# CS221 Section 3: Search

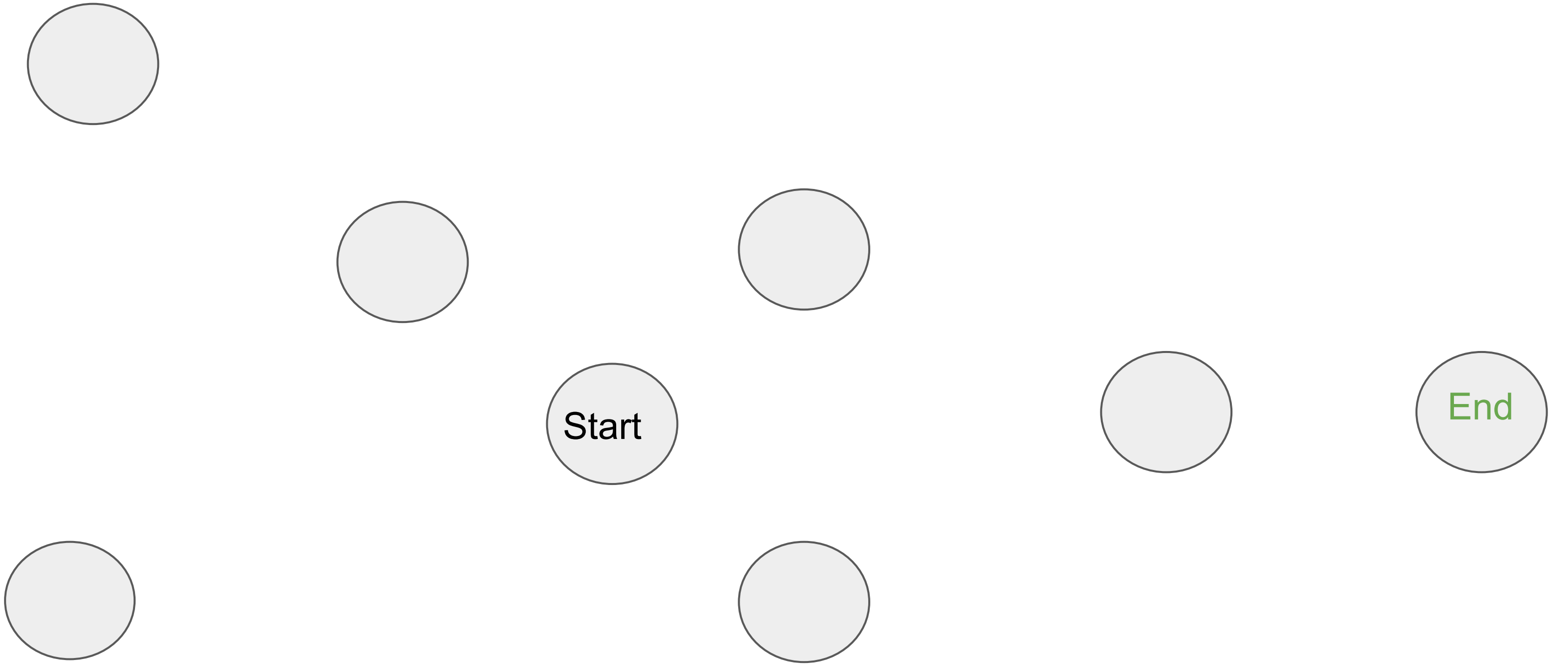
Intuition and Examples

Vig & Rickard

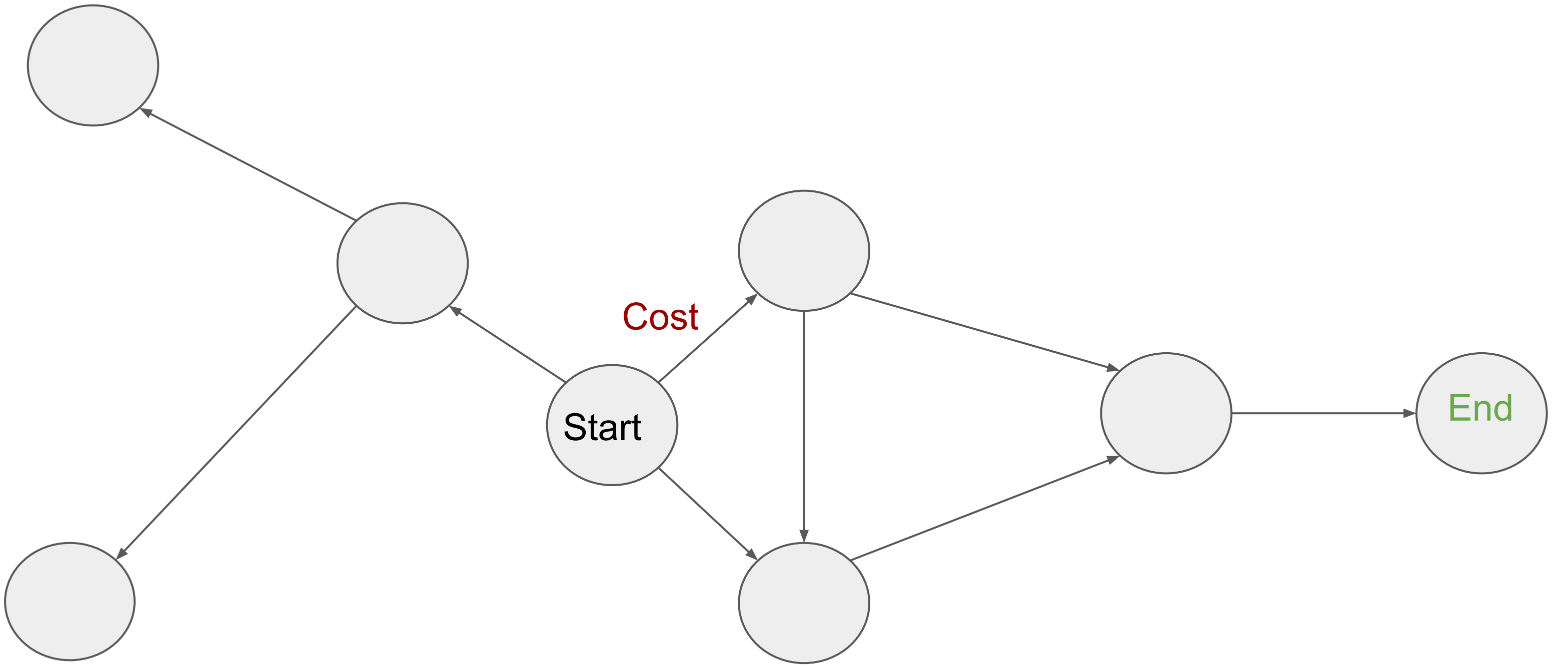
# Search


How to find shortest path
to my destination?
HOME
DESTINATION

# Search



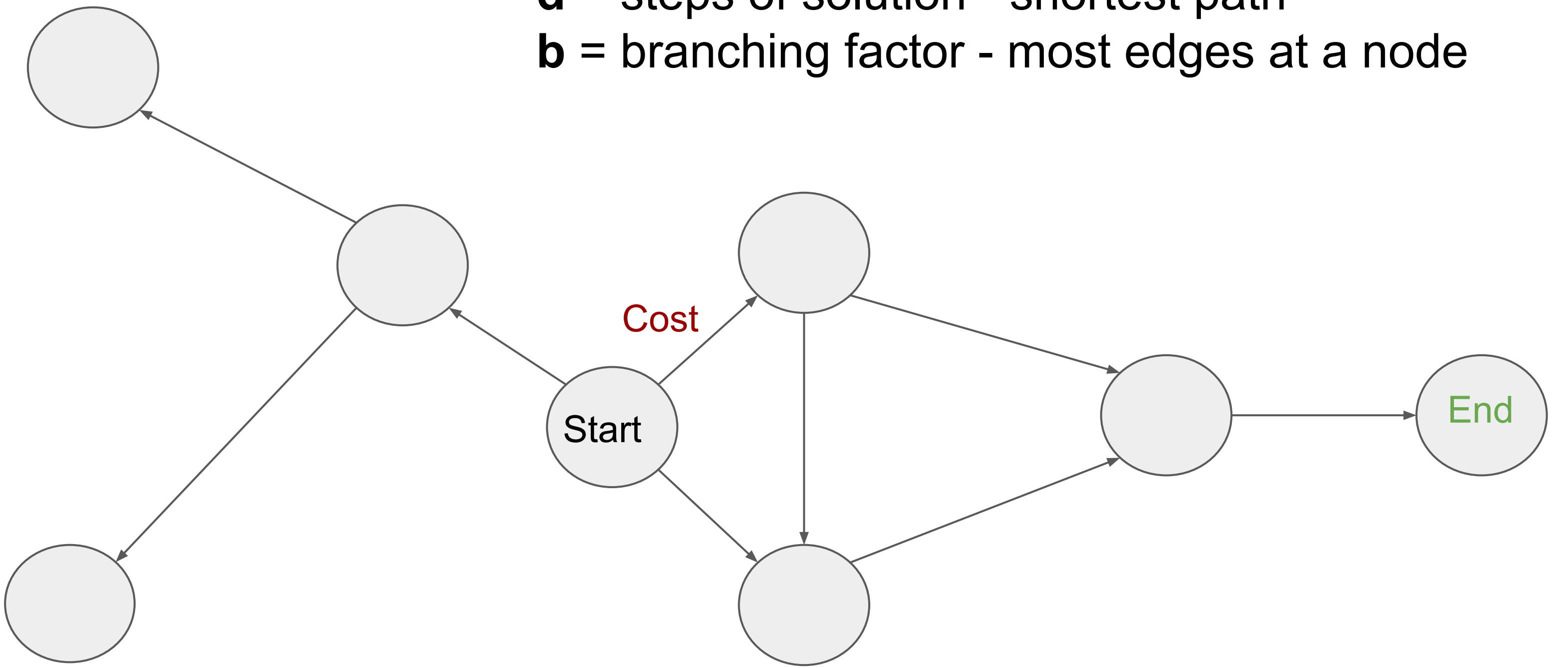
# Search!



