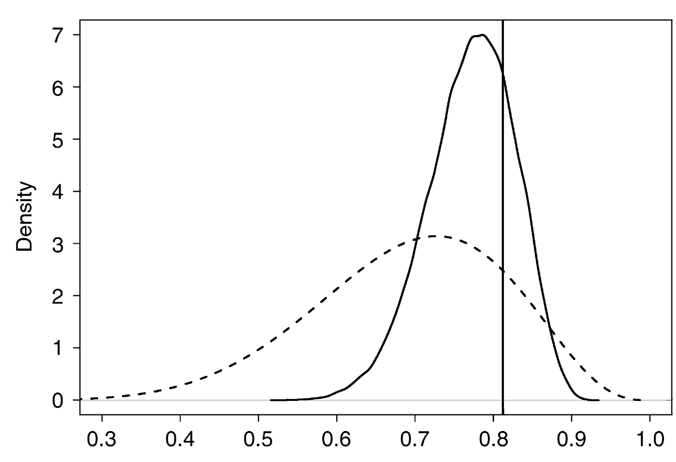

Fig. 6. Posterior distribution for $P(S=1)$: (i) using all data (solid); (ii) using only data observed at S (dashed).

$P(C_2 = 1)$, and $P(S = 1)$, given in Table 2, that we want to estimate. Notice that the incorporation of the component data adds considerable precision to the estimate of the system reliability.

Table 4 provides a numerical summary of the posterior distributions. Even when no data are observed at a particular component, the structure of the BN, the specification of the prior distribution, and the data at the system and the other component allow estimation at that component where no data are observed. Fig. 7 shows the estimate for the reliability at C_1 when the first row of Table 3 is excluded from the analysis (dotted) compared to the Uniform(0,1) prior distribution (dashed) and the estimate when the data are available (solid) (Table 4).

Table 2
Marginal probabilities for Examples 1–3

$P(C_1 = 1)$	0.777
$P(C_2 = 1)$	0.940
$P(S = 1)$	0.812

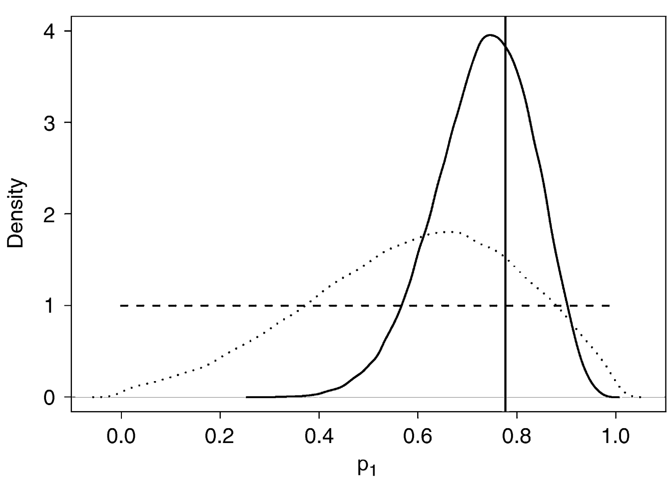

Fig. 7. Posterior distribution for $P(C_1 = 1)$: (i) using all data (solid); (ii) assuming no data was observed at C_1 (dotted); (iii) prior distribution for $P(C_1 = 1)$ (dashed).

Table 3
Data for Examples 1–3

C_1	11/14
C_2	37/41
System (S)	8/11

Table 4
Summary of posterior distributions for Example 1

	Mean	2.5%	97.5%	Actual
p_1	0.73	0.52	0.90	0.777
p_2	0.88	0.77	0.96	0.940
p_S	0.77	0.65	0.87	0.812

Example 2 (*Marginal and conditional distributions unknown*). This example changes the assumptions of Example 1 so that the conditional probabilities, $P(S = 1|C_1 = i, C_2 = j)$, and marginal probabilities, $P(C_1 = i, C_2 = j)$, are unknown. We again have the data in Table 3 and use the product of Uniform(0,1) distributions to specify the prior on $P(C_1 = i, C_2 = j)$. Additionally, we specify informative priors for the conditional distributions:

$P(S = 1|C_1 = 1, C_2 = 1) \sim \text{Uniform}(0.75, 1)$,
$P(S = 1|C_1 = 1, C_2 = 0) \sim \text{Uniform}(0.25, 0.75)$,
$P(S = 1|C_1 = 0, C_2 = 1) \sim \text{Uniform}(0.25, 0.75)$,
$P(S = 1|C_1 = 0, C_2 = 0) \sim \text{Uniform}(0, 0.25)$.

