1. Fishy Computations

$\mathbb{P}[X=i] = \frac{\lambda^i}{i!} e^{-\lambda}$

a) $\mathbb{P}[X=7] = \frac{20^7}{7!} e^{-20}$

b) $\mathbb{P}$[at most once] =

$\lambda = 2x$ per year

$X \sim \text{Pois}(2)$

going at most 1x or just go $x=1$

Let $X = x$ be event we go x times in 2024

We want $\mathbb{P}[X \leq 1]$

$$\left.\begin{aligned} \mathbb{P}[X=1] &= \frac{2^1}{1!} \cdot e^{-2} = 2e^{-2} \\ \mathbb{P}[X=0] &= \frac{2^0}{0!} \cdot e^{-2} = e^{-2} \end{aligned}\right\} \mathbb{P}[X=1] + \mathbb{P}[X=0]$$

$= 2e^{-2} + e^{-2}$

$\boxed{= 3e^{-2}}$

c) $\lambda = 5.7$

$X \sim \text{Pois}(5.7)$

$X = x$ rep. # boats sailing in Laguna

$\mathbb{P}[X \geq 3]$

$$\left.\begin{array}{r} 5.7 \\ \times\ 2 \\ \hline 11.4 \end{array}\right\} X \sim \text{Pois}(11.4)$$

complement

$1 - \mathbb{P}[X < 3]$

$$\left.\begin{aligned} \mathbb{P}[X=0] &= \frac{11.4^0}{0!} \cdot e^{-11.4} = e^{-11.4} \\ \mathbb{P}[X=1] &= \frac{11.4^1}{1!} \cdot e^{-11.4} = 11.4e^{-11.4} \\ \mathbb{P}[X=2] &= \frac{11.4^2}{2!} \cdot e^{-11.4} = 64.98e^{-11.4} \end{aligned}\right\} 77.38e^{-11.4}$$

$\boxed{= 1 - 77.38e^{-11.4}}$

d) $X \sim \text{Pois}(\lambda)$

$$\mathbb{E}[X f(X)] = \lambda \mathbb{E}[f(X+1)]$$

note 16

Law of Unconcious Statician

$$\sum_x x f(x) \mathbb{P}_X[X = x]$$

$$\Downarrow$$

$$\sum x f(x) \frac{\lambda^x}{x!} \cdot e^{-\lambda} = \lambda \mathbb{E}[f(x+1)]$$

$$\sum_{x=0}^{\infty} (x+1) f(x+1) \frac{\lambda^{x+1}}{(x+1)!} \cdot e^{-\lambda} = \lambda \mathbb{E}[f(x+1)]$$

$$\sum_{x=0}^{\infty} \cancel{(x+1)} f(x+1) \frac{\lambda^x \cdot \lambda}{\cancel{(x+1)} x!} \cdot e^{-\lambda} =$$

$$\lambda \sum_{x=0}^{\infty} f(x+1) \frac{\lambda^x}{x!} e^{-\lambda} = \lambda \mathbb{E}[f(x+1)]$$

$$\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$\lambda \sum_{x=0}^{\infty} f(x+1) \cdot \underbrace{\sum_{x=0}^{\infty} \frac{\lambda^x}{x!} e^{-\lambda}}_{=1} = \lambda \mathbb{E}[f(x+1)]$$

$$\lambda \sum_{x=0}^{\infty} f(x+1) \cdot 1 = \lambda \mathbb{E}[f(x+1)]$$

$$\mathbb{E}[X] = \sum a \cdot \mathbb{P}[X = a]$$

$$\lambda \mathbb{E}[f(x+1)] = \lambda \mathbb{E}[f(x+1)]$$

e) $X \sim \text{Pois}(3)$

$$\mathbb{P}[X = 0] = \frac{3^0}{0!} \cdot e^{-3} = e^{-3} \implies \mathbb{P}[\text{no students arrive}] \to \mathbb{P}[X = 0]$$

coin toss = 2 min period

H = > 0 students

T = 0 students

how many H before 1st tails

X = x # students that arrive in 2 min period

find how long you should expect wait until there is a window w/ 0 students?