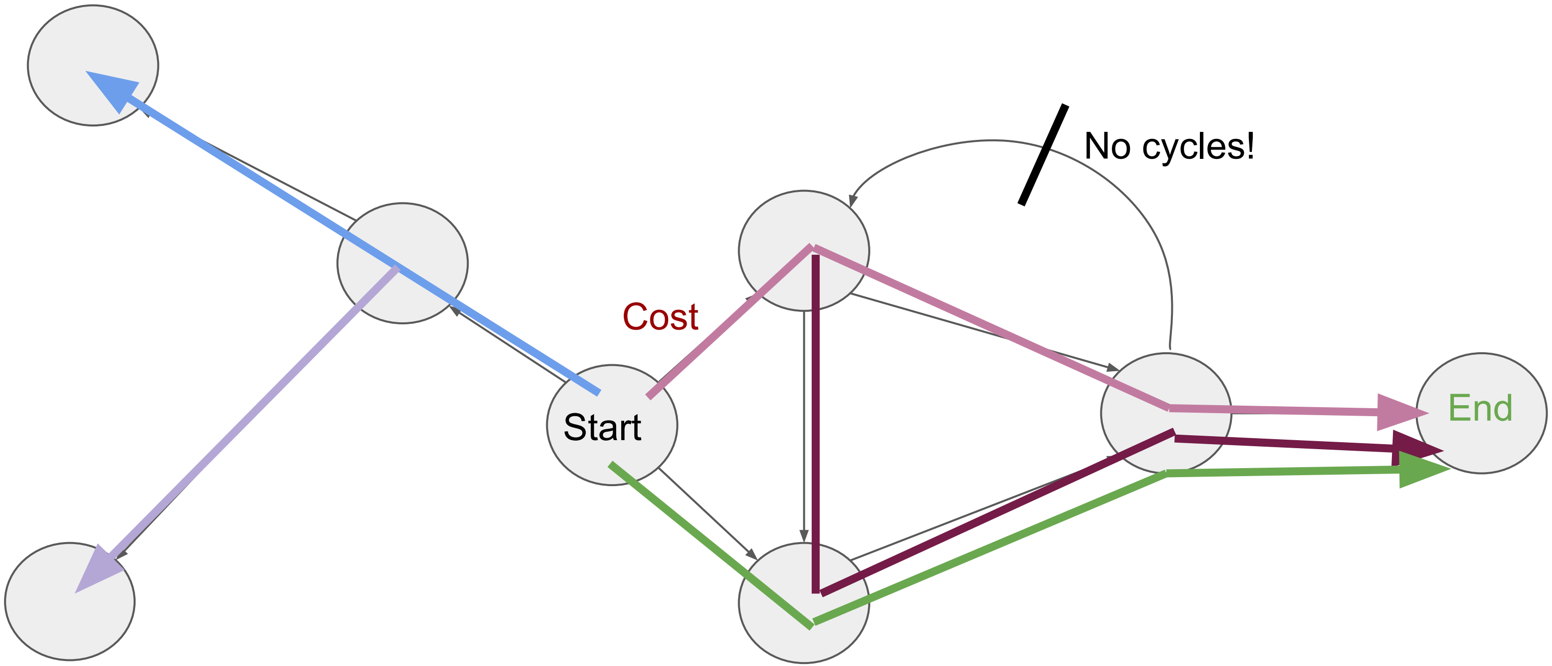
**D** = maximum depth - longest path

**d** = steps of solution - shortest path

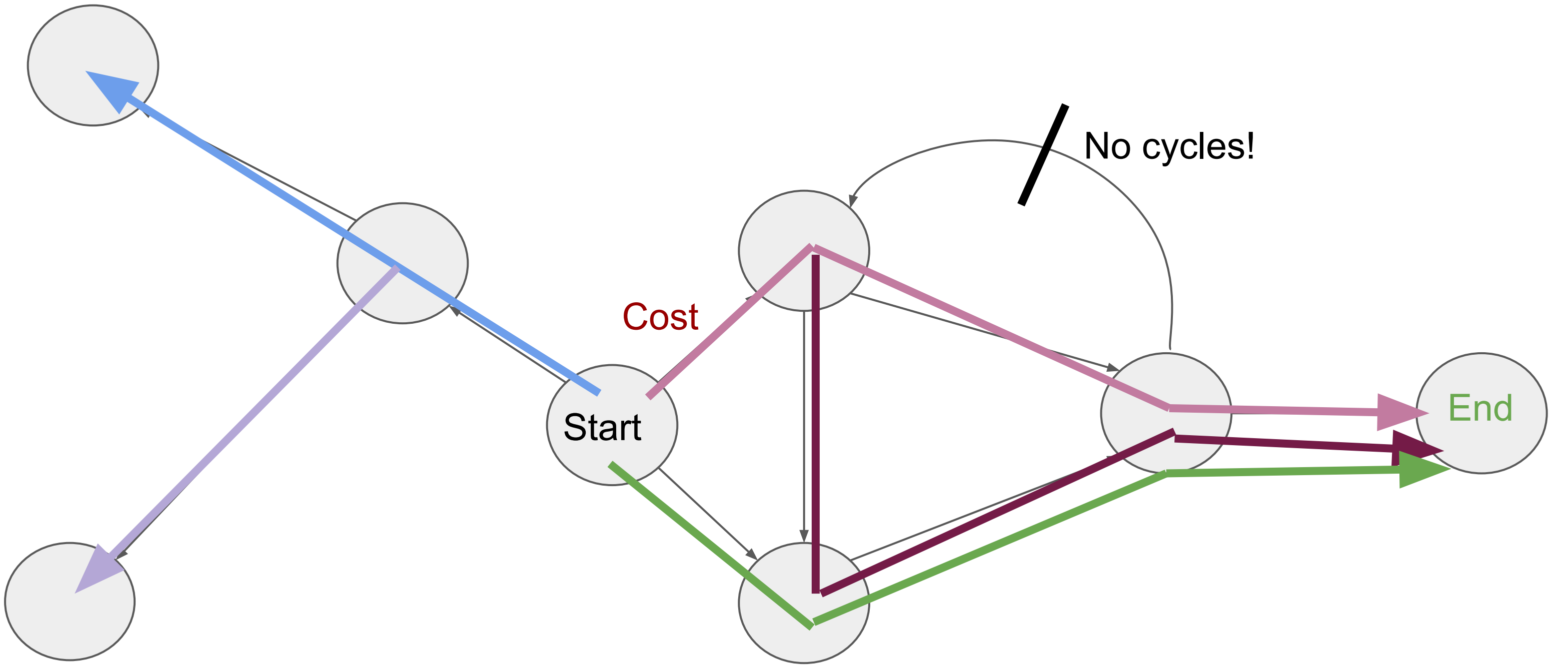
**b** = branching factor - most edges at a node



# Just try them all! Backtracking



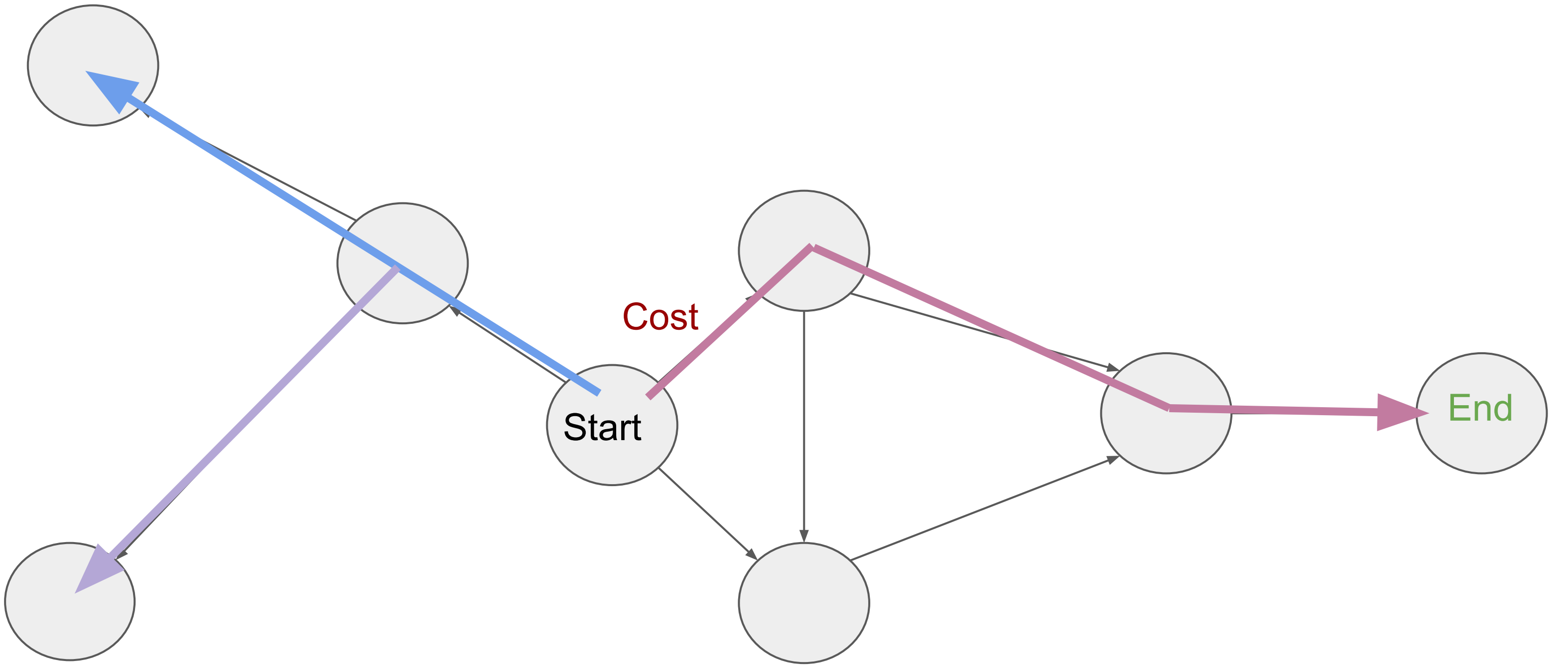
# Just try them all! Backtracking



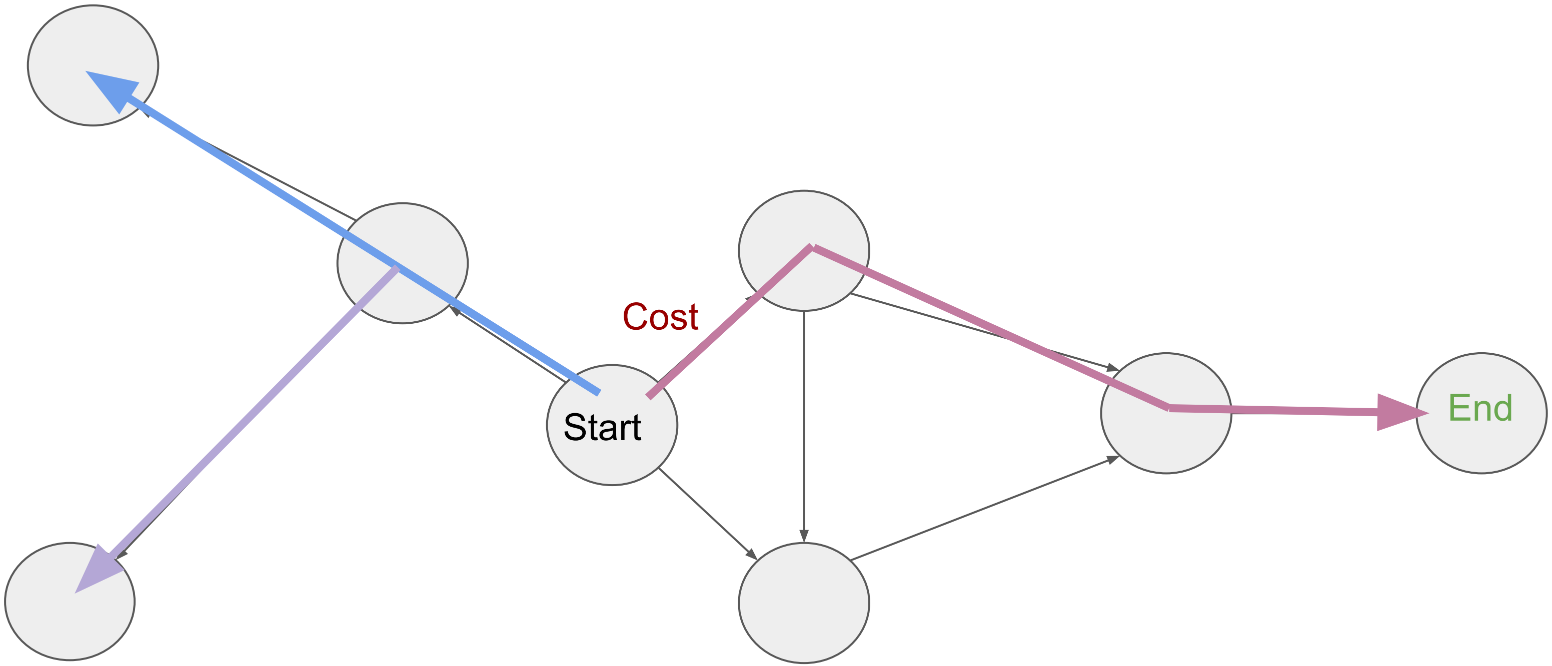

Expensive: O(b^D)

Cheap in space: O(D)

