

Formal Definitions of Unbounded Evolution and Innovation Reveal Universal Mechanisms for Open-Ended Evolution in Dynamical Systems

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One of the most remarkable features of the > 3.5 billion year history of life on Earth is the apparent trend of innovation and open-ended growth of complexity. Similar trends are apparent in artificial and technological systems. However, a general framework for understanding open-ended evolution as it might occur in biological or technological systems has not yet been achieved. Here, we cast the problem within the broader context of dynamical systems theory to uncover and characterize mechanisms for producing open-ended evolution (OEE). We present formal definitions of two hallmark features of OEE: unbounded evolution and innovation. We define unbounded evolution as patterns that are non-repeating within the expected Poincaré recurrence time of an equivalent isolated system, and innovation as trajectories not observed in isolated systems. As a case study, we test three new variants of cellular automata (CA) that implement *time-dependent* update rules against these two definitions. We find that each is capable of generating conditions for OEE, but vary in their ability to do so. Our results demonstrate that *state-dependent* dynamics, widely regarded as a hallmark feature of life, statistically out-perform other candidate mechanisms. It is also the only mechanism to produce OEE in a scalable manner, consistent with notions of OEE as ongoing production of complexity. Our results thereby suggest a new framework for unifying the mechanisms for generating OEE with features distinctive to life and its artifacts, with wide applicability to both biological and artificial systems.

Keywords: open-ended evolution; artificial life; innovation; top-down causation; self-reference; dynamical systems; emergence; creativity

Significance Statement: Open-ended evolution (OEE) is relevant to a variety of biological, artificial and technological systems, but has been challenging to reproduce *in silico*. Most theoretical efforts focus on key aspects of OEE as it appears in biology. We re-cast the problem as a more general one in dynamical systems theory, providing simple criteria for evaluating if a dynamical system is capable of OEE. Comparing three candidate mechanisms against these criteria reveals that *state-dependent* systems statistically out-perform others in generating OEE. Since state-dependent dynamics are widely regarded as a hallmark of life, our results suggest new paths forward in unifying mechanisms for generating OEE with features distinctive to life and its artifacts, with broad applicability to biological and artificial systems.

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I. INTRODUCTION

Many real-world biological and technological systems display rich dynamics, often leading to increasing complexity over time that appears to be limited only by availability of resources. An example is the evolution of biological complexity: the history of life on Earth has displayed a trend of continual evolutionary adaptation and innovation, giving rise to an apparent open-ended increase in the complexity of the biosphere over its > 3.5 billion year history [1]. Other complex systems, from the growth of cities [2], to the evolution of language [3], culture [4, 5] and the Internet [6] appear to exhibit similar trends of innovation and open-ended dynamics. Producing computational models that generate sustained patterns of innovation over time, which elucidate the fundamental mechanisms driving open-ended dynamics in both natural and artificial systems is therefore an important goal in modeling complex systems, with implications for research in diverse fields ranging from biological evolution to artificial life and artificial intelligence.

Despite the significance of realizing open-ended evolution in theoretical models, progress in this direction has been hindered by lack of universally accepted definitions for *open-ended evolution* (OEE). Although relevant to many fields, OEE is most often discussed in the context of

artificial life, where the problem is so fundamental that it has been dubbed a “millennium prize problem” [7]. Many working definitions exist, which can be classified into four hallmark categories as outlined by Banzhaf *et al.* [8]: (1) on-going innovation and generation of novelty [9, 10]; (2) unbounded evolution [1, 11, 12]; (3) on-going production of complexity [13–15]; (4) a defining feature of life [16]. Each of these faces its own challenges, as each is cast in terms of equally ambiguous concepts. For example, “innovation” or “novelty”, “complexity” and “life” are all notoriously difficult to formalize in their own right. It is also not apparent whether “unbounded evolution” is physically possible since real systems are limited in their dynamics by finite resources, finite time, and finite space. A further challenge is identifying whether the diverse concepts of OEE are driving at qualitatively different phenomena, or whether they might be unified within a common conceptual framework. For example, it has been suggested that increasing complexity might not itself be a hallmark of OEE, but instead a consequence of it [9, 16]. Likewise, processes may appear unbounded, even within a finite space if they can continually produce novelty within observable dynamical timescales [17].

Given these limitations, it is unclear at present if OEE is a property unique to life, is inclusive of its artifacts (such as technology), or is an even broader phenomenon that could be a universal property of certain classes of dynamical systems. Many approaches aimed at addressing the hallmarks of OEE have been inspired by biology [17], primarily because biological evolution is the best known example of a real-world system with the potential to be truly open-ended [1]. However, as stated, other examples of potentially open-ended complex systems do exist, such as trends associated with cultural [4, 5] and technological [2, 6] growth. Therefore, we develop a more general framework to seek links between the four aforementioned hallmarks within dynamical systems, while remaining agnostic about biology. Our motivation is to discover universal mechanisms that underlie OEE as it occurs both within and outside of biological evolution.

In dynamical systems theory there exists a natural bound on the complexity that can be generated by a finite deterministic process, which is given by the *Poincaré recurrence time*. Roughly, the Poincaré time is the maximal time after which any finite system returns to its initial state and its dynamical trajectory repeats. Clearly, new dynamical patterns cannot occur past the Poincaré time. To cast the concept of OEE as unbounded evolution firmly within dynamical systems theory, we introduce a formal minimal criteria for *unbounded evolution* (where we stress that here we mean the broader concept of dynamical evolution, not just evolution in the biological sense) in finite dynamical systems: minimally, an unbounded system is one that does *not repeat* within the expected Poincaré time. This definition automatically excludes finite deterministic systems unless they are open to an environment. A corollary is that systems satisfying this minimal definition for unbounded evolution must

also satisfy a formal notion of *innovation*, where we define innovation as dynamical trajectories *not* observed in isolated systems. A system can exhibit OEE if *only if* it is both *unbounded* and *innovative*. Both proposed definitions are *extrinsic*, depending on interaction with an environment, and rely on comparison to counterfactual histories (those of isolated systems).

The utility of adopting these definitions is that they provide a simple way to quantify intuition regarding hallmarks (1) and (2) of OEE for systems of *finite size*, which is applicable to *any* dynamical system. To demonstrate this broad applicability we implement new variants of Cellular Automata (CA), which utilize the rule space of Elementary Cellular Automata (ECA). We introduce these as a case study to explore mechanisms potentially satisfying our formalized definitions of OEE and innovation. Each variant differs in its relative openness to an external environment and implements time-dependent update rules. Of the variants tested, our results indicate that systems that implement rules that are a function of the state of the system are statistically better at satisfying the two criteria for OEE. Given that state-dependent dynamics are often cited as a hallmark feature of life [18–20], this provides a new connection between hallmarks (1), (2) and (4) of OEE. We then explore these state-dependent systems in more depth, calculating general complexity measures including compressibility (based on LZW in [21]) and Lyapunov exponents. We show that the state-dependent systems provide a *scalable* mechanism for generating OEE that includes the capacity for on-going production of complexity by coupling to larger environments. Thus the results connect all four hallmarks of OEE in a new framework that is suggestive of mechanisms that might operate in a diverse range of dynamical systems. The framework therefore holds promise for providing insights into universal mechanisms for generating OEE in dynamical systems, which is applicable to both biological and artificial systems.

II. THEORY

In what follows, we consider discrete dynamical systems u , which implement rules drawn from a set $R = \{r\}$ that operate on a finite set of states $S = \{s\}$. We first introduce a rigorous notion of OEE that relies on formalized definitions of unbounded evolution and innovation, before describing three new variants of cellular automata (CA) with time-dependent rules and varying degrees of openness to an environment, designed to test mechanisms capable of satisfying these formal definitions.

A. A Formal Definition of Open-Ended Evolution in terms of Unbounded Evolution and Innovation

We consider definitions of unbounded evolution and innovation that are applicable to *any* instance of a dynamical system.

cal system u that can be decomposed into two interacting subsystems o and e . Nominally, we refer to these subsystems as the “organism” (o) and “environment” (e). We note that our framework is sufficiently general to apply to systems outside of biology. Here the concept of “organism” is meant to stress that it is only in this subsystem that we expect to potentially observe the rich dynamics intuitively expected of OEE.

Since the Poincaré recurrence time t_P of a finite, deterministic dynamical system provides a natural bound on when one should expect the system to repeat, we minimally define *unbounded evolution* (UE) as occurring only in cases that locally defy this expectation in the subsystem o , and therefore appear unbounded given finite resources:

Definition 1 Unbounded evolution (UE): *A system u , which can be decomposed into subsystems $o \in O$ and $e \in E$ that interact according to a function f , exhibits unbounded evolution if there exists a recurrence time t_r such that $f^{t_r}(\{s_o\}) = \{s_o(t_1), s_o(t_2), s_o(t_3) \dots s_o(t_r)\}$ or a time t'_r such that $f^{t'_r}(\{r_o\}) = \{r_o(t_1), r_o(t_2), r_o(t_3) \dots r_o(t_r)\}$ is non-repeating for $t_r > t_P$ or $t'_r > t_P$, where t_P is the Poincaré recurrence time for an equivalent isolated system $o \in O$, and $\{s_o\}$ and $\{r_o\}$ are the states and rules, respectively, in the dynamical evolution of o .*

Here, t_r and t'_r are the recurrence time of the state-and rule-trajectory of o , respectively (denoted by f), s_o is the state of o , r_o is the rule of o , s_e is the state of e (see Table S1 for full list of model parameters in the Supplement).

By Definition 1, a dynamical system exhibits UE *if and only if* it may be partitioned such that the sequence of one of its subsystems’ states *or* dynamical rules are *non-repeating* within the expected Poincaré recurrence time t_P of an equivalent isolated system. Here, *equivalent* is meant to include *all* systems evolved from any state drawn from the same set possible of states $s_o \in S$ with a fixed rule drawn from the same set of dynamical rules $r_o \in R$, such that $\{s_o\} \subseteq S$ and $\{r_o\} \subseteq R$. We include *dynamic* rules in our definition since, as will we describe below, it is only systems with time-dependent rules that we expect to satisfy Definition 1 *and* Definition 2 for innovation introduced below.

We will describe implementations using explicit examples of Elementary Cellular Automata (ECA) in Section *Model Implementation*, where S is the set of states with the binary alphabet $\{0, 1\}$ and R is the set of ECA rules. ECA are defined as 1-dimensional CA with nearest-neighbor update rules: for an ECA of width w , equivalent isolated systems as defined here include all trajectories evolved with *any* fixed ECA rule from *any* initial state of width w , where t_P is then $t_P = 2^w$ and $w = w_o$, where w_o is the width of o .

Implementing this definition of UE necessarily depends on counterfactual histories of *isolated* systems (*e.g.* of ECA in our examples). These counterfactual systems

cannot, by definition, generate conditions for UE. This suggests as a corollary a natural definition for innovation in terms of comparison to the same set of counterfactual histories:

Definition 2 Innovation (INN): *A system u , which can be decomposed into subsystems $o \in O$ and $e \in E$ that interact according to a function f , exhibits innovation if there exists a recurrence time t_r such that $f^{t_r}(o) = \{s_o(t_1), s_o(t_2), s_o(t_3) \dots s_o(t_r)\}$ is not contained in the set of all possible state-trajectories $\{s_I\}$ for an equivalent isolated (non-interacting) system $o \in O$.*

That is, a subsystem o exhibits INN by Definition 2 if its dynamics are *not* contained within the set of possible trajectories of an equivalent isolated system, as defined above.

