Sundry

1) Countability Basics

a) $f: \mathbb{N} \to \mathbb{N}$, $f(n) = n^2$

↳ one-to-one:

yes: every $x \in \mathbb{N}$ will map to an x^2

assume $f(a) = f(b)$ $\forall$ $a, b \in \mathbb{N}$

$\sqrt{a^2} = \sqrt{b^2}$

$|a| = |b|$

since $a, b \in \mathbb{N}$, they are non-negative $\Rightarrow$ one-to-one

b) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 + 1$

Surjection?

↳ check every $y \in \mathbb{R}$ has pre-image such that $f(x) = y$.

$y = x^3 + 1$

$x^3 = y - 1$

$x = \sqrt[3]{y-1}$

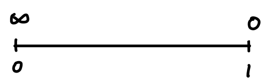
x defined for all $x \in \mathbb{R}$ $\Rightarrow$ surjective

c) $(0, 1)$

$\mathbb{R}_+ = (0, \infty)$

$\to$ $f: (0,1) \to (0, \infty)$

$\boxed{f(x) = \frac{1}{x} - 1}$



$\to$ $g: (0, \infty) \to (0, 1)$

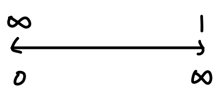
$\boxed{g(x) = \frac{1}{x+1}}$

$x = \frac{1}{y} - 1$

$x + 1 = \frac{1}{y}$

$y(x+1) = 1$

$y = \frac{1}{x+1}$



2) Unprogrammable Programs

a) Proof by Contradiction:

Let $F'(x) = A(F(x))$, where $A(x)$ is a constant function that returns $0 * x$.
Now, we input $P(F', x, 0)$, and we can essentially reduce this to the halting problem.

We can check whether $F'(x) = 0$. This means P will output 0 when x halts. However, this would mean F halts on x, which is a contradiction. Therefore, we have no way of checking if F halts at some point, and this program is uncomputable.

b) Proof by Contradiction:

$P(F, G)$

Assume we have programs F' and G'

$$\left.\begin{array}{l} \hookrightarrow F' \to A(F(x)) \\ G' \to A(G(x)) \end{array}\right\} \text{returns } 0 * x$$

Now, we can pass in F' and G' to P. $P(F', G')$ will return true if F' and G' halt on the same set of inputs, and we will know it halted if either F' and G' return 0. However, this would also mean we solved the halting problem, which is uncomputable. ※

Therefore, this program is uncomputable.

3) Fixed Points

a) Proof by Contradiction:

Suppose we have some constant function $A(x)$ that returns 0, and assume P is computable.

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A(x):
  return 0 * x
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Then, for $P(x)$, we can input $P(A(x))$, which should also always return 0

$P = P(A(x))$ → there are 2 cases:

1. P returns 0
   ↳ this means P halted at some x
2. P does not return 0
   ↳ P did not halt on some x

However, this would mean we solved the halting problem, which we know is uncomputable. Therefore P is also uncomputable. ※

b) Assume we have function $P(x)$, which returns fixed point x if it exists and null otherwise. In part A, we said that in the case that $P = P(A(x))$, if P returned 0 we had a fixed point, and we can just return that fixed point, and if it did not return 0, x did not halt and we just return null. However, we have already shown in part A this leads to a contradiction & is uncomputable, and thus part b is also uncomputable since we can reduce part b → part a.

c) if a fixed point exists & is some natural number, this means we can also represent this in binary. We can simply represent $\mathbb{N}$ in base 10, and we can call $P(x)$ in part b to return true if fixed point exists and has no leading zero's before its most significant bit, and null otherwise, which means that part b is computable. However, we have already shown that part B is uncomputable, so this is a contradiction, which means part c is also uncomputable. ※

4) Probability Warm Up

a)

$$\mathbb{P}[k] = \binom{15}{k} \cdot 0.3^{k} (0.7)^{15-k}$$

our probability p of drawing a green ball is $\frac{30}{100} \to 0.3$

$1-p$ is $100 - \frac{30}{100} = \frac{70}{100}$

we choose k green balls from 15 total balls

b) $$\mathbb{P}[k] = \frac{\binom{70}{15-k} \cdot \binom{30}{k}}{\binom{100}{15}}$$

there are $\binom{30}{k}$ ways to choose k green balls

and $\binom{70}{15-k}$ represents the probability of choosing

an orange ball.

