1. $y=x^2y^3+x^3$
2. $e^{xy}=e^{4x}-e^{5y}$
3. $\cos^2 x + \cos^2 y = \cos(2x+2y)$
4. $x=3+\sqrt{x^2+y^2}$
5. $\dfrac{x-y^3}{y+x^2}=x+2$
6. $\dfrac{y}{x^3}+\dfrac{x}{y^3}=x^2y^4$

1. $y = x^2y^3 + x^3$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2y^3) + \frac{d}{dx}x^3$$

$$\frac{d}{dx}(y) = 1\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(x^2)\ \frac{d}{dx}(y^3)$$

$$= vu' + uv'$$

$$= x^2\left(3y^2\cdot\frac{dy}{dx}\right) + y^3(2x)$$

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{dy}{dx} = 3x^2y^2\cdot\frac{dy}{dx} + 2xy^3 + 3x^2$$

$$\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} = 2xy^3 + 3x^2$$

$$\frac{dy}{dx} = \frac{2xy^3+3x^2}{1-3x^2y^2}$$

2. $e^{xy} = e^{4x} - e^{5y}$

$$\frac{d}{dx}e^{xy} = e^u$$

$$u = xy$$

$$\frac{dy}{dx} = vu' + uv'$$

$$= x\left(1\cdot\frac{dy}{dx}\right) + y(1)$$

$$= x\left(1\cdot\frac{dy}{dx}\right) + y$$

$$\frac{dy}{dx} = e^{xy}\left(x\frac{dy}{dx} + y\right)$$

$$= \frac{d}{dx}e^{4x}$$

$$u = 4x$$

$$\frac{du}{dx} = 4$$

$$\frac{dy}{dx} = e^{4x}\cdot 4$$

$$= 4e^{4x}$$

$$= \frac{d}{dx}e^{5y}$$

$$u = 5y$$

$$\frac{du}{dx} = 5\cdot\frac{dy}{dx}$$

$$\frac{dy}{dx} = e^{5y}\left(5\cdot\frac{dy}{dx}\right)$$

2. $e^{xy} = e^{4x} - e^{5y}$

$\frac{d}{dx}\, e^{xy} = e^{u}$ $\quad = \frac{d}{dx}\, e^{4x}$ $\quad = \frac{d}{dx}\, e^{5y}$

$u = xy$

$\frac{dy}{dx} = vu' + uv'$

$= x(1 \cdot \frac{dy}{dx}) + y(1)$

$= x(1 \cdot \frac{dy}{dx}) + y$

$\frac{dy}{dx} = e^{xy}\left(x \frac{dy}{dx} + y\right)$

$u = 4x$

$\frac{du}{dx} = 4$

$\frac{dy}{dx} = e^{4x} \cdot 4$

$= 4e^{4x}$

$u = 5y$

$\frac{du}{dx} = 5 \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = e^{5y}\left(5 \cdot \frac{dy}{dx}\right)$

$e^{xy}\left(x \frac{dy}{dx} + y\right) = 4e^{4x} - e^{5y}\left(5 \cdot \frac{dy}{dx}\right)$

$ex^2y \frac{dy}{dx} + y(e^{xy}) = 4e^{4x} + 5e^{5y} \cdot \frac{dy}{dx})$

$ex^2y \frac{dy}{dx} + 5e^{5y} \cdot \frac{dy}{dx} = 4e^{4x} - ye^{xy}$

$\frac{dy}{dx} = \frac{4e^{4x} - ye^{xy}}{ex^2y + 5e^{5y}}$

3. $\cos^2 x + \cos^2 y = \cos(2x+2y)$

$$\frac{d}{dx}\cos^2 x + \frac{d}{dx}\cos^2 y = \frac{d}{dx}\cos(2x+2y)$$

$\frac{d}{dx}(\cos x)^2$

$u = \cos x$

$\frac{dy}{du} = u^2$

$= 2u$

$\frac{du}{dx} = \cos x$

$= 1 \cdot -\sin x$

$= -\sin x$

$= 2\cos x \cdot -\sin x$

$= -2\sin\cos x$

$\frac{d}{dx}(\cos y)^2 =$

$u = \cos y$

$\frac{dy}{du} = u^2$

$= 2u$

$\frac{du}{dx} = \frac{d}{dx}(\cos y)$

$= \frac{dy}{dx} \cdot -\sin y$

$= (2\cos(y))(-\frac{dy}{dx}\sin(y))$

$\frac{d}{dx}\cos(2x+2y)$

$u = 2x+2y$

$u = 2 + 2 \cdot \frac{dy}{dx}$

$= 2 + 2 \cdot \frac{dy}{dx} \cdot -\sin(2x+2y)$

$= (2 + 2\frac{dy}{dx})(-\sin(2x+2y))$

$-2\sin x\cos x - 2\cos y\sin y\frac{dy}{dx} = (2 + 2\frac{dy}{dx})(-\sin(2x+2y))$