A motivation for including both Definitions 1 and 2 is that they encompass intuitive notions of “on-going production of novelty” (INN) and “unbounded evolution” (UE), both of which are considered important hallmarks of OEE. We note that the combination of the two definitions does not necessitate that the complexity of individual states increase over time, thus one might observe INN without a corresponding rise in complexity with time.

Importantly, the combination of Definitions 1 and 2 excludes trivial cases. For example, some non-bounded systems could in principle satisfy UE, but not INN since the dynamics might be equivalent to those generated from an appropriately constructed isolated system (*e.g.* a larger ECA in our example). An example is the time evolution of ECA Rule 30 (by Wolfram’s classification [22]), which is known to be a ‘complex’ ECA rule that keeps generating novel patterns in open-boundary conditions. In this case, it should be considered that it is the openness of the system that is generating continual novelty and not an *internal* mechanism. Since we aim to understand the *intrinsic* mechanisms that might drive OEE in real, finite dynamical systems, we require both definitions to be satisfied for a dynamical system to exhibit non-trivial OEE.

B. Model implementation

We evaluate different mechanisms for generating OEE against Definitions 1 and 2, utilizing the rule space of Elementary Cellular Automata (ECA) as a case study. ECA are defined as nearest-neighbor 1-dimensional CA operating on the two-bit alphabet $\{0, 1\}$. There are 256 possible ECA rules, which we numerically label following Wolfram’s heuristic classification [22]. Due to their relative simplicity, ECA represent some of the most widely-studied CA, thus providing a well-characterized foundation for this study. Traditionally, ECA evolve according to a fixed dynamical rule starting from a specified initial state. As such, no isolated finite ECA can meet the criteria laid out in Definitions 1 and 2. An isolated ECA of width w will repeat its pattern of states by the Poincaré

time $t_P = 2^w$ (violating Definition 1). If we instead considered a CA of width w as a subsystem of a larger ECA it would not necessarily repeat within 2^w , but it would *not* be innovative (violating Definition 2). Thus, as stated, we can exclude trivial examples such as ECA Rule 30, or other unbounded but non-innovative dynamical processes, which repeatedly apply the same update rule.

To exclude trivial unbounded cases, Definitions 1 and 2 necessitate that the dynamical rules themselves evolve in time. As we will show, utilizing the set of 256 possible ECA rules as the rule space for CA with *time-dependent* rules makes both UE and INN possible. Rules can be stochastically or deterministically evolved, and we explore both mechanisms here. Specifically, we explore three new CA variants that evolve according to explicitly time-dependent rules, where the time dependence takes on different functional forms for each. We note that there exists a huge number of possible variants one might consider. We therefore focus on three variants that display important mechanisms implicated in generating OEE, including state-dependent dynamics (regarded as a hallmark feature of life [18–20]), openness to an environment [10], and stochasticity. Here openness to an environment is parameterized by the degree to which the rule evolution of o depends on the state (or rule) of o versus the state of e : completely open systems are regarded as depending only on external factors, such that the time-dependence of the rule evolution is *only* a function of the environment.

The first variant, **Case I**, implements *state-dependent* update rules, such that the evolution of o depends on its own state *and* that of its environment. For simplicity, to model a dynamic environment e we utilize the ECA rule space, where e evolves according to a *fixed* rule (such that e is an ECA). The subsystem o , by contrast, updates its own rule according to a function f such that $r_o(t+1) = f(s_o(t), r_o(t), s_e(t))$, where s_o and r_o are the state and rule of the organism and s_e is the state of the environment (See Supplement for details). The dynamics of o are therefore *self-referential* [20], and we regard this case as only partly open to an environment since o depends on its own state. We note that the qualitative features observed are not dependent on the precise functional form of f , so long as f is a function of both o and e (that is, o is self-referential *and* open – see [23] for an example of non-open self-referencing CA that do not display OEE).

The second variant, **Case II**, is completely open to an external environment e , which fully dictates the rule evolution of o . As with Case I, the environment e is an execution of an ECA. The subsystem o updates its rule according to a function f such that $r_o(t+1) = f(s_e(t))$. As such, the rule implemented in o depends *only* on the state of an the external system s_e . We consider o in this example to be more open to its environment than for Case I, as here f does not depend on s_o or r_o .

The final variant, **Case III**, is also open to an external environment, however the environment here is a

TABLE I: Table of cellular automata variants, and the functional form of the rule evolution of o .

CA Variant	“Organism” o Rule Evolution	Environment, e
Case I	$r_o(t+1) = f(s_o(t), r_o(t), s_e(t))$	ECA, varied w_e
Case II	$r_o(t+1) = f(s_e(t))$	ECA, $w_e = 8$
Case III	$r_o(t+1) = f(r_o(t), \xi)$	Heat bath
ECA (Isolated)	$r_o(t+1) = r_o(t)$	None

stochastic “heat bath” and not an ECA. The subsystem o updates its rule according to a function f such that $r_o(t+1) = f(r_o(t), \xi)$, where ξ introduces noise to the system, such that if the noise is above a threshold value μ the rule is “mutated” (see Supplement). The rule evolution of o follows a diffusive random walk through the ECA rule space. In this example, o is also more open to its environment than in Case I (since f does not depend on s_o), but it is less open than Case II since f does depend on r_o .

The three variants are summarized in Table I (additional details in Supplement). Since we restrict the rule space for Cases I–III to that of ECA rules only, the trajectories of ECA with periodic boundary conditions provides a well-defined set of isolated counterfactual trajectories with which to evaluate Definitions 1 and 2. For comparison to isolated systems, we evaluate *all* ECA with widths equivalent to w_o , the width of the “organism” subsystem o . We test the capacity for each of the three cases presented to generate OEE, against Definitions 1 and 2 in a statistically rigorous manner, and compare the efficacy of the different mechanisms implemented in each case.

ECA rules are often categorized in terms of four Wolfram complexity classes [22]. Class I and II are considered simple because all initial patterns evolve quickly into a stable or oscillating, homogeneous state. Class III and IV rules are viewed as generating more complex dynamics. We also analyze the complexity of ECA rules implemented in OEE cases, by considering the frequency of implementation of rules from each class to determine if the complexity of the observed dynamics is an artifact of the ECA rules or *intrinsic* to f .

1. Experimental Methods

For Cases I and II, o and e evolve according to periodic boundary conditions. In Case III, o has periodic boundary conditions. For Case I, where w_e must also be specified, we consider systems with $w_e = \frac{1}{2}w_o$, w_o , $\frac{3}{2}w_o$, $2w_o$ and $\frac{5}{2}w_o$, where w_o is the width o . For Case II, $w_e = 8$ for all simulations, since this permits a 1:1 map from $\{s_e\}$ to the rule space of ECA (because $2^8 = 256$, the total number of ECA.) Results for Case III are given for $\mu = 0.5$, such that each outcome bit in r_o has a 50% probability of flipping at every time step. Other values of μ were explored, with qualitatively similar results (See Supplement).

TABLE II: Percentage of sampled cases displaying OEE (satisfying Definitions 1 and 2) for each CA variant.

w_o	ECA	Case I ($w_o = w_e$)	Case II	Case III
3	0	0.02	42.47	7.42
4	0	0.38	11.54	1.05
5	0	3.41	10.43	2.76
6	0	0.03	0.27	5.2×10^{-3}
7	0	1.06	0.7	4.7×10^{-4}

The number of possible executions grows exponentially large with w_o , limiting the computational tractability of statistically rigorous sampling of the dynamics of each CA variant and of generating the set of counterfactual isolated ECA trajectories. We therefore explored small CA with $w_o = 3, 4, \dots, 7$ and sampled a representative subspace of each (see Supplement). For each system sampled, we measured the recurrence times t'_r and t_r for o . For Case III CA, which are stochastic, all simulations eventually terminated as a random oscillation between the all ‘0’ state and the all ‘1’ state. We therefore used the timescale of reaching this oscillatory attractor as a proxy for t_r for Case III simulations. In cases where t'_r or $t_r > t_P$, where $t_P = 2^w$ for isolated ECA (Definition 1), and the state trajectory was not produced by any ECA of width $w = w_o$ (Definition 2), the system is considered to exhibit OEE. The complexity of the resulting interactions are measured by compressibility and system’s sensitivity based upon Lyapunov exponents [24] (see Supplement for details).

III. RESULTS

The vast majority of executions sampled from all three CA variants were innovative by Definition 2, with $> 99\%$ of Case II and Case III CAs displaying INN, such that their dynamics are not captured in the trajectory of states for *any* isolated ECA with a fixed rule of width $w = w_o$. For Case I CA, the percentage of INN cases increased as a function of both w_o and w_e , ranging from $\sim 30\%$ for the smallest CA explored to $> 99\%$ for larger systems (See Supplement for statistics). OEE cases, by contrast, are much rarer since the number of UE cases is much smaller, typically representing $< 5\%$ of the sampled trajectories. We therefore focus discussion primarily on sampled executions meeting the criteria for OEE, *i.e.* those that satisfied both Definitions 1 and 2.

A. Open-Ended Evolution in CA variants.

The percentage of sampled cases satisfying both Definitions 1 and 2 are shown in Table II, where for purposes of comparison Case I CA statistics are shown only for $w_o = w_e$. As expected (by definition), isolated ECA do

TABLE III: Percentage of sampled cases displaying OEE (satisfying Definitions 1 and 2) for Case I, with varying environment size w_e .

w_o	$\frac{1}{2}w_o$	w_o	$\frac{3}{2}w_o$	$2w_o$	$\frac{5}{2}w_o$
3	0	0.02	6.52	10.81	28.14
4	0	0.38	2.28	2.94	9.65
5	0	3.41	7.04	7.5	8.64
6	0	0.03	2.15	2.64	5.82
7	0	1.06	2.95	4.39	5.34

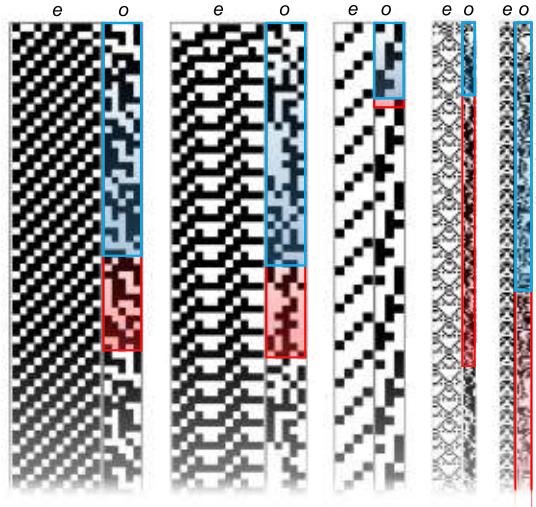


FIG. 1: Five examples of Case I CA, each shown with an environment e on the left, and organism o . For each o , shown is the Poincaré recurrence time (t_P) (blue), and the recurrence time of s_o (t_r) (red). Since $t_r > t_P$ for each o , and none are producible by an ECA, all five o satisfy both Definitions 1 and 2.

not exhibit any OEE cases. However, all three CA variants with time-dependent rules do exhibit examples of OEE, but differ in the percentage of sampled cases. The distributions also differ (see Supplement). Case II and III CA observe steeper drop-offs in the number of OEE cases for increasing w_o . For Case II, no OEE cases are observable for $w_o \geq 8$ (see Supplement) – this represents a sharp cut-off for any Case II-type system: t_r for o is always limited by the dynamics of the environment (here e repeats in a time $t_e \leq 2^w$). For Case II, one could increase w_e , but this in turn requires changing the rule structure of o . We therefore regard Case II to not be a scalable mechanism for generating OEE cases. Case III is likewise not scalable, as the number of OEE cases observed drops off exponentially, with a steeper decline in cases observed for increasing w_o . That is, OEE becomes increasingly rarer for increasing w_o . Observing more OEE cases requires increasing the number of sampled trajectories to observe interesting cases. We therefore focus on Case I in what follows, as the only potentially scalable mechanism for generating OEE of the three mechanisms studied.

