MATH 255 Course Notes

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Winter 2023

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1 Review of Analysis over $\mathbb{R}$

I include all proofs and relevant results from Analysis 1 here for completion. Most of this section is just "bookkeeping."

Point-Set Topology over $\mathbb{R}$

Definition 1.1. The Euclidean distance function on $\mathbb{R}$ (the standard distance between any two points) is $d(x, y) = |y - x|$.

Definition 1.2. An <u>*open neighborhood*</u> or <u>*open ball*</u> of radius ϵ about a point $x \in \mathbb{R}$ is $B_\epsilon(x) = \{y \in \mathbb{R} : d(x, y) = |y - x| < \epsilon\}$.

Note that $B_\epsilon(x) = (x - \epsilon, x + \epsilon)$.

Definition 1.3 (Open Set in $\mathbb{R}$)**.** A set $U \subseteq \mathbb{R}$ is an <u>*open*</u> subset of $\mathbb{R}$ if $\forall x \in U, \exists B_\epsilon(x) \subseteq U$.

Definition 1.4. A set $F \subseteq \mathbb{R}$ is <u>*closed*</u> in $\mathbb{R}$ if its complement F^c is open.

Proposition 1.1.

(a) The union of an abritrary number of open sets is open.

(b) The intersection of a finite number of open sets is open.

(c) The union of a finite number of closed sets is closed.

(d) The intersection if an abritrary number of closed sets is closed.

Proof.

(a) Let I be some index set and for each $i \in I$ let $S_i \subseteq \mathbb{R}$ be open. The let $S = \cup_{i \in I} S_i$ be the intersection.

Let $x \in S$. Then for some i, $x \in S_i$. Since S_i is open, $\exists B_\epsilon(x) \subseteq S_i \subseteq S$.

(b) Let $S = S_1 \cup ... \cup S_n$, where each $S_i \subseteq \mathbb{R}$ is open. Let $x \in S$. Then for each S_i, there exists a $B_{\epsilon_i}(x) subseteq S_i$. Letting $\epsilon = \min\{\epsilon_1, ..., \epsilon_n\}$, we have that $B_\epsilon(x) \subseteq S_i$ for each S_i, so $B_\epsilon(x) \subseteq S$.

(c) Let $S = S_1 \cup ... \cup S_n$, where each $S_i \subseteq \mathbb{R}$ is closed. Then $S^c = S_1^c \cap ... \cap S_n^c$ by De Morgan's laws. Each S_i^c is open, so by (b) S^c is open and thus S is closed.

(d) Let I be some index set and for each $i \in I$ let $S_i \subseteq \mathbb{R}$ be closed. Then let $S = \cap_{i \in I} S_i$ be the intersection. By De Morgan's laws, $S^c = \cup_{i \in I} S_i^c$. Each S_i^c is open, so by (a) S^c is open and thus S is closed.

$\square$

Definition 1.5 (Accumulation/Cluster/Limit Point)**.** For a set $S \subseteq \mathbb{R}$, a point $x \in \mathbb{R}$ is a <u>*cluster point*</u> of S if $\forall B_\epsilon(x), (B_\epsilon(x) \setminus \{x\}) \cap S \neq \emptyset$. That is, every neighborhood of x contains a point of S that is not x.

Proposition 1.2. *For a set $S \subseteq \mathbb{R}$ and a point $x \in \mathbb{R}$, the following are equivalent:*

(a) x is a cluster point of S.

(b) Every $B_\epsilon(x)$ contains infinitely many points of S.

(c) There exists a sequence in $S \setminus \{x\}$ which converges to x.

Proof.

- (a $\implies$ b). Let $B_{\epsilon_1}(x)$. Since x is a cluster point of S, there is a point $x_1 \in (B_\epsilon(x) \setminus \{x\}) \cap S$. Let $\epsilon_2 = d(x_1, x)$, and repeat this process to create $x_2, x_3, ...$. Since $\epsilon_1 > \epsilon_2 > ...$, all $x_n \in B_{\epsilon_1}(x)$.
- (a $\implies$ c). For each $n \in \mathbb{N}^+$, let x_n be a point in $(B_{\frac{1}{n}}(x) \setminus \{x\}) \cap S$, which exists as x is a cluster point of S. Then $d(x_n, x) = \frac{1}{n} \to 0$, so we have created the sequence requored by (c).
- (c $\implies$ b). Let $(x_n) \subset S \setminus \{x\}$ converge to x, which exists by (c). Let $B_\epsilon(x)$. Then for all n greater than some N, $d(x, x_n), \epsilon$, so there are infinitely many points of S in $B_\epsilon(x)$.

□

Proposition 1.3. *A set $F \subseteq \mathbb{R}$ is closed if and only if it contains all of its cluster points.*

Proof. First, assume that F is closed and let $x \in \mathbb{R}$ be a cluster point of F. Assume towards a contradiction that $x \notin F$. Then $x \in F^c$, an open set, so $\exists B_\epsilon(x) \subseteq F^c$. Let $(x_n) \subseteq F \setminus \{x\}$ be a sequence converging to x, which exists by Prop. 1.2. But since each $x_n \in F$, $d(x_n, x) > \epsilon$, as $B_\epsilon(x) \subseteq F^c$, a contradiction.

Now assume that F contains all of its cluster points. Let $x \in F^c$. Consider $l = \inf\{d(x, y) : y \in F\}$. If $l = 0$, then there exists a sequence in F which converges to x, making x a cluster point of F, which is impossible by hypothesis. Then letting $\epsilon = \frac{l}{2}$, clearly $B_\epsilon(x) \cap F = \emptyset$, so $B_\epsilon(x) \subseteq F^c$. Thus, F^c is an open set and F is closed. □

Definition 1.6 (Interior)**.** For a set $S \subseteq \mathbb{R}$, a point $x \in S$ is an <u>*interior point*</u> if there exists some $B_\epsilon(x) \subseteq S$. The set of all interior points S^o is called the <u>interior</u> of S.