$-2\sin x\cos x + 2\cos(y)\sin(y)\frac{dy}{dx} = -2\sin(2x+2y) - 2\frac{dy}{dx}\sin(2x+2y)$

$-\sin(2y)\frac{dy}{dx} + 2\frac{dy}{dx}\sin(2x+2y) = -2\sin(2x+2y) + 2\sin x\cos x$

$$\frac{dy}{dx} = \frac{\sin(2x) - 2\sin(2x+2y)}{2\sin(2x+2y) - \sin(2y)}$$

4. $x = 3 + \sqrt{x^2 + y^2}$

$$\frac{d}{dx}(x) = 3 + \frac{d}{dx}(x^2 + y^2)^{1/2}$$

$$\frac{d}{dx} = (x^2 + y^2)^{1/2}$$

$$x y^{-1/2}$$

$$\frac{dy}{du} = u^{1/2}$$
$$= \frac{1}{2} u^{-1/2}$$

$$\frac{du}{dx} = 2x + 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 y^2)^{-1/2} \left(2x + 2y \cdot \frac{dy}{dx}\right)$$

$$1 = \frac{1}{2} \cdot \frac{1}{\sqrt{x^2 y^2}} \left(2x + 2y \cdot \frac{dy}{dx}\right)$$

$$1 = \frac{1}{2(\sqrt{x^2 y^2})} \left(2x + 2y \cdot \frac{dy}{dx}\right)$$

$$1 = \frac{2x + 2y \cdot \frac{dy}{dx}}{2(\sqrt{x^2 y^2}}$$

$$2(\sqrt{x^2 y^2}) = 2x + 2y \cdot \frac{dy}{dx}$$

$$\frac{2\sqrt{x^2 y^2} - 2x}{2y} = \frac{dy}{dx}$$

$$\frac{\cancel{2}(\sqrt{x^2 y^2} - x)}{\cancel{2}y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2 y^2} - x}{y}$$

5. $\dfrac{x-y^3}{y+x^2} = x+2$

$$\frac{d}{dx}\,\frac{x-y^3}{y+x^2} = \frac{d}{dx}x + \frac{d}{dx}2$$

$$\frac{vu'-uv'}{v^2} \qquad \frac{vu'-uv'}{v^2} = \frac{(y+x^2)(1-3y^2\cdot\frac{dy}{dx}) - (x-y^3)(2x+\frac{dy}{dx})}{(y+x^2)^2}$$

$(y+x^2)(y+x^2)$

$u = x-y^3$

$u' = \dfrac{d}{dx}x - \dfrac{d}{dx}y^3$

$= 1 - 3y^2\cdot\dfrac{dy}{dx}$

$v = y+x^2$

$v' = \dfrac{d}{dx}y + \dfrac{d}{dx}x^2$

$= 1\cdot\dfrac{dy}{dx} + 2x$

$$\frac{(y+x^2)(1-3y^2\cdot\frac{dy}{dx}) - (x-y^3)(2x+\frac{dy}{dx})}{(y+x^2)^2} = 1$$

$$\frac{y-3y^3\frac{dy}{dx}+x^2-3x^2y^2\frac{dy}{dx} - (2x^2+x\frac{dy}{dx}-2xy^3-y^3\frac{dy}{dx})}{y^2+2x^2y+x^4} = 1$$

$$y-3y^3\frac{dy}{dx}+x^2-3x^2y^2\frac{dy}{dx}-2x^2-x\frac{dy}{dx}+2xy^3+y^3\frac{dy}{dx} = y^2+2x^2y+x^4$$

$$y-y^2-2x^2y-x^4+x^2-2x^2+2xy^3 = 3y^3\frac{dy}{dx}+y^3\frac{dy}{dx}+3x^2y^2\frac{dy}{dx}+x\frac{dy}{dx}$$

$$-x^2+y-y^2-x^4-2x^2y+2xy^3 = \frac{dy}{dx}(3y^3-y^3+3x^2y^2+x)$$

$$\frac{dy}{dx} = \frac{-x^2+y-y^2-x^4-2x^2y+2xy^3}{2y^3+x+3x^2y}$$

6. $\frac{y}{x^3} + \frac{x}{y^3} = x^2 y^4$

$$\frac{d}{dx}\frac{y}{x^3} + \frac{d}{dx}\frac{x}{y^3} = \frac{d}{dx} x^2 y^4$$

$$\frac{d}{dx}\frac{y}{x^3} = \frac{vu' - uv'}{v^2} = \frac{(x^3)(\frac{dy}{dx}) - (y)(3x^2)}{x^6}$$