Given the data in Table 3, the likelihood has the form of (1), but it is now a function of p_1, p_2, and the unknown conditional probabilities, τ_1, τ_2, τ_3, and τ_4. The expression for p_S is similar to (2) but now has the additional unknown parameters τ_1, τ_2, τ_3, and τ_4,

$$p_S = \tau_1 p_1 p_2 + \tau_2 p_1(1 - p_2) + \tau_3(1 - p_1)p_2 + \tau_4(1 - p_1)(1 - p_2). \quad (3)$$

Table 5 provides a numerical summary of the posterior distributions. Figs. 8 and 9 show that there is very little difference between the posterior distributions for the component reliabilities with the conditional distributions known or unknown. However, Fig. 10 shows that there is a larger difference in the posterior for the system reliability. The estimate has less precision and a shift in location.

Fig. 11 shows the posterior distribution of τ_1. The data in Table 3 provide virtually no information about the conditional probabilities $P(S = 1|C_1 = i, C_2 = j)$: the posterior distribution is very similar to the Uniform(0.75,1) prior distribution. The posterior distributions for τ_2, τ_3, and τ_4 show that these are also essentially unchanged from their prior distributions and are not presented.

Example 3 (*Marginal and conditional distributions unknown, additional data to estimate conditional distributions*). Suppose that for Example 2, we have the original data from Table 3 and some additional data from Table 6. The data in Table 6 were collected so that we have information about the state of the system and all of its components from the same test. Recall that this is not a

Table 5
Summary of posterior distributions for Example 2

	Mean	2.5%	97.5%	Actual
p_1	0.77	0.53	0.92	0.777
p_2	0.88	0.77	0.96	0.940
p_S	0.73	0.60	0.86	0.812
τ_1	0.87	0.76	0.99	0.95
τ_2	0.50	0.26	0.75	0.70
τ_3	0.50	0.25	0.74	0.40
τ_4	0.13	0.0062	0.24	0.10

fault tree or a series system, so we could have system failures even when both of the components are working (although we did not observe any): Similarly, we can have system failures even when one of the components is working, and system successes even when both components fail.

Given the data in Tables 3 and 6, the likelihood has the form

$$L(p_1, p_2, \tau_1, \tau_2, \tau_3, \tau_4) = p_1^{11}(1-p_1)^3 p_2^{37}(1-p_2)^4 p_S^8(1-p_S)^3 \times \tau_1^{14}\tau_3^5(1-\tau_3)^7\tau_4^2(1-\tau_4)^{16},$$

where the expression for p_S is given in (3).

Using the same prior distributions for p_1, p_2, τ_1, τ_2, τ_3, and τ_4 as in Example 2, we calculate the posterior distributions using MCMC. Table 7 provides a numerical summary of the posterior distributions.

With the data on the conditionals, even though direct data on τ_2 is not available, the posterior distributions have changed from the prior distribution.

Fig. 12 gives the posterior distribution on τ_1. The posterior distribution for the system reliability (dotted), shown in Fig. 13, changes in the expected direction: there is less variability than when the conditionals are unknown and have no data (dashed) and more than when the conditional are completely known (solid).

Example 4 (*Logistic regression*). In this example, we extend the analysis to the case where we have binary data collected over time in the BN. Such situations arise commonly in weapons surveillance programs. Suppose we wish to predict the reliability of a weapon system as a function of time. Suppose further that the system is simple enough that it comprises of two components. Our data represent snapshots across time of system and component successes and failures. Periodically, some number of weapons are removed from the stockpile and tested as full systems. If a failure is observed, it cannot be attributed to a specific component. In addition, other weapons are removed from the stockpile and broken down into components, which are individually tested. The system and component tests generally do not occur at the same points in time, so it is important that the analysis methodology not require concurrent testing. Given such information, we would like to predict the system reliability over time.