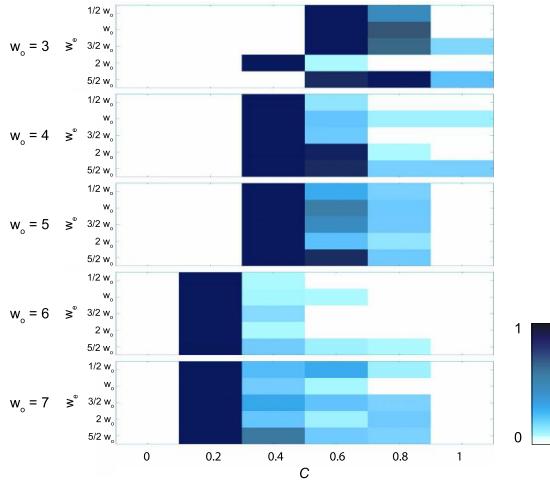


FIG. 2: Heat map of the compressibility values (C) for all sampled Case I CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for C .

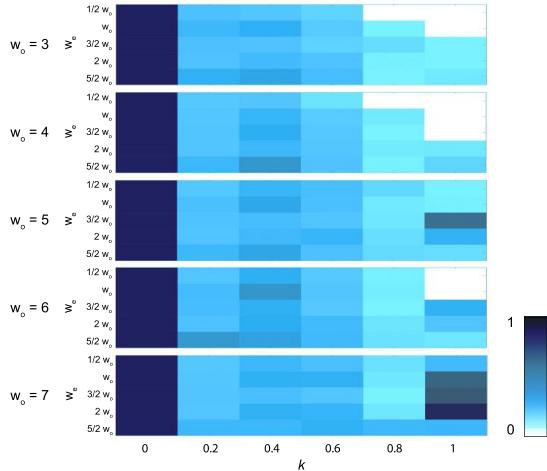


FIG. 3: Heat map of the Lyapunov values (k) for all sampled Case I CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for k .

B. Case I: A Scalable Mechanism for OEE

We explored Case I CA with $w_e = \frac{1}{2}w_o$, w_o , $\frac{3}{2}w_o$, $2w_o$ and $\frac{5}{2}w_o$ to test the scalability of the observed dynamics. Statistics for the percentage of OEE cases sampled are shown in Table III. While increasing w_o in general tends to decrease the percentage of OEE cases observed, the percentage of OEE cases *increases* with increasing w_e . In contrast to Cases II and III, Case I therefore represents a scalable mechanism for OEE [17], as o can be coupled to larger environments and will produce more OEE cases, without changing the structure of o .

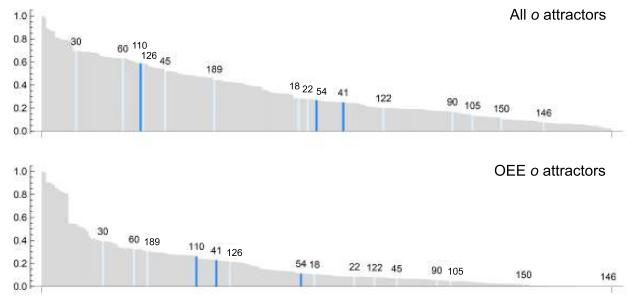


FIG. 4: Frequency distribution of rules used in all sampled Case I CA o attractors with environments $w_o = w_e$. The top panel shows the frequency distribution from all sampled o attractors, while the bottom shows the frequency distribution from only OEE o attractors. Highlighted are Wolfram Class III (light blue) and IV (dark blue). In both samples, Class I and II rules are the most frequently expressed.

C. On-going Generation of Complexity in Case I

We also considered the complexity of Case I CA, relative to isolated ECA as a further test of their scalability and potential to generate complex and novel dynamics. Examples of Case I CA are shown in Fig. 1, which visually demonstrate the novelty of the dynamics generated via the state-dependent mechanism. We characterized the complexity of Case I using the two standard complexity measures (compressibility and Lyapunov exponents) and compared to the same measurements performed on isolated ECA. The trends demonstrate that the compressibility of all sampled trajectories in general decreases with increasing w_o , but increases with increasing w_e (Figure 2). Similar trends are observed for the Lyapunov exponent, as shown in Figure 3, where it is evident that increasing w_o or w_e leads to an increasing number of cases with higher Lyapunov exponent k . OEE cases tend to have the highest k values (see Supplement). We therefore conclude that Case I CA, in general, are more complex than the full repertoire of all ECA evolved according to any fixed dynamical rule, and this is especially true for OEE cases. In addition, we analyzed the ECA rules implemented in sampled Case I trajectories relative to the Wolfram Rule complexity classes, to test whether the complexity of trajectories generated was intrinsic to the state-dependent mechanism, or whether it was attributable to a selection effect favoring complex ECA rules. We find that Case I CA, on average, implement more Class I and II rules than Class III or IV, as shown in the frequency distribution of Fig. 4 for Case I CA with $w_o = w_e$ (see Supplement for data on other w_o and w_e). Thus, we can conclude that the complexity generated by Case I CA is *intrinsic* to the state-dependent mechanism, and is not attributable to a selection effect for complex ECA rules.

IV. DISCUSSION

We have provided formal definitions of *unbounded evolution* (UE) and *innovation* (INN) that can be evaluated in *any* dynamical system which can be decomposed into two interacting subsystems o and e . Systems satisfying both UE and INN we expect to minimally represent mechanisms capable of open-ended evolution (OEE). Testing the criteria for UE and INN against three different CA models with time-dependent rules reveals what we believe to be quite general mechanisms applicable to a broad class of OEE systems.

A. Mechanisms for OEE

Our analysis of CA implementing *time-dependent* update rules with Cases I-III indicates that there are potentially many time-dependent mechanisms that can produce OEE in a subsystem o embedded within a larger dynamical system u , but that some may be more interesting than others. An externally driven time-dependence for the rules of o (Case II), while producing the highest statistics of OEE cases sampled for small o , does not provide a scalable mechanism for producing OEE with increasing system size, unless the structure of o itself is fundamentally altered (such that the rule space changes). Stochastically driven systems display rich dynamics, but for closed subsystems ultimately converge on dynamics with low complexity (Case III). An alternative is to introduce stochasticity to the states, rather than the rules, which would avert this issue. This has the drawback that the mechanism for OEE is then not as clearly mapable to biological processes (or other mechanisms internal to the system), where the genotype (rules) evolve due to random mutations that then dictate the phenotype (states).

We therefore regard Case I CA as the most interesting mechanism explored herein for generating conditions favoring OEE: it is scalable and the dynamics generated are novel. We note that the state-dependent mechanism represents a departure from more traditional approaches to modeling dynamical systems, *e.g.* as occurs in the physical sciences, where the dynamical rule is usually assumed to be fixed. In particular, it represents an explicit form of top-down causation, often regarded as a key mechanism in emergence [18, 25] important in driving major evolutionary transitions [26]. The state-dependent mechanism is also consistent with an important hallmark of biology – that biological systems appear to implement self-referential dynamics such that the “laws” in biology are a function of the states [18–20], which also appears to be characteristic of the evolution of language [27, 28].

B. Applicability to Other Dynamical Systems

We have independently explored openness to an environment, stochasticity and state-dependent dynamics as we expect these to be general and apply to a wide-range of dynamical systems that might similarly display OEE by satisfying Definitions 1 and 2. An important feature of these definitions is that UE and INN must be driven by *extrinsic* factors (an environment), although the mechanisms driving the dynamics characteristic of OEE may be *intrinsic* to the subsystem of interest. OEE can therefore only be a property of a subsystem. We have not explored the case of feedback from o to e that might drive further open-ended dynamics, as characteristic of the biosphere, for example in niche construction [29], but expect even richer dynamics to be observed in such cases.

V. CONCLUSIONS

Our results demonstrate that OEE, as formalized herein, is a general property of dynamical systems with time-dependent rules. This represents a radical departure from more traditional approaches to dynamics where the “laws” remain fixed. This suggests that uncovering the principles governing open-ended evolution and innovation in biological and technological systems requires removing the segregation of states and *fixed* dynamic laws characteristic of the physical sciences for the last 300 years. In particular, state-dependent dynamics have been shown to out-perform other candidate mechanisms in terms of scalability, suggestive of paths forward for understanding OEE. Our analysis connects all four hallmarks of OEE and provides a mechanism for producing OEE that is consistent with the self-referential nature of living systems. By casting the formalism of OEE within the broader context of dynamical systems theory, the proof-of-principle approach presented opens up the possibility of finding unifying principles of OEE that encompass both biological and artificial systems.

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Appendix: Supplementary Material

TABLE IV: Table of terms and model parameters.

Variable Meaning

o	Single organism execution
e	Single environment execution
u	CA execution of coupled o and e
s_o	state of o
r_o	rule of o
s_e	state of e
w_o	width of o
w_e	width of e
f	Interaction function
t_P	Poincaré recurrence time
t_r	Recurrence time of s_o
t'_r	Recurrence time of r_o
O	Set of all possible organism executions
E	Set of all possible environment executions
U	Set of all possible CA executions
t	Time step
w	width of u
μ	Mutation threshold of Case III variant
ξ	random noise for Case III variant, $\xi = [0, 1)$
t_a	Attractor size for o state-trajectory
C	Compressibility
k	Lyapunov exponent

1. Description of Implementations of Time-Dependent Cellular Automata Variants

We consider three variants of traditional Cellular Automata (CA) to identify mechanism(s) that can produce conditions necessary for open-ended evolution (OEE) in bounded regions, subject to the formal criteria for OEE laid out in Definitions 1 and 2 in *A Formal Definition of Open-Ended Evolution in terms of Unbounded Evolution and Innovation*. We consider definitions of unbounded evolution (UE) and innovation (INN) that are applicable to *any* instance of a dynamical system u that can be decomposed into two interacting subsystems o and e . Each CA variant implements *time-dependent* rules for o , with different functional forms f for this time-dependence for each variant considered. Here we describe the implementation of each variant in detail.

a. Case I: Deterministic State-Dependent Rules in Subsystem o

The first variant, Case I, implements *state-dependent* update rules. Case I CA are composed of two spatially separate, fixed-width, 1-dimensional CA: an “organism” o and an environment e . Both o and e are implemented with periodic boundary conditions, and utilize the alphabet $\{0, 1\}$. The environment e is an execution of an ECA, and is evolved according to a fixed rule drawn from the set of 256 possible ECA rules, with periodic boundary conditions.

The subsystem o updates its rule according to a function f such that $r_o(t+1) = f(s_o(t), r_o(t), s_e(t))$, where s_o and r_o are the state and rule of the organism and s_e is the state of the environment. It is also evolved with periodic boundary conditions. The expressed ECA rule of o at time t , $r_o(t)$, is represented by the eight-bit binary representation of its rule table [22], *e.g.* an o implementing Rule 30 at time t will have $r_o(t) = [0, 0, 0, 1, 1, 1, 1, 0]$. We refer to individual bits within the rule by the index i such that $r_o(t)[1] = 0$, $r_o(t)[2] = 0$, $r_o(t)[3] = 0$, $r_o(t)[4] = 1$

etc. for an o implementing Rule 30 at time t . The binary representation of ECA rules are structured such that each successive bit i iterated in this manner represents the output of application of the rule to the ordered set of triplet states $S^3 = [111, 110, 101, 100, 011, 010, 001, 000]$. As an example, the rule table for Rule 30 is shown in Fig. 5.

