skenovi iz predavanja od:

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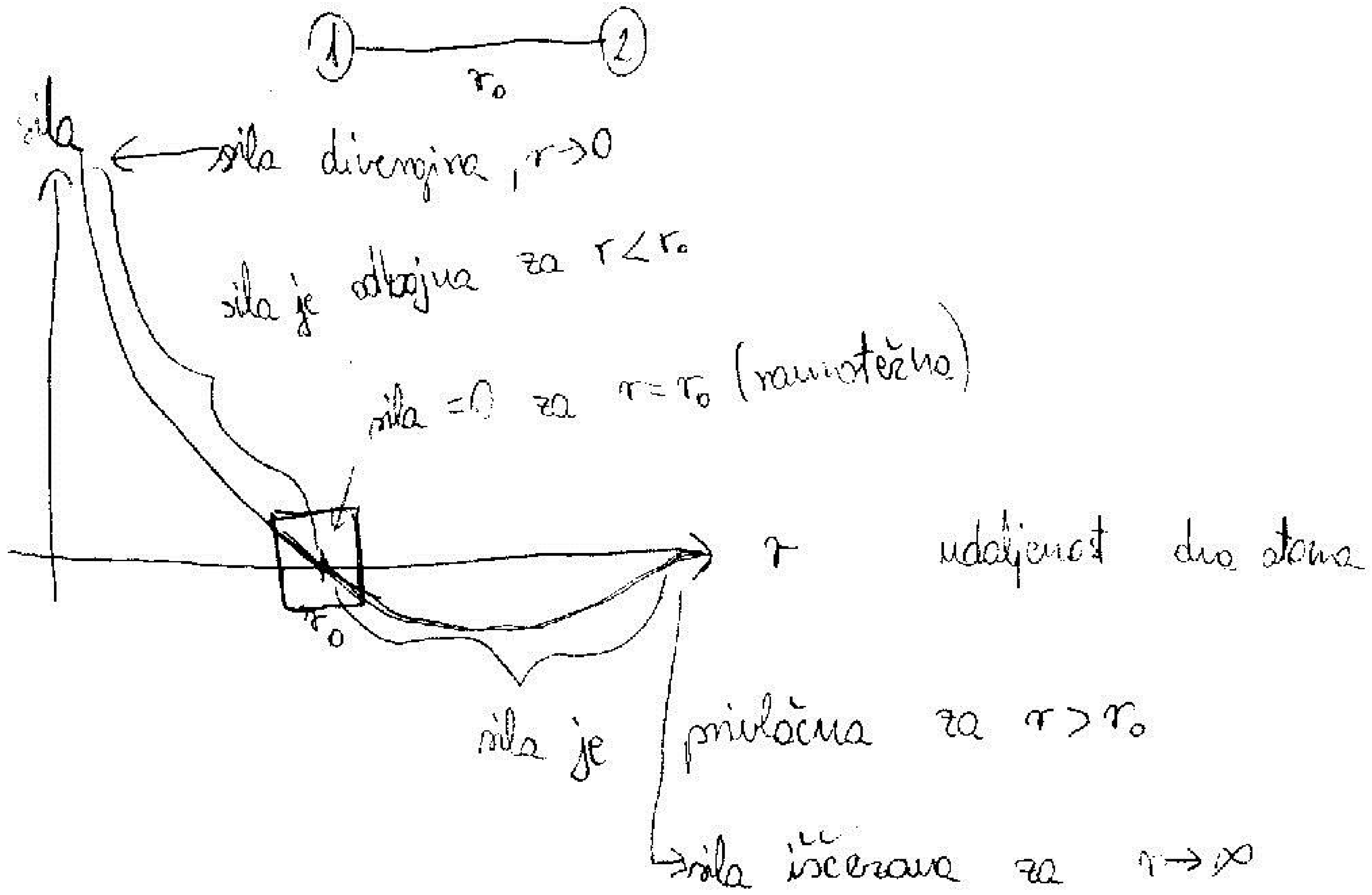
* Čvrsto tijelo

- pod djelovanjem sila može mijenjati oblik

- dvoatomna molekula

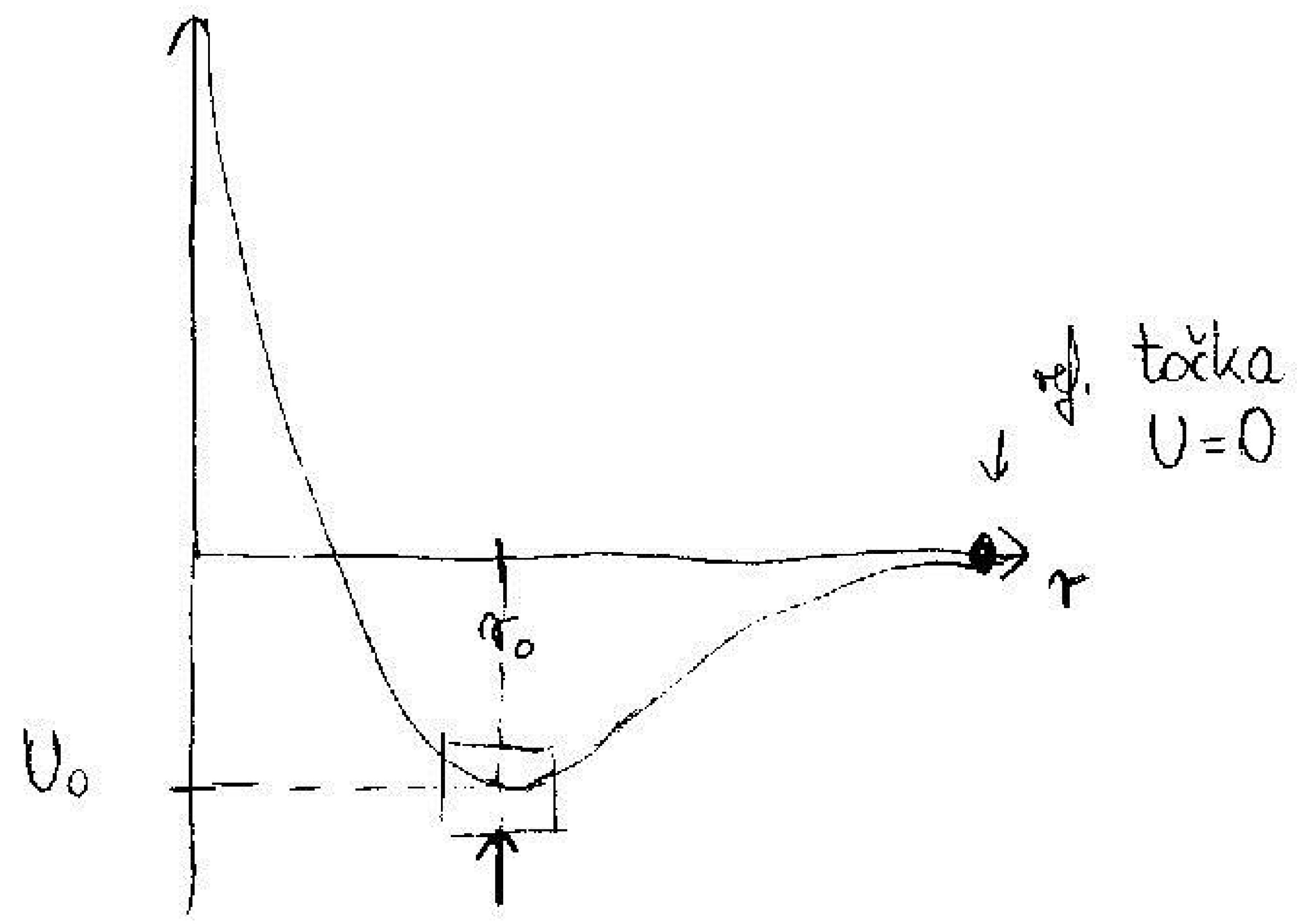


Bitno:

za $r \simeq r_0$

$$\boxed{F(r) = -k(r - r_0)}$$

poten. energija



Bitno:

za $r \simeq r_0$

$$U(r) = U_0 + \frac{k}{2}(r - r_0)^2$$

Podsjetnik

$$F(r) = \frac{\partial}{\partial r} U(r)$$

Elastičnost

(naprezanje) $\Leftrightarrow$ (relativna deformacija)

(omjer $\frac{\text{sila}}{\text{površina}}$) σ

(omjer $\frac{\text{stvarna deformacija}}{\text{ravnotežna dimenzija}}$) δ

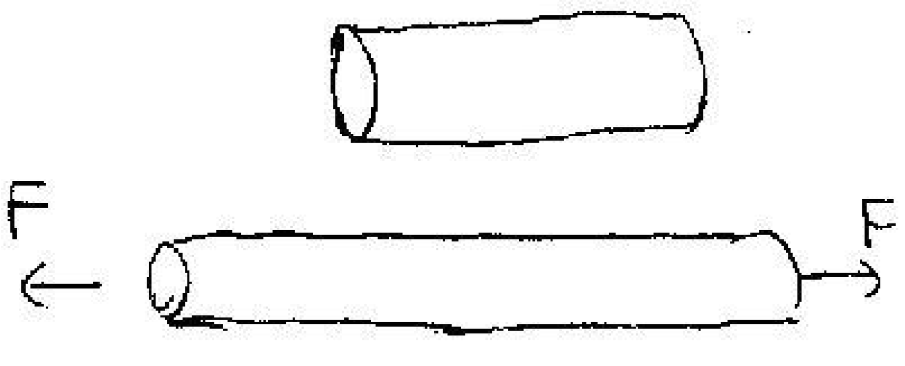
linearno područje (za mala naprezanja, odn. male rel. def.)

$$\sigma = \# \, \delta$$

„modul elastičnosti"

- tlak, vlak, smik (torzija)
- tipovi naprezanja
  - vlačno
  - tlačno
  - smično

* Vlačno naprezanje



$l, r, S = r^2 \pi$

$l \to l + \Delta l$

$r \to r + \Delta r$

naprezanje: $\sigma \equiv \frac{F}{S}$

rel. def:

$$\delta_{uzdužno} = \frac{\Delta l}{l}$$

$$\delta_{poprečno} = \frac{\Delta r}{r}$$

Youngov modul elastičnosti

$$E \equiv \frac{\sigma}{\delta_{uzduž}}$$

Poissonov omjer

$$\mu \equiv -\frac{\delta_\perp}{\delta_\parallel}$$

* rel. promjena volumena

$V = l r^2 \pi$ prije naprezanja

$V \to V + \Delta V = (l+\Delta l)(r+\Delta r)^2 \pi$ nakon naprezanja

$= l r^2 \pi \left(1+\frac{\Delta l}{l}\right)\left(1+\frac{\Delta r}{r}\right)^2$ malo u odnosu na jedinicu

$= V(1+\delta_\parallel)(1+\delta_\perp)^2$

$= V(1+\delta_\parallel)(1+2\delta_\perp+\delta_\perp^2)$ nebitno

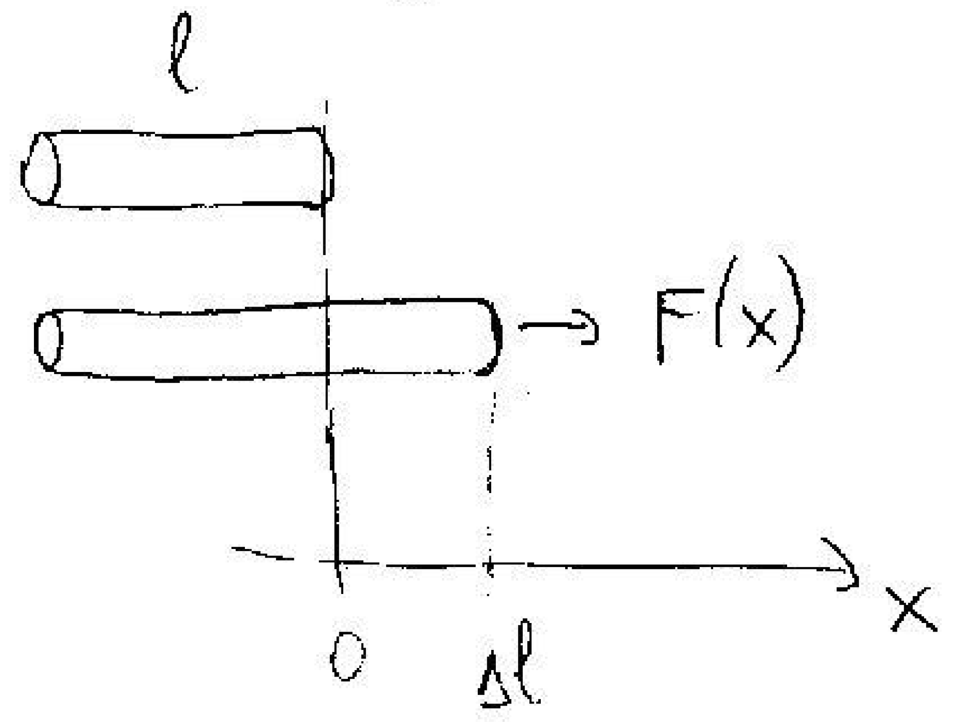
$= V(1+\delta_\parallel+2\delta_\perp+\dots)$

$= V(1+\delta_\parallel-2\mu\delta_\parallel+\dots)$

$= V(1+(1-2\mu)\delta_\parallel+\dots) \simeq \boxed{V\left(1+(1-2\mu)\frac{\sigma}{E}\right)}$

$\Delta V = V(1-2\mu)\frac{\sigma}{E}$, za $\mu = \frac{1}{2}$, $\Delta V = 0$

* elastična energija



$\sigma = E\delta_\parallel$, $\sigma = \frac{E}{S}$, $\delta_\parallel = \frac{x}{l}$

$$\boxed{F(x) = \underbrace{\frac{SE}{l}}_{k} x}$$

$\Delta E_{elast} = \Delta W$

$\Delta W = \int_0^{\Delta l} F(x)\,dx$

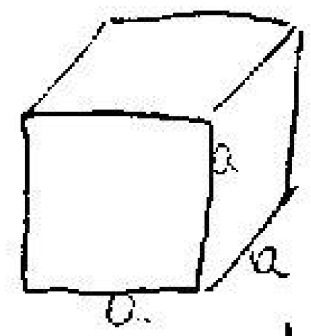
$= \int_0^{\Delta l} \frac{SE}{l} x\,dx = \overbrace{\frac{SE}{l}}^{k} \frac{\Delta l^2}{2}$

- gustoća elast. en.:

$\varepsilon = \frac{\Delta E_{elast}}{V} = \frac{\cancel{S}E}{l}\frac{\Delta l^2}{2}\cdot\frac{1}{\cancel{S}l} =$

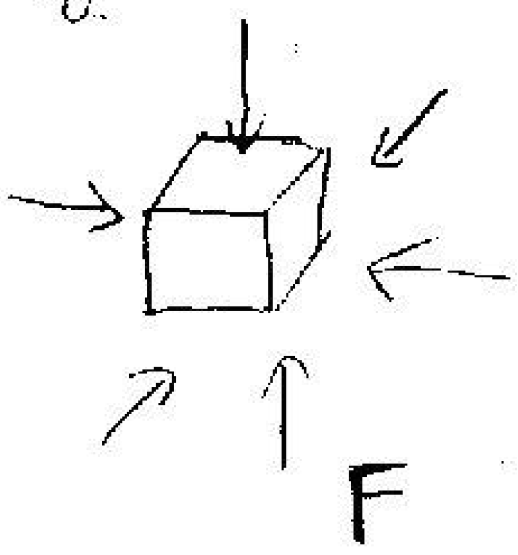
$= \boxed{E\cdot\frac{\delta_\parallel^2}{2}}$

* Tlačno naprezanje



$V = a^3$

$S = a^2$



$V \to V + \Delta V$

- naprezanje: $\sigma = \frac{F}{S}$
- relativna deformacija $\delta_V = \frac{\Delta V}{V}$

- upozorenje:

ako je naprezanje pozitivno, deformacija je negativna i obratno

$$\sigma \gtrless 0 \iff \Delta V \lessgtr 0$$

- volumni modul elastičnosti:

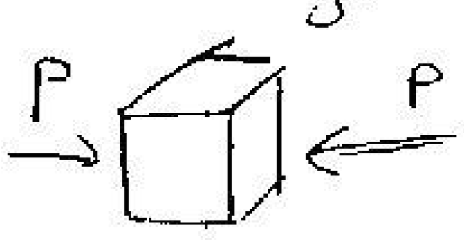
$$\boxed{B \equiv -\frac{\sigma}{\delta_V}}$$

- za tekućine:
  - kompresibilnost:

$$\boxed{K \equiv \frac{1}{B} = -\frac{1}{V}\frac{\partial V}{\partial p}}$$

⇒ Veza E, μ, B

- vlačno (neg.) naprezanje



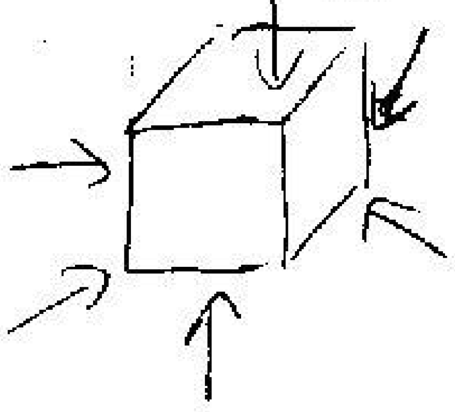
$\sigma = -p$

$\delta_{11} = \frac{\sigma}{E} = -\frac{p}{E}$

$V \to V + \Delta V = V(1 + (1-2\mu)\delta_{11})$

$= V\left(1 - (1-2\mu)\frac{p}{E}\right)$

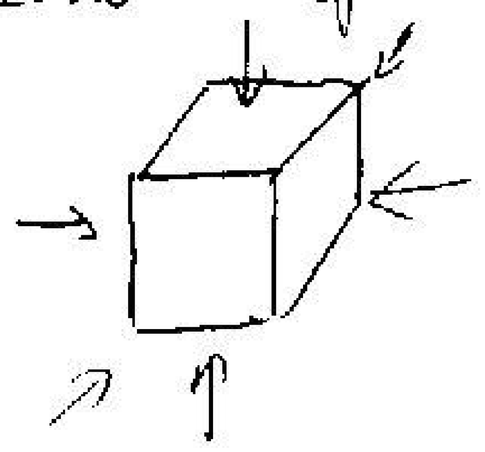
- vlačno (neg.) naprezanje u sva 3 smjera



$V \to V + \Delta V = V\left(1 - (1-2\mu)\frac{p}{E}\right)^3$

$= V\left(1 - 3(1-3\mu)\frac{p}{E}\right)$ (a)

- tlačno naprezanje (pozi)



$$V \to V + \Delta V = V\left(1+\delta_V\right) = V\left(1-\frac{p}{B}\right) \qquad (b)$$

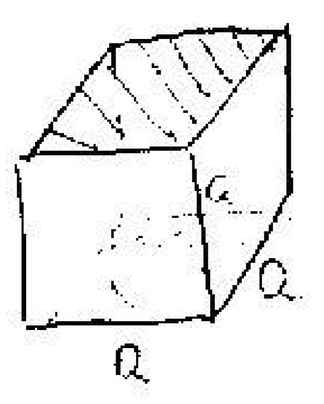
- usporedbom

$$(a) = (b) \quad \Rightarrow \quad \cancel{-}3(1-3\mu)\frac{\cancel{p}}{E} = \cancel{-}\frac{\cancel{p}}{B}$$

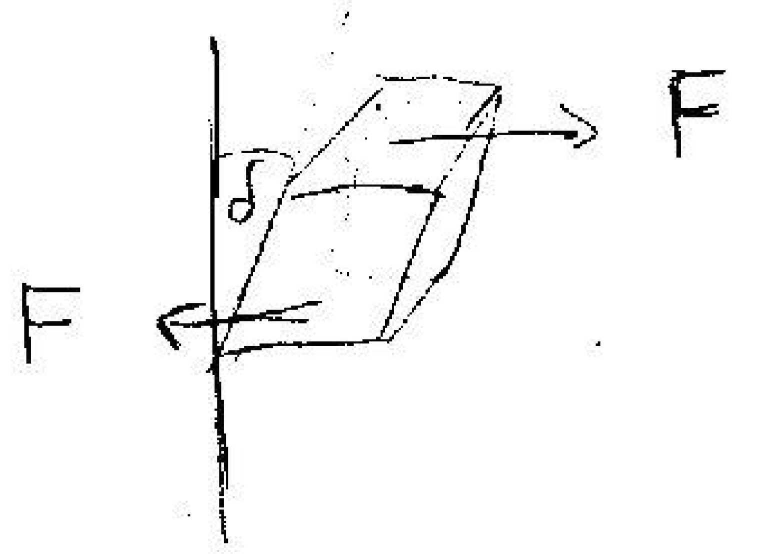
$$E = 3(1-3\mu)\,B$$

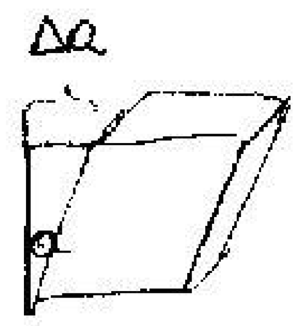
tlak - negativni vlak u sve tri dimenzije

* Smično naprezanje (smik, smicanje, torzija)



$S = a^2$





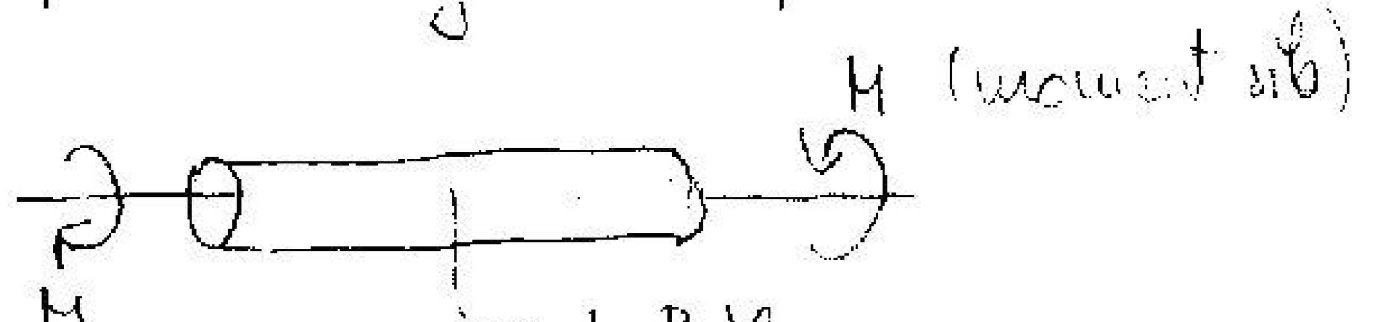
naprezanje:

$$\sigma = \frac{F}{S}$$

relativna deformacija: kut δ

modul smicanja: $\boxed{G \equiv \frac{\sigma}{\delta}}$

* Torzija homogene šipke

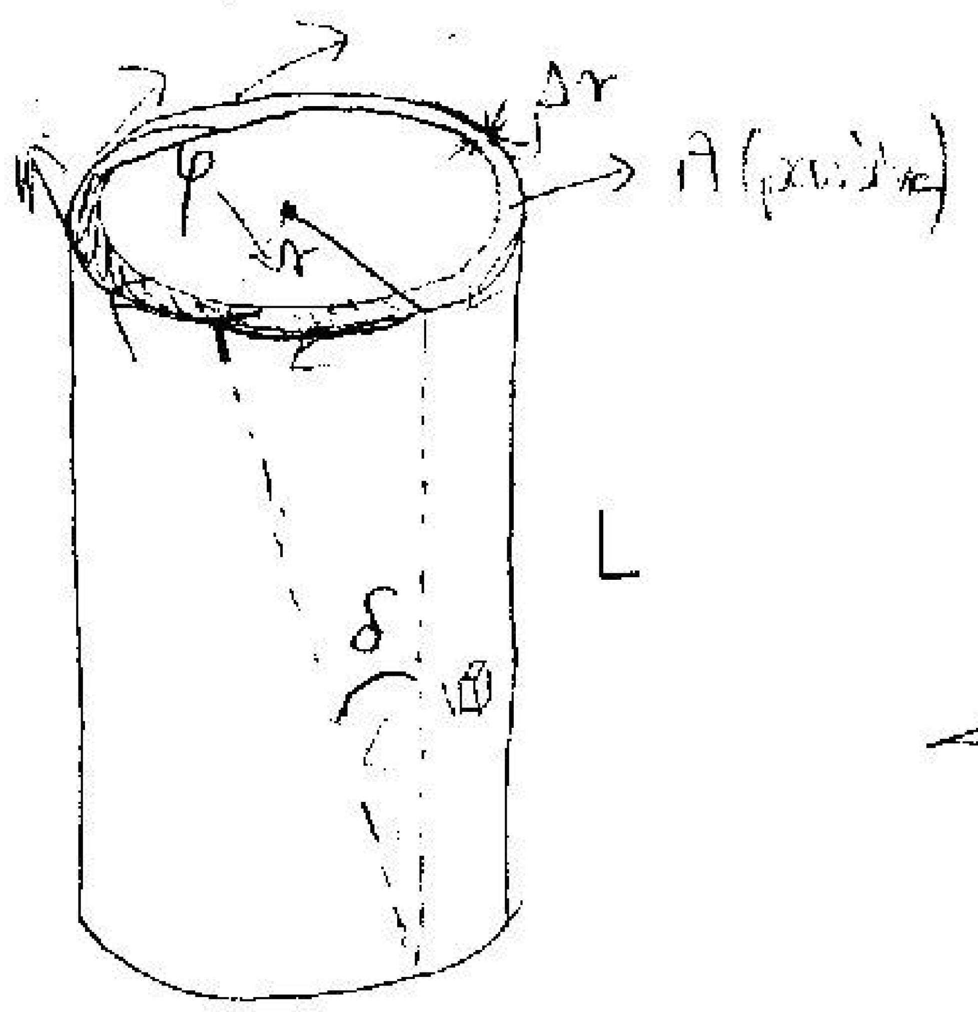


$M = D\varphi$ (D – konstanta torzije)

$$D = \frac{G\pi R^4}{2L}$$

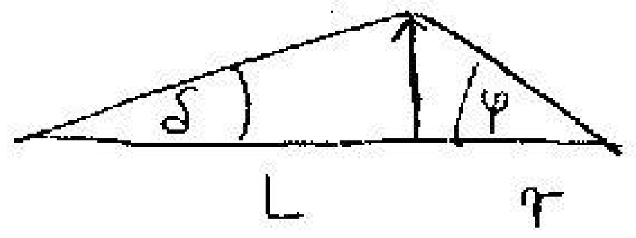
- izvod:

cijev: $L, r, \Delta r, G$ ← svojstvo materijala



kut zakreta: φ

rel. deform.: δ



(mali kutevi)

$$L \cdot \delta = r \cdot \varphi$$

- moment sile:

$$M = F \cdot r = \sigma A r = G \cdot \delta \cdot 2r\pi \Delta r \cdot r = G \cdot \frac{r\varphi}{L} \cdot 2r\pi \Delta r \cdot r =$$

$$= \boxed{\frac{2\pi G r^3 \Delta r}{L}}\, \varphi$$

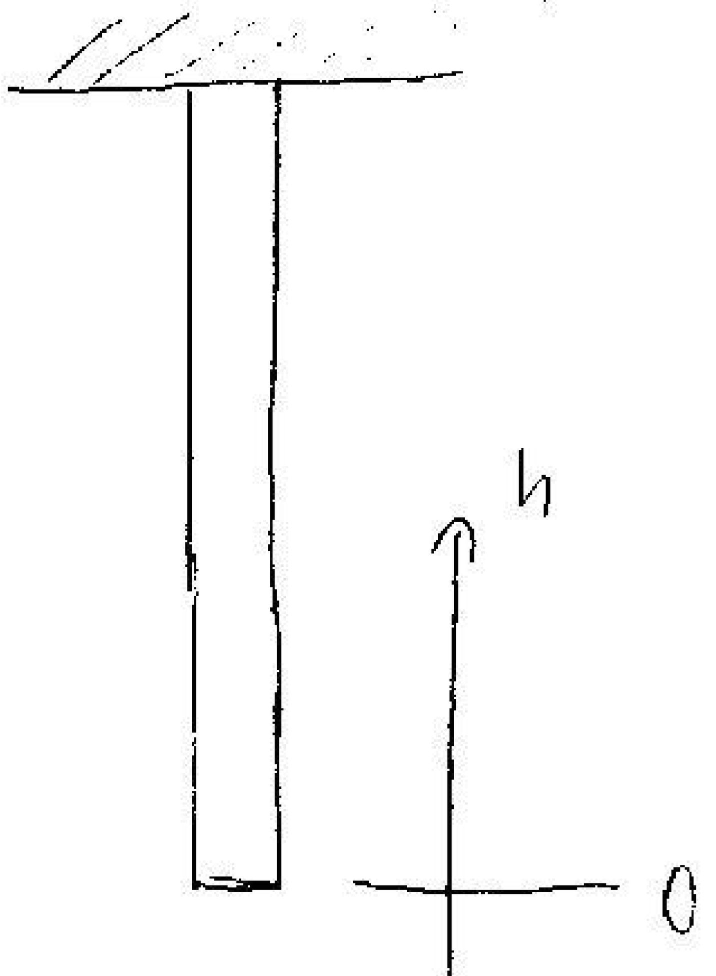
↳ konstanta torzije za cijev

- za šipku:

$$D = \underbrace{\int_0^R 2\pi \frac{G r^3 dr}{L}}_{\text{konst. torzije za cijev debljine } dr}$$

$$= \frac{2\pi G}{L} \frac{R^4}{4} = \frac{G\pi R^4}{2L}$$

Zadatak:



Bakreni štap, $l = 1$ m

$E = 130$ GPa

$\rho = 8960$ kg/m^3

$g = 9.81 \frac{m}{s^2}$

ukupno produljenje?