The data for this analysis are given in Table 8. Notice that in Example 2, we analyzed one column of complete data (e.g., the data in column 3 of Table 8). Here we analyze a sequence of data like that in Table 3 across time and with missing information. As with the previous examples, these data were simulated using the conditional probabilities given in Table 1.

We fit a logistic regression model for C_1 and C_2, with

$$P(C_i = 1) = p_i = \frac{\exp(\alpha_i + \beta_i t)}{1 + \exp(\alpha_i + \beta_i t)}.$$

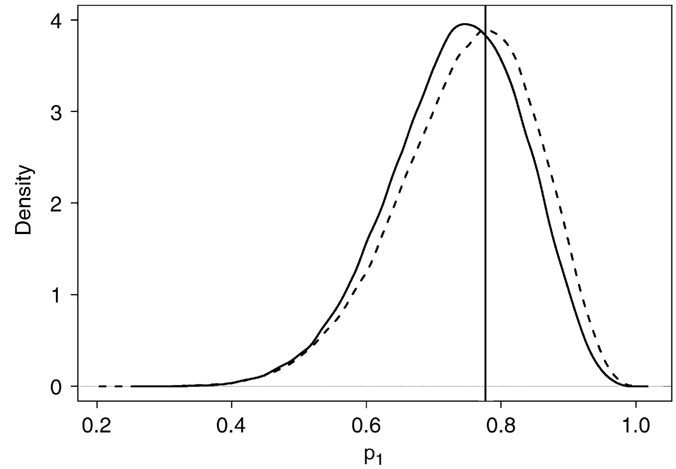

Fig. 8. Posterior distribution for $P(C_1 = 1)$ with: (i) conditional distributions unknown (dashed); (ii) conditionals known as in Example 1 (solid).

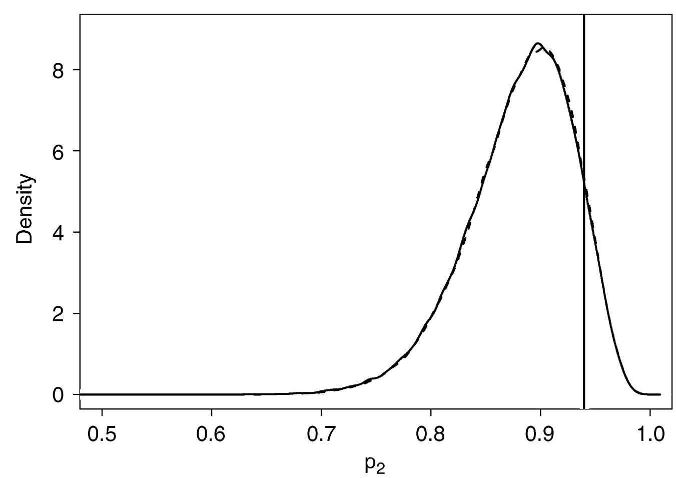

Fig. 9. Posterior distribution for $P(C_2 = 1)$ with: (i) conditional distributions unknown (dashed); (ii) posterior distribution with conditionals known as in Example 1 (solid).

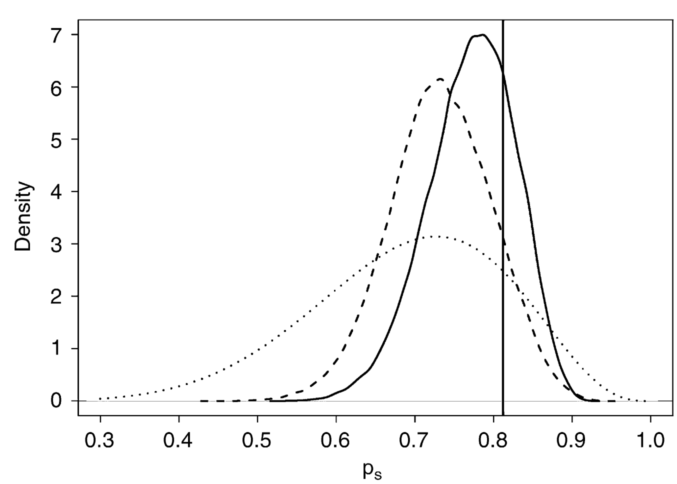

Fig. 10. Posterior distribution for $P(S = 1)$ with: (i) conditional distributions unknown (dashed); (ii) conditionals known as in Example 1 (solid); posterior distribution using only data observed at S (dotted).