- how many 2 min periods until 1st 2 min period w/o students

X ~ # students arriving in 2 min period

Y ~ # 2 min intervals that pass before the first 2 min interval w/no students

$X \sim \text{Pois}(3)$

$Y \sim \text{Geo}(e^{-3})$

Z ~ # students he helps

$$\mathbb{E}[Z] = \mathbb{E}[X] \cdot (\mathbb{E}[Y] - 1)$$

$$= 3 \cdot \left(\frac{1}{e^{-3}} - 1\right)$$

$$\approx 3 \cdot (20 - 1) \approx 3 \cdot 19$$

$$\boxed{\approx 57 \text{ students}}$$

2) Coupon Collector Variance

X - # visits you have to make before you redeem grand prize

$$Var(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$$

indicator variables?

i - # of trips we make to get i^{th} newest card

$$I_k \begin{cases} 1 \text{ if } k^{th} \text{ card is newest card} \\ 0 \text{ otherwise} \end{cases}$$

$\mathbb{E}[I_k] = \frac{1}{n}$ ← all have equal chance to be last card collected

$Var(I_k) = \mathbb{E}[I_k^2] - (\mathbb{E}[I_k])^2$

b/c I_k is indicator variable, $\mathbb{E}[I^2_k] = \mathbb{E}[I_k]$

$$I_k \sim Geo\left(\frac{n-(k-1)}{n}\right)$$

$$Var(X) = \sum_{i=1}^{n} \underbrace{var(X_i)}_{\frac{1-p}{p}} \cdot \underbrace{Var(X)}_{\frac{1}{p}} = \frac{1-p}{p^2}$$

$$= \sum_{i=1}^{n} \frac{1-\frac{n-(i-1)}{n}}{\left(\frac{n-(i-1)}{n}\right)^2} = \sum_{i=1}^{n} \frac{1-\frac{n-i+1}{n}}{\left(\frac{n-i+1}{n}\right)^2}$$

} we hit all values in summation

$$= \sum_{i=1}^{n} \frac{1-\frac{i}{n}}{\left(\frac{i}{n}\right)^2} = \sum_{i=1}^{n} \frac{n(n-i)}{i^2} = \sum_{i=1}^{n} \frac{n^2-ni}{i^2}$$

$$= \sum_{i=1}^{n} \frac{n^2}{i^2} - \sum_{i=1}^{n} \frac{n}{i}$$

since we know n is constant and the different cards are unique, we take it out of the summation

$$\boxed{\therefore Var(X) = n^2\left(\sum_{i=1}^{n} \frac{1}{i^2}\right) - \mathbb{E}[X]}$$

3) Diversify your Hand

$\mathbb{E}[a+b] = \mathbb{E}[a] + \mathbb{E}[b]$

linearity

a) $\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}[X=x]$

X- distinct values

$[2, 3, 4, 5]$

Indicator

i- A, 1, 2, 3 ... 10,

13 possible values

$X_1 = 1$ appears

$X_A = 1$

$X_1 = 0$

$X_2 = 1$

$\vdots$

$X_K = 1$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ add all X_i's

$\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = 0]$

$\mathbb{P}[X_i = 0] = \frac{\binom{48}{5}}{\binom{52}{5}}$

find the complement for $1 - \mathbb{P}[X_i = 0]$: $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$

← we choose 5 from 48 b/c 52-4 cards we don't choose from them

← sample space choose hand of 5 from deck of 52

then we can take the summation:

$\mathbb{E}[X] = \left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right) \cdot 13$

← b/c each X_i is independent of the others

$\mathbb{E}[X] = 0.3412 \times 13$

$\boxed{= 4.436}$

b) $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

↳ $\left(\frac{1 - \binom{48}{5}}{\binom{52}{5}} \cdot 13\right)^2$

→ $X^2 = (X_1 + X_2 + \ldots X_{13})^2$

$= X_1^2 + X_2^2 + \ldots X_{13}^2 + 2\sum_{i \neq j} X_i X_j$

since X_i indicator, $X_i^2 = X_i$

$n \cdot n-1$

$13 \cdot 12 \cdot \mathbb{P}[\text{both } i \text{ and } j]$

for $\mathbb{E}[X_i X_j]$:

$|\Omega| = \binom{52}{5}$

hands w/ both i and j:

$P(\text{no } i \text{ and no } j) = \frac{\binom{44}{5}}{\binom{52}{5}}$

$$P(\text{both } i \text{ and } j) = 1 - \frac{\binom{44}{5}}{\binom{52}{5}}$$

choose 5 from 44 (exclude 4+4 i/j cards)

