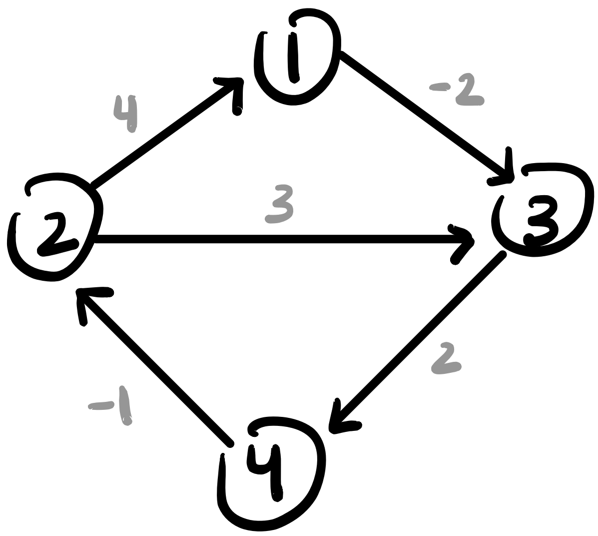
Floyd-Warshall - shortest path btwn. all pairs of vertices, negative edges allowed

time complexity = $O(n^3)$



make an array of min. distances

to vertex

from v.	1	2	3	4
1	0	∞	-2	∞
2	4	0	3	∞
3	∞	∞	0	2
4	∞	-1	∞	0

dist

[3,4]

Fill out the chart with the weights of the graph edges

initialize each nodes path to itself = 0

$dist[i][j] > dist[i][k] + dist[k][j]$

$k = 1\ 2\ 3\ 4$

$i = 1\ 2\ 3\ 4$ to vertex

$j = 1\ 2\ 3\ 4$ from vertex

$K_1 =$

	1	2	3	4
1	0	∞	-2	∞
2	4	0	3	∞
3	∞	∞	0	2
4	∞	-1	∞	0

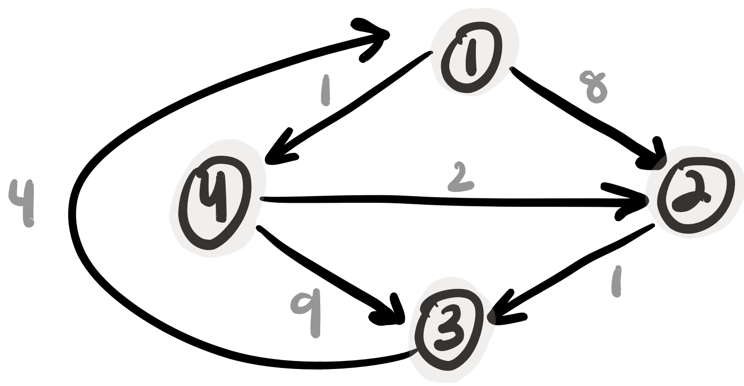
✓

$d[2][1] > dist[2][1] + dist[1][1]$

$$D^0 = W = \begin{matrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{matrix}$$

these don't change!

examples:



write the initial distance matrix

diag = 0, no direct edge = ∞, direct = w

$$D_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 8 & \infty & 1 \\ 2 & \infty & 0 & 1 & \infty \\ 3 & 4 & \infty & 0 & \infty \\ 4 & \infty & 2 & 9 & 0 \end{array}$$

since we have 4 vertices, we will have a total of 4 matrices

↓

$$D_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 8 & \infty & 1 \\ 2 & \infty & 0 & 1 & \infty \\ 3 & 4 & \boxed{\infty} & 0 & \boxed{\infty} \\ 4 & \infty & 2 & 9 & 0 \end{array}$$

transfer the same info, and if i = ∞ the copy the same data from the previous matrix (D_0) and if $j = \infty$ do the same, copy the same data and then for the remaining values indicated by $\square$ you need to perform the algo to replace those #s as k = the number matrix you are on D_1

$$D_1 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & \infty & 1 \\ \infty & 0 & 1 & \infty \\ 4 & \boxed{12} & 0 & \boxed{5} \\ \infty & 2 & 9 & 0 \end{bmatrix} \end{array}$$

$d(2,3) > d(2,1) + d(1,3)$

$\infty > 8 + 4$

$\infty > 12$ TRUE so update $(2,3) = 12$.

but another way you can do it, is by looking at the values above that space for i and j then add those together.

$$D_1 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & \infty & 1 \\ \infty & 0 & 1 & \infty \\ 4 & \boxed{12} & 0 & \boxed{5} \\ \infty & 2 & 9 & 0 \end{bmatrix} \end{array}$$

$d(4,3) > d(4,1) + d(1,3)$

$\infty > 1 + 4$

$\infty > 5$ TRUE so update $(4,3) = 5$

→ and now done with D_1

$$D_2 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & \boxed{\infty} & 1 \\ \infty & 0 & 1 & \infty \\ 4 & 12 & \boxed{0} & 5 \\ \infty & 2 & \boxed{9} & 0 \end{bmatrix} \end{array} \rightarrow \begin{bmatrix} 0 & 8 & 9 & 1 \\ \infty & 0 & 1 & \infty \\ 4 & 12 & 0 & 5 \\ \infty & 2 & 3 & 0 \end{bmatrix} \checkmark$$

$d(3,1) > d(3,2) + d(2,1)$

$0 > 1 + 8 \quad = 9 \quad \checkmark$

$d(3,3) > d(3,2) + d(2,3)$

$0 > 1 + 12 \quad = 13 \quad \times$ but always 0! b/c diag.

