Permutation & Combination

INTRODUCTION Section - 1

1.1 Definition of Factorial

Factorial : The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\underline{|n}$. i.e. $n! = n(n-1)(n-2).....3 \times 2 \times 1$

Thus, $4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

1.2 Properties of factorial

(i) $n!$ is defined for positive integers only.

(ii) Factorials of proper fractions or integers are not defined. Factorial n is defined only for whole numbers.

(iii) $n! = 1 \times 2 \times 3 \times = ... \times (n-1) \times n = [1 \times 2 \times 3 \times = ... \times (n-1)]\, n = (n-1)!\, n$

Thus, $n! = n\,(n-1)!$

(iv) $0! = 1$ (by definition)

(v) If two factorials, *i.e.*, $x!$ and $y!$ are equal, then $x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$.

Illustrating the Concepts :

If $\dfrac{n!}{2!(n-2)!}$ and $\dfrac{n!}{4!(n-4)!}$ *are in the ratio* $2:1$, *then find the value of* n ?

$$\frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2).....3 \times 2 \times 1}{2!(n-2)!}$$

$$= \frac{n(n-1)\{(n-2)(n-3).....3 \times 2 \times 1\}}{2!(n-2)!}$$

$$= \frac{n(n-1)}{2!}\,\frac{(n-2)!}{(n-2)!} = \frac{n(n-1)}{2!}$$

Similarly,

$$\frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)(n-4)!}{4!(n-4)!}$$

$$= \frac{n(n-1)(n-2)(n-3)}{4!}$$

Given: $\dfrac{n!}{2!(n-2)!} : \dfrac{n!}{4!(n-4)!} = 2:1$

$$\Rightarrow \quad \frac{\dfrac{n(n-1)}{2}}{\dfrac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{12}{(n-2)(n-3)} = \frac{2}{1}$$

$\Rightarrow \quad (n-2)(n-3) = 6$

$\Rightarrow \quad n^2 - 5n = 0 \qquad n = 0, 5$

But, for $n = 0$, $(n-2)!$ and $(n-4)!$ are not meaningful.

So, $n = 5$.

Illustration - 1 *The value of* $\frac{(2n)!}{n!}$ is equal to :

(A) $\{1.\,3.\,5.\,.\,.\,.\,.\,.(2n-1)\}2^n$ **(B)** $\{1.\,3.\,5.\,.\,.\,.\,.\,.(2n-1)\}\,2^n\,n!$

(C) $\{1.\,3.\,5.\,.\,.\,.\,.\,.(2n+1)\}2^n$ **(D)** None of these

SOLUTION : (A)

Using the definition of factorials,

$$\frac{(2n)!}{n!} = \frac{1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,.\,.\,(2n-1)\,(2n)}{n!}$$

Separating odd and even terms in numerator, we get :

$$= \frac{\{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,(2n-1)\}\,\{2\,.\,4\,.\,6\,.\,.\,.\,.\,2n\}}{n!} = \frac{\{1\cdot3\cdot5\ldots.(2n-1)\}\,2^n\,n!}{n!}$$

$$= \{1\,.\,3\,.\,5\,.\,.\,.\,.\,.\,.\,.\,(2n-1)\,\}2^n$$

★ 1.3 Exponent of prime *p* in *n*!

Let p be a prime number and n be a positive integer. Then, the last integer amongst $1, 2, 3, \ldots\ldots, (n-1)$, n which is divisible by p is $\left[\frac{n}{p}\right] p$ where $\left[\frac{n}{p}\right]$ denotes the greatest integer less than or equal to n/p.

$E_p\,(n!)$ denote the exponent of p in the positive integer n. Then,

$$E_p\,(n!) = E_p\,(1\,.\,2\,.\,3\,.\,.\,.\,.\,.\,.\,(n-1)\,n)$$

$$= E_p\left(p\,.\,2p\,.\,3p\,.\,.\,.\,.\,.\,.\left[\frac{n}{p}\right]p\right)$$

[As Remaining integers between 1 and n are not divisible by p]

$$= \left[\frac{n}{p}\right] + E_p\left(1\,.\,2\,.\,3\,.\,.\,.\,.\,.\left[\frac{n}{p}\right]\right)$$

Now, the last integer amongst, $1, 2, 3, \ldots\ldots, \left[\frac{n}{p}\right]$ which is divisible by p is :

$$\left[\frac{n/p}{p}\right] p = \left[\frac{n}{p^2}\right] p$$

$$\therefore \qquad E_p\,(n!) = \left[\frac{n}{p}\right] + E_p\left(p\,.\,2p\,.\,.\,.\,.\,.\,.\left[\frac{n}{p^2}\right]p\right)$$

[As Remaining integers between 1 and $[n/p]$ are not divisible by p.]

$$= \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + E_p\left(1\,.\,2\,.\,3\,.\,.\,.\,.\,.\left[\frac{n}{p^2}\right]\right)$$

Continuing in the same manner, we get :

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \,.\,.\,.\,.\,.\, + \left[\frac{n}{p^t}\right]$$

[where t is the largest positive integer such that $p^t \le n \le p^{t+1}$]

Note : This result is not valid for composite numbers.

For Example :

This result cannot be used to find the exponent of 6 in n ! as 6 is composite of 2 and 3.

Illustrating the Concepts :

Find the exponent the 2 in 50 ! ?

$$E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{2^2}\right] + \left[\frac{50}{2^3}\right] + \left[\frac{50}{2^4}\right] + \left[\frac{50}{2^5}\right] = 25 + 12 + 6 + 3 + 1 = 47$$

★

Illustration - 2 *The number of zeroes in* 100! *are :*

(A) 20 **(B)** 24 **(C)** 97 **(D)** 28

SOLUTION : (B)

We know, 10 = 5 × 2

So, to form one 10, we need one 2 and one 5.

Number of 10's will be same as min $\{E_2\,(100!), E_5\,(100!)\}$

$$E_2\,(100!) = \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right]\left[\frac{100}{2^6}\right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

$$E_5\,(100!) = \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] = 20 + 4 = 24$$

$$E_{10}(100!) = min\ \{97, 24\} = 24$$

FUNDAMENTAL PRINCIPLE OF COUNTING Section - 2

2.1. Addition Principle

If a work can be done in m different ways and another work which is independent of first can be done in n different ways. then either of the two operations can be performed in $(m + n)$ ways.

Illustrating the Concepts :

There are 15 gates to enter a city from north and 10 gates to enter the city from east. In how many ways a person can enter the city ?

Number of ways to enter the city from north = 15

Number of ways to enter the city from east = 10

A person can enter the city from north or from east.

Hence, the number of ways to enter the city = 15 + 10 = 25

Illustration - 3 *There are 15 students is a class in which 10 are boys and 5 are girls. The class teacher selects either a boy or a girl for monitor of the class. In how many ways the class teacher can make this selection ?*

(A) 5 **(B)** 10 **(C)** 15 **(D)** 150

SOLUTION : (C)

A boy can be selected for the post of monitor in 10 ways.

A girl can be selected for the post of monitor in 5 ways.

Number of ways in which either a boy or a girl can be selected = 10 + 5 = 15

2.2. Multiplication Principle

If one Operation (I) can be done in m different ways and another Operation (II) can be performed in n different ways, then total number of ways in which both of these can be performed together is $m \times n$. If there are more than two operations to be done, then the total number of different ways to do all of them together will be $m \times n \times p \times$ ....

For example :

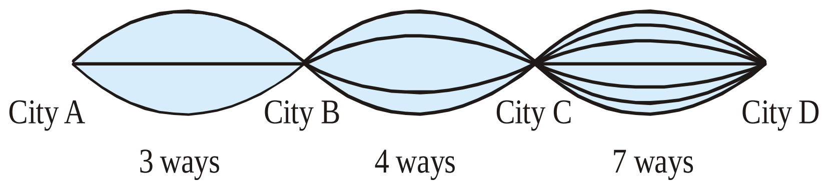
Operation A can be performed in "a, b, c" different ways and Operation B can be performed in p, q different ways, then possible ways to perform Operation A and Operation B together are $a - p$, $a - q$, $b - p$, $b - q$, $c - p$, $c - q$ i.e. each way of Operation A is combined by 2 ways of Operation B.

i.e. Total ways = 2 + 2 + 2 = 2 added 3 times = 2 × 3 = 6.

In general, total ways = $m \times n$ where m and n are individual ways of performing the operations.

Illustrating the Concepts :

There are 3 routes to travel from City A to City B and 4 routes to travel from City B to City C and 7 routes from C to D. In how many different ways (routes) *a man can travel from City A to City D via City B and City C.*



The man can perform the task of traveling from City A to City B in ways = 3.

The man can perform task of traveling from City B to City C in ways = 4.

Similarly from City C to City D in ways = 7.

Using fundamental principle of counting,

Total routes to travel from A to D via B and via C = $m \times n \times p$

$= 3 \times 4 \times 7 = 84$ routes.

Illustration - 4 *A city has 12 gates. In how many ways can a person enter the city through one gate and come out through a different gate ?*

(A) 23 **(B)** 144 **(C)** 132 **(D)** 24

SOLUTION : (C)

There are 12 ways to enter into the city. After entering into the city, the man can come out through a different gate in 11 ways.

Hence, by the fundamental principle of counting,

Total number of ways is $12 \times 11 = 132$ ways.

Illustration - 5 *How many n- digit numbers can be formed using* 1, 2, 3, 7, 9 *without any repetition of digits when :*

(i) $n = 5$

(A) 120 **(B)** 15 **(C)** 6^5 **(D)** 5^6

(ii) $n = 3$

(A) 12 **(B)** 5^3 **(C)** 60 **(D)** 3^5

SOLUTION : (i) . (A) (ii) . (C)

(i) 5-digit numbers

Making a 5-digit number is equivalent to filling 5 places.

Places :	□	□	□	□	□
No. of choices :	1	2	3	4	5

The last place (unit's place) can be filled in 5 ways using any of the five given digits.

The ten's place can be filled in 4 ways using any of the remaining 4 digits.

The number of choices for other places can be calculated in the same way.

Total number of ways to fill all five places = $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

Hence, 120 five-digit numbers can be formed.

(ii) 3-digit numbers

Making a three-digit number is equivalent to filling three places (unit's, ten's, hundred's).

Places :	□	□	□
No. of choices :	3	4	5

Total number of ways to fill all the three places = $5 \times 4 \times 3 = 60$

Hence, 60 three-digit numbers can be formed.

Illustration - 6 *How many 3-letter words can be formed using a, b, c, d, e if :*

(i) *repetition is not allowed*

(A) 60 **(B)** 5^3 **(C)** 3^5 **(D)** 12

(ii) *repetition is allowed?*

(A) 60 **(B)** 5^3 **(C)** 3^5 **(D)** 12

SOLUTION : (i) . (A) (ii) . (B)

(i) Repetition is not allowed :

The number of words that can be formed is equal to the number of ways to fill the three places.

Places :	☐	☐	☐
No. of choices :	5	4	3

First place can be filled in five ways using any of the five letters (a, b, c, d, e).

Similarly second and third places can be filled using 4 and 3 letters respectively.

$\Rightarrow$ Total ways to fill $= 5 \times 4 \times 3 = 60$

Hence 60 words can be formed.

(ii) Repetition is allowed :

The number of words that can be formed is equal to the number of ways to fill the three places.

Places :	☐	☐	☐
No. of choices :	5	5	5

First place can be filled in five ways (a, b, c, d, e).

If repetition is allowed, all the remaining places can be filled in five ways using a, b, c, d, e.

Total ways to fill $= 5 \times 5 \times 5 = 125$

Hence 125 words can be formed.

Illustration - 7 *How many four-digit numbers can be formed using the digits* 0, 1, 2, 3, 4, 5 *?*

(A) 5^4 **(B)** 4^5 **(C)** 300 **(D)** 150

SOLUTION : (C)

For a four-digit number, we have to fill four places and 0 cannot appear in the first place (thousand's place).

Places :				
No. of choices :	5	5	4	3

For the first place, there are five choice (1, 2, 3, 4, 5) ; Second place can then be filled in five ways (0 and remaining four digits) ; Third place can be filled in four ways (remaining four digits) ; Fourth place can be filled in three ways (remaining three digits).

Total No. of ways to fill $= 5 \times 5 \times 4 \times 3 = 300$

Hence, 300 four-digits numbers can be formed.

Illustration - 8 *In how many ways can six persons be arranged in a row ?*

(A) 6! **(B)** 6^6 **(C)** 6^5 **(D)** 5^6

SOLUTION : (A)

Arranging a given set of n different objects is equivalent to filling n places.
So arranging six persons along a row is equivalent to filling 6 places.

Places :						
No. of choices :	6	5	4	3	2	1

Total number of ways to fill all places $= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6\,! = 720$
Hence, 720 arrangements are possible.

Illustration - 9 *How many 5-digit odd numbers can be formed using digits* 0, 1, 2, 3, 4, 5 *without repeating digits?*

(A) $4 \times 4!$ **(B)** 288 **(C)** 5 ! **(D)** 300

SOLUTION : (B)

Making a five digit number is equivalent to filling 5 places.

Places :					
No. of choices :	4	4	3	2	3

To make odd numbers, fifth place can be filled by either of 1, 3, 5 i.e. 3 ways.

First place can be filled in ways = 4 (excluding 0 and two odd number used to fifth place).

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting total ways to fill 5 places

= Total 5-digit odd numbers that can be formed $= 4 \times 4 \times 3 \times 2 \times 3 = 288$ ways.

Illustration - 10 *How many 5-digit numbers divisible by 2 can be formed using digits* 0, 1, 2, 3, 4, 5 *without repetition of digits.*

(A) 120 **(B)** 192 **(C)** 312 **(D)** 208

SOLUTION : (C)

To find 5-digit numbers divisible by 2,

We will make 2 cases. In first case, we will find number of numbers divisible by 2 ending with either 2 or 4. In second case, we will find even numbers ending with 0.

Case-I : Even numbers ending with 2 or 4.

Making a five digit number is equivalent to filling 5 places.

Places :					
No. of choices :	4	4	3	2	2

Fifth place can be filled by 2 or 4 i.e. 2 ways.

First place can be filled in 4 ways (excluding 0 and the digit used to fill fifth place)

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting,

Number of ways to fill 5 places together

$= 4 \times 4 \times 3 \times 2 \times 2 = 192$ ways . . . **(i)**

Case-II : Even numbers ending with 0.

Places :					
No. of choices :	5	4	3	2	1

Making a 5-digit number is equivalent to filling 5 places.

Fifth place is filled by 0, hence can be filled in 1 way.

First place can be filled in 5 ways (Using either of 1, 2, 3, 4, 5).

Similarly places second, third and fourth can be filled in 4, 3, 2 ways respectively.

Using fundamental principle of counting,

Number of ways to fill 5 places together

$= 5 \times 4 \times 3 \times 2 \times 1 = 120$ . . . **(ii)**

Combining (i) and (ii), we get :

Total number of 5 digit numbers divisible by 2 = 192 + 120 = 312

Illustration - 11 *How many 5-digit numbers divisible by 4 can be formed using digits* 0, 1, 2, 3, 4, 5 ?

(A) 144 **(B)** 72 **(C)** 288 **(D)** 312

SOLUTION : (A)

Making a five digit number is equivalent to filling 5 places.

A number would be divisible by 4 if the last 2 places are filled by either of 04, 12, 20, 24, 32, 40, 52.

Case-I :

Last 2 places are filled by either of 04, 20, 40.

Fourth and fifth places can be filled in 3 ways either of 04, 20, 40

Places :	☐	☐	☐	☐	☐
No. of choices :	4	3	2	$\underbrace{\quad\quad}_{3}$	

First place can be filled in 4 ways (excluding the digits used to fill fourth and fifth place).

Similarly second and third place can be filled in 3 and 2 ways respectively.