Sila napetosti:

$$T(h) = \rho S h g$$

rel. def.:

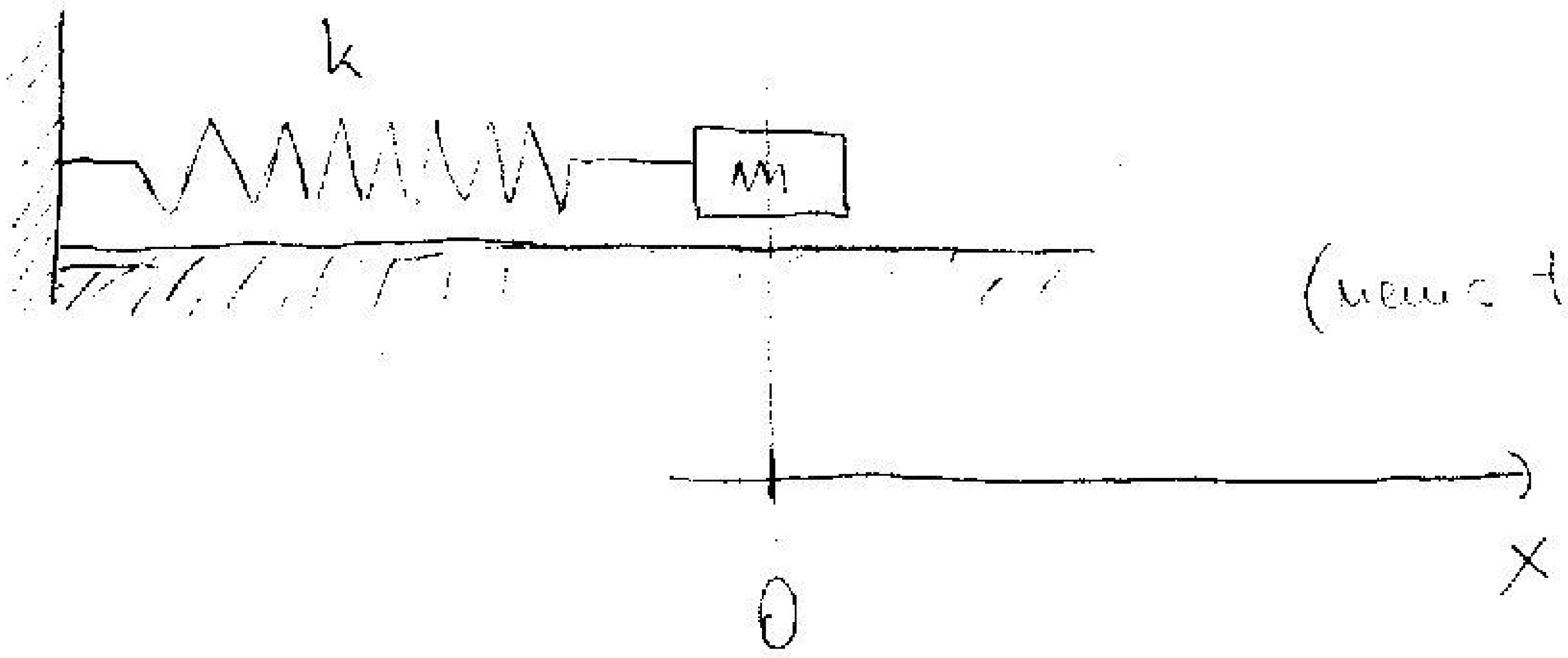
$$\delta_{\parallel}(h) = \frac{\sigma(h)}{E} = \frac{T(h)/S}{E} = \frac{\rho g h}{E}$$

- produljenje elementa duljine dh: $\delta(h)\,dh$ (- produljenje svakog komadića štapa posebno)

$$\Delta l = \int_0^l \delta(h)\,dh = \int_0^l \frac{\rho g h}{E}\,dh = \frac{\rho g}{E}\frac{l^2}{2} = \frac{\rho g l^2}{2E}$$

Jednostavno harmoničko titranje

model



(nema trenja)

- sila: $F(x) = -kx$
- Jedn. gibanja:

$$m \cdot \ddot{x} = -kx$$

$$m \cdot \frac{d^2}{dt^2} x(t) = -k\,x(t)$$

$$m \frac{d^2}{dt} x(t) + kx(t) = 0$$

- pokus

m	k	k/2	k/3	k/4
2				0,488
4				
6				
8	0,502			1,020

$$T \sim \sqrt{\frac{m}{k}}$$

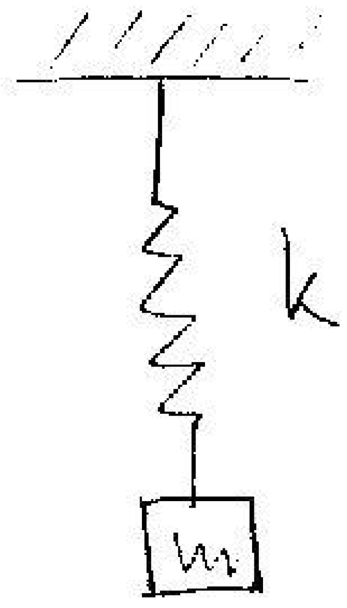
- period ne ovisi o amplitudi

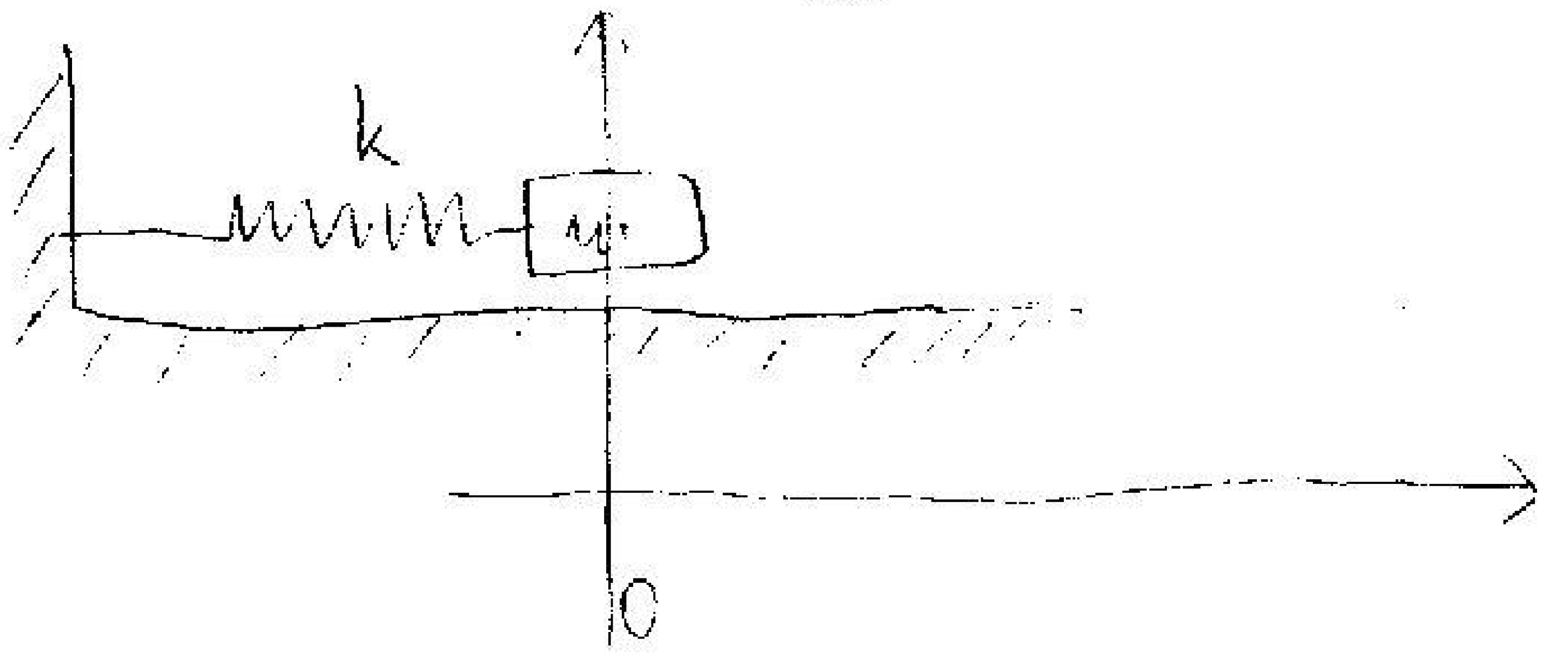
Jednadžba gibanja

$$m\frac{d^2x(t)}{dt^2} = -k\,x(t)$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_0^2 x = 0 \quad \text{gdje} \quad \boxed{\omega_0^2 \equiv \frac{k}{m}}$$





- probno rješenje $\quad x(t) = e^{\lambda t}$

$$\ddot{x}(t) = \lambda^2 e^{\lambda t} = \lambda^2 x(t)$$

$$\lambda^2 x + \omega_0^2 x = 0$$

$$x(\lambda^2 + \omega_0^2) = 0$$

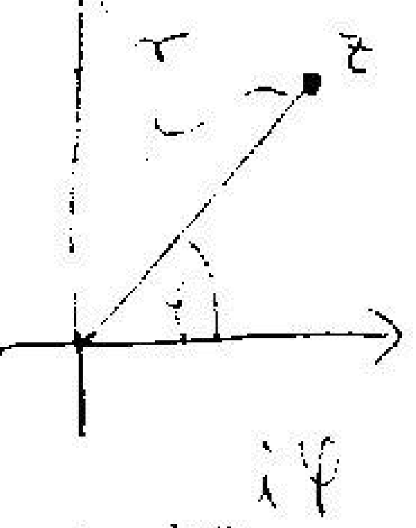
$$\lambda^2 + \omega_0^2 = 0$$

$$\lambda^2 = -\omega_0^2 \quad \Rightarrow \quad \boxed{\lambda_{1,2} = \pm i\omega_0}$$

$$x_{1,2} = e^{\pm i\omega_0 t}$$

- opće rješenje: $\quad x(t) = a_1 \underbrace{e^{i\omega_0 t}}_{x_1(t)} + a_2 \underbrace{e^{-i\omega_0 t}}_{x_2(t)}$

gdje $a_1, a_2 \in \mathbb{C}$

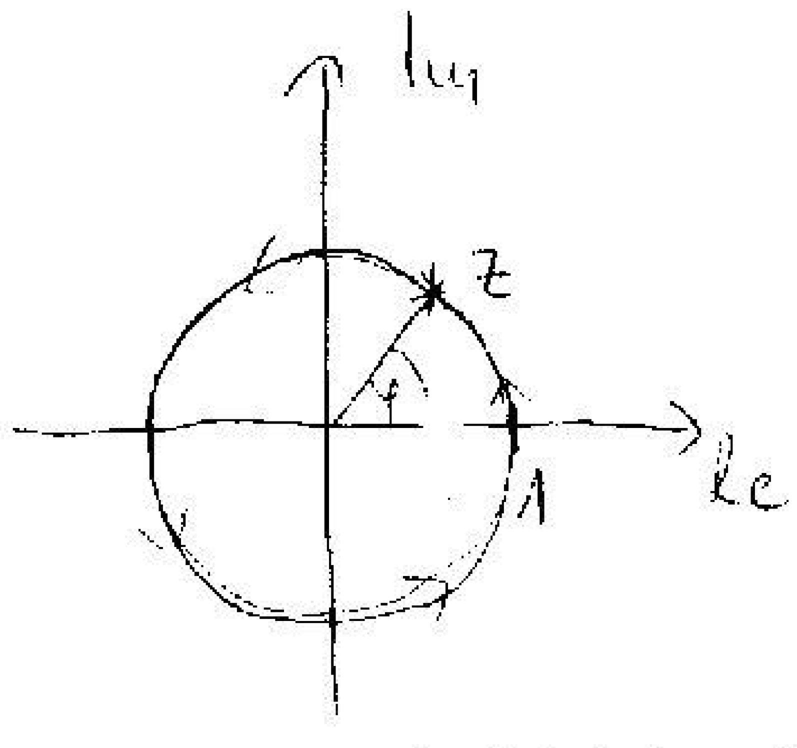


$z = re^{i\varphi}$

$Re = r\cdot\cos\varphi$

$Im = r\cdot\sin\varphi\cdot i$

$1\cdot e^{\overbrace{i\omega t}^{\varphi}}$



Fizikalni zahtjevi: $\underline{x(t) \in \mathbb{R}}$ za $\forall t$ $\Rightarrow$ $a_1 = a_2^*$ $(a_1 = \bar{a}_2)$

$$\equiv \frac{A}{2} \cdot e^{i\varphi} \quad (A, \varphi \in \mathbb{R})$$

$$\Rightarrow x(t) = \boxed{\frac{A}{2} e^{i\varphi}} e^{i\omega_0 t} + \boxed{\frac{A}{2} e^{-i\varphi}} e^{-\omega_0 t} = \frac{A}{2}\left[e^{i\underbrace{(\varphi+\omega_0 t)}_{\psi}} + e^{-i\underbrace{(\varphi+\omega_0 t)}_{\psi}} \right] =$$

(imaginarni dijelovi se krate)

$$= \frac{A}{2}\left[\cos\psi + \cancel{i\sin\psi} + \cos\psi - \cancel{\sin\psi}\right] =$$

$$= A \cdot \cos\psi = \underline{\underline{A \cdot \cos(\omega_0 t + \varphi)}}$$

amplituda

kružna frekvencija

fazni pomak

– početni uvjeti: u $t = t_0$

$$x_0 = x(t_0)$$

$$v_0 = v(t_0) = \dot{x}(t_0)$$

– zbog jednostavnosti $t = t_0 = 0$

$$x_0 = x(0) = A \cdot \cos\varphi \qquad (1)$$

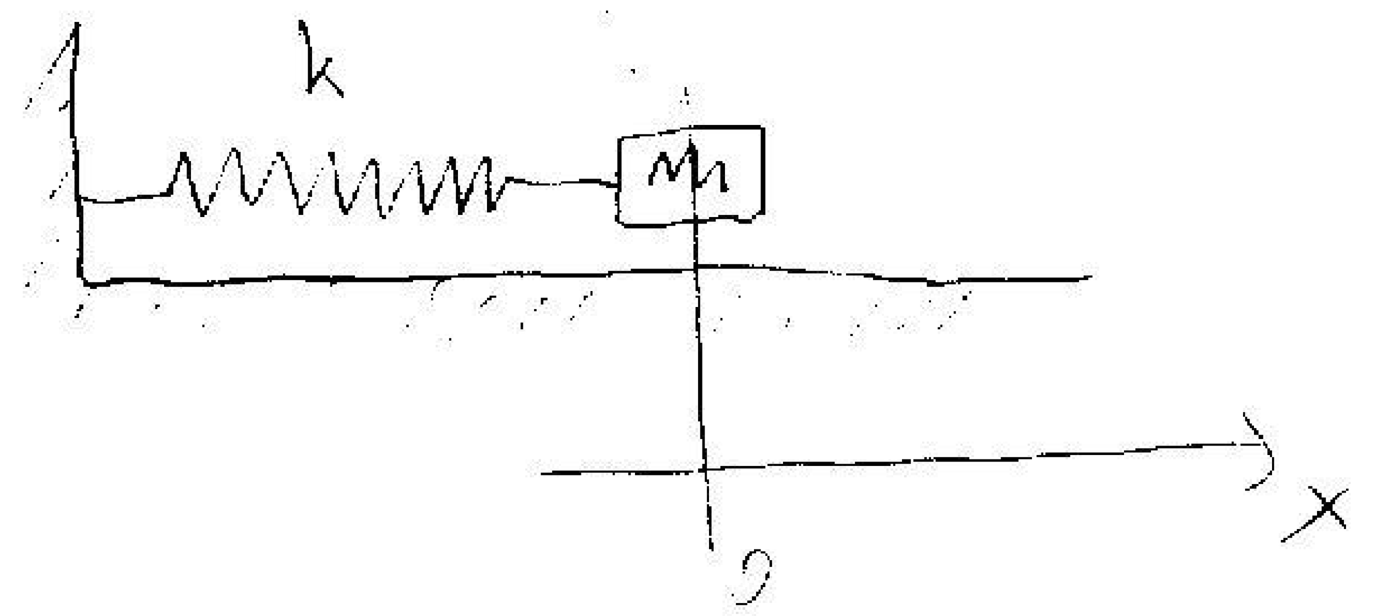
$$\dot{x}(t) = -\omega_0 A \sin(\omega_0 t + \varphi)$$

$$v_0 = \dot{x}(0) = -\omega_0 A \cdot \sin\varphi \qquad (2)$$

iz (1) & (2) $\Rightarrow$

$$\begin{cases} \operatorname{tg}\varphi = -\dfrac{v_0}{x_0 \omega_0} \\ A = \sqrt{x_0^2 + \left(\dfrac{v_0}{\omega_0}\right)^2} \end{cases}$$

Energija titranja



$$\ddot{x} + \omega_0^2 x = 0$$

$$\omega_0^2 = \frac{k}{m}$$

$$x(t) = A \cdot \cos(\omega_0 t + \varphi)$$

$$\boxed{E = \overset{\to E_{kin}}{T} + \overset{\to E_{pot}}{U}}$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

$$= \frac{1}{2} m \left[-\omega_0 A \sin\underbrace{(\omega_0 t + \varphi)}_{\psi}\right]^2 + \frac{1}{2} k \left[A \cos\underbrace{(\omega_0 t + \varphi)}_{\psi}\right]^2$$

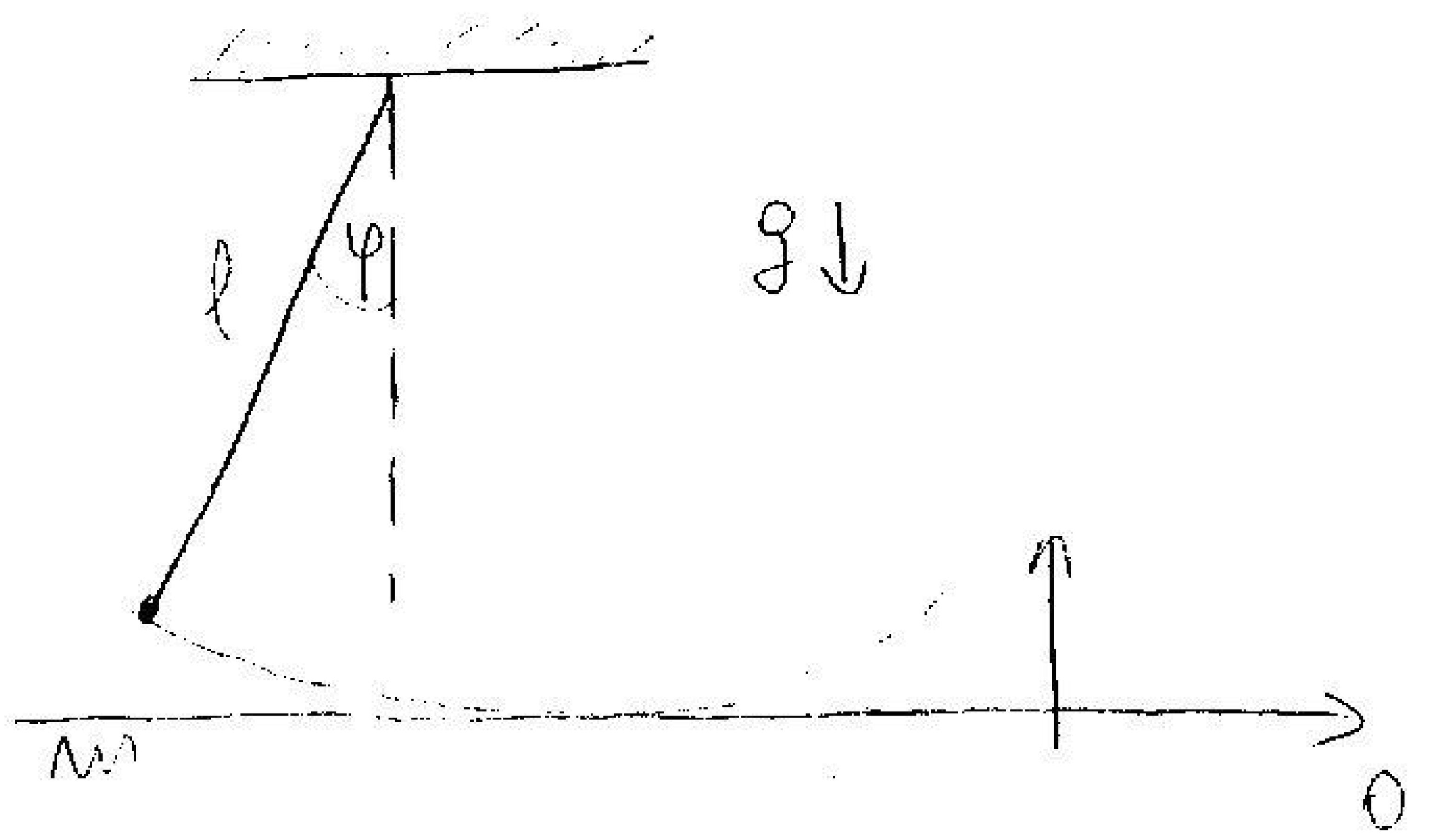
$$= \frac{1}{2} m \underbrace{\omega_0^2}_{=\frac{k}{m}} A^2 \sin^2\psi + \frac{1}{2} k A^2 \cos^2\psi$$

$$= \frac{1}{2} k A^2 \left(\sin^2\psi + \cos^2\psi\right) =$$

$$= \boxed{\frac{1}{2} k A^2}$$

* Matematičko njihalo

-idealizirano



$$T = \frac{1}{2} m v^2 = \frac{1}{2} m (l\dot{\varphi})^2$$

$$U = mgh = mg(l - l\cos\varphi)$$

$$= mgl(1 - \cos\varphi)$$

-iz očuvanja energije slijedi:

$$0 = \frac{dE}{dt} = \frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2} m (l\dot{\varphi})^2 + mgl(1-\cos\varphi)\right)$$

$$= \frac{1}{2} m 2(l\dot{\varphi}) l\ddot{\varphi} + mgl\sin\varphi\,\dot{\varphi}$$

$$= ml^2\dot{\varphi}\underbrace{\left(\ddot{\varphi} + \frac{g}{l}\sin\varphi\right)}_{=0}$$

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0 \quad !!!$$

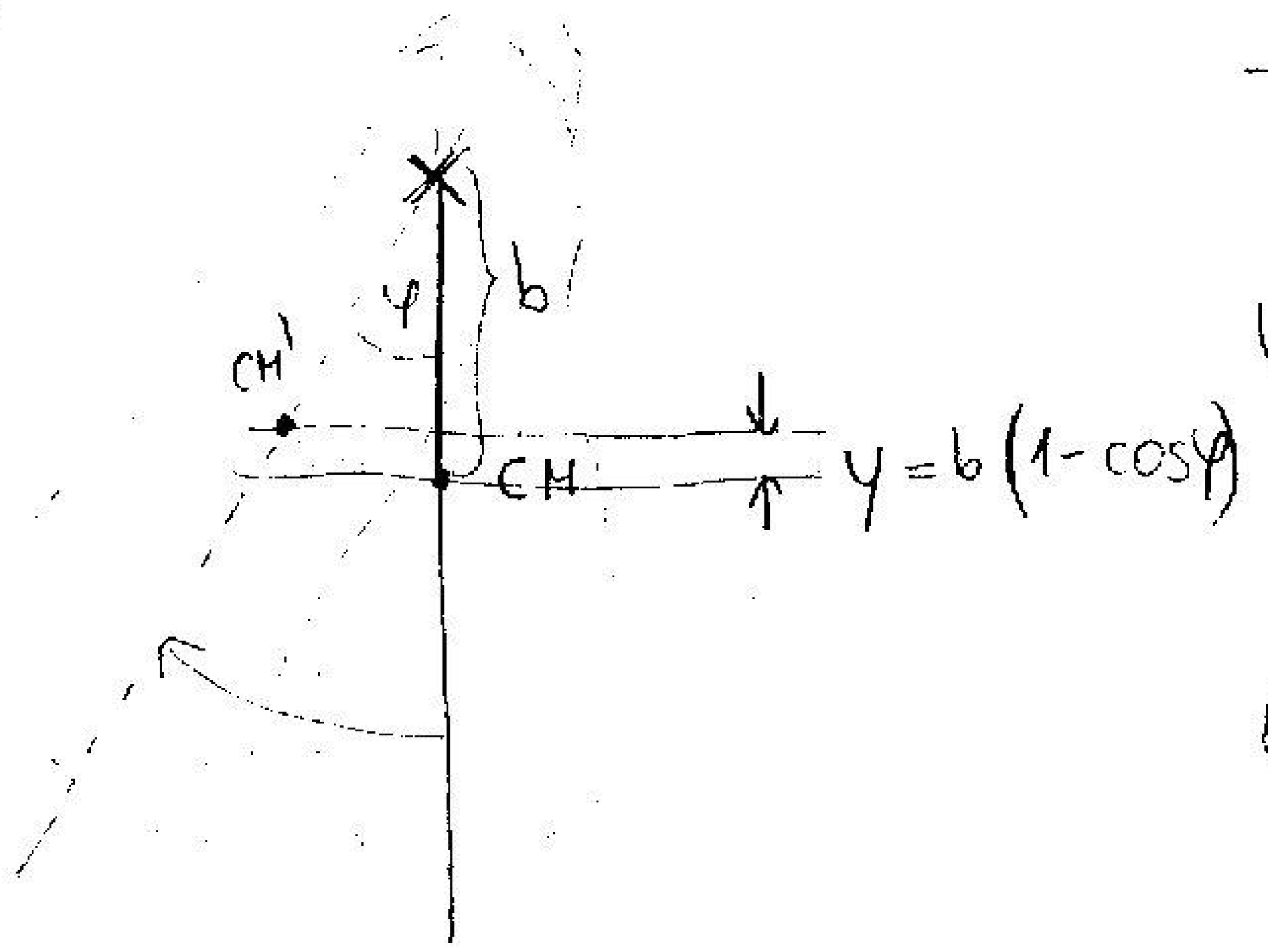
<u>Važno</u> $\sin\varphi \simeq \varphi$ za $\varphi << 1$

$$\Rightarrow \boxed{\ddot{\varphi} + \frac{g}{l}\varphi = 0} \Rightarrow \omega_0^2 = \frac{g}{l}$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

* <u>Fizičko njihalo</u>

!! DD



$$T = \frac{1}{2} I \dot{\varphi}^2$$

$$U = mgy = mgb(1-\cos\varphi)$$

$$0 = \frac{dE}{dt} = \frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2}I\dot{\varphi}^2 + mgb(1-\cos\varphi)\right) =$$

$$= \frac{1}{\cancel{2}} I \cancel{2} \dot{\varphi}\ddot{\varphi} + mgb\sin\varphi \cdot \dot{\varphi} =$$

$$= \dot{\varphi} I \underbrace{\left(\ddot{\varphi} + \frac{mgb}{I}\sin\varphi\right)}_{=0}$$

