Sundry

w/ friends

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1) Probability Potpurri

a) $A, B \in \Omega$

Show $\mathbb{P}[A \setminus B] \geq \mathbb{P}[A] - \mathbb{P}[B]$

set of elements in
A but not in B

$\mathbb{P}[A \setminus B] = \mathbb{P}[A \cap \bar{B}]$

$\downarrow$

$\mathbb{P}[A \cap \bar{B}] = \mathbb{P}[A] - \mathbb{P}[A - B]$

Since the intersection of A & B $\not>$ B
$\mathbb{P}[A \cap B]$ is at most $\mathbb{P}[B]$

$\therefore \mathbb{P}[A] - \mathbb{P}[A \cap B] \geq \mathbb{P}[A] - \mathbb{P}[B]$

b) Prove D is independent of C

$\mathbb{P}[D \mid C] = \mathbb{P}[D \mid \bar{C}]$

$\mathbb{P}[D] = \mathbb{P}[D \mid C] \cdot \mathbb{P}(C) + \mathbb{P}[D \mid \bar{C}] \cdot \mathbb{P}[\bar{C}]$

$= \mathbb{P}[D \mid \bar{C}] \cdot (\mathbb{P}[\bar{C}] + \mathbb{P}[\bar{C}])$

$\mathbb{P}[C] = 1 - \mathbb{P}[\bar{C}]$

$= \mathbb{P}[D] = \mathbb{P}[D \mid C] \cdot 1$

$\therefore$ D is independent of C

c) Counterexample

$\mathbb{P}[A]$ = fair coin lands on heads

$\mathbb{P}[B]$ = fair coin tails

$\Downarrow$

$\mathbb{P}[A \cap B] = 0$

$\mathbb{P}[A \mid B] = 0$

However, since it is a fair coin,
$\mathbb{P}[A] = 0.5$.

$\mathbb{P}[B \mid A] = 0$, but by the same logic,
$\mathbb{P}[B] = 0.5$.

Since $\mathbb{P}[A \mid B] \neq \mathbb{P}[A]$ and $\mathbb{P}[B \mid A] \neq \mathbb{P}[B]$
A and B are not independent just b/c they are disjoint.

2) Independent Complements

Ω - Sample space

$A, B \in \Omega$ - two independent events

a) $\bar{A}$ and $\bar{B}$ must be independent

$\bar{A} = 1 - P[A]$

$\bar{B} = 1 - P[B]$

$\mathbb{P}(A \cap B) = \mathbb{P}(A)P(B)$

$$\mathbb{P}(\bar{A} \cap \bar{B}) = P[\bar{A}]P[\bar{B}]$$

$$= 1 - \mathbb{P}[A] \cdot 1 - \mathbb{P}[B]$$

$\Downarrow$

$$1 - \mathbb{P}[B] - \mathbb{P}[A] + \mathbb{P}[A] * \mathbb{P}[B]$$

LHS:

$$P[\bar{A} \cap \bar{B}]$$

$$1 - \mathbb{P}[\overline{\bar{A} \cap \bar{B}}]$$

$$1 - \mathbb{P}[A \cup B]$$ principle inclusion exclusion

$$1 - (\mathbb{P}[A] + \mathbb{P}[B]) + \mathbb{P}[A \cap B]$$ b/c independent

$$\mathbb{P}[A]\mathbb{P}[B]$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent

b) A and $\bar{B}$ must be independent

$$\mathbb{P}(A \cap \bar{B}) = \mathbb{P}[A] \cdot \mathbb{P}[\bar{B}]$$

$$= \mathbb{P}[A] \cdot (1 - \mathbb{P}[B])$$

$$= \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[B]$$

$$= \mathbb{P}[A] - \mathbb{P}[A \cap B]$$

↑ the part of A that does not contain B

$$= \mathbb{P}[A \cap \bar{B}]$$

$\therefore$ A and $\bar{B}$ are independent

c) A and $\bar{A}$ must be independent

$$\mathbb{P}(A \cap \bar{A}) = \mathbb{P}[A] \cdot \mathbb{P}[\bar{A}]$$

$$= \mathbb{P}[A] \cdot (1 - \mathbb{P}[A])$$

$$= \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[A]$$

$$= \mathbb{P}[A] - \mathbb{P}[A \cap A]$$

$\therefore$ A and $\bar{A}$ are dependent

d) Counterexample:

Let's say for flipping a fair coin, $\mathbb{P}(A] =$ heads and $\mathbb{P}[B] =$ tails.
$P(A \cap A) = \frac{1}{2}$, but $\mathbb{P}(A) \cdot P(A) = \frac{1}{4}$; $\frac{1}{4} \neq \frac{1}{2}$, so $A \neq B$ if they are independent.