Using fundamental principle of counting,

Number of ways to fill 5 places = $4 \times 3 \times 2 \times 3 = 72$ ways . . . (i)

Case-II :

Places :	☐	☐	☐	☐	☐
No. of choices :	3	3	2	$\underbrace{\quad\quad}_{4}$	

Last 2 places are filled by either of 12, 24, 32, 52

Fourth and fifth place can be filled in 4 ways (either of 12, 24, 32, 52).

First place can be filled in 3 ways (excluding 0 and the digits used to fill fourth and fifth place)

Similarly, second and third place can be filled in 3 and 2 ways respectively.

Using fundamental principle of counting,

Total ways to fill 5 places = $3 \times 3 \times 2 \times 4 = 72$ ways. . . . (ii)

Combining (i) and (ii),

Total Number of ways to fill 5 places = Total 5-digit numbers divisible by 4 = 72 + 72 = 144

Illustration - 12 *How many six-digit numbers divisible by* 25 *can be formed using digits* 0, 1, 2, 3, 4, 5 ?

(A) 24 **(B)** 42 **(C)** 256 **(D)** 100

SOLUTION : (B)

Numbers divisible by 25 must end with 25 or 50.

Case-I : Numbers ending with 25

Places :	☐	☐	☐	☐	☐	☐
No. of choices :	3	3	2	1	1	1

Using fundamental principal of counting, total 6 digit numbers divisible by 25 ending with 25

$\Rightarrow$ $3 \times 3! = 18$ numbers are possible.

Case-II : Numbers ending with 50

Places :	☐	☐	☐	☐	☐	☐
No. of choices :	4	3	2	1	1	1

Using fundamental principal of counting, total 6 digit numbers divisible by 25 ending with 50

$\Rightarrow$ $4! = 24$ numbers are possible.

Hence, total numbers of multiples of $25 = 18 + 24 = 42$

NOW ATTEMPT IN-CHAPTER EXERCISE-A BEFORE PROCEEDING AHEAD IN THIS EBOOK

PERMUTATION — Section - 3

3.1 Definition of Permutation

Permutations of objects is defined as various arrangements of objects in a row. These arrangements can be generated by changing the relative positions of objects in the row.

For example :

If 3 objects are represented as *A, B, C,* then permutations (arrangements or orders) of *A, B, C* in a row can be done in the following ways :

ABC, BAC, CAB, ACB, BCA, CBA

It can be observed that these permutations of *A, B, C* in a row are made by changing relative positions of *A, B, C* amongst themselves.

The permutations of *A, B, C* can also be made by taking not all *A, B, C* at a time but by just taking 2 objects at a time. This can be done in the following ways ;

AB, BA, BC, CB, CA, AC

It can be observed that first, 2 objects are selected and then they are permutated (ordered or arranged) in the row by changing their relative positions among themselves.

3.2. Number of Permutations of *n* different objects taken all at a time

Let us consider that we have n different objects say $a_1, a_2, a_3, \ldots\ldots, a_n$. We have to find total number of different permutations (arrangements or orders) of these objects along a row.

Every permutation of n objects is equivalent to filling n boxes with these objects.

Let us consider n boxes

	1	2	3	4	5		n – 1	n
Boxes :	☐	☐	☐	☐	☐	....	☐	☐
No. of ways :	n	n – 1	n – 2	n – 3	n – 4		2	1

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3 \ldots\ldots, a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the object that has been used to fill Box-1).

Similarly, Box-3, Box-4, ..., Box-n can be filled in $(n-2), (n-3), \ldots\ldots, 1$ ways respectively.

Using fundamental principle of counting,

Total number of different ways to fill n boxes

$= $ Total permutations of n objects in a row

$= n(n-1)(n-2) \ldots\ldots\ldots$ 3.2.1

$= \underline{|n}$ [Product of first n natural numbers can be written as $\underline{|n} = n!$, known as factorial]

Note : Total number of ways to permutate (arrange, order) n different objects in a row $= \underline{|n}$

Illustration - 13 *Find number of different words which can be formed using all the letters of the word 'HISTORY'.*

(A) 720 **(B)** 5040 **(C)** 2520 **(D)** 360

SOLUTION : (B)

Every way of arranging letters of the HISTORY will give us a word.

Therefore total number of ways to permutate letters H, I, S, T, O, R, Y, in a row

= Total number of words that can be formed using all letters together = $\lfloor 7$

$= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Illustration - 14 *In how many ways* 5 *different red balls,* 3 *different black balls and* 2 *different white balls can be arranged along a row?*

(A) 10 ! **(B)** 10^{10} **(C)** $10^{10} - 10!$ **(D)** *None of these*

SOLUTION : (A)

Total number of ways to arrange 10 balls along a row

= Number of permutations of 10 different objects in a row

= 10! [Using section 3.2]

Illustration - 15 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places ?*

(A) 6 **(B)** 12 **(C)** 24 **(D)** 48

SOLUTION : (B)

All the letters in the word 'DELHI' are distinct with 2 vowels (E, I) and 3 consonants (D, L, H).

In five letter words, two even places can occupy 'E' and 'I' in 2! ways and remaining 3 places can have consonants D, L, H in 3! ways. Hence, number of words = (3!) × (2!) = 12

Illustration - 16 *Select the correct choice from the given choice of following question.*

(i) *How many words can be made by using letters of the word 'COMBINE' all at a time ?*

(A) 720 **(B)** 5040 **(C)** 2520 **(D)** 360

(ii) *How many of these words begin and end with a vowel ?*

(A) 720 **(B)** 144 **(C)** 5040 **(D)** 360

(iii) *In how many of these words do the vowels and the consonants occupy the same relative positions as in 'COMBINE'?*

(A) 144 **(B)** 720 **(C)** 5040 **(D)** 360

SOLUTION : (i) . (B) (ii) . (A) (iii) . (A)

(i) The total number of words = arrangements of seven letters taken all at a time = 7! = 5040.

(ii) The corresponding choices for all the places are as follows :

Places	vowel						vowel
No. of choices	3	5	4	3	2	1	2

As there are three vowels (O I E), first place can be filled in three ways and the last place can be filled in two ways. The rest of the places can be filled in 5! ways using five remaining letters.

Hence, number of words = $3 \times 5! \times 2 = 720$

(iii) Vowels should be at second, fifth and seventh positions.

They can be arranged in 3 ! ways.

Consonants should be at first, third, fourth and sixth positions.

They can be arranged here in 4 ! ways.

Hence, number of words = $3! \times 4! = 144$

Illustration - 17 *Select the correct choice from the given choice of following questions.*

(i) *How many words can be formed using letters of the word EQUATION taken all at a time ?*

(A) 8! **(B)** $8\times7!$ **(C)** 7! **(D)** $4\times7!$

(ii) *How many of these begin with E and end with N ?*

(A) $2\times6!$ **(B)** 7! **(C)** $2\times7!$ **(D)** 6!

(iii) *How many of these end and begin with a consonant ?*

(A) 4320 **(B)** 720 **(C)** 1440 **(D)** 2880

(iv) *In how many of these, vowels occupy the first, third, fourth, sixth & seventh positions ?*

(A) 360 **(B)** 720 **(C)** 120 **(D)** *None of these*

SOLUTION : (i) . (A) (ii) . (D) (iii) . (A) (iv) . (B)

(i) No. of arrangements taken all at a time

$$= {}^8P_8 = 8\,! = 40320$$

$\Rightarrow$ 40320 words can be formed.

(ii)

Places:	**E**	_	_	_	_	_	_	**N**
choices:	**1**	6	5	4	3	2	1	**1**

No. of words $= 1 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 1$

$= 6! = 720$ words can be formed.

(iii) There are three consonants and five vowels.

Places:	_	_	_	_	_	_	_	_
Choices:	**3**	6	5	4	3	2	1	**2**

- ➢ Ist place can be filled in three ways, using any of the three consonants (T, Q, N).
- ➢ Last place can be filled in two ways, using any of the remaining two consonants.
- ➢ Remaining places can be filled using remaining six letters.

Numbers of words $= 3 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) \times 2$

$= 3 \times (6!) \times 2 = 4320$ words.

(iv) Let **v** : vowels & **c** : consonants

Places :	v	c	v	v	c	v	v	c
Choices:	5	3	4	3	2	2	1	1

- ➢ First, position vowels in the corresponding places in $5 \times 4 \times 3 \times 2 \times 1 = 5!$ ways
- ➢ Put the consonants in remaining three places in $3 \times 2 \times 1 = 3!$ ways

Now, position of words $= 5!\,3! = 120 \times 6 = 720$

3.3. Number of Permutations of *n* different things taken *r* at a time

Let us consider that we have n different objects say $a_1, a_2, a_4, ..., a_n$. We have to find number of different permutations (arrangements or orders) of these objects taken only r at a time. (*i.e.* we have to select r objects and arrange them).

Every arrangement of n objects taken r at a time is equivalent to filling r boxes.

Let us consider r boxes as shown below :

	1	2	3	4	5		r
Boxes :	☐	☐	☐	☐	☐	....	☐
No. of ways :	n	n – 1	n – 2	n – 3	n – 4		n – (r – 1)

Box-1 can be filled in n ways by any of the n objects $a_1, a_2, a_3 ...., a_n$.

Box-2 can be filled in $(n-1)$ ways by any of the remaining $(n-1)$ objects (excluding the one that is used to fill box-1).

Similarly, boxes 3, 4, 5, ..., r^{th} can be filled in $(n-2), (n-3), ..., n-(r-1)$ ways respectively.

Using fumdamental principle of counting,

Total number of ways to fill r boxes = Total permutations of n different objects taken r at a time

$$= n\,(n-1)\,(n-2)\,(n-3)\,..........\,(n-r+1))$$

Multiply and divide by $|\underline{n-r}$ to get, the number of ways to permutate n things taken r at a time

$$= \frac{n\,(n-1)\,(n-2)\,(n-3)\,.......\,(n-r+1))\,|\underline{n-r}}{|\underline{n-r}}$$

$$= \frac{n\,(n-1)\,(n-2)\,.........\,(n+1)\,(n-r+1)\,(n-r)\,(n-r-1)\,.......\,3.2.1}{|\underline{n-r}}$$

$$= \frac{|\underline{n}}{|\underline{n-r}} \qquad \left[\text{Using: } |\underline{n-r} = (n-r)\,(n-r-1)\,......\,3.2.1\right]$$

$= {}^nP_r$ [read it as "$n\;P\;r$"]

⇒ ${}^nP_r$ represents number of permutations (P) of n different objects (n) taken r objects (r) at a time.

$${}^nP_r = \frac{|\underline{n}}{|\underline{n-r}}$$

Illustrating the Concepts :

Find number of different 4 - letter words can be formed using the letters of the word 'HISTORY'.

Making a 4-letter word is equivalent to permutation of letters of the word 'HISTORY' taken 4 at a time.

$\Rightarrow$ Number of 4-letter words using letters of the word 'HISTORY'

= Number of permutation of letters H, I, S, T, O, R, Y taken only 4 at a time

$$= {}^{7}P_{4} = \frac{\lfloor 7}{\lfloor 7-4} = \frac{\lfloor 7}{\lfloor 3}$$

$$= \frac{7 \times 6 \times 5 \times 4 \times \lfloor 3}{\lfloor 3} = 7 \times 6 \times 5 \times 4 = 840$$

Illustration - 18 *In how many ways 5 different red balls, 3 different black balls and 2 different white balls can be placed in 3 different boxes such that each box contains only 1 ball.*

(A) 360 **(B)** 720 **(C)** 1440 **(D)** 1080

SOLUTION : (B)

The placement of 10 balls in 3 different boxes is equivalent to permutations of 10 different balls taken 3 at a time.

This is because every arrangement of 3 balls will give a different way of placing 3 balls in 3 different boxes.

Therefore, total number of ways to place 10 different balls in 3 different boxes

= Number of permutations of 10 different balls taken 3 at a time

$$= {}^{10}P_{3} = \frac{\lfloor 10}{\lfloor 10-3} = \frac{\lfloor 10}{\lfloor 7} = \frac{10 \times 9 \times 8 \times \lfloor 7}{\lfloor 7}$$

$= 10 \times 9 \times 8 = 720$ ways.

3.4 Permutation of objects when not all objects are different

Number of ways to permuate (arrange) n objects out of which p are identical of one kind, q are identical of another kind, r are identical of third kind and rest all are different to each other $= \dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$. . . . (i)

In earlier sections, we discussed how to permutate n different objects either taking all at a time or just r at a time.

In this section, we will discuss how to arrange objects taken all a time when all object are not different from each other.

For example :

If we have to permutate A, A, B (two A letters are identical) then number of permutations would not be same as permutations of 3 distinct objects say $A, B, C.$ This is because two A letters cannot be permutated amongst themselves. Following are the ways to permutate $A, A, B.$

AAB, ABA, BAA *i.e.* 3 ways. This is not equal to $\lfloor 3$.

So, we need to re-define the formula we use to arrange n different objects.

For a case when all objects are not different. The redefined formula is given is (i). *i.e.* the number of ways to permutate n things out of which p are identical of one kind, q are identical of another kind and r are identical of third kind and c, s, t all different $= \dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$.

Numerator of the above formula is factorial of total number of items.

Each terms in derominator is factorial objects which are of same type and identical to each other.

For example :

If we have to arrange letters of the word MATHEMATICS.

Total letters the word = 11

Number of A letters = 2,

Number of M letters = 2 and

Number of T letters = 2

Number of different letters = 5 (H, E, I, C, S)

Numbers of ways to arrange letters of the word MATHEMATICS $= \dfrac{\lfloor 11}{\lfloor 2 \lfloor 2 \lfloor 2}$

[Using the above formula]

Illustrating the Concepts :

How many nine-letter words can be formed by using the letters of the words

(i) *'E Q U A T I O N S'* **(ii)** *'A L L A H A B A D' ?*

(i) All nine letters in the word 'EQUATIONS' are different.

Hence number of words = 9 ! = 362880. [Using section 3.2]

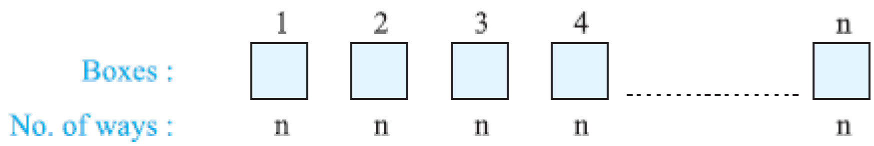
(ii) 'ALLAHABAD' contains 'LL', 'AAAA', 'H', 'B', 'D'. [Using section 3.4]

Hence, number of words $= \dfrac{9!}{2!4!} = \dfrac{9 \times 8 \times 7 \times 6 \times 5}{2} = 7560$

3.5 Arrangement of *n* different objects taken all at a time when repetition of objects is allowed

Here, we have to arrange n different objects in a row where objects can be repeated any number of times i.e. repetition of objects is allowed.

Permutation of n objects in a row is equivalent to filling n boxes. Let us consider n boxes as shown below



Box 1 can be filled in n ways by any of the n objects.

Box 2 can also be filled in n ways as any of the n objects can be used to fill Box 2. This is because we can reuse the object used to fill Box 1 to fill Box 2 as repetition of objects is allowed.

Similarly, Box 3, Box 4, ..., Box n can be filled in n ways each.

Using fundamental principle of counting,

Total number of ways to fill n boxes

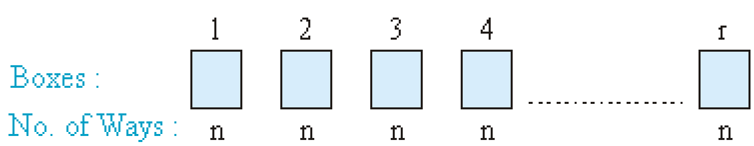
= Total number of ways to permutate n different things taken all at a time when objects can be repeated any number of times

$= n \times n \times n \times n$ ...... n times $= n^n$.

3.6 Arrangement of *n* different objects taken *r* at a time when repetition of objects is allowed

Here we have to arrange n different objects in a row taken only r at a time when objects can be repeated any number of times i.e. repetition of objects is allowed.

