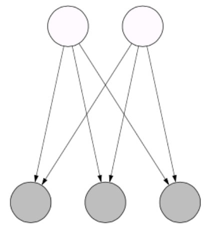
CS221 Section 7
Bayesian Networks

Spring 2018

Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence

Bayesian Networks



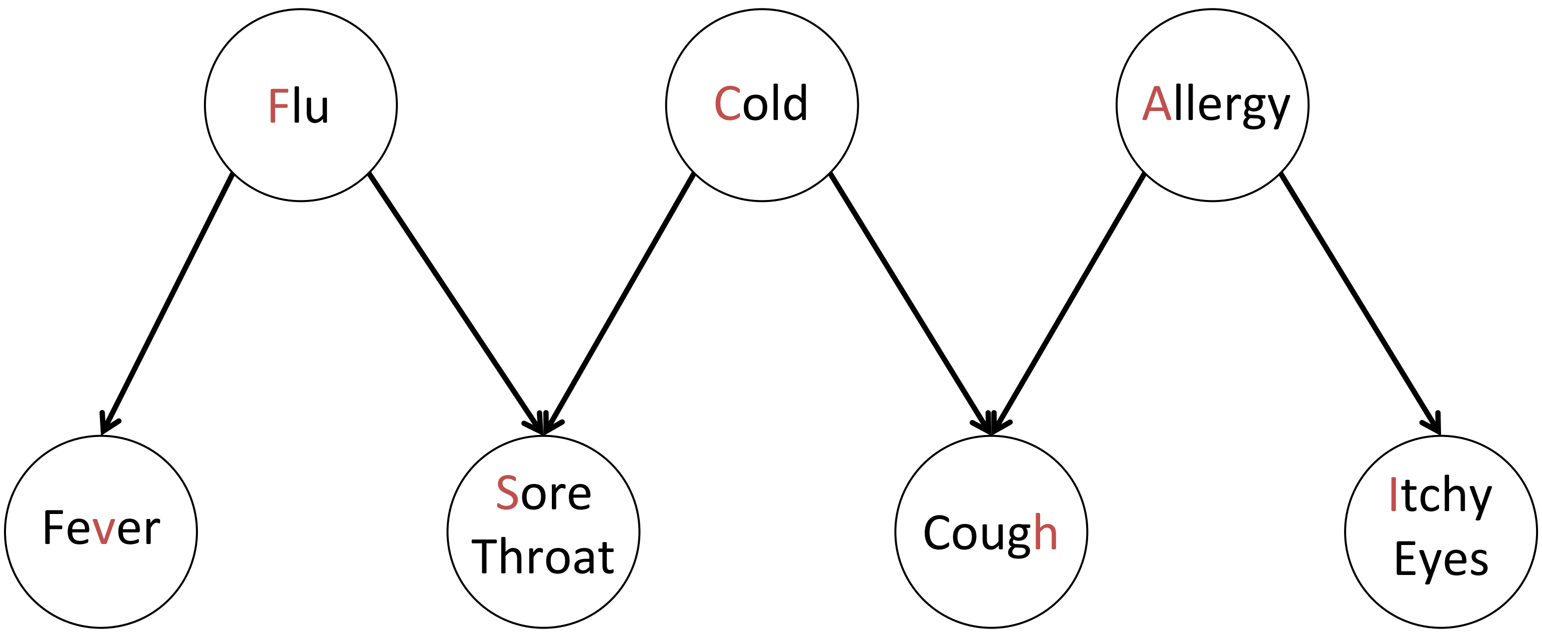
Definition: Bayesian network

Let $X = (X_1, \ldots, X_n)$ be random variables.

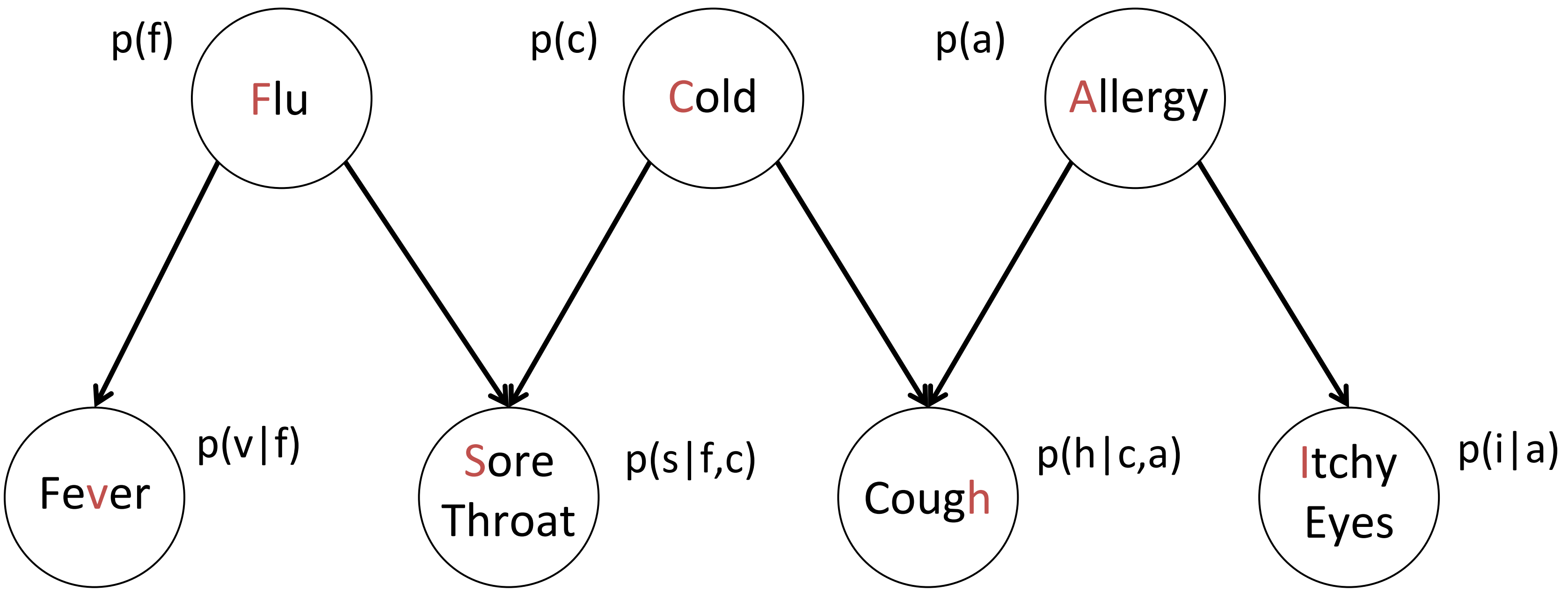
A **Bayesian network** is a directed acyclic graph (DAG) that specifies a joint distribution over X as a product of local conditional distributions, one for each node:

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \prod_{i=1}^{n} p(x_i \mid x_{\text{Parents}(i)})$$

Bayesian Networks



A Bayesian network represents a joint probability distribution.



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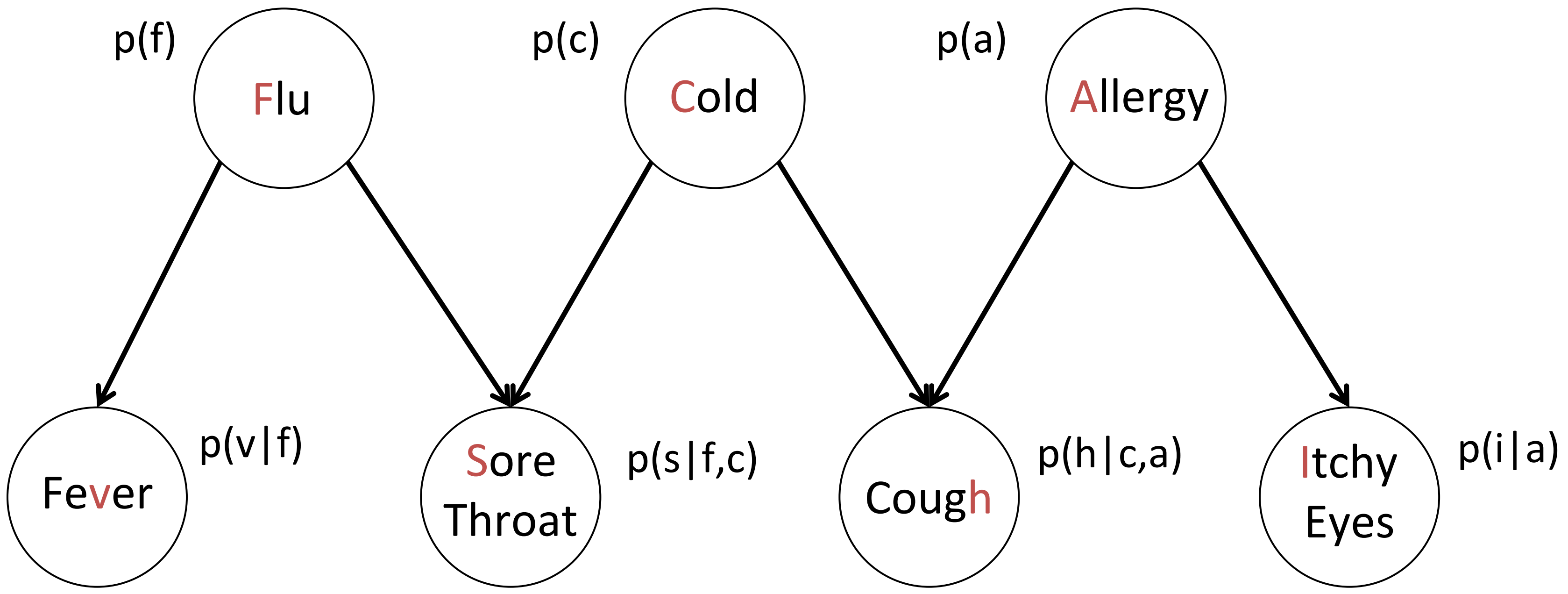


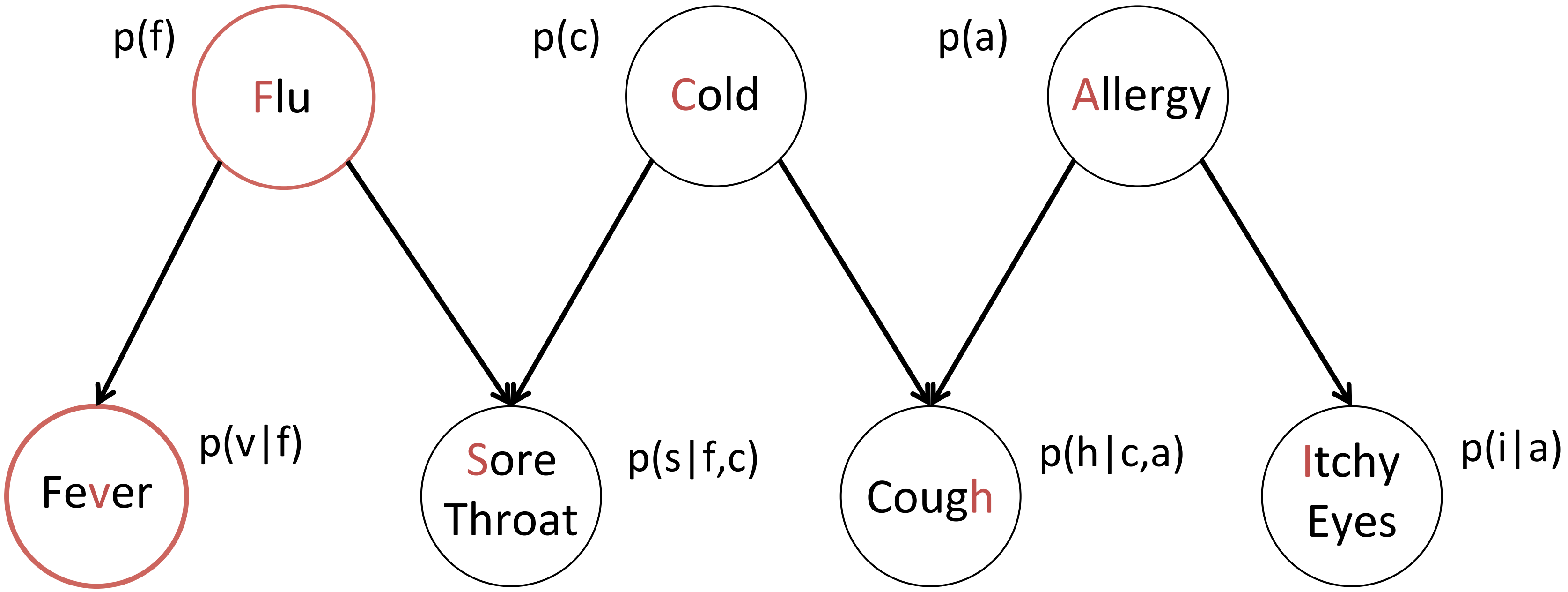
P(F=f, C=c, A=a, V=v, S=s, C=c, I=i) = p(f)p(c)p(a)p(v|a)p(s|f, c)p(h|c,a)p(i|a)

Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence

Probabilistic Queries - Examples

P(F=1|V=1) = ?



Probabilistic Queries - Examples

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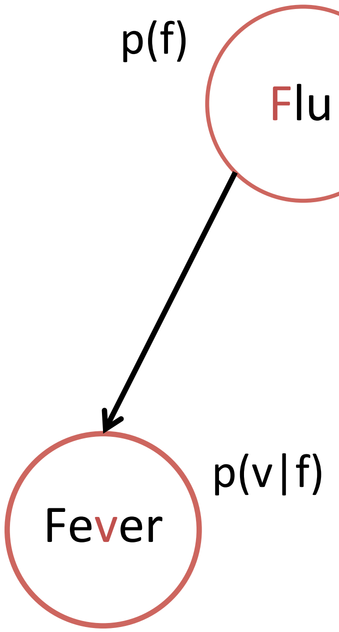
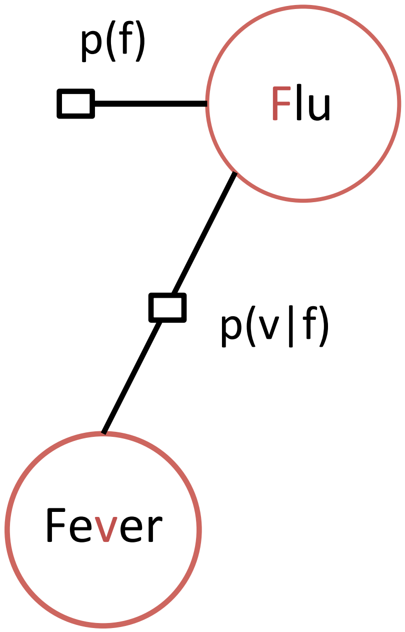


1. Remove (marginalize) variables not ancestors of Q or E.

Probabilistic Queries - Examples

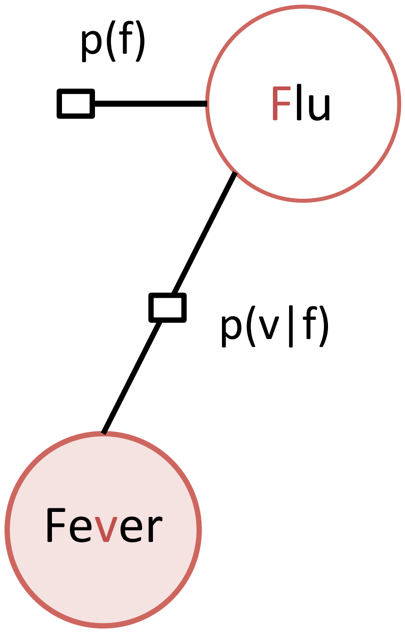
$P(F=1|V=1) = ?$



2. Convert Bayesian network to factor graph.

Probabilistic Queries - Examples

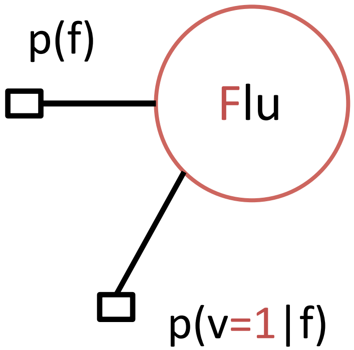
P(F=1|V=1) = ?



3. Condition on E = e.
   3.1 shade nodes

Probabilistic Queries - Examples

$P(F=1|V=1) = ?$

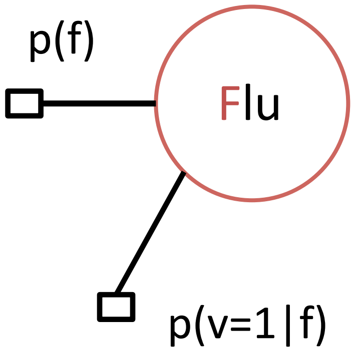


3. Condition on E = e.
    3.2 disconnect

Probabilistic Queries - Examples

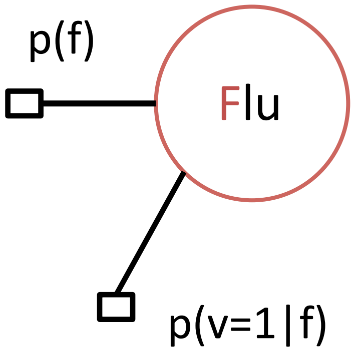
P(F=1|V=1) = ?

4. Remove (marginalize) nodes disconnected from Q.



Probabilistic Queries - Examples

$P(F=1 | V=1) = ?$



5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

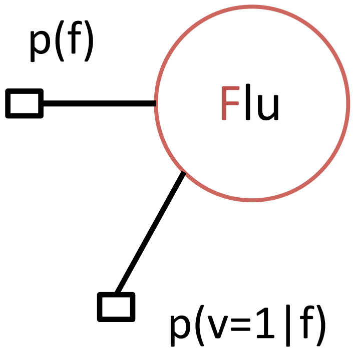
f	p(f)
0	1-α
1	α

f	v	p(v|f)
0	0	0.70
0	1	0.30
1	0	0.20
1	1	0.80

$$P(F=f|V=1) \propto p(f)\, p(v=1|f)$$

Probabilistic Queries - Examples

P(F=1|V=1) = ?



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$$P(F{=}f|V{=}1) \propto p(f)\, p(v{=}1|f) = \begin{cases} (1-\alpha)*0.30, & f=0 \\ & \end{cases}$$

Probabilistic Queries - Examples

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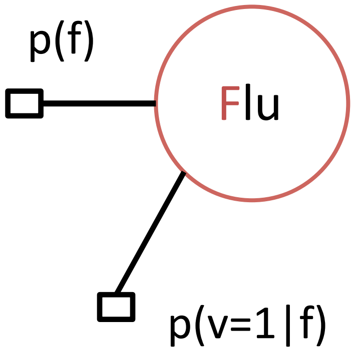

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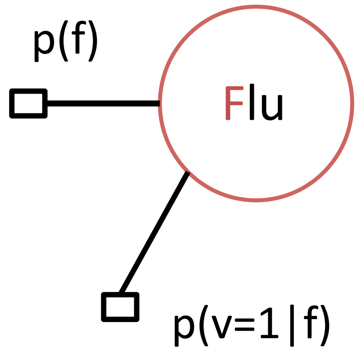
Probabilistic Queries - Examples

P(F=1|V=1) = ?