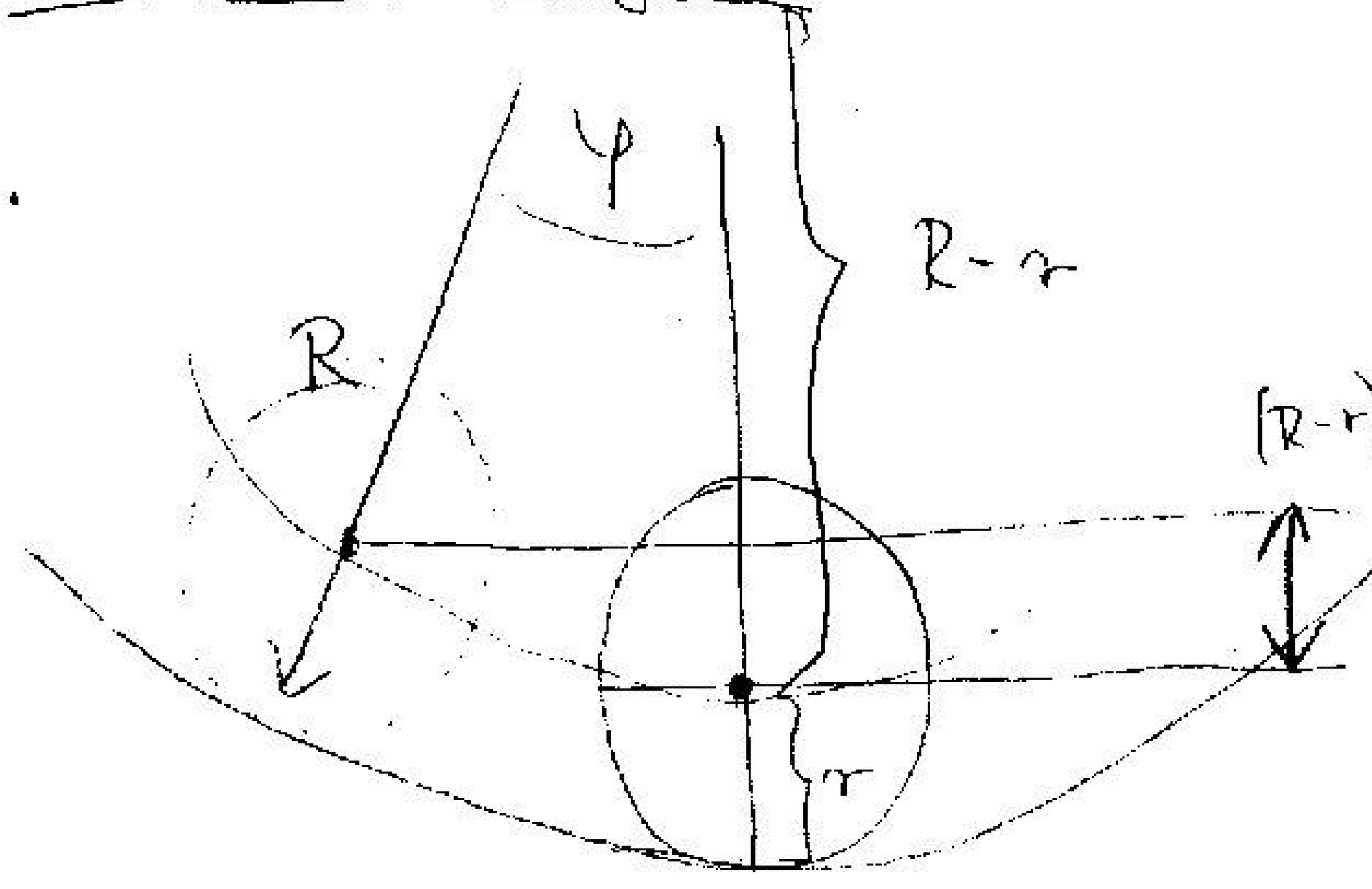
<u>Važno</u> $\varphi << 1 \Rightarrow \sin\varphi \simeq \varphi \Rightarrow \boxed{\ddot{\varphi} + \frac{mgb}{I}\cdot\varphi = 0}$

$$\omega_0^2 = \frac{mgb}{I}, \quad \omega_0 = \sqrt{\frac{mgb}{I}} = \sqrt{\frac{mgb}{I_{CM} + mb^2}}$$

$\Rightarrow$ Reducirana duljina fizičkog njihala:
- ona duljina koju bi moralo imati mat. njihalo da njiše kao fizičko

$$\omega_0^2 = \frac{mgb}{I_{CM} + mb^2} = \frac{g}{l} \Rightarrow \boxed{l_{red} = \frac{I_{CM} + mb^2}{mb}}$$

* kotač u žlijebu

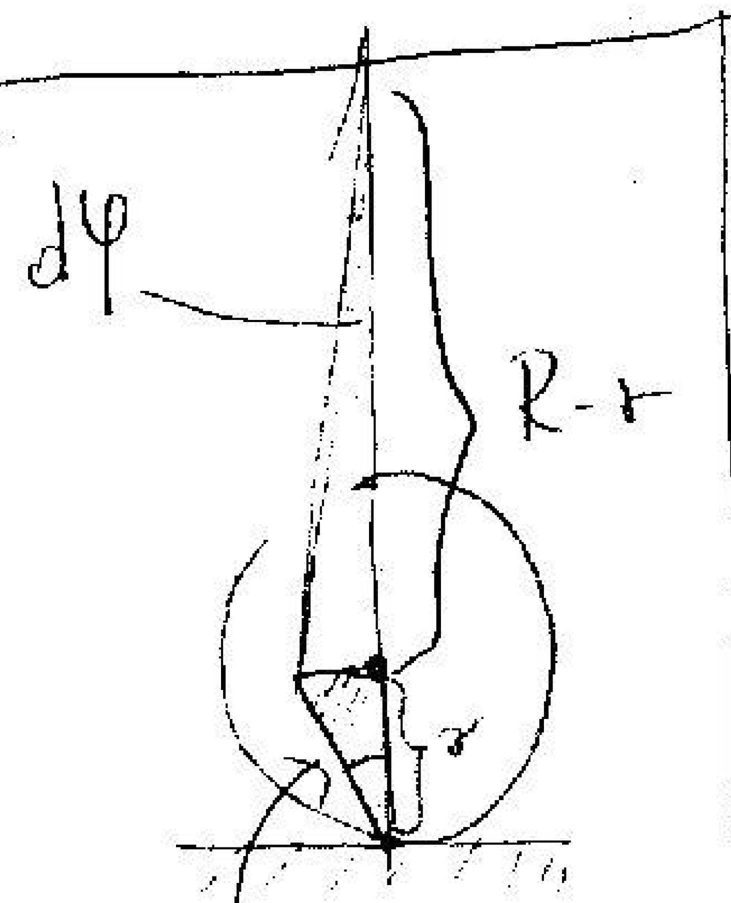


$$U = mg(R-r)(1-\cos\varphi)$$

$$T = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM} \omega_{CM}^2$$

$$= \frac{1}{2} m \left[(R-r)\dot{\varphi}\right]^2 + \frac{1}{2} I_{CM}\left[\frac{R-r}{r}\dot{\varphi}\right]^2$$

$$= \frac{1}{2} m \left[(R-r)\dot{\varphi}\right]^2 + \frac{1}{2}\frac{I_{CM}}{r^2}\left[(R-r)\dot{\varphi}\right]^2$$



$$= \left(\frac{1}{2} m + \frac{1}{2} K \frac{mr^2}{r^2}\right)\left[(R-r)\dot{\varphi}\right]^2$$

dα (kut za koji se kugla zakrene)

$$\frac{d\alpha}{d\varphi} = \frac{R-r}{r} \qquad \omega_{CM} = \frac{d\alpha}{dt} = \frac{R-r}{r}\frac{d\varphi}{dt} = \frac{R-r}{r}\dot{\varphi}$$

$I_{CM} = K m r^2$

$\hookrightarrow \frac{2}{5}$ za kuglu

$\hookrightarrow \frac{1}{2}$ za disk

$$0 = \frac{dE}{dt} = \frac{d}{dt}(T+U)$$

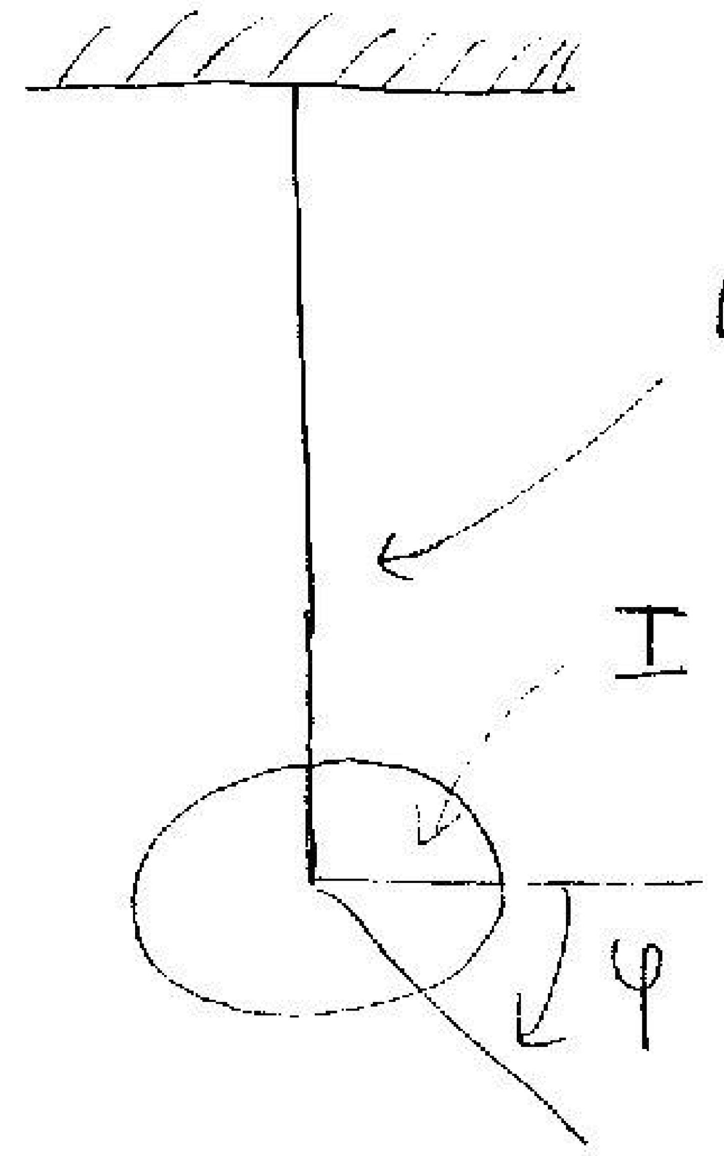
$$= \frac{d}{dt}\left[mg(R-r)(1-\cos\varphi) + \frac{m}{2}(1+K)(R-r)^2\dot{\varphi}^2\right] =$$

$$= mg(R-r)\sin\varphi \cdot \dot{\varphi} + \frac{m}{\cancel{2}}(1+K)(R-r)^2 \cdot \cancel{2}\dot{\varphi}\cdot\ddot{\varphi} =$$

$$= m(R-r)\dot{\varphi}\left[(1+K)(R-r)\ddot{\varphi} + g\underbrace{\sin\varphi}_{\hookrightarrow \varphi}\right] = 0$$

$$\Rightarrow \boxed{\ddot{\varphi} + \frac{g}{(1+K)(R-r)}\varphi = 0}$$

* Torziono njihalo



šipka:

$$M = D\varphi$$

$$\hookrightarrow D = \frac{G\pi R^4}{2L}$$

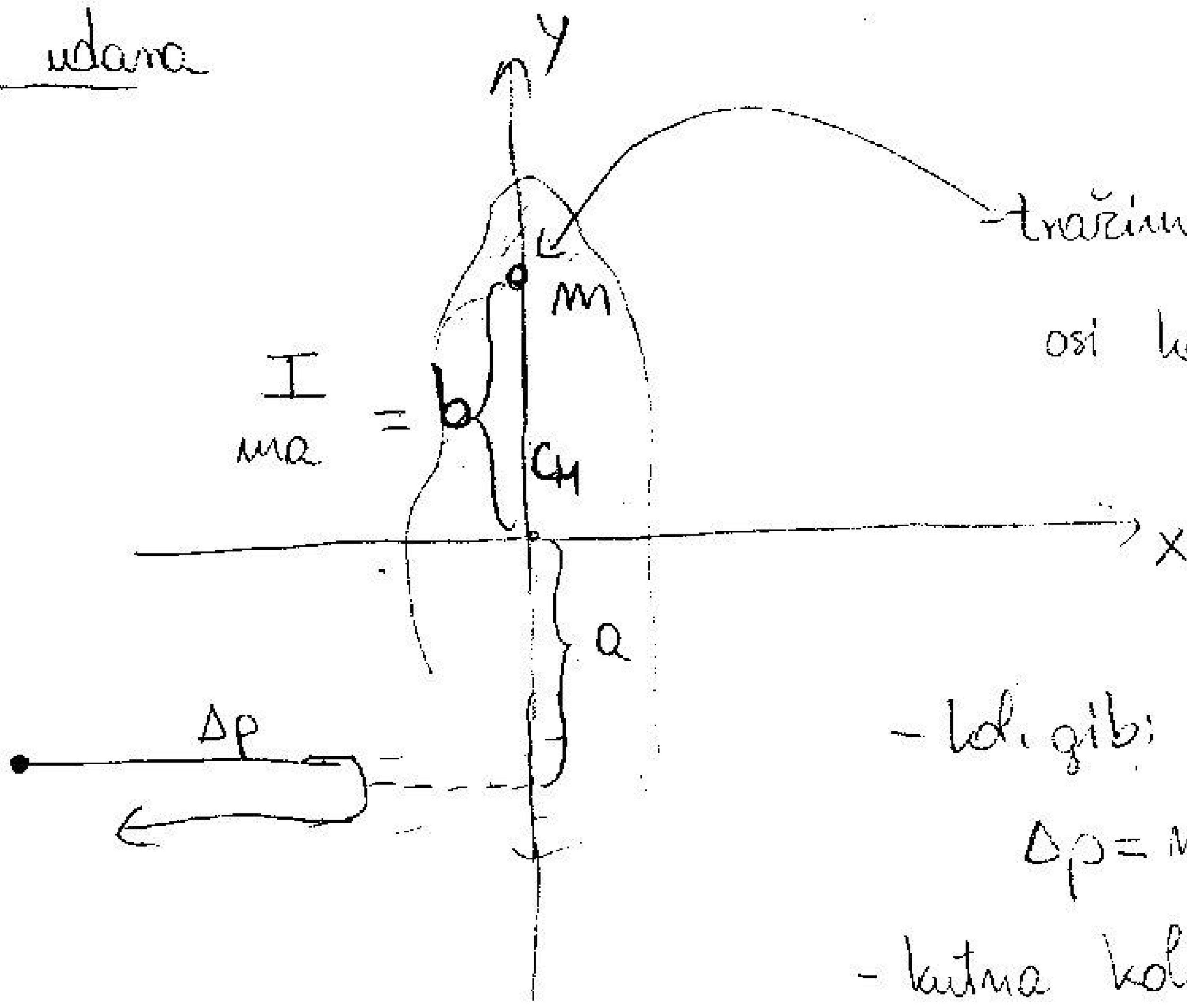
- jednadžba gibanja ploče

$$I\ddot{\varphi} = M \rightarrow \text{pokušava vratiti u ravnotežni položaj}$$

$$= -D\varphi$$

$$\ddot{\varphi} + \frac{D}{I}\varphi = 0, \quad \Rightarrow \quad \omega_0^2 = \frac{D}{I} = \frac{G\pi R^4}{2LI}$$

Centar udara



- tražimo točku na y osi koja miruje nakon udara!

- lin. gib:

$$\Delta p = m \cdot v_{CM}$$

- kutna količina gibanja

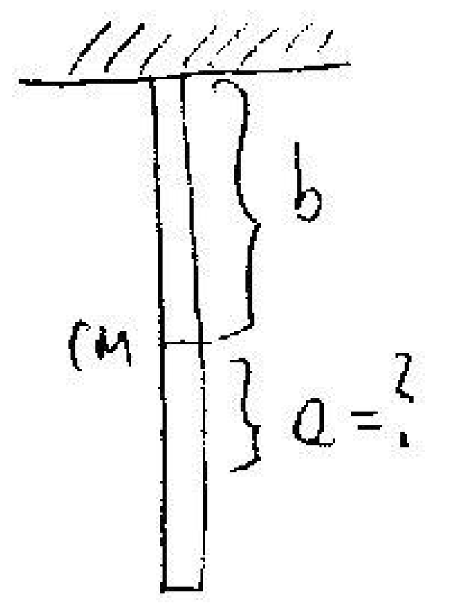
$$\Delta L = a \cdot \Delta p = I_{CM} \cdot \omega_{CM}$$

- točka koja miruje

$$v(y) = v_{CM} - \omega_{CM} y$$

$$0 = v(b) = v_{CM} - \omega_{CM} b \quad \Rightarrow \quad b = \frac{v_{CM}}{\omega_{CM}} = \frac{\frac{\Delta p}{m}}{\frac{a\Delta p}{I}} = \frac{I}{ma}$$

Primjer



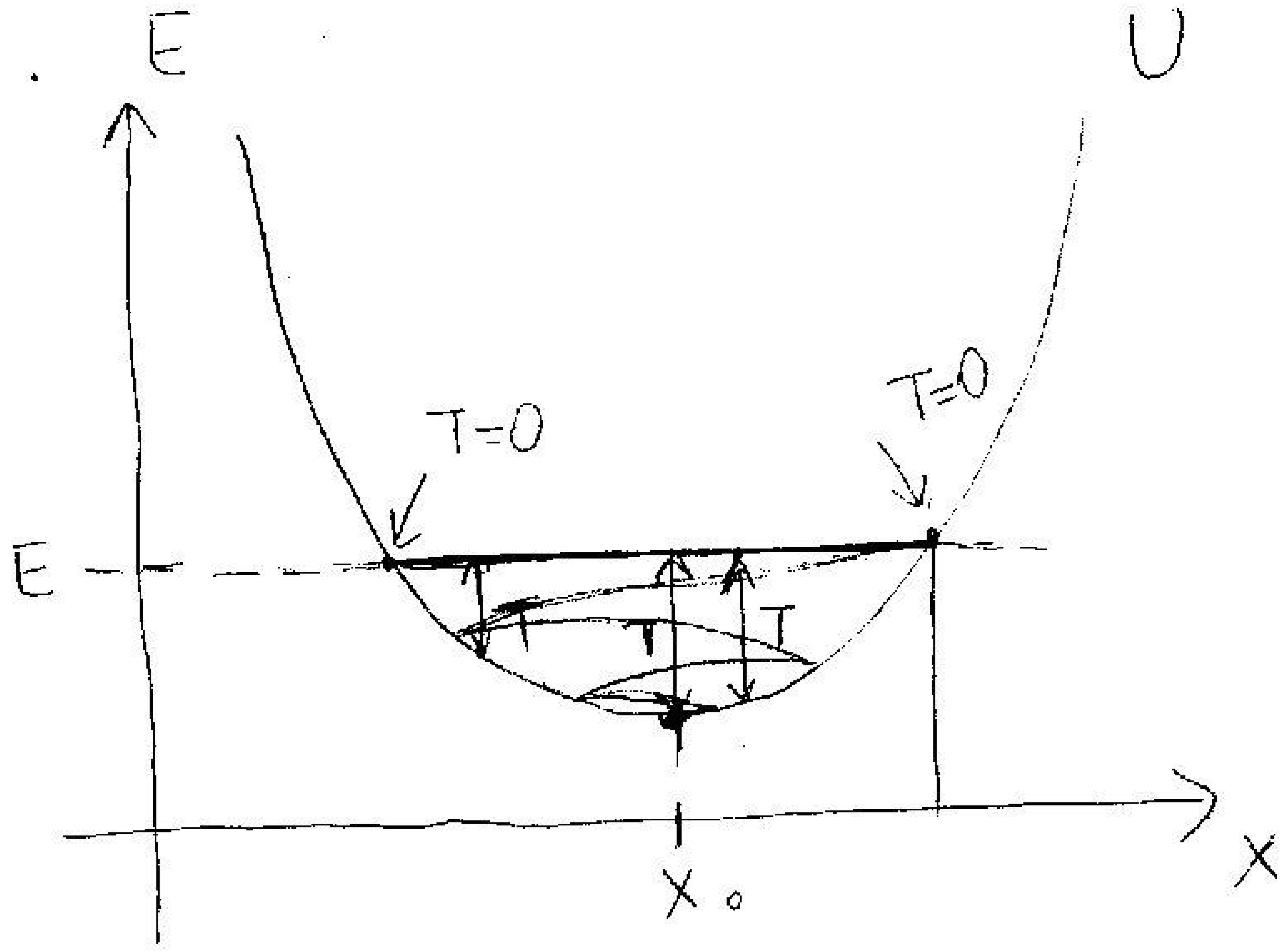
Gdje treba lupiti (a=?) da objesište miruje?

$$b = \frac{I_{CM}}{ma}, \quad \frac{l}{2}, \quad \frac{1}{12} m l^2$$

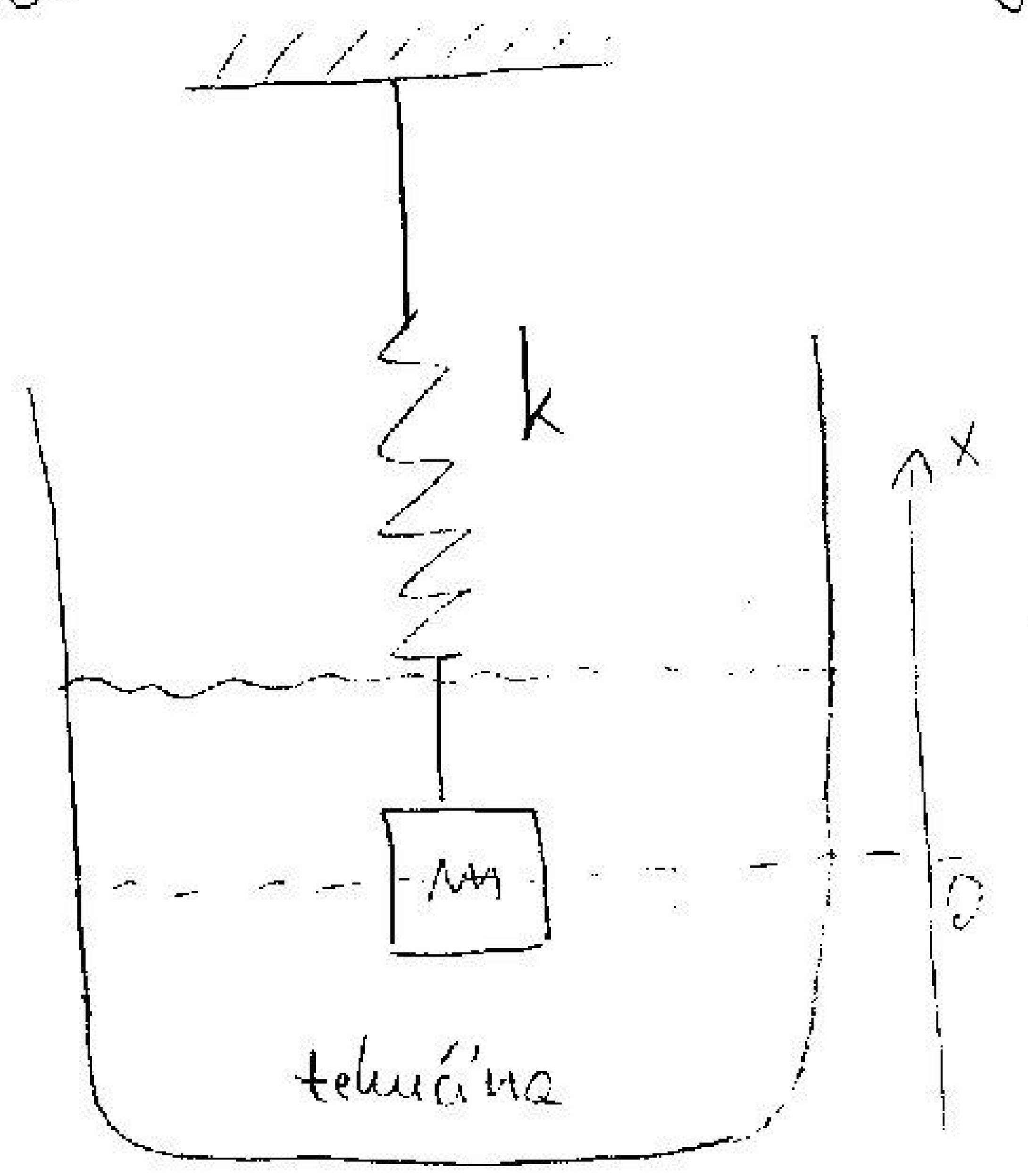
$$\cancel{m} a \frac{l}{2} = \frac{1}{12} \cancel{m} l^2$$

$$\boxed{a = \frac{1}{6} l}$$

Prigušeno titranje



model



- jednadžba gibanja:

$$m\ddot{x}(t) = -k\,x(t) - b\dot{x}(t)$$

↳ akc. ↳ položaj ↳ brzina

→ posebni slučaj: $k=0$

$$x(t) = x_0 + v_0\frac{m}{b}\left(1-e^{-\frac{b}{m}t}\right)$$

→ posebni slučaj: $b=0$

$$x(t) = A\cos(\omega_0 t+\varphi),\quad \omega_0^2=\frac{k}{m}$$

→ općeniti slučaj:

- rješavanje jednadžbe gibanja

$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0 \qquad \text{gdje je } \omega_0^2=\frac{k}{m},\ 2\delta=\frac{b}{m}$$

- probno rješenje: $x(t) = e^{\alpha t}$

$$\Rightarrow \quad \underbrace{\left(\alpha^2 + 2\delta\alpha + \omega_0^2\right)}_{=0} x(t) = 0$$

$$\boxed{\alpha_{1,2} = -\delta \pm \sqrt{\delta^2-\omega_0^2}}$$

3 slučaja:

1) slabo gušenje $\delta < \omega_0$, $\alpha_{1,2} \in \mathbb{C}$

2) snažno gušenje $\delta > \omega_0$, $\alpha_{1,2} \in \mathbb{R}$

3) kritično gušenje $\alpha_1 = \alpha_2 = -\delta$

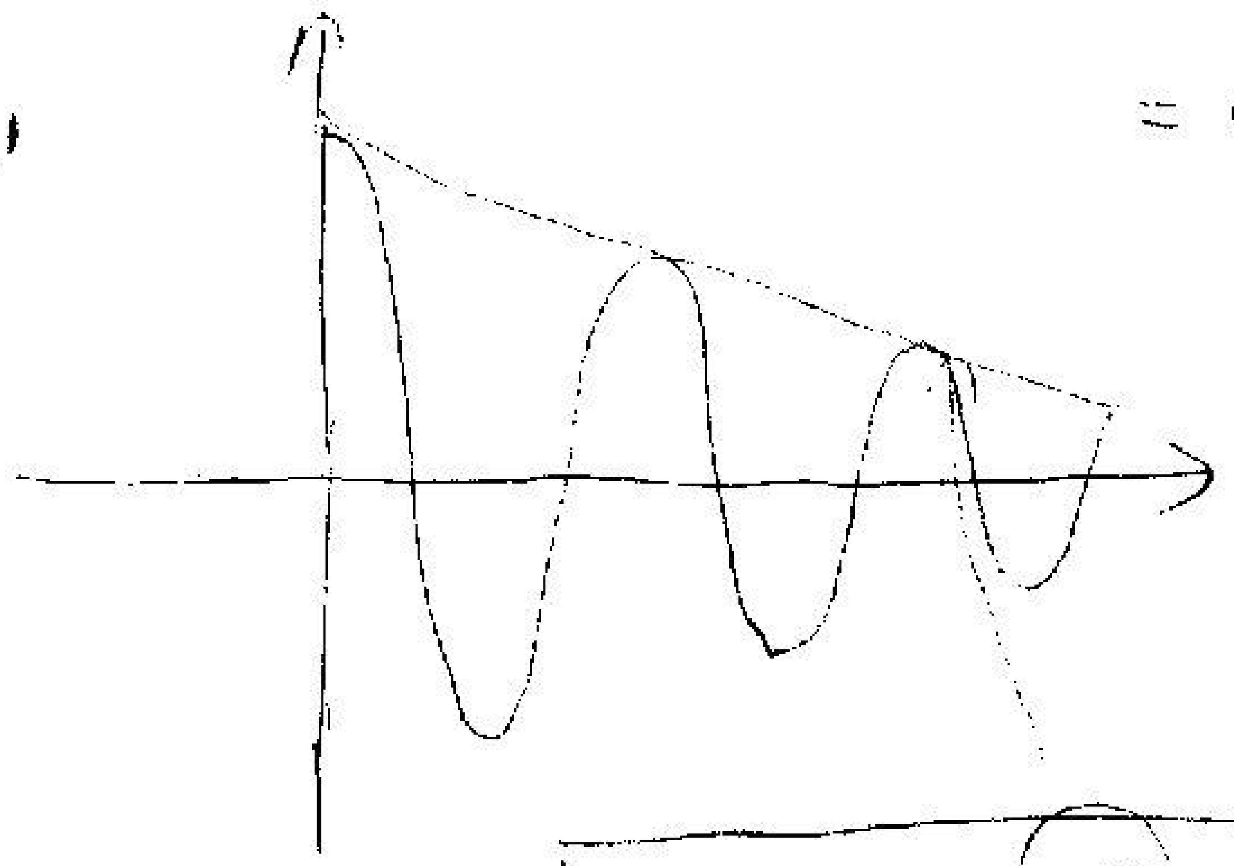
1. slučaj: $\delta < \omega_0$ (slabo gušenje)

$$\alpha_{1,2} = -\delta \pm i\omega, \quad \omega \equiv \sqrt{\omega_0^2 - \delta^2} < \omega_0$$

- opće rješenje:

$$x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}, \quad a_1, a_2 \in \mathbb{C}$$

$$= e^{-\delta t} \underbrace{\left[a_1 e^{i\omega t} + a_2 e^{-i\omega t}\right]}_{*}$$



* (vidjeti izvod za jednostavni harmonički oscilator)

$\boxed{\text{- zahtjev } x(t) \in \mathbb{R}}$ $\quad a_1 = a_2^* = \frac{A}{2} e^{i\varphi}$

$A, \varphi \in \mathbb{R}$

$$\Rightarrow \boxed{x(t) = A e^{-\delta t} \cdot \cos(\omega t + \varphi)}$$

!!! !!!