we divide by the sample space, which is $\binom{100}{15}$ since

we choose 15 balls from 100 total.

c) $\mathbb{P}[\text{at least 1 repeated value}] = 1 - \frac{5}{162}$

complement:

all 5 rolls are unique

$$\left.\begin{aligned}
&\mathbb{P}[\text{1st roll unique}] = \frac{6}{6} \\
&\mathbb{P}[\text{2nd roll unique}] = \frac{5}{6} \\
&\mathbb{P}[\text{3rd roll unique}] = \frac{4}{6} \text{ or } \frac{2}{3} \\
&\text{4th} \quad = \frac{3}{6} \text{ or } \frac{1}{2} \\
&\text{5th} \quad = \frac{2}{6} \text{ or } \frac{1}{4}
\end{aligned}\right\} 1 \cdot \frac{5}{6} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{5}{162}$$

5) Five Up

a) $\Omega = 2^5 = 32$

b) $\mathbb{P}[\text{exactly 3 heads}] = \binom{5}{3} = \frac{5!}{(5-3)!\,3!} = 10$

$\binom{5}{3}$ - choose 3 tosses out of 5 to be heads

c) $\mathbb{P}[\text{3 or more heads}] = \binom{5}{3} + \binom{5}{4} + 1$

$\binom{5}{3}$ - choose 3 tosses out of 5 to be heads

$\binom{5}{4}$ - choose 4 tosses to be heads

$\binom{5}{5} = 1$ way to have all 5 tosses be heads

d) $\mathbb{P}[HHHTT]$ -

prob. head → p
prob. tail → $1-p$

$p \cdot p \cdot p \cdot (1-p)(1-p)$

$= 0.5^5$

$\mathbb{P}[HHHHT]$

$p \cdot p \cdot p \cdot p\,(1-p)$
$= p^4(1-p)$
$= 0.5^5$

both cases are 0.5^5 since we are flipping a fair coin

e) $\mathbb{P}[\text{at least 1 head}] = 1 - 0.5^5$

complementary - all tails/no heads
$= 0.5^5$

f) $\mathbb{P}[\text{more heads than tails}] = \binom{5}{3} \cdot 0.5^5 + \binom{5}{4} \cdot 0.5^5 + 0.5^5$

heads → at least 3

ways to choose 3 heads from 5 coin flips

3 heads + 1 tail : $\binom{5}{3} \cdot 0.5^5$

each of the sequences has 0.5^5 probability

4 heads + 1 tail: $\binom{5}{4} \cdot 0.5^5$

5 heads : $\binom{5}{5} = 1 \cdot 0.5^5 = 0.5^5$

g) heads - $p = \frac{2}{3}$

tails - $1-p = \frac{1}{3}$

$\mathbb{P}[HHHTT] = p \cdot p \cdot p \cdot (1-p)(1-p)$
$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3}$
$\boxed{= \frac{8}{243}}$

$\mathbb{P}[HHHHT] = p \cdot p \cdot p \cdot p \cdot (1-p)$
$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3}$ $\boxed{= \frac{16}{243}}$

h) $\mathbb{P}[\text{at least 1 head}] = 1 - \left(\frac{1}{3}\right)^5$

complementary – all tails

$= \left(\frac{1}{3}\right)^5$

$\frac{1}{3}$ chance for tail, 5 flips $\Rightarrow \left(\frac{1}{3}\right)^5$

i) $\mathbb{P}[\text{more heads than tails}] = \binom{5}{3}\left(\left(\frac{2}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^2\right) + \binom{5}{3}\left(\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)\right) + \left(\frac{2}{3}\right)^5$

3 H + 2T: $\binom{5}{3} \cdot \left(\left(\frac{2}{3}\right)^3 \cdot \left(\frac{1}{3}\right)^2\right) \longrightarrow$ for example: HHHTT can have diff. combinations (THHHT, HHTHT, etc.)

so we have $\binom{5}{3}$ ways to select 3 heads

4H + 1T: $\binom{5}{4} \cdot \left(\left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)\right)$

then we have $p = \frac{2}{3}$ for 3 heads $\rightarrow \left(\frac{2}{3}\right)^3$

$1-p = \frac{1}{3}$ for 2 tails $\rightarrow \left(\frac{1}{3}\right)^2$

5T: $\binom{5}{5} = 1 \cdot \left(\frac{2}{3}\right)^5 = \left(\frac{2}{3}\right)^5$

and we apply this same logic to the other 2 scenarios