Sundry:

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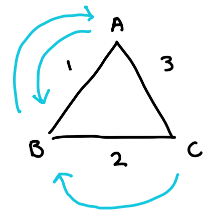
1) Airport

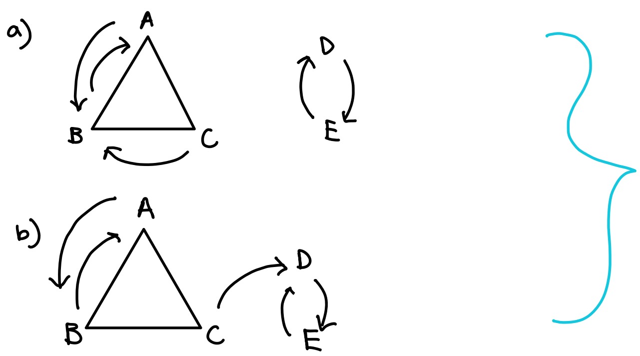
base case: $n=1$; when $n=1$, the claim holds since there will be $2(1)+1 = 3$ airports. two airports a and b will have the closest distance, thus, their planes will always fly to each other. airport c will fly to their closest airport, but since it is furthest away from the two other airports, there will be no plane flying to airport c.



inductive hypothesis: for some value of $n=k$, where $k \in \mathbb{Z}^+$ if there are $2k+1$ airports, the statement holds true

inductive step: for $n=k+1$, we can show the following:

$$2(k+1)+1 = 2k+2+1 = 2k+3$$



case 5 airports

when we add in 2 more airports for $2k+3$, there are 2 scenarios: airport D & E are closest to each other, and send airplanes to each other cyclically (shown in a) or results in a remapping (b).

the pigeonhole principle states that if there are n boxes & k items, and $k>n$, then there must be at least 1 box w/ 2 items. Similarly, when we add 2 airports ($2k+1 \rightarrow 2k+3$), each of the 2 new airports must have an airplane already destined for it, and we know from our inductive hyp. that the $2k+1$ airports will always have an airport w/ no destined planes.

proven by induction.

2) Proving Inequality

base case: $n=1$; when $n=1$, the claim holds since $\frac{1}{3^1} = \frac{1}{3} < \frac{1}{2}$

inductive hypothesis: for all $n > 1$ and $\leq k$, where $k \in \mathbb{Z}^+$,
the statement holds true for $\frac{1}{3^1} + \dots \frac{1}{3^k} < \frac{1}{2}$

inductive step: for $n=k+1$:

$$\frac{1}{3^1} + \dots \frac{1}{3^{k+1}} < \frac{1}{2}$$

$$\frac{1}{3^1} + \dots \frac{1}{3^k} + \frac{1}{3^{k+1}} < \frac{1}{2}$$

consider $n=4$: $\frac{(a_1 + a_2 + \dots a_4)}{4} \geq \sqrt[4]{a_1 \cdot a_2 \cdot \dots a_4}$

$$= \frac{(a_1 + a_2)}{2} + \frac{(a_3 + a_4)}{2} \geq \sqrt{a_1 \cdot a_2} + \sqrt{a_3 \cdot a_4}$$

$$= \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \quad \text{and} \quad \frac{a_3 + a_4}{2} \geq \sqrt{a_3 \cdot a_4}$$

$$= \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \frac{\sqrt{a_1 a_2} + \sqrt{a_3 a_4}}{2}$$

↳ Left side equals sum of first 4 terms / 4.

Proven by induction

3) AM GM

$(\sqrt{a}-\sqrt{b})^2 = a - 2\sqrt{ab} + b$

$\hookrightarrow a - \sqrt{ab^2} + b$

$= a - 2\sqrt{ab} + b$

Show $\dfrac{a_1 + \dots a_n}{n} \geq \sqrt[n]{a_1 \dots a_n}$

a) base case: $n=2$, $a_1, a_2 \in \mathbb{R}^+$ $\Big\}$ $(\sqrt{a}-\sqrt{b})^2 = a - 2\sqrt{ab} - b$

substituting in a_1 for a and a_2 for b, we get:

$$(\sqrt{a_1} - \sqrt{a_2})^2 = a_1 - 2\sqrt{a_1 a_2} - a_2$$

we know this to be true from the initial equation

for $n=2$: $2\left(\dfrac{a_1 + a_2}{2}\right) \geq (\sqrt{a_1 a_2})2$

$\hookrightarrow a_1 + a_2 \geq 2\sqrt{a_1 a_2}$

$\hookrightarrow a_1 - 2\sqrt{a_1 a_2} + a_2 \geq 0$

since we know the left hand side of the inequality $= (\sqrt{a_1} - \sqrt{a_2})^2$ is some number squared, it must be ≥ 0.
therefore, the equation holds true for $n=2$.

