Sundry

Worked on w/ tutor Uri Kreindler & at office hours and w/friend

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1) Tellers

$X \sim$ # customers that enter bank at given hour

a) S_n = tellers?

$|\hat{p} - p| \leq \varepsilon$

$\hat{p} = \frac{1}{n} S_n$

confidence $1 - \delta = 95\%$

$\mathbb{P} \leq 0.05$

$\mathbb{P}[X \geq c] \leq \frac{\mathbb{E}[X]}{c}$

$\mathbb{P}[X \geq c] \leq \frac{5}{c}$ $\qquad \frac{5}{100}$

fail at most 5

$\mathbb{P}[< 0.05]$

$\boxed{c = 100}$

b) Chebyshev's

$\mathbb{P}[|X - \mu| \geq c] \leq \frac{Var(X)}{c^2}$

$\mathbb{P}[|X - 5| \geq c] \leq \frac{5}{c^2}$

$c = 10 \rightarrow \boxed{c = 15}$

we add 5 to c b/c $\mathbb{P}[|X - 5|]$

2) Just one Tail, Please

a) Show:

$$\mathbb{P}[X \geq a] \leq \frac{\sigma^2 + c^2}{(a+c)^2}$$

given: $\mathbb{E}[X] = 0$

$a > 0$

$Var(X) = \sigma^2$

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[\varphi(X)]}{\varphi(a)}$$

$$\leq \frac{\mathbb{E}[\varphi(X)]}{(a+c)^2} \leftarrow \text{replace } x \text{ w/ } a$$

$\mathbb{E}[\varphi(X)] \begin{matrix} = \\ < \end{matrix} \Big\}$ show one or both

since we know $Var(x) = \sigma^2$

$$\sigma^2 + c^2 = Var(x) + c^2$$

$$= \mathbb{E}[x^2] - \mathbb{E}[x]^2 + c^2$$

since we know: $\varphi(x) = (x+c)^2$

$$\mathbb{E}[\varphi(X)] = \mathbb{E}[(x+c)^2]$$

$$= \mathbb{E}[x^2 + 2cx + c^2]$$

$$= \mathbb{E}[x^2 + 2cx] + c^2$$

now, we just want to show $\mathbb{E}[x^2 + 2cx]$ either is less than or equals $Var(x)$

$$\mathbb{E}[x^2 + 2cx] \leq \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

$$\mathbb{E}[\cancel{x^2}] + \mathbb{E}[2cx] \leq \mathbb{E}[\cancel{x^2}] - \mathbb{E}[x]^2$$

$$\mathbb{E}[2cx] \leq -\mathbb{E}[x]^2$$

$$\mathbb{E}[2cx] \leq 0$$

$$2c\,\mathbb{E}[x] \leq 0$$

$$\mathbb{E}[x] \leq 0$$

since $\mathbb{E}[x] = 0$, we showed $\mathbb{E}[x] \leq 0$

b) $\mathbb{P}[X \geq a] \leq \frac{\sigma^2 + c^2}{(a+c)^2}$

differentiate

$$0 \leq \frac{(\sigma^2 + c^2)\frac{d}{dc}(a+c)^2 - (a+c)^2 \frac{d}{dc}\sigma^2 + c^2}{[(a+c)^2]^2}$$

$$0 = \frac{[(\sigma^2 + c^2)\cdot 2(a+c)] - [(a+c)^2 \cdot 2c]}{[(\cancel{a+c})^2]^2} \rightarrow$$

$$0 = \frac{[(\sigma^2+c^2)(2a+2c)-(a^2+2ac+c^2)2c]}{[(a+c)^2]^2}$$

$$0=\frac{[2a\sigma^2+2c\sigma^2+2ac^2+2c^3-(2ca^2+4ac^2+2c^3)]}{[(a+c)^2]^2} \to 0=\frac{[2a\sigma^2+2c\sigma^2+\underline{2ac^2}+\cancel{2c^3}-2ca^2-\underline{4ac^2}-\cancel{2c^3}]}{[(a+c)^2]^2}$$

$$0=\frac{-2ac^2-2ca^2+2a\sigma^2+2c\sigma^2}{[(a+c)^2]^2} \qquad 0=\frac{-2ac(c+a)+2\sigma^2(a+c)}{[(a+c)^2]^2}$$

num = 0

$$0=\frac{-2ac+2\sigma^2\cancel{(a+c)}^2}{[(a+c)^2]^{\cancel{2}}} \to 0=\frac{-2ac+2\sigma^2}{(a+c)^2}$$

the numerator must $=0$ and the denominator $\neq 0$

solving for c:

