# ST 740: Multiparameter Inference

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# Joint as Conditional times Marginal

$$
\begin{aligned}
\pi(\theta_1 \mid \mathbf{y}) &= \int \pi(\theta_1, \theta_2 \mid \mathbf{y}) d\theta_2 \\
&= \int \pi(\theta_1 \mid \theta_2, \mathbf{y}) \pi(\theta_2 \mid \mathbf{y}) d\theta_2
\end{aligned}
$$

# Posterior Predictive Distribution

$$
\begin{aligned}
f(\tilde{y} \mid y) &= \int f(\tilde{y} \mid \theta)\pi(\theta \mid \mathbf{y})d\theta \\
&= \int f(\tilde{y} \mid \theta, \mathbf{y})\pi(\theta \mid \mathbf{y})d\theta \\
&= \int f(\tilde{y}, \theta \mid \mathbf{y})d\theta
\end{aligned}
$$

# Conjugate Prior for Normal Model

Using a noninformative prior, we found that

$$\pi(\mu \,|\, \sigma^2, \mathbf{y}) \quad \propto \quad \text{Normal}(\bar{y}, \sigma^2/n)$$
$$\pi(\sigma^2 \,|\, \mathbf{y}) \quad \propto \quad \text{InverseGamma}(\frac{n-1}{2}, \frac{(n-1)s^2}{2})$$

Factoring $\pi(\mu, \sigma^2) = \pi(\mu \,|\, \sigma^2)\pi(\sigma^2)$, the conjugate prior for $\sigma^2$ would also be inverse gamma, and the conjugate prior for $\mu$ (conditional on $\sigma^2$) would be normal.

Consider

$$\mu \,|\, \sigma^2 \quad \sim \quad \text{Normal}(\mu_0, \sigma^2/\kappa_0)$$
$$\sigma^2 \quad \sim \quad \text{InverseGamma}(\nu_0, \eta_0)$$

# Conjugate Prior for Normal Model

The joint prior distribution has the form

$$\pi(\mu, \sigma^2) \propto \frac{1}{\sigma} \exp(-\frac{\kappa_0}{2\sigma^2}(\mu - \mu_0)^2)(\frac{1}{\sigma^2})^{\nu_0+1} \exp(-\eta_0/\sigma^2)$$

Note that $\mu$ and $\sigma^2$ are not independent a priori.

The general form of a Normal-Inverse Gamma$(\mu_0, \kappa_0, \nu_0, \eta_0)$ distribution is

$$f(\mu, \sigma^2 \mid \mu_0, \kappa_0, \nu_0, \eta_0) = \\ \frac{{\eta_0}^{\nu_0}}{\Gamma(\nu_0)} \frac{\kappa_0}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(\kappa_0(\mu - \mu_0)^2 + 2\eta_0\right)\right)(\frac{1}{\sigma^2})^{\nu_0+1}$$

# Conjugate Prior for Normal Model

The posterior density for $(\mu, \sigma^2)$ has the form

$$\pi(\mu, \sigma^2 \mid \mathbf{y}) \propto \frac{1}{\sigma} \exp(-\frac{\kappa_0}{2\sigma^2}(\mu - \mu_0)^2)(\frac{1}{\sigma^2})^{\nu_0+1} \exp(-\eta_0/\sigma^2)$$
$$\sigma^{-n} \exp(-\frac{1}{2\sigma^2}[(n-1)s^2 + n(\bar{y} - \mu)^2])$$
$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\nu_0+1} \left(\frac{1}{\sigma}\right) \exp(-\frac{\eta_0}{\sigma^2} - \frac{1}{2\sigma^2}(n-1)s^2)$$
$$\exp(-\frac{\kappa_0}{2\sigma^2}(\mu - \mu_0)^2 - \frac{n}{2\sigma^2}(\bar{y} - \mu)^2)$$