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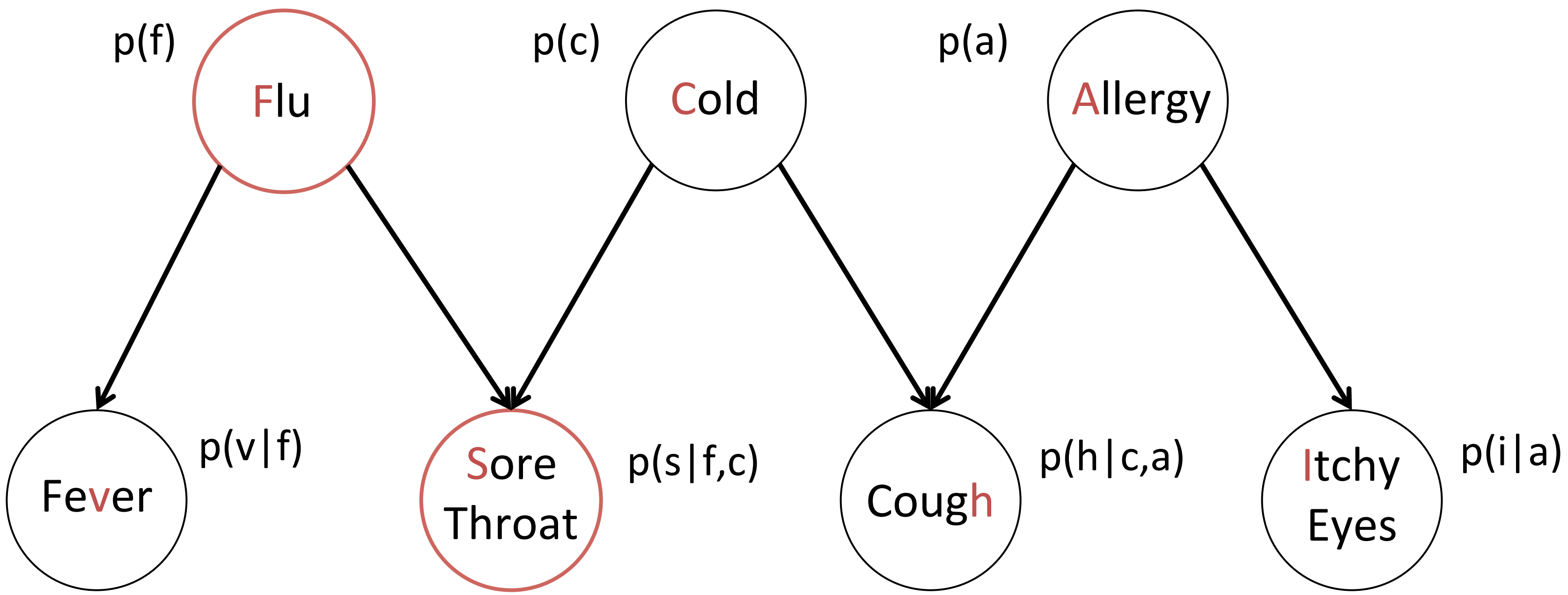
$$P(F=f|V=1) \propto p(f)\,p(v=1|f) = \begin{cases} (1-\alpha)*0.30, & f=0 \\ \alpha*0.80, & f=1 \end{cases}$$

$$P(F=1|V=1) = \frac{\alpha*0.80}{\alpha*0.80+(1-\alpha)*0.30}$$

Probabilistic Queries - Examples

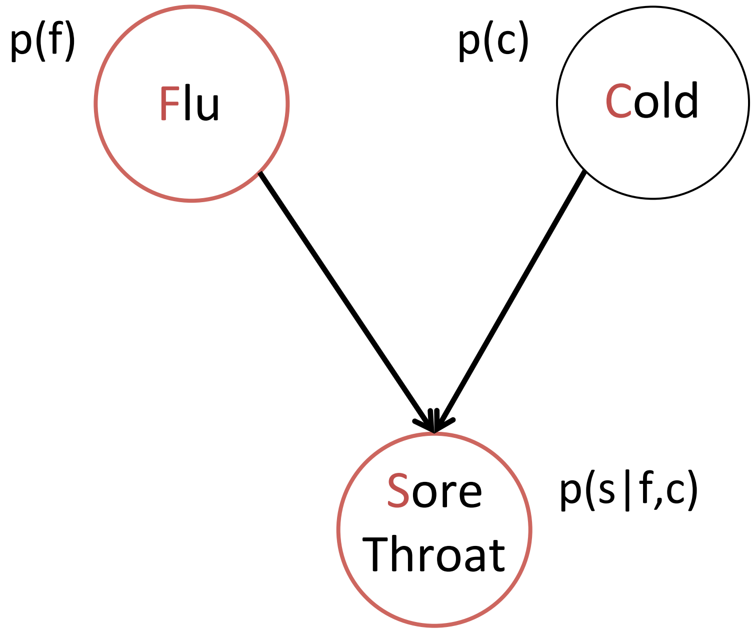
$P(F=1|S=1) = ?$



Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

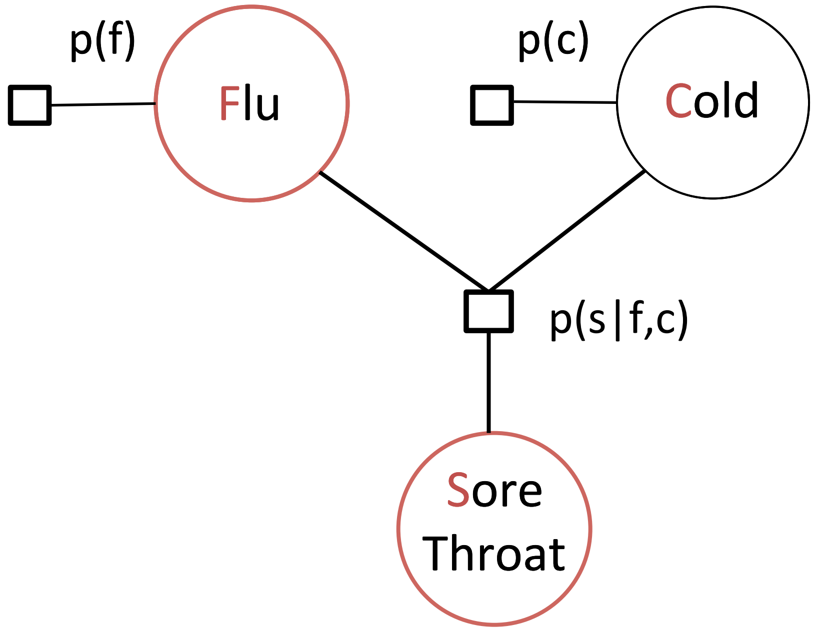
1. Remove (marginalize) variables not ancestors of Q or E.



Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

2. Convert Bayesian network to factor graph.

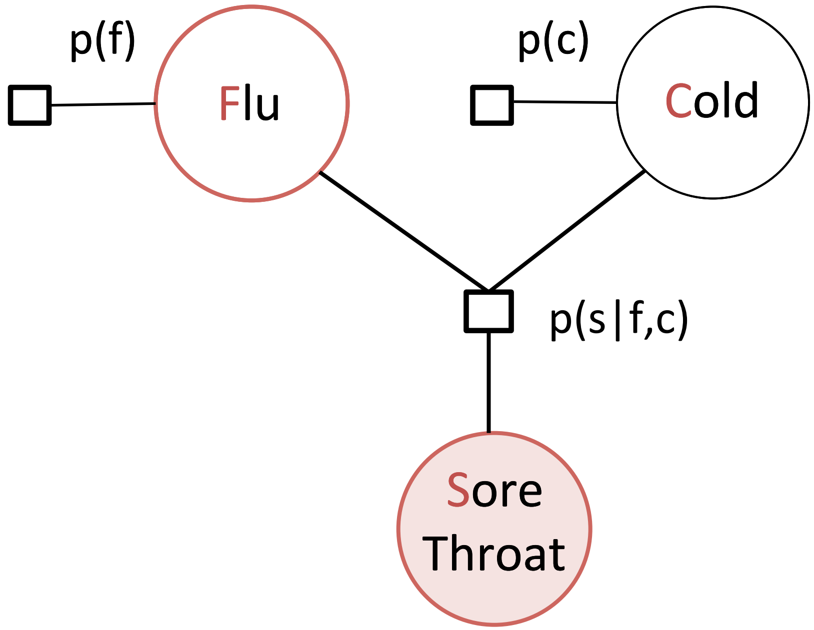


One factor per variable!

Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

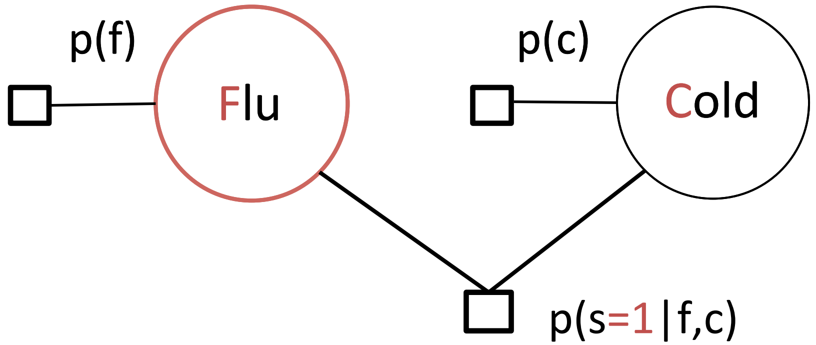
3. Condition on E = e.
   3.1 shade nodes



Probabilistic Queries - Examples

$P(F=1|S=1) = ?$

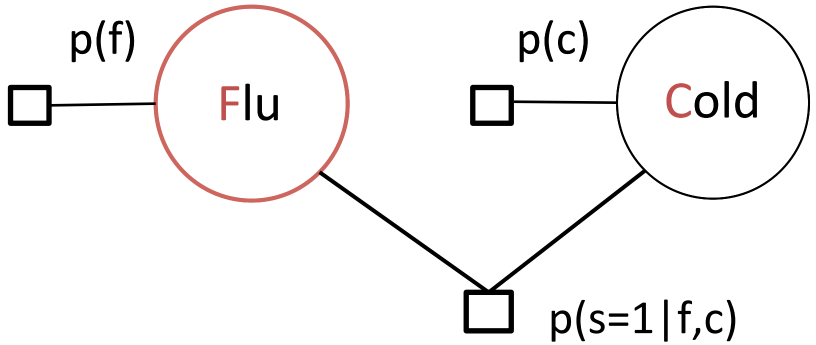
3. Condition on E = e.
   3.2 disconnect



Probabilistic Queries - Examples

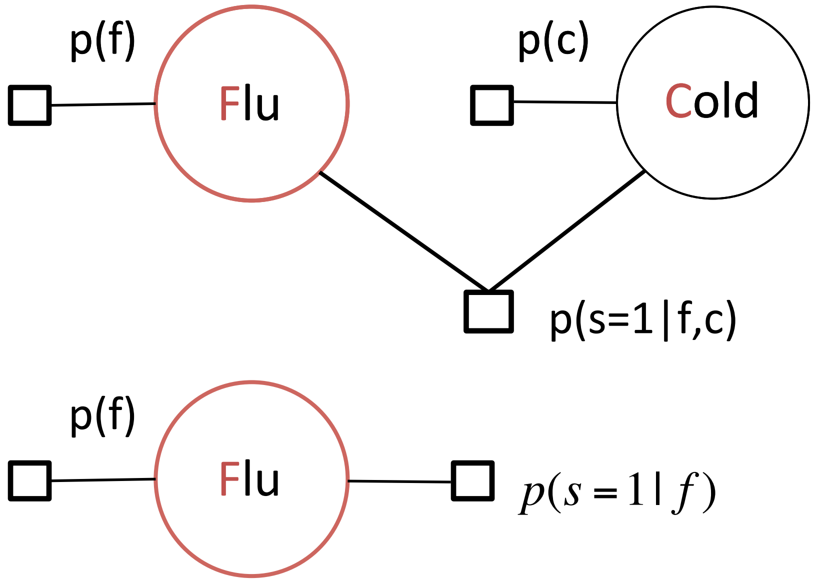
P(F=1|S=1) = ?

4. Remove (marginalize) nodes disconnected from Q.



Probabilistic Queries - Examples

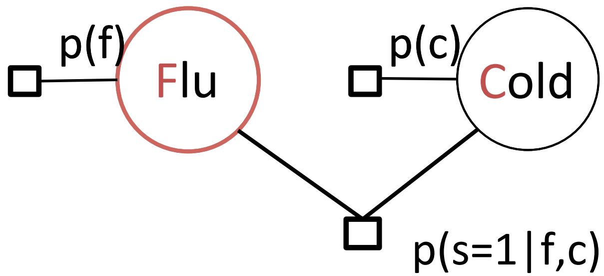
P(F=1|S=1) = ?

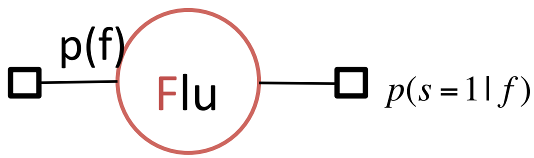


5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Probabilistic Queries - Examples

P(F=1|S=1) = ?





$$p(s=1|f)$$

$$=\sum_c p(c)p(s=1|f,c)$$

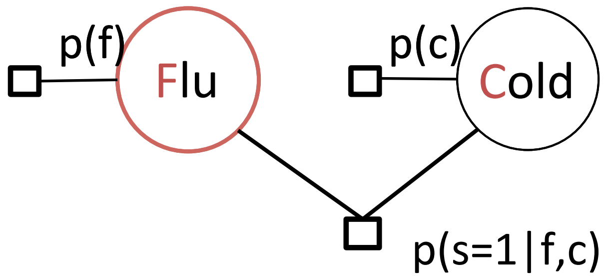
$$=p(c=0)p(s=1|f,c=0)+p(c=1)p(s=1|f,c=1)$$

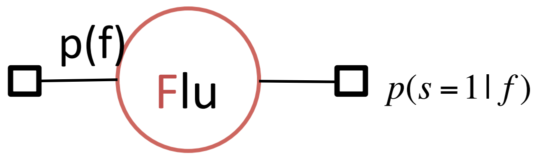
f	p(s=1,f)
0	?
1	?

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Probabilistic Queries - Examples

P(F=1|S=1) = ?





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5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

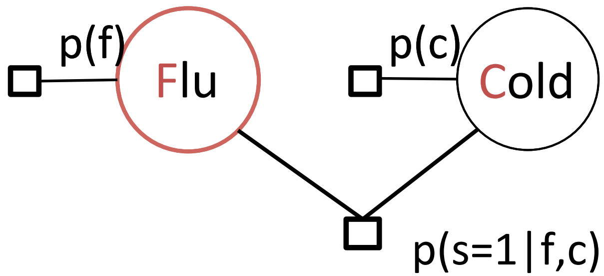
c	p(c)
0	1-β
1	β

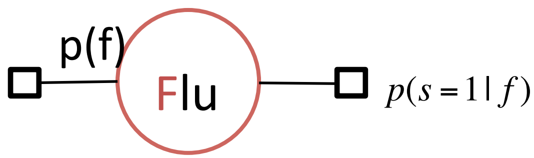
s	f	c	p(s|f,c)
0	0	0	1.00
1	0	0	0
0	1	0	0.30
1	1	0	0.70
0	0	1	0.25
1	0	1	0.75
0	1	1	0.10
1	1	1	0.90

f	p(s=1,f)
0	β*0.75
1	?

Probabilistic Queries - Examples

P(F=1|S=1) = ?





$$p(s=1|f)$$
$$=\sum_c p(c)p(s=1|f,c)$$
$$=p(c=0)p(s=1|f,c=0)+p(c=1)p(s=1|f,c=1)$$
$$=\begin{cases} (1-\beta)*0+\beta*0.75, & f=0 \\ (1-\beta)*0.70+\beta*0.9, & f=1 \end{cases}$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

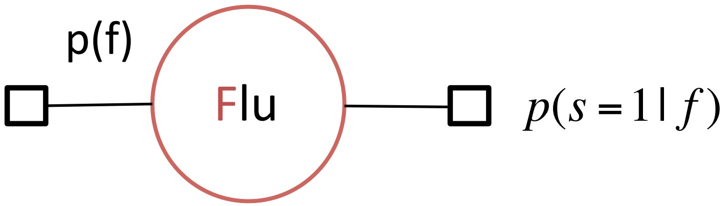
c	p(c)
0	1-β
1	β

s	f	c	p(s|f,c)
0	0	0	1.00
1	0	0	0
0	1	0	0.30
1	1	0	0.70
0	0	1	0.25
1	0	1	0.75
0	1	1	0.10
1	1	1	0.90

f	p(s=1,f)
0	β*0.75
1	((1-β)*0.7+β*0.9)

Probabilistic Queries - Examples

P(F=1|S=1) = ?



$$P(F=f \mid S=1)$$
$$\propto p(f)p(s=1 \mid f)$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

f	p(s=1|f)
0	β*0.75
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Probabilistic Queries - Examples

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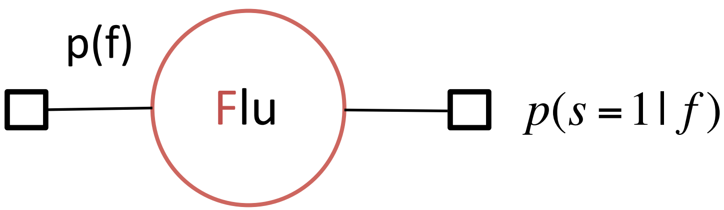

$$P(F=f\mid S=1)$$
$$\propto p(f)p(s=1\mid f)$$
$$=\begin{cases} (1-\alpha)\beta*0.75, & f=0 \\ & \end{cases}$$

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f	p(f)
0	1-α
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f	p(s=1|f)
0	β*0.75
1	((1-β)*0.7+β*0.9)

Probabilistic Queries - Examples

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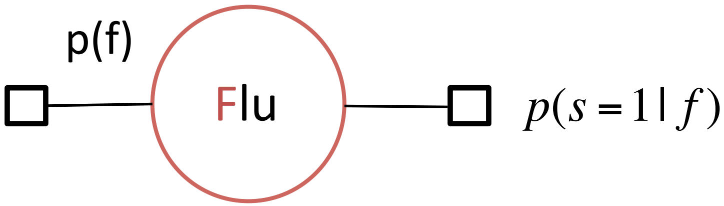

$$P(F=f\,|\,S=1)$$

$$\propto p(f)p(s=1\,|\,f)$$

$$= \begin{cases} (1-\alpha)\beta * 0.75, & f=0 \\ \alpha((1-\beta)*0.70+\beta*0.9), & f=1 \end{cases}$$

5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

f	p(f)
0	1-α
1	α

f	p(s=1|f)
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1	(1-β)*0.7+β*0.9

Probabilistic Queries - Examples

P(F=1|S=1) = ?

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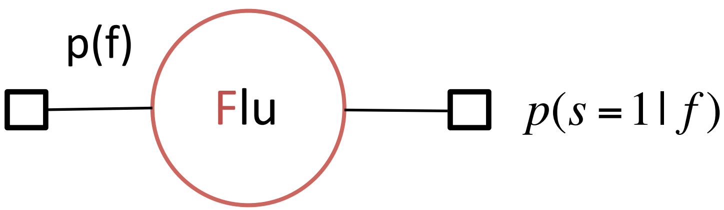


f	p(f)
0	1-α
1	α

f	p(s=1|f)
0	β*0.75
1	(1-β)*0.7+β*0.9

$$P(F=f\,|\,S=1)$$

$$\propto p(f)p(s=1\,|\,f)$$

$$= \begin{cases} (1-\alpha)\beta * 0.75, & f=0 \\ \alpha((1-\beta)*0.70+\beta*0.9), & f=1 \end{cases}$$

$$P(F=1\,|\,S=1) = \frac{p(f=1)p(s=1\,|\,f=1)}{p(f=1)p(s=1\,|\,f=1)+p(f=1)p(s=1\,|\,f=1)}$$

$$= \frac{\alpha((1-\beta)*0.70+\beta*0.9)}{(1-\alpha)\beta*0.75+\alpha((1-\beta)*0.70+\beta*0.9),}$$

Probabilistic Queries – Cookbook

Given a query $P(Q|E=e)$

1. Remove (marginalize) variables not ancestors of Q or E.
2. Convert Bayesian network to factor graph.
3. Condition (shade nodes / disconnect) on E = e.
4. Remove (marginalize) nodes disconnected from Q.
5. Run probabilistic inference algorithm (manual, variable elimination, Gibbs sampling, particle filtering).