# Try them all until you find one! DFS

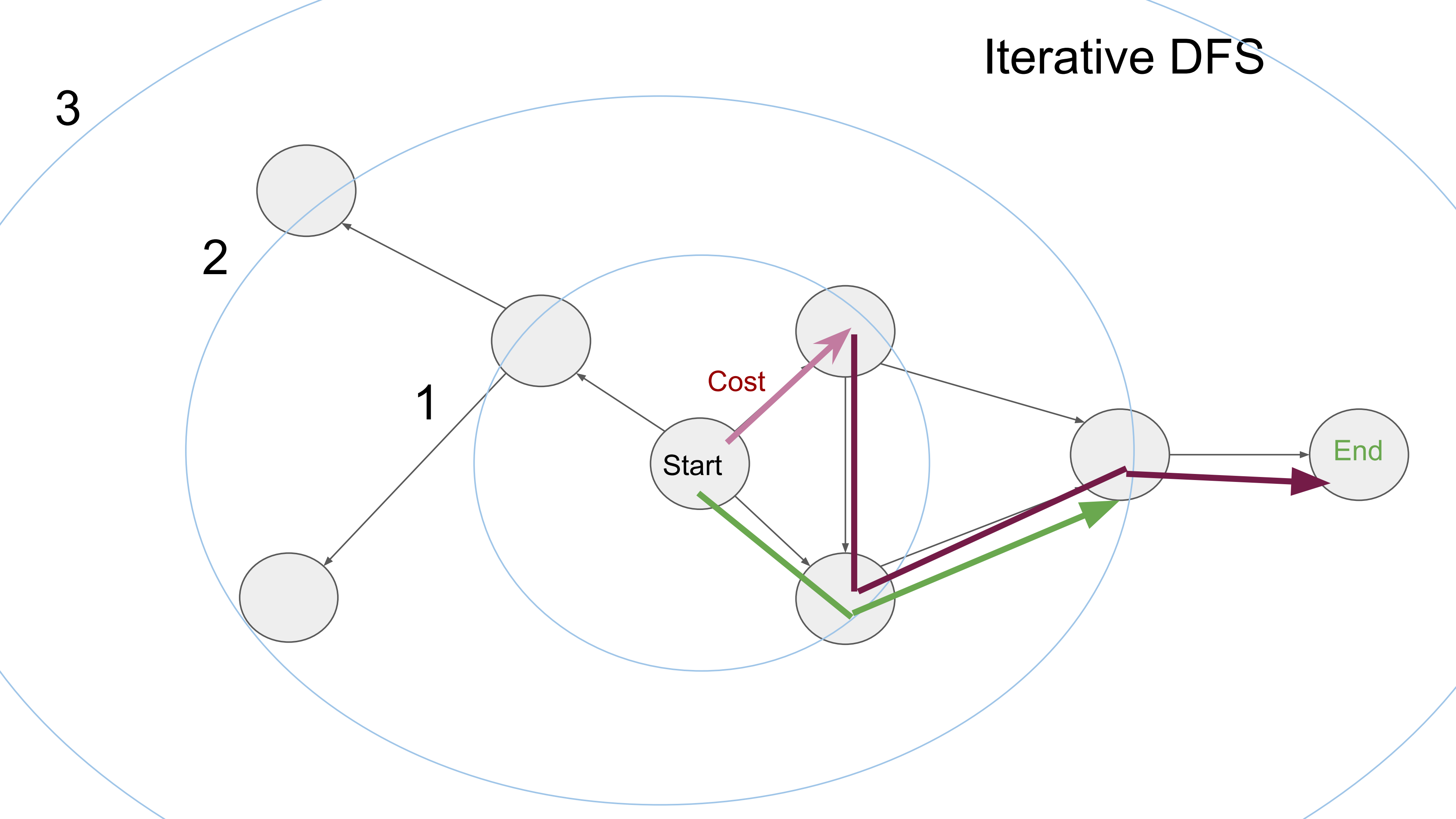


## Try them all until you find one! DFS

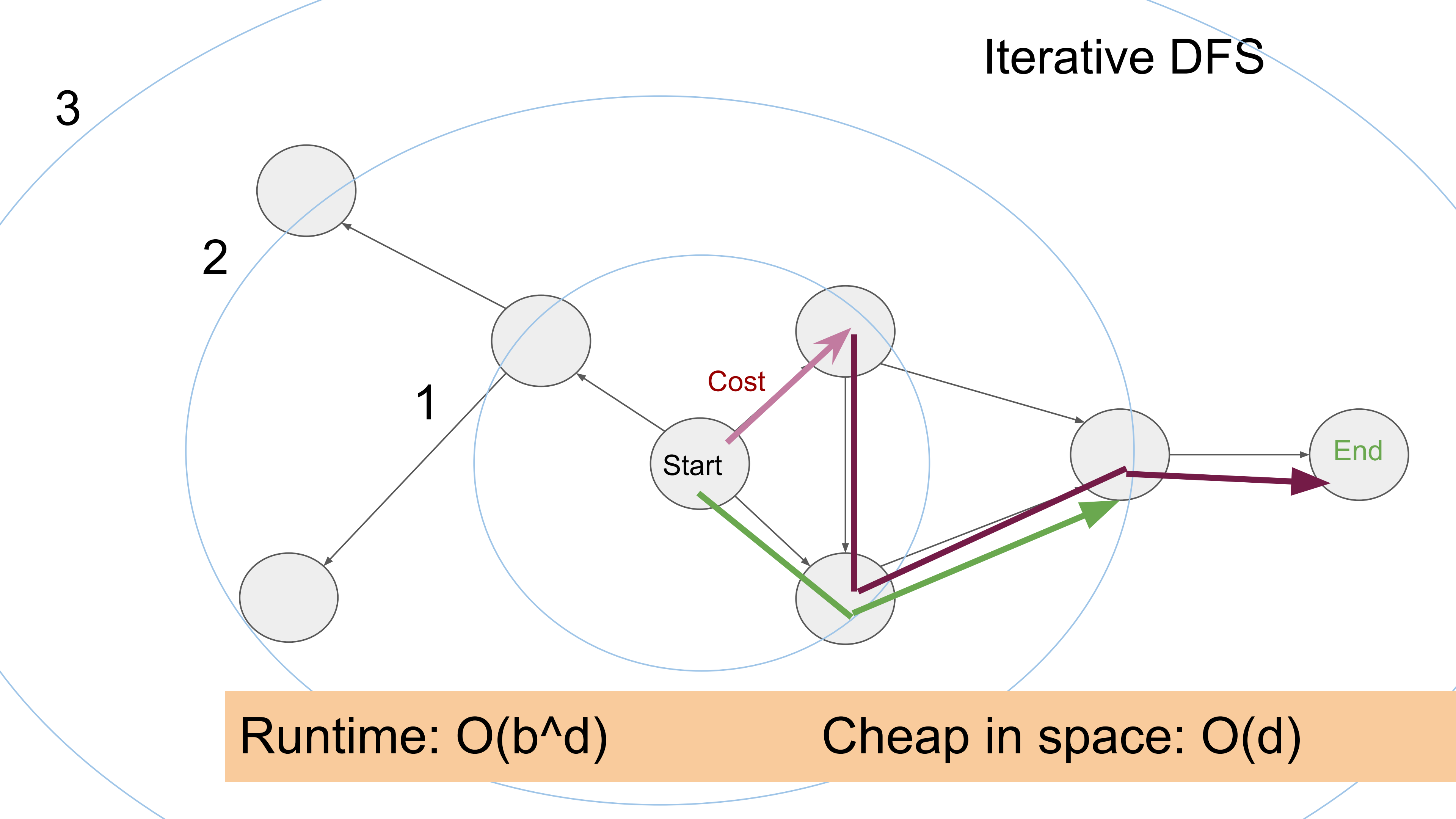



Expensive: O(b^D)    Cheap in space: O(D)

# Iterative DFS



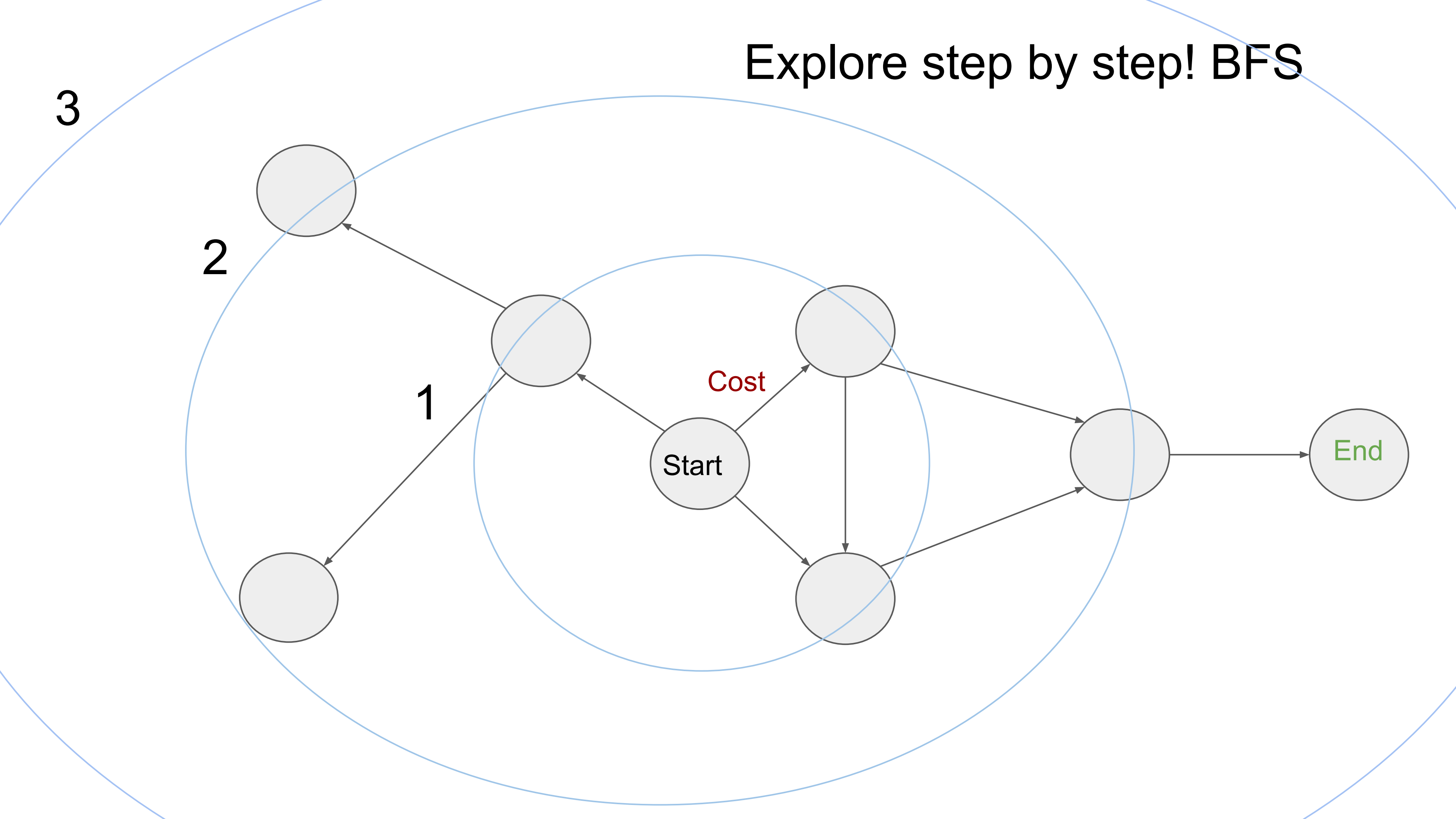
# Iterative DFS



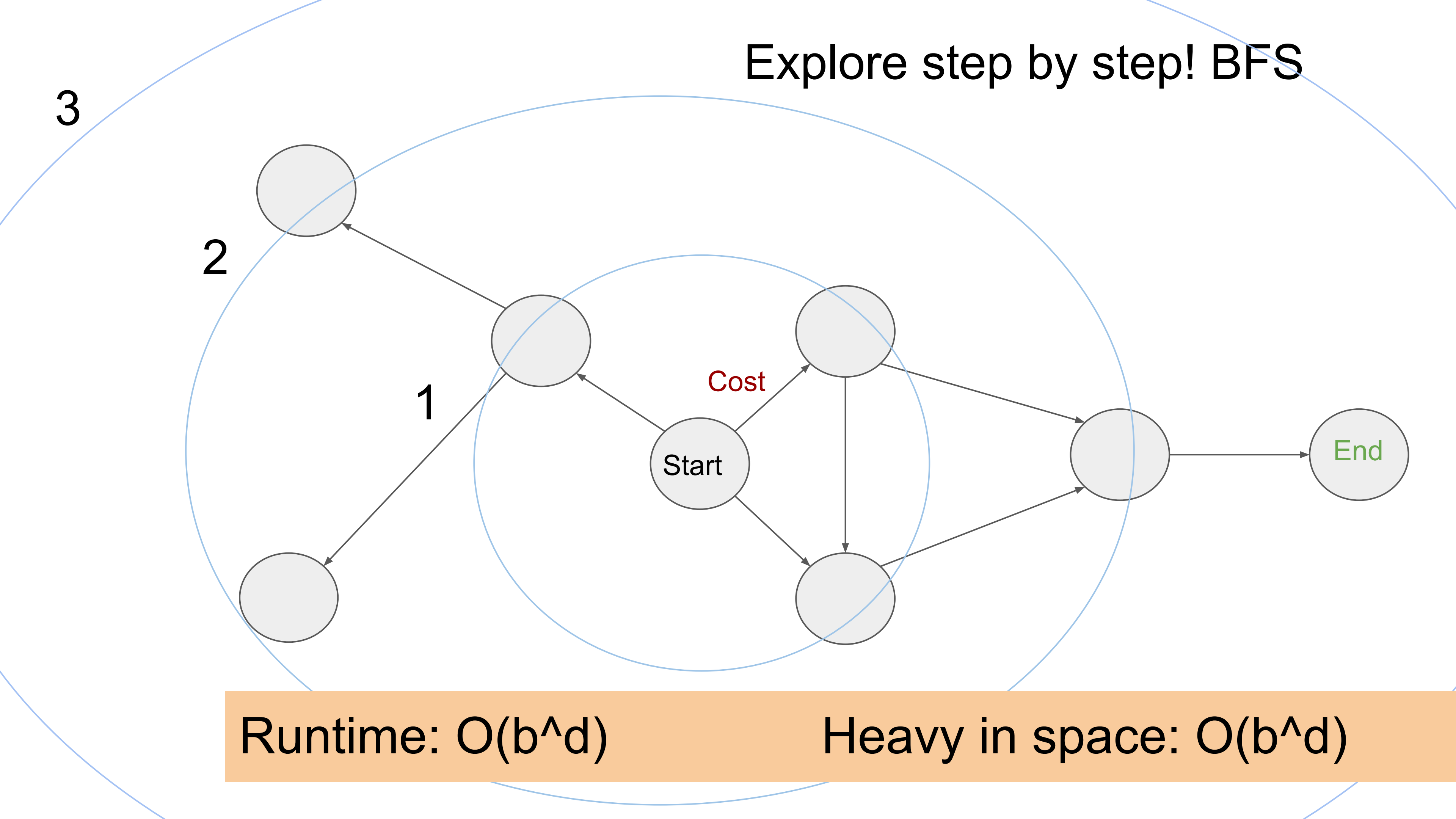

Runtime: O(b^d)