3. slučaj: $\delta = \omega_0$ (kritično prigušenje)

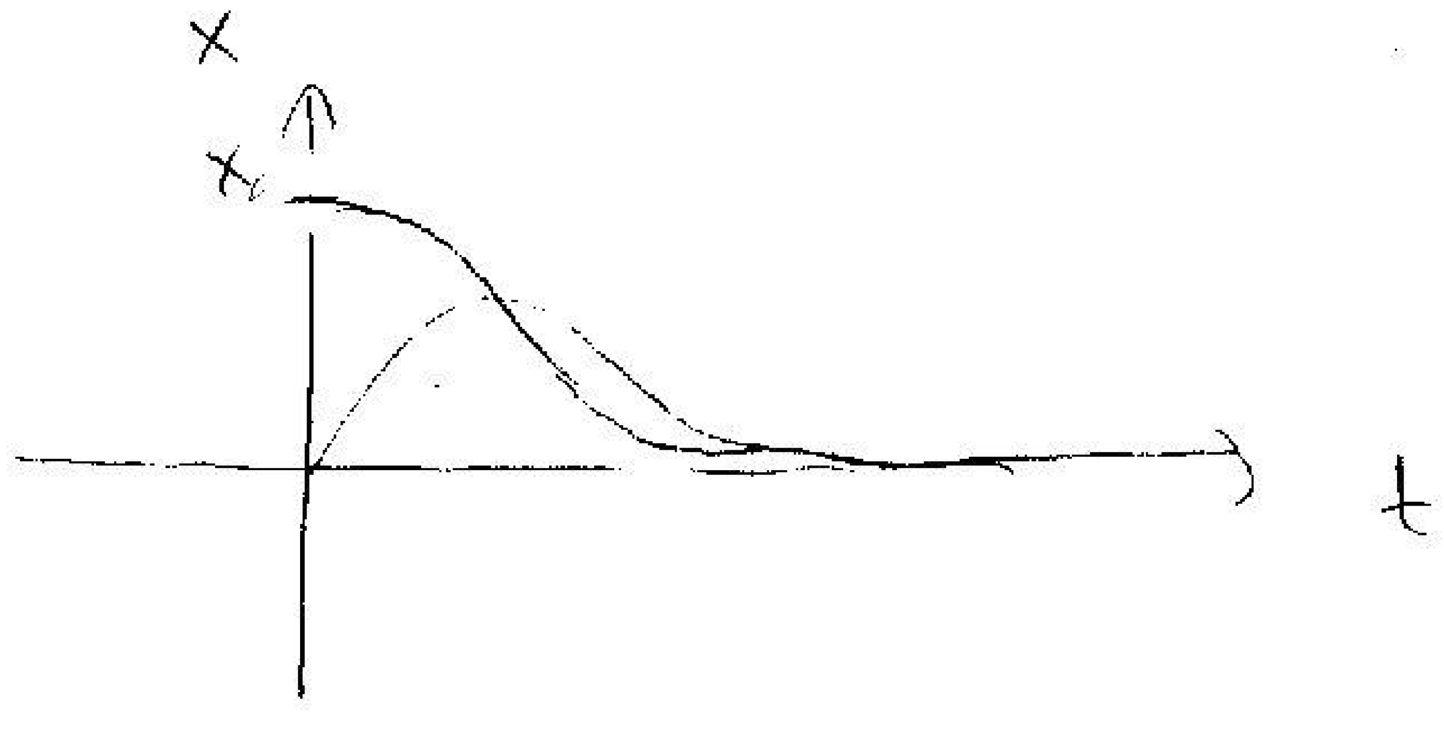
- opće rj.: $x(t) = (a_1 + a_2 t) e^{-\delta t}$

- početni uvjeti u $t=0$: $x_0 = x(0)$, $v_0 = \dot{x}(0)$

- rješenje:

$$x(t) = e^{-\delta t}\big(\overbrace{x_0 + (v_0 + x_0\delta) t}^{\text{linearna funkcija}}\big)$$

početni uvjeti: x_0, v_0, x_0



2. smaćno gušenje, $\delta > \omega_0$

$$\alpha_{1,2} = -\delta \pm \mathcal{g}$$

$$\mathcal{g} \equiv \sqrt{\delta^2 - \omega_0^2}$$

- opće rješenje

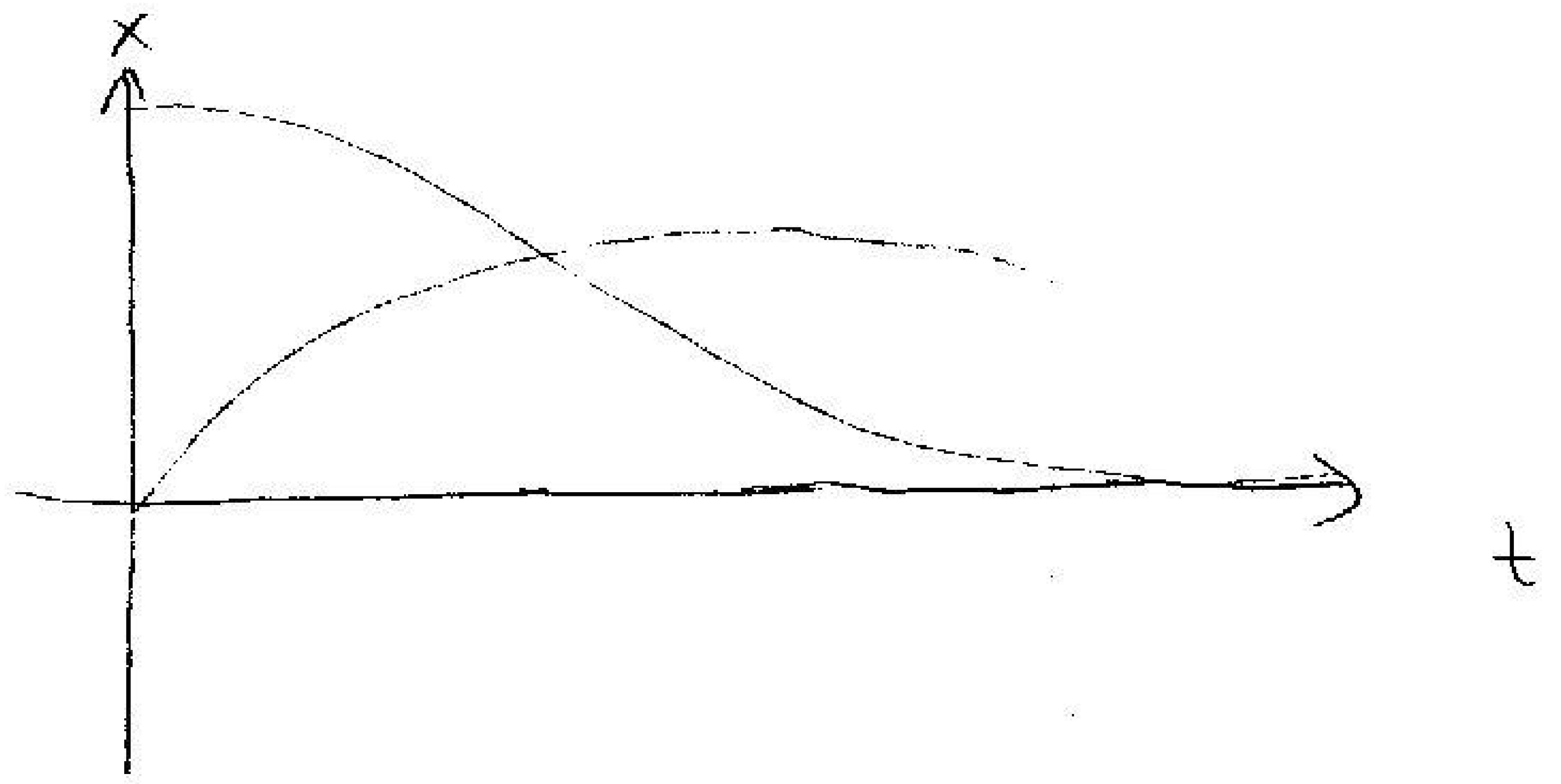
$$x(t) = a_1 e^{\alpha_1 t} + a_2 e^{\alpha_2 t}$$
$$= e^{-\delta t}\left[a_1 e^{\mathcal{g}t} + a_2 e^{-\mathcal{g}t}\right]$$
$$= e^{-\delta t}\left[b_1 \operatorname{ch}(\mathcal{g}t) + b_2 \operatorname{sh}(\mathcal{g}t)\right]$$

- početni uvjeti:

$$x_0 \equiv x(0)$$
$$v_0 \equiv \dot{x}(0)$$

- rješenje:

$$x(t) = e^{-\delta t}\left[x_0 \operatorname{ch}(\mathcal{g}t) + \frac{v_0 + x_0\delta}{\mathcal{g}} \operatorname{sh}(\mathcal{g}t)\right]$$



Energija prigušenog titranja

E_{kin} $\quad T = \frac{1}{2} m v^2$

$E_{poten.}$ $\quad U = \frac{1}{2} k x^2$

$$E_{uk} = T + U$$

promjena E u vremenu:

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) = \frac{d}{dt}\left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} k x^2\right) = m\dot{x}\ddot{x} + kx\dot{x} =$$

$$= m\dot{x}\left(\ddot{x} + \frac{k}{m}x\right) = m\dot{x}\left(\ddot{x} + \omega^2 x\right)$$

(sjetimo se j. gibanja $\left(\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0\right)$)

$$\Rightarrow \; = m\dot{x}\left(-2\delta\dot{x}\right) = -2\delta m\dot{x}^2 = -4\delta\frac{m\dot{x}^2}{2} = \boxed{-4\delta T}$$

* mjere slabog prigušenja

- logaritamski dekrement prigušenja

$$\lambda \equiv \ln\frac{x(t)}{x(t+\tau)}$$

$\tau \longrightarrow$ period prigušenog titranja

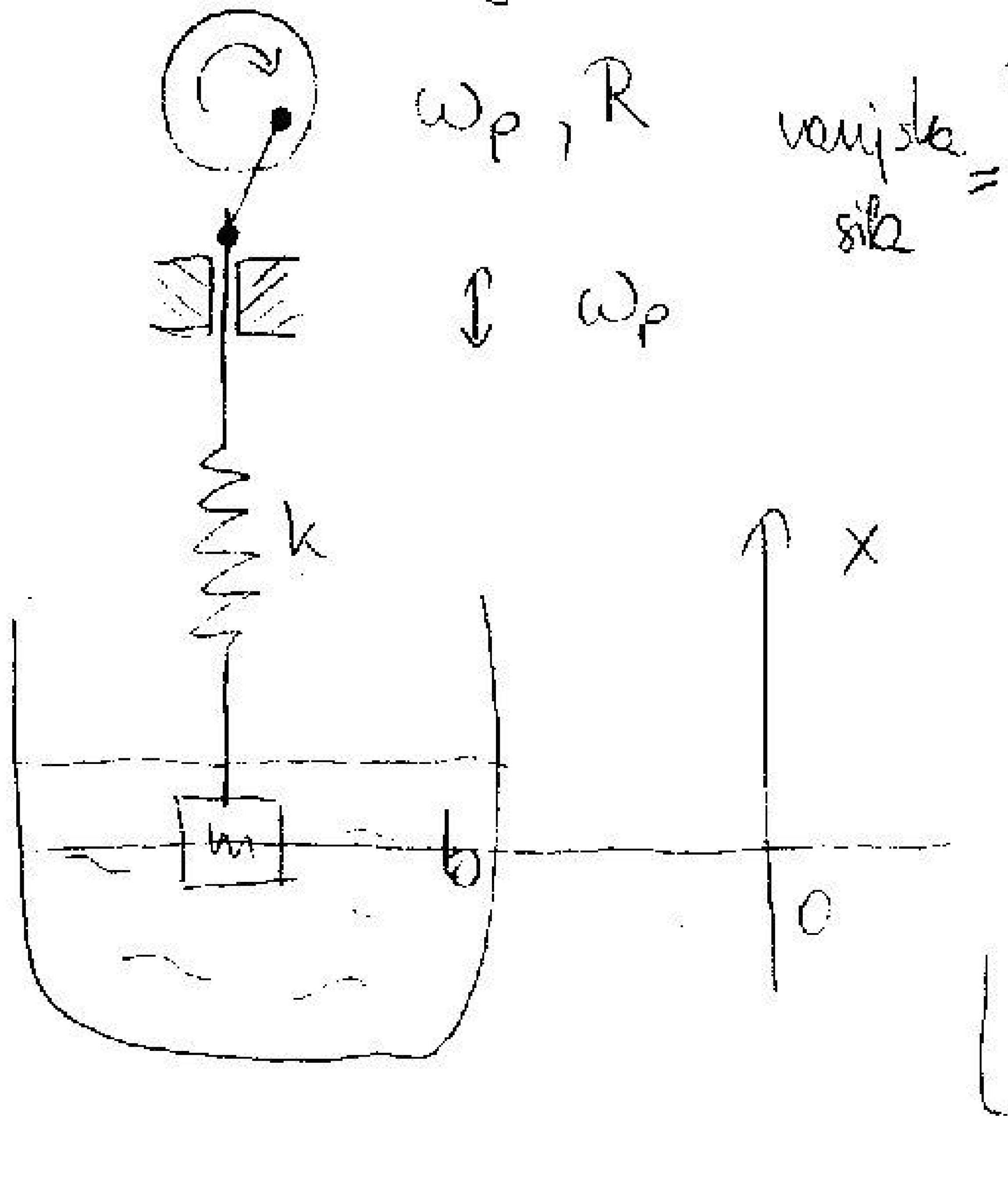
$$= \ln\frac{\cancel{A}e^{-\delta t}\cancel{\cos(\omega t+\varphi)}}{\cancel{A}e^{-\delta(t+\tau)}\cos(\underbrace{\omega(t+\tau)}_{(2\pi)}+\varphi)} = \ln e^{\delta\tau} = \delta\tau =$$

$$= \delta\frac{2\pi}{\omega} = 2\pi\frac{\delta}{\omega} \simeq \boxed{2\pi\frac{\delta}{\omega_0}}$$

- faktor dobrote, Q-faktor

$$Q^{-1} = -\frac{1}{2\pi}\left(\frac{\Delta E}{E}\right)_{ciklus} = \dots = \frac{2\delta}{\omega} \simeq \frac{2\delta}{\omega_0}$$

* Prisilno titranje



F_p - amplituda sile

vanjska sila $= \boxed{kR}\cos(\omega_p t)$

- jednadžba gibanja

$$m\ddot{x} = -kx - b\dot{x} + F_p\cos(\omega_p t)$$

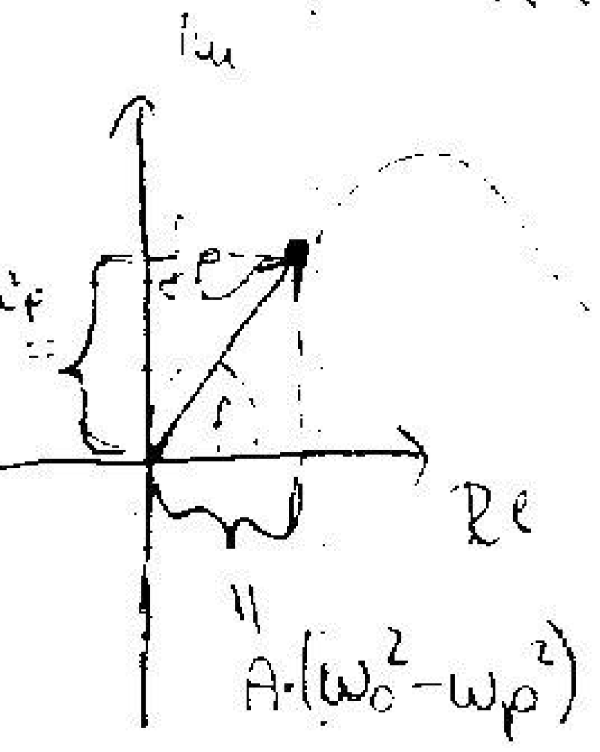
$$\omega_0^2 = \frac{k}{m}, \quad 2\delta = \frac{b}{m}, \quad f_p \equiv \frac{F_p}{m}$$

$$\boxed{\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = f_p\cos(\omega_p t)} \quad (*)$$

-očekujemo rješenje oblika

$$x(t) = A(\omega_p)\cdot\cos\left(\omega_p t - \varphi(\omega_p)\right)$$

$$x(t) = A e^{i(\omega_p t-\varphi)} \rightarrow (*) \Rightarrow$$



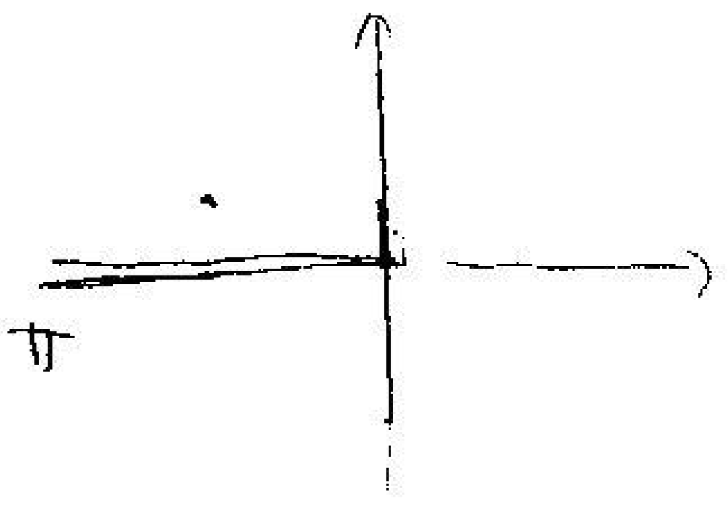
$$\left(-\omega_p^2 + 2\delta(i\omega_p)+\omega_0^2\right) A \cancel{e^{i(\omega_p t)}} e^{-i\varphi} = f_p \cancel{e^{i(\omega_p t)}}$$

$$\left[(\omega_0^2-\omega_p^2) + i2\delta\omega_p\right]A = f_p\cdot e^{i\varphi}$$

$$\Rightarrow tg\,\varphi = \frac{2\cancel{A}\delta\omega_p}{\cancel{A}(\omega_0^2-\omega_p^2)} = \frac{2\delta\omega_p}{(\omega_0^2-\omega_p^2)}$$

- slučaj $\omega_p \gg \omega_0$, $tg\,\varphi \to -\frac{2\delta}{\omega_p}$ π

  $\varphi \Rightarrow \pi$



- slučaj $\omega_p \to 0$

  $\varphi \to 0$

$$A^2 = \frac{f_p^2}{\underbrace{(\omega_0^2-\omega_p^2)^2 + 4\delta^2\omega_p^2}_{M}}$$

Rezonancija amplitude prisilnog titranja

$$0 = \frac{d}{d\omega_p}M = 2\left(\omega_0^2-\omega_p^2\right)(-2\omega_p) + 8\delta^2\omega_p$$

$$= 4\omega_p\underbrace{\left(-\omega_0^2+\omega_p^2+2\delta^2\right)}_{0} \Rightarrow \boxed{\omega_{p,rez}^2 = \omega_0^2 - 2\delta^2}$$

<u>Snaga:</u>

$$P(t) = F_p(t)\,v(t)$$

$$= \underbrace{F_p\cos(\omega_p t)}_{sila}\cdot\underbrace{A(\omega_p)\left[-\omega_p\sin(\omega_p t-\varphi)\right]}_{brzina}$$

$$= F_p A(\omega_p)\left[\cos^2(\omega_p t)\sin\varphi + \sin(\omega_p t)\cos(\omega_p t)\cos\varphi\right]$$

$\to$ [illegible]

Srednja snaga:

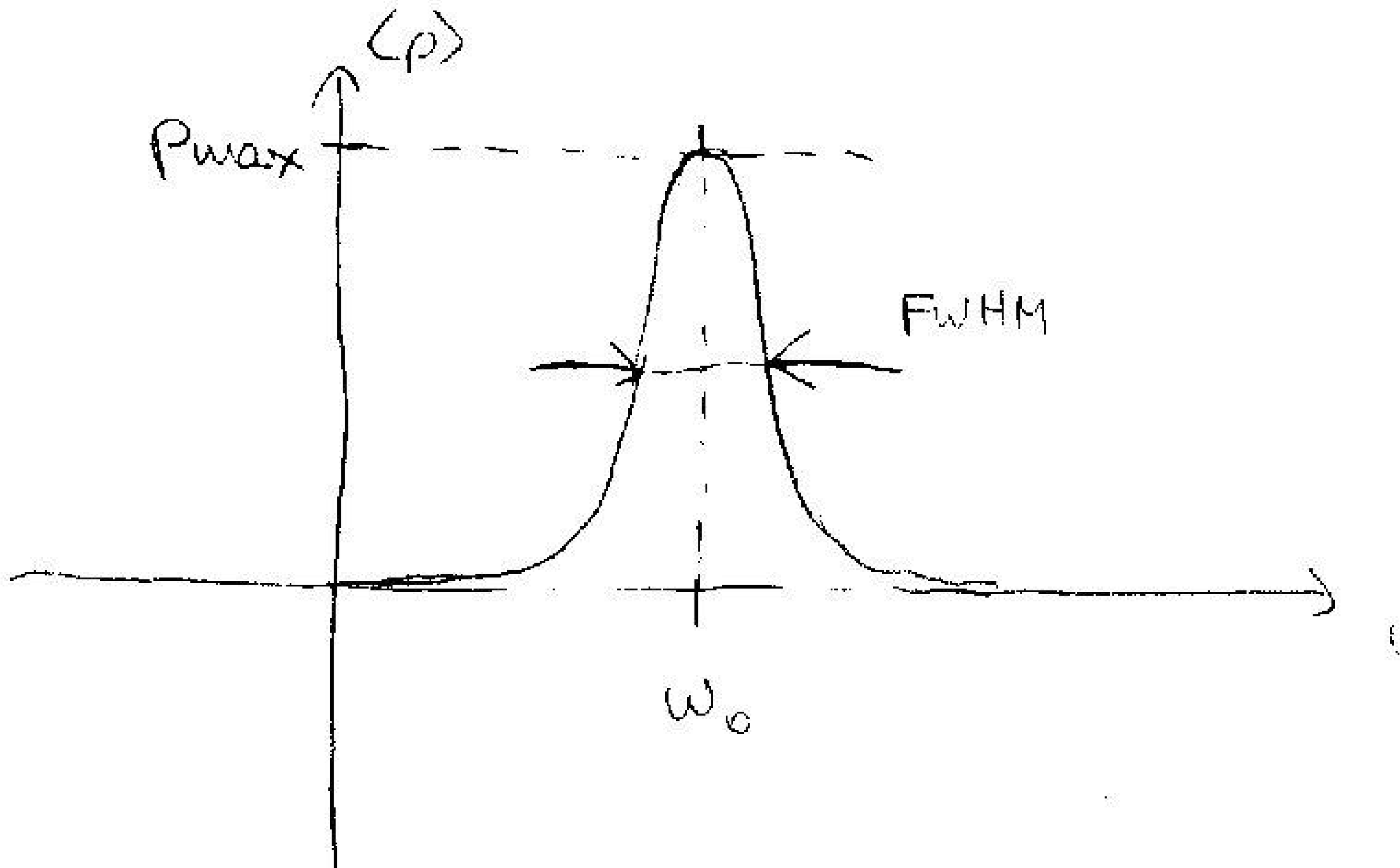
$$\langle P \rangle = \frac{1}{\tau}\int_0^{\tau} P(t)\,dt = \frac{\omega_p}{2\pi} F_p A(\omega_p) \int_0^{\tau} \left[\cos^2(\omega_p t)\sin\varphi + \sin(\omega_p t)\cos(\omega_p t)\cos\varphi\right] dt =$$

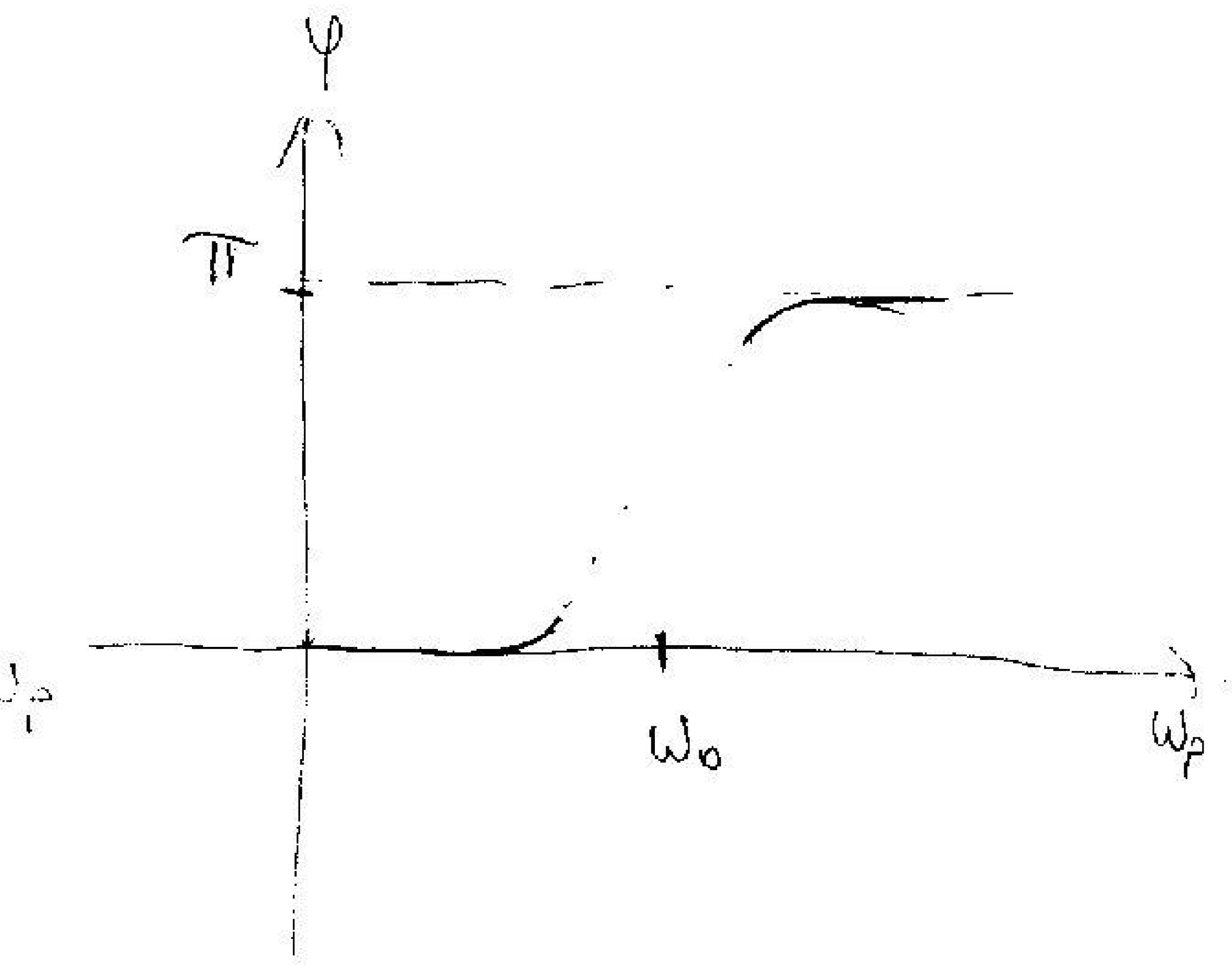
$$= \frac{\omega_p}{2\cancel{\pi}} F_p A(\omega_p)\, \cancel{\pi} \sin\varphi = \frac{1}{2}\omega_p F_p A(\omega_p) \sin\varphi =$$

$$= m\delta f_p^2 \frac{\omega_p^2}{(\omega_0^2-\omega_p^2)^2 + 4\delta^2\omega_p^2}$$