Roadmap

- Bayesian Networks Introduction
- Probabilistic Queries
- Conditional Independence

Conditional Independence

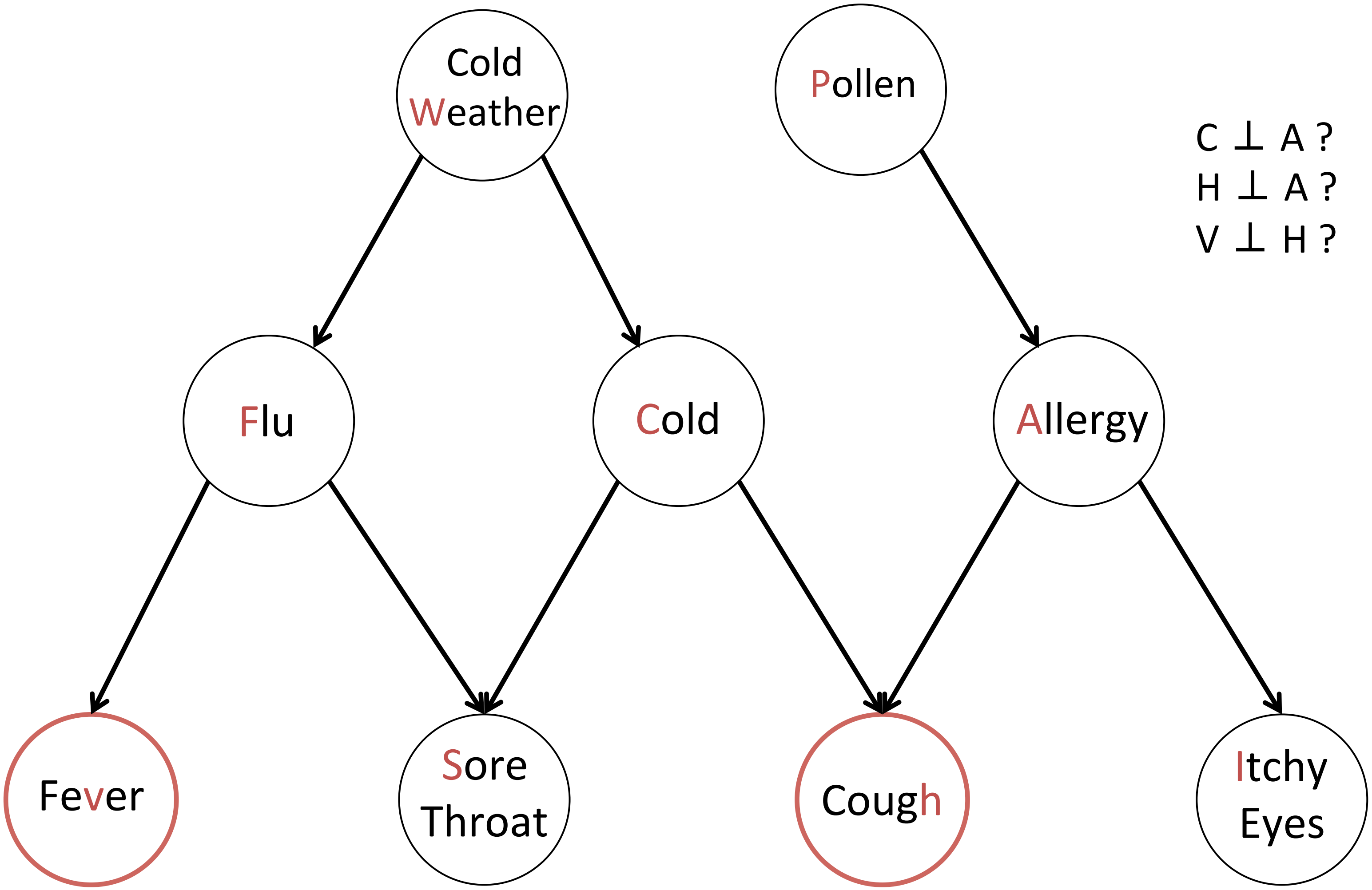
Definition: conditional independence

- Let A, B, C be a partitioning of the variables.
- We say A and B are **conditionally independent** given C if conditioning on C produces a graph in which A and B are independent.
- In symbols: $A \perp\!\!\!\perp B \mid C$.

How to determine if A and B are independent, given C?

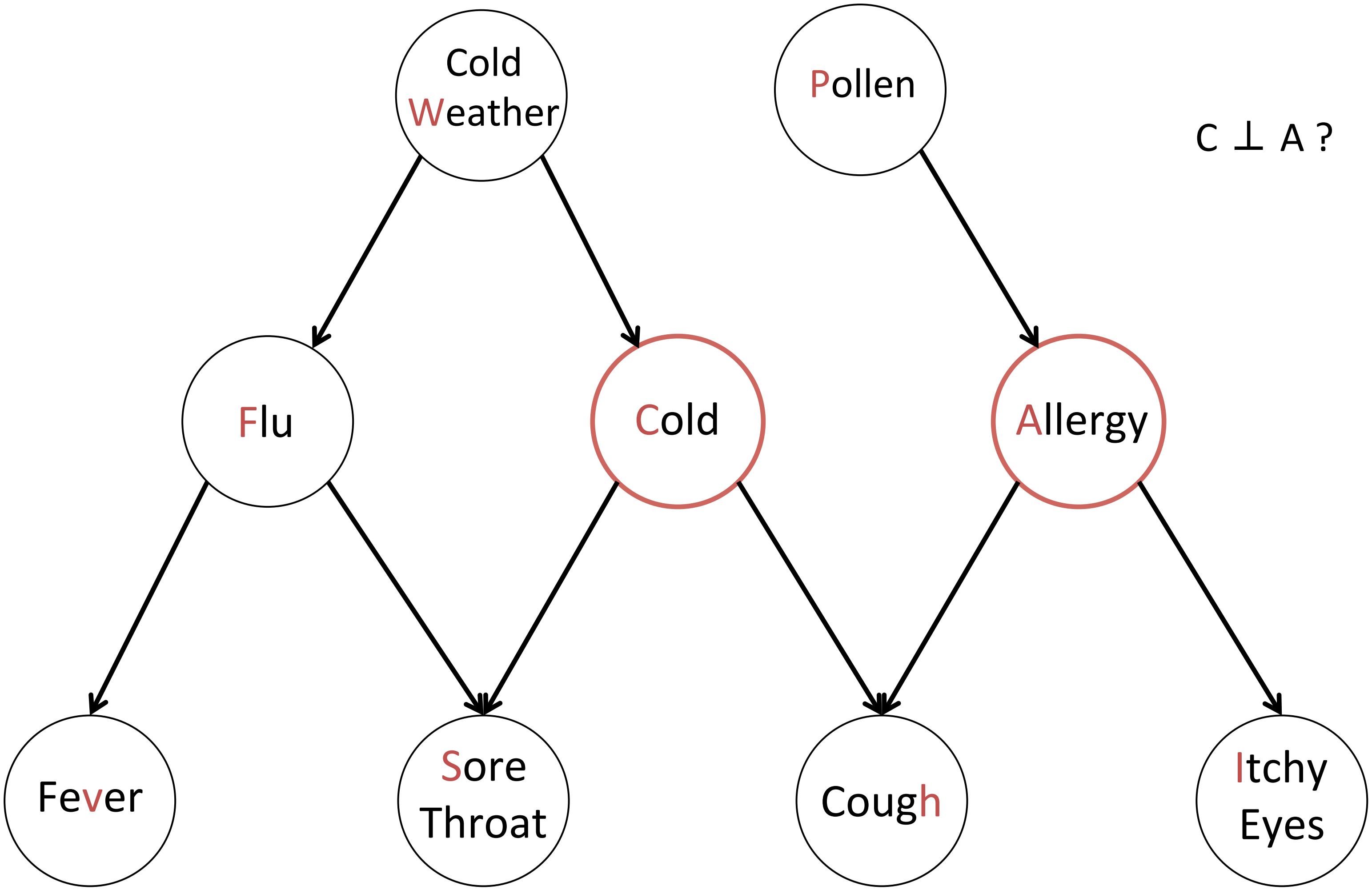
- If every undirected path from X to Y **is blocked** by C, then X and Y are conditional independent given C
- Paths between X and Y are blocked if:
  - X -> C -> Y
  - X <- C -> Y

Conditional Independence



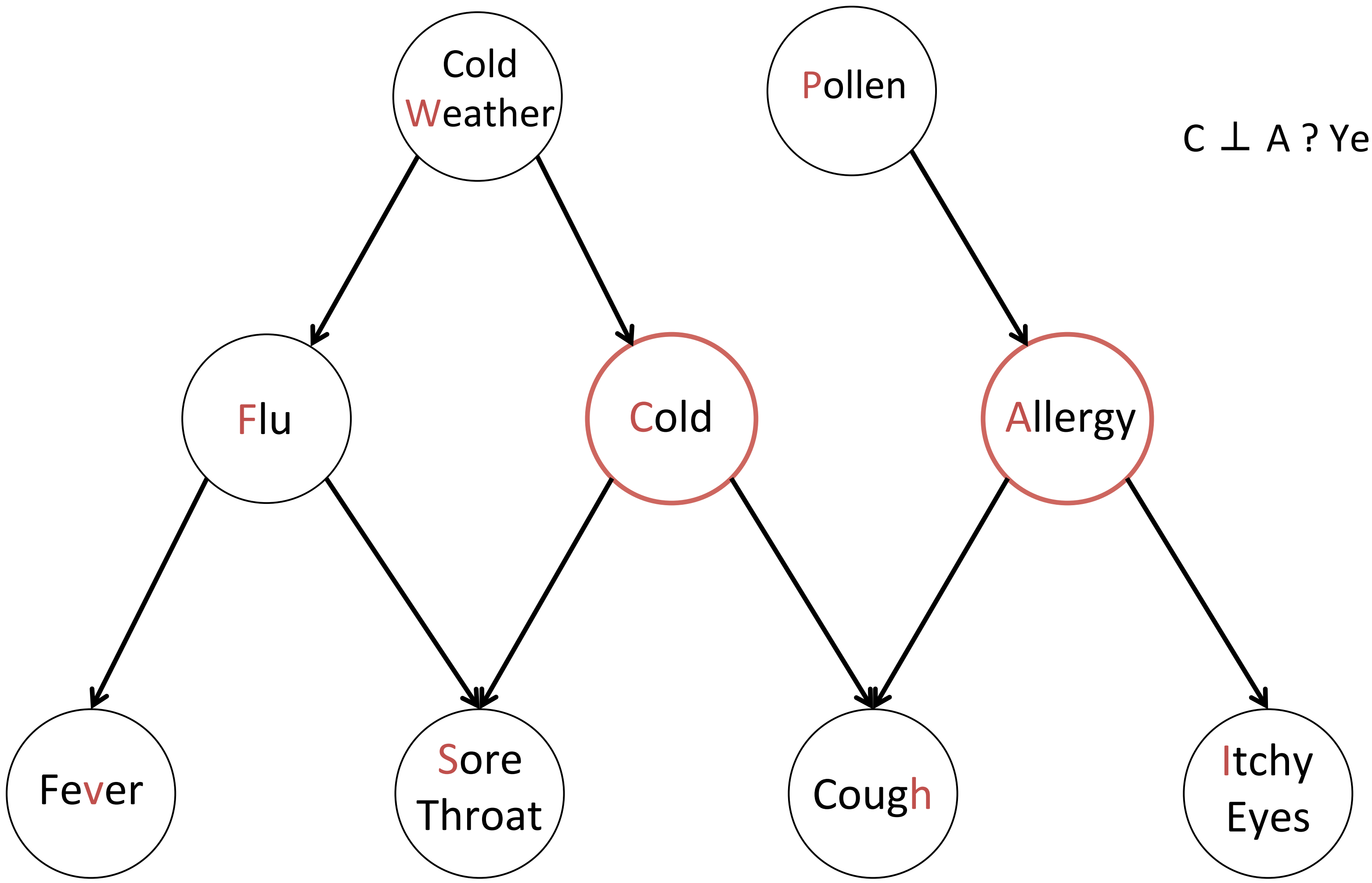
$C \perp A$?
$H \perp A$?
$V \perp H$?

Conditional Independence



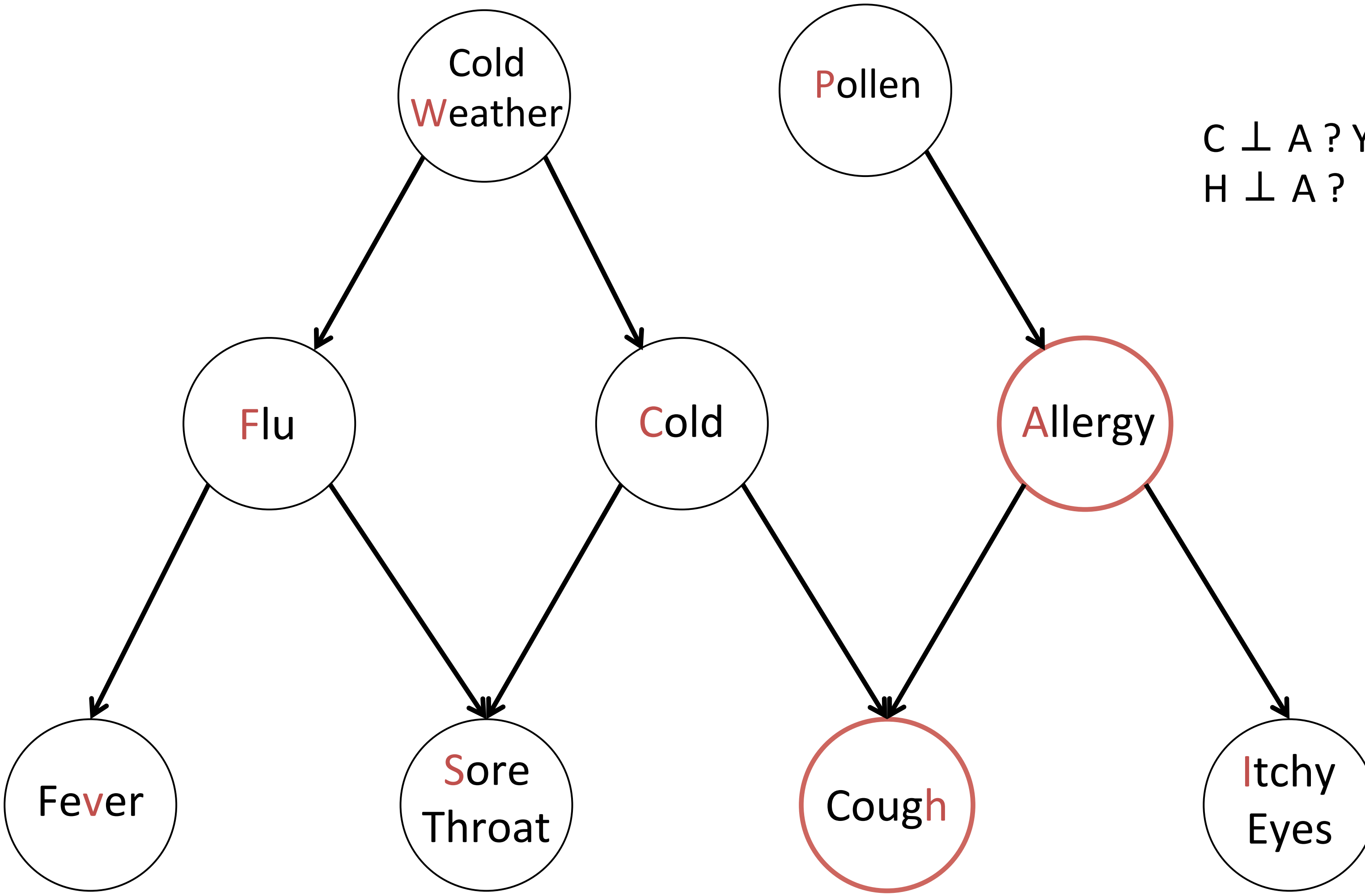
$C \perp A$?

Conditional Independence



C ⊥ A ? Yes

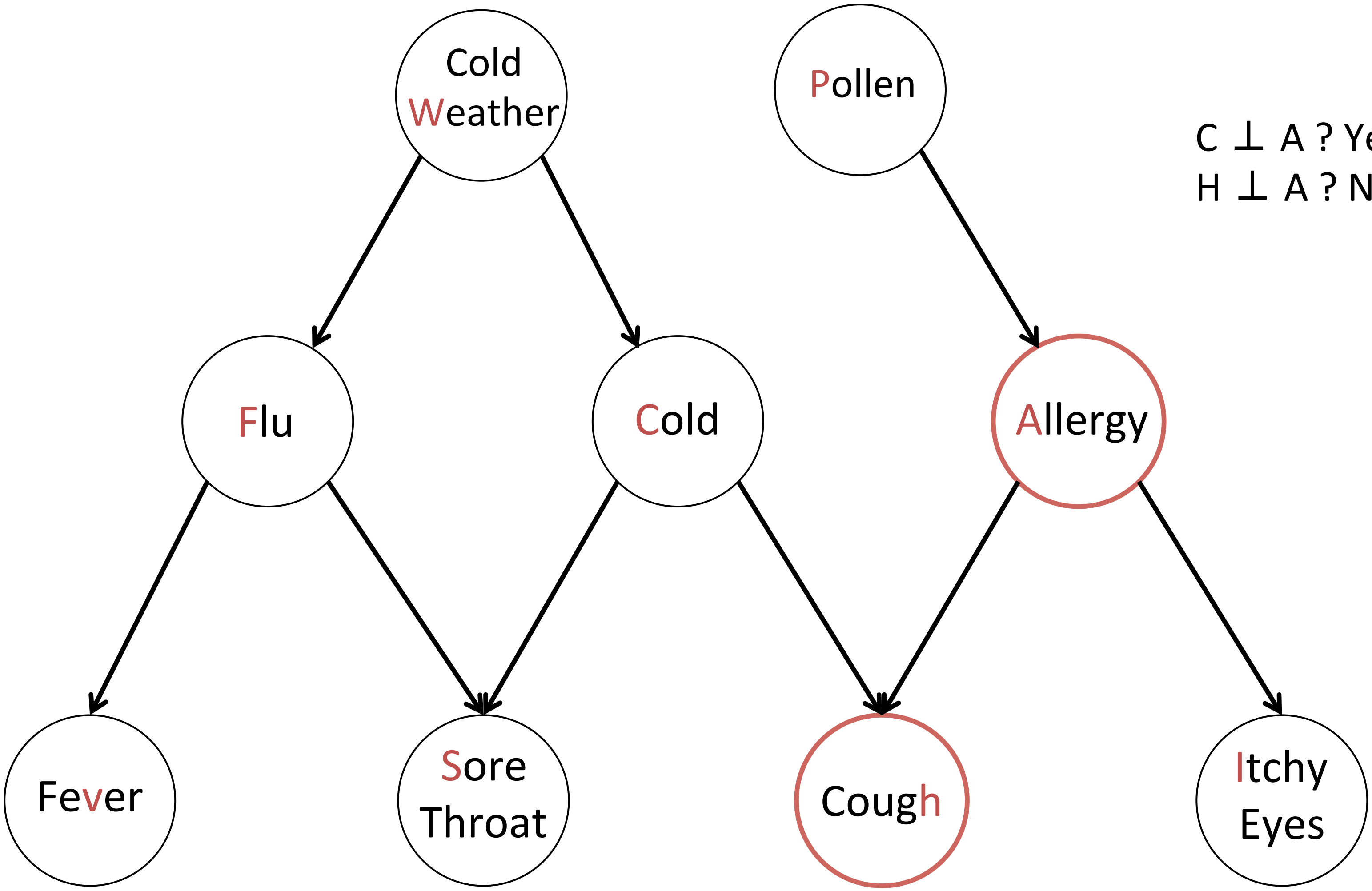
Conditional Independence



$C \perp A$? Yes

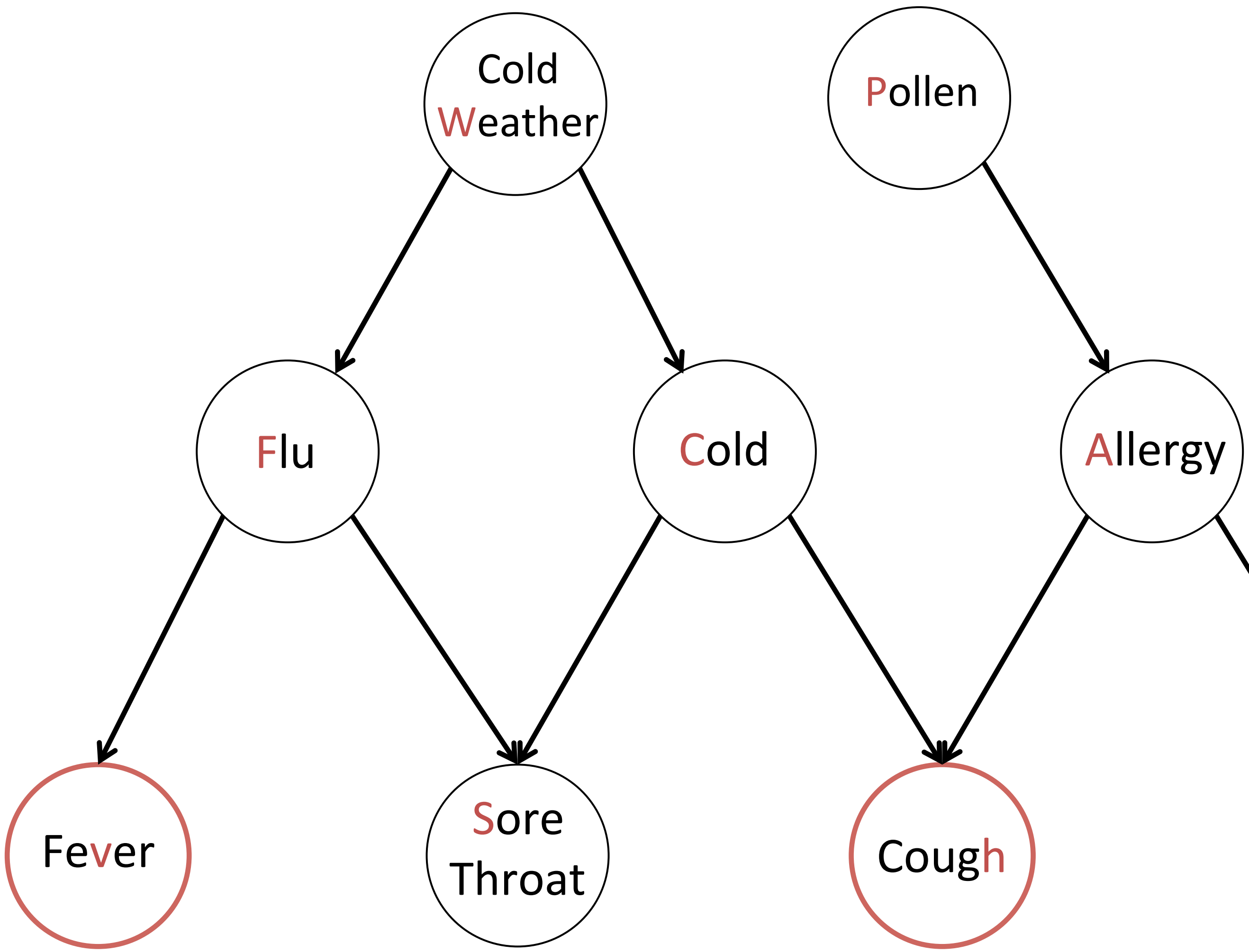
$H \perp A$?