Definition 1.7. For a set $S \subseteq \mathbb{R}$, a point $x \in S$ is an <u>*isolated point*</u> of S if there exists some $B_\epsilon(x)$ such that $B_\epsilon(x) \cap S = \{x\}$.

Definition 1.8 (Boundary)**.** For a set $S \subseteq \mathbb{R}$, a point $x \in \mathbb{R}$ is a <u>*boundary point*</u> of S if $\forall B_\epsilon(x)$, both $B_\epsilon(x) \cap S \neq \emptyset$ and $B_\epsilon(x) \cap S^c \neq \emptyset$. The set of all boundary points ∂S is called the <u>boundary</u> of S.

Definition 1.9. The closure of a set $S \subseteq \mathbb{R}$ is $\overline{S} = S \cup \partial S$.

Proposition 1.4. *Let $S \subseteq \mathbb{R}$. Then*

(a) S is open if and only if $S = S^o$.

(b) S is closed if and only if $S = \overline{S}$.

Proof. Let $S \subseteq \mathbb{R}$. Then

(a) First, assume that S is open. Obviously $S^o \subseteq S$, so we prove the reverse inclusion. Let $x \in S$. Then $\exists B_\epsilon(x) \subseteq S$, so $x \in S^o$.

Now, assume that $S = S^o$. Let $x \in S$. Since $x \in S^o, \exists B_\epsilon(x) \subseteq S$, so S is open.

(b) First, assume that S is closed. Clearly, $S \subseteq \overline{S}$, so we prove the reverse inclusion. It suffices to show that $\partial S \subseteq S$. Let x $\in \partial S$. Then $\forall B_\epsilon(x), S \cap B_\epsilon(x) \neq \emptyset$. But then $x \notin S^c$, an open set, as then there would be an open ball entirely inside S^c.

Now, assume that $S = \overline{S}$. Let $x \in S^c$. Since, $x \notin \overline{S}$, $\exists B_\epsilon(x)$ such that $B_\epsilon(x) \cap S = emptyset \implies B_\epsilon(x) \subseteq S^c$, so S^c is open.

□

Proposition 1.5. *For a set $S \subseteq \mathbb{R}$, every point $x \in S$ is either an interior point or a boundary point.*

Proof. Let $x \in S$ not be a boundary point. We will show that $x \in S^o$. Since $x \notin \partial S$, there exists a $B_\epsilon(x)$ such that $B_\epsilon(x) \cap S^c = \emptyset$. But then $B_\epsilon(x) \subseteq S$, so $x \in S^o$. □

Proposition 1.6. *For a set $S \subseteq \mathbb{R}$, if $x \in S$ is not an isolated point then it is a cluster point.*

Proof. Let $x \in S$ not be a cluster point. Then $\exists B_\epsilon(x)$ such that $B_\epsilon(x) \cap S = \{x\}$, as if every ball contained some other point in S, x would be a cluster point. Thus x is isoated by definition. □

Definition 1.10. A set $S \subseteq \mathbb{R}$ is <u>bounded</u> if $S \subseteq B_M(0)$ for some $M > 0$.

Definition 1.11 (Dense Set)**.** A set $S \subseteq \mathbb{R}$ is <u>dense</u> in $\mathbb{R}$ if $\overline{S} = \mathbb{R}$.

Definition 1.12 (Perfect Set)**.** A set $S \subseteq \mathbb{R}$ if it is closed and every point of S is a cluster point.

The Cantor Set

Definition 1.13 (Cantor Set)**.** Let the Cantor Set C be the intersection $\cap_{n\in\mathbb{N}} C_n$. We define C_n inductively. Let $C_0 = [0,1]$. We then remove the middle third to define $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Continue removing the middle third of each subinterval to define each C_n.

The Cantor set is closed as an intersection of finite unions of closed sets.

Note that $C \subset [0,1]$. We can also see that C is nonempty. Clearly it contains 0 and 1, but additionally the endpoints of the removed middle thirds are always present in the set. What's interesting, however, is that the total length of the removed intervals is 1 (proof by sum of a geometric series).

The Cantor set is the set of numbers in $[0,1]$ with ternary expansions that contain no ones. This can lead to an argument that the Cantor set is uncountable.

Compactness in $\mathbb{R}$

Definition 1.14 (Open Cover)**.** Let I be an index set and for all $i \in I$ let $G_i \subseteq \mathbb{R}$ be an open set. Then $\mathcal{O} = \{G_i\}_{i \in I}$ is an <u>*open cover*</u> of a set $S \subseteq \mathbb{R}$ if $S \subseteq \cup_{i \in I} G_i$.

A subcover of some open cover $\mathcal{O}$ is a subset of $\mathcal{O}$ that is also an open cover.

Example 1.1. Let $S \subseteq \mathbb{R}$. The following are open covers of S:

(a) $\{\mathbb{R}\}$

(b) $\{(-n, n)\}_{n \in \mathbb{N}^+}$

(c) $\{(x-1, x+1)\}_{x \in \mathbb{R}}$

Definition 1.15 (Compact Set)**.** A set $K \subseteq \mathbb{R}$ is <u>*compact*</u> if every open cover of K has a finite subcover.

Theorem 1.7. *For a set $K \subseteq \mathbb{R}$, the following are equivalent:*

(a) K is compact.

(b) K is closed and bounded.

(c) Every sequence in K has a convergent subsequence with limit in K.