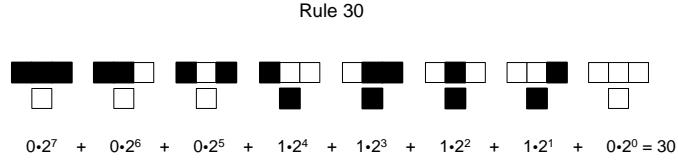


FIG. 5: Example of an ECA rule table for Rule 30. The cell representation of the ordered triplet $S^3 = [111, 110, 101, 100, 011, 010, 001, 000]$ is shown in the top row, with the cell representation of $r_o = [0, 0, 0, 1, 1, 1, 1, 0]$ (Rule 30) shown below. Each ECA Rule has a different sequence of colored cells in the second row. The bottom row is the conversion from binary to decimal representation of Rule 30.

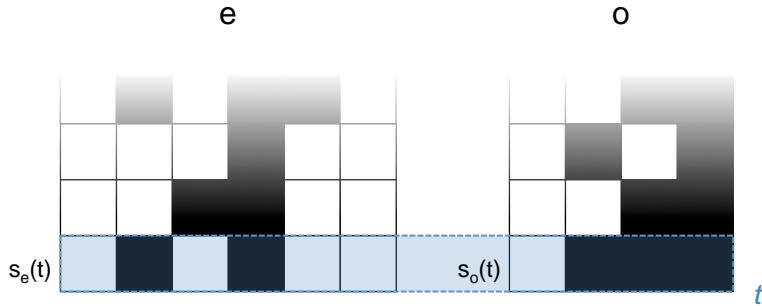


FIG. 6: Example of an “organism” o with $w_o = 4$, coupled to an environment e with $w_e = 6$, where $r_o(t) = [0, 0, 0, 1, 1, 1, 1, 0]$. In this example s_o and s_e are compared at time-step t to update $r_o(t) \rightarrow r_o(t+1)$.

The function f for our example implementation of state-dependent CA is constructed such that at each time-step t it compares the relative frequency of each triplet i in S^3 in the state of o and e , $s_o(t)$ and $s_e(t)$, respectively, and flips the corresponding bit i in $r_o(t)$ if i is expressed in s_o and the relative frequency of the triplet in $s_o(t)$ meets or exceeds the relative frequency in $s_e(t)$. That is, at each time-step t , a bit i in $r_o(t)$ will flip $0 \leftrightarrow 1$ if $n_i(s_o(t)) \geq n_i(s_e(t))$, where n_i counts the relative frequency of triplet i . Formally,

$$r_o(t+1)[i] = \begin{cases} \overline{r_o(t)[i]} & \text{if } n_i(s_o(t)) \geq n_i(s_e(t)) \\ r_o(t)[i] & \text{if } n_i(s_o(t)) < n_i(s_e(t)) \end{cases} \quad (\text{A.1})$$

where the overbar represents logical negation. An example implementation of this update function is shown in Fig. 6, where an “organism” o with $w_o = 4$ is coupled to an environment e with $w_e = 6$, and $r_o(t) = [0, 0, 0, 1, 1, 1, 1, 0]$. Table V shows the relative frequency of each triplet in S^3 in $s_o(t)$ and $s_e(t)$ for the example in Fig. 6. Only for $i = 3$, corresponding to the triplet $\{1, 0, 1\}$, is $n_3(s_o(t)) \geq n_3(s_e(t))$. Therefore, $r_o(t+1)[3] = \overline{r_o(t)[3]} = \bar{0} = 1$, as shown schematically in Fig. 7. In this example, the interaction of o and e under f changes r_o from Rule 30 at time-step t to Rule 62 at $t+1$.

We note that the qualitative features of the state-dependent dynamics reported here are not sensitive to the details of the function f in Eq. A.1, and only depend on its arguments such that we expect similar results for any mapping $r_o(t+1) = f(s_o(t), r_o(t), s_e(t))$, which is history and state-dependent: that is, that depends on the state of o and e and the history of past rules implemented by o . A schematic comparison of ECA and Case I CA are shown in Fig. 8, where the rule updating schemes highlighting the differences are shown with colored arrows.

b. Case II: Deterministic Time-Dependent Rules in Subsystem o

The second variant, Case II, is similarly composed of two spatially separate, fixed-width, 1-dimensional CA: an “organism” o and an environment e . As with Case I, both o and e are implemented with periodic boundary conditions,

$$\begin{array}{l} i \quad S^3[i] \quad n_i(s_o) \quad n_i(s_e) \end{array}$$

1	111	1/4	-
2	110	1/4	-
3	101	1/4	1/6
4	100	-	1/6
5	011	1/4	-
6	010	-	2/6
7	001	-	1/6
8	000	-	1/6

TABLE V: Relative frequency distribution for triplets in S^3 drawn from $s_o(t)$ and $s_e(t)$ in Fig. 6.

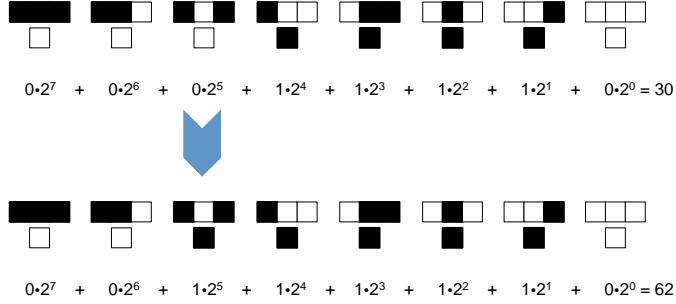


FIG. 7: Update of r_o from Rule 30 to Rule 62, based on the frequency of triplets in Table V.

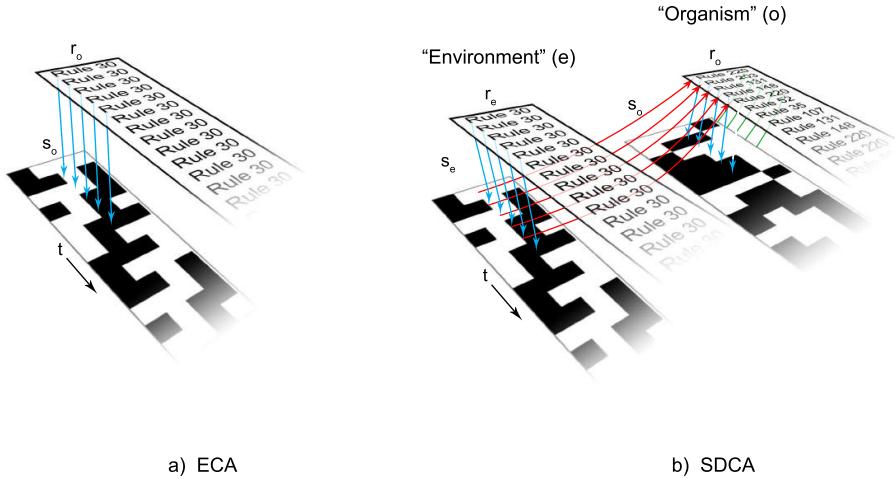


FIG. 8: Comparison of (a) an ECA update scheme and (b) the state-dependent update scheme of a Case I CA. For an ECA the update rule is fixed, with the same rule implemented at each time step ((a) blue arrows). The environment e of a Case I state-dependent CA behaves exactly like an ECA and likewise evolves according to a fixed update rule ((b) blue arrows). The “organism” o of the Case I state-dependent CA, by contrast, updates its rule at each time-step depending on its rule at the previous time-step, its own state ((b) green arrows) and the state of e ((b) red arrows). The new rule for o is then implemented to update the state of o ((b) blue arrows).

and utilize the alphabet $\{0, 1\}$. The environment e is an execution of an ECA, and is evolved according to a fixed rule drawn from the set of 256 possible ECA rules, just as in Case I.

The key difference between Case I and Case II CA is that for Case II, the subsystem o updates its rule according

to a function f such that $r_o(t+1) = f(s_e(t))$. That is, for Case II the update rule of o depends *only* on the state of the external environment s_e and is independent of the current state or rule of o (that is, o is *not* self-referential). Formally,

$$r_o(t+1)[i] = s_e(t)[i] \quad (\text{A.2})$$

Here $r_o(t)$ is determined uniquely by $s_e(t)$, such that the binary representation of each possible state of the environment uniquely maps to one ECA rule according to Wolfram's binary classification scheme [22]. For this implementation the environment must be of width $w_e = 8$ to mediate a bijective map between $\{s_e\}$ and $\{r_o\}$. Case II CA emulate systems where the rules for dynamical evolution are modulated exclusively by the time evolution of an external system.

c. Case III: Stochastic Time-Dependent Rules in Subsystem o

The final variant, Case III, is composed of a single, fixed-width, 1-dimensional CA: the “organism” o . Like Case II, the rule evolution of Case III is driven *externally* and does not depend on s_o . However, here the external environment e is a stochastic heat bath and not an ECA. In Case III CA, the subsystem o updates its rule according to a function f such that $r_o(t+1) = f(r_o(t), \xi)$, where ξ introduces random fluctuations in the implemented rule of o by stochastically flipping bits in r_o . Formally,

$$r_o(t+1)[i] = \begin{cases} \overline{r_o(t)[i]} & \text{if } \xi < \mu \\ r_o(t)[i] & \text{if } \xi \geq \mu \end{cases} \quad (\text{A.3})$$

where μ is a fixed threshold for flipping between $[0, 1]$, and ξ is a random number drawn from the interval $[0, 1)$. This implements a diffusive-random walk through ECA rule space. Since the rule of o at time $t+1$, $r_o(t+1)$, depends on the rule at time t , $r_o(t)$, the dynamics of Case III CA are path-dependent in a similar manner to Case I (both rely on flipping bits in $r_o(t)$, where Case I do so deterministically as a function of s_o and s_e and Case III do so stochastically).

2. Experimental Methods

The number of possible executions grows exponentially large with w_o , limiting the computational tractability of statistically rigorous sampling of the dynamics of each CA variant and of generating the set of counterfactual isolated ECA trajectories. We therefore explored small CA with $w_o = 3, 4, \dots, 7$ and sampled a representative subset of all possible trajectories for each w_o . We then generated statistics on the number of sampled trajectories satisfying Definitions 1 and 2 for unbounded evolution and innovation, respectively.

a. Case I Experiments

For Case I, w_e must be specified in addition to w_o . We consider systems with $w_e = \frac{1}{2}w_o$, w_o , $\frac{3}{2}w_o$, $2w_o$ and $\frac{5}{2}w_o$. For comparison to Case II and Case III CA, $w_e = w_o$ statistics are used. For each w_o and w_e , the initial state of o , $s_o(0)$, the initial state of e , $s_e(0)$, the initial rule of o , $r_o(0)$ and the rule of e , r_e , are drawn at random. For $r_o(0)$ and r_e , we only consider the 88 non-equivalent ECA rules, which dramatically reduces the number of possible cases, but still covers the full spectrum of complexity in initial configurations. We then permit r_o to evolve into any of the 256 possible ECA rules. We also ensure that no two cases sampled are initialized with the same tuple $\{s_o(0), s_e(0), r_o(0), r_e\}$.

The space of all possible Case I CA executions is too large to explore the full space computationally. Since each e and o are each initiated with a state and a rule, the size of U ($\#U$) for Case I CA is defined by

$$U = E \times O = \#R_e \#S_e \times \#R_o \#S_o = 2^{8w_e} \times 2^{8w_o} \quad (\text{A.4})$$

where $\#R$ and $\#S$ is the number of possible rules and states, respectively. For $w_o = 3$ and 4 , exploring the full space of all possible initial conditions for state-dependent CA is computationally tractable, and verified that the statistics reported for all reported OEE and innovative u are characteristic of the full computational space. The number of sampled cases for Case I CA (sampled at random) is given in Table VI.