$$-2ac+2\sigma^2=0$$
$$-2ac=-2\sigma^2$$
$$ac=\sigma^2$$
$$c=\frac{\sigma^2}{a}$$

from part a:

$$\frac{\sigma^2+c^2}{(a+c)^2} = \frac{\sigma^2+\left(\frac{\sigma^2}{a}\right)^2}{\left(a+\left(\frac{\sigma^2}{a}\right)\right)^2} = \frac{\sigma^2+\frac{\sigma^4}{a^2}}{\left(\frac{a^2+\sigma^2}{a}\right)^2} \to \frac{\frac{\sigma^2a^2+\sigma^4}{\cancel{a^2}}}{\frac{(a^2+\sigma^2)^2}{\cancel{a^2}}}$$

$$=\frac{\sigma^2a^2+\sigma^4}{(a^2+\sigma^2)^2} = \frac{\sigma^2\cancel{(a^2+\sigma^2)}}{(a^2+\sigma^2)^{\cancel{2}}} \quad \boxed{=\frac{\sigma^2}{a^2+\sigma^2}}$$

c) i) $\mathbb{P}[X \geq \mathbb{E}[X]+a] > \frac{Var(x)}{2a^2}$

or $\mathbb{P}[X \leq \mathbb{E}[X]-a] > \frac{Var(x)}{2a^2}$

$X \sim$ avg customers per hour

$X\sim Pois(3) \quad \mathbb{E}[X]=3 \quad Var[X]=3$

$a\sim 2$

$$\mathbb{P}[X\leq 3-2] > \frac{3}{2(2)^2}$$

$$\mathbb{P}[X\leq 1] > \frac{3}{8} \approx 0.375$$

however, $\mathbb{P}[X=1] = \frac{3^1}{1!}e^{-3}$

$$=3e^{-3}$$

$$\mathbb{P}[X=0]=\frac{3^0}{0!}e^{-3} = 1e^{-3}$$

so $\mathbb{P}[X\leq 1] = 3e^{-3} \approx 0.149$

- using the bound, we get $\mathbb{P}[X\leq 1] > 3/8$ which is about 0.375, but using the probability for $X\sim$ Poisson we get $\mathbb{P}[X\leq 1]\approx 0.149$, and

$$\mathbb{P}[X\leq 1] > 0.375 \neq \mathbb{P}[X\leq 1]\approx 0.149$$

$\therefore$ therefore, our method is not correct, and this also makes intuitive sense because we can't expect the distribution around the expectation to be completely uniform/equal, for example this method would not hold true for a geometric or exponential distribution.

d) $\mathbb{E}[X] = 3$
$Var(X) = 2$

i) $\mathbb{P}[X \geq c] \leq \frac{3}{c}$

$\therefore \mathbb{P}[X \geq 5] \leq \frac{3}{5}$

ii) $\mathbb{P}[|X-\mu| \geq c] \leq \frac{Var(X)}{c^2}$

$\mathbb{P}[|X-3| \geq c] \leq \frac{2}{c^2}$

since we start w/ $\mathbb{P}[X \geq 5]$ and we want form $\mathbb{P}[|X-\mu| \geq c]$

$\mathbb{P}[|X-3| \geq 5-3] = \mathbb{P}[|X-3| \geq 2]$

then, $c=2$ and:

$\mathbb{P}[|X-3| \geq c] \leq \frac{2}{2^2} \rightarrow \frac{2}{4} = \frac{1}{2}$

$\therefore \mathbb{P}[|X-3| \geq 2] \leq \frac{1}{2}$

iii) $Y \sim X-3 \rightarrow$ s.t. $\mathbb{E}[Y] = 0$

$\mathbb{P}[Y \geq a] \leq \frac{\sigma^2}{a^2+\sigma^2}$ $\quad Var(Y) = Var(X)$

since $\mathbb{E}[X]=3$ and we start w/ $\mathbb{P}[X \geq 5]$ and $Y = X-3$

$\mathbb{P}[Y \geq 2] \leq \frac{2}{2^2+2} \rightarrow \frac{2}{4+2} \rightarrow \frac{2}{6} \rightarrow \frac{1}{3}$

$\therefore \mathbb{P}[Y \geq 2] \leq \frac{1}{3}$

3) Short Answer

a) $X \sim [0, 2]$ ← uniform → cdf: $F(X) \begin{cases} 0 & , x < 0 \\ x/2 & , 0 \le x \le 2 \\ 0 & , x > 2 \end{cases}$ $f_X(x) \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x \le 2 \\ 0, & x > 2 \end{cases}$