Permutation of n objects in a row taken r at a time is equivalent to filling r boxes. Let us consider r boxes as shown below :



Box 1 can be filled in n ways by any of the n objects.

Box 2 can also be filled in n ways as any of the n objects can be used to fill Box 2. This is because, we can reuse the object used to fill Box 1 to fill Box 2 as repetition of objects is allowed.

Similarly, Box 3, Box 4, ... , Box r can be filled in n ways each.

Using fundamental principle of counting,

Total number of way to fill n boxes

= Total number of ways to permutate n different things taken r at a time when objects can be repeated any number of times

$= n \times n \times n$ .... r times $= n^r$

Illustration - 19 *In how many ways can 5 letters be posted in 4 letter boxes ?*

(A) 4^5 **(B)** 5^4 **(C)** $5!$ **(D)** $4!$

SOLUTION : (A)

Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

Illustration - 20 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin :*

(i) *at any one of the 7 floors*

(A) 5^7 **(B)** 7^5 **(C)** $5!$ **(D)** $7!$

(ii) *at different floors*

(A) $7!$ **(B)** 5^7 **(C)** $7!/2!$ **(D)** 7^5

SOLUTION : (i) . (B) (ii) . (C)

Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4 and A_5 can leave the cabinin 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$

Illustration - 21 *There are 6 single choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each ?*

(A) 15625 **(B)** 8000 **(C)** 4000 **(D)** 4096

SOLUTION : (B)

Here, we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences = $4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

NOW ATTEMPT IN-CHAPTER EXERCISE-B BEFORE PROCEEDING AHEAD IN THIS EBOOK

COMBINATIONS Section - 4

4.1 Definition of Combination

Let A, B, C be three different objects. The number of different ways in which we can select two objects out of A, B, C are $AB, BC, CA.$

These ways of selection of two objects from three different objects are also known as combinations of A, B, C taken two at a time or we can say grouping of A, B, C taken two at a time.

In permutation, we had seen that by changing the relative positions of the objects in the row we could generate new permutations i.e. permutation AB is different from BA. But when making combinations (groups selections), by changing the relative positions of objects, we don't get new combinations.

For example :

Combination (selection or group) of objects A, B is same as combination of objects B, A. Thus we treat AB and BA as same combination (selection or group).

4.2 Combination of *n* different objects taken *r* at a time without repetition of objects

We need to find number of ways to select (combine or group) r objects from n different objects.

Let n different objects be $a_1, a_2, a_3, \ldots\ldots, a_n$.

Let x be no. of ways to select r different objects. i.e. x represents number of different groups (combinations) of r objects that can be formed using n different objects.

Each group contains r different objects.

Now if we decide to permutate (arrange) r objects in a group, then it can be done in $\lfloor r$ ways.

Permutations of r different objects in x groups can be done in $x\lfloor r$ ways . . . (i)

But we know permutation of n different objects taken r at a time = ${}^nP_r$ . . . (ii)

Combining (i) and (ii), we get : $x\lfloor r = {}^nP_r$

$$\Rightarrow \quad x = \frac{{}^nP_r}{\lfloor r} = \frac{\lfloor n}{\lfloor n-r \, \lfloor r} = {}^nC_r \qquad \text{[Read it } {}^nC_r\text{]}$$

Number of ways to select r objects from n different objects = ${}^nC_r = \dfrac{\lfloor n}{\lfloor r \, \lfloor n-r}$.

Note : In new notation system, ${}^nC_r$ is also written as $C(n;\, r)$ or $\begin{pmatrix} n \\ p \end{pmatrix}$

4.3 Properties of ^nC_r

(i) $^nC_0 = {^nC_n} = 1$

(ii) $^nC_r = {^nC_{n-r}}$

(iii) If $^nC_r = {^nC_k}$, then $r = k$ or $n - r = k$

(iv) $^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$

(v) $r \cdot {^nC_r} = n\, {^{n-1}C_{r-1}}$

(vi) $\dfrac{1}{r+1}\, {^nC_r} = \dfrac{1}{n+1}\, {^{n+1}C_{r+1}}$

(vii) $\dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}$

(viii) (a) If n is even, ^nC_r is greatest for $r = n/2$.

(b) If n is odd, ^nC_r is greatest for $r = \dfrac{n-1}{2}, \dfrac{n+1}{2}$.

Illustration - 22 *Select the correct choice from the given choices of following questions.*

(i) *How many triangles can be formed by joining the vertices of a hexagon ?*

(A) 10 **(B)** 20 **(C)** 30 **(D)** 60

(ii) *How many diagonals are there in a polygon with n sides ?*

(A) $\dfrac{n(n-1)}{2}$ **(B)** $\dfrac{n(n+1)}{2}$ **(C)** $\dfrac{n(n-3)}{2}$ **(D)** $\dfrac{n(n+3)}{2}$

SOLUTION : (i) . (B) (ii) . (C)

(i) Let $A_1, A_2, A_3, \ldots\ldots, A_6$ be the vertices of the hexagon. One triangle is formed by selecting a group of 3 points from 6 given vertices.

Number of triangles = Number of groups of 3 each from 6 points.

$$= {^6C_3} = \frac{6!}{3!3!} = 20$$

(ii) Number of lines that can be formed by using the given vertices of a polygon

= Number of groups of 2 points each selected from the n points.

$$= {^nC_2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

➢ Out of $^nC_2$ lines, n are the sides of the polygon and remaining $^nC_2 - n$ are the diagonals.

So, number of diagonals $= \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$

Illustration - 23 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicketkeepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounders, 2-bowlers and 1 wicketkeeper.*

(A) $\frac{10!}{5!}$ (B) $\frac{14\times 10!}{5!}$ (C) $\frac{14\times 10!}{3\times 5!}$ (D) $\frac{10!}{3\times 5!}$

SOLUTION : (C)

Divide the selection of team into four operation.

I : Selection of batsman can be done (5 from 10) in ${}^{10}C_5$ ways.

II : Selection of bowlers can be done (2 from 8) in ${}^{8}C_2$ ways.

III : Selection of all-rounders can be done (3 from 5) in ${}^{5}C_3$ ways.

IV : Selection of wicketkeeper can be done (1 from 2) in ${}^{2}C_1$ ways.

Hence, the team can be selected in $= {}^{10}C_5 \times {}^{8}C_2 \times {}^{5}C_3 \times {}^{2}C_1$ ways $= \frac{14\times 10!}{3\times 5!}$ = ways.

Illustration - 24 *A man has 7 relatives, 4 of them ladies and 3 gentlemen ; his wife has 7 relatives, 3 of them are ladies and 4 gentlemen. In how many ways can he invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives?*

(A) 144 (B) 720 (C) 485 (D) 340

SOLUTION : (C)

The possible ways of selecting 3 ladies and 3 gentleman for the party can be analysed with the help of the following table.

Man's relative		Wife's relative		Number of ways
Ladies (4)	Gentlemen (3)	Ladies (3)	Gentlemen (4)	
3	0	0	3	${}^4C_3\ {}^3C_0\ {}^3C_0\ {}^4C_3 = 16$
2	1	1	2	${}^4C_2\ {}^3C_1\ {}^3C_1\ {}^4C_2 = 324$
1	2	2	1	${}^4C_1\ {}^3C_2\ {}^3C_2\ {}^4C_1 = 144$
0	3	3	0	${}^4C_0\ {}^3C_3\ {}^3C_3\ {}^4C_0 = 1$

Total number of ways to invite = 16 + 324 + 144 + 1 = 485

Illustration - 25 *A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated ?*

(A) $\frac{8!\,10!}{4!\,6!}$ (B) $\frac{8!\,8!\,10!}{4!\,6!}$ (C) $\frac{8!\,8!}{4!\,6!}$ (D) $\frac{8!\,8!}{6!}$

SOLUTION : (B)

Let $A_1, A_2, A_3, \ldots\ldots, A_{16}$ be the sixteen persons. Assume that A_1, A_2, A_3, A_4 want to sit on side 1 and A_5, A_6 want to sit on side 2.

The persons can be made to sit if we complete the following operations.

(i) Select 4 chairs from the side 1 in $^8C_4$ ways and allot these chairs to A_1, A_2, A_3, A_4 in 4 ! ways.

(ii) Select two chairs from side 2 in $^8C_2$ ways and allot these two chairs to A_5, A_6 in 2 ! ways.

(iii) Arrange the remaining 10 persons in remaining 10 chairs in 10 ! ways.

$\Rightarrow$ Hence the total number of ways in which the persons can be arranged

$$= \left({}^8C_4 4!\right)\left({}^8C_2 2!\right)(10!) = \frac{8!}{4!\,4!}4! \times \frac{8!\,2!}{2!\,6!}10! = \frac{8!\,8!\,10!}{4!\,6!}$$

Illustration - 26 *A mixed doubles tennis game is to be arranged from 5 married couples. In how many ways the game be arranged if no husband and wife pair is included in the same game?*

(A) 60 (B) 30 (C) 120 (D) 240

SOLUTION: (A)

To arrange the game we have to do the following operations.

(i) Select two men from from 5 men in $^5C_2$ ways.

(ii) Select two women from 3 women excluding the wives of the men already selected. This can be done in $^3C_2$ ways.

(iii) Arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2 , this can be done in 2 ways :

M_1W_1 play against M_2W_2

M_2W_1 play against M_1W_2

Hence the number of ways to arrange the game = $^5C_2\ {}^3C_2\ (2) = 10 \times 3 \times 2 = 60$

NOW ATTEMPT IN-CHAPTER EXERCISE-C BEFORE PROCEEDING AHEAD IN THIS EBOOK

TYPICAL PROBLEM CATEGORIES Section - 5

Most of the typical problems in permutation and combination can be categorised in various types. To solve these typical problems, it is advisable to apply standard ways or methods available.

Let us call these categories of problems as Typical Problem Categories (TPC).

5.1 TPC - 1 : Always including particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should always be included in the selection = ${}^{n-p}C_{r-p}$.

Logic :

We can select p particular objects in 1 way. Now from remaining $(n-p)$ objects we select remaining $(r-p)$ objects in ${}^{n-p}C_{r-p}$ ways.

Using fundamental principle of counting, number of ways to select r objects where p particular objects are always included = $1 \times {}^{n-p}C_{r-p} = {}^{n-p}C_{r-p}$.

Illustrating the Concepts :

In how many ways a team of 11 players be selected from a list of 16 players where two particular players should always be included in the team.

Using formula given in TPC-1, number of ways to make a team of 11 players from 16 players always including 2 particular players = ${}^{16-2}C_{11-2} = {}^{14}C_9$

5.2 TPC-2 : Always excluding *p* particular objects in the selection

The number of ways to select r objects from n different objects where p particular objects should never be included in the selection = ${}^{n-p}C_r$.

Logic :

As p particular objects are never to be selected, selection should be made from remaining $(n - p)$ objects. Therefore r objects can be selected from $(n-p)$ different objects in ${}^{n-p}C_r$ ways.

Illustrating the Concepts :

In how many ways a team of 11 players can be selected from a list of 16 player such that 2 particular players should be included in the selection.

Using the formula given in TPC-2, the number of ways to select a team of 11 players from a list of 16 players, always excluding 2 particular players = ${}^{16-2}C_{11} = {}^{14}C_{11}$

5.3 TPC-3 : Always including *p* particular objects in the arrangement

The number of ways to select and arrange (permutate) r objects from n different objects such that arrangement should always include p particular objects $= {}^{n-p}C_{r-p}\,\underline{\lfloor r}$.

Logic :

First select p particular objects which should always be included in 1 way. **. . . (i)**

Then select remaining $(r-p)$ objects from remaining $(n-p)$ objects in ${}^{n-p}C_{r-p}$ ways. **. . . (ii)**

Finally arrange r selected objects in $\underline{\lfloor r}$ ways. **. . . (iii)**

Using fundamental principle of countings operations **(i)**, **(ii)** and **(iii)** can be performed together in ways.

$$= 1 \times {}^{n-p}C_{r-p} \times \underline{\lfloor r} \text{ ways.}$$

5.4 TPC-4 : Always excluding *p* particular objects in the arrangement

The number of ways to select and arrange r objects from n different objects such that p particular objects are always excluded in the selection $= {}^{n-p}C_r\,\underline{\lfloor r}$.

Logic :

First exclude p particular objects from n different objects.

Then select r objects from $(n-p)$ different objects in ${}^{n-p}C_r$ ways. **. . . (i)**

Then permutate r selected objects in $\underline{\lfloor r}$ ways. **. . . (ii)**

Using fundamental principle of counting operations (i) and (ii) can be performed together in ways

$$= {}^{n-p}C_r\,\underline{\lfloor r} \text{ ways.}$$

Illustrating the Concepts :

How many three-letter words can be made using the letters of the words 'SOCIETY', so that

(i) *'S' is included in each word?* **(ii)** *'S' is not included in any word?*

(i) To include S in every word, we have two cases.

Step - I :

Select the remaining two letters from remaining 6 letters i.e.

O, C, I, E, T, Y in ${}^{7-1}C_{3-1} = {}^{6}C_2$ ways.

Step - II :

Include S in each group and then arrange each group of three in 3! ways.

$\Rightarrow$ Number of words = ${}^6C_2\ 3\,! = 90$

(ii) If S is not to be included, then we have to make all the three words from the remaining 6.

$\Rightarrow$ Number of words = ${}^6C_3\ 3\,! = 120$

5.5 TPC-5 : *p* particular objects always together in the arrangement.

The number of ways to arrange *n* different objects such that *p* particular objects remain together in the arrangement

$$= \lfloor(n-p+1)\ \lfloor p$$

Logic :

Make a group of p particular objects that should remain together. Arrange this group of *p* particular objects and remaining $(n-p)$ objects in $\lfloor n-p+1$ ways. **. . . (i)**

Finally arrange *p* particular objects amongst themselves in $\lfloor p$ ways. **. . . (ii)**

Using fundamental principle of counting operations (i) and (ii) can be performed together in ways

$$= \lfloor(n-p+1)\ \lfloor p\,.$$

Illustrating the Concepts :

How many words can be formed using the letters of the word 'TRIANGLE' so that

(a) *'A' and 'N' are always together ?* **(b)** *'T', 'R', 'I 'are always together ?*

(a) Assume (*AN*) as a single letter. Now there are seven letters in all :

(*AN*), T, R, I, G, L, E

Seven letters can be arranged in 7 ! ways.

All these 7! words will contain *A and N* together. *A and N* can now be arranged among themselves in 2! ways *(AN and NA)*.

Hence total number of words = 7! 2! = 10080

(b) Assume (*TRI*) as a single letter.

- The letters : (*TRI*), *A*, *N*, *G*, *L*, *E* can be rearranged in 6! ways.
- *TRI* can be arranged among themselves in 3! ways.

Total number of words = 6! 3! = 4320

Illustration - 27 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together in every word?*

(A) 240 **(B)** 1440 **(C)** 720 **(D)** 480

SOLUTION : (B)

There are 5 vowels and 3 consonants in 'EQUATION'. To form the words we have three cases.

Step I : Select vowels (3 from 5) in ${}^5C_3$ ways.

Step II : Select consonants (2 from 3) in ${}^3C_2$ ways.

Step III : Arrange the selected letters (3 vowels and 2 consonants always together) in $4! \times 2!$ ways.

Hence the no. of words $= {}^5C_3\, {}^3C_2\, 4!\, 2!$

$= 10 \times 3 \times 24 \times 2 = 1440$

5.6 TPC-6 : p particular objects always separated in the arrangement.

The number of ways to arrange n different objects such that p particular objects are always separated

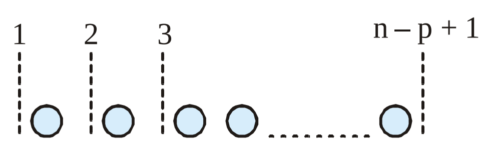
$$= {}^{n-p+1}C_p \; \underline{|n-p} \; \underline{|p}$$

Logic :

We have to place p particular objects between $(n-p)$ remaining objects so that all p particular objects must be separated from each other.