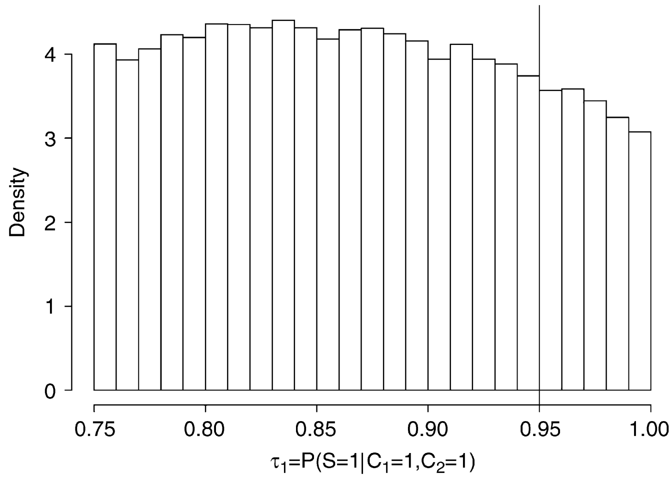

Fig. 11. Posterior distribution for $\tau_1 = P(S = 1|C_1 = 1, C_2 = 1)$ with conditional distributions unknown.

Table 6
Additional data for conditional distributions for Example 3

$S = 1\|C_1 = 1, C_2 = 1$	14/14
$S = 1\|C_1 = 0, C_2 = 1$	5/12
$S = 1\|C_1 = 0, C_2 = 0$	2/18

Table 7
Summary of posterior distributions for Example 3

	Mean	2.5%	97.5%	Actual
p_1	0.74	0.52	0.91	0.777
p_2	0.88	0.77	0.96	0.940
p_S	0.76	0.63	0.87	0.812
τ_1	0.94	0.80	1.0	0.95
τ_2	0.50	0.26	0.74	0.70
τ_3	0.44	0.27	0.67	0.40
τ_4	0.13	0.033	0.24	0.10

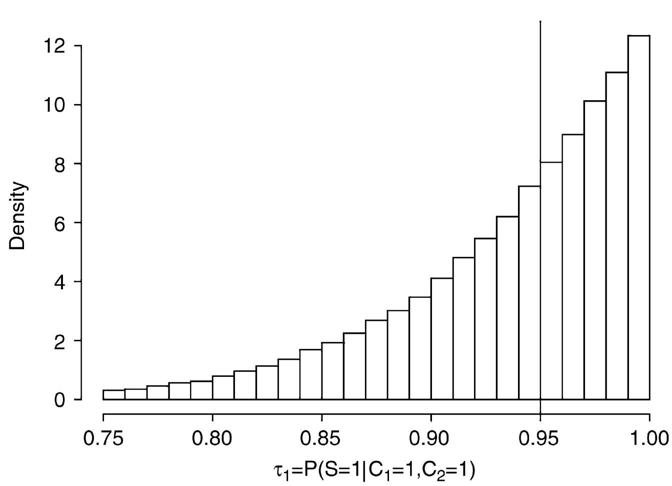

Fig. 12. Posterior distribution for $\tau_1 = P(S = 1|C_1 = 1, C_2 = 1)$ with additional data from Table 4.