Conditional Independence



C ⊥ A ? Yes
H ⊥ A ? No

Conditional Independence

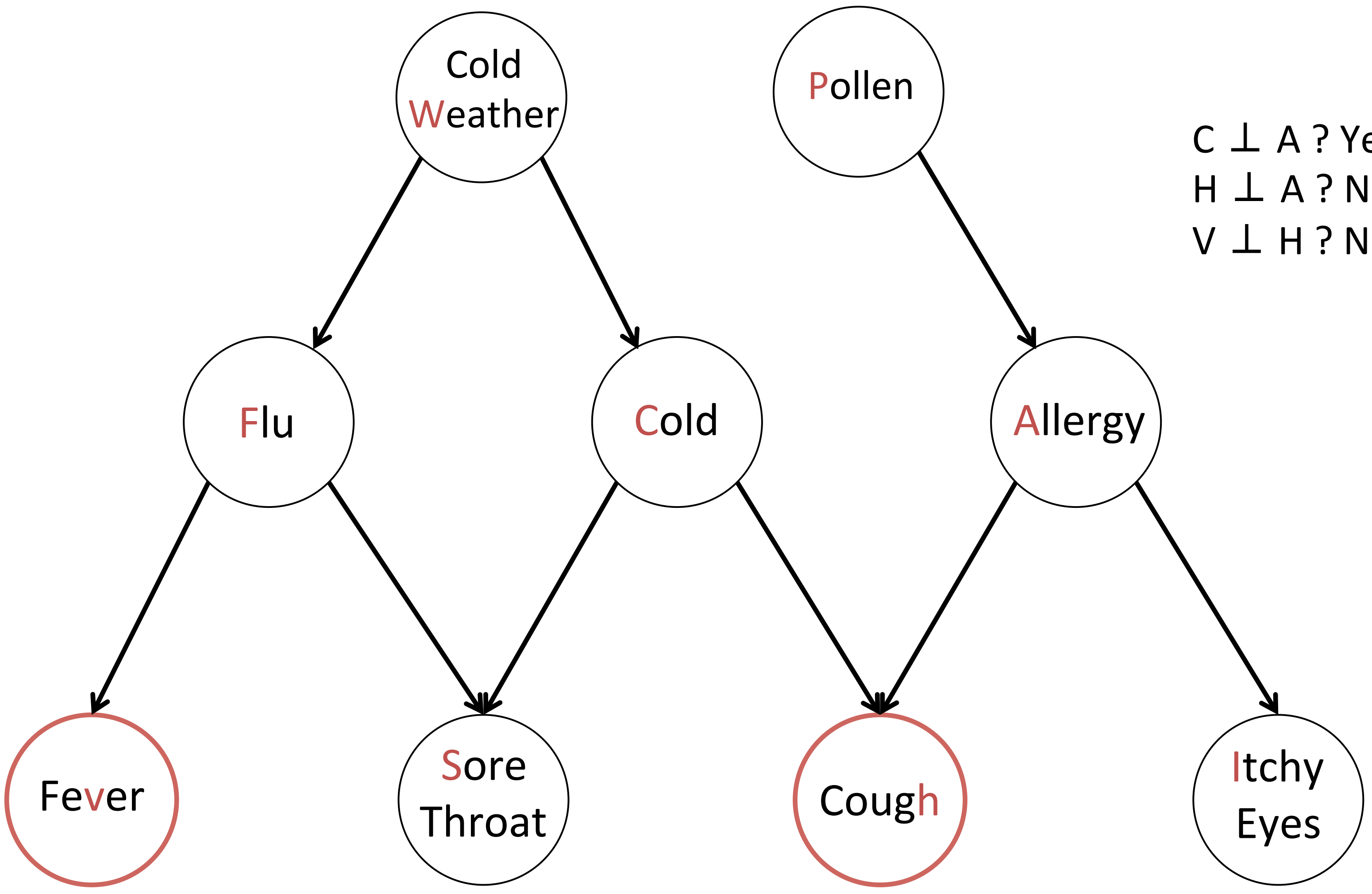


$C \perp A$? Yes

$H \perp A$? No

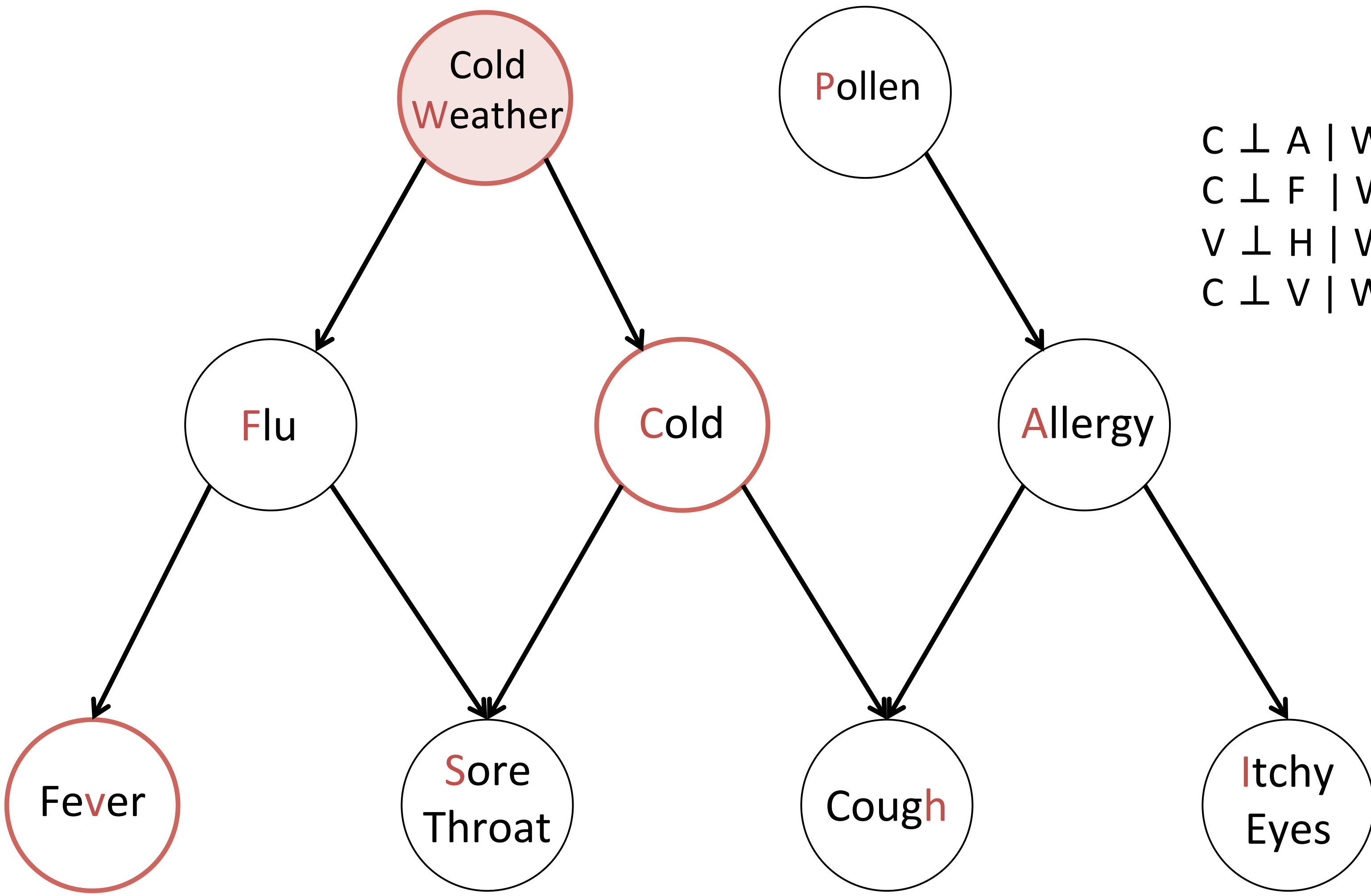
$V \perp H$?

Conditional Independence



$C \perp A$? Yes
$H \perp A$? No
$V \perp H$? No

Conditional Independence



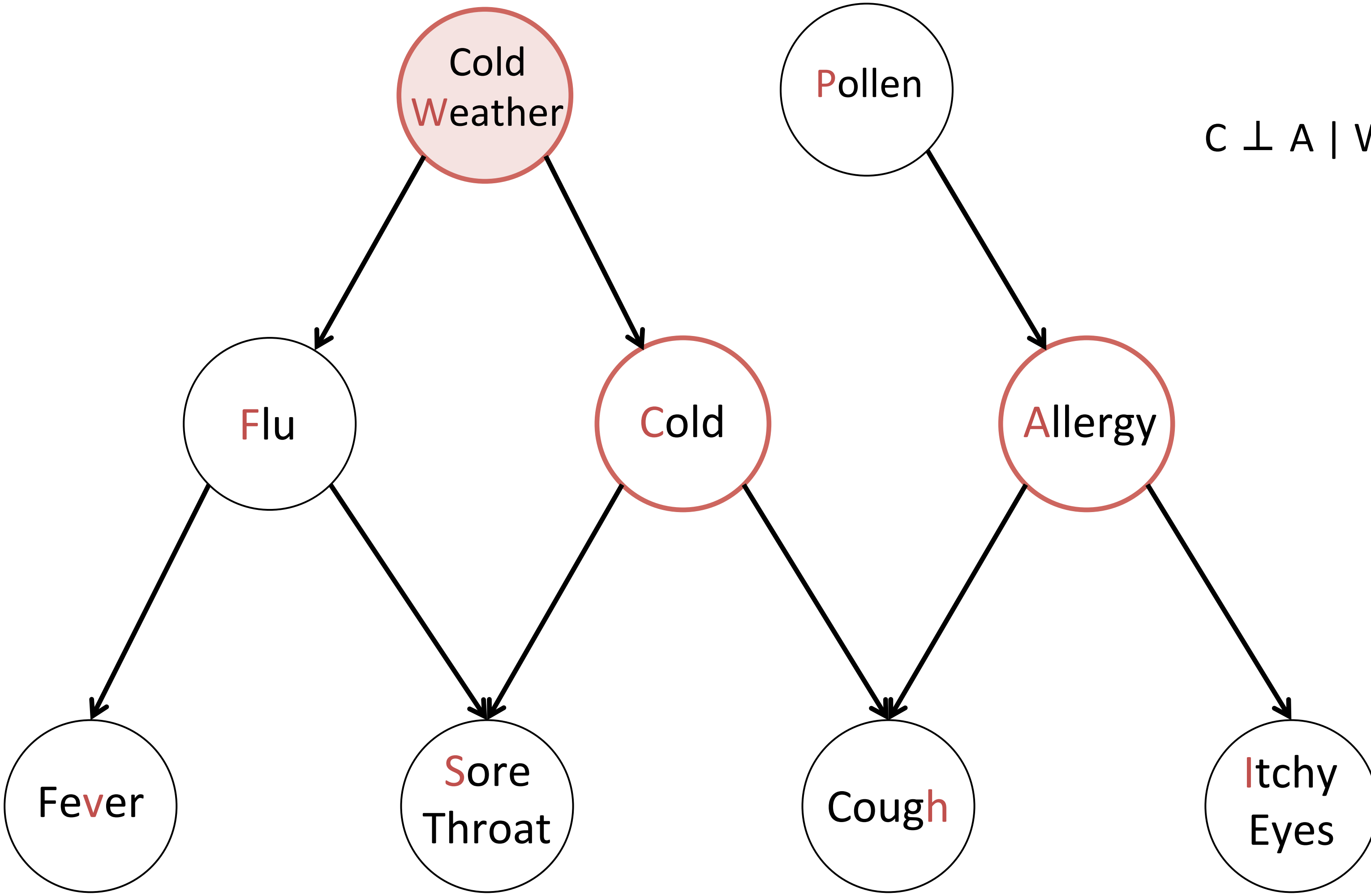
$C \perp A \mid W?$

$C \perp F \mid W?$

$V \perp H \mid W?$

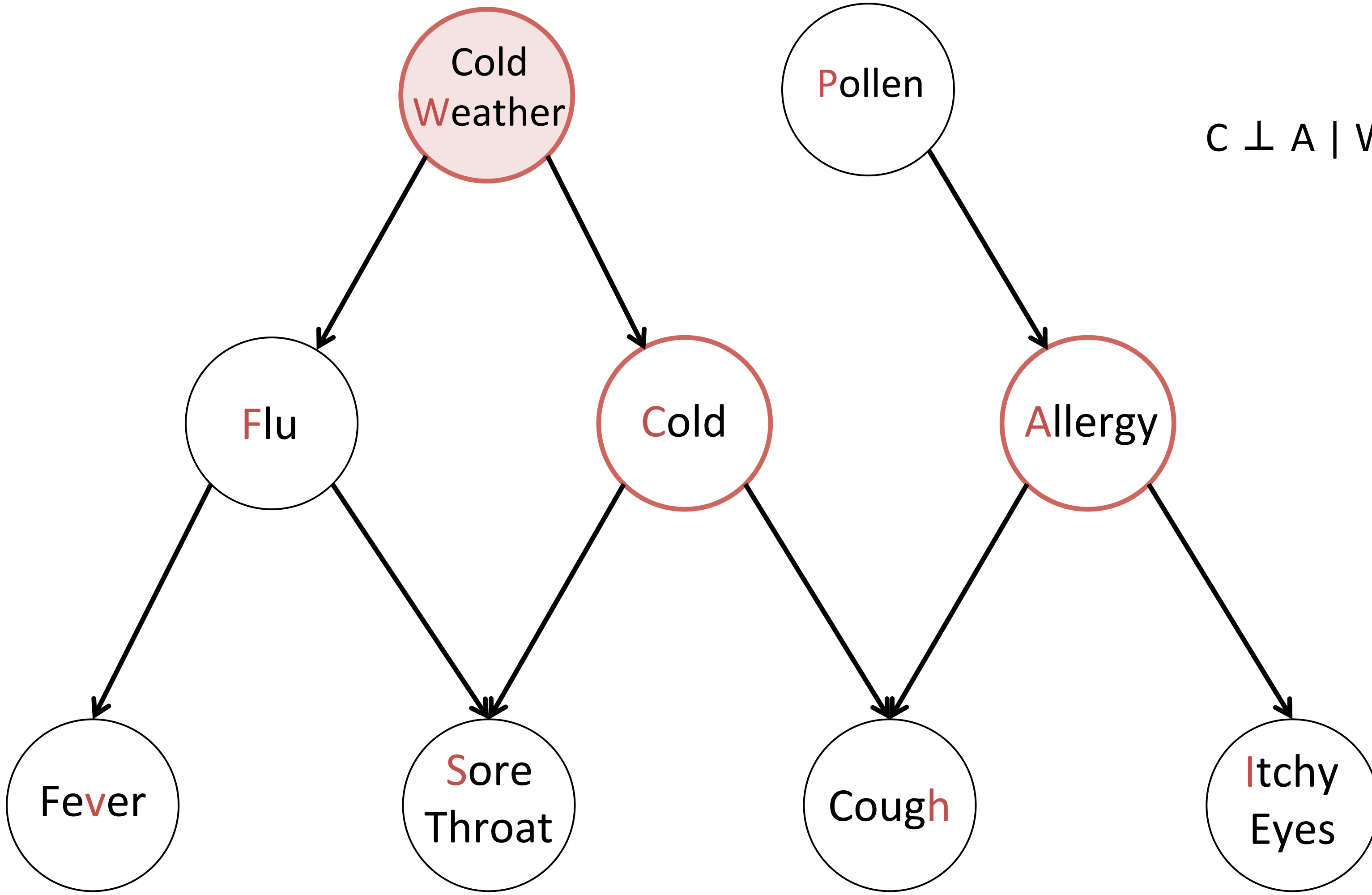
$C \perp V \mid W?$

Conditional Independence



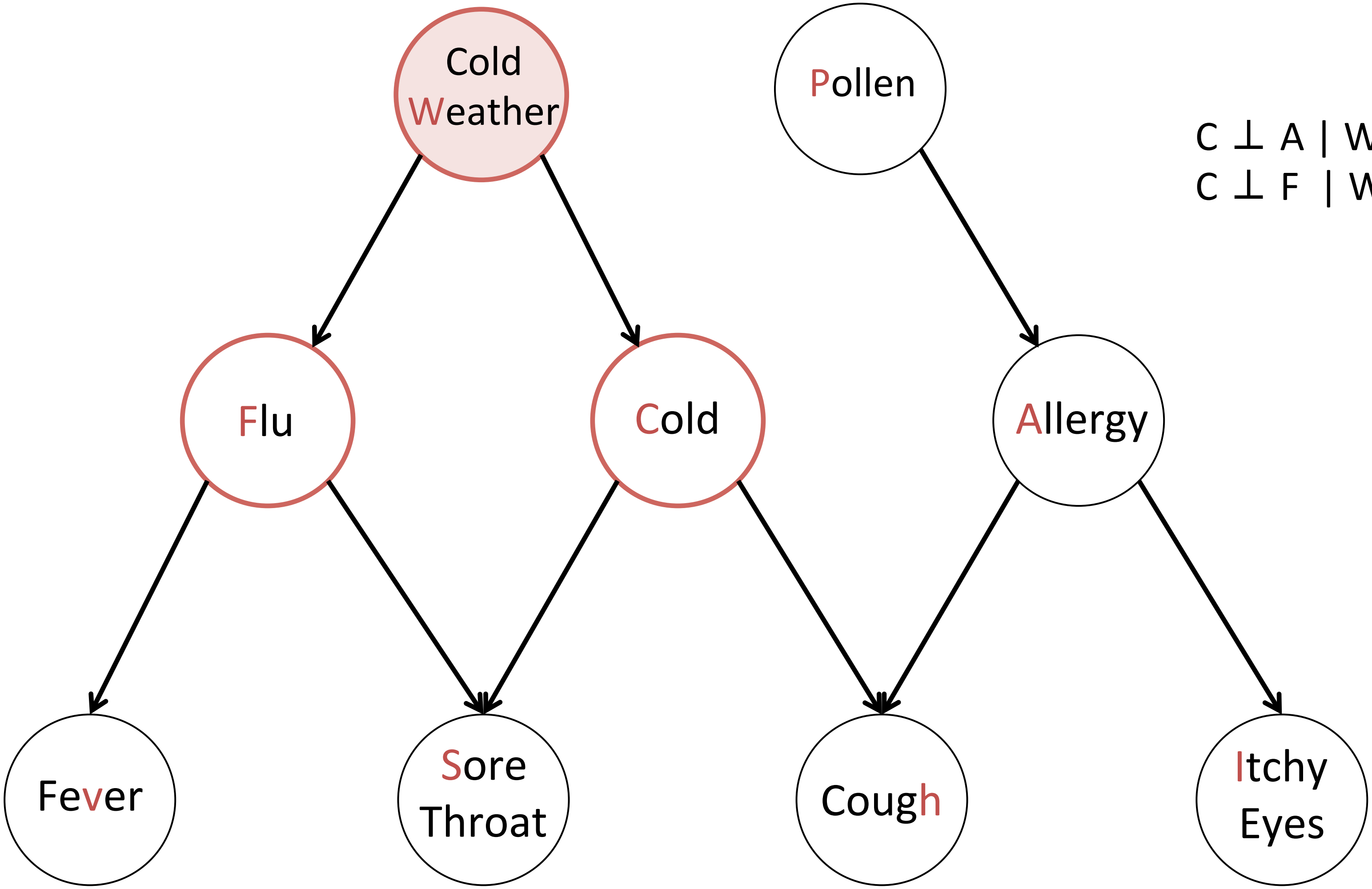
C ⊥ A | W?