3) Monty Hall's Revenge

a) $i, j, k \in \{1, 2, \dots n\}$ prize door

$\{\underbrace{\frac{1}{n} \times \frac{1}{n} \times \dots}_{n-1} 1\}$ winning if you

i - prize door $\rightarrow n$
j - door choose $\rightarrow n$
k - door Carol opens $\rightarrow n-1$ if $j=i$
$\rightarrow n-2$ if $j \neq i$

$\Omega = \{n \cdot n \cdot n-1\}$

$= \{n \cdot n \cdot n-2\}$

P-S : probability winning if switch
switch if $i \neq j$

$P[i \neq j] = \frac{n-1}{n}$ $\rightarrow \frac{1}{n}$ probability that $i=j$

P[switching to correct door] $= \frac{1}{n-2}$ $\rightarrow$ $n-1$ -1
door carol chose, door you just chose

prob winning if we switch $- \left(\frac{n-1}{n} \cdot \frac{1}{n-2}\right)$

prob not switching $- \frac{1}{n}$

$$\underbrace{\frac{n-1}{n} \cdot \frac{1}{n-2}}_{\text{prob winning if switch}} > \underbrace{\frac{1}{n}}_{\text{prob. not switching}}$$

b) 10 doors
reveal 8 doors
2 doors left

switch $- \left(\frac{n-1}{n} \cdot \frac{1}{2}\right)$

$n-(n-2)$
$n-n+2$
↳ $\frac{1}{2}$

not switch $- \frac{1}{n}$

$$\underbrace{\frac{n-1}{n} \cdot \frac{1}{2}}_{\text{switching}} \geq \underbrace{\frac{1}{n}}_{\text{not switching}}$$

↳ switching will increase odds

c) $k < n-1$ cars
$n-k$ goats $\rightarrow n > 2$ doors

$$\underbrace{\frac{k}{n}}_{\text{car on first pick}} \cdot \underbrace{\frac{k-1}{n-1-j}}_{\text{car on 2nd given car on 1st}} + \left(\underbrace{\frac{k}{n-1-j}}_{\text{car on second given goat on first}} \cdot \underbrace{\frac{n-k}{n}}_{\text{n-k goats from n doors}}\right) > \underbrace{\frac{k}{n}}_{\text{-probability not switching}}$$

$k-1$
↑ we already picked 1 car out of k cars

$n-1-j \rightarrow$ n doors minus j doors revealed + previous door

$\therefore$ k should be small $\rightarrow$ the fewer the amount of cars, the bigger the advantage of switching.

j should be large $\rightarrow$ the more goats are revealed, the better the chances are for switching.

4) Cliques in Random Graphs

a)

$2^{n(n-1)/2}$ ↑ total # possible edges

$\frac{n(n-1)}{2}$

b) total edges: $2^{k(k-1)/2}$

↳ $\frac{1}{2^{k(k-1)/2}}$ ← either on/off $\frac{1}{2}$ chance

c) condition

1) share 0 edges

2) share 1 vertex

for a graph w/ lower amount of edges, it has higher likelihood of being connected than a graph w/ higher amount of edges.

d) $\binom{n}{k}$

$$\frac{n!}{(n-k)!k!} \leq n^k$$

$$\frac{n \cdot n-1 \cdot n-2 \cdot \ldots n-k+1}{k!} \leq n \cdot n \cdot n \ldots \cdot n \text{ (k times)}$$

↳ the numerator on the left is a product of decreasing terms, which is already going to be smaller than n^k, which can be expressed as the product of n k-times.

e) A_i - specific k-clique exists

$\binom{n}{k}$ - how many indiv. k-cliques exist

$$\binom{n}{k} \cdot \frac{1}{2^{k(k-1)/2}} \leq \frac{1}{n}$$

└ using union bound: our n depends on how many k-cliques we can even form

$$\binom{n}{k} \cdot \frac{1}{2^{k(k-1)/2}} \leq n^k \cdot \frac{1}{2^{k(k-1)/2}} \leq \frac{1}{n}$$

↓

$$n^{4\log_2 n+1} \cdot \frac{1}{2^{4\log_2 n+1((4\log_2 n+1)-1))/2}} \leq \frac{1}{n}$$

$$n \cdot n \ldots \cdot n \text{ (}4\log_2 n+1\text{ times)} \cdot \frac{1}{2^{4\log_2 n+1((4\log_2 n+1)-1))/2}} \leq \frac{1}{n}$$