From figure we can see there are $(n-p+1)$ places between $(n-p)$ objects where we can place p particular objects such that p objects are separated from each other.

Select p places from $(n-p+1)$ places for p particular objects in ${}^{n-p+1}C_p$ ways.



(n – p) remaining objects

Now place and arrange p objects in p selected places in $\underline{|p}$ ways. If all p particular objects are not different, then we use formula given in 3.4 to arrange p objects.

Finally, arrange $(n-p)$ remaining objects in $\underline{|n-p}$ ways. If $(n-p)$ objects are not different, then use formula given in section 3.4 to arrange them.

Using fundamental principle of counting, all operations can be done together in

$$= {}^{n-p+1}C_p \; \underline{|p} \; \underline{|n-p} \text{ ways.}$$

Illustrating the Concepts :

There are 9 candidates for an examination out of which 3 are appearing in Mathematics and remaining 6 are appearing in different subjects. In how many ways can they be seated in a row so that no two Mathematics candidates are together?

Divide the work in two cases.

Step - I :

First, arrange the remaining candidates in 6 ! ways.

Step - II :

Place the three Mathematics candidates in the row of six other candidates so that no two of them are together.

X : Places available for Mathematics candidates.

O : Others.

X	O	X	O	X	O	X	O	X	O	X	O	X

In any arrangement of 6 other candidates (O), there are seven places available for Mathematics candidates so that they are not together. Now 3 Mathematics candidates can be placed in these 7 places in ${}^7P_3$ ways.

Hence total number of arrangements

$$= 6!\,{}^7P_3 = 720 \times \frac{7!}{4!} = 151200$$

Illustration - 28 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus (–) signs are together?*

(A) ${}^8C_5$ **(B)** ${}^8C_5 \times 7! \times 5!$ **(C)** ${}^8C_5 \times 5!$ **(D)** ${}^8C_5 \times 7!$

SOLUTION : (A)

Step - I :

The plus signs can be arranged in one way (because all are identical).

	+		+		+		+		+		+		+

A blank box shows available spaces for the minus signs.

Step - II :

The 5 minus (–) signs are now to be placed in the 8 available spaces so that no two of them are together.

(i) Select 5 places for minus signs in ${}^8C_5$ ways.

(ii) Arrange the minus signs in the selected places in 1 way (all signs being identical).

Hence number of possible arrangements = $1 \times {}^8C_5 \times 1 = 56$

5.7 TPC - 7 : Problems based on atleast or at most constraint

There are problems in which constrants are to select minimum (at least) or maximum (at most) objects in the selection.

In these problems, we should always make cases to select objects. If we don't make cases, we will be get wrong answer. Following illustrations will show you how to make cases to solve problems of this type.

Illustrating the Concepts :

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour?

The selection of balls from 5 red and 6 white balls will consist of any of the following possibilities.

Red Balls (out of 5)	2	3	4
White Balls (out of 6)	4	3	2

- If the selection contains 2 red & 4 white balls, then it can be done in ${}^5C_2\ {}^6C_4$ ways.
- If the selection contains 3 red & 3 white balls then it can be done in ${}^5C_3\ {}^6C_3$ ways.
- If the selection contains 4 red & 2 white balls then it can be done in ${}^5C_4\ {}^6C_2$ ways.

Any one of the above three cases can occur. Hence the total number of ways to select the balls

$$= {}^5C_2\ {}^6C_4 + {}^5C_3\ {}^6C_3 + {}^5C_4\ {}^6C_2 = 10\,(15) + 10\,(20) + 5\,(15) = 425$$

Illustration - 29 *In how many ways a team of 5 members can be selected from 4 ladies and 8 gentlemen such that selection includes at least 2 ladies ?*

(A) 336 **(B)** 448 **(C)** 449 **(D)** 456

SOLUTION : (D)

As the selection includes 'at least' constraint, we make cases to find total number of teams.

Ladies in the team(4)	Gentlemen in the team (8)	Number of ways to select team of 5
2	3	${}^4C_2 \times {}^8C_3$
3	2	${}^4C_3 \times {}^8C_2$
4	1	${}^4C_4 \times {}^8C_1$

Combining all cases shown in the table = total number of ways to select a team of 5 members

$$= {}^4C_2 \times {}^8C_3 + {}^4C_3 \times {}^8C_2 + {}^4C_4 \times {}^8C_1 = 456$$

★ 5.8 TPC-8 : Permutations of *n* objects taken *r* at a time when all *n* objects are not different

In this section we will discuss how to arrange (permutate) n objects taken or at a time where all n objects are not different. For example arrangements of letters $AABBBC$ taken 3 at a time.

To find such arrangements, it is not possible to derive a formula that can be applied in all such cases.

So, we will discuss a method (or procedure) that should be applied to find arrangements. The method involves making cases based on alike items we choose in the arrangement. You should read the following illustrations to learn how to apply this "method of cases" to find arrangements of n objects taken r at a time when all objects are not different.

Illustrating the Concepts :

In how many ways we can arrange letters A,A,B,B,B,C taken 3 at a time.

The given letters include AA, BBB, C *i.e.* $2A$ letters, $3B$ letters and $1C$ letter.

To find arrangements of B letters, we will make following cases based on alike letters we choose in the arrangement.

Case - 1 : All 3 letters are alike

- 3 alike letters can be selected from given letters in only 1 way i.e. BBB.

Further 3 selected letters can be arranged among themselves in $\dfrac{\lfloor 3}{\lfloor 3} = 1$ way.

$\Rightarrow$ Total number of arrangement with all letters alike = 1 **. . . (i)**

Case - 2 : 2 alike and 1 different

- 2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.
- 1 different letter (different from alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (C, A or B either).

Further 2 alike and 1 different selected letters can be arranged among themselves in $\dfrac{\lfloor 3}{\lfloor 2}$ ways.

$\Rightarrow$ Total number of arrangements with "2 alike and 1 different letter"

$$= {}^2C_1 \times {}^2C_1 \times \frac{\lfloor 3}{\lfloor 2} = 2 \times 2 \times 3 = 12 \qquad \textbf{. . . (ii)}$$

Case - 3 : All different letters

- All 3 letters different can be selected from 3 different letters (A, B, C) in 1 way.

Further 3 different letters can be arranged among themselves in $\lfloor 3$ ways.

$\Rightarrow$ Total number of arrangements with all 3 letters different $= 1 \times \lfloor 3 = \lfloor 3 = 6$ **. . . (iii)**

Combining (i), (ii) and (iii),

Total number of permutations of A, A, B, B, B, C taken 3 at a time = 1 + 12 + 6 = 19

Illustration - 30 *How many four-letter words can be formed using the letters of the word 'INEFFECTIVE' ?*

(A) 840 **(B)** 1380 **(C)** 1422 **(D)** *None of these*

SOLUTION : (C)

'*INEFFECTIVE*' contains 11 lettres : *EEE, FF, II, C, T, N, V.*

As all letters are not different, we cannot use ^nP_r. The four-letter words will be from any one of the following categories.

1. 3 alike letters, 1 different letter.
2. 2 alike letters, 2 alike letters.
3. 2 alike letters, 2 different letters.
4. All different letters.

1. 3 alike, 1 different :

3 alike can be selected in one way i.e. *EEE.*

Different letters can be selected from *F, I, T, N, V, C* in $^6C_1$ ways.

$\Rightarrow$ Number of groups $= 1 \times {}^6C_1 = 6$ $\Rightarrow$ Number of words $= 6 \times \dfrac{4!}{3! \times 1!} = 24$

2. 2 alike, 2 alike :

Two sets of 2 alike can be selected from 3 sets *(EE, II, FF)* in $^3C_2$ ways.

$\Rightarrow$ Number of groups $= {}^3C_2$ $\Rightarrow$ Number of words $= {}^3C_2 \times \dfrac{4!}{2! \times 2!} = 18$

3. 2 alike, 2 different :

$\Rightarrow$ Number of groups $= ({}^3C_1) \times ({}^6C_2) = 45$ $\Rightarrow$ Number of words $= 45 \times \dfrac{4!}{2!} = 540$

4. All different :

$\Rightarrow$ No. of groups $= {}^7C_4$ (out of *E, F, I, T, N, V, C*) $= 35$ $\Rightarrow$ No. of words $= 35 \times 4! = 840$

Hence total four-letter words $= 24 + 18 + 540 + 840 = 1422$

★ 5.9 TPC-9 : Selection of *r* objects from *n* objects when all *n* objects are not different.

In this problem type we will discuss how to select *r* objects from *n* objects when all *n* objects are not different.

For example selection of 3 letters from letters *AABBBC.*

To find number of ways to select, it is possible to derive a formula that can be applied in all such cases.

Instead of formula, we will discuss a method (procedure) that should be applied to find selections.

The method involves making cases based on alike items in the selection. You should be through the following illustrations to learn how to apply this "method of cases" to find selections of *r* objects from *n* objects when all *n* objects are not different.

Illustrating the Concepts :

In how many ways 3 letters can be selected from letters A,A,B,B,B,C.

The given letters include AA, BBB, C i.e. 2A letters, 3B letters and 1C letter.

To find number of selections, we will make the following cases based on alike letters we choose in the selection.

Case - 1 : All 3 letters are alike

- 3 alike letters can be selected from given letters in only 1 way i.e. BBB.

$\Rightarrow$ The number of selections with all 3 letters alike = 1 **. . . (i)**

Case - 2 : 2 alike and 1 different letter

- 2 alike letters can be selected from 2 sets of alike letters (AA, BB) in ${}^2C_1$ ways.
- 1 different letter (different from alike letters) can be selected from remaining letters in ${}^2C_1$ ways. (either A or B).

Using fundamental principle of counting,

Total number of selections with 2 alike and 1 different letter = ${}^2C_1 \times {}^2C_1 = 4$ ways **. . . (ii)**

Case - 3 : All letters different

- All 3 letters different can be selected from 3 different letters (A, B, C) in 1 ways.

$\Rightarrow$ Total number of ways to select 3 different letters in 1 way **. . . (iii)**

Combining (i), (ii) and (iii),

Total number of ways to select 3 letters from given letters = 1 + 4 + 1 = 6.

Illustration - 31 *In how many ways 4 letters can be selected from letters of the word 'INEFFECTIVE' ?*

(A) 80 **(B)** 89 **(C)** 51 **(D)** *None of these*

SOLUTION : (B)

INEFFECTIVE contains 11 letters : EEE, FF, II, C, T, N, V

We will make following cases to select 4 letters.

Case - 1 : 3 alike and 1 different

- 3 alike letters can be selected from 1 set of 3 alike letters (EEE) in 1 way.

$\Rightarrow$ The number of ways to select 3 alike letters = 1

$\Rightarrow$ The number of ways to select 1 different letters = 6 $\Rightarrow$ Total ways = 6 × 1 = 6 **. . . (i)**

Case - 2 : 2 alike and 2 alike

- '2 alike and 2 alike' means we have to select 2 groups of 2 alike letters (EE, FF, II) in ${}^3C_2$ ways.

$\Rightarrow$ The number of ways to select "2 alike and 2 alike" letters = ${}^3C_2 = 3$.

Case - 3 : 2 alike and 2 different

- 1 group of 2 alike letters can be selected from 3 sets of 2 alike letters *(EE, FF, II)* in ${}^3C_1$ ways.
- 2 different letters can be selected from 6 different letters (*C, T, N, V,* remaining 2 sets of two letters alike) in ${}^6C_2$ ways.

$\Rightarrow$ The number of ways to select "2 alike and 2 different letters" ${}^3C_1 \times {}^6C_2 = 3 \times 15 = 45$ . . . (ii)

Case - 4 : All different letters

- All different letters can be selected from 7 different letters *(I, E, F, N, C, T, V)* in ${}^7C_4$ ways.

$\Rightarrow$ The number of ways to select all different letters = ${}^7C_4 = 35$ . . . (iii)

Combining (i), (ii), (iii), we get

Total number of ways to select 4 letters from letters of the word *INEFFECTIVE*

= 6 + 3 + 45 + 35 = 89.

★ 5.10 TPC-10 : Selection of one or more objects

(A) Selection of one or more objects from *n* different objects

The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects = $2^n - 1$.

Logic :

The numbers of ways to select 1 object from n different objects = ${}^nC_1$

The number of ways to select 2 objects from n different objects = ${}^nC_2$

- - - - - -　　- - - - - -　　- - - - - -

- - - - - -　　- - - - - -　　- - - - - -

- - - - - -　　- - - - - -　　- - - - - -

The number of ways to select n objects from n different objects = ${}^nC_n$

Combining all above cases, we get

The number of ways to select at least one (one or more) object from n different objects

$= {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots\ldots + {}^nC_n$

$= 2^n - 1$ [Using : sum of binomial coefficients in the expansion of $(1+x)^n = 2^n$]

Alternate Logic :

Let us assume $a_1, a_2, a_3, \ldots\ldots, a_n$ be n different objects.

We have to make our selection from these n objects.

Objects	a_1	a_2	a_3	a_4	. . . .	a_n
Ways	2	2	2	2	. . . .	2

We can make our selection from a_1 object in 2 ways.

This is because either we will choose a_1 or we would not choose a_1. Similarly selection of $a_2, a_3, \ldots, a_n$ can be done in 2 ways each.

Using fundamental principle of counting,

The total number of ways to make selection of at least one object from $a_1, a_2, \ldots, a_n$

$$= 2 \times 2 \times 2 \times 2 \ldots\ldots\ n \text{ times}$$

$$= 2^n$$

But the above selection includes a case where we have not selected any object. On subtracting this case from 2^n we get, the number of ways to select atleast one (one or more) object from n different objects $= 2^n - 1$

Note :

(a) The number of ways to select 0 or more objects from n different objects $= 2^n$

(b) The number of ways to select at least 2 objects from n different objects $= 2^n - 1 - {}^nC_1$

(c) The number of ways to select at least r objects from n different objects

$$= 2^n - 1 - {}^nC_1 - {}^nC_2 - {}^nC_3 - \ldots\ldots - {}^nC_{r-1}$$

(B) Selection of one or more objects from *n* identical objects

The number of ways to select one or more objects (or at least one object) from n identical objects $= n$

Logic :

To select r objects from n identical objects, we can not use ${}^nC_r$ formula here as all objects are not different. In fact, all objects all identical. It means we can not choose objects. It does not matter which r objects we take as all objects are identical.

The number of ways to select 1 object from n identical objects = 1

The number of ways to select 2 objects from n identical objects = 1

- - - - - -   - - - - - -   - - - - - -

- - - - - -   - - - - - -   - - - - - -

- - - - - -   - - - - - -   - - - - - -

The number of ways to select n objects from n identical objects = 1

Combining all above cases, we get

Total number of ways to select 1 or more objects from n identical objects

$$= 1 + 1 + \ldots\ldots\ n \text{ times}$$

$$= n$$

Note :

(a) The number of ways to select 0 or more objects from n identical objects = $n + 1$

(b) The number of ways to select at least 2 objects from n identical objects = $n - 1$

(c) The number of ways to select r objects from n identical objects is 1

(d) The total number of selections of some or all out of $(p + q + r)$ objects where p are alike of one kind, q are alike of second kind and rest r are alike of third kind is

$(p + 1)(q + 1)(r + 1) - 1$ [Using fundamental principle of counting]

(C) Selection of one or more objects from objects which are not all different from each other.

The number of ways to select one or more objects from $(p + q + r \ldots\ldots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, ... and remaining n are different from each other $= [(p + 1)(q + 1)(r + 1) \ldots\ldots\ldots 2^n] - 1$.