Cheap in space: O(d)

# Explore step by step! BFS



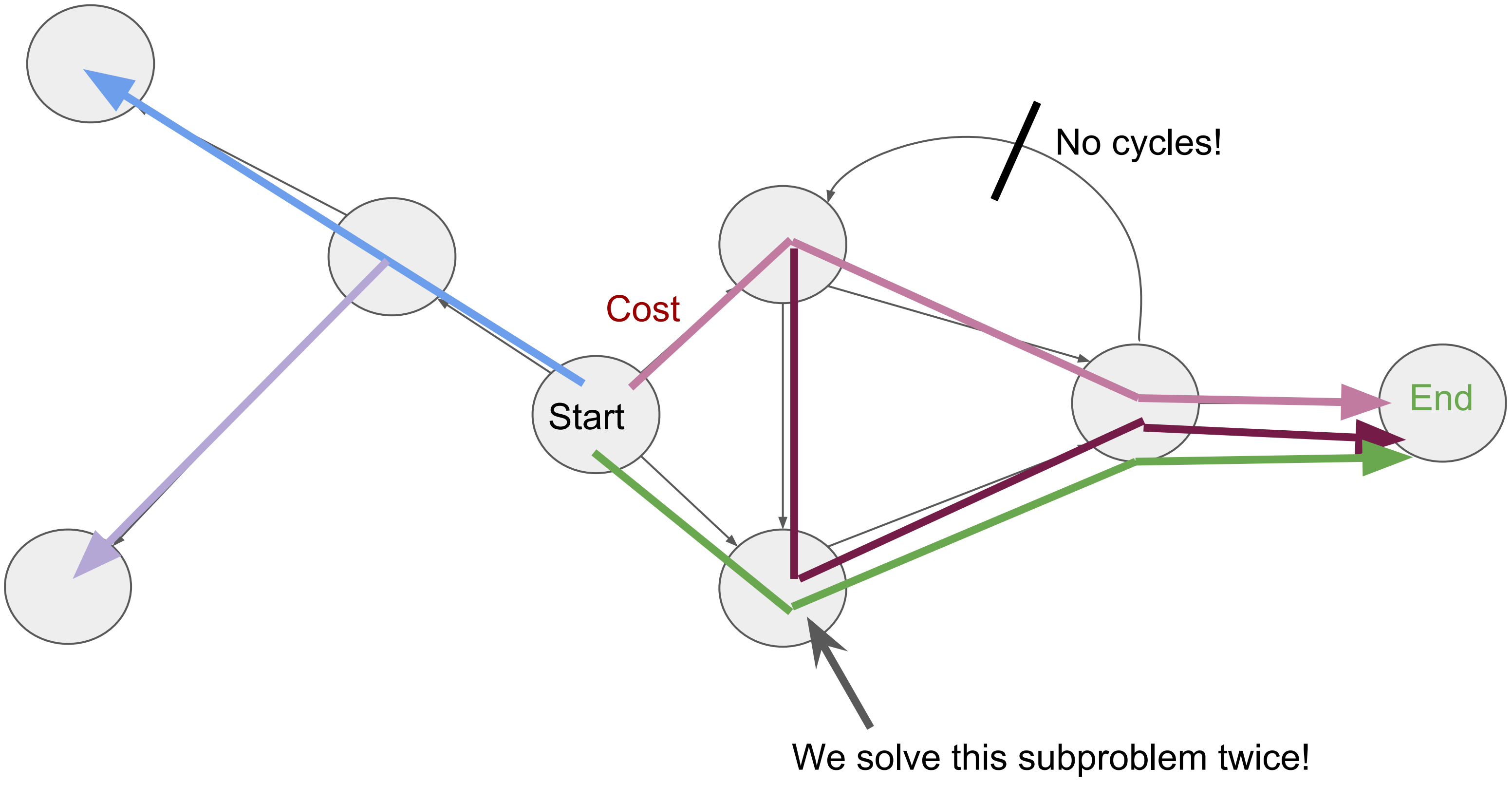
# Explore step by step! BFS



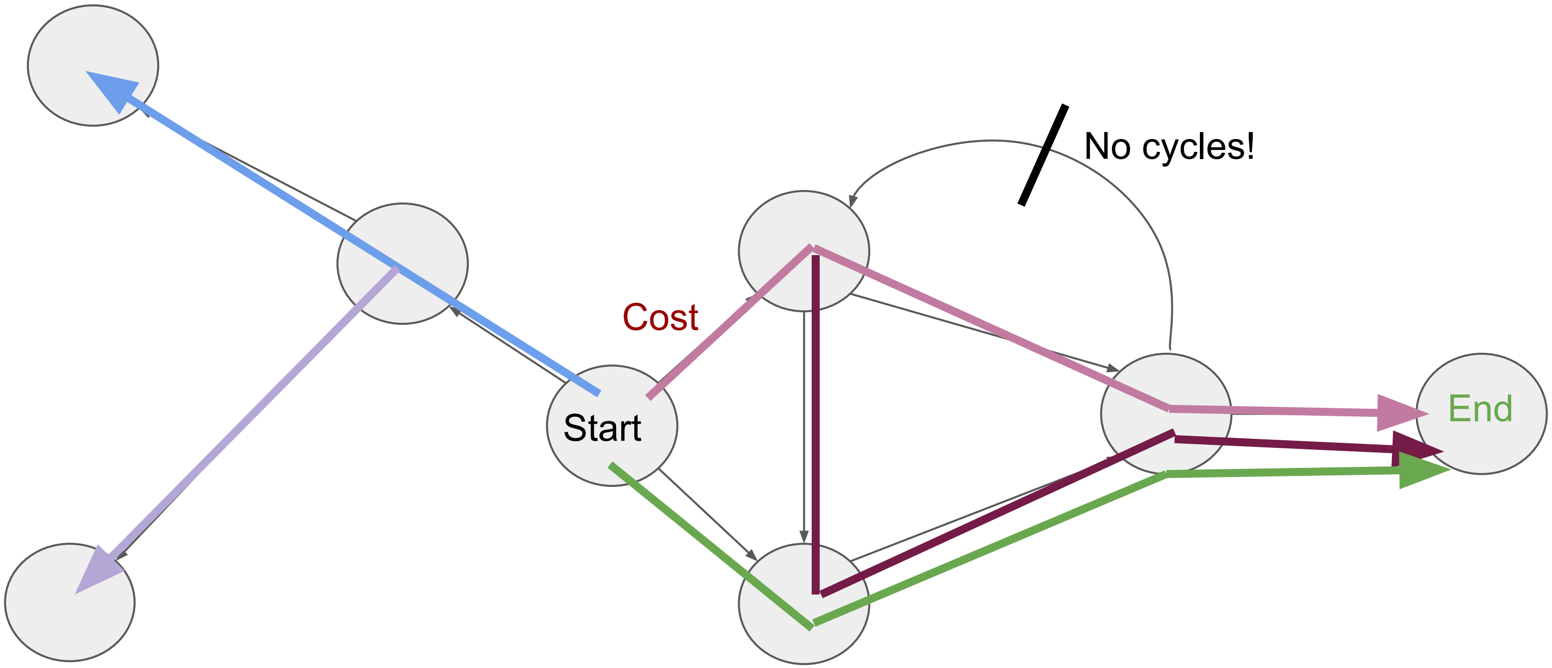

Runtime: O(b^d)

Heavy in space: O(b^d)

# Remember subproblems! DP



# Remember subproblems! DP

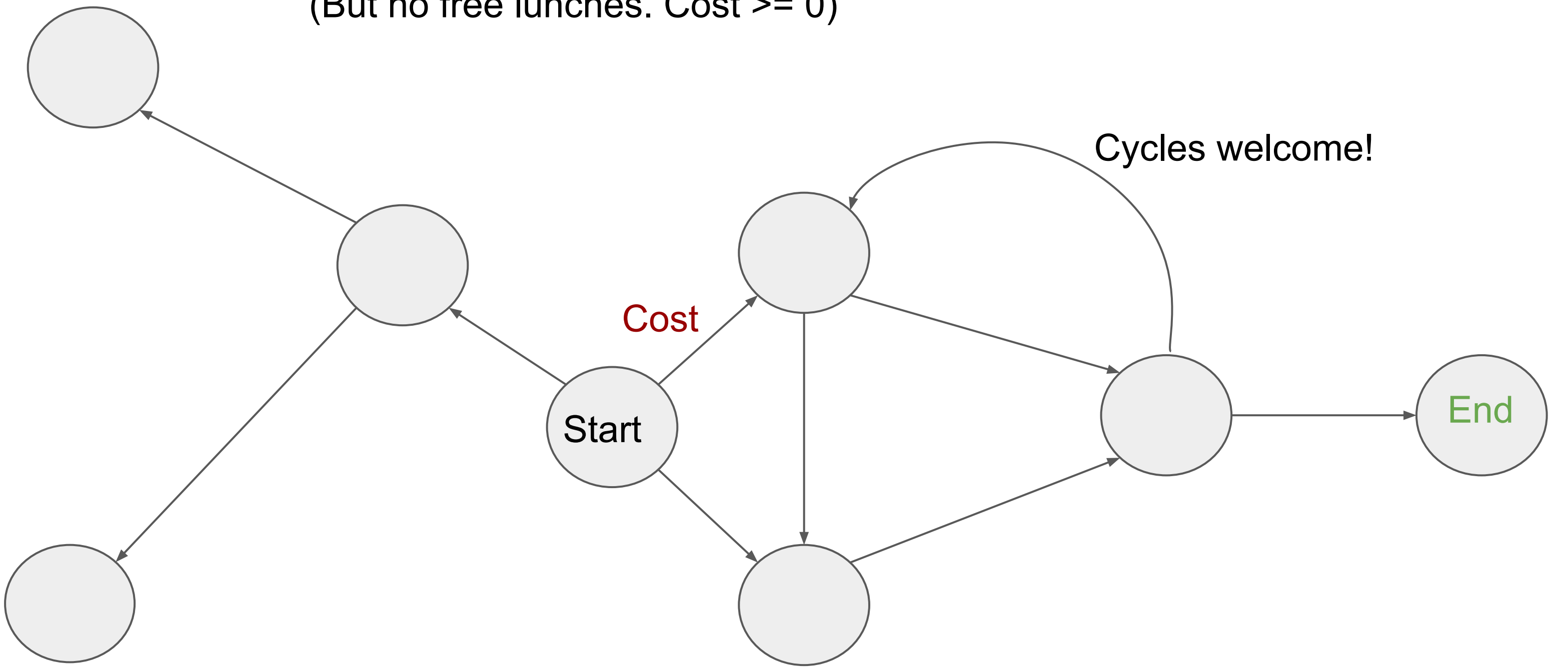



Runtime: O(#states * b)