Conditional Independence



C ⊥ A | W? Yes

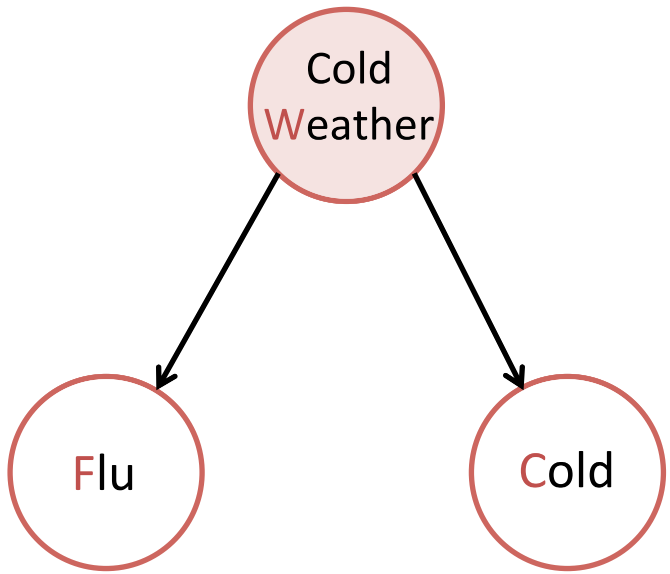
Conditional Independence



$C \perp A \mid W$? Yes

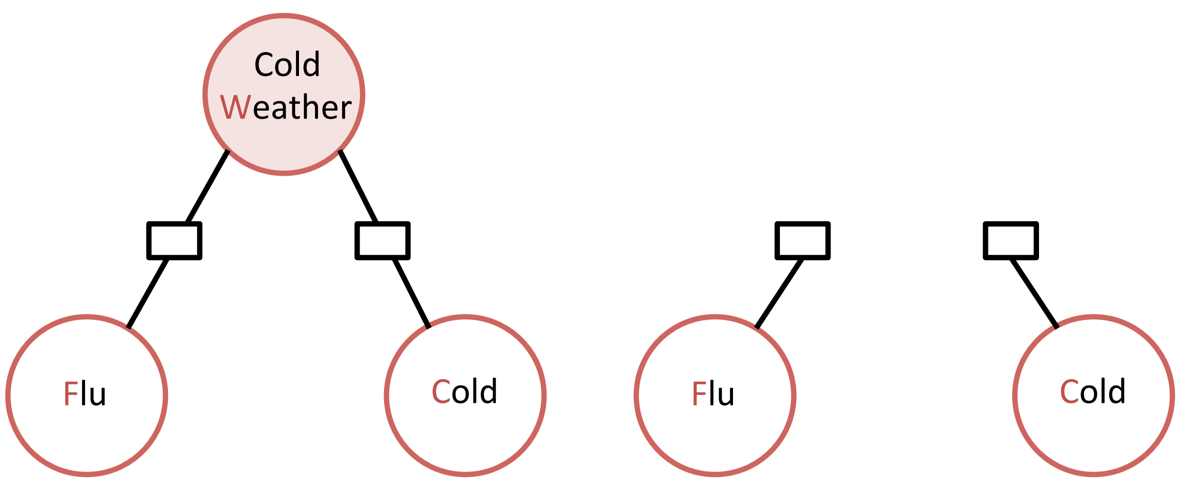
$C \perp F \mid W$?

Conditional Independence

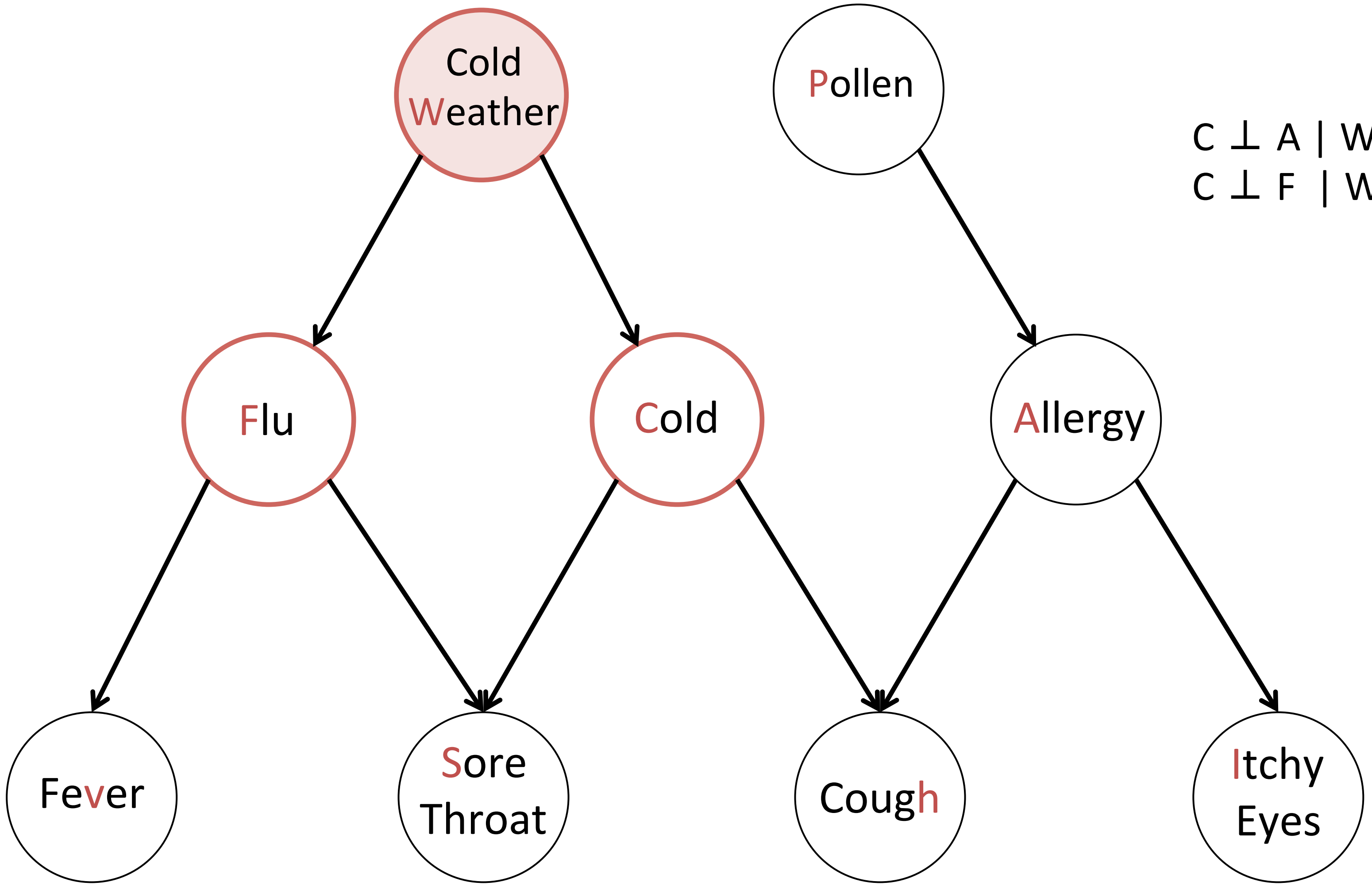


$C \perp A \mid W$? Yes

$C \perp F \mid W$?

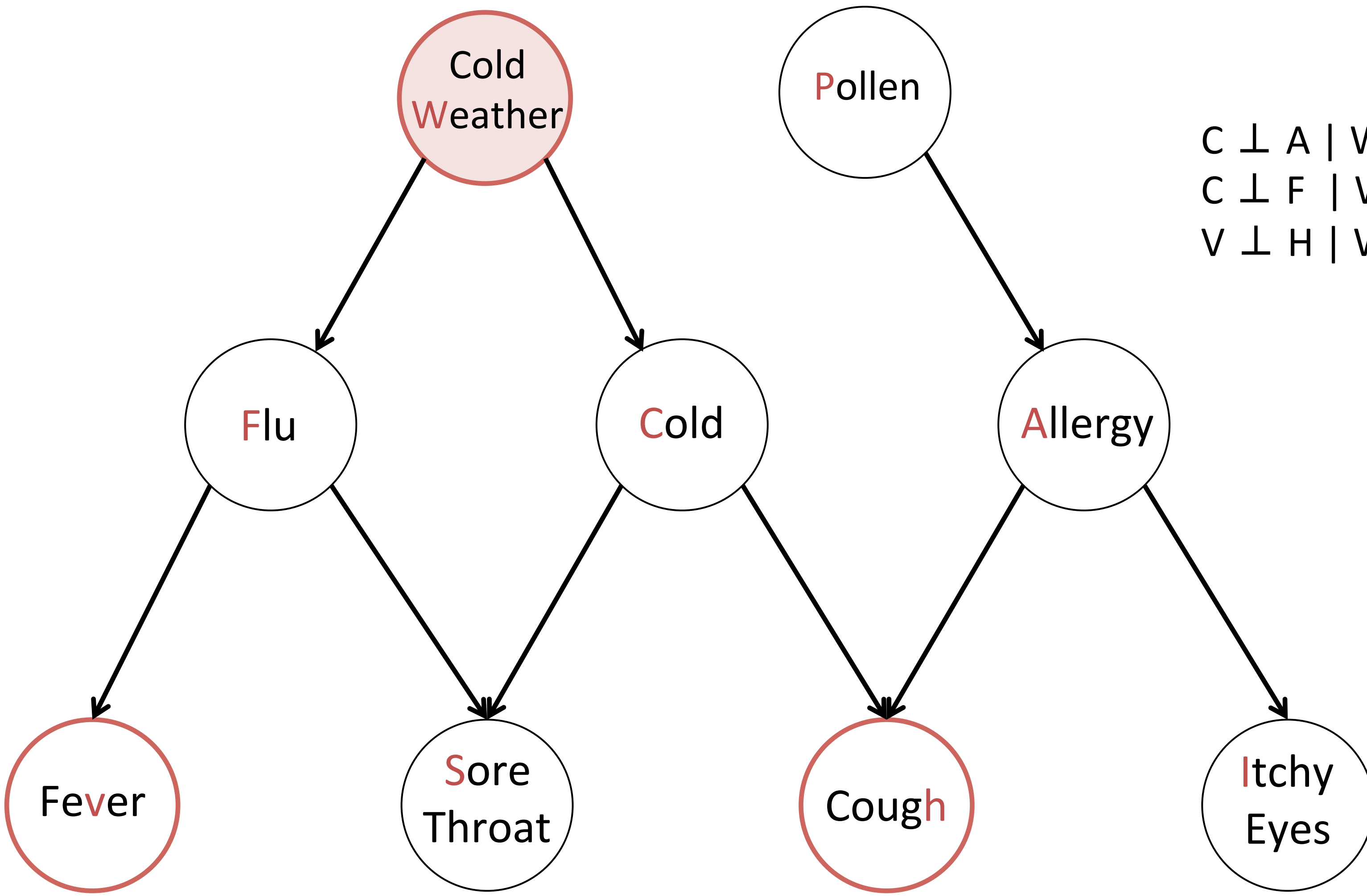


Conditional Independence



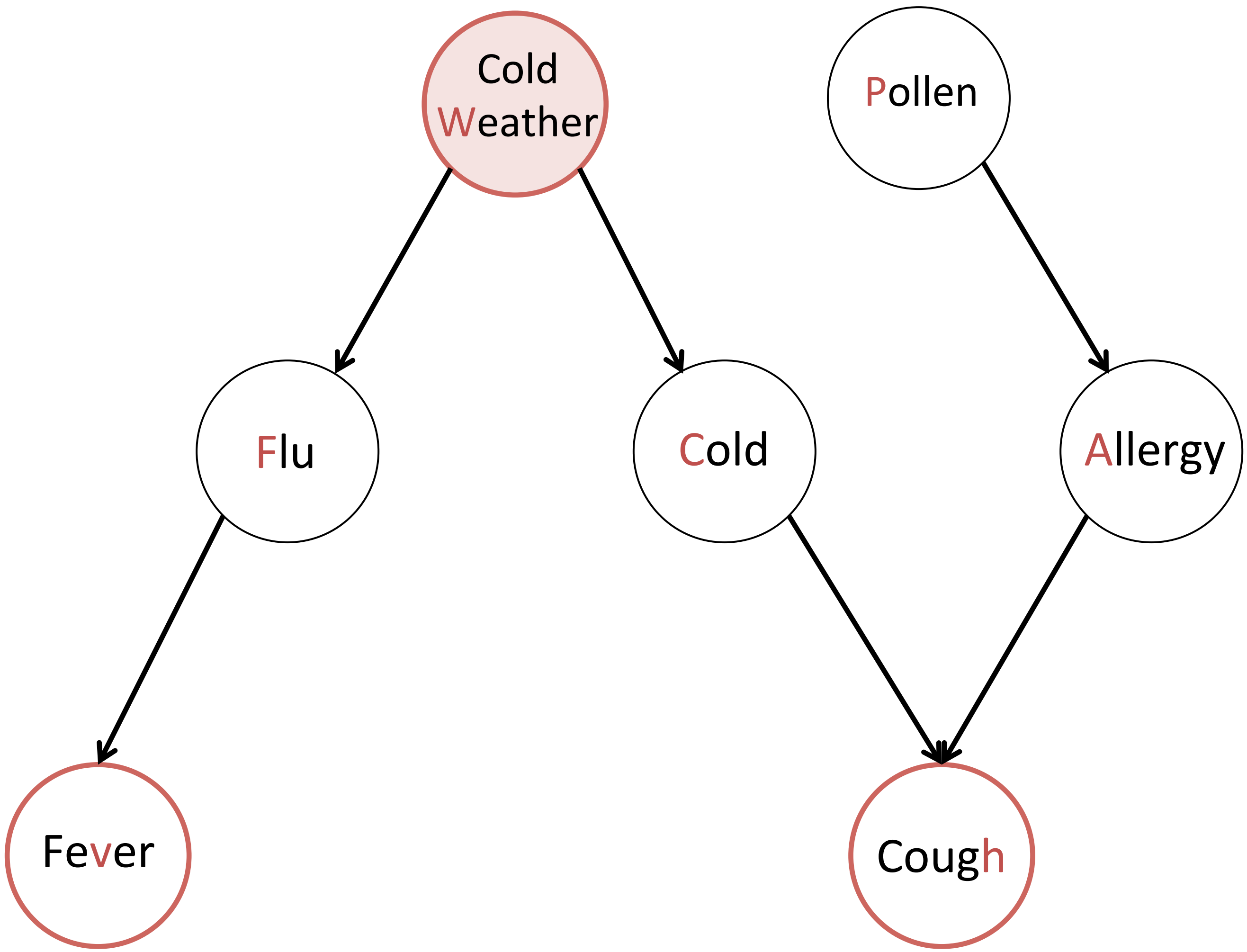
C ⊥ A | W? Yes
C ⊥ F | W? Yes

Conditional Independence



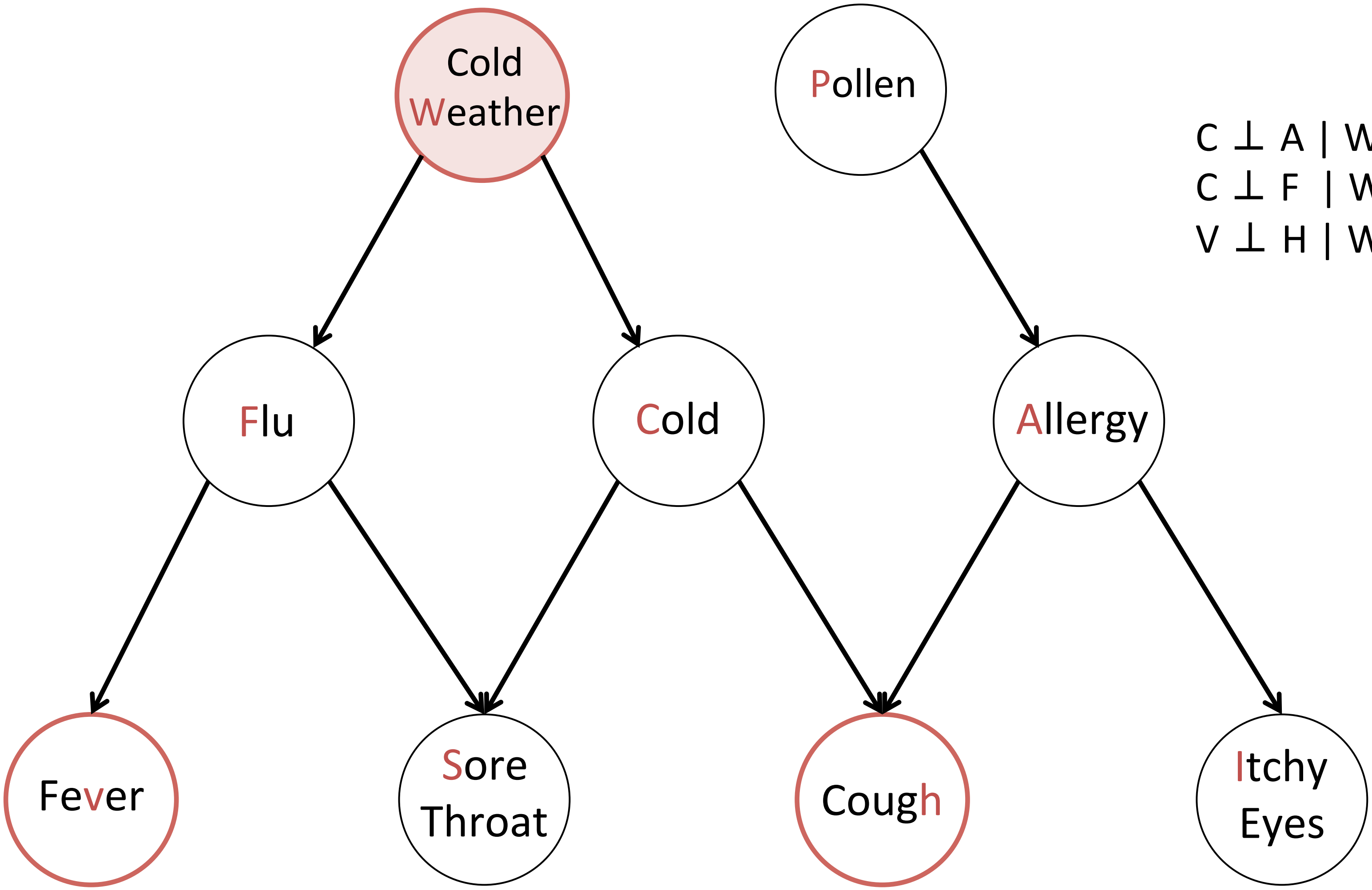
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$?