$y = f_X(X) = 4X^2 + 1$

CDF : $\mathbb{P}[y \le y] = \mathbb{P}[4X^2 + 1 \le y]$

$\mathbb{P}[4X^2 \le y - 1$

$X^2 \le \frac{y-1}{4}$

$\mathbb{P}[X \le \frac{1}{2}\sqrt{y-1}]$

plug in for X

CDF: $f(x) = \frac{d}{dx} \frac{1}{2}\sqrt{y-1}$

use chain rule

$= \frac{1}{2} \cdot \frac{1}{2\sqrt{y-1}} = \boxed{\frac{1}{4\sqrt{y-1}}}$

$F_X(x) \begin{cases} 0 & , \frac{1}{2}\sqrt{y-1} < 0 \\ \frac{1}{4}\sqrt{y-1} & , 0 \le \frac{1}{2}\sqrt{y-1} \le 2 \\ 0 & , \frac{1}{2}\sqrt{y-1} > 2 \end{cases}$

$f_X(x) = \begin{cases} 0 & , \frac{1}{2}\sqrt{y-1} < 0 \\ 1/2 & , 0 \le \frac{1}{2}\sqrt{y-1} \le 2 \\ 0 & , \frac{1}{2}\sqrt{y-1} > 2 \end{cases}$

↓

$f_X(x) = \begin{cases} 0 & , \frac{1}{2}\sqrt{y-1} < 0 \\ \frac{1}{8\sqrt{y-1}} & , 0 \le \frac{1}{2}\sqrt{y-1} \le 2 \\ 0 & , \frac{1}{2}\sqrt{y-1} > 2 \end{cases}$

$\mathbb{E}[y] = \int_{-\infty}^{\infty} y f_y(y)\, dy$

$\mathbb{E}[y] = \underbrace{\int_1^{17}}_{} y\, f(y)$

b/c our bounds are $4(0)^2 + 1 = 1$

$4(2)^2 + 1 = 17$

$= \int_1^{17} \frac{y}{8\sqrt{y-1}}$

$\mathbb{E}[y] = E[4X^2 + 1]$] b/c $\mathbb{E}[1] = 1$

$= 4E[X^2] + 1$

$= 4\left(\frac{4}{3}\right) + 1$

$= \boxed{\frac{19}{3}}$

$\mathbb{E}[X^2] = \int_0^2 x^2 f(x)\, dx$

$= \int_0^2 x^2 \cdot \frac{1}{2} dx$

$= \frac{1}{6} x^3 \Big]_0^2$

$= 8/6 \rightarrow \boxed{4/3}$

$\text{Var}(y) = \text{Var}(4X^2 + 1)$

$= \text{Var}(4X^2)$

$= 4^2\, \text{Var}(X^2)$

$= 16\, \text{Var}(X^2)$

$= 16(32/10 - 16/9)$

$\text{Var}(X^2) = \mathbb{E}[X^4] - \mathbb{E}[X^2]^2$

$= \mathbb{E}[X^4] - \left(\frac{4}{3}\right)^2$

$= \mathbb{E}[X^4] - 16/9$

$= 32/10 - 16/9$

$\mathbb{E}[X^4] = \int_0^2 x^4 \cdot f(x)\, dx$

$= \int_0^2 x^4 - \frac{1}{2} dx$

$= \frac{1}{10} x^5 \Big]_0^2 = 32/10$

b) $f(x, y) = \begin{cases} cxy + 1/4 \\ 0 \end{cases}$ $x \in [1, 2]$, $y \in [0, 2]$

$\mathbb{P}[1 \le cxy + \frac{1}{4} \le 2,\ 0 \le cxy + \frac{1}{4} \le 2]$

$= \int_0^2 \int_1^2 (cxy + \frac{1}{4})\, dx\, dy$

$= \int_0^2 \frac{1}{2} cyx^2 + \frac{1}{4} dy$

$= \frac{3}{4} cy^2 + \frac{1}{4} y \Big]_0^2 = 3c + 1/2$

c) $X \sim Exp(3)$

i) $\mathbb{P} : X \in [0, 1]$

$\mathbb{P}[0 \le X \le 1] = CDF = F(1) - F(0)$

$= \mathbb{P}[X \le 1] - \mathbb{P}[X \le 0]$

$= (1 - e^{-3 \cdot 1}) - (1 - e^{-3 \cdot 0})$

$\boxed{= 1 - e^{-3}}$

ii) floor $(5.8) = 5$

$\mathbb{P}[y = k] = \mathbb{P}[k \le X \le k+1]$

$= F[k+1] - F(k)$

$= (1 - e^{-3(k+1)}) - (1 - e^{-3k})$

$= e^{-3k} - e^{-3k-3}$

$\boxed{= e^{-3k}(1 - e^{-3})}$

d) $X_i \sim Exp(\lambda)$

CDF of $U = P[u \le t]$

Calculate complement: we also know that for $V > t$, all the exp. variables $> t$, i.e. $e^{-\lambda t}, e^{-\lambda t}$, etc.