Memory: O(#states)

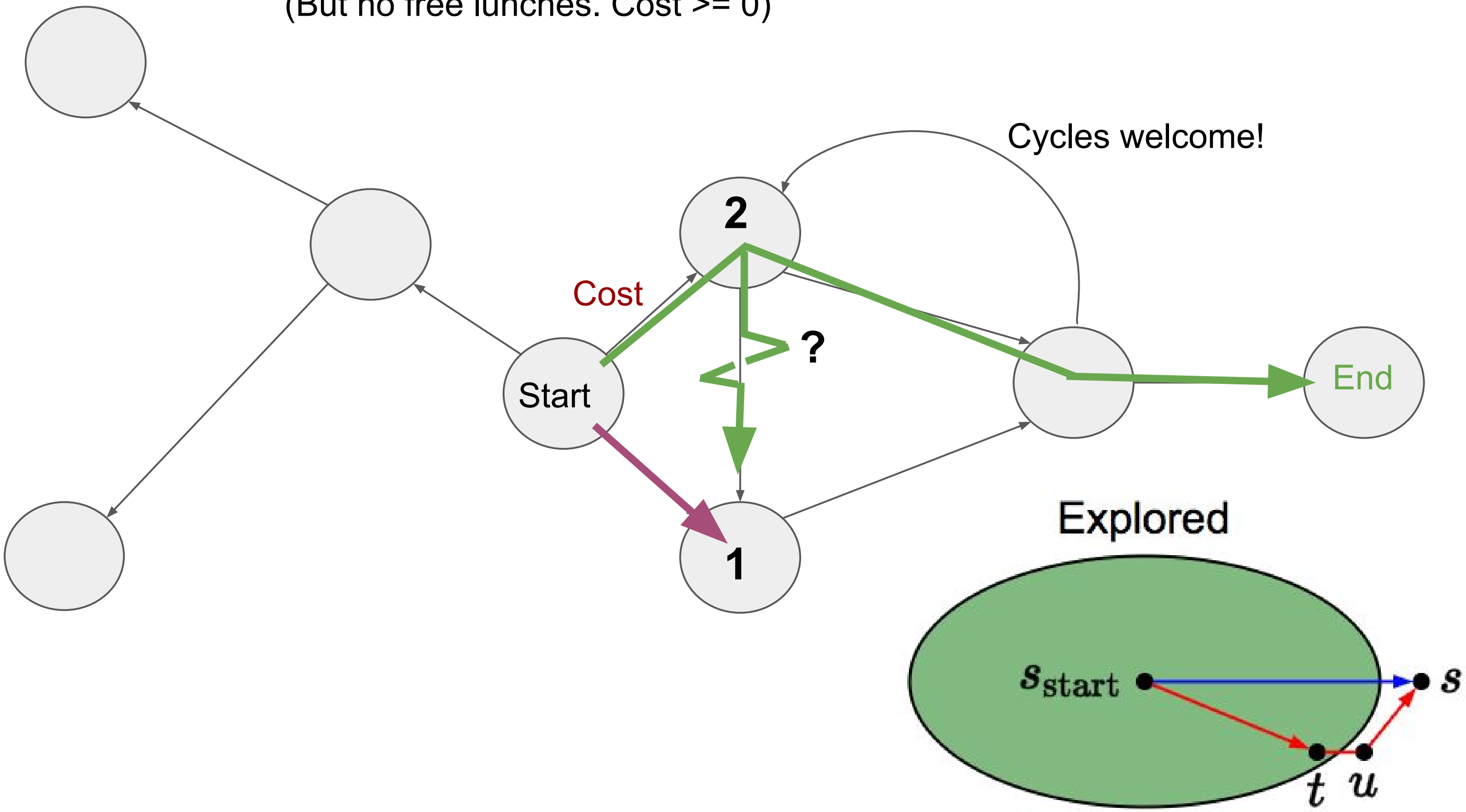
# Path of least resistance with memory! UCS

(But no free lunches. Cost >= 0)



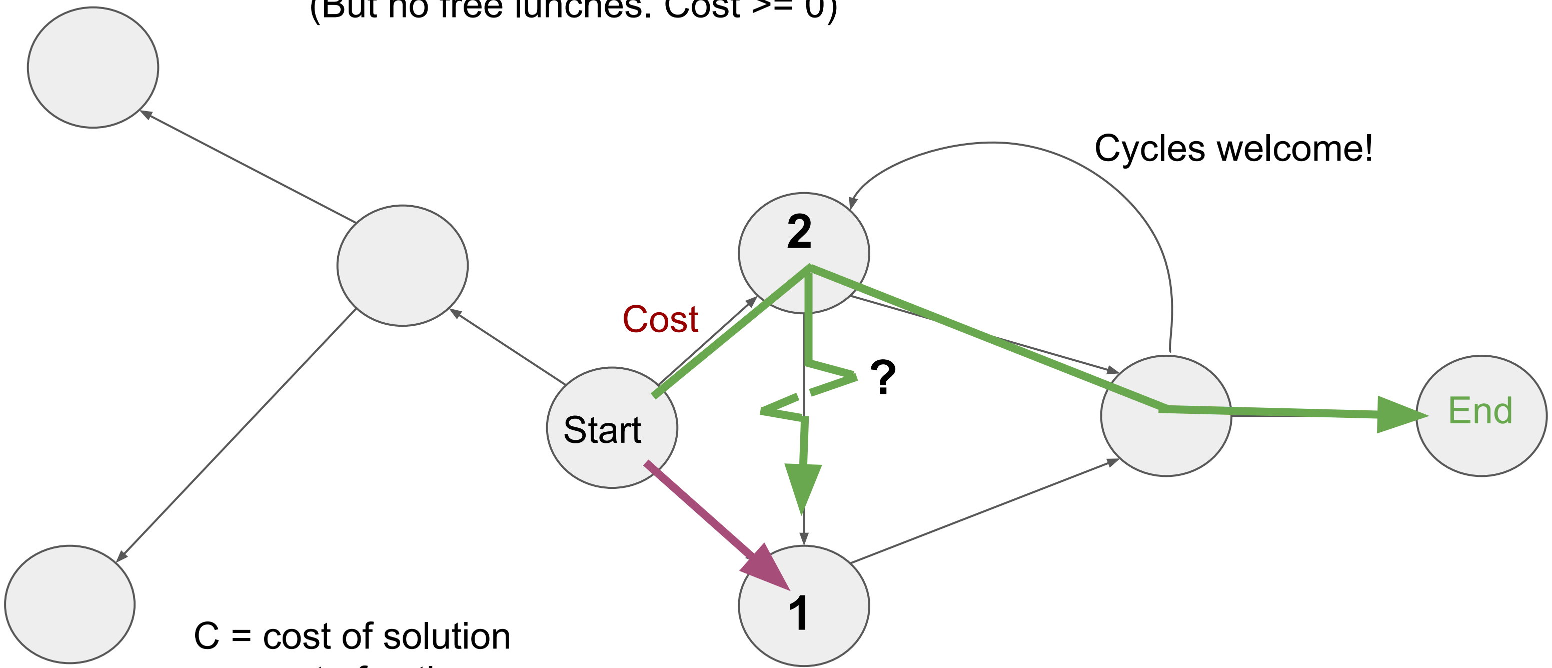
# Path of least resistance with memory! UCS

(But no free lunches. Cost >= 0)



# Path of least resistance with memory! UCS

(But no free lunches. Cost >= 0)




C = cost of solution
e = cost of action

Runtime: ~ O(min(b^(C/e), #states * b))     Memory = Runtime

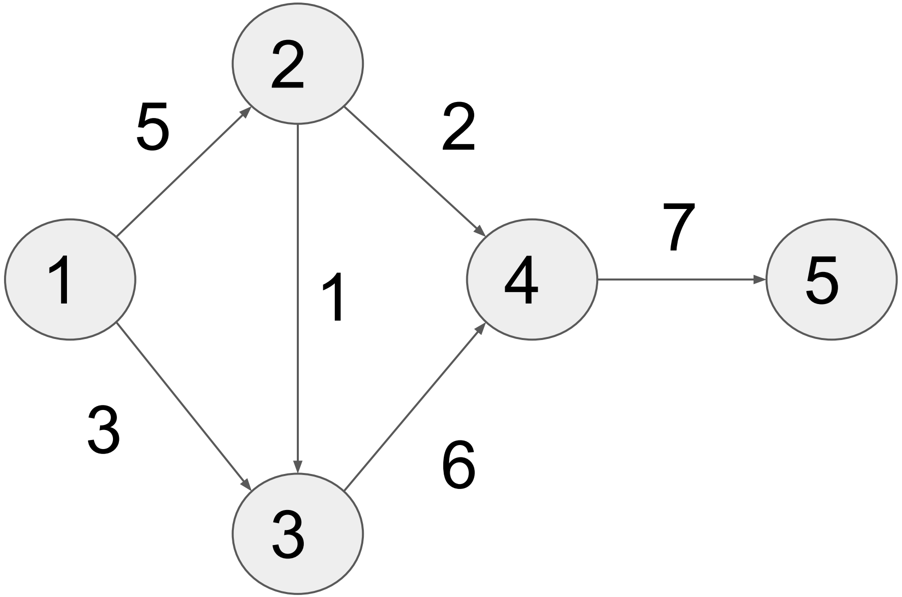
# Problem

There exists N cities, conveniently labelled from 1 to N.

There are roads connecting some pairs of cities. The road connecting city i and city j takes c(i,j) time to traverse. However, one can only travel from a city with smaller label to a city with larger label (i.e. each road is one-directional).

From city 1, we want to travel to city N. What is the shortest time required to make this trip, given the additional constraint that we should visit more odd-labeled cities than even labeled cities?

# Example




Best path is [1, 3, 4, 5] with cost 16.

[1, 2, 4, 5] has cost 14 but visits equal number of odd and even cities.

# State Representation

**Key idea: state**

A **state** is a summary of all the past actions sufficient to choose future actions **optimally**.

# State Representation

We need to know where we are currently at: **current_city**

We need to know how many odd and even cities we have visited thus far: **#odd, #even**

State Representation: (**current_city, #odd, #even**)

Total number of states: $O(N^3)$

# Can We Do Better?

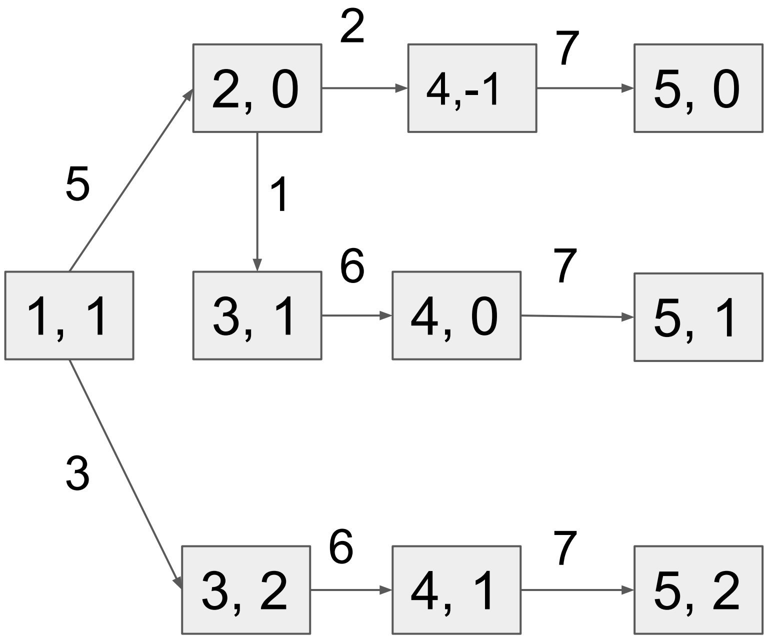
Check if all the information is really required

We store **#odd** and **#even** so that we can check whether **#odd** - **#even** > 0 at (N, **#odd**, **#even**)

Why not store **#odd - #even** directly instead?

(**current_city, #odd - #even**)    --  $O(N^2)$ states

# State Graph



# Precise Formulation of Problem

Let E be the set of roads between cities

State s = (i, d) -- (current_city, #odd - #even)

Actions(s): {move(j) for (i,j) $\in$ E}

Cost(s, move(j)): c(i,j)

Succ(s, a): (j, d + 1) if j is odd, (j, d - 1) otherwise

Start: (1, 1)

isGoal(s): i = N and d > 0

# Solving the Problem

Since we are computing shortest path, which is some form of optimization, we consider DP and UCS.