Logic :

The numbers of ways to select 0 or more objects from p alike objects of one kind = $p + 1$

The number of ways to select 0 or more objects from q alike objects of second kind = $q + 1$

The number of ways to select 0 or more objects from r alike objects of third kind = $r + 1$

- - - - - - - - - - - - - - - - - -
- - - - - - - - - - - - - - - - - -
- - - - - - - - - - - - - - - - - -

The number of ways to select 0 or more objects from n different objects = 2^n

Combining all cases and using fundamental princple of counting, we get :

Total number of ways to select 0 or more objects = $[(p + 1)(q + 1)(r + 1) \ldots\ldots 2^n]$ **. . . (i)**

But above selection includes a case where we have not selected any object. So we need to subtract 1 from the above result if we want to select at least one object.

Therefore, the total number of ways to select one or more objects (at least one) from p alike of one kind, q alike of another kind, r alike of third kind, ..., and n different objects

$$= [(p + 1)(q + 1)(r + 1) \ldots\ldots 2^n] - 1$$

Note :

(a) The number of ways to select 0 or more objects from p alike of one kind, q alike of second kind, r alike of third kind and n different objects = $(p + 1)(q + 1)(r + 1)\, 2^n$

(b) The number of ways to select objects from p alike of one kind, q alike of second kind and r alike of third kind and n different objects such that selection includes at least one object of each kind = $pqr\,(2^n - 1)$

Illustrating the Concepts :

Find the number of ways in which one or more letters can be selected form the letters :

A, A, A, A, B, B, B, C, D, E.

The given letters can be divided into five following categories : (*AAAA*), (*BBB*), *C*, *D*, *E*.

To select at least one letter, we have to take five decisions - one for every category.

Selections from (*AAAA*) can be made in 5 ways :

Include no A, include one *A*, include *AA*, include *AAA*, include *AAAA*.

Similarly, selections from (*BBB*) can be made in 4 ways, and selections from *C, D, E* can be made in $2 \times 2 \times 2$ ways.

$\Rightarrow$ Total number of selection

$= 5 \times 4 \times (2 \times 2 \times 2) - 1 = 159$ (excluding the case when no letter is selected).

★

Illustration - 32 *A man has 5 friends. In how many ways can he invite one or more of them to a party?*

(A) 32 **(B)** 31 **(C)** 30 **(D)** 16

SOLUTION : (B)

If he invites one person to the party,

Number of ways = ${}^5C_1$

If he invites two persons to the party,

Number of ways = ${}^5C_2$

Proceeding on the similar pattern,

Total number of ways to invite

$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$

$= 5 + 10 + 10 + 5 + 1 = 31$

Alternate Approach :

To invite one or more friends to the party, he has to take 5 decisions – one for every friend.

Each decision can be taken in two ways - invited or not invited.

Hence the number of ways to invite one or more

= (number of ways to make 5 decisions – 1)

$= 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1 = 31$

Note that we have subtract 1 to exclude the case when all are not invited.

★

Illustration - 33 *The question paper in the examination contains three sections - A, B, C. There are 6, 4, 3 questions in sections A, B, C respectively. A student has the freedom to answer any number of questions attempting at least one from each section. In how many ways can the paper be attempted by a student?*

(A) 8192 **(B)** 6615 **(C)** 7168 **(D)** *None of these*

SOLUTION : (B)

There are three possible cases :

Case - I :

Section A contains 6 questions. The student can select at least one from these in $2^6 - 1$ ways.

Case - II :

Section B contains 4 questions. The student can select at least one from these in $2^4 - 1$ ways.

Case - III :

Section C can similarly be attempted in $2^3 - 1$ ways.

Hence total number of ways to attempt the paper

$= (2^6 - 1)(2^4 - 1)(2^3 - 1)$

$= 63 \times 15 \times 7 = 6615$

★

Illustration - 34 *Find the number of factors (excluding 1 & the expression itself) of the product of $a^7 b^4 c^3 d e f$ where a, b, c, d, e, f are all prime numbers.*

(A) 1280 **(B)** 1279 **(C)** 2178 **(D)** 1260

SOLUTION : (C)

A factor of expression $a^7 b^4 c^3 d e f$ is simply the result of selecting one or more letters from $7\ a's,\ 4\ b's,\ 3\ c's,\ d, e, f$.

The collection of letters can be observed as a collection of 17 objects out of which 7 are alike of one kind ($a's$), 4 are of second kind ($b's$), 3 are of third kind ($c's$) and 3 are different (d, e, f).

The number of selections

$= (1 + 7)(1 + 4)(1 + 3)\, 2^3 = 8 \times 5 \times 4 \times 8$

$= 1280.$

But we have to exclude two cases :

(i) When no letter is selected,

(ii) When all letters are selected.

Hence the number of factors = 1280 – 2 = 1278

★ 5.11 TPC-11 : Dearrangement Theorem

If n distinct objects are to be arranged in a row such that no object occupies its original place, then to find number of ways to arrange them, we use dearrangement theorem i.e.

$$\text{Number of ways to dearrange} = \underline{|n}\left[1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \ldots\ldots (-1)^n \frac{1}{\underline{|n}}\right]$$

Let S_1, S_2, S_3 are three slots where objects A, B, C should be placed.

Number of ways to place A, B, C in S_1, S_2, S_3 such that A goes to S_1, B goes to S_2 and C goes to S_3 i.e. all objects are placed in their correct places = 1.

Number of way to place only one object in a wrong slot is not possible because if A is placed in say S_2, then B, whose correct slot is S_2, would take either S_1 or S_3. It means B is also placed in the wrong slot. So it is not possible to place object in wrong slot.

To place objects A, B, C in S_1, S_2, S_3 such that all objects are placed in wrong slots, we use dearrangement theorem *i.e.*

Number of ways to place A, B, C all in wrong slots $= \underline{|3}\left[1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}\right] = 2$ ways.

★

Illustration - 35 *There are 5 boxes of 5 different colors. Also there are 5 balls of colors same as those of the boxes. In how many ways we can place 5 balls in 5 boxes such that*

(i) *all balls are placed in the boxes of colors not same as those of the ball.*

(A) 44 **(B)** 45 **(C)** 48 **(D)** 60

(ii) *at least 2 balls are placed in boxes of the same color.*

(A) 32 **(B)** 31 **(C)** 76 **(D)** 75

SOLUTION : (i) . (A) (ii) . (B)

(i) All the balls should be placed in the wrong boxes

i.e. boxes not of the color same as balls.

Using dearrangement theorem, number of ways in which this can be done.

$$= \underline{|5}\left[1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\frac{1}{\underline{|5}}\right]$$

$$= 120\left[1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\right] = 60 - 20 + 5 - 1 = 44$$

(ii) Atleast 2 balls are placed in the correct boxes i.e. boxes of the color same as ball

= Total number of ways to place balls in boxes – No. of ways to place balls such that all balls are placed in wrong boxes – No. of ways to place balls in boxes such that 1 ball is placed in the correct box (i.e. boxes of the same color as balls).

= $\underline{|5} - 44 -$ No. of ways to select a ball that will be in correct box × No. of ways in which remaining 4 balls can be placed in 4 boxes such that all balls go in wrong boxes (boxes of color different from balls).

$$= \underline{|5} - 44 - {}^5C_1 \times \underline{|4}\left[1-\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}\right]$$

$= 120 - 44 - 5 \times 9$ [Using answer of (i) part and dearrangement theorem]

$= 120 - 44 - 45 = 31$

★ 5.12 TPC-12 : Sum of the numbers

In this TPC, we will learn how to find sum of all the numbers that can be formed using the given digits.

The following illustrations will help you learn how to find the sum of the numbers.

Illustrating the Concepts :

Find the sum of all five-digit numbers that can be formed using digits 1, 2, 3, 4, 5 *if repetition is not allowed?*

There are 5! = 120 five digit numbers and there are 5 digits. Hence by symmetry or otherwise we can see that each digit will appear in any place *i. e.*

(unit's or ten's or . . . .) $\frac{5!}{5}$ times.

$\Rightarrow$ X = sum of digits in any place

$$\Rightarrow X = \frac{5!}{5}\times 5 + \frac{5!}{5}\times 4 + \frac{5!}{5}\times 3 + \frac{5!}{5}\times 2 + \frac{5!}{5}\times 1$$

$$\Rightarrow \quad X = \frac{5!}{5}\times(5+4+3+2+1) = \frac{5!}{5}(15)$$

$\Rightarrow$ Hence, the sum of all numbers

$= X + 10X + 100X + 1000X + 10000X$

$= X(1 + 10 + 100 + 1000 + 10000)$

$= \frac{5!}{5}(15)(1+10+100+1000+10000)$

$= 24\,(15)\,(11111) = 3999960$

5.13 TPC-13 : Rank of a word in the dictionary

In this problem type, dictionary of words is formed by using the letters of the given word. The dictionary format means words are arranged in the alphabetical order. You will be supposed to find the rank (position) of the given word or some other word in the dictionary.

Following illustrations will help you learn how to find the rank in the dictionary.

Illustrating the Concepts :

If all the letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary, then find the rank of the word 'RANDOM' in the dictionary.

In a dictionary the words at each stage are arranged in alphabetical order. In the given problem, we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. D, M, N, O will occur in the first place the same number of times.

Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120
Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120

After this, words beginning with *RA* must follow.

Number of words beginning with *RAD* or *RAM* = 3!

Now the words beginning with *RAN* must follow.

First one is *RANDMO* and the next one is *RANDOM*.

$\therefore$ Rank of *RANDOM* = 5 (5!) + 2 (3!) + 2 = 614

Illustration - 36 *Find the rank of the word "TTEERL" in the dictionary of words formed by using the letters of the word 'LETTER'.*

(A) 168 **(B)** 170 **(C)** 169 **(D)** 171

SOLUTION : (B)

In the dictionary of words formed, we need to count words before the word *'TTEERL'* in the dictionary. To count such words, we need to first count words starting with *E, L, R, TE, TL, TR* and then add 2 to the count for words *'TTEELR'* and *'TTEERL'*.

Number of words starting with *E* = Arrangement of letters *E, T, T, R, L* = $\dfrac{\lfloor 5}{\lfloor 2}$

Number of words starting with *L* = Arrangement of letters *E, T, T, E, R* = $\dfrac{\lfloor 5}{\lfloor 2\lfloor 2}$

Number of words starting with *R* = Arrangement of letters *E, T, T, E, L* = $\dfrac{\lfloor 5}{\lfloor 2\lfloor 2}$

Number of words starting with *TE* = Arrangement of letters *T, E, R, L* = $\lfloor 4$

Number of words starting with *TL* = Arrangement of letters *E, T, E, R* = $\dfrac{\lfloor 4}{\lfloor 2}$

Number of words starting with *TR* = Arrangement of letters *T, E, E, L* = $\dfrac{\lfloor 4}{\lfloor 2}$

$$\text{Rank of TTEERL} = \frac{\lfloor 5}{\lfloor 2} + \frac{\lfloor 5}{\lfloor 2\lfloor 2} + \frac{\lfloor 5}{\lfloor 2\lfloor 2} + \lfloor 4 + \frac{\lfloor 4}{\lfloor 2} + \frac{\lfloor 4}{\lfloor 2} + 2 = 170$$

★ 5.14 TPC-14 : Selection of r objects from n objects when all n objects are not different using 'Integral equation method'

In this problem type we will how to select r objects from n objects when all n objects are not different. We discussed the same problem type earlier in TPC-9 ...... The method we discussed in TPC-9 .... was the "cases method". i.e. we make "cases" based on alike objects in the selection. For example, if we have to select 4 objects, we make cases 'all alike', '3 alike 1 different', '2 alike 2 alike', '2 alike 2 different' and 'all different' cases.

This 'cases' method can be used only if we have to select few objects say 3, 4, 5. For large number of selection of objects we use 'Integral Equation Method'. In this method we group alike objects together and with each group we define a variable representing number of objects selected from the group. Then we add all variables and equate the sum to the total objects to be selected.

For example if we have to select 3 objects from $AAAAABBBBCCC$ objects, then we make groups of identical objects, group of all A objects, group of all B objects and group of all C objects. Let x_1, x_2, x_3 be the number of A, B, C objects selected respectively.

As total number of objects to be selected is 3, we can make following integral equation :

$$x_1 + x_2 + x_3 = 3 \qquad \text{[where } 0 \le x_i \le 3 \ i = 1, 2, 3]$$

Number of solutions of the above integral equation is same number of ways to select 3 objects from the given objects. This is because every solution of the equation is a way to select 3 objects.

Number of solutions of the equation

$$= \text{Coefficient of } x^{\text{RHS}} \text{ in } \left[x^{\min(x_1)} + x^{\min(x_1)+1} + \ldots\ldots x^{\max(x_1)}\right] \times$$
$$\left[x^{\min(x_2)} + x^{\min(x_2)+1} + \ldots\ldots x^{\max(x_2)}\right] \times \left[x^{\min(x_3)} + x^{\min(x_3)+1} + \ldots\ldots x^{\max(x_3)}\right]$$

Note : RHS represents right hand side of the equation. For each variable x_1, x_2, x_3 a bracket is formed using the values the variable can take. The derivation of the above method is out of syllabus of the JEE preparation.

$\Rightarrow$ Number of solutions

$$= \text{coefficient of } x^3 \text{ in } (x^0 + x^1 + x^2 + x^3)^3$$

$$= \text{coefficient of } x^3 \text{ in } \left[\frac{1-x^4}{1-x}\right]^3 = \text{coefficient of } x^3 \text{ in } (1-x^4)^3 (1-x)^{-3}$$

$$= \text{coefficient of } x^3 \text{ in } ({}^3C_0 - {}^3C_1 x^4 + {}^3C_2 x^8 - {}^3C_3 x^{12})(1-x)^{-3}$$

$$= \text{coefficient of } x^3 \text{ in } (1-x)^{-3} \qquad [\because \text{ other terms cannot generate } x^3 \text{ term}]$$

$$= {}^{3+3-1}C_3 = {}^5C_3 = 10 \qquad [\text{Using : Coefficient of } x^r \text{ in } (1-x)^{-n} = {}^{n+r-1}C_r]$$

★ **Illustration - 37** *In a box there are* 10 *balls;* 4 *red,* 3 *black,* 2 *white and* 1 *yellow. In how many ways can a child select* 4 *balls out of these* 10 *balls?* (*Assume that the balls of the same colour are identical*)

(A) 20 **(B)** 18 **(C)** 19 **(D)** 17

SOLUTION : (A)

Let x_1, x_2, x_3 and x_4 be the number of red, black, white, yellow balls selected respectively.

Number of ways to select 4 balls = Number of integral solution of the equation $x_1 + x_2 + x_3 + x_4 = 4$

Conditions on x_1, x_2, x_3 and x_4

The total number of red, black, white and yellow balls in the box are 4, 3, 2 and 1 respectively.

So we can take : Max $(x_1) = 4$, Max $(x_2) = 3$, Max $(x_3) = 2$, Max $(x_1) = 1$

There is no condition on minimum number of red, black, white and yellow balls selected, so take :

Min $(x_i) = 0$ for $i = 1, 2, 3, 4$

Number of ways to select 4 balls

$= \text{coeff of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4) \times (1 + x + x^2 + x^3) \times (1 + x + x^2) \times (1 + x)$

$= \text{coeff of } x^4 \text{ in } (1 - x^5)(1 - x^4)(1 - x^3)(1 - x^2)(1 - x)^{-4}$

$= \text{coeff of } x^4 \text{ in } (1 - x)^{-4} - \text{coeff. of } x^2 \text{ in } (1 - x)^{-4} - \text{coeff of } x^1 \text{ in } (1 - x)^{-4} - \text{coeff. of } x^0 \text{ in } (1 - x)^{-4}$

$= {}^7C_4 - {}^5C_2 - {}^4C_1 - {}^3C_0$

$= \dfrac{7 \times 6 \times 5}{3!} - 10 - 4 - 1 = 35 - 15 = 20$

Thus, number of ways of selecting 4 balls from the box subjected to the given conditions is 20.

Another Approach : (using TPC-9 i.e. 'cases' method)

The 10 balls are *RRRR BBB WW Y* (where *R, B, W, Y* represent red, black, white and yellow balls respectively).