$\mathbb{P}[X \le t] = 1 - e^{-\lambda t}$

$\mathbb{P}[X > t] = e^{-\lambda t}$

CDF of u:

$\mathbb{P}[u \le t] = 1 - e^{-\lambda_1 t} e^{\lambda_2 t} \ldots e^{\lambda_n t}$

$= 1 - e^{-t(\lambda_1 + \lambda_2 + \ldots \lambda_n)}$

$=$ CDF where $\mu = \lambda_1 + \lambda_2 + \ldots \lambda_n$

4) Uniform Distribution

ℓ - circumference = 10

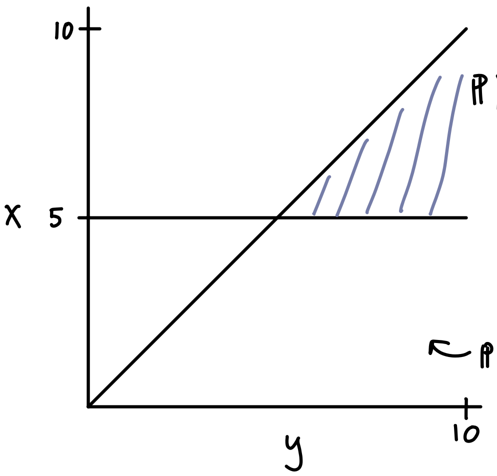
$X \sim$ position of first spinners mark

$Y \sim$ pos. of second spinners mark

} indep.

$\mathbb{P}[X \geq 5]$ given $Y \geq X$

$$\hookrightarrow \mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[X \geq 5 \cap Y \geq X]}{\mathbb{P}[Y \geq X]}$$



$\mathbb{P}[X \geq 5 \cap Y \geq X] = \frac{1}{8}$

$\mathbb{P}[Y \geq X] = \frac{1}{2}$

$$\frac{\frac{1}{8}}{\frac{1}{2}} = \boxed{\frac{1}{4}}$$

5) Darts with Friends

Michelle - $[0, 1]$
$X \sim$ distance of M's throw

Alex - $[0, 2]$
$Y \sim$ dist. A's throw

a) i) $a = \pi$
dist $\to 0, 1$
$F(x) = \mathbb{P}[X \leq 1]$

$$\frac{\pi x^2}{\pi(1)^2} = \boxed{x^2}$$

probability for x to be some value within $[0, 1]$ is just the area x could take on $\to \pi x^2$

$|\Omega| = \pi r^2 = \pi$

$$\begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

ii) $a = \pi(2)^2$
$= 4\pi$

$F(y) = \mathbb{P}[y \leq 2]$

$$\frac{\pi y^2}{4\pi} = \boxed{\frac{y^2}{4}}$$

$|\Omega| = \pi(2)^2 = 4\pi$

iii) PDF - x

$f(x) = \frac{d}{dx} x^2$
$\boxed{= 2x}$

$$\begin{cases} 0, & x \leq 0 \\ 2x, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

iv) PDF - y

$f(y) = \frac{d}{dx} \frac{y^2}{4}$
$= \frac{1}{4}(y^2)\frac{d}{dy} = \frac{1}{4} \cdot 2y$
$= \frac{1}{2}y = \boxed{\frac{y}{2}}$

b) $\mathbb{P}$[Michelle's throw closer] $= \mathbb{P}[x < y \mid y < 1] = 1/2$
since $x < y$
always true

$\mathbb{P}[x < y \mid 1 < y < 2] = 1$
$\mathbb{P}[y < 1] = 1/4$
$\mathbb{P}[1 < y < 2] = 3/4$

$\mathbb{P}[x < y \mid y < 1]\,\mathbb{P}[y < 1] + \mathbb{P}[x < y \mid 1 < y < 2]\,\mathbb{P}[1 < y < 2]$
$= 1/2 \cdot 1/4 + 1 \cdot 3/4$
$= 1/8 + 3/4 = \boxed{7/8}$

$\longrightarrow$ $\mathbb{P}$[Alex's throw closer] $= \mathbb{P}[y < x]$
$= 1 - 7/8 = \boxed{1/8}$

c) CDF of $U = \max(X, Y)$

$$\mathbb{P}[U \leq a] = \underbrace{\mathbb{P}[X \leq a]\ \mathbb{P}[Y \leq a]}_{\text{cdf}}$$