Conditional Independence



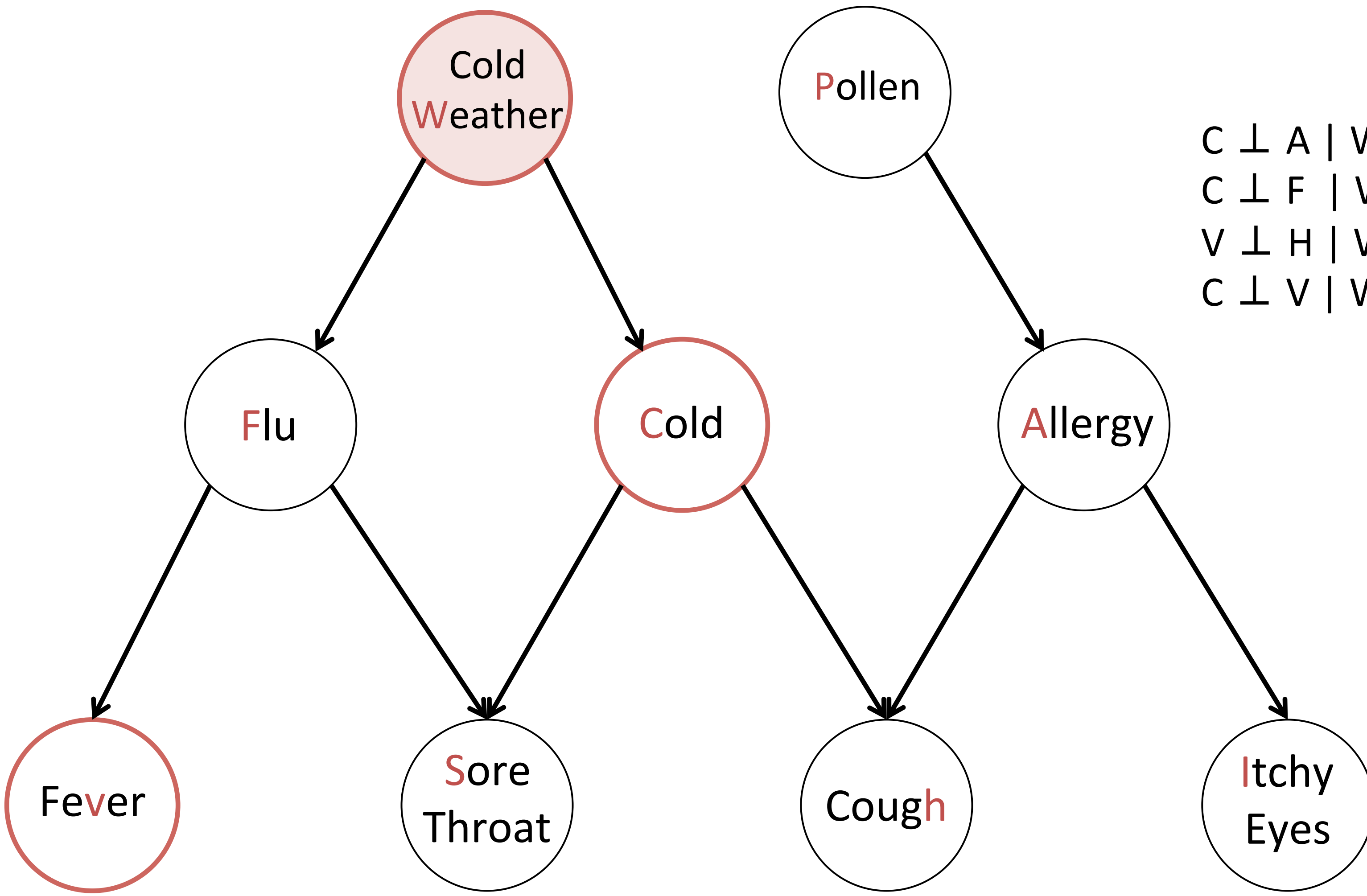
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$?

Conditional Independence



$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes

Conditional Independence



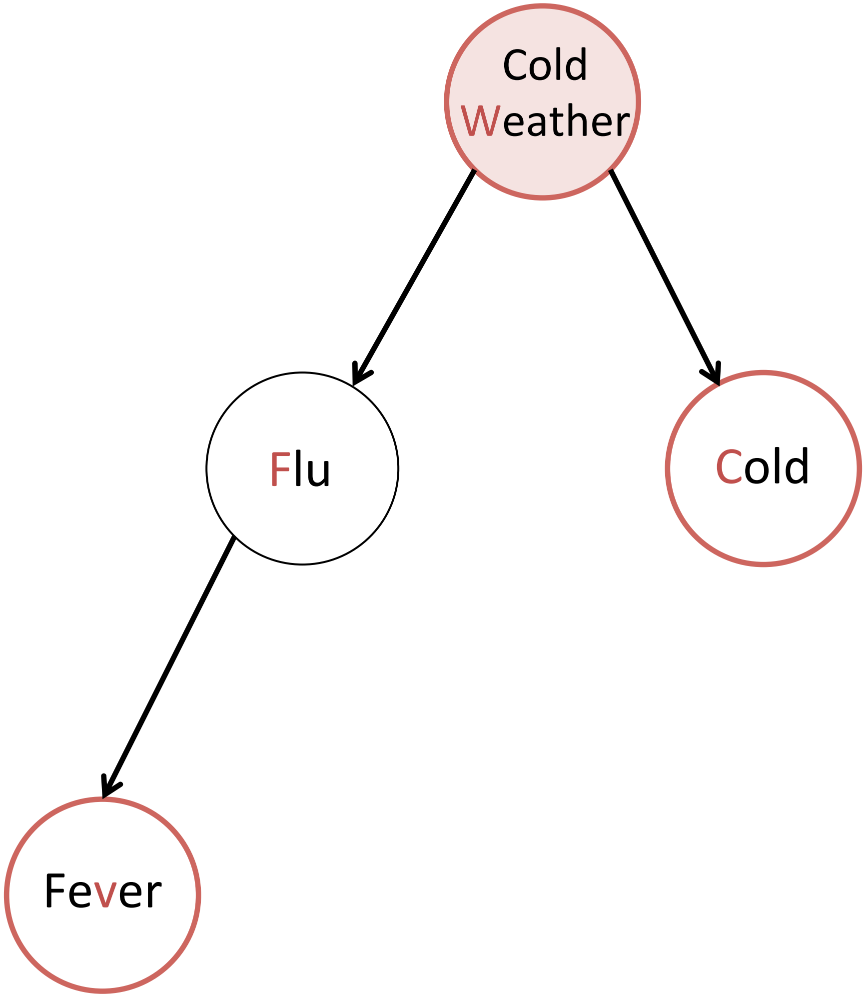
C ⊥ A | W? Yes

C ⊥ F | W? Yes

V ⊥ H | W? Yes

C ⊥ V | W?

Conditional Independence



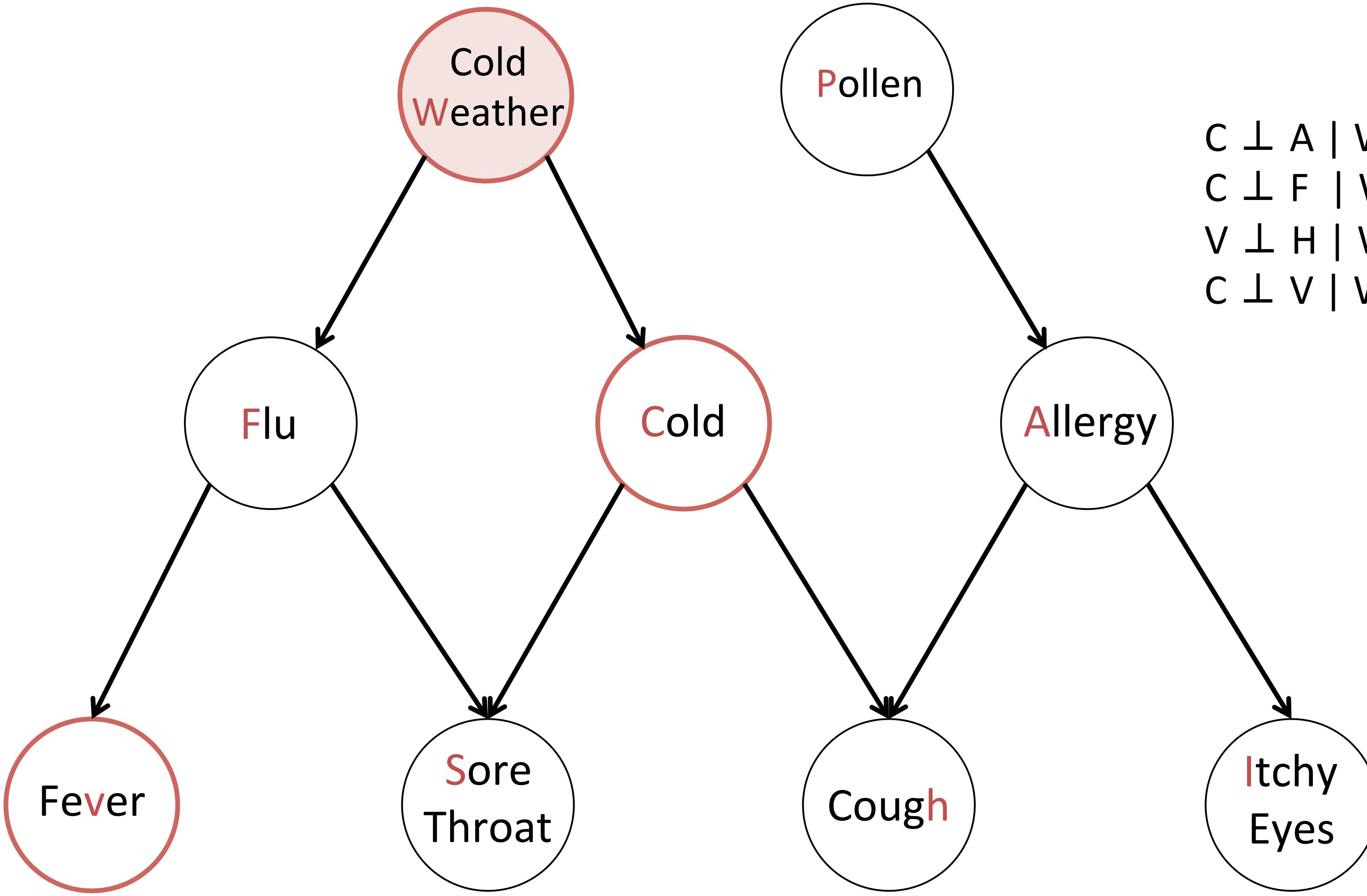
C ⊥ A | W? Yes

C ⊥ F | W? Yes

V ⊥ H | W? Yes

C ⊥ V | W?

Conditional Independence



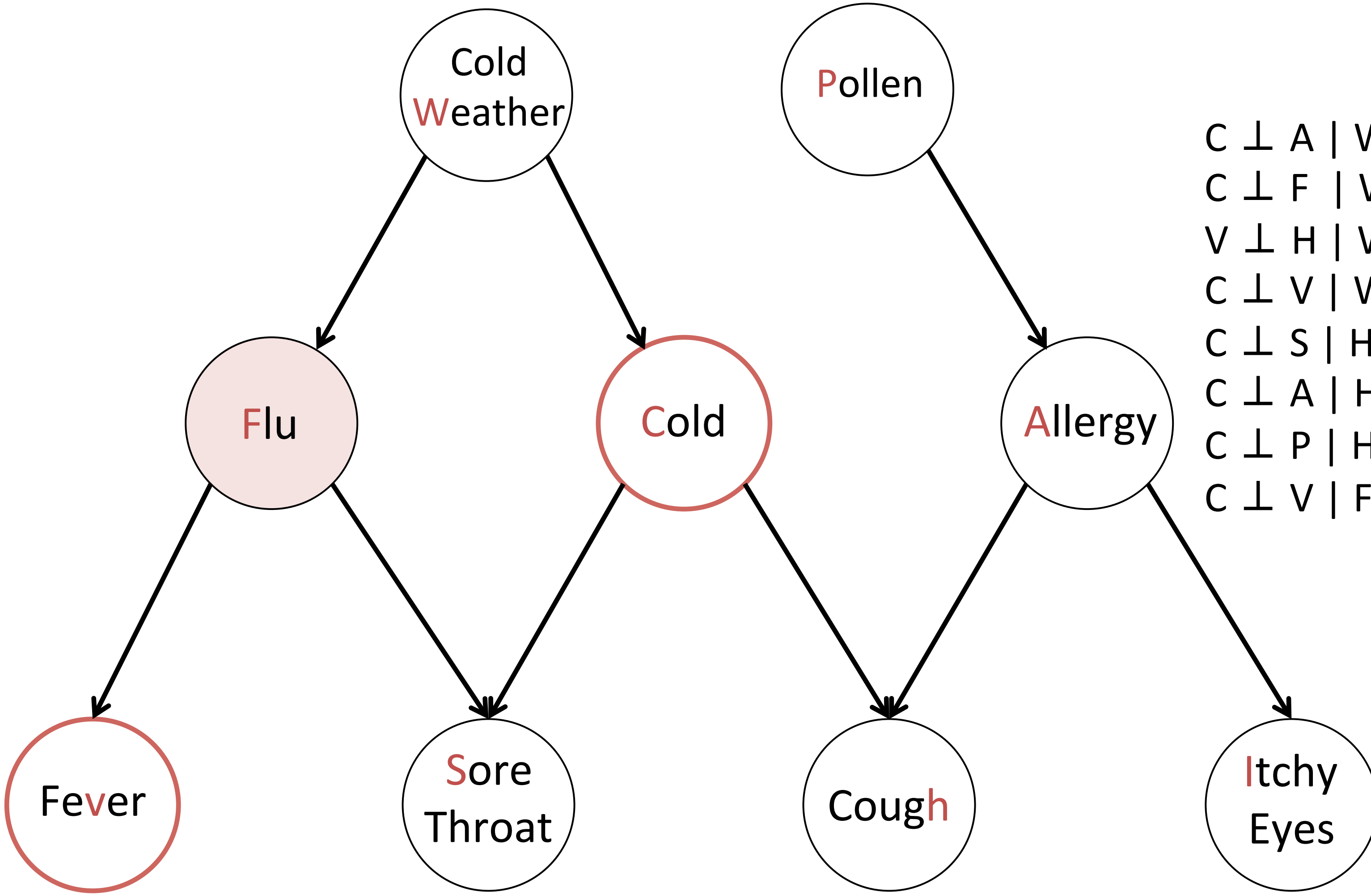
$C \perp A \mid W$? Yes

$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

Conditional Independence



$C \perp A \mid W$? Yes

$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

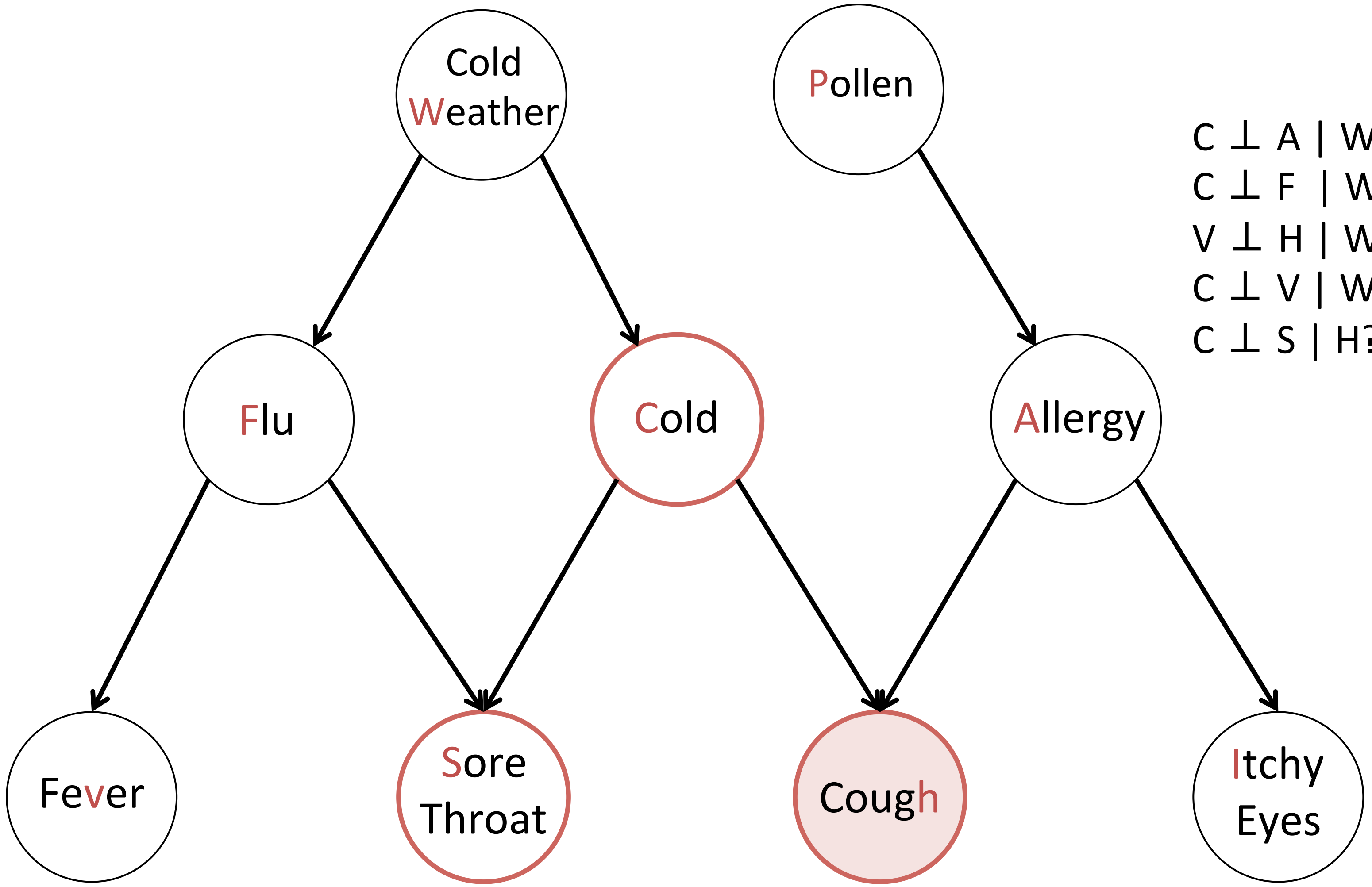
$C \perp S \mid H$?

$C \perp A \mid H$?

$C \perp P \mid H$?

$C \perp V \mid F$?