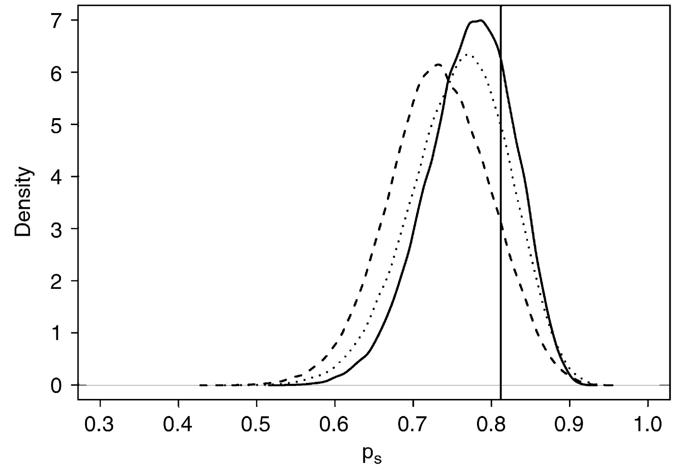

Fig. 13. Posterior distribution for $P(S = 1)$ with: (i) conditional distributions unknown (dashed); (ii) posterior distribution with conditionals known as in Example 1 (solid); (iii) posterior distribution with additional data on conditionals (dotted).

This implies that the reliability of the system at time t has the following form:

$$
\begin{aligned}
p_S = {} & \tau_1 \frac{\exp(\alpha_1 + \alpha_2 + (\beta_1 + \beta_2)t)}{(1 + \exp(\alpha_1 + \beta_1 t))(1 + \exp(\alpha_2 + \beta_2 t))} \\
& + \tau_2 \frac{\exp(\alpha_1 + \beta_1 t)}{(1 + \exp(\alpha_1 + \beta_1 t))(1 + \exp(\alpha_2 + \beta_2 t))} \\
& + \tau_3 \frac{\exp(\alpha_2 + \beta_2 t)}{(1 + \exp(\alpha_1 + \beta_1 t))(1 + \exp(\alpha_2 + \beta_2 t))} \\
& + \tau_4 \frac{1}{(1 + \exp(\alpha_1 + \beta_1 t))(1 + \exp(\alpha_2 + \beta_2 t))}.
\end{aligned}
$$

While each component is modeled by a logistic regression, this structure does not carry over to the system.

Fig. 14 shows the posterior mean and 90% credible interval for system reliability. Table 9 provides a numerical summary of the posterior distributions. This is often the stated goal of analysis in a weapon surveillance context. This information can be useful in developing tactics. Suppose that the system requires 80% reliability. As the system ages and its reliability drops, it may be necessary to use two systems instead of one to achieve the requirement. However, in addition to the full system summary, it can be informative to consider the reliability over time for each component. Fig. 15 shows the posterior mean and 90% credible interval for component 1 reliability. When considering a life extension program, where a particular component might be changed to improve reliability, predicting reliability into the future provides useful information.

4. Discussion

In this paper we provide a generalization of Hamada et al. [2], which discusses how to model multilevel binary data for the special case of fault trees. Fault trees do not always capture the complexities of a system. We suggest the use of

Table 8
Data for Example 4

	1	2	3	4	5	6	7	8	9	10
C_1	19/19	—	16/19	12/12	—	—	9/13	—	—	3/10
C_2	35/35	47/48	37/38	—	44/45	35/37	—	33/42	—	30/39
System (S)	15/16	14/14	12/14	—	13/14	11/12	—	5/16	12/19	8/14

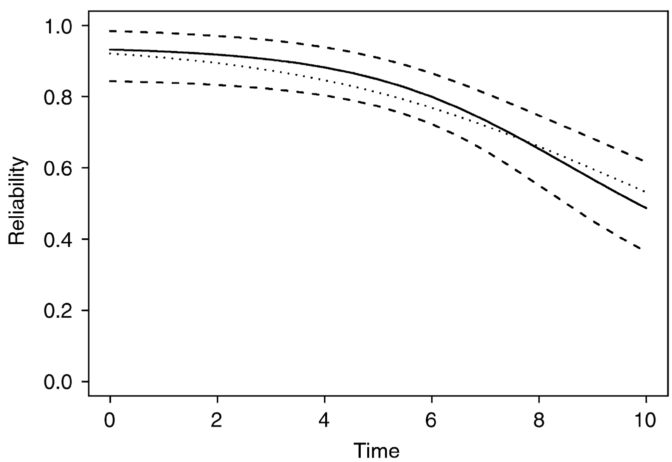

Fig. 14. Posterior distribution of $P(S=1)$ with a 90% credible interval, actual value plotted as dotted line.

