STA257

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the "negative binomial" distributions

Consider a Bernoulli(p) process. Count the number of trials until the r^{th} "success".

This is a random variable. Call it X.

What is the p.m.f. of X?

$$p(k) = P(X = k) = \begin{cases} \binom{k-1}{r-1} p^k (1-p)^{k-r} & x \in \{r, r+1, r+2, \ldots\} \\ 0 & \text{otherwise} \end{cases}$$

Is this a valid pmf? Yes (a little obscure to figure out)

We say X has a negative binomial distribution with paramters p and r, or $X \sim \mathrm{NegBin}(p, r)$.

the "hypergeometric distributions"

Many examples from Ch. 1 (quality control, Lotto, some of the balls in urns etc.)

Consider a Bernoulli process, stopped after n trials in which there were r "successes". Let X be the number of successes in the "first" m out of n trials.

What is the p.m.f. of X?

$$p(k) = P(X = k) = \begin{cases} \dfrac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}} & : k \in \{0, \ldots, r\} \\ 0 & : \text{ otherwise} \end{cases}$$

Is this a valid pmf? Yes. (Problem 34 from Chapter 1)

We say X has a hypergeometric distribution with paramters r, n, and m.

digression — constants as "random variables"

In calculus etc. you may have (unconsciously) considered things like $f(x) + a$ for real constant a. This could have a few interpretations, such as:

1. The sum of the numbers $f(x)$ and a.
2. The value of the function $f + g$ evaluated at x, in which it happens that $g(x) = a$ for all x.

We do this in probability as well. We can allow a random variable X to be some constant a no matter what. Then X is discrete with :

$$p(x) = P(X = x) = \begin{cases} 1 & : \; x = a \\ 0 & : \; \text{otherwise,} \end{cases}$$

$$F(x) = P(X \leq x) = \begin{cases} 0 & : \; x < a \\ 1 & : \; x \geq a. \end{cases}$$

the Poisson distributions - I

Named after a French guy called Poisson.

Can be defined completely abstractly just by declaring X has a Poisson distribution with parameter λ, or X ~Poisson(λ), if X has p.m.f:

$$p(k) = P(X = k) = \begin{cases} \frac{\lambda^k e^{-\lambda}}{k!} & : k \in \{0, 1, 2, \dots\} \\ 0 & : \text{otherwise.} \end{cases}$$

Is this a valid p.m.f.? Yes.

But this does not come close to explaining the importance of the Poisson distributions.

Bernoulli process — with a time scale

Consider a Binomial(n, p) distribution. Let's introduce a time scale to the underlying Bernoulli(p) process.

Step 1. Take a fixed time interval $(0, t)$ and divide it into $n_1 = n$ subintervals and let $p_1 = p$. Let's say one Bernoulli(p_1) trial happens inside each subinterval, and we keep track of the number of "successes".

A few "general" observations:

- Only one success can happen inside each subinterval.
- Results in non-overlapping collections of subintervals are independent.
- Successes happen at a "rate" of about $n \cdot p$ (intuitively) over the whole time $(0, t)$, and the number of k successes has a Binomial(n_1, p_1) distribution.
- t didn't matter — if we double it to $2t$ the rate doubles also to $2np$.

double the number of intervals

Step 2. Divide the same interval into $n_2 = 2n$ subintervals.

We want a trial to happen inside each subinterval, *but we want the same overall "rate" of success.*

So now we allow a Bernoulli(p_2) trial with $p_2 = p/2$ to occur inside each subinterval.

The same "general" observations continue to hold, except now the number of successes has a Binomial$(2n, p/2)$ distribution.

Double the number of intervals again… (so now $(0, t)$ has $4n$ intervals with a Bernoulli($p/4$) trial in each.)

pass to the limit

Define λt ("rate") to be fixed and always equal to np. This implies $p = \frac{\lambda t}{n}$.

What happens to Binomial$\left(n, \frac{\lambda t}{n}\right)$ distributions as $n \to \infty$?

λ is the rate of occurrences per unit time.

Examples and more discussion next time.