TABLE VI: The size of the randomly sampled subspace for Case I CA for each w_o and w_e explored.

CA Variant	w_o	#u	% Explored	CA Variant	w_o	#u	% Explored
Case I: $w_e = \frac{1}{2}w_o$	3	2.1×10^6	1.25	Case I: $w_e = 2w_o$	3	3.36×10^7	6.92×10^{-2}
	4	4.19×10^6	1.25		4	2.68×10^8	1.73×10^{-2}
	5	1.68×10^7	0.62		5	2.15×10^9	4.69×10^{-3}
	6	3.36×10^7	0.62		6	1.72×10^{10}	3.16×10^{-4}
	7	1.34×10^8	0.31		7	1.37×10^{11}	7.75×10^{-5}
	3	4.19×10^6	0.63		3	6.71×10^7	3.91×10^{-2}
	4	1.68×10^7	0.31		4	1.074×10^9	4.88×10^{-2}
Case I: $w_e = w_o$	5	6.71×10^7	0.16		5	8.59×10^8	1.22×10^{-3}
	6	2.68×10^8	7.81×10^{-2}		6	1.37×10^{11}	1.53×10^{-4}
	7	1.07×10^9	3.91×10^{-2}		7	1.1×10^{12}	3.81×10^{-5}
	3	8.34×10^6	0.28				
	4	6.71×10^7	6.92×10^{-2}				
	5	2.68×10^8	3.75×10^{-2}				
	6	2.15×10^9	2.52×10^{-3}				
	7	8.59×10^9	1.24×10^{-3}				

b. Case II Experiments

For Case II, $w_e = 8$ for all simulations, since this permits a bijective map from $\{s_e\}$ to the rule space of ECA and thus the set of rules $\{r_o\}$. As with Case I CA, executions are initialized with a randomized tuple $\{s_o(0), s_e(0), r_o(0), r_e\}$, ensuring that no two experiments are initialized with the same tuple. Cases with $w_e > 8$ could be considered, but would remove o from exploring the ECA rule space in its rule evolution (thus altering the structure of o), and would not be directly comparable to our Case I and Case III CA. We therefore restrict attention only to $w_e = 8$ for Case II experiments in this study. The number of sampled cases for Case II CA (sampled at random) is given in Table VII.

TABLE VII: The size of the randomly sampled subspace for Case II CA for each w_o explored.

CA Variant	w_o	#u	% Explored
Case II: $w_e = 8$	3	1.34×10^8	2.31×10^{-2}
	4	2.68×10^8	2.1×10^{-2}
	5	5.37×10^8	2.1×10^{-2}
	6	1.07×10^9	2.1×10^{-2}
	7	2.15×10^9	1.98×10^{-2}

c. Case III Experiments

For Case III, a threshold μ for stochastic flipping of the bits in the rule table of o must be set. Results for Case III are given for $\mu = 0.5$ in the main paper, such that each outcome bit in the rule table at every time step $r_o(t)$ has a 50% probability of flipping. Results for other values are reported in Sec. S 6. Since we evolve only the subsystem o for Case III CA, executions are initialized with a random tuple $\{s_o(0), r_o(0)\}$. We do not restrict sampled executions to unique tuples, since a different random seed is set for each execution. The number of sampled cases for Case III CA (sampled at random) is given in Table VIII. Since the rule evolution for these CA is stochastic, this sample is taken over the possible initial conditions.

TABLE VIII: The size of the randomly sampled subspace for Case III CA for each w_o explored.

CA Variant	w_o	#u	% Explored
Case III:	3	5.24×10^5	10
Random	4	1.05×10^6	5
	5	2.1×10^6	5
	6	4.19×10^6	5
	7	8.39×10^6	5

3. Calculating Recurrence Time, Compressibility and Lyapunov Exponent

The complexity of the state trajectory $\{s_o(0), s_o(1), \dots, s_o(t_r)\}$ was measured by means of its *compressibility* C , and calculation of the *Lyapunov exponent* k . These measures are compared to the same measures taken on ECA executed with the same initial state and rule, which are considered the null case for the coupled execution.

a. Recurrence Time

For Cases I and II, we measured the recurrence times t'_r and t_r for o , for both the rule evolution $\{r_o(t_1), r_o(t_2), r_o(t_3) \dots r_o(t'_r)\}$ and the state evolution $\{s_o(t_1), s_o(t_2), s_o(t_3) \dots s_o(t_r)\}$, respectively. Recurrence times were calculated by determining the time t_r or t'_r when the sequence of states or rules of o , respectively, repeated. In general, t_r and t'_r for o are not the same as for the full system u (or as each other, such that often $t_r \neq t'_r$). We therefore first determined when u repeated the tuple $\{s_o, s_e, r_o\}$ such that $\{s_o(t'), s_e(t'), r_o(t')\} = \{s_o(t), s_e(t), r_o(t)\}$, where $t < t'$. We then determined the t_r such that $\{s_o(t_r), s_o(t_r+1), \dots, s_o(t')\} = \{s_o(t_i), s_o(t_i+1), \dots, s_o(t_r)\}$ for $t_i < t_r$ (and likewise for t'_r with the replacement r_o for s_o). The time step t_i is identified as initiation of the attractor dynamics for o . In many cases, we find attractors that are unbounded and innovative by Definitions 1 and 2, in addition to full trajectories up to recurrence. An example illustrating the expected Poincaré time for o , t_P , its recurrence time for the state trajectory t_r and the attractor size for the full system u (up to the recurrence time t' for the full system) is shown in Fig. 9).

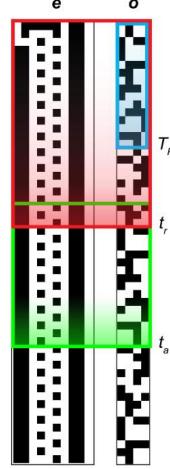


FIG. 9: Relevant timescales for describing the dynamics of o embedded in u . Shown are the Poincaré recurrence time t_P (blue) for an isolated ECA of the same width w_o as o , the state-trajectory recurrence time t_r of o (red), and attractor size of the full system u (green).

Since Case III CA are stochastically evolved, their dynamics do not repeat with a unique recurrence time t_r for o . However, all executions sampled eventually terminated in an oscillation between the two homogeneous states (all-'0's or all-'1's). These states are attractors for every fixed rule ECA evolved under periodic boundary conditions, so once a Case III CA evolves to either homogeneous state, no heterogeneity will ever be produced (the dynamics behave somewhat like dissipation of the heterogeneity in the initial state). We therefore consider it more meaningful to

calculate the number of time steps before convergence to this oscillatory attractor in place of the recurrence time t_r . This captures the timescale of relevance for all interesting (and potentially complex) dynamics, which occur in the transient before converging to this attractor.

b. Compressibility

The Kolmogorov-Chaitin complexity of string s is defined as the size of the shortest computer program p running on a universal Turing machine U that produces the string s :

$$K_U(s) = \min\{|p|, U(p) = s\} \quad (\text{A.5})$$

Although it cannot be computed exactly, it is lower semi-computable and can be approximated by using a general lossless compression algorithm L [30]. This upper-bound approximation of the Kolmogorov-Chaitin complexity is normalized according to a normalized compression measure C :

$$C(s) = \frac{L(s)}{\max(C_i(s), \text{length}(s))} \quad (\text{A.6})$$

Throughout this paper C_i is the Compress algorithm based on the LZW algorithm [21] but it can be replaced by any other compression algorithm and the measure is therefore a family of possible indexes approximating K . We use C as measure over the state-trajectory of the organism o for each execution u , as an approximation of the characteristic complexity of o in the limit of large times $t \rightarrow \infty$.

Large values of C indicate low Kolmogorov-Chaitin complexity, meaning the output was produced by a simple (short) program p . The normalization constant $\max(C_i(s), \text{length}(s))$ was calculated by measuring the number of bits resulting from a generalized compression algorithm for the Poincaré recurrence time of the entire system u , not an isolated organism, in order to normalize the observed C to its maximum possible value for an organism coupled to an environment. This closely approximates an upper limit in C for the longest possible non-repeating trajectory for any given o .

In order to ensure the normalization constant for an organism of width w_o is a close approximation to the maximal value, $C_i(s)$ was calculated for 10^7 randomly generated ECA of $\text{length}(s) = 2^{2w_o}$, where w is the width of u . The maximum of this set was used as the normalization constant $\max(C_i(s), \text{length}(s))$.

c. Lyapunov Exponent

The Lyapunov exponent k captures the speed at which a perturbation moves throughout a system [24], thereby quantifying sensitivity to initial conditions. In CA, k can be, in general, measured by perturbing a single bit in the initial condition, and counting how many bits differ compared to the unperturbed time evolution in each time step:

$$y(t) = \text{HammingDistance}[s(t), s_o(t)] \quad (\text{A.7})$$

The resulting time series of $y(t)$ values can be approximated as an exponential function, $y(t) = e^{kt}$, where k is estimated numerically. High values of k indicate sensitivity to perturbations throughout the time evolution, which is typically associated with complex dynamical systems, such as those that occur in deterministic chaos [31].

4. Statistics of Sampled Trajectories Displaying Innovation (INN)

Tables IX and X show the statistics for sampled o that were found to be innovative according to Definition 2. Innovative o were identified by having a state-trajectory s_o that cannot be reproduced by any closed, fixed rule ECA of width $w = w_o$.

TABLE IX: Percentage of sampled cases displaying INN for each CA variant.

w_o	ECA	Case I ($w_o = w_e$)	Case II	Case III
3	0	54.62	99.98	99.82
4	0	74.66	99.97	99.87
5	0	92.56	99.97	99.92
6	0	88.14	99.97	99.94
7	0	97.14	99.97	99.97

TABLE X: Percentage of sampled cases displaying INN for Case I, with varying environment size w_e .

w_o	$w_e = \frac{1}{2}w_o$	$w_e = w_o$	$w_e = \frac{3}{2}w_o$	$w_e = 2w_o$	$w_e = \frac{5}{2}w_o$
3	30.72	54.62	70.10	86.04	93.29
4	33.32	74.66	86.57	95.52	97.47
5	32.42	92.56	96.22	98.32	98.72
6	35.64	88.14	97.03	98.91	99.29
7	52.92	97.14	97.43	99.51	99.63

5. Distributions of Recurrence Times

Figure 10 shows box-whisker plots of recurrence times for each CA variant explored as described in Section 3 a. In addition to these statistics, $w_o = 8$ cases were explored for Case I $w_o = w_e$, Case II, and Case III CA. None of these statistics showed examples of OEE o , but larger w_e in Case I CA did display few cases of OEE o for $w_o = 8$ (data not shown do to small data set size).