# Conjugate Prior for Normal Model

## Collecting Terms

$$
\begin{aligned}
\pi(\mu, \sigma^2 \mid \mathbf{y}) \propto & \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\nu_0+1} \exp(-\frac{1}{\sigma^2}(\eta_0 + \frac{(n-1)s^2}{2})) \\
& \left(\frac{1}{\sigma}\right) \exp\left(-\frac{1}{2\sigma^2}\left(\kappa_0\mu^2 - 2\kappa_0\mu_0\mu + \kappa_0\mu_0^2 + n\bar{y}^2 - 2n\bar{y}\mu + n\mu^2\right)\right) \\
\propto & \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\nu_0+1} \exp(-\frac{1}{\sigma^2}(\eta_0 + \frac{(n-1)s^2}{2})) \\
& \left(\frac{1}{\sigma}\right) \exp\left(-\frac{\kappa_0+n}{2\sigma^2}\left(\mu^2 - 2\frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0+n}\mu + \frac{\kappa_0\mu_0^2 + n\bar{y}^2}{\kappa_0+n}\right)\right)
\end{aligned}
$$

# Conjugate Prior for Normal Model

## Completing the Square

$$
\begin{aligned}
\pi(\mu, \sigma^2 \mid \mathbf{y}) \propto & \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\nu_0+1} \exp(-\frac{1}{\sigma^2}(\eta_0 + \frac{(n-1)s^2}{2})) \\
& \left(\frac{1}{\sigma}\right) \exp\left(-\frac{\kappa_0 + n}{2\sigma^2}(\mu - \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n})^2\right) \\
& \exp\left(-\frac{\kappa_0 + n}{2\sigma^2}\left(\frac{\kappa_0\mu_0^2 + n\bar{y}^2}{\kappa_0 + n} - \left(\frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n}\right)^2\right)\right) \\
\propto & \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+\nu_0+1} \exp(-\frac{1}{\sigma^2}(\eta_0 + \frac{(n-1)s^2}{2})) \exp\left(-\frac{\kappa_0 n}{2\sigma^2}(\bar{y} - \mu_0)^2)\right) \\
& \left(\frac{1}{\sigma}\right) \exp\left(-\frac{\kappa_0 + n}{2\sigma^2}(\mu - \frac{\kappa_0\mu_0 + n\bar{y}}{\kappa_0 + n})^2\right)
\end{aligned}
$$

# Conjugate Prior for Normal Model

Complete the square for $\mu$, and we see that the posterior distribution has the form:

$$\begin{aligned}
\mu \,|\, \sigma^2, \mathbf{y} &\sim \text{Normal}\left(\frac{\kappa_0}{\kappa_0 + n}\mu_0 + \frac{n}{\kappa_0 + n}\bar{y}, \frac{\sigma^2}{\kappa_0 + n}\right) \\
\sigma^2 \,|\, \mathbf{y} &\sim \text{InverseGamma}\left(\frac{n}{2} + \nu_0, \eta_0 + \frac{(n-1)s^2}{2} + \frac{\kappa_0 n(\bar{y} - \mu_0)^2}{2(\kappa_0 + n)}\right)
\end{aligned}$$

# Multinomial Model

- Generalization of the binomial model, for the case where observations can have more than two possible values.
- Sampling distribution: multinomial with parameters $(\theta_1, ..., \theta_k)$, the probabilities associated to each of the $k$ possible outcomes.
- Example: In a survey, respondents may: Strongly Agree, Agree, Disagree, Strongly Disagree, or have No Opinion when presented with a statement such as "8am is too early to have class".

# Multinomial Model

## Sampling Distribution

Formally

- $y = (y_1, y_2, \ldots, y_k)$, a k-vector of counts of the number of observations for each outcome
- $\theta_j$: probability of $j$th outcome
- $\sum_{j=1}^{k} \theta_j = 1$ and $\sum_{j=1}^{k} y_j = n$

The sampling distribution has the form

$$f(\mathbf{y} \mid \theta) \propto \Pi_{j=1}^{k} \theta_j^{y_j}$$