$\leftarrow |\Omega|$

$$\mathbb{E}[X^2] = \mathbb{E}[X_1^2] + \mathbb{E}[X_2^2] + \dots \mathbb{E}[X_{13}^2] + 2\sum_{i \neq j} \mathbb{E}[X_i X_j]$$

only = 0 or 1

$$= \frac{1-\binom{48}{5}}{\binom{52}{5}} \cdot 13 + 13 \cdot 12 \cdot 1 - \frac{\binom{44}{5}}{\binom{52}{5}}$$

probability both i & j appear in hand at same time

$$= 4.463 + (156 \cdot 0.5821)$$

$$= 4.463 + 90.8076 \approx \boxed{95.2706}$$

4) Double Check your Intuition Again

a) i) $Cov(X+Y, X-Y)$ X - first result   Y - second result

$= \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y]\,\mathbb{E}[X-Y]$

$= \mathbb{E}[X^2 - Y^2] - \mathbb{E}[X+Y]\,\mathbb{E}[X-Y]$

since X & Y are independent,

$\cancel{\mathbb{E}[X^2]} - \cancel{\mathbb{E}[Y^2]} - (\mathbb{E}[X] + \mathbb{E}[Y])(\mathbb{E}[X] - \mathbb{E}[Y])$

$\mathbb{E}[X^2] = \mathbb{E}[Y^2]$

$= -(\mathbb{E}[X] + \mathbb{E}[Y])(\mathbb{E}[X] - \mathbb{E}[Y])$

$= [(3.5 + 3.5)\underbrace{(3.5 - 3.5)}_{=0}$

$\boxed{= 0}$

ii) $\mathbb{P}[X+Y=j \cap X-Y=k] = \overset{2}{\mathbb{P}}[X+Y=j] \cdot \mathbb{P}[X-Y=k]$

Let $j = 2$, $k = 0$

$\mathbb{P}[X+Y=2] = 1/36$

$\mathbb{P}[X-Y=0] = 1/6$

1 $\mathbb{P}[X+Y=j \cap X-Y=k] = 1/36$, which is not equal to $1/36 \cdot 1/6$

since 1 $\neq$ 2, $(X+Y)$ & $(X-Y)$ are not independent.

b) X r.v.

$Var(X) = 0$

True; $Var(X) = \mathbb{E}[(X-\mu)^2]$

Let $Y = (X-\mu)^2$

$\mathbb{E}[Y] = \sum_{y \in Y} y \cdot \underbrace{\mathbb{P}[Y=y]}_{\text{between 0 and 1}}$

$\underbrace{y \in (X-\mu)^2}$

both always positive, y is squared and $\mathbb{P}[Y=y]$ in the range $(0, 1)$

$\therefore$ for $\mathbb{E}[(X-\mu)^2] = 0$, we would have to have X be a constant b/c there are only non-negative terms in $Var(X)$ calculation, and we sum up all X's, therefore if $X = \mu$ then we get $Var(X) = 0$

c) $Var(cX) = c\,Var(X)$

False;

$Var(X) = \mathbb{E}[(X-\mu)^2]$

$Var(cX) = \mathbb{E}[(cX-\mu)^2]$

$= \mathbb{E}[c^2X^2 - 2cX\mu + \mu^2]$ using linearity

$= \mathbb{E}[c^2X^2] - \mathbb{E}[2cX\mu] + \mu^2$

$\mu = \mathbb{E}[cX]$

$\mathbb{E}[2cX\mu] = 2\mu \underbrace{\mathbb{E}[cX]}_{\mu} = 2\mu^2$

$= \mathbb{E}[c^2x^2] - 2\mu^2 + \mu^2$

$= \mathbb{E}[c^2x^2] - \mu^2$

$= c^2\,\mathbb{E}[x^2] - \mu^2$

$= c^2\,\mathbb{E}[x^2] - \underbrace{\mathbb{E}[cX]^2}_{(c\,\mathbb{E}[x])^2} \quad = c^2\,\mathbb{E}[x^2] - c^2\,\mathbb{E}[x]^2$

$= c^2\underbrace{(\mathbb{E}[x^2] - \mathbb{E}[x]^2)}_{\mathrm{Var}(X)}$

$\therefore \mathrm{Var}(cX) = c^2\,\mathrm{Var}(x) \neq c\,\mathrm{Var}(x)$

d) $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sigma(X)\,\sigma(Y)}$

if X, Y independent, then $\mathrm{Cov}(X,Y) = 0$; converse is not true

Let A be r.v. taking values -1 and 1 w.p. $1/2$
and B be r.v. $B = A^2$; B will always be 1 b/c it'll be $(-1)^2$ or 1^2

A & B are dependent, and $\mathrm{Corr}(A,B) = 0$ since B is a constant
and covariance between a r.v. and a constant is 0.