Recall

- DP can handle negative edges but works only on DAGs
- UCS works on general graphs, but cannot handle negative edges

Since we have a DAG and all edges are positive, both work!

# Solving the Problem: Uniform Cost Search

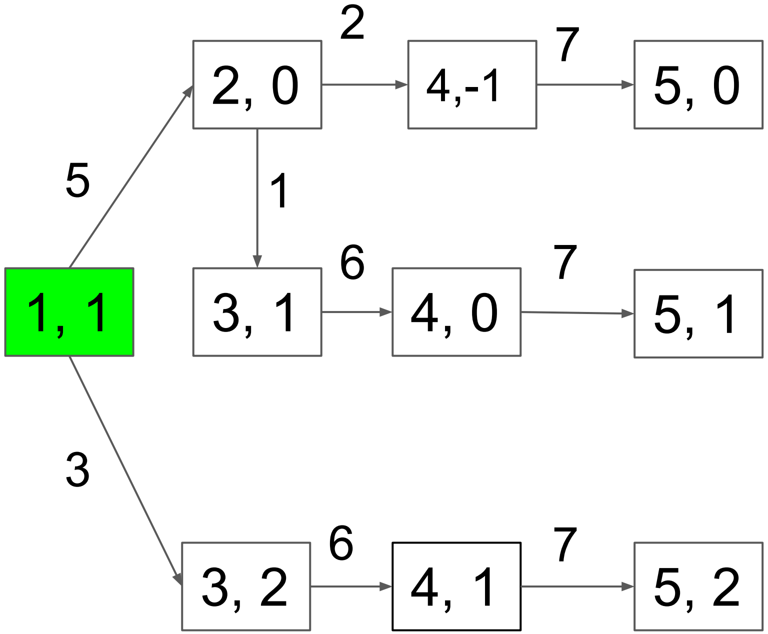
**Algorithm: uniform cost search [Dijkstra, 1956]**

Add $s_{\text{start}}$ to **frontier** (priority queue)

Repeat until frontier is empty:

- Remove $s$ with smallest priority $p$ from frontier
- If IsGoal$(s)$: return solution
- Add $s$ to **explored**
- For each action $a \in$ Actions$(s)$:
  - Get successor $s' \leftarrow$ Succ$(s, a)$
  - If $s'$ already in explored: continue
  - Update **frontier** with $s'$ and priority $p +$ Cost$(s, a)$

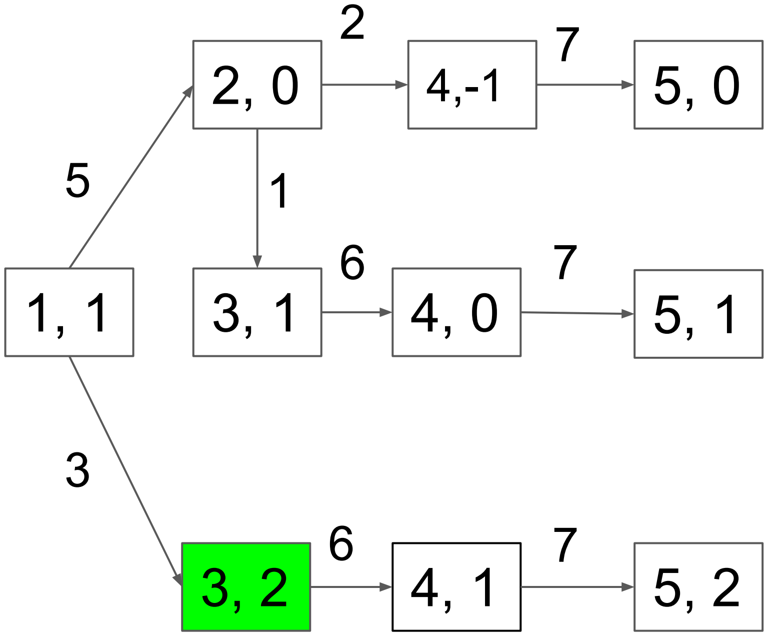
# Simulation of UCS




Explored:
(1, 1) : 0

Frontier:
(2, 0) : 5
(3, 2) : 3

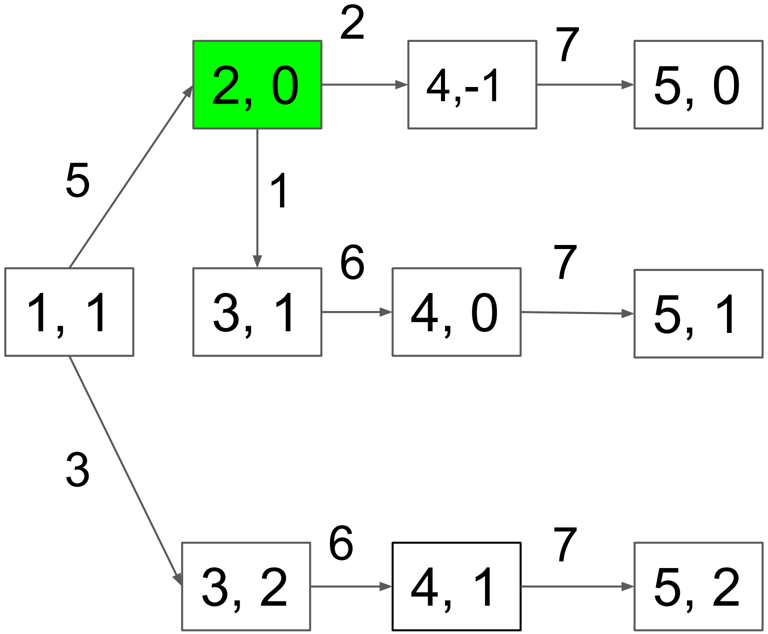
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3

Frontier:
(2, 0) : 5
(4, 1) : 9

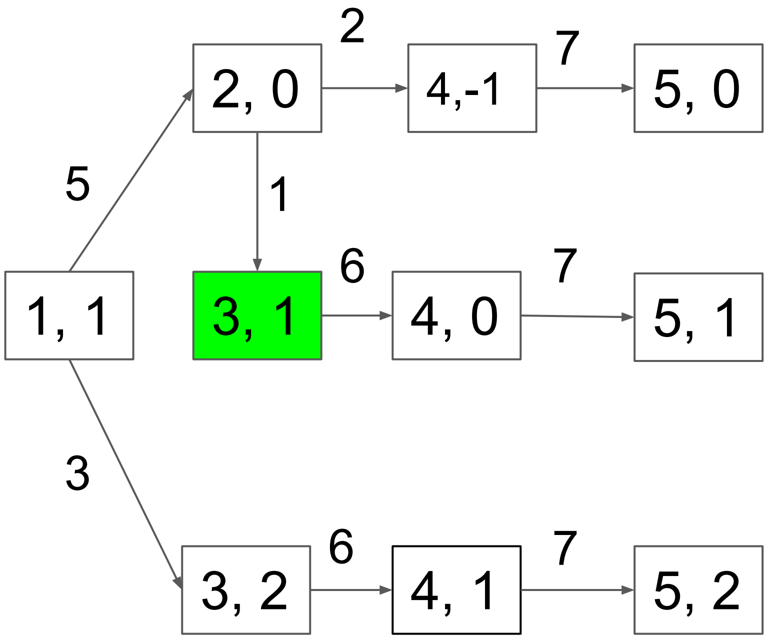
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5

Frontier:
(4, 1) : 9
(4, -1) : 7
(3, 1) : 6

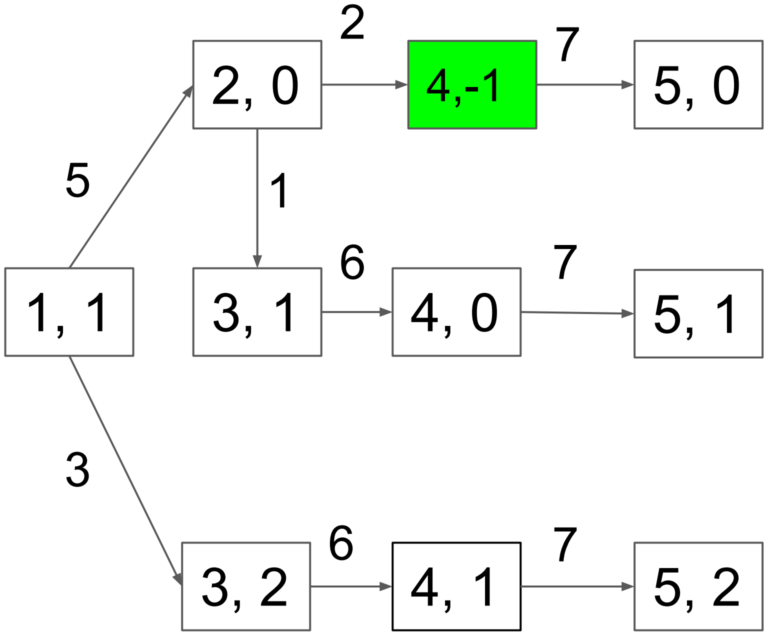
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5
(3, 1): 6

Frontier:
(4, 1) : 9
(4, -1) : 7
(4, 0): 12

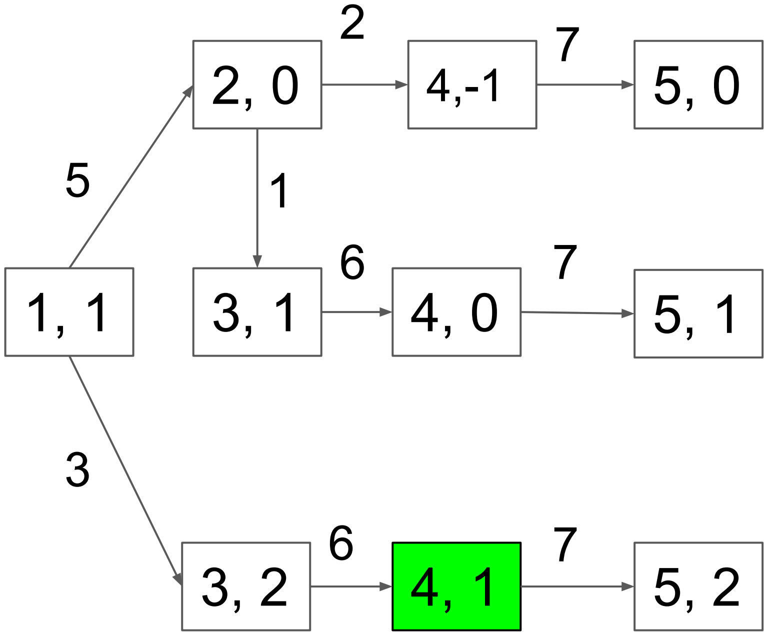
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5
(3, 1) : 6
(4, -1) : 7

Frontier:
(4, 1) : 9
(4, 0) : 12
(5, 0) : 14

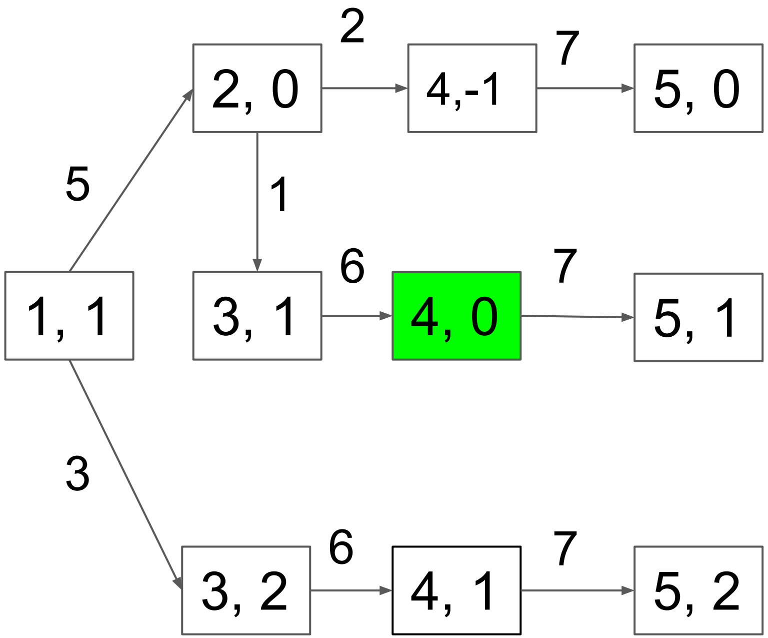
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5
(3, 1) : 6
(4, -1) : 7
(4, 1) : 9