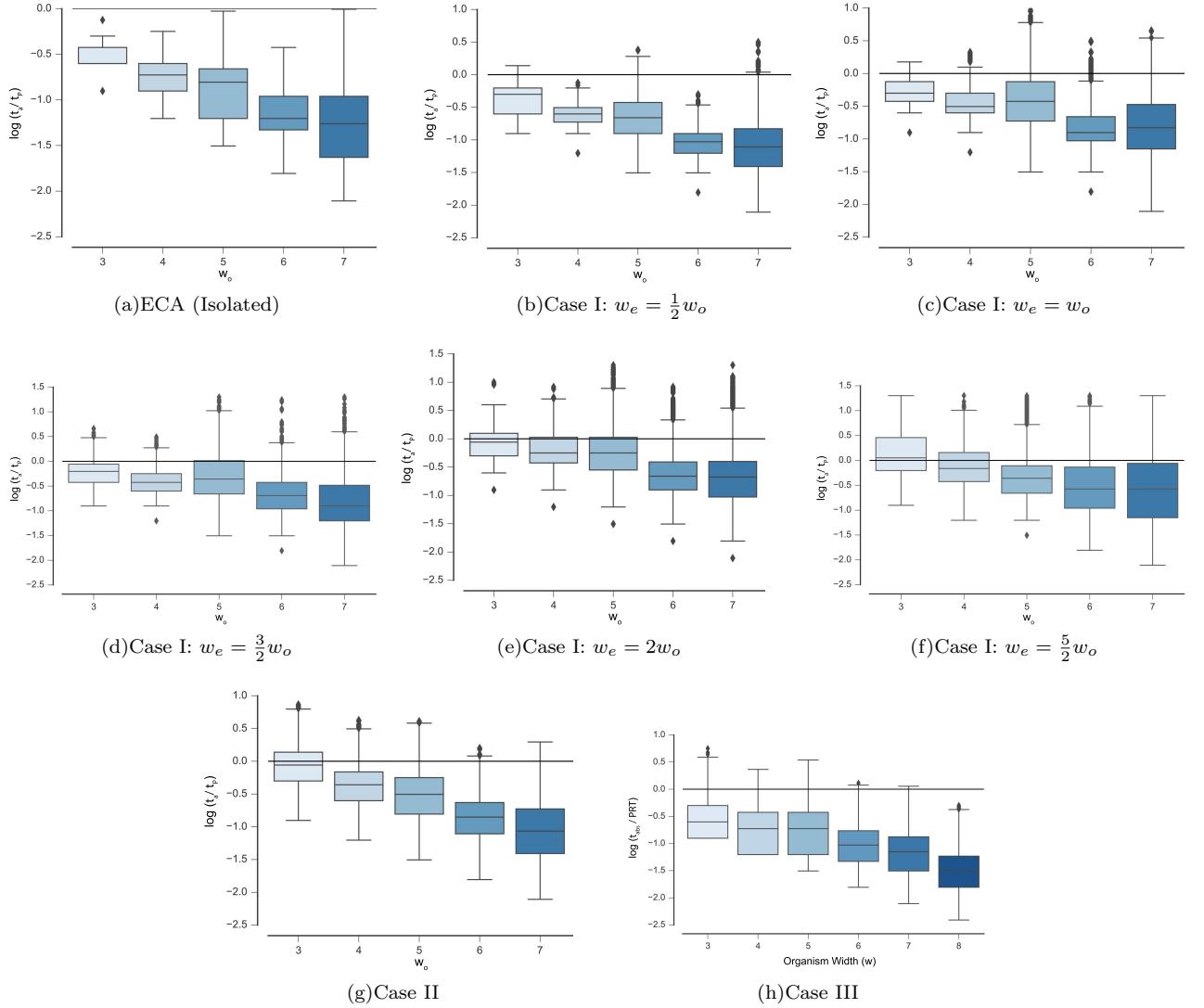


FIG. 10: Distribution of recurrence times t_r for the state trajectory of o , for ECA, Case I, Case II, and Case III. Recurrence times are normalized to the Poincaré time $t_P = 2^{w_o}$ for an isolated ECA, where the black horizontal line indicates where $t_r/t_P = 1$ (shown on a log scale). Sample trajectories displaying *unbounded evolution* (UE) occur for $t_r/t_P > 1$. Results for Case III are shown for $\mu = 0.5$ and indicate the timescale to reach the a homogenous attractor state.

6. Scalability of Case II and III CA

Case III CA generate fewer OEE cases than Case I as the width of the organism increases, with no cases observed in our statistical sample for $w_o = 8$. Case I SDCA also lack OEE cases for $w_o = 8$ but the fraction of OEE cases drops off less dramatically as a function of w_o than as observed for either Case II or Case III CA. The frequency distribution of recurrence times observed in our sampled data sets for Case I, Case III, and Case II CA are shown in Figures 11 and 12, respectively. The data indicates that for equivalently sized ensembles of sampled trajectories for Case I, Case II, and Case III CA, the Case I CA generate OEE cases with higher statistical certainty than either the Case II or Case III CA.

In comparing these results to the scalability of the three cases we note that Case II is not scalable since it would require changing the structure of the rules of the organism o . The mutation rate μ is important in determining the distribution of t_a , and thus the scalability for Case III. For smaller and smaller μ the distribution of timescales t_a increases as shown in Figure 13. In the context of biological evolution, the mechanism for increasing the number of OEE cases under Case III would therefore be for systems to evolve toward *slower* mutation rates over time. However, because the distributions are exponentially distributed, the number of OEE cases is always a small tail of

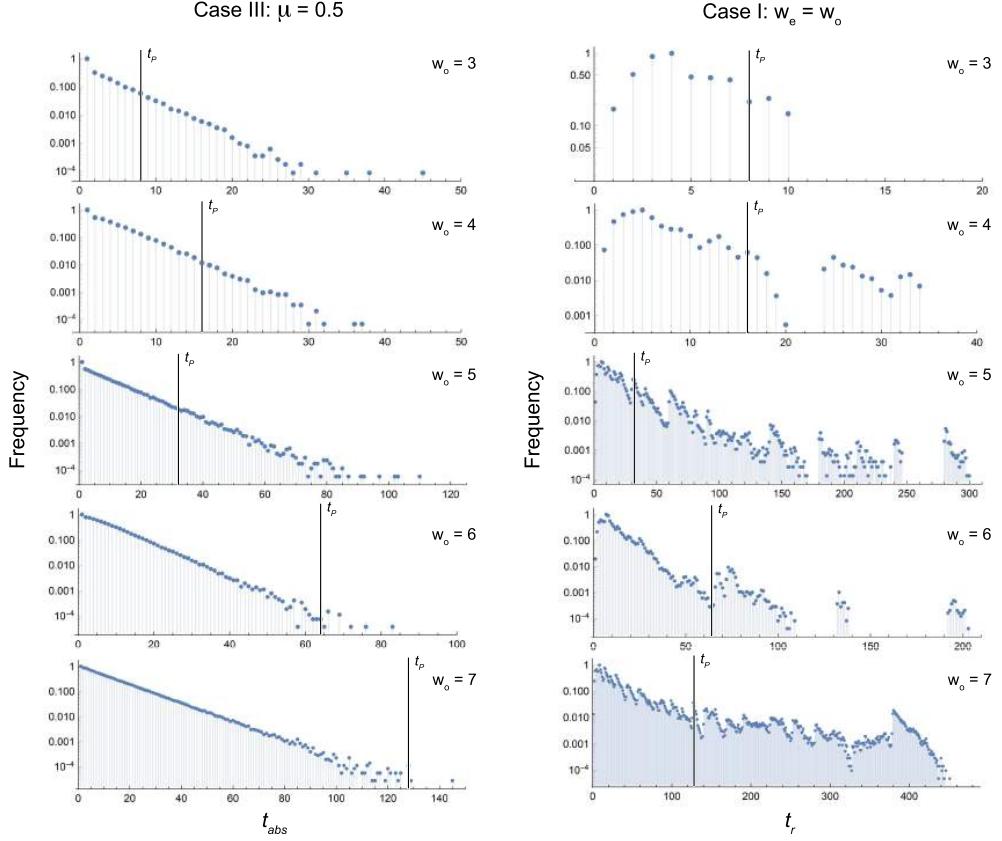


FIG. 11: Frequency distributions of recurrence times t_{abs} and t_r for Case III ($\mu = 0.5$) and Case I ($w_e = w_o$) CA, respectively. For comparison, each panel shows distributions for a given w_o . On each distribution, the Poincaré recurrence time is shown as t_P . t_{abs} is the approximate recurrence time of a CA with randomly changing update rules. This is determined by the time it takes for the states to reach an all-1 or all-0 state.

the distribution. A fixed width w_o could always be found in a *large enough* statistical sample such that $t_a > t_r$, where t_r is the recurrence time of Case I CA with an equivalent organism width w_o . In general, this statistical sample will be much larger for Case III CA of a given mutation rate μ than for Case I CA which are coupled to a large environment (Figure 11(b)). OEE cases are therefore much rarer for Case III than Case I CA. Additionally, once Case III reaches the terminal attractor their dynamics are not complex, whereas Case I CA will repeat an attractor state that is in general complex and is often times open-ended (such that the attractor itself satisfies Definitions 1 and 2).

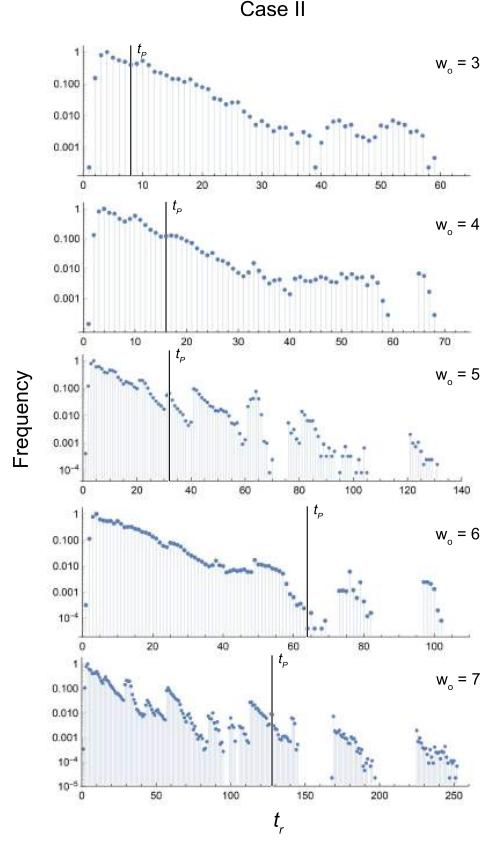


FIG. 12: Frequency distributions of recurrence times t_r for Case II CA. For comparison, each panel shows distributions for a given w_o . On each distribution, the Poincaré recurrence time is shown as t_P .

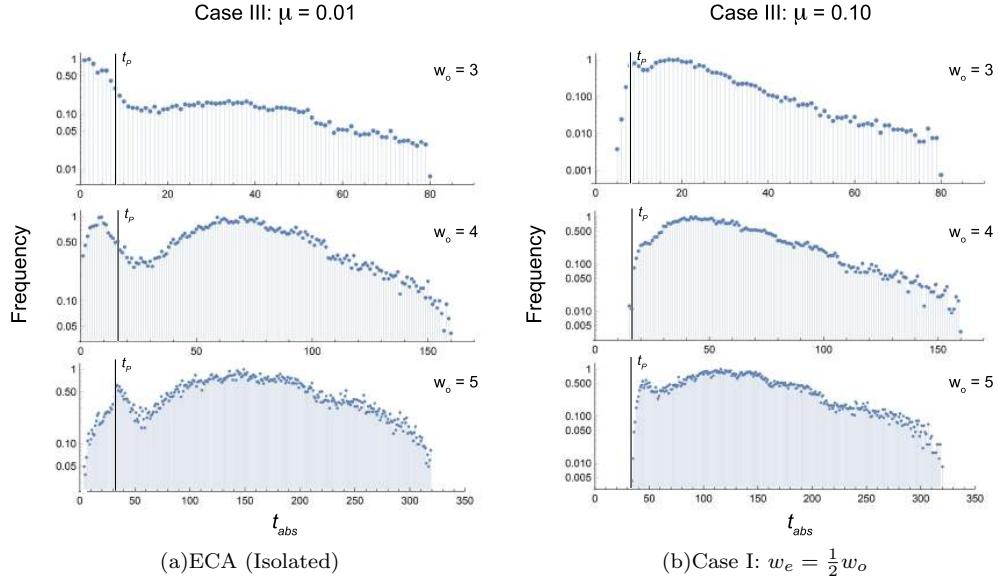


FIG. 13: Frequency distributions of recurrence times t_{abs} for Case III CA with $\mu = 0.01$ (left) and $\mu = 0.1$ (right). For comparison, each panel shows distributions for a given w_o . On each distribution, the Poincaré recurrence time is shown as t_P . t_{abs} is the approximate recurrence time of a CA with randomly changing update rules. This is determined by the time it takes for the states to reach and all-1 or all-0 state.