The work of selection of the balls from the box can be divided into following categories.

Case - I : All alike

Number of ways of selecting all alike balls = ${}^1C_1 = 1$

Case - II : 3 alike and 1 different

Number of ways of selecting 3 alike and 1 different balls = ${}^2C_1 \times {}^3C_1 = 6$

Case - III : 2 alike and 2 alike

Number of ways of selecting 2 alike and 2 alike balls = ${}^3C_2 = 3$

Case - IV : 2 alike and 2 different

Number of ways of selecting 2 alike and 2 different balls = ${}^3C_1 \times {}^3C_2 = 9$

Case - V : All different

Number of ways of selecting all different balls = ${}^4C_4 = 1$

Total number of ways to select 4 balls = 1 + 6 + 3 + 9 + 1 = 20

★ 5.15 TPC-15 : Points of Intersection between geometrical figures

We can use ${}^nC_r$ (number of ways to select r objects from n different objects) to find points of intersection between geometrical figures.

For example :

(a) Number of points of intersection between n non-concorrent and non parallel lines is ${}^nC_2$.

Logic :

When two lines intersect, we get a point of intersection. Two lines from n different lines can be selected in ${}^nC_2$ ways. Therefore, number of points of intersection is ${}^nC_2$.

(b) Number of lines that can be drawn using n points such that no three of them are collinear is ${}^nC_2$.

Logic :

A lines can be drawn through two points. Two points can be selected from n different points in ${}^nC_2$ ways. Therefore, number of lines that can be drawn is ${}^nC_2$.

(c) Number of triangles that can formed using n points such that no three of them are collinear is ${}^nC_3$.

Logic :

A triangle is formed using 3 different points. Three points can be selected from n different points is ${}^nC_3$ ways. Therefore, we can form ${}^nC_3$ triangles using n different points.

(d) Number of diagonals that can be drawn in a n sided polygon is $\dfrac{n(n-3)}{2}$.

Logic :

There are n vertices in a n sided polygon. When two vertices are joined (excluding the adjacent vertices), we get a diagonal. The number of ways to select 2 vertices from n vertices is ${}^nC_2$. But this also includes n sides (when adjacent vertices are selected). Therefore number of diagonals

$$= {}^nC_2 - n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

Illustrating the Concepts :

There are 10 *points in a plane, no three of which are in the same straight line, excepting* 4 *points, which are collinear. Find the* (*i*) *number of straight lines obtained from the pairs of these points;* (*ii*) *number of triangles that can be formed with the vertices as these points.*

(i) Number of straight lines formed joining the 10 points, taking 2 at a time = $^{10}C_2 = \dfrac{10!}{2!8!} = 45$

Number of straight lines formed by joining the four points (which are collinear), taking 2 at a time

$= {}^4C_2 = \dfrac{4!}{2!2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

So, required number of straight lines = 45 – 6 + 1 = 40

(ii) Number of triangles formed by joining the points, taking 3 at a time = $^{10}C_3 = \dfrac{10!}{3!7!} = 120$

Number of triangles formed by joining the 4 points (which are collinear), taken 3 at a time

$= {}^4C_3 = 4$

Also, 4 collinear points cannot form a triangle when taken 3 at a time.

So, required number of triangles = 120 – 4 = 116

★

Illustration - 38 *In a plane there are* 37 *straight lines, of which* 13 *pass through the point A and* 11 *pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

(A) 533 **(B)** 536 **(C)** 535 **(D)** 530

SOLUTION : (C)

The number of points of intersection of 37 straight lines is $^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point *A*. Therefore instead of getting $^{13}C_2$ points, we get merely one point *A*. Similarly, 11 straight lines out of the given 37 straight lines intersect at point *B*. Therefore instead of getting $^{11}C_2$ points, we get only one point *B*. Hence, the number of instersection points of the lines is

$$^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535$$

Illustration - 39 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

(A) m^2n^2 (B) $\frac{m^2n^2}{4}$ (C) $\frac{m(m-1)\ n\ (n-1)}{4}$ (D) $m(m-1)\ n\ (n-1)$

SOLUTION : (C)

A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be choosen in ${}^mC_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^nC_2$ ways. Hence, the number of parallelograms formed.

$$= {}^mC_2 \times {}^nC_2 = \frac{m\,(m-1)}{2} \times \frac{n\,(n-1)}{2} = \frac{mn\,(m-1)\,(n-1)}{4}$$

Illustration - 40 *There are n concurrent lines and another line parallel to one of them. The number of different triangles that will be formed by the (n + 1) lines, is*

(A) $\frac{(n-1)\,n}{2}$ (B) $\frac{(n-1)(n-2)}{2}$ (C) $\frac{n\,(n+1)}{2}$ (D) $\frac{(n+1)\,(n+2)}{2}$

SOLUTION : (B)

The number of triangles = number of selections of 2 lines from the $(n-1)$ lines which are cut by the last line

$$= {}^{n-1}C_2 = \frac{(n-1)!}{2!\,(n-3)!} = \frac{(n-1)\,(n-2)}{2}$$

Illustration - 41 *There are p points in a plane, no three of which are in the same straight line with the exception of q, which are all in the same straight line. Find the number of*

(i) *straight lines which can be formed by joining them.*

(A) ${}^pC_2 - {}^qC_2$ **(B)** ${}^pC_2 - {}^qC_2 + 1$ **(C)** ${}^pC_2 + {}^qC_2$ **(D)** ${}^pC_2 + {}^qC_2 + 1$

(ii) *triangles which can be formed by joining them.*

(A) ${}^pC_3 - {}^qC_3 + 1$ **(B)** ${}^pC_3 - {}^qC_3 - 1$ **(C)** ${}^pC_3 - {}^qC_3$ **(D)** ${}^pC_3 + {}^qC_3$

SOLUTION : (i) . (B) (ii) . (C)

(i) If no three of the p points were collinear, the number of straight lines = number of groups of two that can be formed from p points $= {}^pC_2$.

Due to the q points being collinear, there is a loss of ${}^qC_2$ lines that could be formed from these points.

But these points are giving exactly one straight line passing through all of them.

Hence the number of straight lines $= {}^pC_2 - {}^qC_2 + 1$

(ii) If no three points were collinear, the number of triangles $= {}^pC_3$

But there is a loss of ${}^qC_3$ triangles that could be formed from the group of q collinear points.

Hence the number of triangles formed $= {}^pC_3 - {}^pC_3$

5.16 TPC-16 : Formation of Subsets

In this problem type, we select elements from a given set to form subsets. We are Supposed to form subsets under constraints. From example, two subset P and Q are to be formed such that $P \cup Q$ has all elements, $P \cap Q$ has no elements *etc*. To understand the problems based on this type, read the following illustrations carefully.

Illustration - 42 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random then find the number of ways to form sets such that* $P \cup Q = X$.

(A) 3^n **(B)** 2^n **(C)** $2^n - 1$ **(D)** $3^n - 1$

SOLUTION : (A)

As $P \cup Q = X$, it means every elements would be either included in P or *in* Q or both. So for every element, there are 3 choices.

$\Rightarrow$ Number of ways to select P and Q such that $(P \cup Q = X) = 3^n$

Illustration - 43 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to chosen P and Q such that $P \cup Q$ contains exactly r elements.*

(A) 3^r **(B)** ${}^nC_r 3^r$ **(C)** 3^n **(D)** 2^n

SOLUTION : (B)

$P \cup Q$ has r elements. It means r elements out of n elements should be present in either P or in Q or both.

r elements out of n elements can be selected in ways = ${}^nC_r$

Each of these r elements has 3 choices

$\Rightarrow$ No. of ways to select elements for P and $Q = 3^r$

Rest $(n - r)$ elements has 1 choice i.e. neither go in P nor in Q $\Rightarrow$ No. of ways = 1^{n-r}.

$\Rightarrow$ Number of ways to select P and Q such that $P \cup Q$ has exactly r elements = ${}^nC_r\, 3^r\, (1)^{n-r} = {}^nC_r\, 3^r$

Illustration - 44 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P \cap Q$ is empty i.e. $P \cap Q = \phi$.*

(A) 3^n **(B)** 2^n **(C)** $2^n - 1$ **(D)** $3^n - 1$

SOLUTION : (A)

$P \cap Q = \phi$. It means P and Q should be disjoint sets. That is there is no elements common in P and Q.

$\Rightarrow$ For every elements in set X there are 3 choices. Either it is selected in P but not in Q or selected in Q but not in P or not selected in both P and Q.

$\Rightarrow$ Number of ways to select P and Q such that $P \cap Q = \phi = 3^n$

Illustration - 45 *Let X is a set containing n elements. A subset P of set X is chosen at random. The set X is then reconstructed by replacing the elements of set P and another set Q is chosen at random. Find number of ways to select P and Q such that $P = \bar{Q}$.*

(A) 3^n **(B)** 2^n **(C)** $2^n - 1$ **(D)** $3^n - 1$

SOLUTION : (B)

$P = \bar{Q}$. It means P and Q are complementary sets i.e. every element present in X is either present in P or Q

$\Rightarrow$ For every element there are 2 choices to select. Either it will be selected for P or it will be selected for Q.

$\Rightarrow$ No. of ways to select = 2^n

NOW ATTEMPT IN-CHAPTER EXERCISE-D BEFORE PROCEEDING AHEAD IN THIS EBOOK

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DIVISION AND DISTRIBUTION OF NON-IDENTICAL ITEMS Section - 6

6.1 Case - I : Unequal division and distribution of non-identical objects

In this section we will discuss ways to divide non-identical objects into groups. For example, if we have to divide three different balls (b_1, b_2, b_3) among 2 boys (B_1 *and* B_2) such that B_1 gets 2 balls and B_2 gets 1 ball, then

Number of ways to divide balls among boys is 3 ways as shown in the following table.

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2

Instead of writing all ways and counting them, we can make a formula to find number of ways.

First select 2 balls for B_1 in ${}^3C_2$ and then remaining 1 ball for B_2 in ${}^1C_1$ ways.

Total number of ways, using fundamental principle of counting, is $= {}^3C_2 \times {}^1C_1 = 3 \times 1 = 3$ ways.

If we have to divide 3 non-identical balls among 2 boys such that one boy should get 2 and other boy should get 1, then following are the ways :

B_1	B_2
b_1, b_2	b_3
b_2, b_3	b_1
b_3, b_1	b_2
b_3	b_1, b_2
b_1	b_2, b_3
b_2	b_3, b_1

Distribution of above 3 ways among 2 boys. You can observe that entries are interchanged, between B_1 and B_2.

$\Rightarrow$ Total ways to distribute = 6.

Instead of writing all ways and counting them, we can just find number of ways using fundamental principle of counting.

First select 2 balls for B_1 in ${}^3C_2$ ways, then select 1 remaining ball for B_2 in ${}^1C_1$ ways, finally distribute the way to divide among 2 boys in $\lfloor 2$ ways (ball given to B_1 and B_2 are interchange) because any boy can get 2 balls and the other 1 ball.

Using fundamental principle of counting, total number of ways = ${}^3C_2 \times {}^1C_1 \times \lfloor 2 = 3 \times 1 \times 2 = 6$ ways.

Now generalising the above cases, we can write the following formula :

(a) Number of ways in which $(m+n+p)$ different objects can be divided into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects $= {}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^{p}C_p$

$$= \frac{(m+n+p)!}{m!\,n!\,p!}$$

(b) Number of ways in which $(m+n+p)$ different objects can be divided and distributed (entries are distributed among groups) into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects

= No. of ways to divide $(m+n+p)$ objects in 3 groups × No. of ways to distribute 'division-ways' among groups

= No. of ways to divide $(m+n+p)$ objects in 3 groups × (Number of groups) ! $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Above formulae are written for dividing objects into 3 groups but in case groups are more, then also we follow the same approach. For example,

Number of ways to divide non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups G_1, G_2, G_3, G_4 gets 1, 2, 3, 4 objects respectively $= \dfrac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4}$.

Number ways to divide 10 non-identical objects in 4 groups (G_1, G_2, G_3, G_4) such that groups get objects in number 1, 2, 3, 4 (i.e. any group can get 1 object or 2 objects or 3 objects or 4 objects).

= Number of ways to divide and distribute 10 objects in 4 groups containing 1, 2, 3, 4 objects

$$= \frac{\lfloor 10}{\lfloor 1 \lfloor 2 \lfloor 3 \lfloor 4} \times \lfloor 4$$

6.2 Case - II : Equal division and distribution of non-identical objects

Here, we will see formulae to divide and distribute non-identical objects equally in groups i.e. each group gets equal numbers of objects. For example dividing 6 objects in 3 groups such that each group gets 2 objects,

(a) Number of ways to divide (mn) objects equally in m group (each group gets n objects)

$$= \frac{\lfloor mn}{(\lfloor n)^m} \frac{1}{\lfloor m}$$

(b) Number of ways to distribute and (mn) objects equally in m group (each group gets n objects)

$$= \frac{\lfloor mn}{(\lfloor n)^m}$$

6.3 Case - III : Equal as well as Unequal Division and Distribution of non identical objects

Here, we will see formulae to divide and distribute non-identical objects into groups such that not all groups contain equal or unequal number of objects i.e. some groups get equal and some get unequal number of objects. For example division of 6 objects in 4 groups containing 1, 1, 2, 2 objects.

(a) Number of ways to divide $(m + 2n + 3p)$ objects in 6 groups containing m, n, n, p, p, p objects is :

$$= \frac{\lfloor m+2n+3p}{\lfloor m\,(\lfloor n)^2\,(\lfloor p)^3} \times \frac{1}{\lfloor 2} \times \frac{1}{\lfloor 3}$$

Note : We divide by $\lfloor 2$ because there are two groups containing n objects each (equal number of objects).

We divide by $\lfloor 3$ because these are 3 groups containing p objects each (equal number of objects).

(b) Number of ways to divide and distribute $(m + 2n + 3p)$ objects in 6 groups containing m, n, n, p, p, p objects

= [Number of ways to divide $(m + 2n + 3p)$ objects in 6 groups] × (Number of groups) !

$$= \left[\frac{\lfloor m+2n+3p}{\lfloor m\,(\lfloor n)^2\,(\lfloor p)^3} \frac{1}{\lfloor 2} \times \frac{1}{\lfloor 3}\right] \times \lfloor 6$$

(c) Number of ways to divide and distribute $(m + 2n + 3p)$ objects in 6 groups containing m, n, n, p, p, p objects such that $(m + 2n + 3p)$ objects are distributed among equal groups only $= \dfrac{\lfloor m+2n+3p}{\lfloor m\,(\lfloor n)^2\,(\lfloor p)^3}$

Above formulae is defined for dividing objects in 6 groups but we can make similar formulae for other cases.

For example :

Number of ways to divide 10 objects in 4 groups containing 3, 3, 2, 2 objects $= \dfrac{\lfloor 10}{(\lfloor 2)^2\,(\lfloor 3)^2}\,\dfrac{1}{\lfloor 2}\,\dfrac{1}{\lfloor 2}$

Number of ways to divide and distribute completely 10 objects in 4 groups containing 3, 3, 2, 2 objects $= \left[\dfrac{\lfloor 10}{(\lfloor 2)^2}\,\dfrac{1}{(\lfloor 3)^2}\,\dfrac{1}{\lfloor 2}\,\dfrac{1}{\lfloor 2}\right] \times \lfloor 4$

Number of ways to divide and 'distribute among equal groups' 10 objects containing 2, 2, 3, 3 objects $= \dfrac{\lfloor 10}{\lfloor 2\,\lfloor 2\,\lfloor 3\,\lfloor 3}$

6.4 Other Formulae :

(a) Number of ways to divide n non-identical objects in r groups such that each group gets 0 or more number of objects (empty groups are allowed) $= r^n$.