Frontier:
(4, 0) : 12
(5, 0) : 14
(5, 2) : 16

# Simulation of UCS




Explored:

(1, 1) : 0

(3, 2) : 3

(2, 0) : 5

(3, 1) : 6

(4, -1) : 7

(4, 1) : 9

(4, 0) : 12

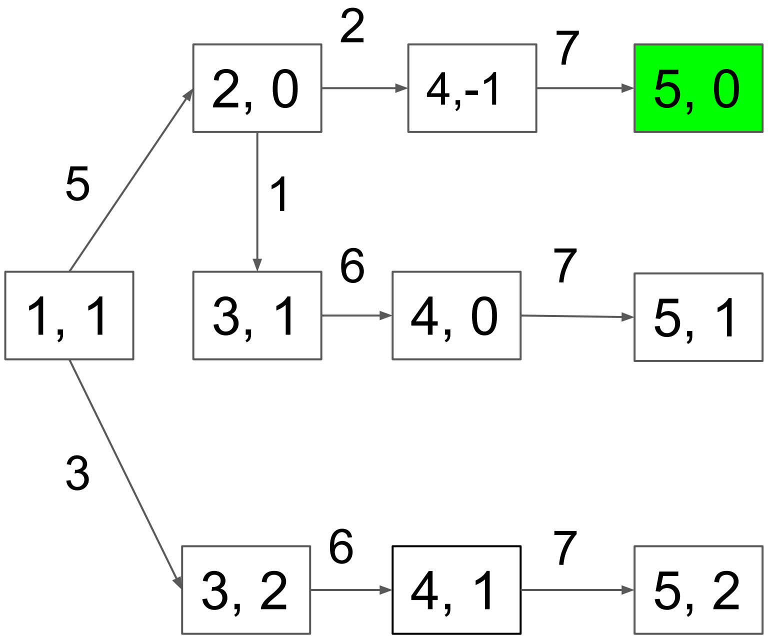
Frontier:

(5, 0) : 14

(5, 2) : 16

(5, 1) : 19

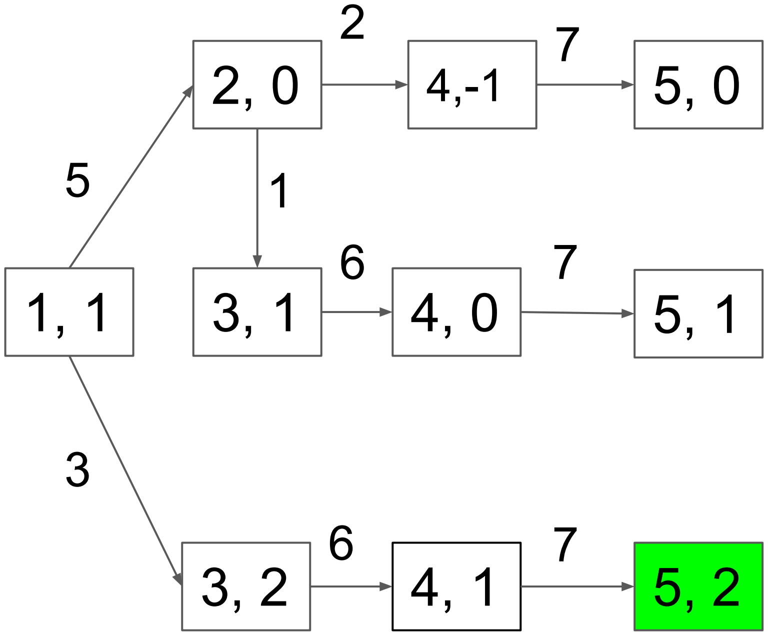
# Simulation of UCS




Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5
(3, 1) : 6
(4, -1) : 7
(4, 1) : 9
(4, 0) : 12
(5, 0) : 14

Frontier:
(5, 2) : 16
(5, 1) : 19

# Simulation of UCS

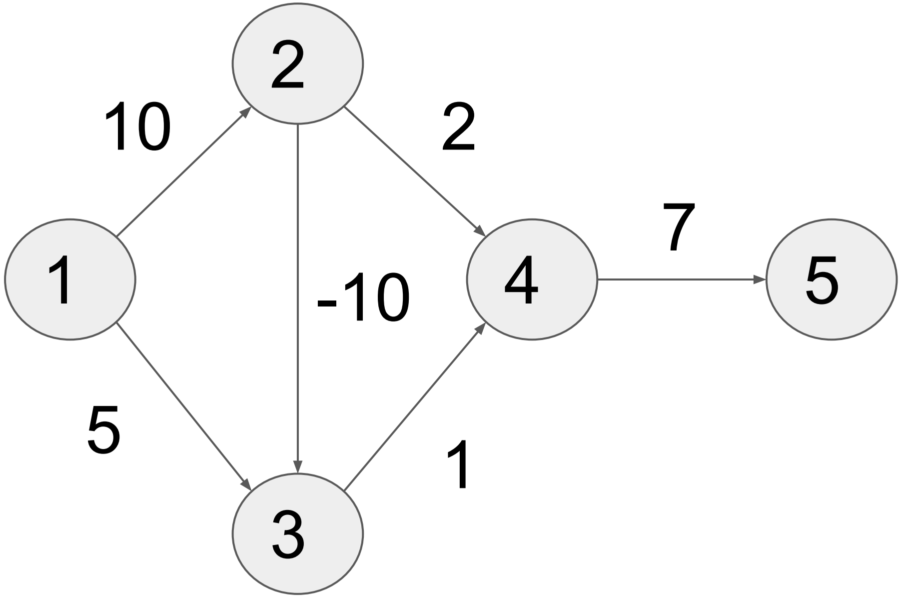



Explored:
(1, 1) : 0
(3, 2) : 3
(2, 0) : 5
(3, 1) : 6
(4, -1) : 7
(4, 1) : 9
(4, 0) : 12
(5, 0) : 14
(5, 2) : 16

Frontier:
(5, 1) : 19

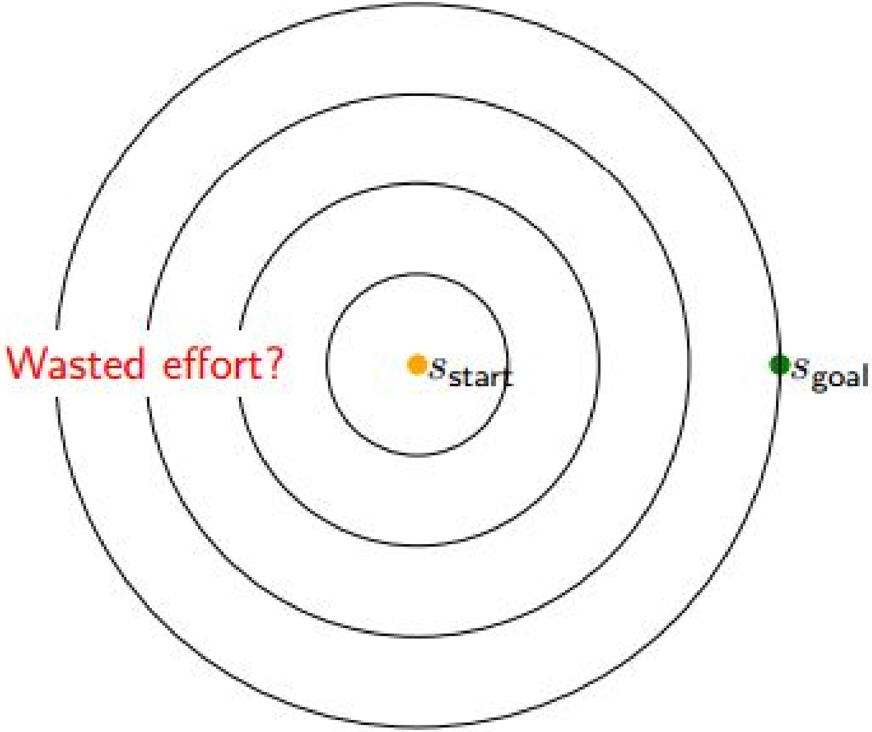
**STOP!**

# UCS cannot handle negative edge weights




Best path is [1,2,3,4,5] with cost of 8, but UCS will output [1,3,4,5] with cost of 13 because 3 is set as ‘explored’ before 2.

# Improve UCS: A* Search



# Recap of A* Search

- Modify the cost of edges and run UCS on the new graph
  - Cost’(s, a) = Cost(s, a) + h(Succ(s, a)) - h(s)
- h(s) is a heuristic that is our estimate of FutureCost(s)
- If h(s) is consistent then the modified edge weights will return min cost path
- One can find a good consistent h by performing relaxation
- If c is min cost on original graph, c’ is min cost on modified graph, then c’ = c + h(s_goal) - h(s_start)

# Relaxation

A good way to come up with a reasonable heuristic is to solve an easier (less constrained) version of the problem

For example, we can remove the constraint that we visit more odd cities than even cities.

$h(s) = h((i, d))$ = length of shortest path from city $i$ to city $N$

# Note on Relaxation

The main point of relaxation is to attain a problem that **can be solved more efficiently**.

In our case, the modified shortest path problem has O(N) states instead of O(N^2) can thus can be solved more efficiently

# Checking consistency

- Cost(s, a) + h(Succ(s, a)) - h(s) ≥ 0 (Triangle Inequality)
  - Suppose s = (i, d) and Succ(s, a) = (j, d')
  - Note that h((i, d)) - h((j, d')) ≤ c(i, j) = Cost(s, a)
- h((N, d)) = 0

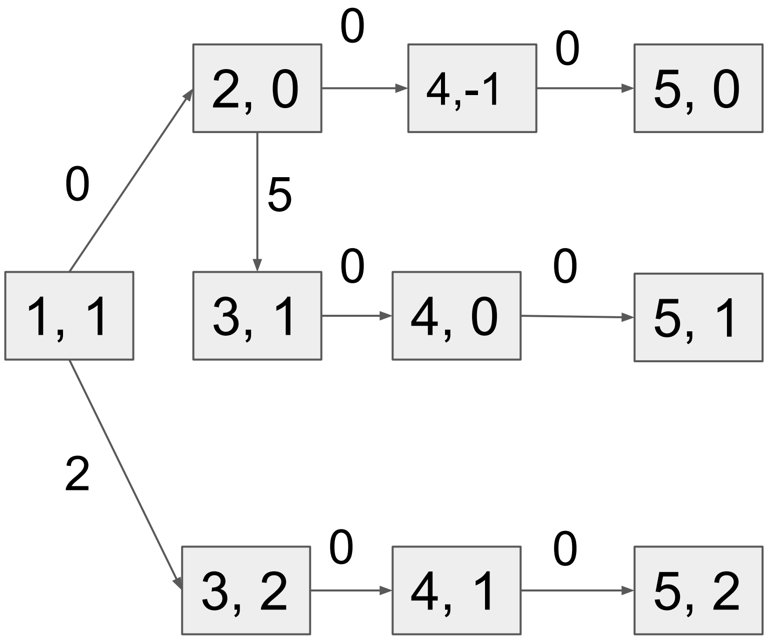
# How to compute h?

We can reverse the direction of all edges, and then perform UCS starting from city N, and our goal state is city 1.