7. ECA Rule Complexity of Case I CA

To determine if the complexity observed in Case I CA is *intrinsic* to the state-dependent mechanism, or is an artifact of a selection-effect favoring complex rules, we determined the frequency of rules implemented in Case I CA utilizing the Wolfram classification scheme for Elementary Cellular Automata [22]. Because these were sampled across a collection of many o from a given environment e , we nominally call this sampling the “metagenomes”. There are four Wolfram Classes: Class I and II the least complex, often generating simple repeating patterns. Class III rules are more complex displaying random patterns, and Class IV are regarded as the most complex, displaying rich dynamical structure (for example, ECA Rule 110, which is known to be Turing Universal [32] is a Class IV CA). The resulting rule frequency distribution is shown in Figure 15 for all Case I CA of a given w_o and separately for the OEE cases in Figure 14. The OEE o attractor statistics in Figure 14 are the cases above the black line shown in Figure 16. The results show that Case I CA primarily implement Class I and II rules, indicative that the complexity observed is *intrinsic* to the state-dependent mechanism.

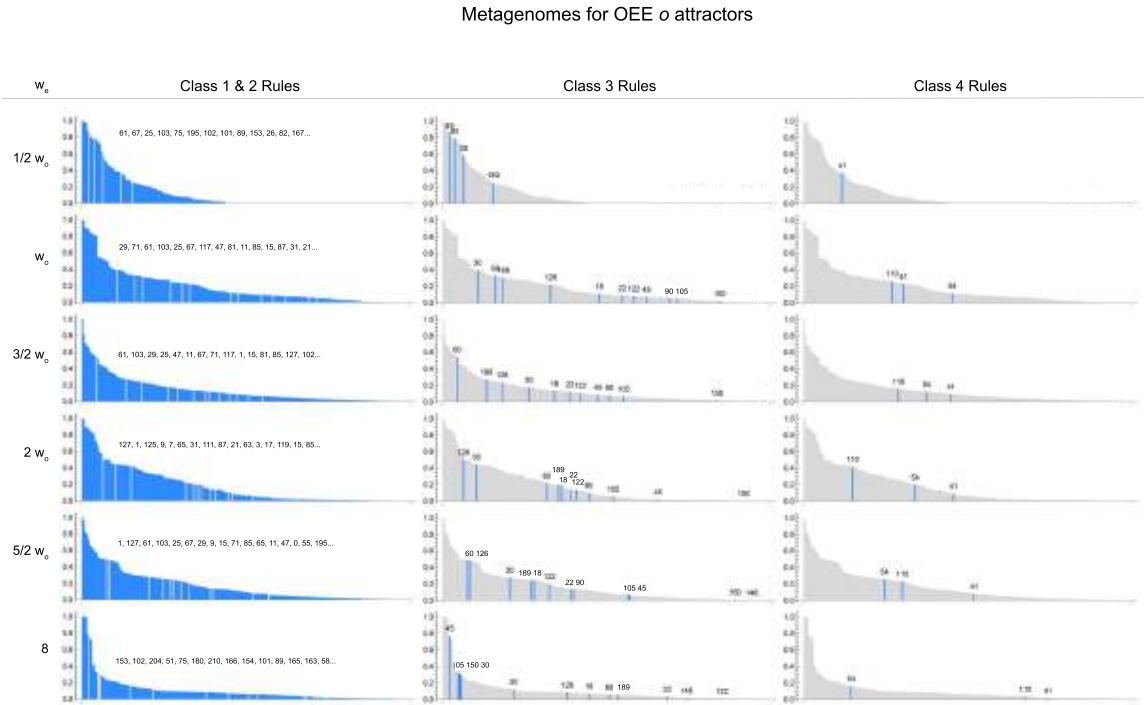


FIG. 14: Frequency distributions of rules (“metagenomes”) used in OEE o attractors. Each column shows the same distribution, but highlighted in a different manner to show particular rule Wolfram classes. Each row represents metagenomes for a different e . Case I CA mainly use Class I and II CA rules in their attractors and is no different for OEE o attractors.

Metagenomes for all o attractors

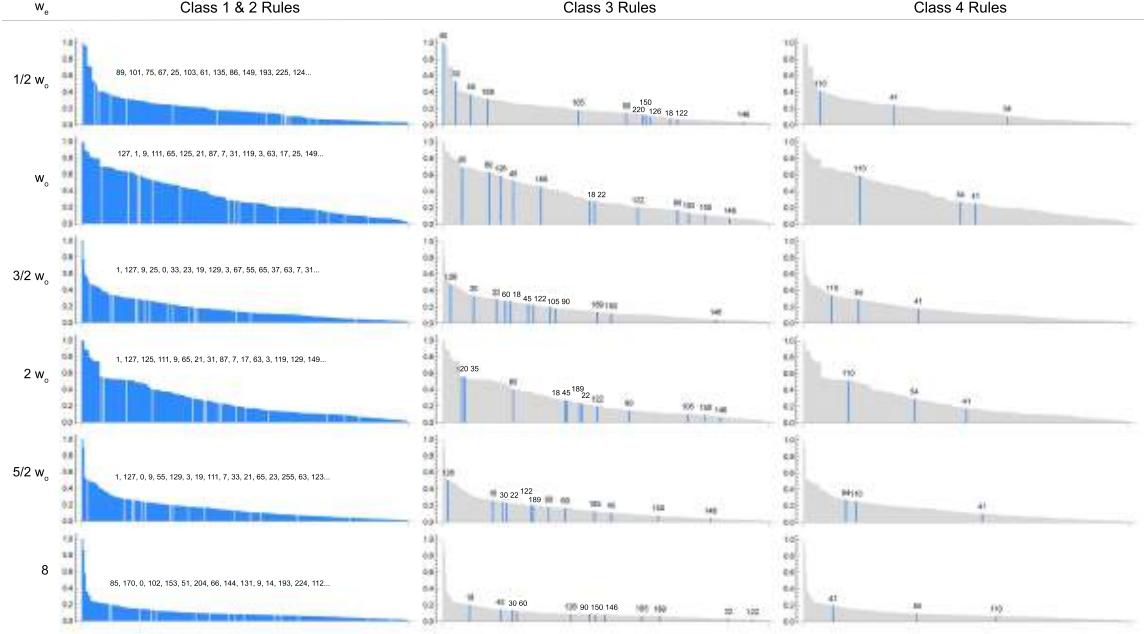


FIG. 15: Frequency distributions of rules (“metagenomes”) used in non-OEE σ attractors. Each column shows the same distribution, but highlighted in a different manner to show particular rule Wolfram classes. Each row represents metagenomes for a different e . Case I CA mainly use Class I and II CA rules in their attractors.

8. Distributions of Attractor Sizes

Figure 16 shows box-whisker plots of attractor sizes for each CA variant explored as described in Section 3 a.

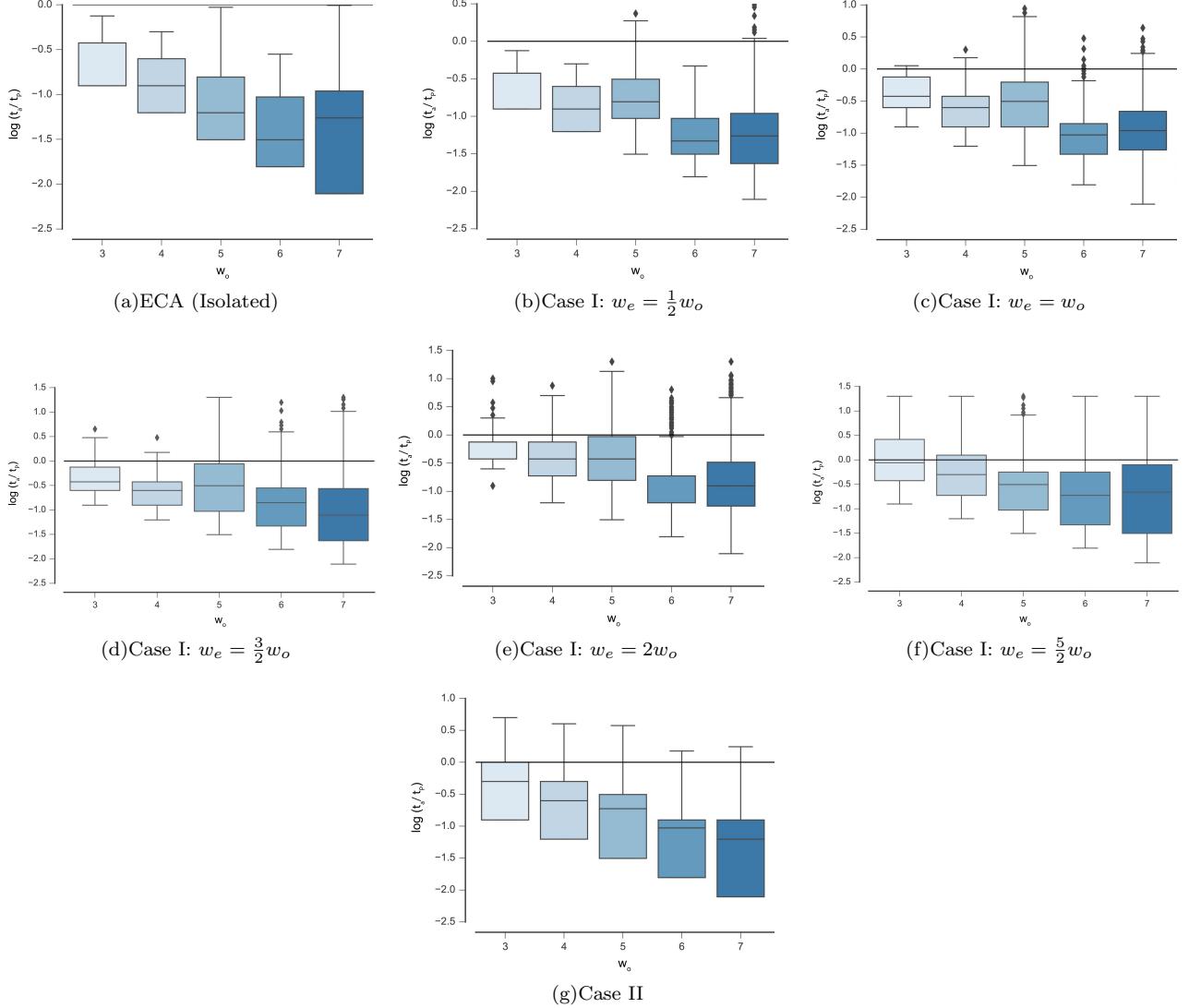


FIG. 16: Distribution of attractor sizes t_a for the state trajectory of o , for ECA, Case I, and Case II CA. Attractor sizes are normalized to the Poincaré time $t_P = 2^{w_o}$ for an isolated ECA, where the black horizontal line indicates where $t_a/t_P = 1$ (shown on a log scale). Sample trajectories displaying *unbounded evolution* (UE) occur for $t_a/t_P > 1$.