b) for $n = k \in \mathbb{Z}^+$; suppose $n = 2^k$. Show AM-GM holds for $n = 2^{k+1}$.

inductive step: 2^{k+1}

$\hookrightarrow 2 \cdot 2^k$

1) $a_1 + a_2 + \dots a_n = b_1$ $\to c_1$
2) $a_{n+1} + \dots a_{2n} = b_2$ $\to c_2$

$\Big\}$ multiplying $\to$

$$\frac{b_1 + b_2}{2} \geq \sqrt{c_1 c_2}$$

AM-GM inequality holds

c) for $n = k \geq 2$, $k \in \mathbb{Z}^+$, suppose the AM-GM inequality holds for $n=k$. prove $n = k-1$.

inductive step: $n = k-1$

$$a_k = \frac{a_1 + \dots + a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 \dots a_{k-1}}$$

$\Big\}$ strong induction

d) For part a, we've established the base case. In part b, we've done the inductive step, which proves the statement holds true for $n=k$, and in part c, we used strong induction to show the statement holds true. Since we've proven true for $n=2$ (base case), $n=2^k$, and $n = 2^{k+1}$, we can prove this statement is true for all $n \geq 2$.

4) A Coin Game

prove total score $= \frac{n(n-1)}{2}$

base case: $n=2$

Player A:

n=2 → Stack 1: Stack 2: } Score: $1*1=1$

total score: $\frac{n(n-1)}{2}$

$= \frac{2(2-1)}{2} = \frac{2(1)}{2} = \boxed{1}$

↳ the claim holds for $n=2$.

inductive hypothesis: for all $n=k$, where $k \geq 2$ and $k \in \mathbb{Z}^+$, the claim holds for $\frac{k(k-1)}{2}$.

inductive step: prove $k+1$

consider stack $m+1$; we split

1) k 2) $m+1-k$

the score for that turn will be $\frac{k*(m+1-k)}{2}$

thus, total score = (score $k=2$) + (score $k=3$) + ... (score $k=m$) + (score $k=m+1$)

$= \frac{2(m+1-2)}{2} + \frac{3(m+1-3)}{2} + \ldots \frac{m(m+1-m)}{2} + \frac{m+1(m+1-1)}{2}$

$= \frac{2(m-1)}{2} + \frac{3(m-2)}{2} + \ldots \frac{m(1)}{2} + \frac{m+1(m)}{2}$

$= \frac{1}{2}\left(2(m-1) + 3(m-2) + \ldots m\right) + \frac{m(m+1)}{2}$

last term = last turn's score
in same format as $\frac{k(k-1)}{2}$ / $\frac{k(k+1)}{2}$

since the last term will always be $\frac{k(k+1)}{2}$, the original claim holds true.
proven by induction.

5) Pairing up

base case: $n=2$;

Jobs	Candidates
1	A , B
2	A , B

Candidates	Jobs
A	1 , 2
B	1 , 2

Day	1	2
A	1, 2	1
B		2

Matching: (A, 1) (B,2)
no rogue couple

$2^{n/2} \rightarrow 2^{2/2} \rightarrow 2$ distinct stable matchings

inductive hypothesis: for some value $n=k$, where $k \in \mathbb{Z}^+$,
the statement holds true for $2^{k/2}$ distinct matchings

inductive step: $n = k+2$

$= 2^{k/2} \cdot 2^{2/2}$

$= 2^{k/2 + 2/2}$

$= 2^{k+2/2}$

in the form of $2^{k/2}$

therefore, the original claim holds, there are at least $2^{n/2}$ distinct stable matchings.
proven by induction.

6) A Better Stable Pairing

a) $R \wedge R' = \{(A,3), (B,4), (C,1), (D,2)\}$

A	1	2
B	1	2

1	A	B
2	A	B

R: $\{(A,1), (B,2)\}$
R': $\{(A,2), (B,1)\}$ $\rightarrow (A,1), (B,2)$

No rogue couples.

b) No job prefers R or R' to $R \wedge R'$
in $R \wedge R'$, we always get the best pairing; as R represents all pairings in R and R' represents all pairings in R'.

c) j prefers R to R' and c prefers R' to R

or j prefers R' to R and c prefers R to R'
consider: if j & c do not prefer the same candidate, this would result in a rogue couple in either R or R', and this is a contradiction b/c we assume already that R' and R are stable.

d) i) $R \wedge R'$ is a pairing
↳ given, each candidate and their jobs are unique.

ii) $R \wedge R'$ is stable

R:
1) $\{(A,1) (B,2)\}$

$\{(A,1), (B,2)\}$

R':
$\{(A,1), (B,2)\}$
$\{(A,2), (B,1)\}$

As A & B both prefer job 1, $R \wedge R'$ is bound to have a rogue couple. Proven by contradiction