Perturbative solution for $\Lambda(f_0)$ for neutral loci

In the case where $\gamma_A, \gamma_B, \gamma_{AB}$ are small compared to both 1 and ρ, the characteristic $\psi(\tau', x, y, z)$ follows from

$$\frac{\partial \psi}{\partial \tau'} = -\rho\psi - \psi^2 + \frac{\rho y}{1 + y\tau'} + \frac{\rho x}{1 + x\tau'}, \quad \psi(0, x, y, z) = z. \tag{1}$$

This equation is difficult to solve in the general case. To make progress, we consider a perturbation expansion around $x = 1 + \delta x$, $y = 1 + \delta y$, $z = 2 + \delta z$, defining

$$\psi(\tau', x, y, z) = \sum_{i,j,k=0}^{\infty} \delta_x^i \delta_y^j \delta_z^k \psi_{i+j+k}^{x^i y^j z^k}(\tau'). \tag{2}$$

Substituting the above series expansion into Eq. (9) and matching coefficients in front of δx, δy, δz, we obtain a system of ordinary differential equations that can be solved numerically,

$$\begin{cases} \partial_{\tau'}\psi_0 = -\rho\psi_0 - \psi_0^2 + \frac{2\rho}{1+\tau'}, \quad \psi_0(0) = 2; \\ \partial_{\tau'}\psi_1^x = -\rho\psi_1^x - 2\psi_0\psi_1^x + \frac{\rho}{(1+\tau')^2}, \quad \psi_1^x(0) = 0; \\ \partial_{\tau'}\psi_1^y = -\rho\psi_1^y - 2\psi_0\psi_1^y + \frac{\rho}{(1+\tau')^2}, \quad \psi_1^y(0) = 0; \\ \partial_{\tau'}\psi_1^z = -\rho\psi_1^z - 2\psi_0\psi_1^z, \quad \psi_1^z(0) = 1; \\ \partial_{\tau'}\psi_2^{xy} = -\rho\psi_2^{xy} - 2\psi_0\psi_2^{xy} - 4\psi_1^x\psi_1^y, \quad \psi_2^{xy}(0) = 0; \\ \partial_{\tau'}\psi_2^{xz} = -\rho\psi_2^{xz} - 2\psi_0\psi_2^{xz} - 4\psi_1^x\psi_1^z, \quad \psi_2^{xz}(0) = 0; \\ \partial_{\tau'}\psi_2^{yz} = -\rho\psi_2^{yz} - 2\psi_0\psi_2^{yz} - 4\psi_1^y\psi_1^z, \quad \psi_2^{yz}(0) = 0; \\ \partial_{\tau'}\psi_3^{xyz} = -\rho\psi_3^{xyz} - 2\psi_0\psi_3^{xyz} - 6\psi_1^x\psi_2^{yz} - 6\psi_1^y\psi_2^{xz} - 6\psi_1^z\psi_2^{xy}, \quad \psi_3^{xyz}(0) = 0. \end{cases} \tag{3}$$

We can see that solving the system above is enough to compute Λ. To find the numerator, we need to evaluate

$$\left\langle f_{Ab} f_{aB} f_{AB} \cdot e^{-\frac{f_A + f_B}{f_0}} \right\rangle \approx -\theta^2 f_0^4 \int_0^{\tau} d\tau' \, \partial_x \partial_y \partial_z \psi \left[\Phi_x + \Phi_y - \rho \Phi_x \Phi_y\right] \Bigg|_{\substack{x=1 \\ y=1 \\ z=2 \\ \tau=\infty}}, \tag{4}$$

where

$$\Phi_x(\tau') = \sum_{i=0}^{\infty} \delta_x^i \Phi_i^x(\tau'), \tag{5a}$$

$$\Phi_y(\tau') = \sum_{j=0}^{\infty} \delta_y^j \Phi_j^y(\tau'). \tag{5b}$$

The denominator will be dominated by

$$\left\langle f_{Ab}^2 f_{aB}^2 \cdot e^{-\frac{f_A+f_B}{f_0}} \right\rangle \approx \theta^2 f_0^4 \partial_x^2 \partial_y^2 H_A H_B \Big|_{\substack{x=1\\y=1\\\tau=\infty}} = \theta^2 f_0^4 \frac{1}{x^2y^2}\Big|_{\substack{x=1\\y=1}} = \theta^2 f_0^4. \quad (6)$$

Substitution the series expansions in Eqs. (2, 5) into Eq. (11) and dividing it by Eq. (6), we find Λ up to the lowest order in δx, δy, and δz as

$$\begin{aligned}\Lambda \approx\ & \left[\rho \int_0^\tau d\tau'\, \psi_1^z \Phi_1^x \Phi_1^y - \frac{1}{2}\int_0^\tau d\tau'\, \psi_2^{xz} \Phi_1^y \left(1-\rho\Phi_0^x\right)\right. \quad (7)\\ & \left.- \frac{1}{2}\int_0^\tau d\tau'\, \psi_2^{yz} \Phi_1^x \left(1-\rho\Phi_0^y\right) + \frac{1}{6}\int_0^\tau d\tau'\, \psi_3^{xyz} (\Phi_0^x + \Phi_0^y - \rho\Phi_0^x\Phi_0^y)\right]\Bigg|_{\tau=\infty}\\ =\ & \int_0^\infty d\tau' \left[\rho\psi_1^z - \frac{1}{2}\left[1+\rho(1+\tau')\right]\left[\psi_2^{xz}+\psi_2^{yz}\right] + \frac{1}{6}(1+\tau')\left[2-\rho(1+\tau')\right]\psi_3^{xyz}\right],\end{aligned}$$

where we have used

$$\Phi_0^x(\tau')\Big|_{\tau=\infty} = \Phi_0^y(\tau')\Big|_{\tau=\infty} = -(1+\tau'), \quad (8a)$$

$$\Phi_1^x(\tau')\Big|_{\tau=\infty} = \Phi_1^y(\tau')\Big|_{\tau=\infty} = 1. \quad (8b)$$