9. Compressibility and Lyapunov Exponent Values for Case II CA

As a final comparison between Case I and Case II CA, the C and k values, as defined in Sections 3, are shown in a heat map in Figures 18 and 19, and Figures 21 and 22, respectively. In addition, values for o that displayed OEE are shown in Figures 17 and 20.

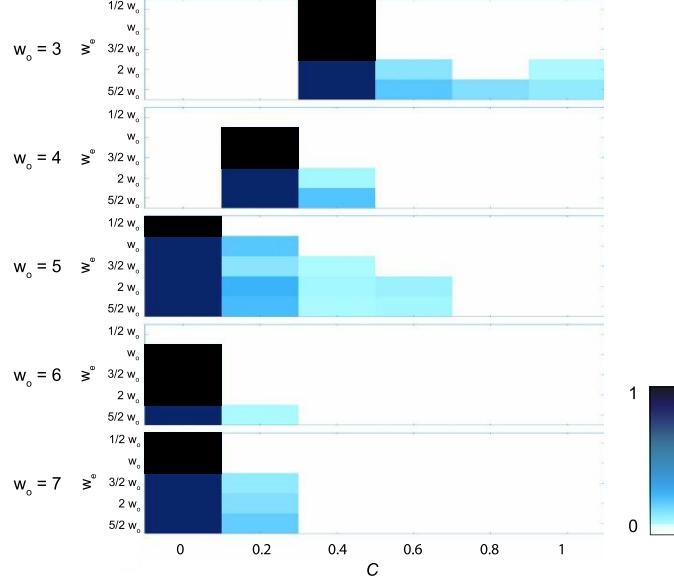


FIG. 17: Heat map of the compression values (C) for only OEE Case I CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for C .

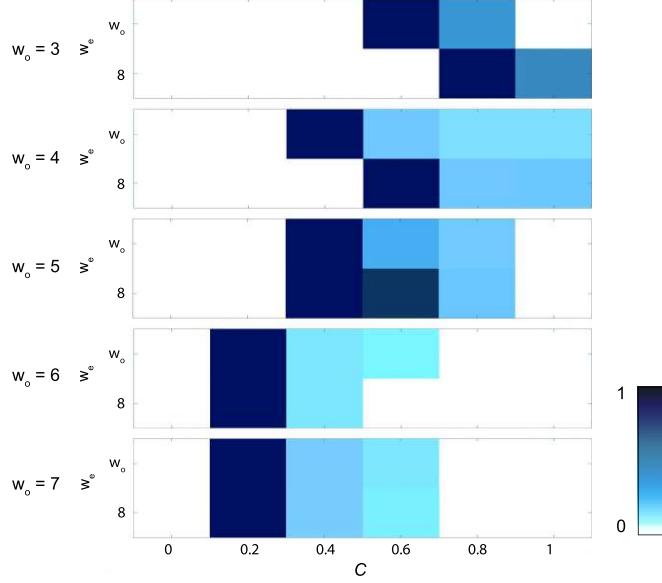


FIG. 18: Heat map of the compression values (C) for all sampled Case I CA o ($w_e = w_o$) and Case II CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for C .

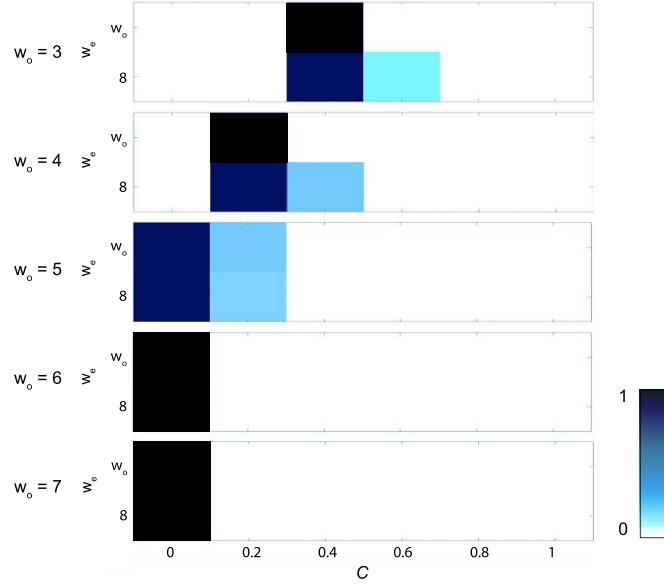


FIG. 19: Heat map of the compression values (C) for only OEE Case I CA o ($w_e = w_o$) and Case II CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for C .

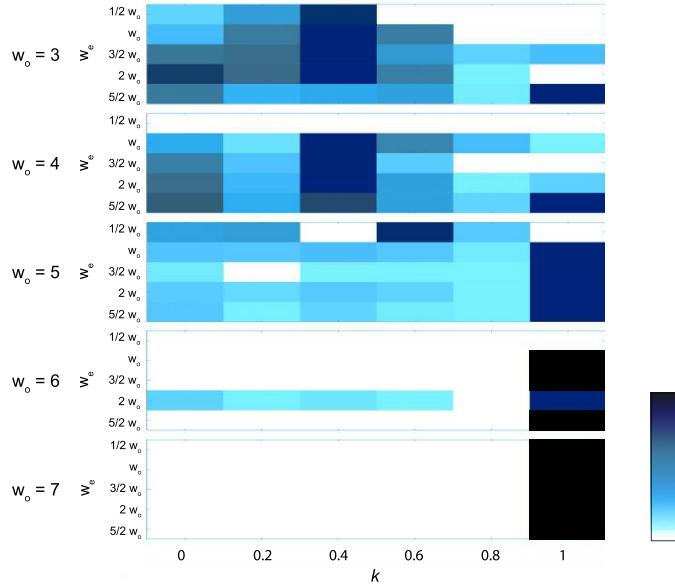


FIG. 20: Heat map of the Lyapunov exponent values (k) for only OEE Case I CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for k .

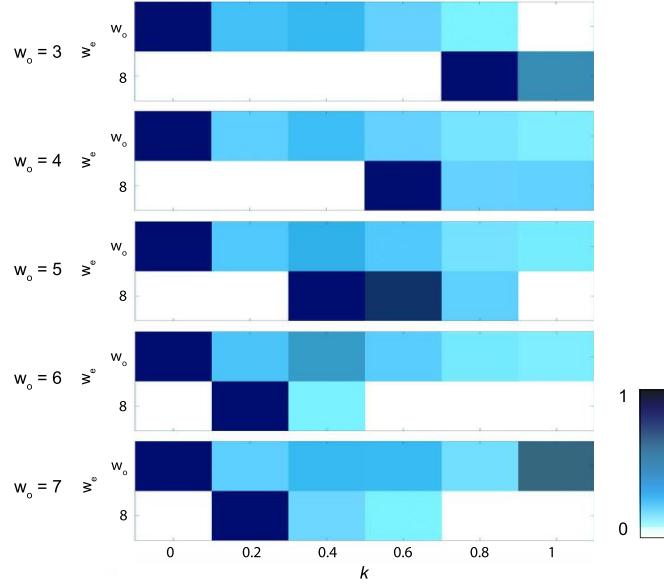


FIG. 21: Heat map of the Lyapunov exponent values (k) for all sampled Case I CA o ($w_e = w_o$) and Case II CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for k .

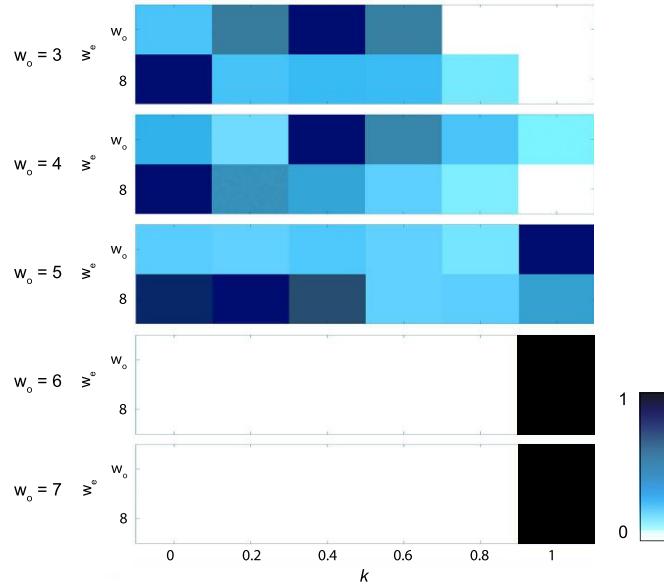


FIG. 22: Heat map of the Lyapunov exponent values (k) for only OEE Case I CA o ($w_e = w_o$) and Case II CA o . Each panel shows values for a given w_o where the value for w_e varies. The x-axis shows the distribution of values for k .

10. Larger Systems

Figure 23 shows example executions of Case I state-dependent CA with large organism width $w_o = 101$, which visually demonstrate that the novelty of the dynamics reported herein scale to large system sizes.

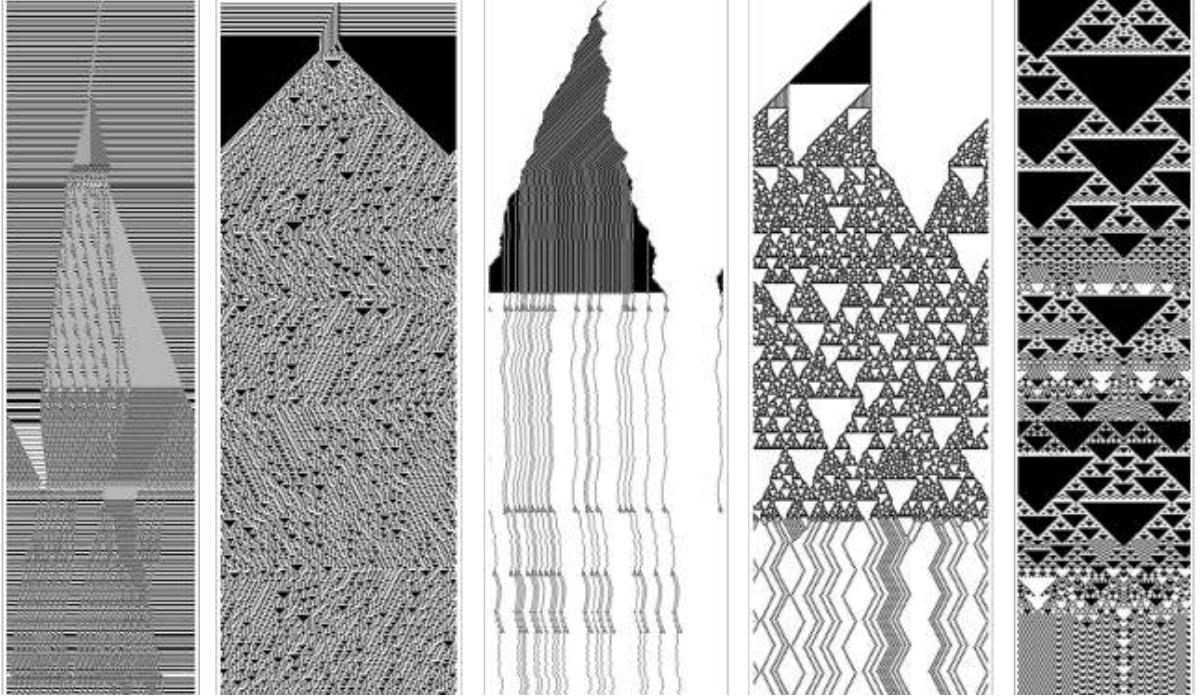


FIG. 23: Relevant timescales for describing the dynamics of o embedded in u . Shown are the Poincaré recurrence time t_P (blue) for an isolated ECA of the same width w_o as o , the state-trajectory recurrence time t_r of o (red), and attractor size t_a of the full system u (green).