↑ since this is smaller than n, the product of LHS < $\frac{1}{n}$

5) Symmetric Marbles

- 4 red & 4 blue marbles

Rachel : R > B

Brooke : B > R

Tie : B=R

a) A_1 - first marble is red

A_2 - second marble is red

} No, they are not independent b/c we are drawing w/o replacement. Thus, A_2 is dependent on the outcome of A_1.

b) Probability rachel wins

$|\Omega| - \binom{8}{4}$ choose 4 balls out of 8 to be red

how many of these does rachel win in

3R + 1B $\Rightarrow \left(\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{4}{5}\right)4$

↑ 4 possible reorderings

or

4R + 0B $\Rightarrow \frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}$

$= \frac{96}{1680}\cdot 4 = \frac{2}{35}\cdot 4 = \frac{8}{35}$

$= \frac{24}{1680} = \frac{3}{210} = \frac{1}{70}$

$$= \frac{\frac{8}{35}+\frac{1}{70}}{\binom{8}{4}} = \boxed{\frac{\frac{17}{70}}{\binom{8}{4}}}$$

c) $P[RRRR]$: $\left(\frac{1}{2}\right)\cdot\left(\frac{3}{7}\right)\cdot\left(\frac{1}{3}\right)\cdot\left(\frac{1}{5}\right) = \frac{3}{210} = \boxed{\frac{1}{70}}$

2/6 red marbles

4/8 = 1/2 chance of grabbing 1st red marble

3/7 red marbles/ total marbles

1/5 red marbles

if rachel wins, all marbles are red = 100%

$$\mathbb{P}(\text{All red} \mid \text{Rachel Wins}) = \frac{\mathbb{P}(R\cap W)}{\mathbb{P}[W]} = \frac{\mathbb{P}[W\mid R] * \mathbb{P}[R]}{\mathbb{P}[W]}$$

$$= \frac{1 * \left[\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{2}{6}\cdot\frac{1}{5}\right]}{\frac{\frac{17}{70}}{\binom{8}{4}}} = \boxed{\frac{1}{70}\cdot\frac{\binom{8}{4}}{\frac{17}{70}}}$$

8 red marbles
4 blue marbles

d) probability 3rd marble is red?

Disc 9A : symmetry : probability
of picking first marble + it being red
is the same as third

$\rightarrow \frac{8}{12} = \boxed{\frac{2}{3}}$

e) $K=0$ - 0 $K=2$ - $2/4$ $K=4 \rightarrow 1$
$K=1$ - $1/4$ $K=3$ - $3/4$

↳ $\boxed{\frac{K}{4}}$

f) P(Rachel Wins | Third marble is red)

↳ complement of rachel winning

3B + 1R

$\frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9} = \frac{24}{990} = \frac{4}{165}$

↑11 balls,
12-1 red ball

$1 - \frac{4}{165} = \frac{161}{165}$

P[Rachel Wins | Third marble] = $\frac{161}{165}$

b) Socks

a) $|\Omega| = 2n!$

$\uparrow$ 2n! length

$\mathbb{P}[w \in \Omega] = \frac{1}{2n!}$

b) $\mathbb{P}[A_i] = (2n-1)! \cdot 2$

we have 2n! socks, consider -1 sock then you have either right/left (represented as $\cdot 2$)

$$\frac{2(2n-1)!}{2n!} = \frac{2(2n-1)!}{2n \cdot (2n-1)!}$$

$\uparrow$ divide over sample space

$\therefore P[A_i] = \frac{1}{n}$

c) $\mathbb{P}[A_1 \cap \ldots \cap A_K]$

$$\therefore \frac{(2n-k)! \cdot 2^k}{2n!}$$

$\uparrow$ for each of the k socks, either L or R, so 2 choices each $\rightarrow 2^k$

d) $|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1}$

using inclusion/exclusion:

all double

$$\sum_{k=1}^{n} A_i - \sum (A_i \cap A_j) + \sum (A_i \cap A_j \cap A_K)$$

$$\hookrightarrow \sum (A_i \cap A_j \cap A_h \cap A_b) \ldots + \sum (A_i \cap A_2 \ldots A_n)$$

$$= \frac{\left(\sum_{i=0}^{n-2} \frac{2^{1+2i}(2n-(1+2i))!}{2n!} - \frac{(2^{(2+2i)})(2n-(2+2i))}{2n!} \right) + \frac{2^n(2n-n)!}{2n}}{2n!}$$

e) complement of d

$$1 - \frac{\left(\sum_{i=0}^{n-2} \frac{2^{1+2i}(2n-(1+2i))!}{2n!} - \frac{(2^{(2+2i)})(2n-(2+2i))}{2n!} \right) + \frac{2^n(2n-k)!}{2n}}{2n!}$$