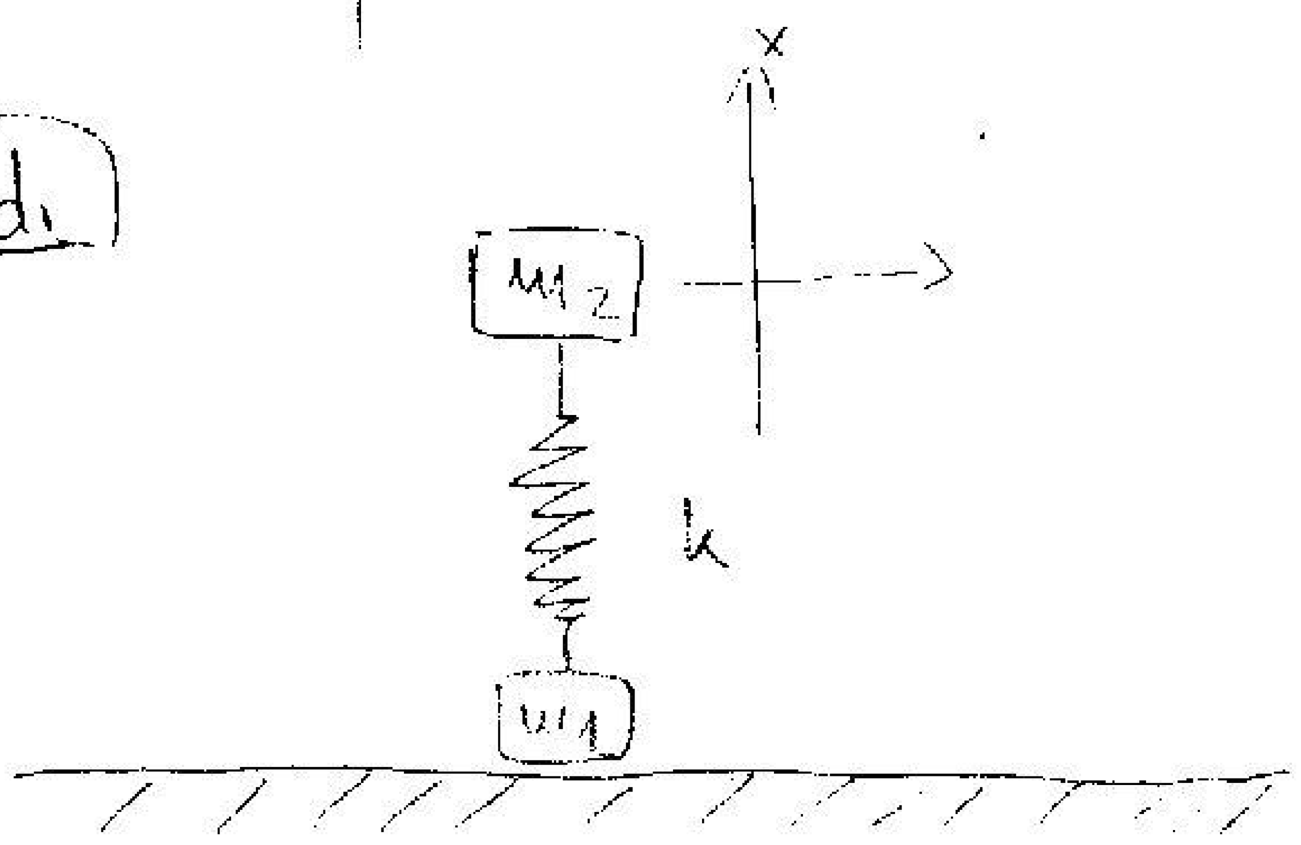
- maksimum $\langle p \rangle$ je pri $\boxed{\omega_p = \omega_0}$

$$\boxed{\langle p \rangle_{max} = \frac{m}{4\delta} f_p^2}$$





(Zad.)



* Ako m_1 fiksiram

$$F_1 = kx = k\,a\cos(\omega_0 t + \varphi)$$

(zab. gravitaciju!) ↳ amplituda titranja

$$F_1 = -m_2 g + kx = -m_2 g + ka\cos(\omega_0 t + \varphi)$$

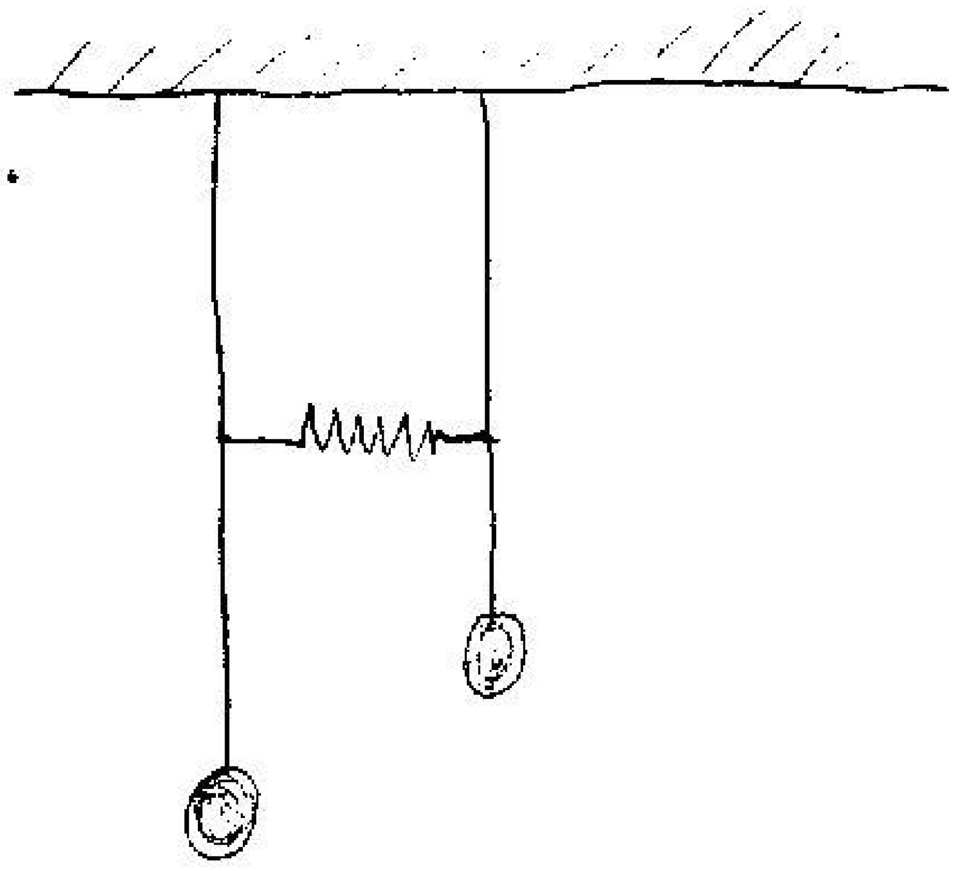
$$F \in \left[-m_2 g - ka,\ \underbrace{-m_2 g + ka}\right]$$

Ako m_1 slobodan, odvaja se pri $F_1 = m_1 g = -m_2 g + ka$

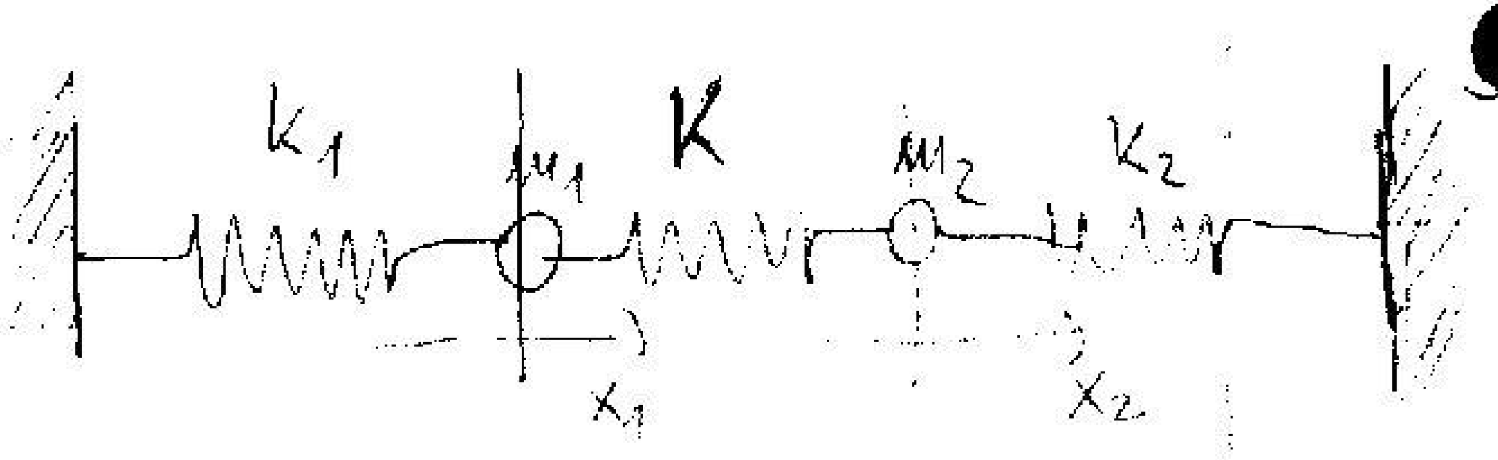
$$ka = (m_1 + m_2) g$$

$$\boxed{a = \frac{m_1 + m_2}{k} g}$$

* <u>Oberbeckovo njihalo</u>



$\Longleftrightarrow$



- jednostavnije:

$$m_1 = m_2 = m$$

$$k_1 = k_2 = k \neq K$$

$$\left.\begin{aligned} m\ddot{x}_1 &= -kx_1 - K(x_1 - x_2) \quad (1) \\ m\ddot{x}_2 &= -kx_2 - K(x_2 - x_1) \quad (2) \end{aligned}\right\}$$

- pogađanje u fazi:

$$\omega_1^2 = \frac{k}{m}$$

- u protufazi:

$$\omega_2^2 = \frac{k + 2K}{m}$$

- <u>račun:</u>

$$\xi_{1,2} = x_1 \pm x_2$$

$$\boxed{(1) + (2)} \Rightarrow \quad m\,\underbrace{(\ddot{x}_1 + \ddot{x}_2)} = -k(x_1 + x_2)$$

$$m\ddot{\xi}_1 = -k\xi_1$$

$$\ddot{\xi}_1 + \frac{k}{m}\xi_1 = 0$$

$$\ddot{\xi}_1 + \omega_1^2\xi_1 = 0$$

$$\xi_1(t) = A_1\cos(\omega_1 t + \varphi_1)$$

titranje u fazi, $\omega_1^2 = \frac{k}{m}$

$\boxed{(1)-(2)} \Rightarrow$

$$m\underbrace{(\ddot{x}_1-\ddot{x}_2)}=-k\underbrace{(x_1-x_2)}-2K\underbrace{(x_1-x_2)}$$

$$m\ddot{\xi}_2=-k\xi_2-2K\xi_2$$

(zadatak, ispit)

$$\ddot{\xi}_2+\frac{k+2K}{m}\xi^2=0$$

$$\ddot{\xi}_2+\omega_2^2\xi_2=0$$

$\xi_2(t)=A_2\cos(\omega_2 t+\varphi_2)$ titranje u protufazi;

$$\omega_2^2=\frac{k+2K}{m}$$

$$\begin{pmatrix} x_1=\frac{1}{2}(\xi_1+\xi_2) \\ x_2=\frac{1}{2}(\xi_1-\xi_2) \end{pmatrix}$$

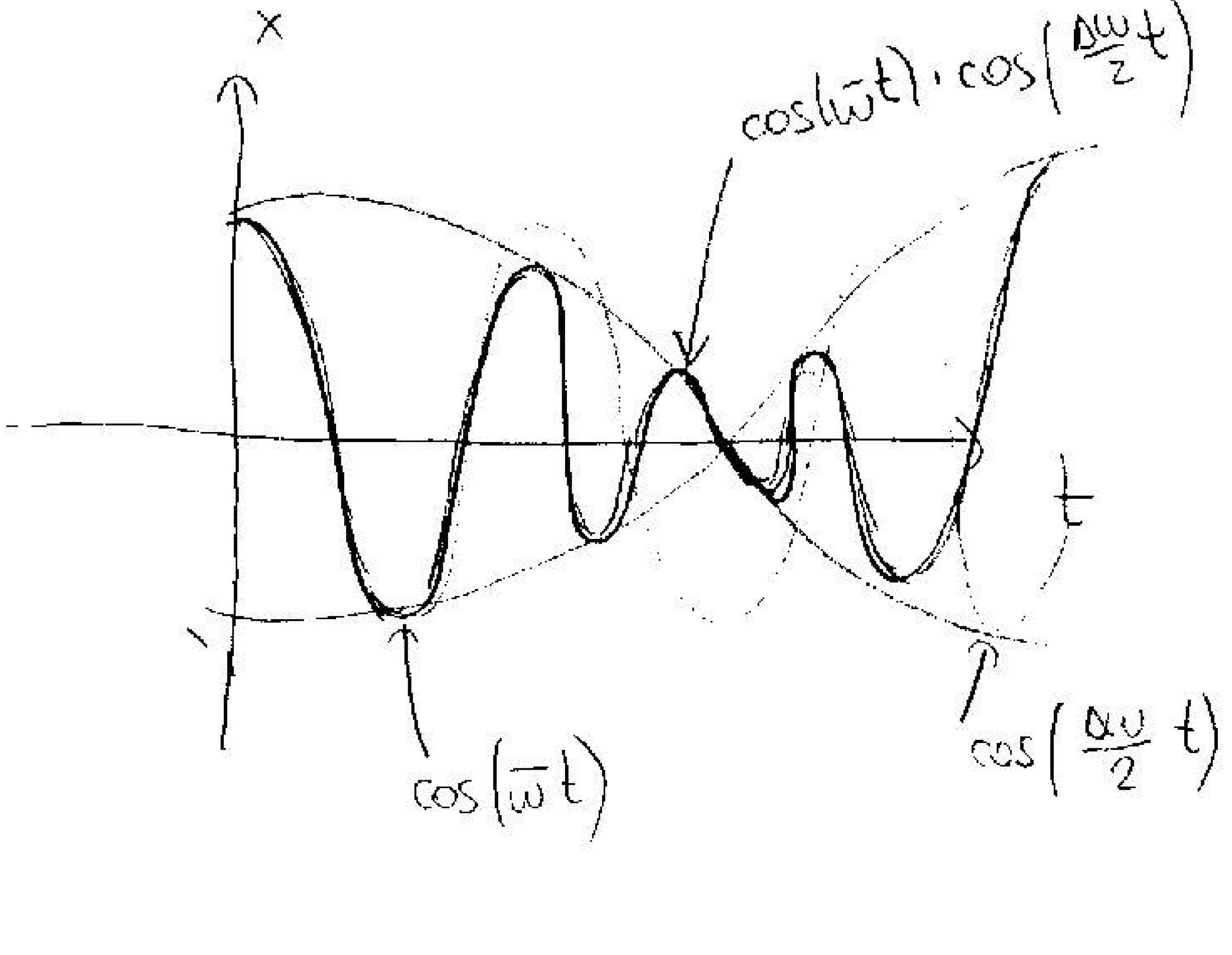
$$x=\frac{A_1}{2}\cos(\omega_1 t+\varphi_1)+\frac{A_2}{2}\cos(\omega_2 t+\varphi_2)$$

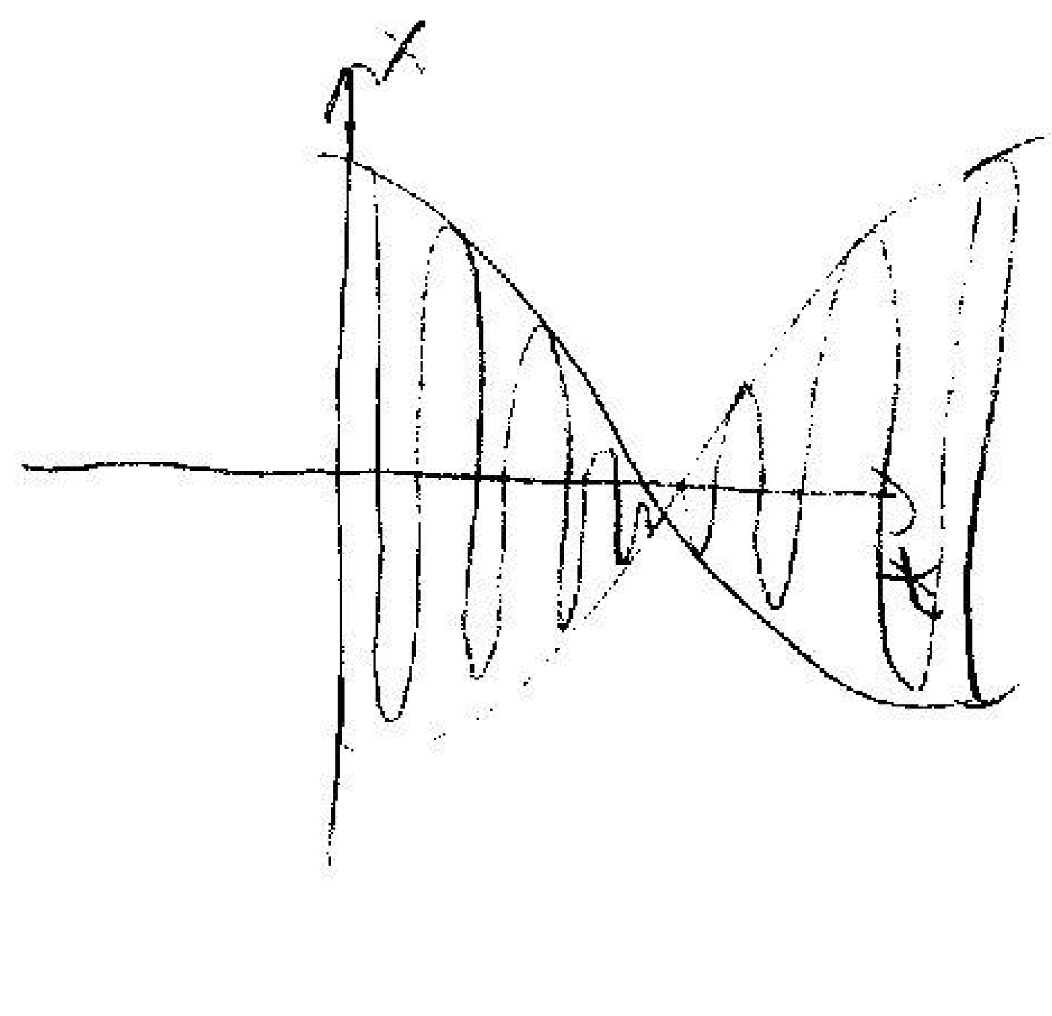
- posebni slučaj: $K \ll k \Rightarrow \omega_1 \approx \omega_2$, $\omega_{1,2}=\overline{\omega}\mp\frac{\Delta\omega}{2}$, $\Delta\omega=\omega_2-\omega_1$

$\overline{\omega}=\frac{1}{2}(\omega_1+\omega_2)$

$A_1=A_2$, $\varphi_1=\varphi_2=0$

$$\boxed{x=A\underbrace{\cos(\overline{\omega}t)}_{\text{brzo titranje}}\underbrace{\cos\left(\frac{\Delta\omega}{2}t\right)}_{\text{titranje frekvencijom koja je duplo manja od razlike}}}$$





x,t)

VALOVI

- valovi → Meh., EM

- oblik → kuglasti (kružni), ravni

- širenje → progresivni, stojni

- način titranja → long., trans.

- izvor $\ast \xrightarrow{\quad r \quad} \circ$ detektor

$f(t) = A_0 \cos(\omega t)$ $\qquad$ $A(r)\cos(\omega t - \varphi(r))$

"frekvencija"

→ kašnjenje u fazi

$\varphi(r) = kr$ → "valni vektor"

- "valna duljina" - najmanja udaljenost za koju vrijedi:

$$\cos(\omega t - kx) = \cos(\omega t - k(x+\lambda))$$

$$k\lambda = 2\pi \Rightarrow \lambda = \frac{2\pi}{k}$$

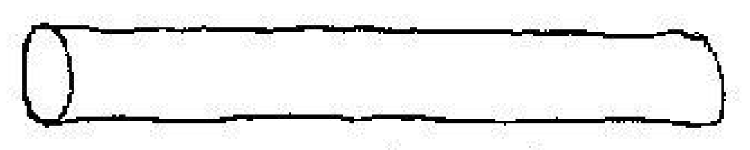
- "fazna brzina": $v_f = \frac{\lambda}{T} = \frac{\omega}{k}$

- načini pisanja

$$y(t,x) = A\cos(\omega t - kx) = A\cos[k(vt - x)] = A\cos\left[2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\right]$$

* Izvod valne jednadžbe

- fizikalni model

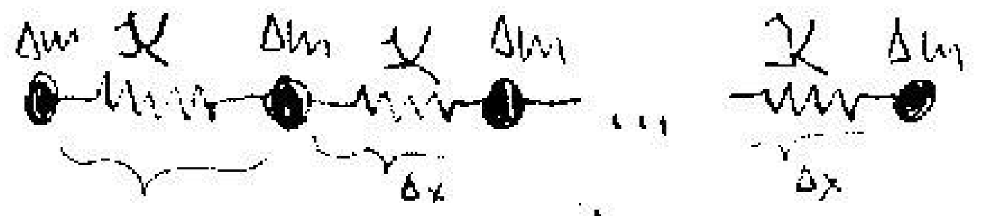


M, L, S, E, T

- ekvivalentno

$M, L, k = \frac{SE}{L}, T$

- ekvivalentno, N-broj čestica



$\Delta x \qquad \Delta m_1 = \frac{M}{N}, \qquad \Delta x = \frac{L}{N}, \qquad K = Nk$

$$k_k = K \frac{X}{N}$$

$K = Nk = N \frac{SE}{L}$, prisutna je napetost niti T

<u>Deriviranje</u>

$$\frac{d}{dx} f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\frac{\Delta x}{2}) - f(x - \frac{\Delta x}{2})}{\Delta x}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = f''(x)$$

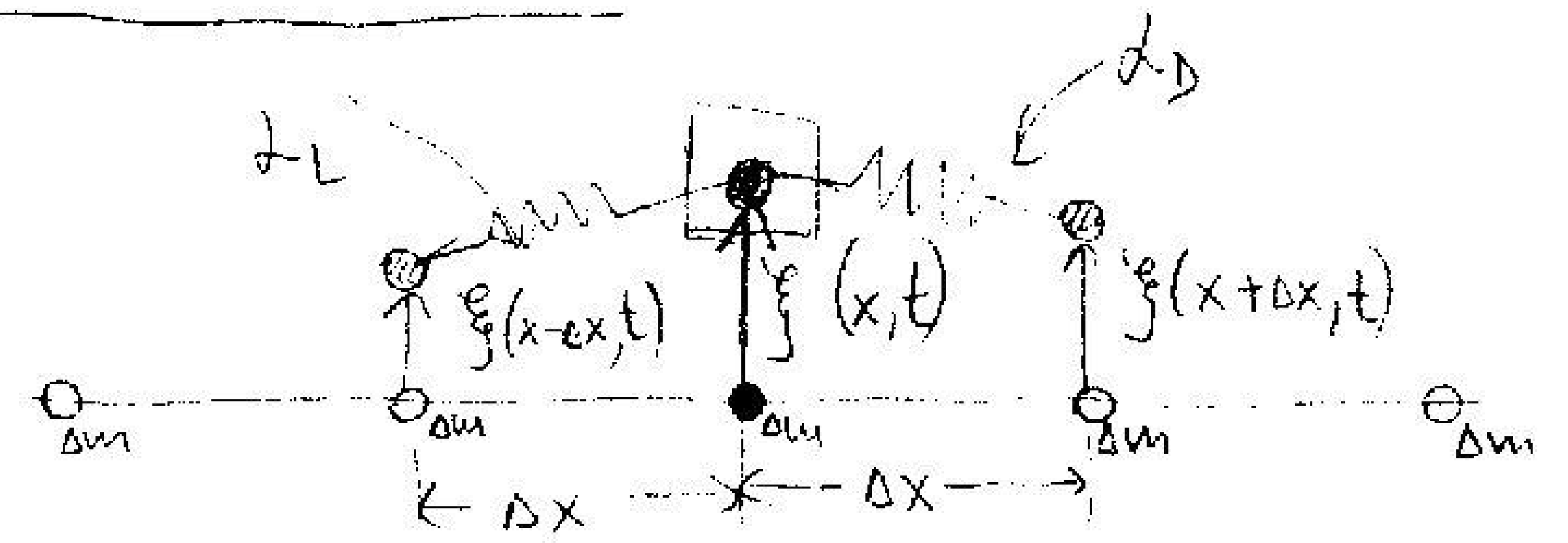
$$= \lim_{\Delta x \to 0} \frac{f'(x+\frac{\Delta x}{2}) - f'(x-\frac{\Delta x}{2})}{\Delta x}$$