$d(3,4) > d(3,2) + d(2,4)$

$9 > 1 + 2 \quad = 3 \quad \checkmark$

$$D_3 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & 9 & 1 \\ \infty & 0 & 1 & \infty \\ 4 & 12 & 0 & 5 \\ \infty & 2 & 3 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & 9 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 12 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix} \end{array} \checkmark$$

$d(2,1) > d(2,3) + d(3,1)$

$8 > 12 + 9 \quad \times$ Not true, don't update

$d(4,1) > d(4,3) + d(3,1)$

$1 > 5 + 9 \quad \times$ Not true, don't update

$d(1,2) > d(1,3) + d(3,2)$

$\infty > 4 + 1 \quad \checkmark \quad = 5$

$d(4,2) > d(4,3) + d(3,2)$

$\infty > 5 + 1$ ✓ $= 6$

$d(1,4) > d(1,3) + d(3,4)$

$\infty > 4 + 3$ ✓ $= 7$

$d(2,4) > d(2,3) + d(3,4)$

$2 > 12 + 3$ ✗ not true, don't update

$$D_4 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 8 & 9 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 12 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 3 & 4 & 1 \\ 5 & 0 & 1 & 6 \\ 4 & 7 & 0 & 5 \\ 7 & 2 & 3 & 0 \end{bmatrix} \end{array}$$ ✓

$d(2,1) > d(2,4) + d(4,1)$

$8 > 2 + 1$ ✓ $= 3$

$d(3,1) > d(3,4) + d(4,1)$

$9 > 3 + 1$ ✓ $= 4$

$d(1,2) > d(1,4) + d(4,2)$

$5 > 7 + 6$ ✗

$d(3,2) > d(3,4) + d(4,2)$

$1 > 3 + 6$ ✗

$d(1,3) > d(1,4) + d(4,3)$

$4 > 7 + 5$ ✗

$d(2,3) > d(2,4) + d(4,3)$

$12 > 2 + 5$ ✓ $= 7$

final matrix =

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 4 & 1 \\ 2 & 5 & 0 & 1 & 6 \\ 3 & 4 & 7 & 0 & 5 \\ 4 & 7 & 2 & 3 & 0 \end{array}$$

✓

all in all, you have a matrix consiting of weights btwn two vertices (i and j) and the rest are ∞.

for D_1 you will keep the data the same for the entire row 1 and col 1 (with diag all 0's) and if any ∞ is part of those rows/cols. you keep that data the same too. otherwise run $d(i,j) > d(i,k) + d(k,j)$ with k being the # matrix you are in $D_\#$

$$D_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \infty & 13 & \infty \\ 2 & 9 & 0 & 14 & 15 \\ 3 & 10 & 12 & 0 & 16 \\ 4 & 11 & \infty & 17 & 0 \end{array}$$

$$D_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \infty & 13 & \infty \\ 2 & 9 & 0 & 14 & 15 \\ 3 & 10 & 12 & 0 & 16 \\ 4 & 11 & \infty & 17 & 0 \end{array}$$

$$D_3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \infty & 13 & \infty \\ 2 & 9 & 0 & 14 & 15 \\ 3 & 10 & 12 & 0 & 16 \\ 4 & 11 & \infty & 17 & 0 \end{array}$$

$$D_4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & \infty & 13 & \infty \\ 2 & 9 & 0 & 14 & 15 \\ 3 & 10 & 12 & 0 & 16 \\ 4 & 11 & \infty & 17 & 0 \end{array}$$

current row/col data stays the same

diag of all 0's stay the same

where the ∞'s were in the row/cols

Floyd-Warshall shortest path btwn. all pairs of vertices $O(n^3)$

$$\begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 6 \\ \infty & 8 & 0 & 5 \\ 2 & 6 & 5 & 0 \end{bmatrix}$$

The diaganol never changes, it is ALWAYS 0.

Then D_x has row x and col x stay the same from the previous D.

Then if ∞ is in the D_x row or col. then the #'s in that other row and col. also stay the same as the previous D.

EX:

$$D_1 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 6 \\ \infty & 8 & 0 & 5 \\ 2 & 6 & 5 & 0 \end{bmatrix} \end{array}$$

then the other #'s need to be updated based on:

$d(i,j) > d(i,k) + d(k,j)$

$d(4,2) > d(4,1) + d(1,2)$

$6 > 2 + 3$ True so update!

$d(2,4) > d(2,1) + d(1,4)$

$6 > 3 + 2$ True so update!

$$D_1 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 5 \\ \infty & 8 & 0 & 5 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array}$$

Then Repeat!

$$D_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 5 \\ \infty & 8 & 0 & 5 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 11 & 2 \\ 3 & 0 & 8 & 5 \\ 11 & 8 & 0 & 11 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array}$$

$$D_3 = \begin{array}{c} \\ \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 5 \\ 11 & 8 & 0 & 11 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 11 & 2 \\ 3 & 0 & 8 & 5 \\ 11 & 8 & 0 & 11 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array}$$

stays the same

$$D_4 = \begin{array}{c} \\ \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & \infty & 2 \\ 3 & 0 & 8 & 5 \\ 11 & 8 & 0 & 11 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 3 & 7 & 2 \\ 3 & 0 & 8 & 5 \\ 7 & 8 & 0 & 11 \\ 2 & 5 & 5 & 0 \end{bmatrix} \end{array}$$