Conditional Independence



$C \perp A \mid W$? Yes

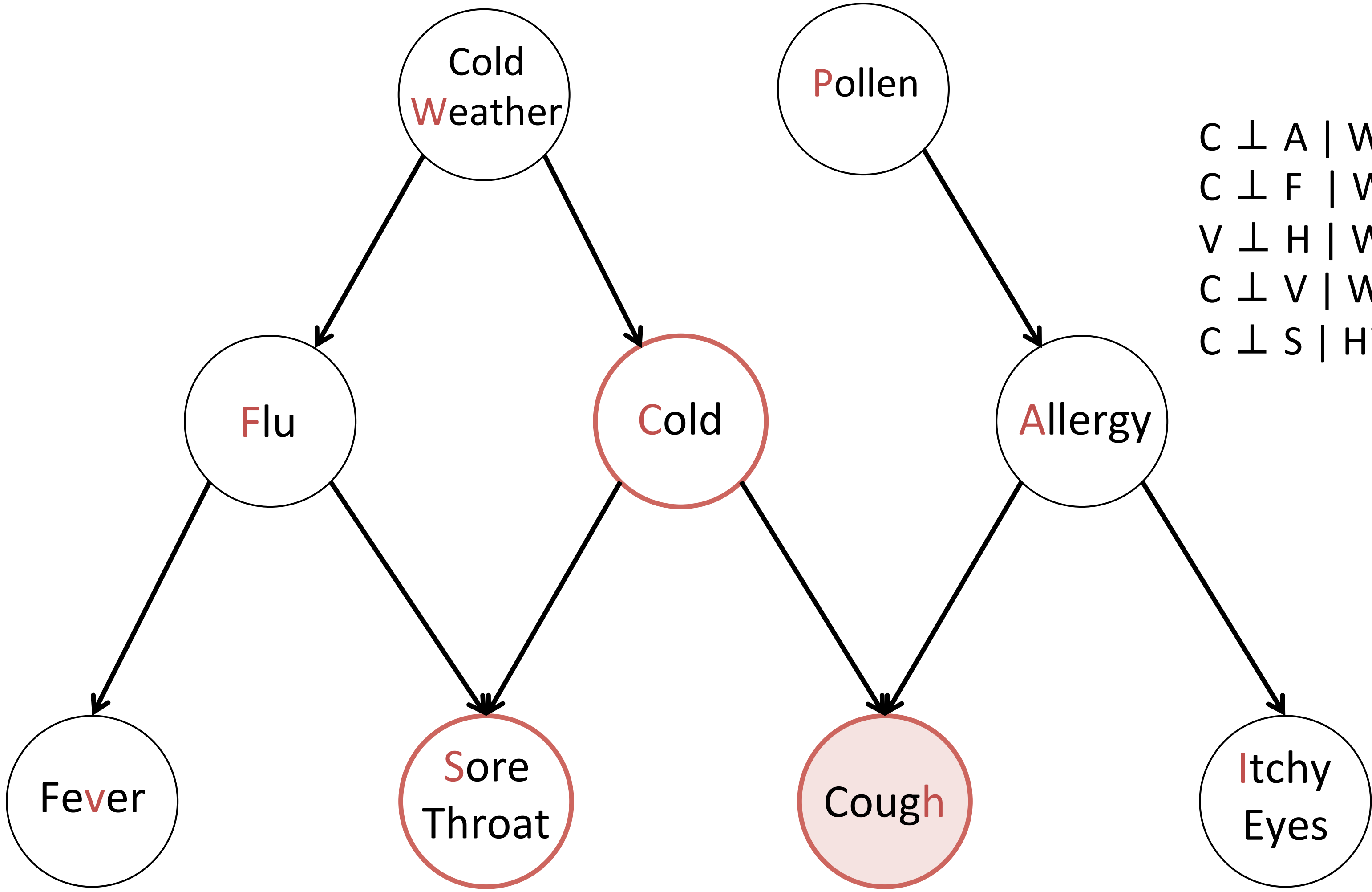
$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

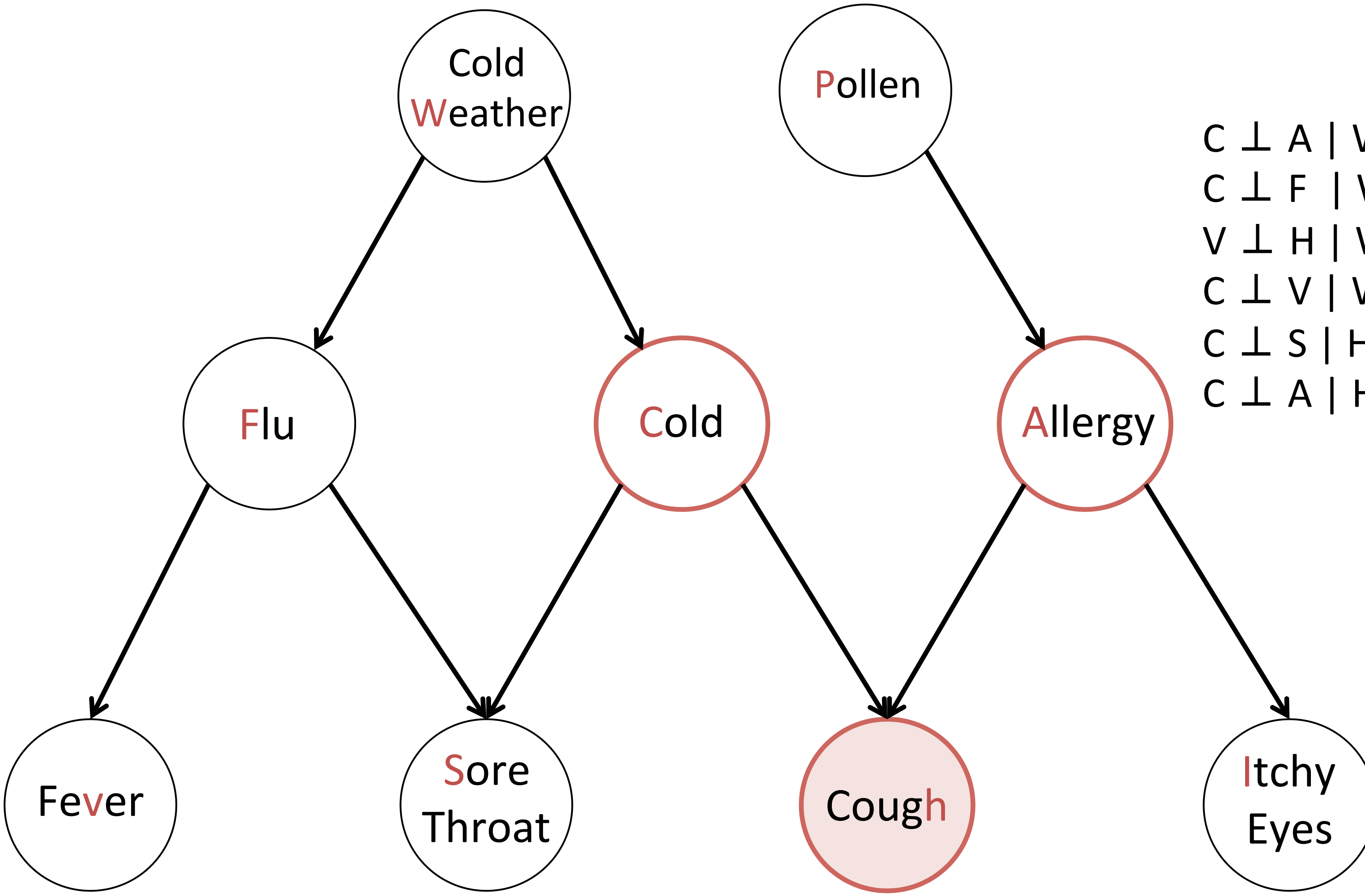
$C \perp S \mid H$?

Conditional Independence



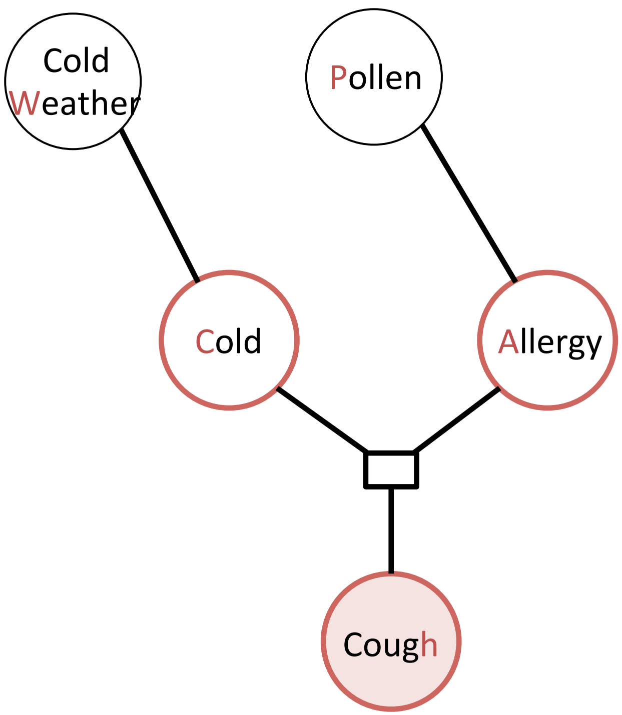
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No

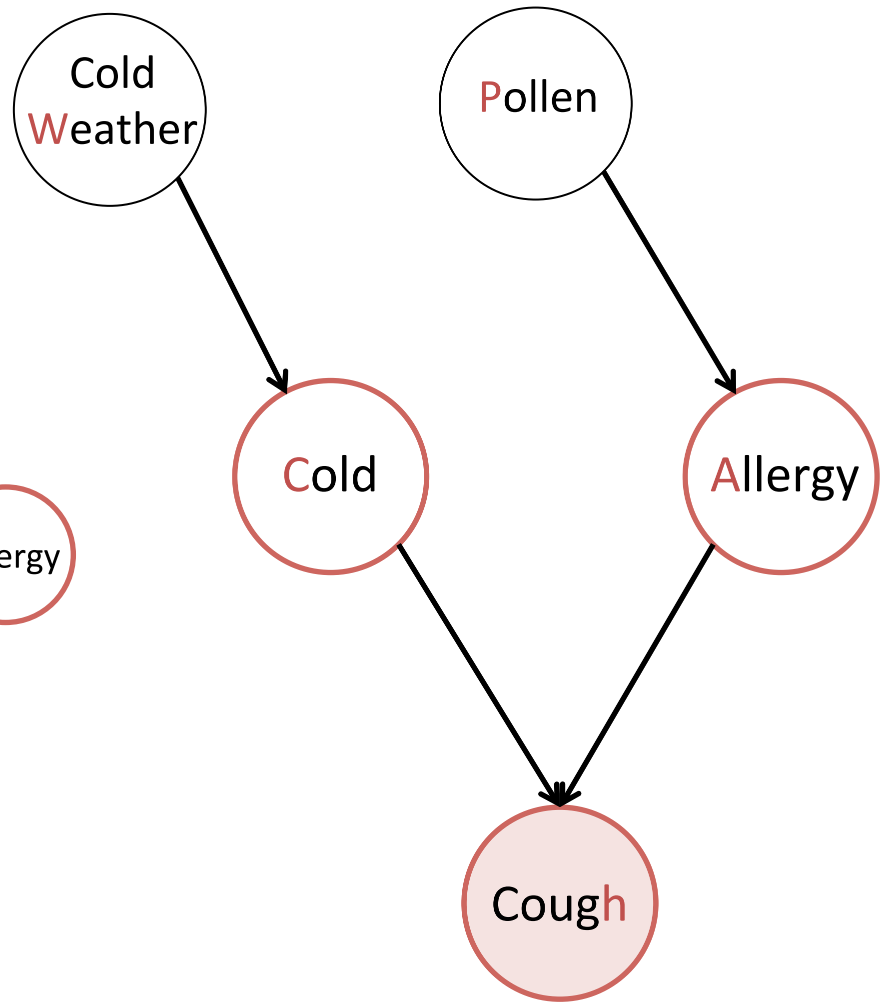
Conditional Independence



C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H?

Conditional Independence





$C \perp A \mid W$? Yes

$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

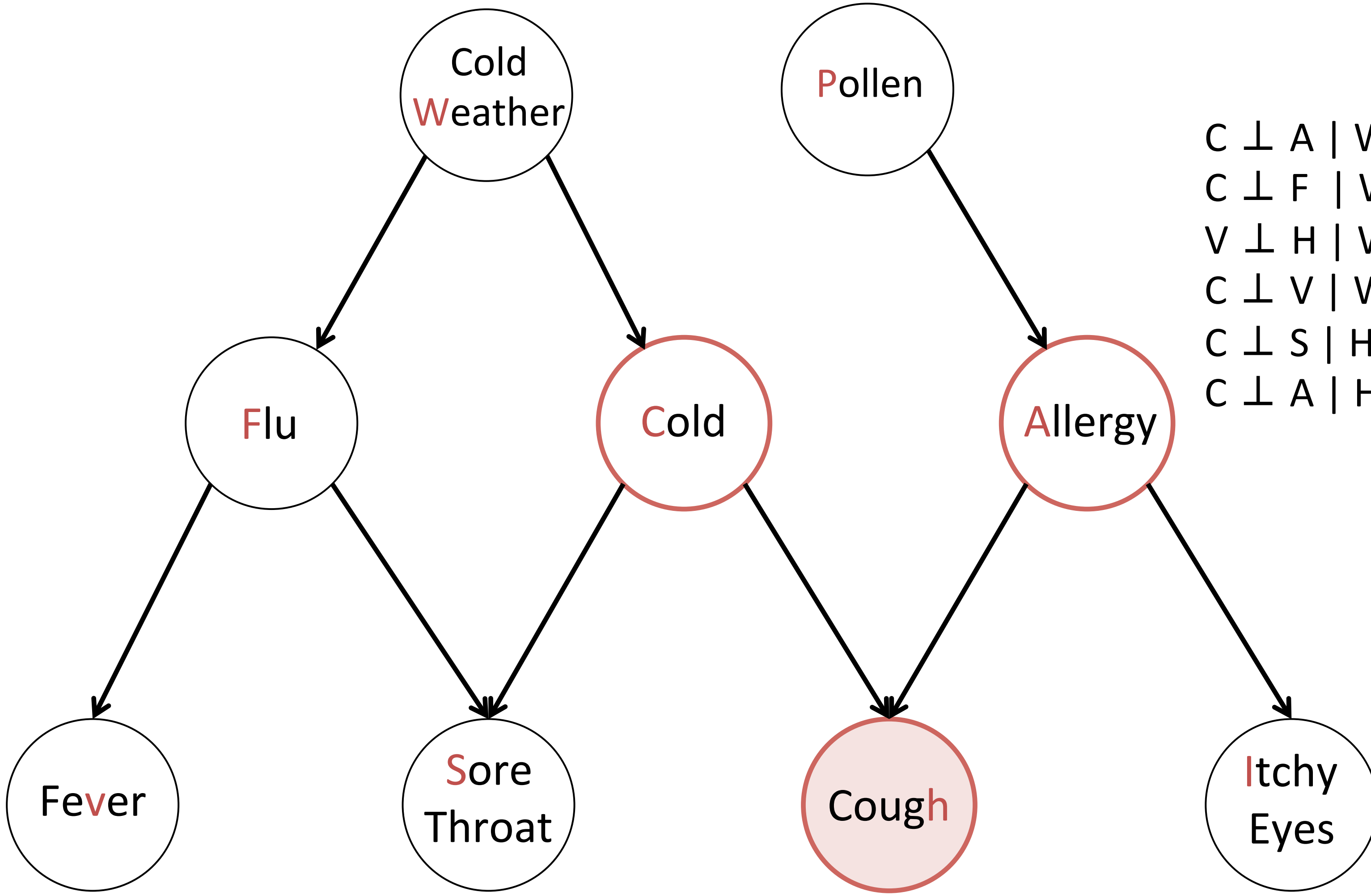
$C \perp V \mid W$? Yes

$C \perp S \mid H$? No

$C \perp A \mid H$?

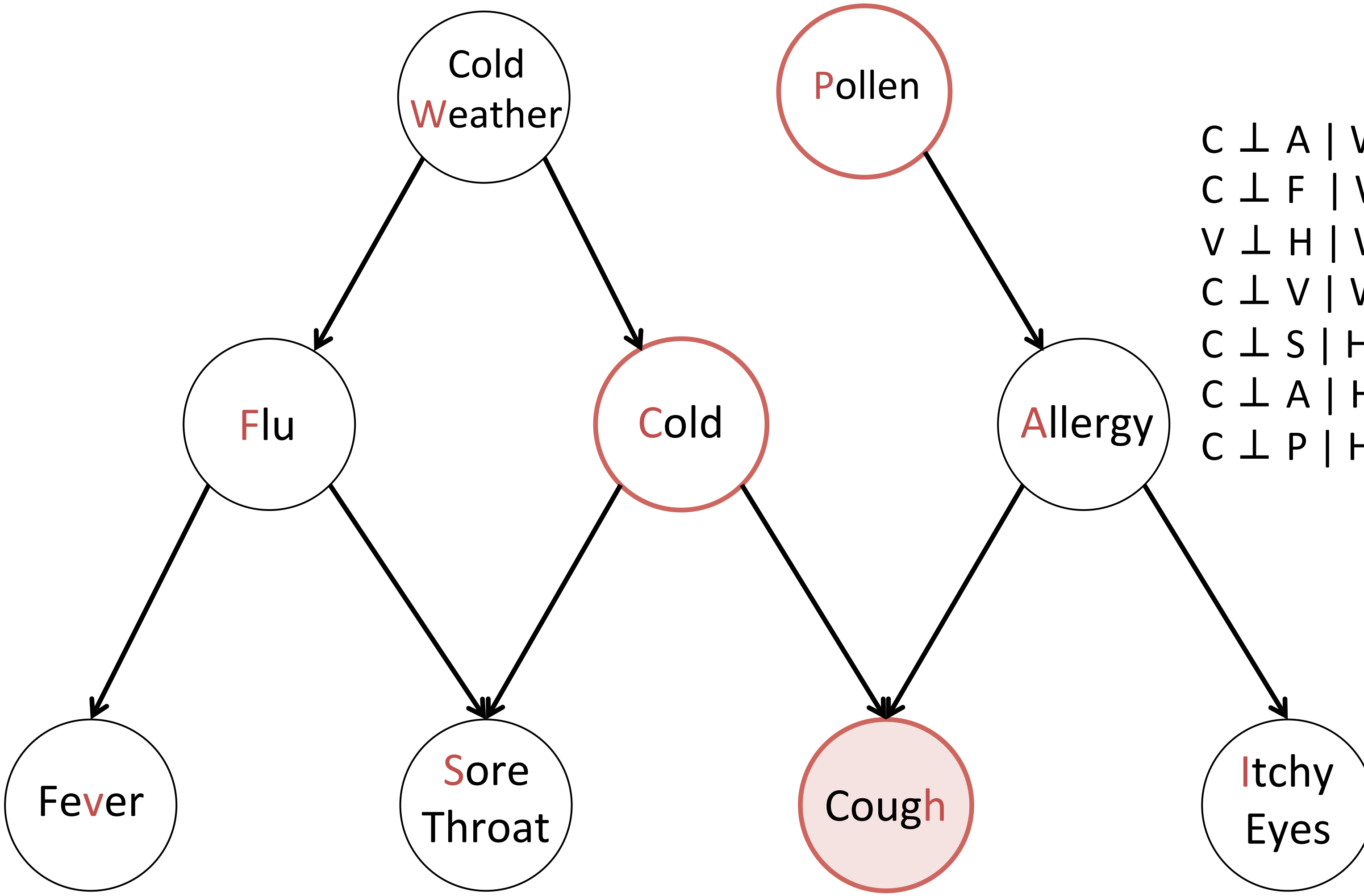
Explaining Away!