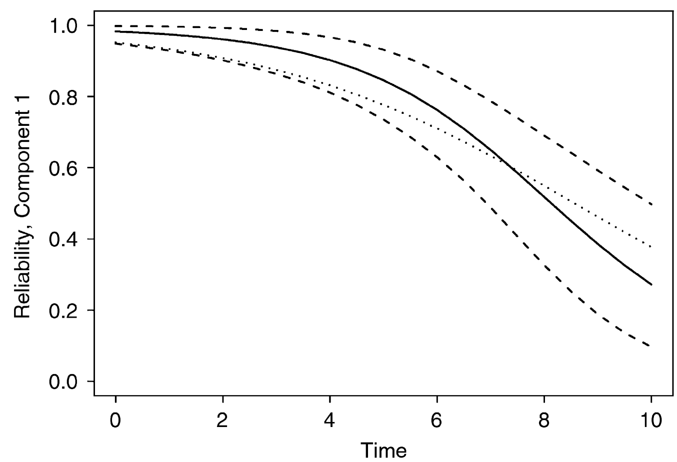

Fig. 15. Posterior distribution of $P(C_1=1)$ with a 90% credible interval, actual value plotted as dotted line.

Table 9
Summary of posterior distributions for Example 4

	Mean	2.5%	97.5%	Actual
α_1	4.58	2.92	6.60	3.0
α_2	5.15	3.76	6.70	4.5
β_1	−0.56	−0.85	−0.33	−0.35
β_2	−0.42	−0.60	−0.24	−0.35
τ_1	0.94	0.85	1.0	0.95
τ_2	0.49	0.26	0.74	0.70
τ_3	0.45	0.26	0.70	0.40
τ_4	0.12	0.0066	0.24	0.10

BNs to model multilevel system reliability. Our focus is on using information at all levels and in all directions of the BN.

Example 1 considers the case of a BN where the conditional probabilities are known, but the component and system reliabilities need to be estimated. Example 2 extends Example 1 to consider the case where the conditional probabilities are also unknown. Example 3 shows the impact of adding additional data to Example 2 to improve the estimation of the conditional probabilities. Example 4 is the culmination of Examples 1 and 2 where we consider component and system data collected over time.

All of the analyses considered here are fully Bayesian, as they specify prior distributions for the unknown parameters and use Bayes' Theorem to calculate the posterior distributions. Although this paper considers uniform prior distributions, other prior distributions work equally well within this framework.

Further work concerns multinomial data. While the generalizations of the likelihoods in this case appears straightforward, for a system with a large number of nodes (e.g., Wilson et al. [6]), the implementation is problematic. In a large system, analytical expressions for system reliability are intractable. Other problems include prior specification, efficient computation, and information integration.

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References

[1] Bobbio A, Portinale L, Minichino M, Ciancamerla E. Improving the analysis of dependable systems by mapping fault trees into Bayesian networks. Reliab Eng Syst Safe 2001;71:249–60.
[2] Hamada M, Martz H, Reese CS, Graves T, Johnson V, Wilson A. A fully Bayesian approach for combining multilevel failure information in fault tree quantification and optimal follow-on resource allocation. Reliab Eng Syst Safe 2004;86:297–305.
[3] Lee B. Using Bayesian belief networks in industrial FMEA modeling and analysis. In: Proceedings of the annual reliability and maintainability

symposium, international symposium on product quality and integrity, Philadelphia, 2001, January 2001. p. 7–15.

[4] Portinale L, Bobbio A, Montani S. From artificial intelligence to denpendability: modeling and analysis with Bayesian networks. In: Wilson AG, Limnios N, Keller-McNulty SA, Armijo YM, editors. Modern statistical and mathematical methods in reliability. Singapore: World Scientific; 2005. p. 365–81.

[5] Sigurdsson J, Walls L, Quigley J. Bayesian belief nets for managing expert judgement and modeling reliability. Qual Reliab Eng Int 2001;17:181–90.

[6] Wilson A, McNamara L, Wilson G. Information integration for complex systems. Reliab Eng Syst Saf 2007;92:121–30.

[7] Wood AP. Multistate block diagrams and fault trees. IEEE Trans Reliab 1985;34:236–40.