(b) Number of ways to divide n non-identical objects in r groups such that each group gets at least one object (empty groups are not allowed)

$$= r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - {}^rC_3 (r-3)^n + \ldots\ldots (-1)^{r-1}\, {}^rC_{r-1} 1^n$$

Illustrating the Concepts :

➢ *In how many ways can 12 books be equally distributed among 3 students ?*

In this question, we have to divide books equally among 3 students. So we will use formulae given in section 6.2.

where we divided non-dentical objects equally among groups.

Therefore, number of ways to divide and distribute 12 non-identical objects among 3 students equally

$$= \frac{\lfloor 12}{(\lfloor 4)^3}$$

➢ *In how many ways we can divide 52 playing cards ?*

(i) *among 4 players equally ?* **(ii)** *in 4 equal parts ?*

(i) 52 cards are to be divided equally among 4 players. Each player will get 13 cards. If we distribute a way of division among players, it a different division. It means we should apply distribution formula. Using formula given in section 6.2 (b), we get:

Number of ways to divide playing cards $= \dfrac{\lfloor 52}{(\lfloor 13)^4}$

(ii) As we have to make 4 equals parts, each part consists of 13 cards. We will apply division formula (not distribution) becuause if we distribute a way of division among various parts, then it will be the same division. Using formula used in section 6.2 (a), we get:

Number of ways to divide 52 cards in 4 parts $= \dfrac{\lfloor 52}{(\lfloor 13)^4} = \dfrac{1}{\lfloor 4}$

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Illustration - 46 *In how many ways can 7 departments be divided among 3 ministers such that every minister gets at least one and atmost 4 departments to control ?*

(A) 630 **(B)** 1050 **(C)** 1680 **(D)** *None of these*

SOLUTION : (C)

Let 3 minister be M_1, M_2, M_3.

Following are the ways in which we can divide 7 departments among 3 ministers such that each minister gets at least one and atmost 4.

S.No.	M_1	M_2	M_3
1	4	2	1
2	2	2	3
3	3	3	1

Note : If we have a case (2, 2, 3), then there is no need to make cases (3, 2, 2) or (2, 3, 2) because we will include them when we apply distribution formula to distribute ways of division among ministers.

Case - I :

We divide 7 departments among 3 ministers in number 4, 2, 1 i.e. unequal division. As any minister can get 4 departments, any can get 2 any can get 1 department, we should apply distribution formula. Using formula given in section 6.1 (b), we get :

Number of ways to divide and distribute departments in number 4, 2, 1

$$= \left[\frac{\underline{|7}}{\underline{|4}\,\underline{|2}\,\underline{|1}}\right]\times 3! = 630 \qquad \ldots \text{(i)}$$

Case - II :

It is 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula given in section 6.3 (b), we get :

Number of ways to divide departments in number 2, 2, 3,

$$= \left[\frac{\underline{|7}}{\underline{|2}\,\underline{|2}\,\underline{|3}}\,\frac{1}{\underline{|2}}\right]\times \underline{|3} = 630 \qquad \ldots \text{(ii)}$$

Case - III :

It is also 'equal as well as unequal' division. As any minister can get any number of departments, we use complete distribution formula. Using formula given in section 6.3 (c), we get :

Number of ways to divide and distribute departments in number 3, 3, 1

$$= \left[\frac{\underline{|7}}{(\underline{|3})^2\,(\underline{|2})}\,\frac{1}{\underline{|2}}\right]\times \underline{|3} = 420 \qquad \ldots \text{(iii)}$$

Combining (i), (ii) and (iii), we get number of ways to divide 7 departments among 3 ministers

$$= 630 + 630 + 420 = 1680$$

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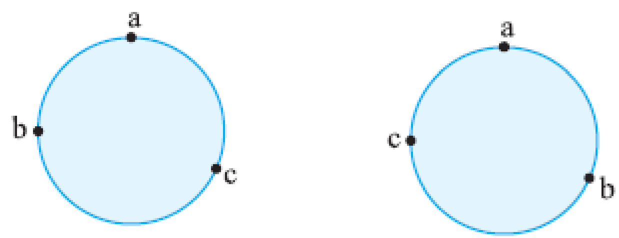
CIRCULAR PERMUTATION Section - 7

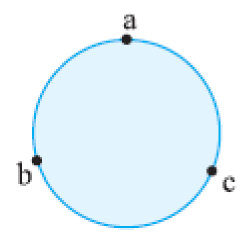
7.1 Introduction to Circular Permutation

When objects are to be arranged (ordered) in a circle instead of a row, it is known as Circular Permutation. For example, three objects a, b, c can be permutated in a circle as shown below :

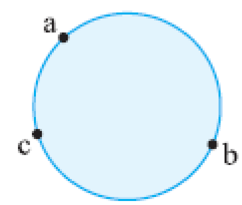


Number of ways to arrange a, b, c in circle is not same as number of ways to arrange a, b, c in a row.

This is because arrangements *abc, bca, cab* in a row are same in circle as shown in the figure on right.



Similarly arrangements *acb, cba, bac* in a row are same in circle shown in the figure on right.

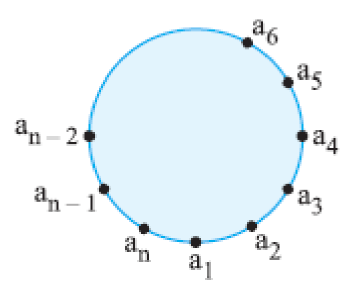


Therefore,

Number of ways to arrange n different objects in a circle

= Number of circular permutations of n objects = $\underline{|n-1}$.

Proof :



Let $a_1, a_2, a_3, \ldots\ldots a_{n-1}, a_n$, be n distinct objects. Let the total number of circular permutations be x. Consider one of these x permutations as show in Figure.

Clearly, this circular permutation provides n linear permutations as given below

$$
\begin{array}{l}
a_1, a_2, a_3, \ldots\ldots\ldots\, a_{n-1}, a_n \\
a_2, a_3, a_4, \ldots\ldots\, a_n, a_1 \\
a_3, a_4, a_5, \ldots\ldots\, a_n, a_1, a_2 \\
a_4, a_5, a_6, \ldots\ldots\, a_n, a_1, a_2, a_3 \\
\quad\;\ldots \qquad\qquad \ldots \\
\quad\;\ldots \qquad\qquad \ldots \\
a_n, a_1, a_2, a_3, \ldots\ldots, a_{n-1}
\end{array}
$$

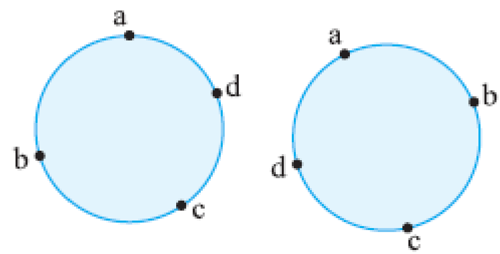
Thus, each circular permutation gives n linear permutations. But there are x circular permutations. So, that number of linear permutations is xn. But the number of linear permutations of n distinct objects is $n\,!$.

$$\therefore \quad xn = n\,! \quad \Rightarrow \quad x = \frac{n!}{n} = (n-1)\,!$$

7.2 Difference between Clockwise and Anti Clock wise

In some of the problems, we need to consider clockwise and anti-clockwise arrangements of objects as same arrangements.

See the following circular permutations.



There is a difference of just the cyclic order. In first arrangement a, b, c, d are arranged in anti-clockwise order where as in second there are arranged in clockwise order.

If we have to consider these arrangements same (for example arrangement of flowers in garland or arrangement of beads in a necklace), then we need to divide total circular permutation by 2.

Therefore,

Number of circular permutations of n different objects such that clockwise and anticlockwise arrangements of objects are same $= \dfrac{\underline{|n-1}}{2}$.

Illustrating the Concepts :

➢ *In how many ways can a party of 4 men and 4 women be seated at a circular table so that no two women are adjacent?*

(A) 144 **(B)** 24 **(C)** 72 **(D)** 288

The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men in $(4-1)! = 3!$ ways. Now, 4 vacant seats can be occupied by 4 women in 4! ways.

Hence, the required number of seating arrangements $= 3! \times 4! = 144$

➢ *In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements?*

(A) 360 **(B)** 180 **(C)** 720 **(D)** 90

Clearly, 7 persons can sit at a round table in $(7-1)! = 6!$ ways. But, in clockwise and anticlockwise arrangements, each person will have at the same neighbours.

So, the required number of ways $= \dfrac{1}{2}(6!) = 360$

★

Illustration - 47 *There are 20 persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.*

(A) 18! **(B)** $2\times18!$ **(C)** 17! **(D)** $2\times17!$

SOLUTION : (B)

Let B_1 and B_2 be two brothers among 20 persons and let M be a person that will sit between B_1 and B_2. The person M can be chosen from 18 persons (excluding B_1 and B_2) in 18 ways. Considering the two brothers B_1 and B_2 and person M as one person, we have 18 persons in all. These 18 persons can be arranged around a circle in $(18 - 1)! = 17!$ ways.

But B_1 and B_2 can be arranged among themselves in 2! ways.

Hence, the total number of ways $= 18 \times 17! \times 2! = 2 \times 18!$.

★

Illustration - 48 *A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates are*

(i) *always together?*

(A) 18! **(B)** $2 \times 18!$ **(C)** 17! **(D)** $2 \times 17!$

(ii) *never together?*

(A) $17 \times 18!$ **(B)** $18 \times 18!$ **(C)** $15! \times 18!$ **(D)** $16! \times 18!$

SOLUTION : (i) . (B) (ii) . (A)

(i) Let D_1 and D_2 be two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. These 19 delegates can be seated round a circular table in $(19 - 1)! = 18!$ ways. But two particular delegates can arrange among themselves in 2! ways ($D_1 D_2$ and $D_2 D_1$).

Hence, the total number of ways $= 18! \times 2! = 2\,(18!)$.

(ii) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of seating arrangements of 20 persons around the round table. Clearly 20 persons can be seated around a circular table in $(20 - 1)! = 19!$ ways.

Hence, the required number of seating arrangements $= 19! - 2 \times 18! = 17\,(18!)$.

Another Approach :

First arrange remaining 18 persons in $(18 - 1)! = 17!$ ways.

Then select two places out of 18 places in ${}^{18}C_2$ ways and arrange the two in 2! ways.

No. of ways $= 17! \times {}^{18}C_2 \times 2! = 17\,(18!)$

★

Illustration - 49 *There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together ?*

(A) 2880 **(B)** 1440 **(C)** 720 **(D)** 360

SOLUTION : (A)

Five gentlemen can be seated at a round table in $(5-1)! = 4!$ ways. Now, 5 places are created in which 4 ladies are to be seated. Select 4 seats for 4 ladies from 5 seats in ${}^5C_4$ ways. Now 4 ladies can be arranged on the 4 selected seats in 4! ways.

Hence, the total number of ways in which no two ladies sit together$= 4! \times {}^5C_4 \times 4! = (4!)\,5\,(4!) = 2880$

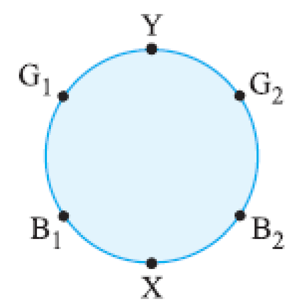
★

Illustration - 50 *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*

(A) 4 **(B)** 8 **(C)** 2 **(D)** 16

SOLUTION : (A)

Let B_1, B_2 and X be three boys and G_1, G_2 and Y be three girls. Since the boy X does not want any girl neighbour. Therefore boy X will have his neighbours as boys B_1 and B_2 as shown in Fig. Similarly, girl Y has her neibhbours as girls G_1 and G_2 as shown Fig. But the boys B_1 and B_2 can be arranged among themselves in 2! ways and the girls G_1 and G_2 can be arranged among themselves in 2! ways.



Hence, the required number of arrangements $= 2! \times 2! = 4$

★

Illustration - 51 *Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.*

(A) 288 **(B)** 144 **(C)** 72 **(D)** 576

SOLUTION : (A)

Considering 4 particular flowers as one group of flower, we have five flowers (one group of flowers and remaining four flowers) which can be strung to form a garland in 4!/2 ways. But 4 particular flowers can be arranged amongst themselves in 4! ways. Thus, the required number of ways $= \dfrac{4! \times 4!}{2} = 288$

★★

DIVISION OF IDENTICAL OBJECTS INTO GROUPS Section - 8

8.1 Introduction

In this section, we will discuss how to find number of ways to divide identical objects into groups.
For example, if we have to divide 5 identical copies of a book among 3 boys such that each boy gets at least 1 copy, then it can be achieved as shown below :

Boy 1	Boy 2	Boy 3
3	1	1
1	3	1
1	1	3
2	2	1
1	2	2
2	1	2

i.e. 5 identical copies can be divided in 6 ways.
We can study following formulae to find number of ways to divide them instead of writing ways and counting them.

8.2 Formulae

(a) The number of ways to divide n identical objects into r groups (different) such that each group gets 0 or more objects (empty groups are allowed) = ${}^{n+r-1}C_{r-1}$.

Proof :
Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, . . . , r respectively.
As total objects to be divided is n, we can take

Sum of the objects given to all groups = n

$\Rightarrow$ $x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n$.

This equation is known as integral equation as all variables are integer.

As each group can get 0 or more, following are constraints on integer variables.

$0 \le x_1 \le x$; $0 \le x_2 \le n, \ldots, \; 0 \le x_r \le n$ i.e. $0 \le x_i \le n$ $i = 1, 2, 3, \ldots, r$.

We can observe that number of integral solutions of the above equation is equal to number of ways to divide n identical objects among r groups such that each gets 0 or more.

How to find number of solutions ?

To find number of solutions, following method can be used. The derivation of this method is out of scope of JEE preparations.

Consider r brackets corresponding to r groups. In each bracket, take an expression given by

$x^0 + x + x^2 + \ldots + x^n$. Here the various powers of x, i.e. 0, 1, 2, . . . , n correspond to the number of objects each group can take in the division.

Since the total number of objects is n. So, the required number of ways is the coefficient of x^n in the product

$$(x^0 + x + x^2 + \ldots + x^n)(x^0 + x^1 + \ldots + x^n) \ldots (x^0 + x^1 + x^2 + \ldots + x^n)$$

(r *brackets*)

Thus, the required number or ways

$= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + x^2 \ldots + x^n)^r$

$= \text{Coefficient of } x^n \text{ in } \left(\dfrac{1 - x^{n+1}}{1 - x}\right)^r$

$= \text{Coefficient of } x^n \text{ in } (1 - x^{n+1})^r \ (1 - x)^{-r}$

$= \text{Coefficient of } x^n \text{ in } (1 - x)^{-r}$ [$\because$ x^{n+1} cannot be used to generate x^n term].

$= {}^{n+r-1}C_n = {}^{n+r-1}C_{r-1}$.

[Using : coefficient of x^r in $(1 - x)^{-n} = {}^{n+r-1}C_r$ and ${}^nC_r = {}^nC_{n-r}$].

(b) The number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are not allowed).

$= {}^{n-1}C_{r-1}$.

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, . . . r respectively.

As total objects to be divided is n, we can take sum of the objects given to all groups = n

$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n$ . . . **(i)**

This is an integral equation as all variables are integer. Each group should get at least one object.

$\Rightarrow \quad 1 \le x_1 \le n \,;\, 1 \le x_2 \le n, \ldots, \ 1 \le x_r \le n$

i.e. $\quad 1 \le x_i \le n \qquad i = 1, 2, 3, 4, \ldots, r$.

We can observe that number of solutions of obove equation is equal to number of ways to divide n objects among r groups such that each group gets at least one.