Conditional Independence



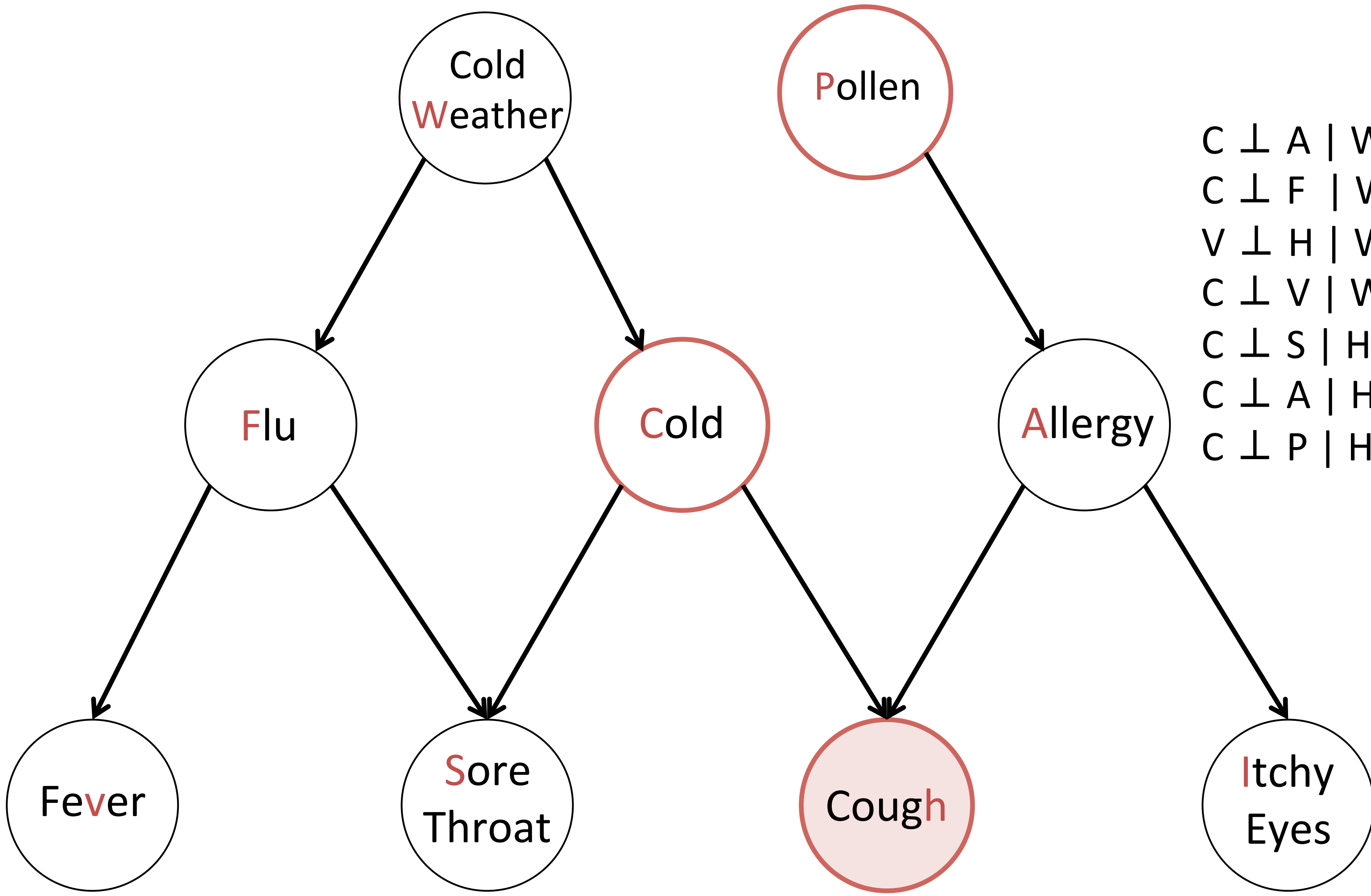
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No

Conditional Independence



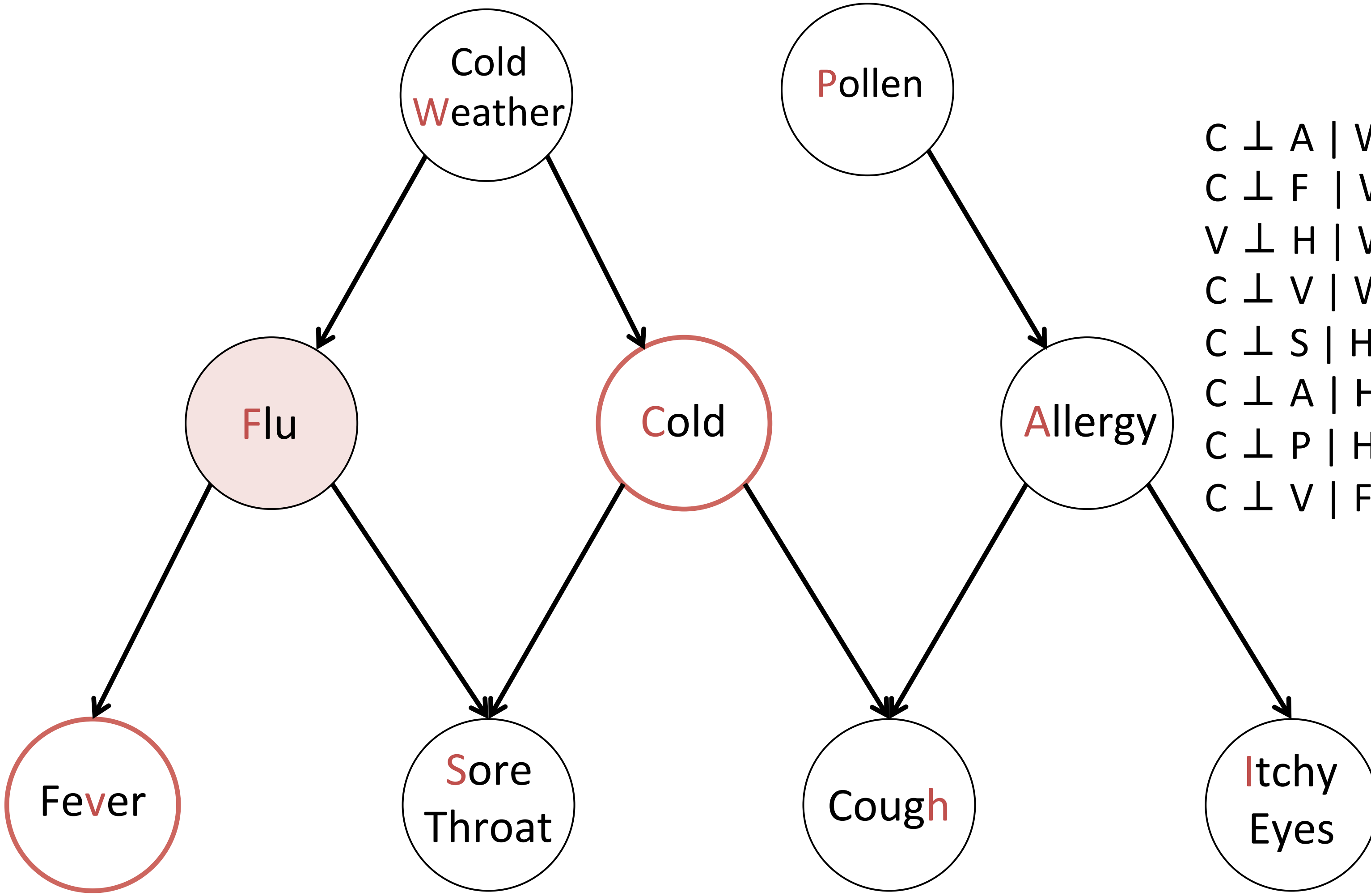
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$?

Conditional Independence



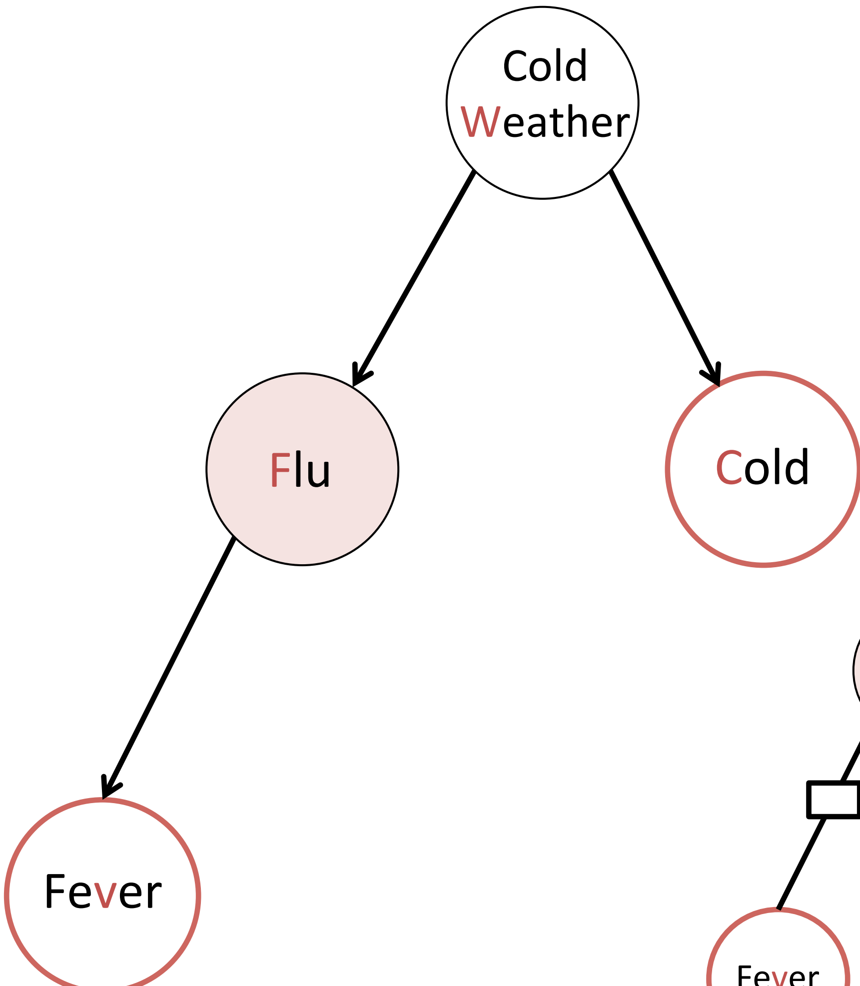
$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No

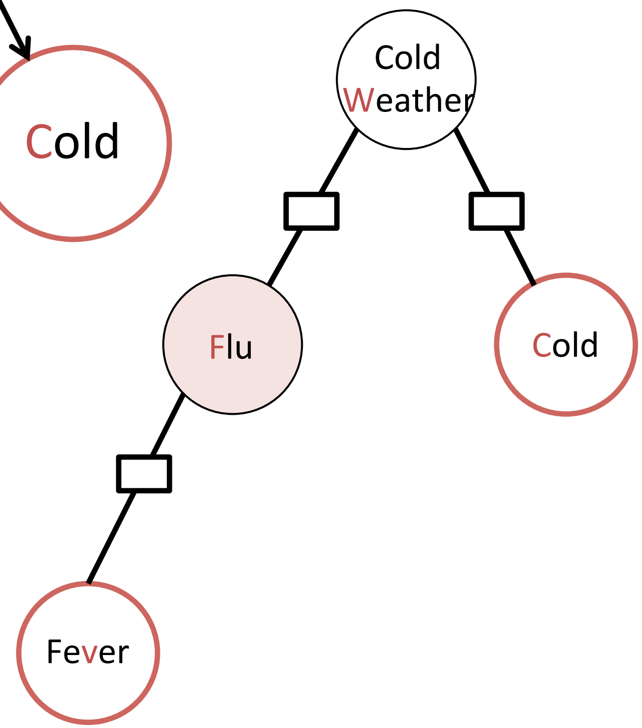
Conditional Independence



C ⊥ A | W? Yes
C ⊥ F | W? Yes
V ⊥ H | W? Yes
C ⊥ V | W? Yes
C ⊥ S | H? No
C ⊥ A | H? No
C ⊥ P | H? No
C ⊥ V | F?

Conditional Independence





$C \perp A \mid W$? Yes

$C \perp F \mid W$? Yes

$V \perp H \mid W$? Yes

$C \perp V \mid W$? Yes

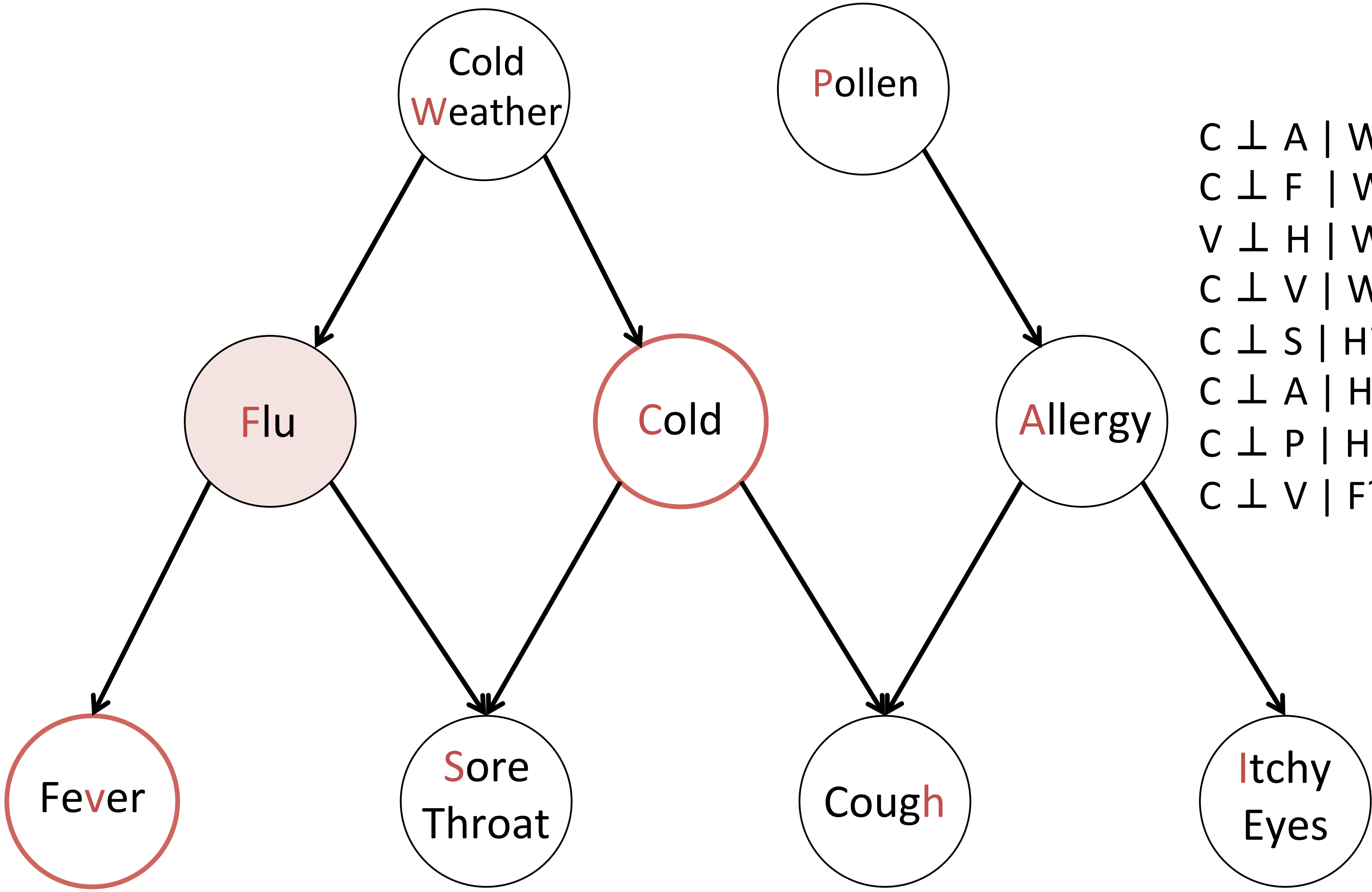
$C \perp S \mid H$? No

$C \perp A \mid H$? No

$C \perp P \mid H$? No

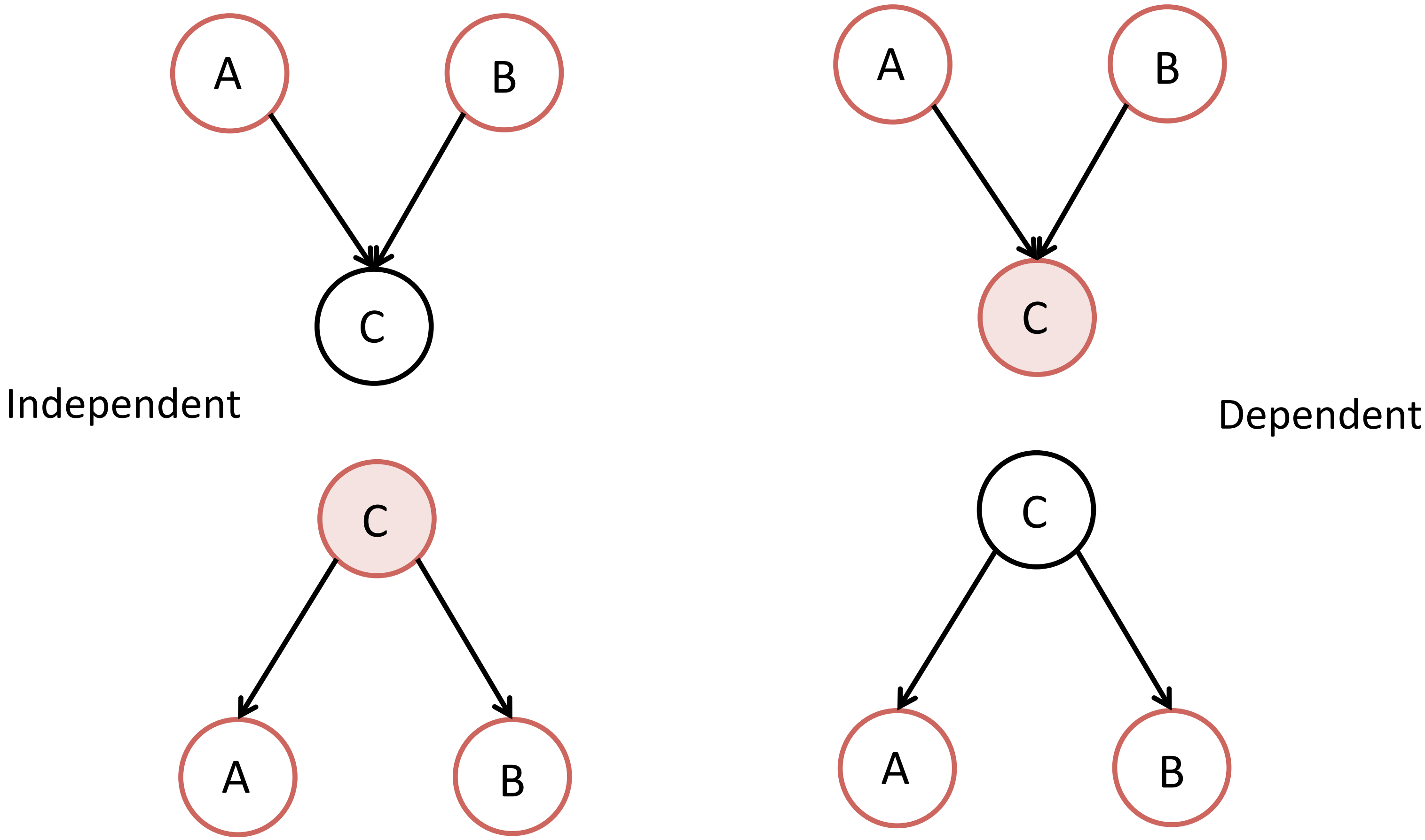
$C \perp V \mid F$?

Conditional Independence

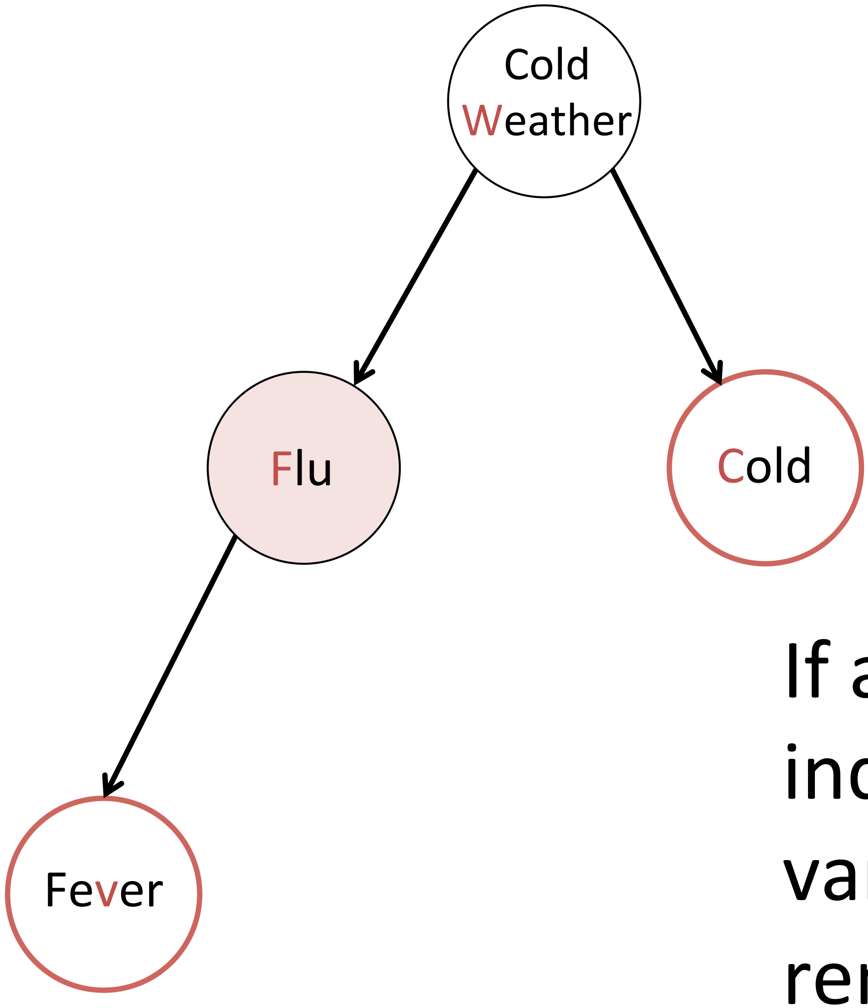


$C \perp A \mid W$? Yes
$C \perp F \mid W$? Yes
$V \perp H \mid W$? Yes
$C \perp V \mid W$? Yes
$C \perp S \mid H$? No
$C \perp A \mid H$? No
$C \perp P \mid H$? No
$C \perp V \mid F$? Yes

Patterns



Conditional Independence



$P(C=c|F=f) = ?$

If a variable (Fever) is independent of the Query variable Q (Cold), we can remove(marginalize) it.

Questions?