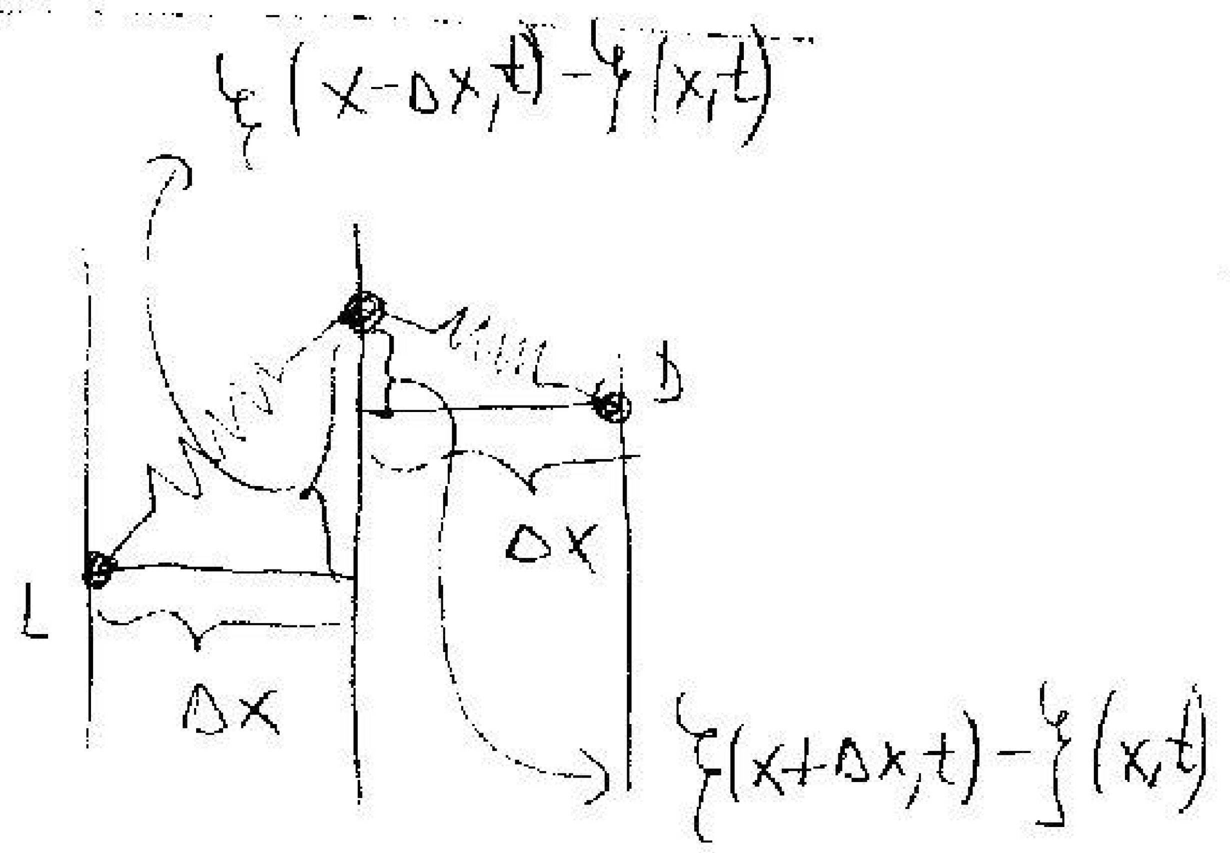
$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{f(x) - f(x-\Delta x)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$ druga derivacija

---

<u>Transverzalni val</u>





$\xi(x,t)$: "elongacija"

-jednadžba gibanja:

$\simeq$ sin

$$\Delta m \frac{\partial^2}{\partial t^2} \xi(x,t) = T\,tg\alpha_D + T\,tg\alpha_L$$

$$= T\frac{\xi(x+\Delta x,t) - \xi(x,t)}{\Delta x} + T\frac{\xi(x-\Delta x,t) - \xi(x,t)}{\Delta x} =$$

$$= T\frac{[\xi(x+\Delta x,t) - 2\xi(x,t) + \xi(x-\Delta x,t)]}{\Delta x} =$$

$$\Delta m \cdot \ddot{\xi} - T\frac{[\ ]}{\Delta x} = 0$$

$$\Delta m = \frac{M}{N} = \frac{M}{L/\Delta x} = \frac{M}{L}\Delta x$$

$\mu = \frac{M}{L}$ .. linijska gustoća mase

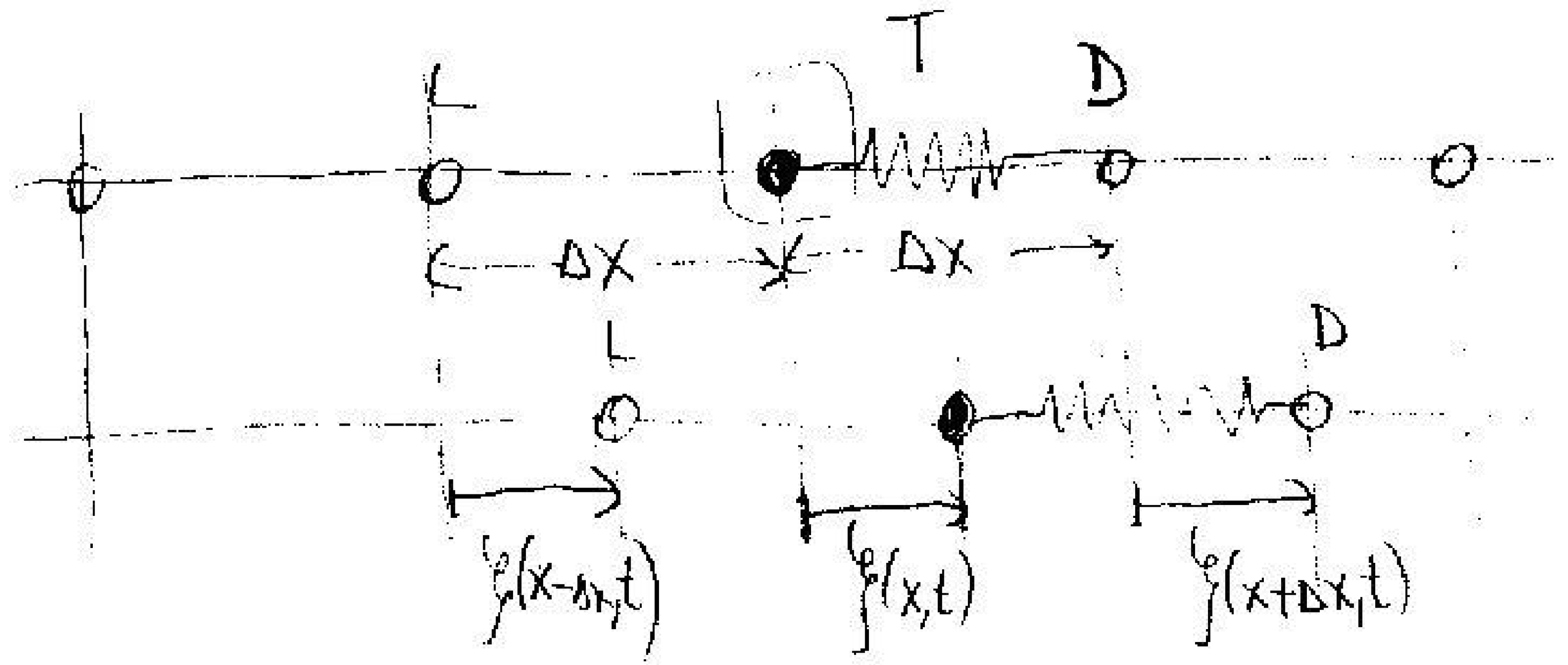
$$\ddot{\xi} - \frac{T}{\mu}\frac{[\ ]}{(\Delta x)^2} = 0$$

$$\boxed{\ddot{\xi} - v^2 \xi'' = 0 \qquad v^2 \equiv \frac{T}{\mu}}$$

Longitudinalni val

- ravn. pol.

- pot.



- jedn. gibanja

$$\Delta m \frac{\partial^2}{\partial t^2}\xi(x,t) = \underbrace{\Big[T + K\overbrace{\big(\xi(x+\Delta x,t) - \xi(x,t)\big)}^{\text{produljenje opruge u odnosu na ravn. pol.}}\Big]}_{\text{desni susjed}} +$$

$$\underbrace{\Big[-T + K\big(\xi(x-\Delta x,t) - \xi(x,t)\big)\Big]}_{\text{lijevi susjed}} =$$

$$= K\big[\xi(x+\Delta x,t) - 2\xi(x,t) + \xi(x-\Delta x,t)\big]$$

$$\Delta m = \frac{M}{N} = \frac{M}{\frac{L}{\Delta x}} = \frac{M}{L}\Delta x$$

$$K = Nk = N\frac{SE}{L} = \frac{L}{\Delta x}\frac{SE}{L} = \frac{SE}{\Delta x}$$

$$\frac{M}{L}\Delta x\,\ddot{\xi} = \frac{SE}{\Delta x}[\;]$$

$$\ddot{\xi} - \frac{L}{M}\,\frac{SE\,[\;]}{(\Delta x)^2} = 0$$

$$\ddot{\xi} - \frac{SEL}{M}\,\xi'' = 0$$

$$\boxed{\ddot{\xi} - v^2\,\xi'' = 0 \qquad v^2 = \frac{SEL}{M} = \frac{V}{M}E = \frac{E}{\rho}}$$

Rješenje valne jednadžbe:

$$\frac{\partial^2}{\partial t^2}\xi(x,t) - v^2\,\frac{\partial^2}{\partial x^2}\xi(x,t) = 0 \qquad (*)$$

$$\begin{cases} v^2 = \dfrac{T}{\mu} & \text{trans.} \\ v^2 = \dfrac{E}{\rho} & \text{long.} \end{cases}$$

$$\boxed{\ddot{\xi} - v^2\xi'' = 0} \qquad (*)$$

Rješenje je svaka funkcija $\xi(x,t) = f(x \pm vt)$ "Bilo što"

DOKAZ:

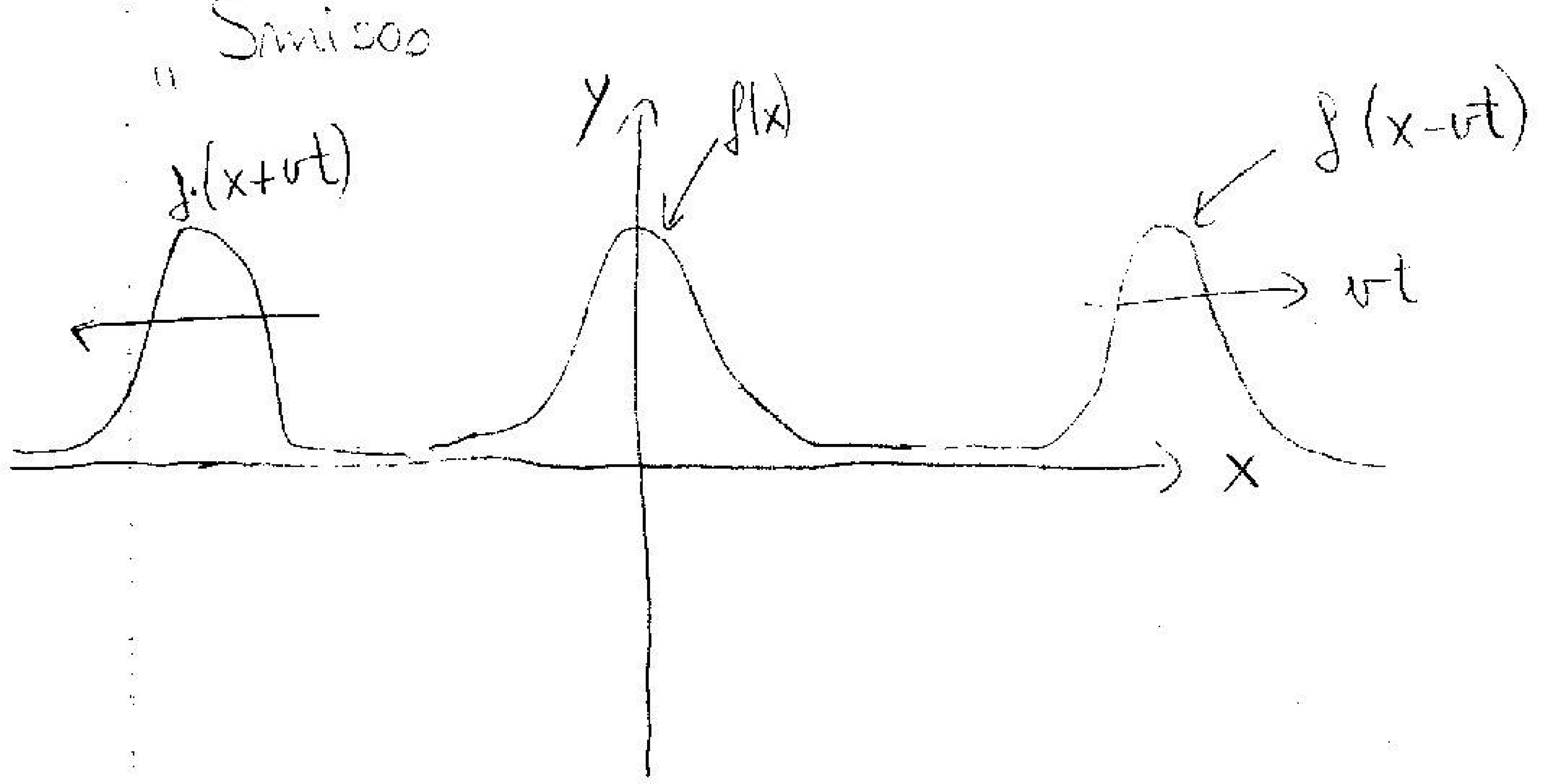
$$\ddot{\xi} = \frac{\partial^2}{\partial t^2}\xi(x,t) = \frac{\partial^2}{\partial t^2}f(x\pm vt) = \frac{\partial}{\partial t}f'(x\pm vt)\cdot(\pm v) = f''(x\pm vt)(\pm v)^2 =$$

$$= v^2 f''(x\pm vt)$$

$$\xi'' = \frac{\partial^2}{\partial t^2}\xi(x,t) = \frac{\partial^2}{\partial t^2}f(x\pm vt) = f''(x\pm vt)$$

$$\Rightarrow (*) \Rightarrow \qquad 0 = 0$$

„Smisao"



Primjer: $f(x) = \cos(kx)$

$k \Rightarrow$ valni broj (vektor) $\left[\frac{rad}{m}\right]$

$\lambda \equiv \frac{2\pi}{k}$ ... valna duljina $[m]$

rješenje valne j-e $\quad \xi(x,t) = \cos[k(x \pm vt)] = \cos[kx \pm \omega t]$

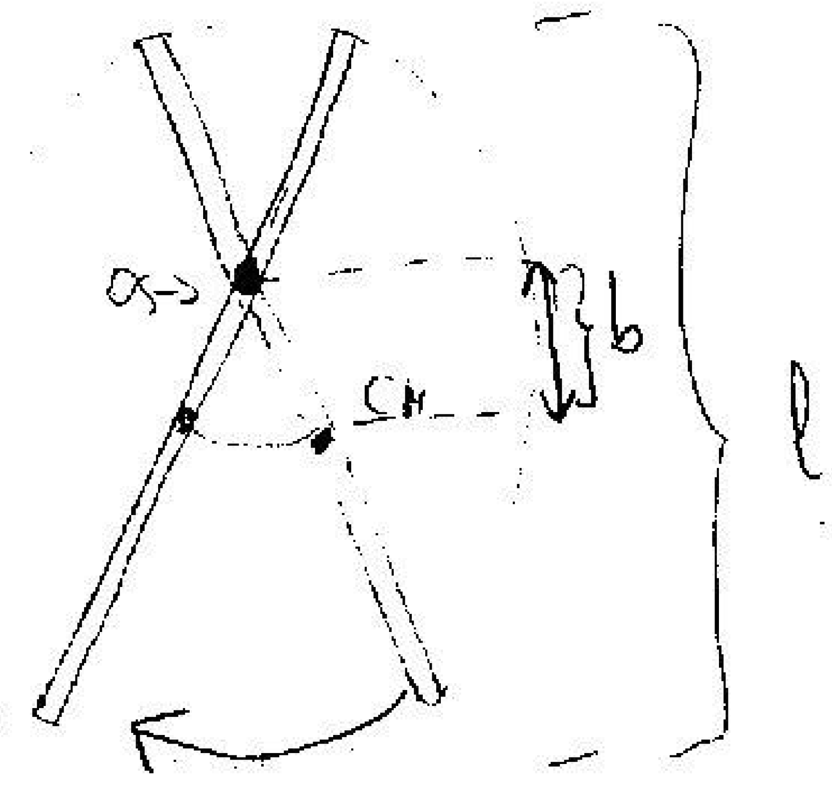
$\omega \equiv kv$ ... frekvencija $\left[{rad}/{s}\right]$ (kružna)

$f = \frac{\omega}{2\pi}$ ... frekvencija $\left[\frac{1}{s} = Hz\right]$

također:

$\xi(x,t) = \cos\left[2\pi\left(\frac{x}{\lambda} \pm ft\right)\right]$

Zadatak: Odredite udaljenost od središta homogenog štapa duljine l za kojeg bi trebalo postaviti vodoravnu os da bi period malog titranja oko te osi bio najkraći.



period: $T = 2\pi\sqrt{\frac{I_{CM} + mb^2}{mgb}}$ IZVOD?

Ekstrem:

$$0 = \frac{d}{db}T = 2\pi \cdot \frac{1}{2\sqrt{\ }} \cdot \left(\frac{2mb}{mgb} - \frac{I_{CM} + mb^2}{(mgb)^2} mg\right) =$$

$$= \frac{2\pi}{2\sqrt{\ }} \frac{1}{mg}\left(2mb - \frac{I_{CM}+mb^2}{\cancel{mg}b}\cancel{mg}\right) = \frac{\pi}{\sqrt{\ }}\left(2mb - \frac{I_{CM}}{b} - mb\right) =$$

$$= \frac{\pi}{\sqrt{\ }}\underbrace{\left(mb - \frac{I_{CM}}{b}\right)}_{=0} \qquad I_{CM} = \frac{1}{12}ml^2$$

$$\Rightarrow \quad mb = \frac{1}{b}\frac{1}{12}ml^2$$

$$b^2 = \frac{l^2}{12}$$

$$b = \frac{l}{\sqrt{12}} = \frac{l}{2\sqrt{3}}$$

Superpozicija valova

- zbrajamo valove:

$$\xi_1(x,t) = A_1\cos(k_1x - \omega_1t + \varphi_1)$$
$$\xi_2(x,t) = A_2\cos(k_2x - \omega_2t + \varphi_2)$$

- zbog jednostavnosti $A_1 = A_2$
- oznake: $\bar{k} = \frac{k_1+k_2}{2}$, $\Delta k = k_2 - k_1$

$$\bar{\omega} = \frac{\omega_1+\omega_2}{2}, \quad \Delta\omega = \omega_2 - \omega_1$$

$$\bar{\varphi} = \frac{\varphi_1+\varphi_2}{2}, \quad \Delta\varphi = \varphi_2 - \varphi_1$$

"brza oscilacija"

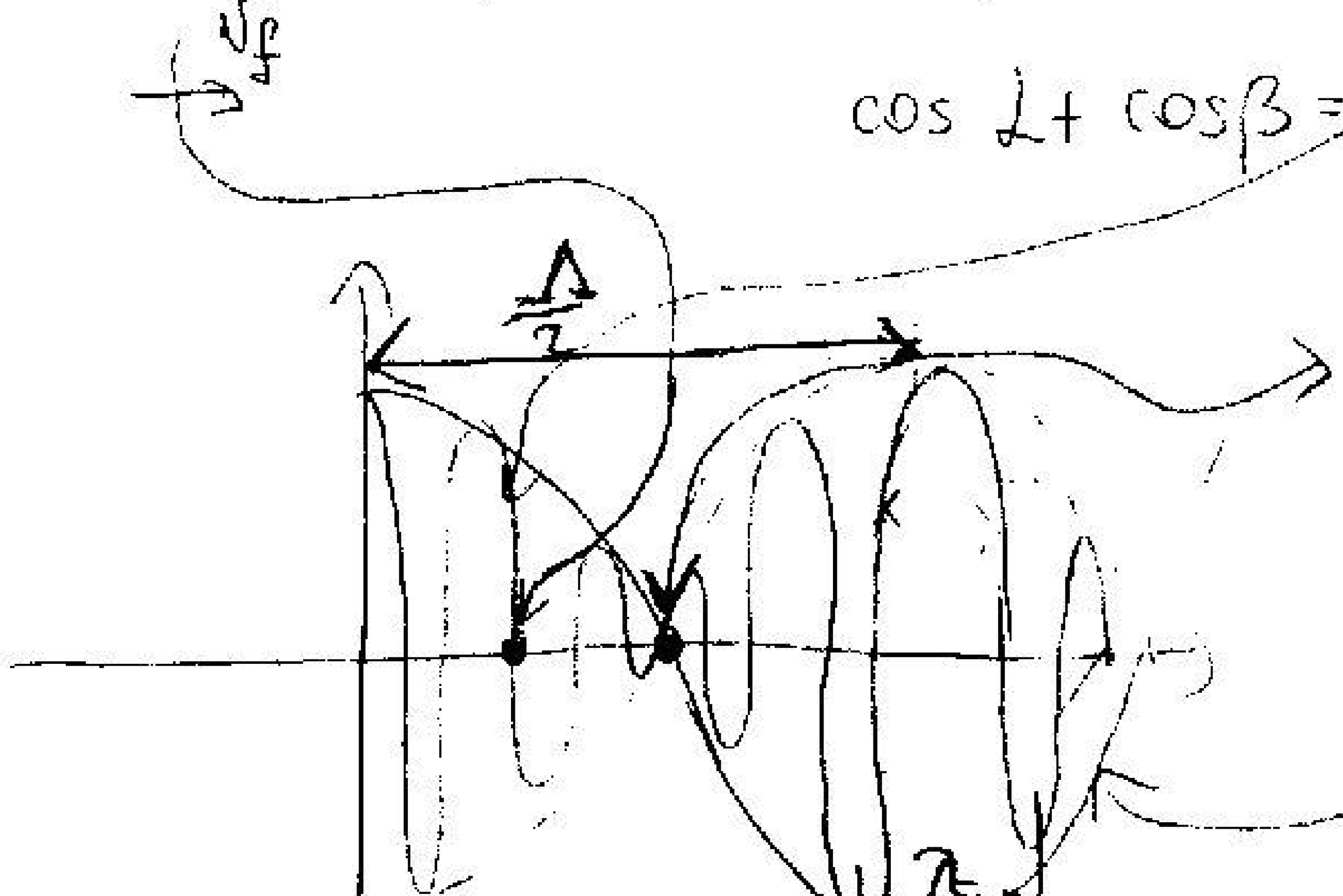
$\lambda = \frac{2\pi}{\bar{k}}$

fazna brzina $v_f = \frac{\bar{\omega}}{\bar{k}}$

$$\xi(x,t) = \xi_1(x,t) + \xi_2(x,t) = 2A\boxed{\cos(\bar{k}x - \bar{\omega}t + \bar{\varphi})}\boxed{\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t + \frac{\Delta\varphi}{2}\right)}$$

gdje smo koristili:

$$\cos\alpha + \cos\beta = 2\cos\frac{(\alpha+\beta)}{2}\cos\frac{(\alpha-\beta)}{2}$$



"modulacija" (envelopa)

$$\Lambda = \frac{2\pi}{\frac{\Delta k}{2}}$$

$$v_g = \text{grupna brzina} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Uobičajeno je!

$$v_g < v_f$$

<u>Posebni slučajevi:</u>

a) $\Delta\omega = 0$, $\Delta k = 0$, $\Delta\varphi = 0$

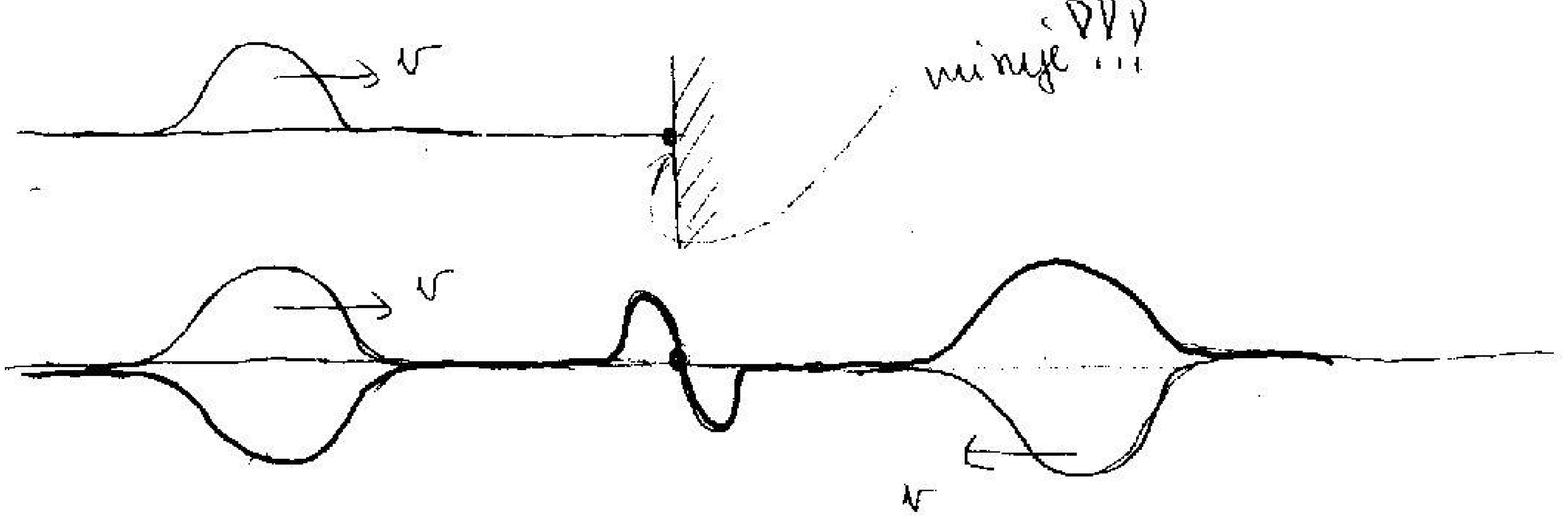
$$\psi(x,t) = 2\cdot\psi_1(x,t) = 2\cdot\psi_2(x,t)$$

„konstruktivna interferencija"

b) $\Delta\omega = 0$, $\Delta k = 0$, $\Delta\varphi = \pi$

$\psi(x,t) = 0$ „destruktivna interferencija"

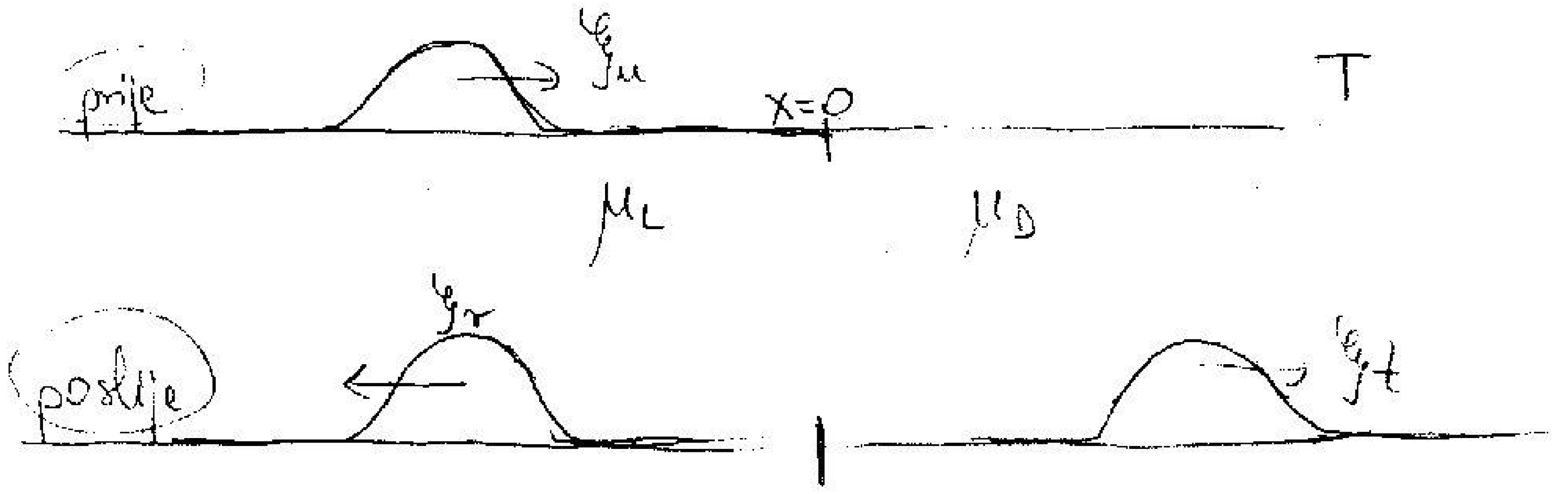
* Odbijanje (refleksija) i prijenos (transmisija) valova



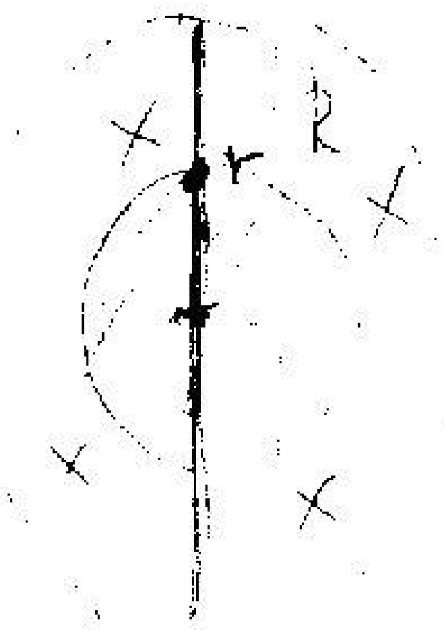
<u>Model:</u> napeto uže, T

promjena μ @ $x=0$

$$\mu_L \neq \mu_D$$



Zad 22.