$u = y$

$u' = \frac{dy}{dx}$

$v = x^3$

$v' = 3x^2$

$$\frac{d}{dx} x^2 y^4 = vu' + uv'$$

$u = x^2$

$u' = 2x$

$v = y^4$

$v' = 4y^3 \cdot \frac{dy}{dx}$

$= 2xy^4 + 4x^2y^3 \cdot \frac{dy}{dx}$

$$\frac{d}{dx}\frac{x}{y^3} = \frac{vu' - uv'}{v^2} = \frac{(y^3)(1) - (x)(3y^2 \cdot \frac{dy}{dx})}{y^6}$$

$u = x$

$u' = 1$

$v = y^3$

$v' = 3y^2 \cdot \frac{dy}{dx}$

$$= \frac{(x^3)(\frac{dy}{dx}) - (y)(3x^2)}{x^6} + \frac{(y^3)(1) - (x)(3y^2 \cdot \frac{dy}{dx})}{y^6} = 2xy^4 + 4x^2y^3 \cdot \frac{dy}{dx}$$

$$= x^3\frac{dy}{dx} - 3yx^2 + \frac{x^6(y^3 - 3xy^2 \cdot \frac{dy}{dx})}{y^6} = 2x^7y^4 + 4x^8y^3 \cdot \frac{dy}{dx}$$

$$y^6x^3\frac{dy}{dx} - 3y^7x^2 + x^6y^3 - 3x^7y^2\frac{dy}{dx} = 2x^7y^{10} + 4x^8y^9 \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(y^6x^3 - 3x^7y^2 - 4x^8y^9\right) = 2x^7y^{10} + 3y^7x^2 - x^6y^3$$

$$\frac{dy}{dx} = \frac{2x^7y^{10} + 3y^7x^2 - x^6y^3}{y^6x^3 - 3x^7y^2 - 4x^8y^9} = \frac{\cancel{x^2y^3}(2x^5y^7 + 3y^4 - x^4)}{\cancel{y^2x^3}(y^4 - 3x^4 - 4x^5y^7)}$$

$$\frac{dy}{dx} = \frac{2x^5y^8 + 3y^5 - x^4y}{xy^4 - 3x^5 - 4x^6y^7}$$

Past year question.

1 Use differentials to estimate the value of $(\sqrt{64.04})^3 - \frac{2}{\sqrt{64.04}}$ to two decimal places.

(6 marks)

$x^3 = (\sqrt{64.04})^3$

$x = \sqrt{64.04}$

$x_0 \approx 8$

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(x_0) = f'(8) = 3(8)^2 = 192$

$f(x_0) = f(8) = 8^3 = 512$

$$(\sqrt{64.04})^3 - \frac{2}{\sqrt{64.04}} \approx \left[512 + 192(\sqrt{64.04} - 8)\right] - \frac{2}{\sqrt{64.04}}$$

$$\approx 512.25$$

3 Given $f(x) = \sqrt{3x-2}$. Use differentials to estimate the change in $f(x)$ as x changes from $x = 2$ to $x = 2.03$.

(6 marks)

$f(x) = \sqrt{3x-2}$

$= (3x-2)^{1/2}$

$f'(x) = \frac{3}{2\sqrt{3x-2}}$

$f'(x_0) = \frac{3}{2\sqrt{3(2)-2}}$

$= \frac{3}{4}$

$f(x_0) = \sqrt{3(2)-2}$

$= 2$

$u = 3x-2$

$= u^{1/2}$

$= \frac{1}{2}u^{-1/2}$

$= \frac{1}{2}(3x-2)^{-1/2} \cdot 3$

$= \frac{3}{2} \cdot \frac{1}{\sqrt{3x-2}}$

$= \frac{3}{2\sqrt{3x-2}}$

$$\sqrt{3x-2} \approx 2 + \frac{3}{4}(2.03 - 2)$$

$$\approx 2.0225$$

(6 marks)

5 Use a linear approximation to estimate $\cos(29°)$. Give your answer to four (4) decimal places. $\left[\text{Hint}: f(x_0+\Delta x)\approx f(x_0)+f'(x_0)\Delta x\right]$

(5 marks)

6

degree → rad.

$29 \times \frac{\pi}{180} = \frac{29\pi}{180}$

$x_0 = 0.5$

$f\left(\frac{29\pi}{180}\right) = \cos\left(\frac{29\pi}{180}\right)$

$\Delta x = x - x_0$

$\Delta x = \frac{29\pi}{180} - 0.5$

$= 0.0061$

$f(x) = \cos(x)$

$f'(x) = -\sin(x)$

$f(x_0) = \cos(0.5) = 0.8776$

$f'(x_0) = f'(0.5) = -\sin(0.5) = -0.4794$

$\cos(29°) = \cos\left(\frac{29\pi}{180}\right) = 0.8776 + [-0.4794(0.0061)]$

≈ 0.8747