$\therefore$ counterexample: even though A & B dependent, $\mathrm{Corr}(A,B) = 0$.

e) is $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$?

counterexample: Extending same idea from 4d

$X \sim$ r.v. taking -1 or 1 w.p $1/2$
$Y \sim$ r.v. $Y = X^2$ (always 1)

In this case, we already know X & Y are not independent, and both have nonzero standard deviations.
They also have zero correlation since Y is constant.

$\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$ b/c $\mathrm{Cov}(X,Y) \neq 0$ since X & Y are dependent.

5) Dice Games

a) $X \sim$ # total rolls (including last one)
(until she gets a 1)

$y \sim$ # rolls on which she gets even #

$\mathbb{E}[Y \mid X = x]$ $X \sim Geo(\frac{1}{6})$ $y \sim Bin(x-1, \frac{5}{6})$

↳ her last roll will always be one

$\mathbb{P}[\text{roll even}] = \frac{1}{2}$

$\mathbb{P}[X=i] = (1-p)^{x-1} \cdot p$
$= (\frac{5}{6})^{x-1} \cdot \frac{1}{6}$

if she gets a 1 at k^{th} roll, then sum previous k-1 rolls over $X \sim Geo(\frac{1}{6})$

using law of iterated expectations from n20:

$\mathbb{E}[y] = \mathbb{E}[\mathbb{E}[y \mid X]]$

$= \sum_n^{\infty} \mathbb{E}[X \mid y = y]\, \mathbb{P}[y = y]$

$X_i \begin{cases} 1 & \text{if roll even} \\ 0 & \text{otherwise} \end{cases}$

$= \sum_n^{\infty} n\, \mathbb{E}[y_1]\, \mathbb{P}[X = x]$

$= \mathbb{E}[y_1] \sum_n n\, \mathbb{P}[X = x] = \sum_x^{\infty} \frac{3}{5}(x-1)\, \mathbb{P}[X=x]$

$= \sum_x^{\infty} \frac{3}{5}x - \frac{3}{5} \cdot \mathbb{P}[X=x]$

$= \underbrace{\sum_x^{\infty} \mathbb{P}[X=x]}_{\mathbb{E}[X]} \frac{3}{5}x - \frac{3}{5}$

$Geo(\frac{1}{6})$
$\mathbb{E}[X] = 6$

$= 6 \cdot \sum_x^{\infty} \frac{3}{5}x - \frac{3}{5}$

$\boxed{= 3}$ ← used infinite series sum calculator

b) Bob - 1 die
if odd → add # that die equals to collection
if even → remove

$\mathbb{E}[X_i]$ # of dice after i time steps

$\mathbb{E}[X_1] = 1$; initially start w/ one die

find $\mathbb{E}[X_k \mid X_{k-1}]$

$\mathbb{P}[odd] = \mathbb{P}[even] = \frac{1}{2}$
↓
$\mathbb{E}[\text{\# dice added}] = \frac{1+3+5}{3} = \frac{9}{3} = 3$

$\mathbb{E}[\text{change in dice}] = (\frac{1}{2} \cdot 3) - (\frac{1}{2} \cdot 1) = \frac{2}{2} = \boxed{1}$

↳ so, $\mathbb{E}[X_k \mid X_{k-1}]$

$\mathbb{E}[X_k] = X_{k-1} + 1$

$\mathbb{E}[X_k] = \mathbb{E}[X_1] + n - 1$
$= 1 + n - 1$
$\boxed{= n}$

∴ expected # dice after n steps = n

scratch work

$$\mathbb{P}[X=x] = \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6}$$

$$\mathbb{E}[Y \mid X=x] = (x-1) \cdot \frac{1}{2}$$

$$\mathbb{E}[Y] = \sum_{x=1}^{\infty} \mathbb{E}[Y \mid X=x] \cdot \mathbb{P}[X=x]$$

$$= \sum_{x=1}^{\infty} (x-1) \cdot \frac{1}{2} \cdot \left(\frac{5}{6}\right)^{x-1} \cdot \frac{1}{6}$$

$$= \frac{1}{12} \sum_{x=1}^{\infty} (x-1) \left(\frac{5}{6}\right)^{x-1}$$

$$= \frac{1}{12} \sum_{x=0}^{\infty} (x) \left(\frac{5}{6}\right)^{x}$$

$$= \frac{1}{12} \cdot 30 = 2.5$$