$$F(r) = -G\frac{m\,M(r)}{r^2} = -G m \frac{\frac{4}{3} r^3 \pi \rho}{r^2} = -\frac{4}{3} G \pi \rho r m$$

$$m\,\ddot{r} = -\underbrace{\frac{4\pi}{3} G \rho}_{k} r m$$

$$m\ddot{r} + k r = 0 \quad /:m$$

$$\ddot{r} + \frac{k}{m} r = 0$$

$$\boxed{\omega_0^2 = \frac{k}{m} = G\frac{4\pi}{3}\rho}$$

$$\xi_u = a_u \cos(k_L x - \omega t)$$

$$\xi_r = a_r \cos(k_L x + \omega t)$$

$$\xi_t = a_t \cos(k_D x - \omega t)$$

Uvjet 1.: (uvjet spojenosti)

$$\xi_u + \xi_r = \xi_t \quad @ \quad x=0, \quad \text{za } \forall t$$

$$a_u \cos(-\omega t) + a_r \cos(+\omega t) = a_t \cos(-\omega t)$$

$$\boxed{a_u + a_r = a_t}$$

Uvjet 2.: (uvjet glatke spojenosti)

$$\xi_u' + \xi_r' = \xi_t' \quad @ \quad x=0, \quad \text{za } \forall t$$

$$-k_L a_u \sin(-\omega t) - k_L a_r \sin(+\omega t) = -k_D a_t \sin(-\omega t)$$

$$\boxed{k_L a_u - k_L a_r = k_D a_t}$$

$a_u + a_r = a_t$

$k_L a_u - k_L a_r = k_D(a_u + a_r)$

$-k_L a_r - k_D a_r = k_D a_u - k_L a_u$

$$\boxed{a_r = -\frac{a_u(k_D-k_L)}{(k_L+k_D)} = \frac{k_L-k_D}{k_L+k_D}\, a_u}$$

$$\boxed{a_t = \frac{2k_L}{k_L+k_D}\, a_u}$$

$$a_r = \frac{\sqrt{\mu_L}-\sqrt{\mu_D}}{\sqrt{\mu_L}+\sqrt{\mu_D}}\, a_u$$

$$a_t = \frac{2\sqrt{\mu_L}}{\sqrt{\mu_L}+\sqrt{\mu_D}}\, a_u$$

-općenito:

$$k = \frac{\omega}{v} = \omega\sqrt{\frac{\mu}{T}}$$

- posebni slučajevi:

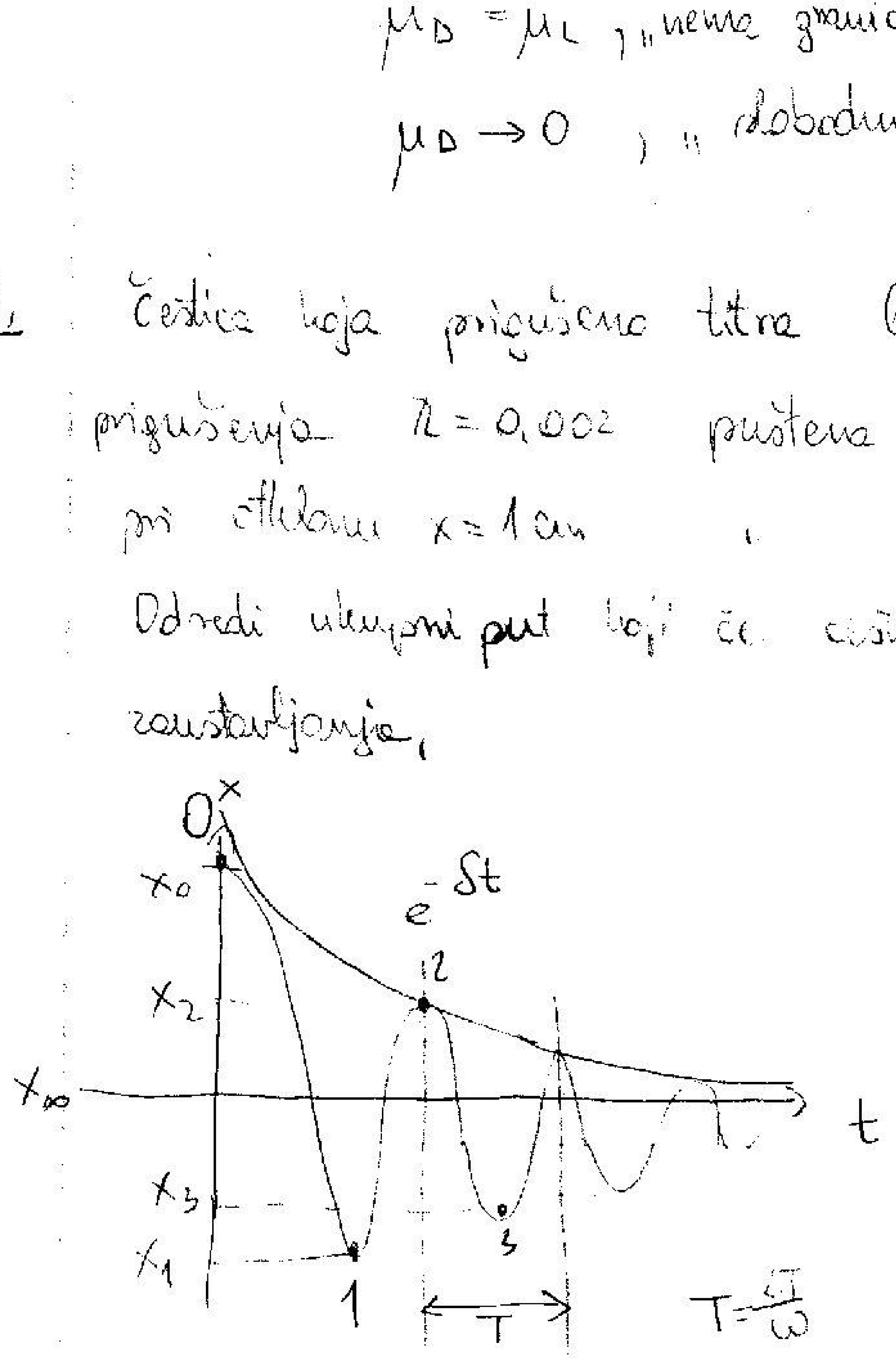
$\mu_D \to \infty$, „čvrsti kraj", $a_r = -a_u$

$\mu_D = \mu_L$, „nema granice", $a_r = 0$

$\mu_D \to 0$, „slobodni kraj", $a_r = a_u$

Zad. Čestica koja prigušeno titra logaritamskim dekrementom prigušenja $\lambda = 0{,}002$ puštena je u gibanje iz mirovanja pri otklonu $x = 1$ cm.

Odredi ukupni put koji će čestica preći do konačnog zaustavljanja.



$x(t) = Ae^{-\delta t}\cos(\omega t + \varphi)$

x_n ... n-ti krajnji položaj

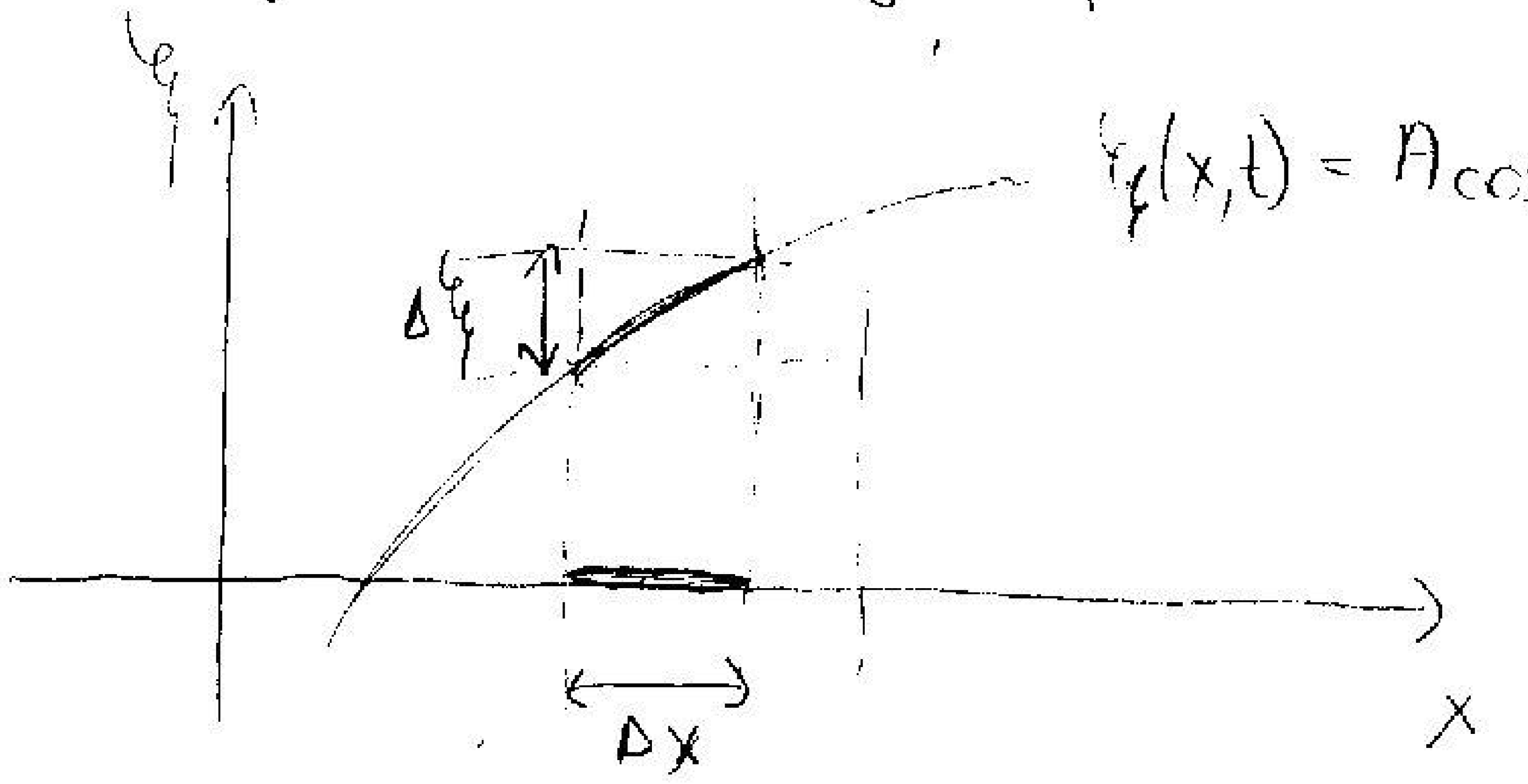
$$\frac{x_{n+1}}{x_n} = -e^{-\delta(T/2)} = -e^{-\frac{\lambda}{2}}$$

$$s = |x_0| + 2|x_1| + 2|x_2| + 2|x_3| + \dots$$
$$= x_0 + 2x_0 e^{-\lambda/2} + 2x_0\left(e^{-\lambda/2}\right)^2 + 2x_0\left(e^{-\lambda/2}\right)^3 + \dots$$
$$= x_0\left[-1 + 2\sum_{k=0}^{\infty}\left(e^{-\lambda/2}\right)^k\right]$$

$$\sum_{k=0}^{\infty}\left(e^{-\lambda/2}\right)^k = \frac{1}{1-e^{-\lambda/2}}$$

$$\boxed{s = x_0\left(\frac{2}{1-e^{-\lambda/2}} - 1\right)}$$

* Energija i snaga pri transverzalnom valnom gibanju



$$\xi(x,t) = A\cos(kx - \omega t)$$

žica: μ, T

$$\underbrace{v = \sqrt{\frac{T}{\mu}} = \frac{\omega}{k}}_{Tk^2 = \mu\omega^2}$$

$$\Delta E_{kin} = \frac{1}{2}\Delta m v^2 = \frac{1}{2}\mu\,\Delta x\,\dot{\xi}^2 = \frac{1}{2}\mu\,\Delta x\left(\omega A\sin(kx-\omega t)\right)^2 =$$
$$= \frac{1}{2}\mu\omega^2 A^2 \sin^2(kx-\omega t)\,\Delta x$$

$$\Delta E_{pot} = T\Delta l = T\Big(\overbrace{\sqrt{(\Delta x)^2 + (\Delta\xi)^2} - \Delta x}^{\text{produljenje žice}}\Big) = T\left(\sqrt{1+\left(\frac{\Delta\xi}{\Delta x}\right)^2} - 1\right)\Delta x =$$
$$= T\left(\sqrt{1+(\xi')^2} - 1\right)\Delta x = T\frac{1}{2}(\xi')^2\Delta x = \frac{1}{2}T(-kA\sin())^2 =$$
$$= \frac{1}{2}Tk^2A^2\sin^2()\,\Delta x$$

$x \ll 1$

$\sin x \simeq x$

$\cos x \simeq 1 - \frac{x^2}{2}$

$\ln(1+x) \simeq x$

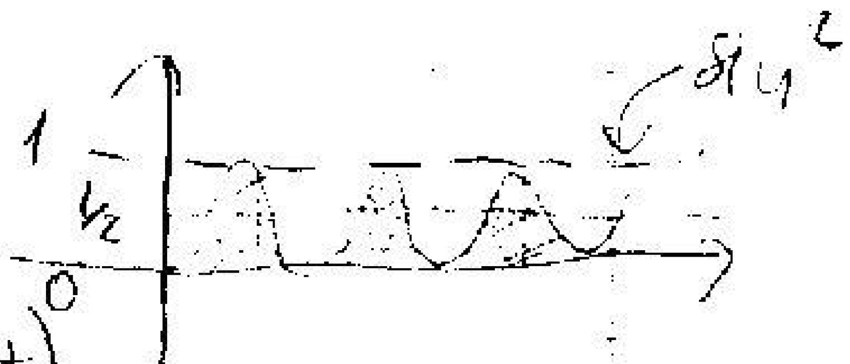
$\frac{1}{1+x} \simeq 1 - x$

$\boxed{\sqrt{1+x} \simeq 1 + \frac{x}{2}}$

$e^x \simeq 1 + x$

$$\Delta E = E_{pot} + E_{kin} = \frac{1}{2}A^2\big(\overbrace{Tk^2 + \mu\omega^2}^{=2\mu\omega^2}\big)\sin^2(kx-\omega t)\,\Delta x =$$
$$= A^2\mu\omega^2\sin^2(kx-\omega t)\,\Delta x$$

Gustoća ukupne energije



$$\frac{\Delta E}{\Delta x} = \frac{1}{\Delta x}\left(\Delta E_{pol} + \Delta E_{kin}\right) = A^2\mu\omega^2 \sin^2(kx-\omega t)$$

↳ srednja vrijednost: $= \frac{1}{2}$

-usrednjeno u vremenu:

$$\left\langle \frac{\Delta E}{\Delta x} \right\rangle = \frac{1}{2} A^2\mu\omega^2 = \left(\frac{\Delta E_{kin}}{\Delta x}\right)_{max} = \left(\frac{\Delta E_{pot}}{\Delta x}\right)_{max}$$

Snaga

$$\overline{P} = \left\langle \frac{dE}{dt} \right\rangle = \left\langle \frac{dE}{dx} \right\rangle \overbrace{\frac{dx}{dt}}^{=v} = \frac{1}{2}\mu\omega^2 A^2 v$$

Refleksija i transmisija snage

$$\overline{P_u} = \overline{P_r} + \overline{P_t} \qquad (ZOE)$$

koef.:

$$R = \frac{\overline{P_r}}{\overline{P_u}}$$

$$T = \frac{\overline{P_t}}{\overline{P_u}}$$

$$\overline{P_u} = \frac{1}{2}\mu_L\omega^2 a_u^2 v_L = \frac{1}{2}\mu_L\omega^2 a_u^2\sqrt{\frac{T}{\mu_L}} = \frac{1}{2}\omega^2 a_u^2\sqrt{\mu_L T}$$

$$\overline{P_t} + \overline{P_r} = \frac{1}{2}\omega^2 a_t^2\sqrt{\mu_D T} + \frac{1}{2}\omega^2 a_r^2\sqrt{\mu_L T}$$

$$a_r = \frac{\sqrt{\mu_L} - \sqrt{\mu_D}}{\sqrt{\mu_L} + \sqrt{\mu_D}}\, a_u \qquad a_t = \frac{2\sqrt{\mu_L}}{\sqrt{\mu_L} + \sqrt{\mu_D}}\, a_u$$

$$= \dots = \frac{1}{2}\omega^2 a_u^2 \sqrt{\mu_L T} = P_u$$

- $$R = \left(\frac{a_r}{a_u}\right)^2 = \left(\frac{\sqrt{\mu_L} - \sqrt{\mu_D}}{\sqrt{\mu_L} + \sqrt{\mu_D}}\right)^2$$

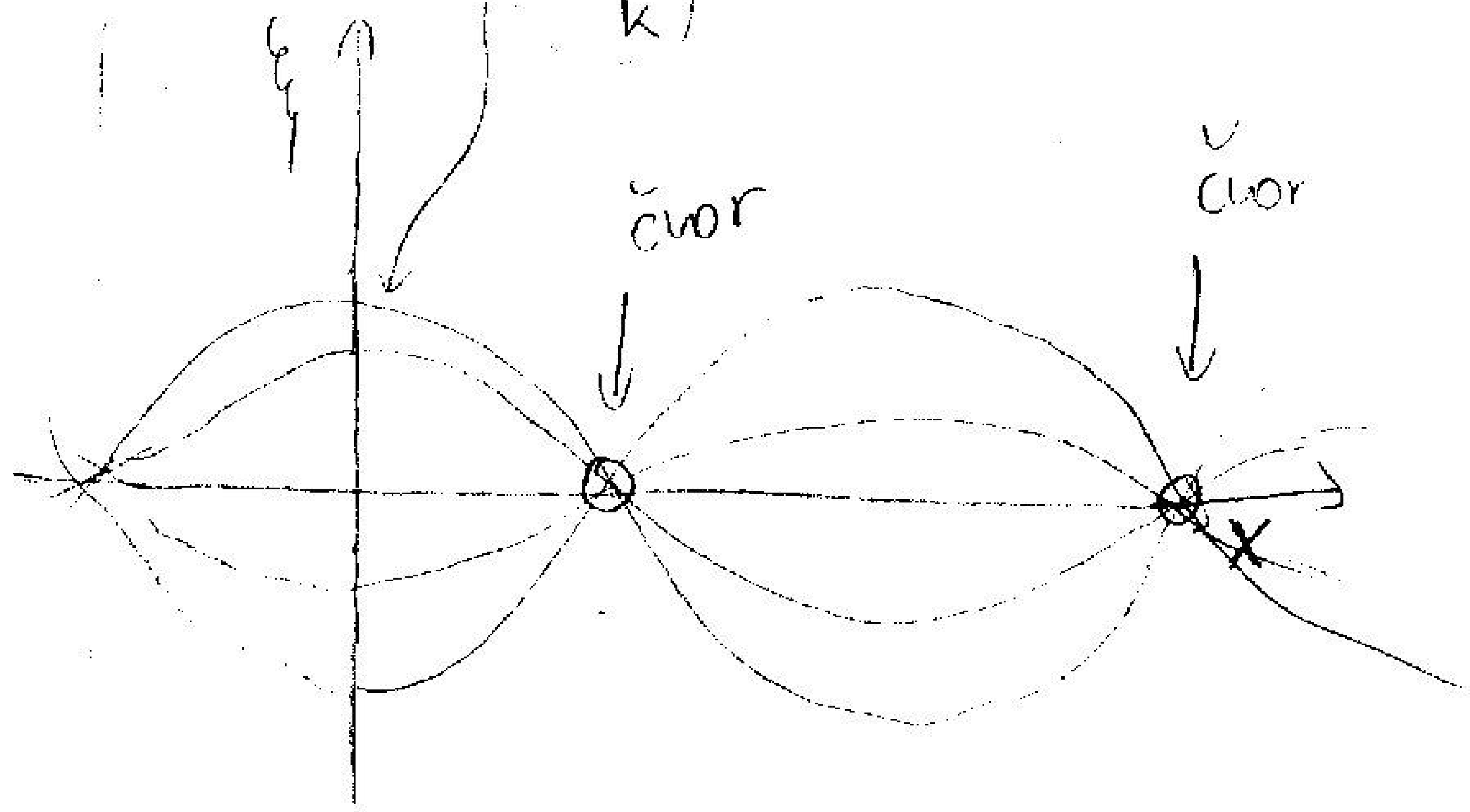
$$T = 1 - R$$

Stojni valovi

$$\xi(x,t) = A\cos(k(x-vt)) + A\cos(k(x+vt))$$

$$= 2A \underbrace{\cos(kx)}_{\substack{\text{oscilacija} \\ \text{u prostoru}}} \underbrace{\cos(\overset{\omega}{kvt})}_{\substack{\text{oscilacija} \\ \text{u vremenu}}}$$

$\left(\lambda = \frac{2\pi}{k}\right)$ $\quad$ $\omega = kv$



* Slučaj

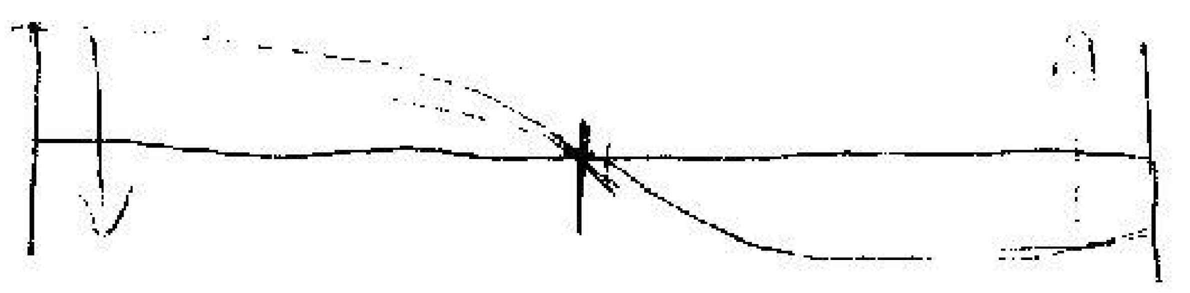
- Oba kraja čvrsta (žica gitare, cijev zatvorena na oba kraja)

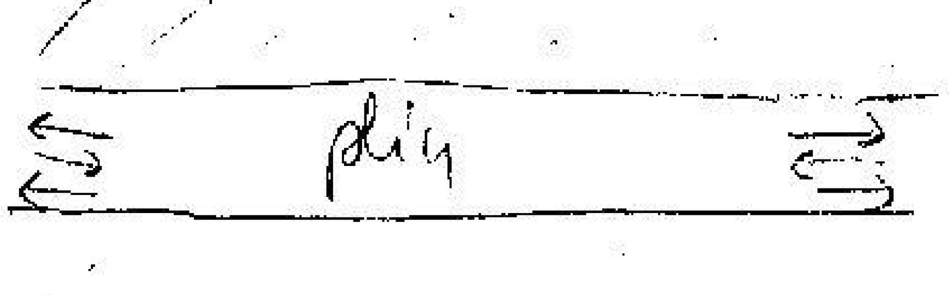
$$L = n\frac{\lambda}{2} \Longrightarrow k_n = \frac{2\pi}{\lambda_n} = 2\pi\frac{n}{2L} = \frac{n\pi}{L}$$

$n = 1, 2, 3, \dots,$

$$\omega_n = k_n v = \frac{n\pi}{L} v.$$

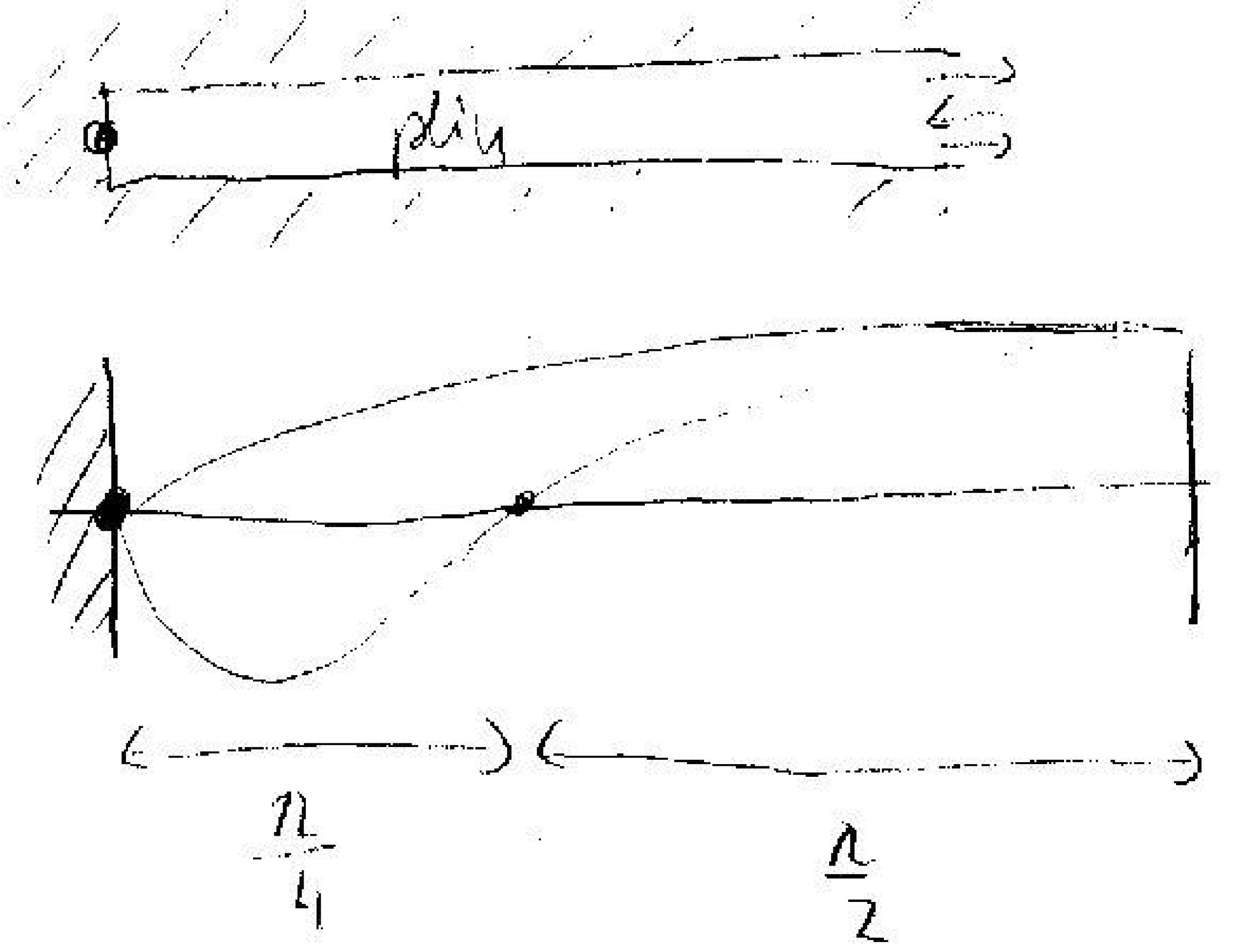
- Oba kraja slobodna





- ... isto ko i slučaj a)

c) jedan kraj slobodan, a drugi kraj čvrst



$$L = \frac{\lambda}{4} + n\frac{\lambda}{2} = (2n+1)\frac{\lambda}{4}, \quad \lambda_n = \frac{4L}{2n+1}, \quad k_n = \frac{2\pi}{\lambda_n} = 2\pi\frac{2n+1}{4L} = (2n+1)\frac{\pi}{2L}$$

$$\omega_n = k_n v$$

Zvuk (long. valovi u plinovima)

$$v^2 = \frac{E}{\rho} \rightarrow \frac{B}{\rho} = \frac{\gamma p}{\rho}$$

B.. volumni modul stlačivosti

γ ... adijabatska konstanta

$$\gamma = \frac{f+2}{f}, \quad f = 3, 5, 6$$

p ... tlak

$$\kappa = \frac{1}{B} = -\frac{1}{V}\frac{\partial V}{\partial p} \text{ ... kompresibilnost}$$