This takes O(n log n) time, where n is the number of states whose distance to city N is no farther than the distance of city 1 to city N

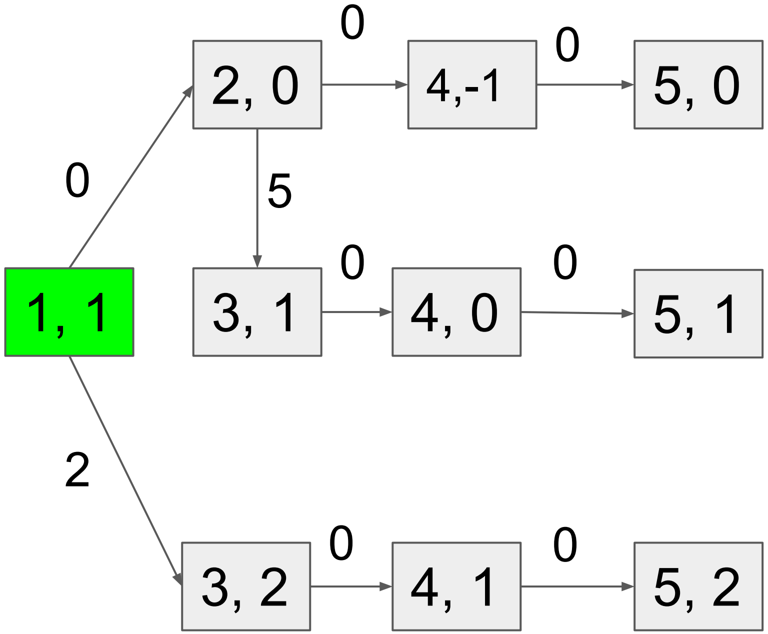
| city | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| h | 14 | 9 | 13 | 7 | 0 |

# Modified State Graph




Cost(s, a) + h(Succ(s, a)) - h(s)

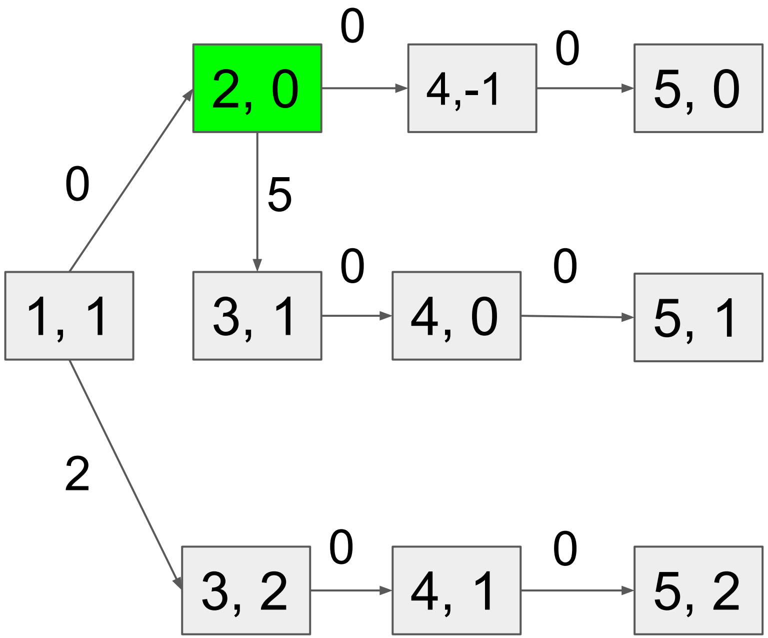
# Simulation of UCS (A*)




Explored:
(1, 1) : 0

Frontier:
(2, 0) : 0
(3, 2) : 2

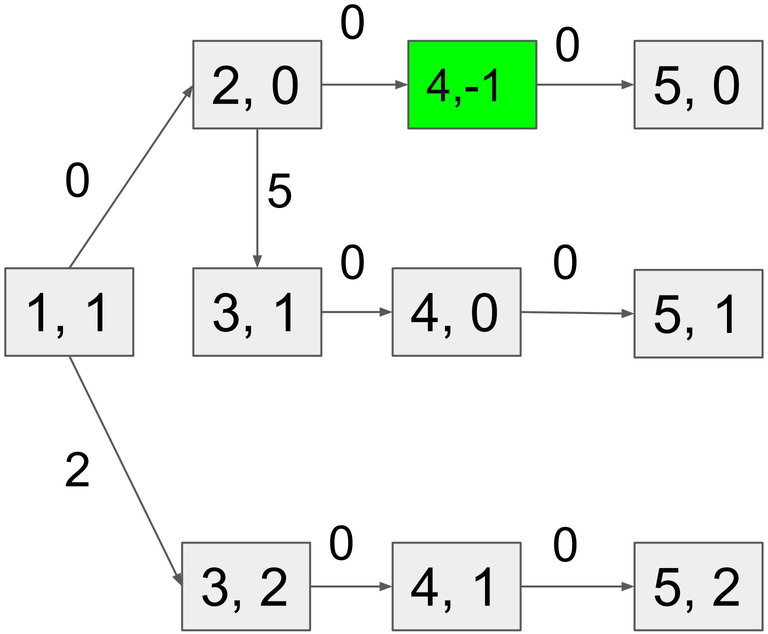
# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0

Frontier:
(3, 2) : 2
(3, 1) : 3
(4, -1) : 0

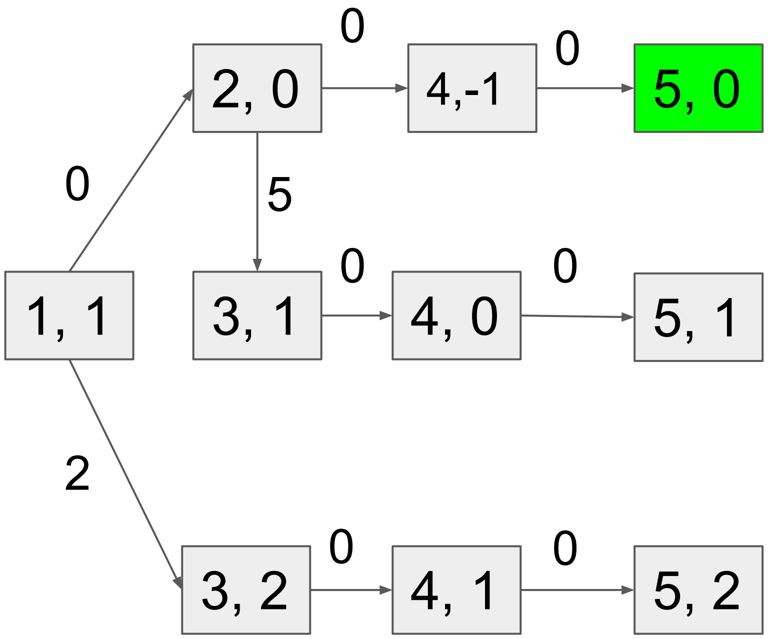
# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0
(4, -1) : 0

Frontier:
(3, 2) : 2
(3, 1) : 3
(5, 0) : 0

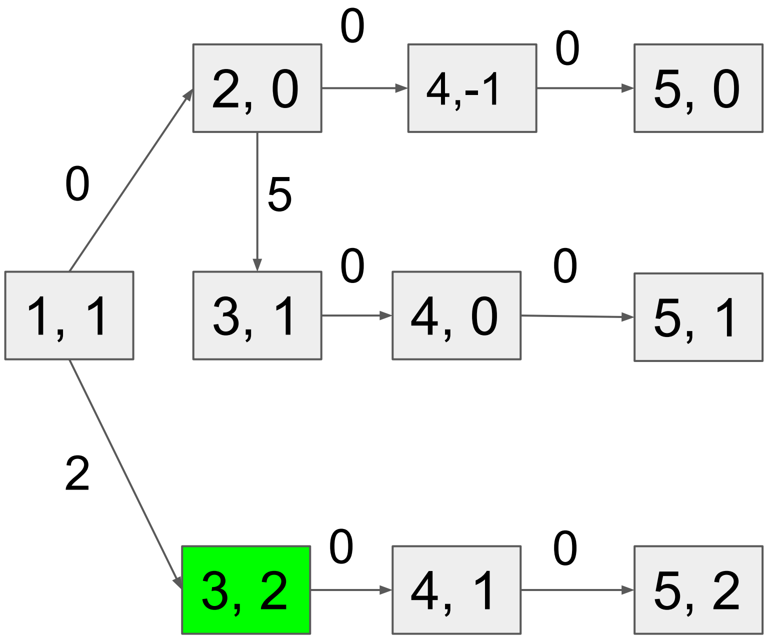
# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0
(4, -1) : 0
(5, 0) : 0

Frontier:
(3, 2) : 2
(3, 1) : 3

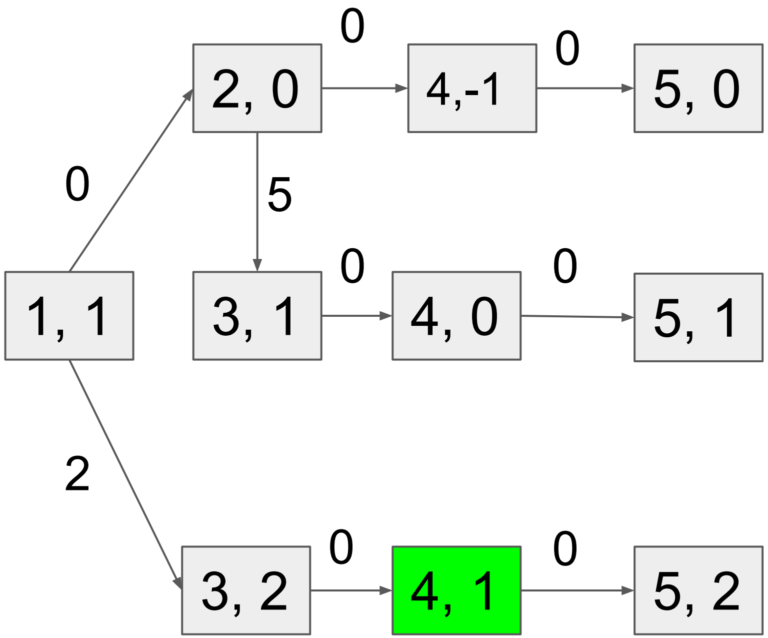
# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0
(4, -1) : 0
(5, 0) : 0
(3, 2) : 2

Frontier:
(3, 1) : 3
(4, 1) : 2

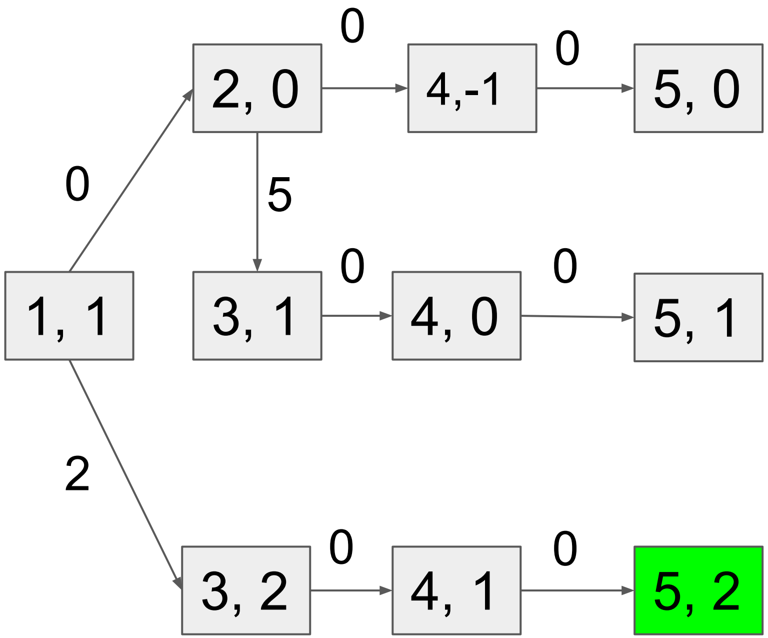
# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0
(4, -1) : 0
(5, 0) : 0
(3, 2) : 2
(4, 1): 2

Frontier:
(3, 1) : 3
(5, 2) : 0

# Simulation of UCS (A*)




Explored:
(1, 1) : 0
(2, 0) : 0
(4, -1) : 0
(5, 0) : 0
(3, 2) : 2
(4, 1) : 2
(5, 2) : 2

Frontier:
(3, 1) : 3

**STOP!**

Actual Cost is 2 + h(1) = 2 + 14 = 16

# Comparison of States visited

| **UCS** | | **UCS(A*)** | |
|---|---|---|---|
| Explored: | Frontier: | Explored: | Frontier: |
| (1, 1) : 0 | (5, 1) : 19 | (1, 1) : 0 | (3, 1) : 3 |
| (3, 2) : 3 | | (2, 0) : 0 | |
| (2, 0) : 5 | | (4, -1) : 0 | |
| (3, 1) : 6 | | (5, 0) : 0 | |
| (4, -1) : 7 | | (3, 2) : 2 | |
| (4, 1) : 9 | | (4, 1) : 2 | |
| (4, 0) : 12 | | (5, 2) : 2 | |
| (5, 0) : 14 | | | |
| (5, 2) : 16 | | | |

# Summary

- States Representation/Modelling
  - make state representation as compact as possible, remove unneccesary information
- DP
  - underlying graph cannot have cycles
  - visit all reachable states
- UCS
  - actions cannot have negative cost
  - visit only a subset of states
- A*
  - ensure that relaxed problem can be solved more efficiently