Proof :

The number of solutions of integral equation (i)

$= \text{coeff. of } x^n \text{ in } (x + x^2 + \ldots + x^n)^r$

Now, Coeff. of x^n in $(x + x^2 + x^3 + \ldots + x^n)^r$

$= \text{Coeff. of } x^n \text{ in } x^r (1 + x^2 + \ldots x^{n-1})^r$

$= \text{Coeff. of } x^{n-r} \text{ in } (1 + x + x^2 + \ldots x^{n-1})^r$

$= \text{Coeff. of } x^{n-r} \text{ in } \left(\frac{1-x^n}{1-x}\right)^r$

$= \text{Coeff. of } x^{n-r} \text{ in } (1-x)^{-r}$ [using : x^n cannot be used to generate x^{n-r}]

$= {}^{n-r+r-1}C_{n-r} = {}^{n-1}C_{n-r} = {}^{n-1}C_{r-1}$

(c) The number of ways to divide n identical objects in r groups (different) such that each group gets minimum m objects and maximum k objects

$= \text{Coefficient of } x^n \text{ in } (x^m + x^{m+1} + \ldots\ldots + x^k)^r$

The logic of the above can be understood after reading proofs of (a) and (b) results.

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Illustration - 52 *Find the number of ways of distributing* 5 *identical balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.*

(A) 12 **(B)** 6 **(C)** 24 **(D)** 72

SOLUTION : (B)

Let x_1, x_2 and x_3 be the number of balls into three boxes so that no box is empty and each box being large enough to accommodate all the balls.

The number of ways of distributing 5 balls into Boxes 1, 2 and 3 is the number of integral solutions of the equation $x_1 + x_2 + x_3 = 5$ subjected to the following conditions on x_1, x_2, x_3. **. . . (i)**

Conditions on x_1, x_2 and x_3 :

According to the condition that the boxes should contain at least one ball, we can find the range of x_1, x_2 and x_3 i.e.

Min $(x_i) = 1$ and Max $(x_i) = 3$ for $i = 1, 2, 3$ [using : Max $(x_1) = 5 -$ Min $(x_2) -$ Min (x_3)]

or $1 \le x_i \le 3$ for $i = 1, 2, 3$

So, number of ways of distributing balls

= number of integral solutions of (i)

$= \text{coeff. of } x^5 \text{ in the expansion of } (x + x^2 + x^3)^3$

$= \text{coeff. of } x^5 \text{ in } x^3 (1 - x^3)(1 - x)^{-3}$

$= \text{coeff. of } x^2 \text{ in } (1 - x^3)(1 - x)^{-3}$

$= \text{coeff. of } x^2 \text{ in } (1 - x)^{-3}$ [as x^3 cannot generate x^2 terms]

$= {}^{3+2-1}C_2 = {}^4C_2 = 6$

Another Approach :

The number of ways to divide n identical objects into r groups so that no group remains empty

$= {}^{n-1}C_{r-1}$ [using result 8.2 (b)]

$= {}^{5-1}C_{3-1} = {}^{4}C_{2} = 6$

★

Illustration - 53 *How many integral solutions are there to $x + y + z + t = 29$, when $x \geq 1$, $y \geq 2$, $z \geq 3$ and $t \geq 0$?*

(A) 5200 **(B)** 2600 **(C)** 1300 **(D)** 650

SOLUTION : (B)

We have,

$x \geq 1, y \geq 2, z \geq 3$ and $t \geq 0$, where x, y, z, t are integers

Let $u = x - 1, v = y - 2, w = z - 3.$

Then, $x \geq 1 \Rightarrow u \geq 0$; $y \geq 2 \Rightarrow u \geq 0$; $z \geq 3 \Rightarrow w \geq 0$

Thus, we have

$u + 1 + v + 2 + w + 3 + t = 29 \Rightarrow u + v + w + t = 23$ [where $u \geq 0$; $v \geq 0$; $w \geq 0$]

The number of solutions of above equation is equal to number of ways to divide 23 identical objects among 4 groups such that gets 0 or more.

$\Rightarrow$ The total number of solution of this equation is

$= {}^{23+4-1}C_{4-1} = {}^{26}C_{3} = 2600$ [Using result given in 8.2]

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Illustration - 54 *Find the number of ways of distributing 10 identical balls in 3 boxes so that no box contains more than four balls and less than 2 balls*

(A) 6 **(B)** 12 **(C)** 24 **(D)** 36

SOLUTION : (A)

Let x_1, x_2 and x_3 be the number of balls placed in Boxes 1, 2 and 3 respectively.

Number of ways of distributing 10 balls in 3 boxes

= Number of integral solutions of the equation $x_1 + x_2 + x_3 = 10$ . . . (i)

Conditions on x_1, x_2 and x_3

As the boxes should contain atmost 4 ball and at least 2 balls, we can make

Max $(x_i) = 4$ and Min $(x_i) = 2$ for $i = 1, 2, 3$

or $2 \le x_i \le 4$ for $i = 1, 2, 3$

So the number of ways of distributing balls in boxes

= number of integral solutions of equation (i)

= coeff of x^{10} in the expansion of $(x^2 + x^3\ \ x^4)^3$

= coeff of x^{10} in $x^6 (1 - x^3)^3 (1 - x)^{-3}$

= coeff of x^4 in $(1 - x^3)^3 (1 - x)^{-3}$

= coeff of x^4 in $(1 - {}^3C_1\, x^3 + {}^3C_2\, x^6 + \ldots)(1 - x)^{-3}$

= coeff of x^4 in $(1 - x)^{-3}$ – coeff of x in ${}^3C_1 (1 - x)^{-3}$

$= {}^{4+3-1}C_4 - 3 \times {}^{3+1-1}C_1 = {}^6C_4 - 3 \times {}^3C_1 = 15 - 9 = 6$

★

Illustration - 55 *The number of ways in which 30 marks can be alloted to 8 questions if each question carries atleast 2 marks,is :*

(A) ${}^{21}C_7$ **(B)** ${}^{21}C_8$ **(C)** ${}^{13}C_7$ **(D)** *None of these*

SOLUTION : (A)

Let, the marks given in each question be ;

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8,$ [where x_i's ≥ 0 $(i = 1, 2, \ldots, 8)$]

and $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 = 30$

Let $x_i = y_i + 2\ \forall\ i = 1, 2, 3 \ldots\ldots, 8$

$\therefore\ y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 30 - 16 \Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 = 14$

where $y_i \ge 0\ \forall\ i = 1, 2, 3, \ldots, 8$

$\Rightarrow$ Number of solutions of the above equation

= Number of ways to divide 14 identical objects among 8 groups such that each group gets 0 or more.

$\Rightarrow$ Number of solutions $= {}^{14+8-1}C_{8-1} = {}^{21}C_7$ [Using result given in 8.2]

NOW ATTEMPT IN-CHAPTER EXERCISE-E BEFORE PROCEEDING AHEAD IN THIS EBOOK

NOW ATTEMPT OBJECTIVE WORKSHEET TO COMPLETE THIS EBOOK

THINGS TO REMEMBER

1. Definition of Factorial

Factorial : The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\lfloor n$.

i.e. $n! = n(n-1)(n-2).....3 \times 2 \times 1$

Thus $4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

2. Properties of factorial

(i) $n!$ is defined for positive integers only.

(ii) Factorials of proper fractions or integers are not defined. Factorial n is defined only for whole numbers.

(iii) $n! = 1 \times 2 \times 3 \times = ... \times (n-1) \times n$

$= [1 \times 2 \times 3 \times = ... \times (n-1)]\, n = (n-1)!\, n$

Thus, $n! = n\,(n-1)!$

(iv) $0! = 1$ (by definition)

(v) If two factorials, i.e., $x!$ and $y!$ are equal, then $x = y$ or $x = 0, y = 1$ or $x = 1, y = 0$.

3. Addition Principle

If a work can be done in 'm' different ways and another work which is independent of first can be done in 'n' different ways. Then either of the two operations can be performed in $(m + n)$ ways.

4. Permutation

(i) Total number of ways to permutate (arrange, order) n different objects in a row = $\lfloor n$.

(ii) Number of permutations (P) of n different objects (n) taken r objects (r) at a time is ${}^nP_r = \dfrac{\lfloor n}{\lfloor n-r}$.

(iii) Number of ways to permuate (arrange) n objects out of which p are identical of one kind, q are identical of another kind, r are identical of third kind and rest all are different to each other $= \dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$.

(iv) Total number of ways to permutate n different things taken all at a time when objects can be repeated any number of times is n^n.

(v) Total number of ways to permutate n different things taken r at a time when objects can be repeated any number of times is n^r.

5. Combination

(i) Number of ways to select r objects from n different objects = ${}^nC_r = \dfrac{\lfloor n}{\lfloor r \lfloor n-r}$.

(ii) Number of ways to select r objects from n different objects where each object can be selected any number of times is ${}^{n+r-1}C_r$.

6. Properties of ${}^nC_r$

(i) ${}^nC_0 = {}^nC_n = 1$

(ii) ${}^nC_r = {}^nC_{n-r}$

(iii) If ${}^nC_r = {}^nC_k$, then $r = k$ or $n - r = k$

(iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(v) $r \cdot {}^nC_r = n\ {}^{n-1}C_{r-1}$

(vi) $\dfrac{1}{r+1}\ {}^nC_r = \dfrac{1}{n+1}\ {}^{n+1}C_{r+1}$

(vii) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(viii) (a) If n is even, ${}^nC_r$ is greatest for $r = n/2$.

(b) If n is odd, ${}^nC_r$ is greatest for $r = \dfrac{n-1}{2}, \quad \dfrac{n+1}{2}$

7. Typical Problem Categories (TPC)

TPC-1 : The number of ways to select r objects from n different objects where p particular objects should always be included in the selection = ${}^{n-p}C_{r-p}$.

TPC-2 : The number of ways to select r objects from n different objects where p particular objects should never be included in the selection = ${}^{n-p}C_r$.

TPC-3 : The number of ways to select and arrange (permutate) r objects from n different objects such that arrangement should always include p particular objects = ${}^{n-p}C_{r-p} \lfloor r$.

TPC-4 : The number of ways to select and arrange r objects from n different objects such that p particular objects are always excluded in the selection = ${}^{n-p}C_r \lfloor r$.

TPC-5 : The number of ways to arrange n different objects such that p particular objects remain together in the arrangement = $\lfloor (n-p+1) \lfloor p$.

TPC-6 : The number of ways to arrange n different objects such that p particular objects are always separated = ${}^{n-p+1}C_p \lfloor n-p \lfloor p$

TPC-7 : There are problems in which constraints are to select minimum (at least) or maximum (at most) objects in the selection. (See approach in chapter)

TPC-8 : Permutations of n objects taken r at a time when all n objects are not different.

TPC-9 : Selection of r objects from n objects when all n objects are not different.

TPC-10 : Selection of one or more objects :

- **(i)** The number of ways to select one or more objects from n different objects or we can say, selection of at least one object from n different objects = $2^n - 1$.
- **(ii)** The number of ways to select one or more objects (or at least one object) from n identical objects = n
- **(iii)** The number of ways to select one or more objects from $(p + q + r \ldots\ldots + n)$ objects where p objects are alike of one kind, q are alike of second kind, r are alike of third kind, ... and remaining n are different from each other

  $= [(p + 1)(q + 1)(r + 1) \ldots\ldots\ldots 2^n] - 1$

TPC-11 : If n distinct objects are to be arranged in the row such that no object occupies its original place, then to find number of ways to arrange them, we use dearrangement theorem i.e.

Number of ways to dearrange = $n!\left[1 - \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \ldots\ldots (-1)^n \frac{1}{\underline{|n}}\right]$

TPC-12 : This TPC deals with how to find sum of all the numbers that can be formed using the given digits.

TPC-13 : This TPC deals with dictionary related question of words formed by using the letters of the given word.

TPC-14 : Selection of r objects from n objects when all n objects are not different using 'Integral equation method'

TPC-15 : Points of Intersection between geometrical figures

- **(i)** Number of points of intersection between n non-concorrent and non parallel lines is ${}^nC_2$
- **(ii)** Number of lines that can be drawn using n points such that no three of them are collinear is ${}^nC_2$
- **(iii)** Number of triangles that can formed using n points such that no three of them are collinear is ${}^nC_3$
- **(iv)** Number of diagonals that can be drawn in a n sided polygon is $\frac{n(n-3)}{2}$

TPC-16 : This TPC deals with selecting elements from a given set to form subsets.

8. Division and Distribution

Case - I : Unequal division and distribution of non-identical objects

(i) Number of ways in which $(m+n+p)$ different objects can be divided into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects $= {}^{m+n+p}C_m \, {}^{n+p}C_n \, {}^{p}C_p$

$$= \frac{(m+n+p)!}{m!\,n!\,p!}.$$

(ii) Number of ways in which $(m+n+p)$ different objects can be divided and distributed (entries are distributed among groups) into 3 unequal groups (groups contain unequal number of objects) containing m, n, p objects

= No. of ways to divide $(m+n+p)$ objects in 3 groups × No. of ways to distribute 'division-ways' among groups

= No. of ways to divide $(m+n+p)$ objects in 3 groups × (Number of groups) ! $= \dfrac{(m+n+p)!}{m!\,n!\,p!} \times 3!$

Case - II : Equal division and distribution of non-identical objects

(i) Number of ways to divide (mn) objects equally in m group (each group gets n objects)

$$= \frac{\underline{|mn}}{(\underline{|n})^m} \frac{1}{\underline{|m}}.$$

(ii) Number of ways to divide and (mn) objects equally in m group (each group gets n objects)

$$= \frac{\underline{|mn}}{(\underline{|n})^m}.$$

Case - III : Equal as well as Unequal Division and Distribution of non identical objects

(i) Number of ways to divide $(m+2n+3p)$ objects in 6 groups containing m, n, n, p, p, p objects in 6 groups containing m, n, n, p, p, p objects

$$= \frac{\underline{|m+2n+3p}}{\underline{|m}\,(\underline{|n})^2\,(\underline{|p})^3} \times \frac{1}{\underline{|2}} \times \frac{1}{\underline{|3}}.$$

(ii) Number of ways to divide and distribute $(m+2n+3p)$ objects in 6 groups containing m, n, n, p, p, p objects = [Number of ways to divide $(m+2n+3p)$ objects in 6 groups] × (Number of groups) !

$$= \left[\frac{\underline{|m+2n+3p}}{\underline{|m}\,(\underline{|n})^2\,(\underline{|p})^3} \frac{1}{\underline{|2}} \times \frac{1}{\underline{|3}}\right] \times \underline{|6}$$

(iii) Number of ways to divide and distribute among equal groups $(m+2n+3p)$ objects in 6 groups containing m, n, n, p, p, p objects such that $(m+2n+3p)$ objects are distributed among equal groups only $= \dfrac{\lfloor m+2n+3p}{\lfloor m\,(\lfloor n)^2\,(\lfloor p)^3}$.

9. Other Formulae :

(i) Number of ways to divide n non-identical objects in r groups such that each group gets 0 or more number of objects (empty groups are allowed) $= r^n$.

(ii) Number of ways to divide n non-identical objects in r groups such that each group gets at least one object (empty groups are not allowed)

$= r^n - {}^rC_1\,(r-1)^n + {}^rC_2\,(r-2)^n - {}^rC_3\,(r-3)^n + \ldots\ldots (-1)^{r-1}\,{}^rC_{r-1}\,1^n$.

10. Circular Permutation

(i) Number of ways to arrange n different objects in a circle = Number of circular permutations of n objects $= \lfloor n-1$.

(ii) Number of circular permutations of n different objects such that clockwise and anticlockwise arrangements of objects are same $= \dfrac{\lfloor n-1}{2}$.

11. Division of Identical Objects into Groups

(i) The number of ways to divide n identical objects into r groups (different) such that each group gets 0 or more objects (empty groups are allowed) $= {}^{n+r-1}C_{r-1}$.

(ii) The number of ways to divide n identical objects into r groups (different) such that each group receives at least one object (empty groups are not allowed).

$= {}^{n-1}C_{r-1}$.

(iii) The number of ways to divide n identical objects in r groups (different) such that each group gets minimum m objects and maximum k objects

$=$ Coefficient of x^n in $(x^m + x^{m+1} + \ldots\ldots + x^k)^r$

My Chapter Notes