Dokaz

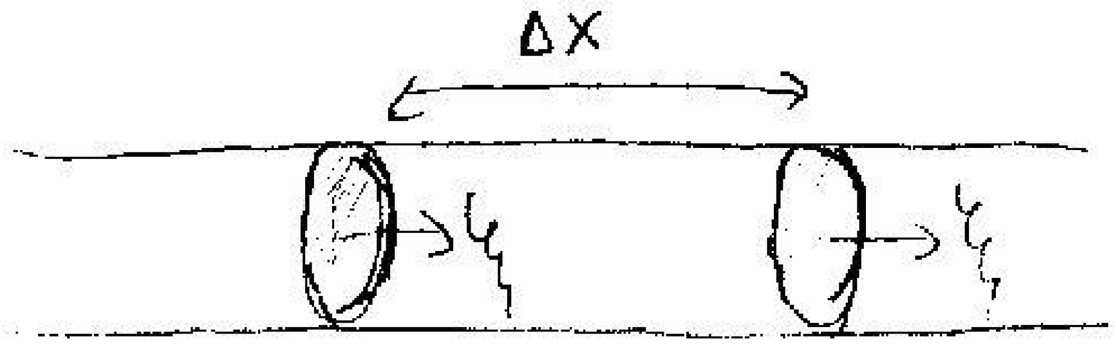
$$B = -\frac{\Delta p}{\frac{\Delta V}{V}} = -V\frac{dp}{dV}$$

Adijabatski proces:

$$pV^{\gamma} = \text{konst}$$

$$dp\,V^{\gamma} + p\gamma V^{\gamma-1}\,dV = 0$$

$$\frac{dp}{dV} = -\frac{\gamma p}{V} \Rightarrow \boxed{B = \gamma p}$$



$$\xi(x,t) = \xi_0 \cos(kx - \omega t)$$

Energija (analogijom)

$$\Delta E = (\Delta E_{kin})_{max} = \frac{1}{2}\Delta m v^2 = \frac{1}{2}\rho S\Delta x\,\omega^2 \xi_0^2 \underbrace{\sin^2(kx-\omega t)}_{1}$$

Smaga

$$P = \frac{\Delta E}{\Delta t} = \frac{\Delta E}{\Delta x}\underbrace{\frac{\Delta x}{\Delta t}}_{v}$$

$$\boxed{P = \frac{1}{2}\rho S \xi_0^2 \omega^2 v}$$

Intenzitet

$$I = \frac{P}{S} = \frac{1}{2}\rho\xi_0^2\omega^2 v$$

Razina buke

$$D = 10\log_{10}\frac{I}{I_0}$$

$$I_0 \equiv 10^{-12}\ \frac{W}{m^2}$$

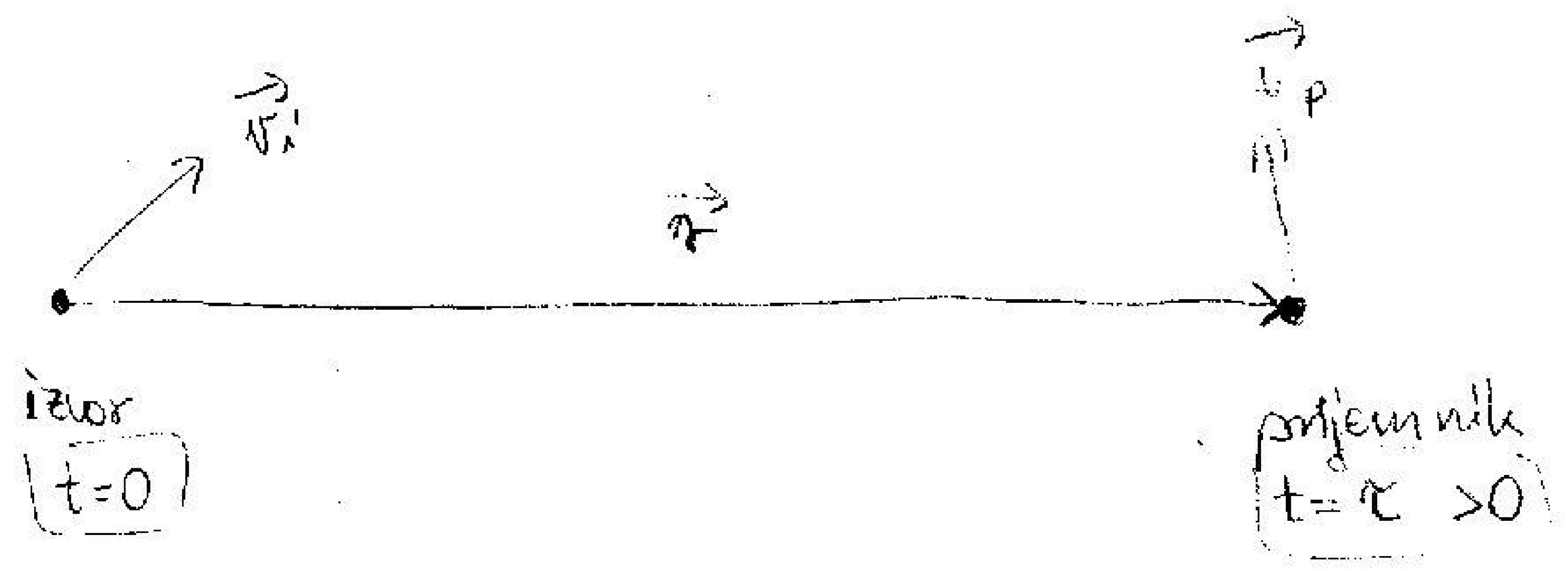
Amplituda tlaka

$$\Delta p = -B\frac{\Delta V}{V} = -\rho v^2 \frac{S\cdot\xi'\cancel{\Delta x}}{S\cancel{\Delta x}} =$$

$$= +\underbrace{\rho v^2 \xi_0 k}\sin(kx-\omega t)$$

$$= \underbrace{(\Delta p)_{max}}_{\text{amplituda oscilacija tlaka}}\sin(kx-\omega t)$$

* Dopplerov efekt



$$\frac{f_p}{f_i} = \frac{v_z - \vec{v}_p \cdot \hat{r}_{ip}}{v_z - \vec{v}_i \cdot \hat{r}_{ip}}$$

→ jedinični vektor od izvora prema prijemniku

$$\boxed{\tau = \frac{x}{v_z}}$$

→ .... duljina puta koju je zvuk prevalio

→ brzina zvuka

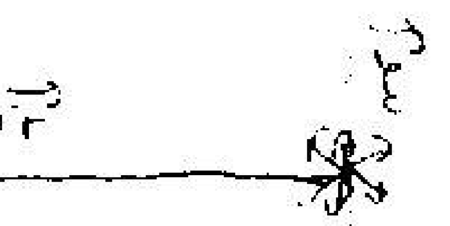
→ trajanje putovanja zvuka

- promatramo signal odaslan u $t=\Delta t$
- taj signal putuje: $\tau + \Delta\tau$ → produljenje trajanja putovanja
- stiže u prijemnik:

$$t = \tau + \Delta t + \Delta\tau$$

→ jer je kasnije poslan

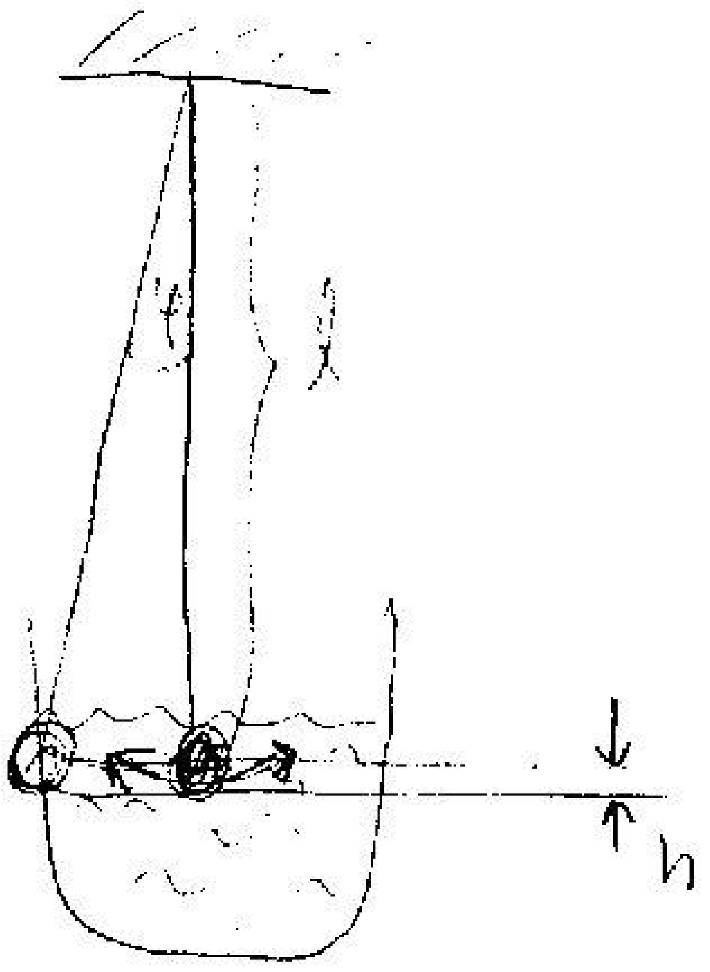
$$\tau + \Delta\tau = \frac{\left|\vec{r} + \vec{v}_p(\Delta t + \Delta\tau) - \vec{v}_i \Delta t\right|}{v_z} = \frac{1}{v_z}\left(r + \vec{v}_p \cdot \hat{r}\,(\Delta t + \Delta\tau) - \vec{v}_i \cdot \hat{r}\,\Delta t\right)$$



$$|\vec{r} + \vec{\xi}| \approx r + \vec{\xi} \cdot \hat{r}$$

Zad.

kuglica



$2r = 1\,cm$

$l = 1\,m$

$\rho_{Al} = 2{,}7$

$\rho_v = 1$

$g = 9{,}81\,m/s$

$\eta =$

$F_\eta = 6\pi \eta r v$

ZoE → j. gib. → $\omega_0^2,\ 2\delta \Rightarrow \omega^2, \lambda$

$\to v = l \cdot \dot\varphi$

$T = \frac{1}{2} m_{Al} v^2 = \frac{1}{2} m_{al} l^2 \dot\varphi^2$

$m_{Al} \to \frac{4}{3} r^3 \pi \cdot \rho_{Al}$

$l(1-\cos\varphi)$ $\qquad \frac{\varphi^2}{2}$

$U = m_{al} g h - m_v g h = (m_{al} - m_v) g h = (m_{al} - m_v) g l (1-\cos\varphi) =$

$= (m_{al} - m_v) g l \frac{\varphi^2}{2}$

promjena energije $E = T + U$

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) = \left(m_{al} l^2 \dot\varphi \ddot\varphi + (m_{al} - m_v) g l \varphi \dot\varphi\right) =$$

$$= m_{Al} l^2 \dot\varphi \left[\ddot\varphi + \frac{(m_{al} - m_v) g l}{m_{Al} l^2} \varphi\right]$$

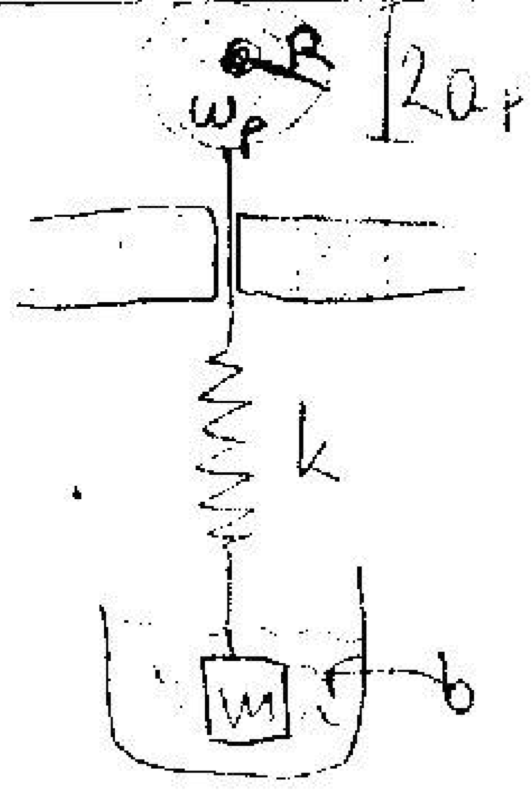
$$= m_{Al} l^2 \dot\varphi \left[\ddot\varphi + \left(1 - \frac{\rho_v}{\rho_{Al}}\right)\frac{g}{l} \varphi\right] \quad (*)$$

- nije 0, gubi se energija
- mora biti jednaka snazi koju vanjska sila koči, trenje (otpor)

$$P_\eta = \vec F_\eta \cdot \vec v = -(6\pi \eta r v) v = -6\pi\eta r l^2 \dot\varphi^2 \quad (**)$$

$$\frac{dE}{dt} = P_\eta \Rightarrow \boxed{\ddot\varphi + \underbrace{\frac{6\pi\eta r}{m_{Al}}}_{=2\delta} \dot\varphi + \underbrace{\left(1 - \frac{\rho_v}{\rho_{Al}}\right)\frac{g}{l}}_{=\omega_0^2} \varphi = 0}$$

Iω.



prisila

$$F_p(t) = k a_p \cos(\omega_p t)$$

$\hookrightarrow (F_p)_{max}$

Želimo da max. amplitude oscilacija ne bude veća od a_p.

$$A_p = \frac{f_p}{\sqrt{(\omega_0^2-\omega_p^2)^2+4\delta^2\omega_p^2}}$$

$$\omega_0^2 = \frac{k}{m} \quad , \quad f_p = \frac{(F_p)_{max}}{m} = \frac{k a_p}{m} = \omega_0^2 a_p$$

$$2\delta = \frac{b}{m}$$

zahtjev: $A_p \le a_p$ za $\forall \omega_p$

Rezonancija amplitude: $\omega_{rez} = \sqrt{\omega_0^2 - 2\delta^2}$

$$A_{rez} = \frac{a_p\omega_0^2}{\sqrt{4\delta^4+4\delta^2(\omega_0^2-2\delta^2)}} = \frac{a_p\omega_0^2}{\sqrt{4\delta^2\omega_0^2-4\delta^4}} =$$

$$= \frac{a_p}{2\sqrt{\frac{\delta^2}{\omega_0^2}-\frac{\delta^4}{\omega_0^4}}} = \frac{a_p}{2\frac{\delta}{\omega_0}\sqrt{1-\frac{\delta^2}{\omega_0^2}}} = \frac{a_p\omega_0}{2\delta\sqrt{1-\frac{\delta^2}{\omega_0^2}}}$$

$$A \le a_p$$

granični δ: $\frac{A_{rez}}{a_p} = 1 \Rightarrow \frac{\omega_0}{2\delta\sqrt{1-\frac{\delta^2}{\omega_0^2}}} = 1$

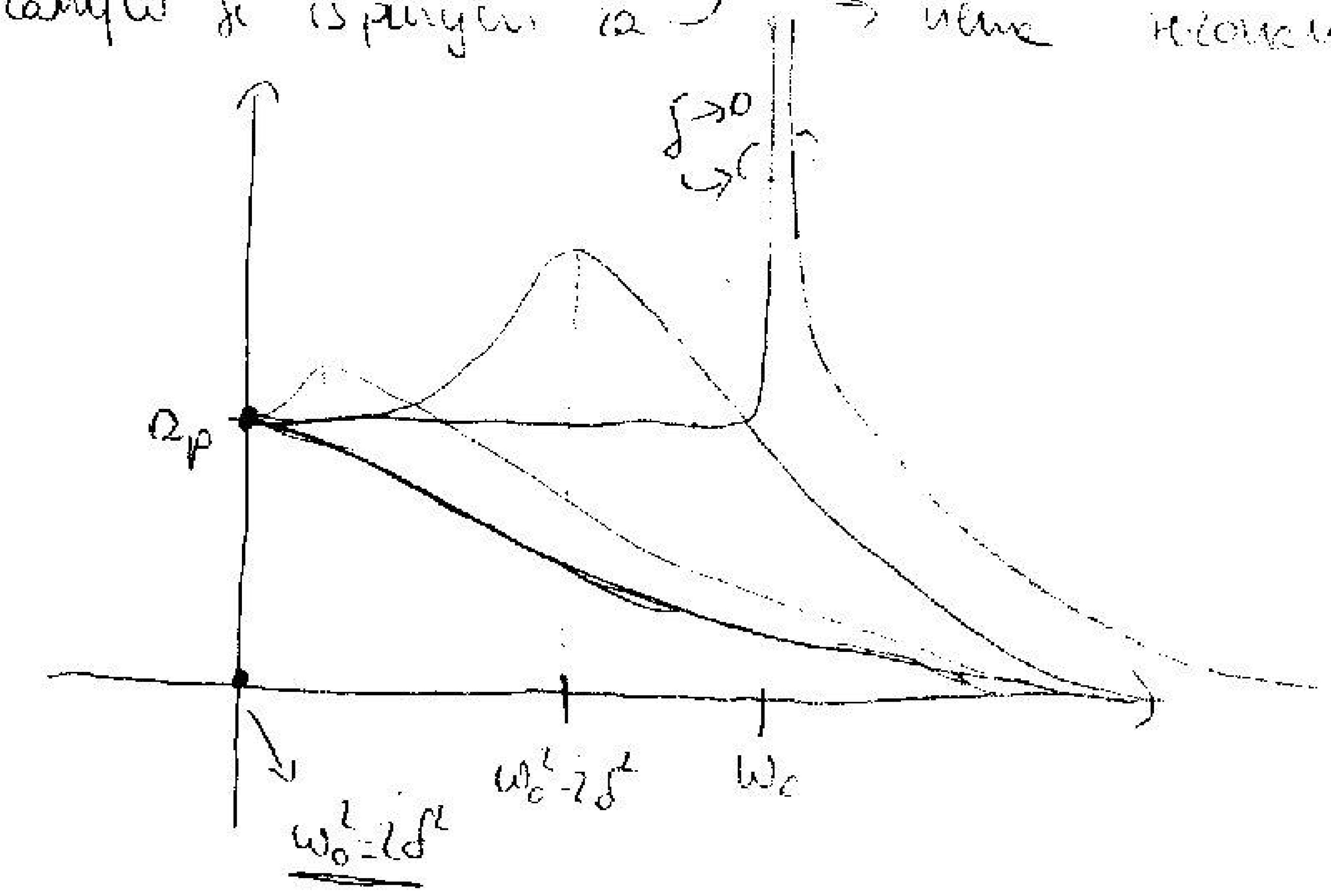
$$\omega_0^2 = 4\delta^2\left(1-\frac{\delta^2}{\omega_0^2}\right)$$

$$\omega_0^2 - 4\delta^2 + 4\frac{\delta^4}{\omega_0^2} = 0$$

$$(\delta^2)_{1,2} = \frac{4}{2\cdot\frac{4}{\omega_0^2}} \pm \frac{\overset{\nearrow 0}{\sqrt{16-16\cdot\frac{1}{\omega_0^4}\cdot\omega_0^4}}}{2\cdot\frac{4}{\omega_0^2}} = \frac{\omega_0^2}{2}$$

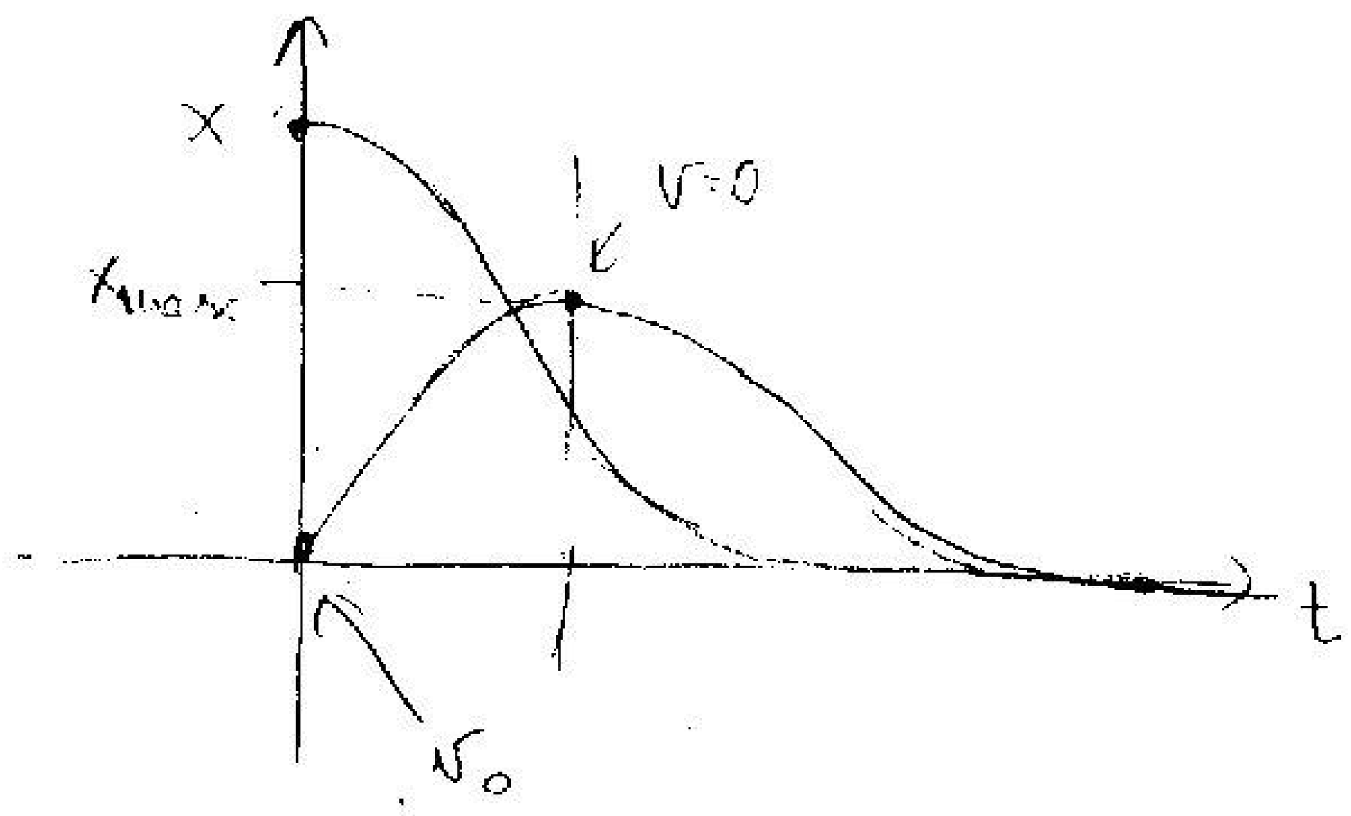
$$\delta_{1,2} = \frac{\omega_0}{\sqrt{2}}$$

$\Rightarrow$ zahtjev je ispunjen za $\rightarrow$ nema rezonancije kad je $2\delta^2 \cdot \omega_0^2$



Zad. Kritično prigušeni oscilator vlastite frekv. ω_0 pokrenut je u gibanje iz ravnotežnog gibanja brzinom v_0. Odredi max otklon u odnosu na početni položaj.

$x_{max} = ?$



- kritično gušenje:

$\delta = \omega_0$

$$x(t) = (C_1 + C_2 t)\, e^{-\omega_0 t}$$

početni uvjeti: $x(0) = x_0 = 0$, $v(0) = v_0 > 0$

$$x(0) = (C_1 + C_2 0)\, e^{-\omega_0 0} = \boxed{C_1 = 0}$$

$$\dot{x}(t) = C_2 e^{-\omega_0 t} - \omega_0 (C_1 + C_2 t) e^{-\omega_0 t} = (C_2 - \omega_0 C_1 - \omega_0 C_2 t)\, e^{-\omega_0 t}$$
$$= C_2 (1 - \omega_0 t)\, e^{-\omega_0 t}$$

$$v(0) = C_2 = v_0$$

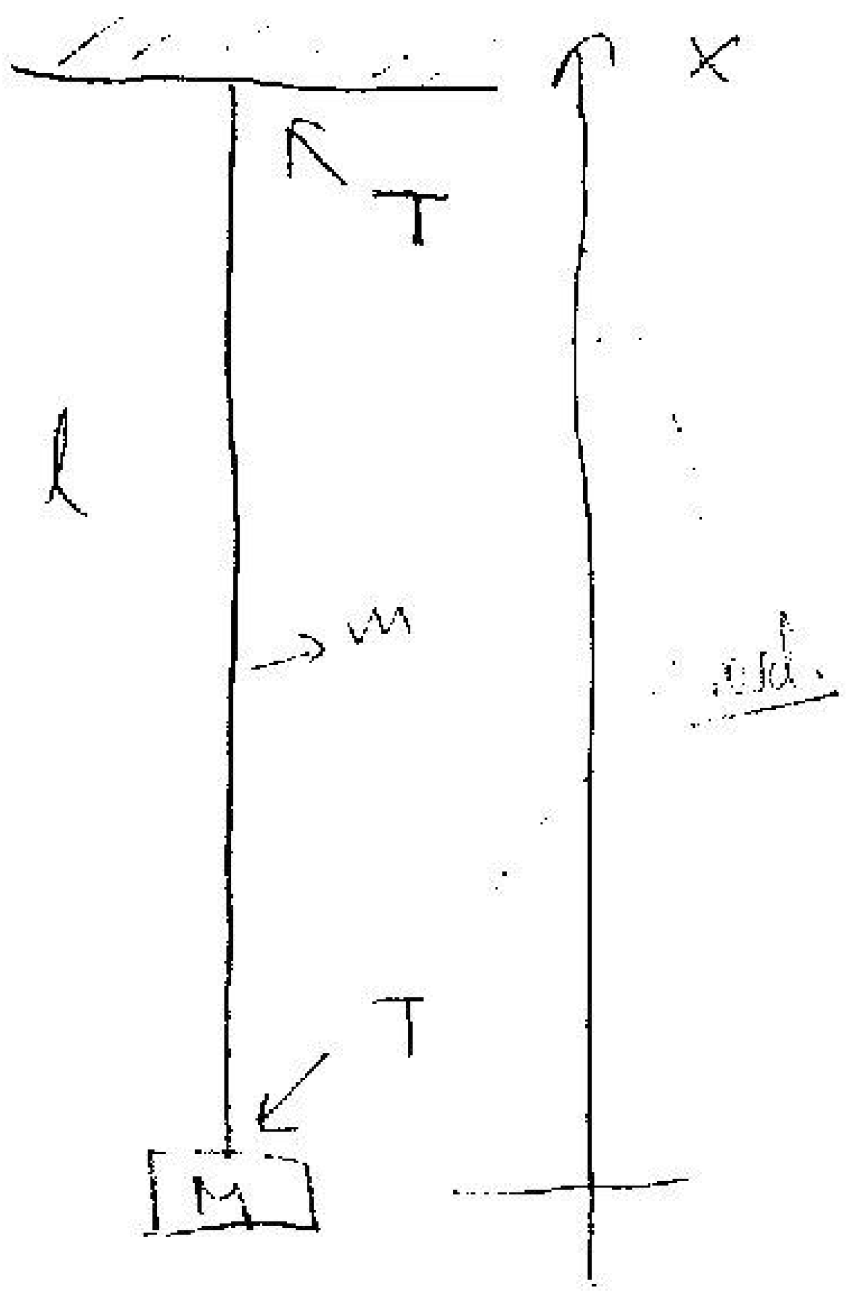
$$\boxed{x(t) = v_0 t e^{-\omega_0 t}}$$

$x_{max} \Leftrightarrow v(t) = 0$

$v = 0 \Rightarrow \boxed{1 = \omega_0 t}, \; t = \frac{1}{\omega_0}$

$$x_{max} = x\left(\frac{1}{\omega_0}\right) = \frac{v_0}{\omega_0} e^{-1} = \frac{v_0}{e\,\omega_0}$$

Zad. Žica na kojoj visi uteg da bi ju napunio ... možemo poslati transverzalni val



ako $m \approx M$

– napetost žice ovisi o putovanju vala

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{Mg + \frac{x}{l} mg}{\frac{m}{l}}}$$

– trajanje putovanja vala ...

$$v = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{v}$$

$$t = \int_0^l \frac{dx}{v(x)} = \int_0^l \sqrt{\frac{\frac{m}{l}}{Mg + \frac{x}{l} mg}}\, dx$$