Perturbative solution for $\Lambda(f_0)$ in the general case

In the general case, the characteristic $\psi(\tau', x, y, z)$ follows from

$$\frac{\partial\psi}{\partial\tau'} = -(\gamma_{AB}+\rho)\psi - \psi^2 + \rho\frac{\gamma_A x e^{-\gamma_A\tau'}}{\gamma_A + x(1-e^{-\gamma_A\tau'})} + \rho\frac{\gamma_B y e^{-\gamma_B\tau'}}{\gamma_B + y(1-e^{-\gamma_B\tau'})}, \quad \psi(0,x,y,z) = z. \quad (9)$$

Expanding around $x = 1+\delta x$, $y = 1+\delta y$, $z = 2+\delta z$, we obtain a system of ordinary differential equations, which reduces to that in Eq. (3) if we set $\gamma_A = 0$, $\gamma_B = 0$, $\gamma_{AB} = 0$,

$$\begin{cases} \partial_{\tau'}\psi_0 = -(\rho+\gamma_{AB})\psi_0 - \psi_0^2 + \frac{\rho\gamma_A e^{-\gamma_A\tau'}}{1+\gamma_A-e^{-\gamma_A\tau'}} + \frac{\rho\gamma_B e^{-\gamma_B\tau'}}{1+\gamma_B-e^{-\gamma_B\tau'}}, \quad \psi_0(0) = 2; \\ \partial_{\tau'}\psi_1^x = -(\rho+\gamma_{AB})\psi_1^x - 2\psi_0\psi_1^x + \frac{\rho\gamma_A^2 e^{-\gamma_A\tau'}}{(1+\gamma_A-e^{-\gamma_A\tau'})^2}, \quad \psi_1^x(0) = 0; \\ \partial_{\tau'}\psi_1^y = -(\rho+\gamma_{AB})\psi_1^y - 2\psi_0\psi_1^y + \frac{\rho\gamma_B^2 e^{-\gamma_B\tau'}}{(1+\gamma_B-e^{-\gamma_B\tau'})^2}, \quad \psi_1^y(0) = 0; \\ \partial_{\tau'}\psi_1^z = -(\rho+\gamma_{AB})\psi_1^z - 2\psi_0\psi_1^z, \quad \psi_1^z(0) = 1; \\ \partial_{\tau'}\psi_2^{xy} = -(\rho+\gamma_{AB})\psi_2^{xy} - 2\psi_0\psi_2^{xy} - 4\psi_1^x\psi_1^y, \quad \psi_2^{xy}(0) = 0; \\ \partial_{\tau'}\psi_2^{xz} = -(\rho+\gamma_{AB})\psi_2^{xz} - 2\psi_0\psi_2^{xz} - 4\psi_1^x\psi_1^z, \quad \psi_2^{xz}(0) = 0; \\ \partial_{\tau'}\psi_2^{yz} = -(\rho+\gamma_{AB})\psi_2^{yz} - 2\psi_0\psi_2^{yz} - 4\psi_1^y\psi_1^z, \quad \psi_2^{yz}(0) = 0; \\ \partial_{\tau'}\psi_3^{xyz} = -(\rho+\gamma_{AB})\psi_3^{xyz} - 2\psi_0\psi_3^{xyz} - 6\psi_1^x\psi_2^{yz} - 6\psi_1^y\psi_2^{xz} - 6\psi_1^z\psi_2^{xy}, \quad \psi_3^{xyz}(0) = 0. \end{cases} \tag{10}$$

Similarly to the neutral case above, we obtain for Λ

$$\begin{aligned} \Lambda \approx \Bigg[\rho\int_0^\tau d\tau'\,\psi_1^z\Phi_1^x\Phi_1^y - \frac{1}{2}\int_0^\tau d\tau'\,\psi_2^{xz}\Phi_1^y\,(1-\rho\Phi_0^x) \\ - \frac{1}{2}\int_0^\tau d\tau'\,\psi_2^{yz}\Phi_1^x\,(1-\rho\Phi_0^y) + \frac{1}{6}\int_0^\tau d\tau'\,\psi_3^{xyz}(\Phi_0^x+\Phi_0^y-\rho\Phi_0^x\Phi_0^y)\Bigg]\Bigg|_{\tau=\infty}, \end{aligned} \tag{11}$$

where

$$\Phi_0^x(\tau')\Bigg|_{\tau=\infty} = -\frac{1+\gamma_A-e^{-\gamma_A\tau'}}{\gamma_A(1+\gamma_A)}, \quad \Phi_0^y(\tau')\Bigg|_{\tau=\infty} = -\frac{1+\gamma_B-e^{-\gamma_A\tau'}}{\gamma_B(1+\gamma_B)}, \tag{12a}$$

$$\Phi_1^x(\tau')\Bigg|_{\tau=\infty} = \frac{e^{-\gamma_A\tau'}}{(1+\gamma_A)^2}, \quad \Phi_1^y(\tau')\Bigg|_{\tau=\infty} = \frac{e^{-\gamma_B\tau'}}{(1+\gamma_B)^2}. \tag{12b}$$