Prigušeno titranje

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad T = \frac{2\pi}{\omega_0}$$

$\vec{F}_{tr} = -b\vec{v}$ $\quad b > 0$ - konstanta prigušenja

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{k}{m} = \omega_0^2 \qquad 2\delta = \frac{b}{m}$$

δ - faktor prigušenja

$$\frac{d^2x}{dt^2} + 2\delta\frac{dx}{dt} + \omega_0^2 x = 0$$

$t = 0$
$x = x_0$
$v = v_0$

$x = Be^{\lambda t}$

$$Be^{\lambda t}\lambda^2 + 2\delta Be^{\lambda t}\lambda + Be^{\lambda t}\omega_0^2 = 0$$

$$Be^{\lambda t}\underbrace{\left(\lambda^2 + 2\delta\lambda + \omega_0^2\right)}_{\ddot{O}} = 0$$

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

Slabo prigušenje

$\frac{\delta^2}{\omega_0^2} < 1$ - sila prigušenja je slaba

$\omega = \sqrt{\omega_0^2 - \delta^2}$ $\quad \lambda_{12} = -\delta \pm i\omega$

$x = Ce^{-\delta t} e^{i\omega t} + De^{-\delta t} e^{-i\omega t}$

$t=0$

$x_0 = C + D \rightarrow Im(C) = -Im(D)$ $\longrightarrow$ $D = C^*$

$v = C(i\omega - \delta)e^{-\delta t} e^{i\omega t} + D(-i\omega - \delta)e^{-\delta t} e^{-i\omega t}$

$t=0 \quad v_0 = C(i\omega + \delta) + D(-i\omega - \delta)$

$v_0 = -\delta(C+D) + i\omega(C-D) = -\delta x_0 + i\omega(C-D)$ $\qquad Re(C) = Re(D)$

$C = Ee^{i\varepsilon}$

$x = Ee^{-\delta t} e^{i(\omega t + \varepsilon)} + Ee^{-\delta t} e^{-i(\omega t + \varepsilon)} = 2Ee^{-\delta t}\cos(\omega t + \varepsilon) = A_0 e^{-\delta t}\sin(\omega t + \psi)$

$x = A_0 e^{-\delta t}\sin(\omega t + \psi)$

$$x = A_0 e^{-\delta t} \sin(\omega t + \varphi)$$

$$A(t) = A_0 e^{-\delta t}$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}}$$

Logaritamski dekrement prigušenja

$$\lambda = \ln \frac{A(t)}{A(t+\tau)} = \ln \frac{A_0 e^{-\delta t}}{A_0 e^{-\delta(t+T)}} = \ln e^{\delta T} = \delta T$$

$$\lambda = \delta T$$

Q - faktor dobrote / kvalitete

$$Q = 2\pi \frac{\langle E \rangle}{|\Delta E|}$$

$\langle E \rangle$ - prosječna energija oscilatora u jednom periodu

$$\langle E \rangle = \frac{1}{T} \int_t^{t+T} E(t)\, dt$$

$\frac{2\pi}{Q} = \frac{|\Delta E|}{\langle E \rangle}$ - relativni gubitak E po periodu

ΔE - gubitak energije u jednom periodu

$$\Delta E = T \frac{d\langle E \rangle}{dt}$$

$$\frac{\delta^2}{\omega^2} << 1$$

$$E(t) = E_k + E_p = \frac{mv^2}{2} + \frac{k}{2}x^2$$

$$\langle E \rangle = e^{-2\delta t} \cdot \text{konst.}$$

$$\ln \langle E \rangle = -2\delta t + \ln(\text{konst.})$$

$$\frac{d(\ln\langle E\rangle)}{dt} = -2\delta = \frac{d \ln\langle E\rangle}{d\langle E\rangle} \cdot \frac{d\langle E\rangle}{dt} = \frac{1}{\langle E\rangle} \cdot \frac{d\langle E\rangle}{dt}$$

$$\Delta E = T \cdot (-2\delta)\langle E\rangle$$

$$Q = 2\pi \frac{\langle E\rangle}{T 2\delta \langle E\rangle} = \frac{\pi}{\delta T} = \frac{\pi}{\Lambda}$$

$$\boxed{Q = \frac{\pi}{\Lambda}}$$

- za jako slabo prigušenje

- relaksacijsko vrijeme

$$t^* = \frac{1}{\delta}$$

$$t = t^*$$

$$e^{-\delta t} \cdot \sin(\omega t + \varphi)$$

$$e^{-2\delta t} \to e^{-2\delta(t+T)} = e^{-2\delta t} \underbrace{e^{-2\delta T}}_{\approx 1}$$

$$e^{-2\delta T}$$

$$2\delta T = 2\delta \frac{2\pi}{\omega} = 4\pi \frac{\delta}{\omega} << 1$$

$t^* = \frac{1}{\delta}$

$t = t^*$

$A(t^*) = \frac{A_0}{e} = 0.368\, A_0$

$t^* = \frac{1}{\delta} = \frac{T}{\delta T} = T\frac{Q}{\pi}$

$\frac{t^*}{T} = \frac{Q}{\pi}$

Aperiodično gibanje

$\lambda_{1,2} = -\delta \mp \sqrt{\delta^2 - \omega_0^2}$

$\frac{\delta^2}{\omega_0^2} > 1 \qquad \omega' = \sqrt{\delta^2 - \omega_0^2} \qquad \lambda_{1,2} = -\delta \pm \omega'$

$x = Ce^{-\delta t} e^{\omega' t} + De^{-\delta t} e^{-\omega' t}$

$t = 0$
$x = x_0$
$v = v_0$
$\vdots$

$C, D \in \mathbb{R}$

$ch(\omega' t) = \frac{e^{\omega' t} + e^{-\omega' t}}{2} \qquad sh(\omega' t) = \frac{e^{\omega' t} - e^{-\omega' t}}{2}$

$e^{\omega' t} = ch\,\omega' t + sh\,\omega' t$

$e^{-\omega' t} = ch\,\omega' t - sh\,\omega' t$

$$x = Ce^{-\delta t} \operatorname{ch} \omega' t + Ce^{-\delta t} \operatorname{sh} \omega' t + De^{-\delta t} \operatorname{ch} \omega' t - De^{-\delta t} \operatorname{sh} \omega' t$$

$$x = e^{-\delta t} \left[\underbrace{\operatorname{ch} \omega' t \,(C + D)}_{A} + \underbrace{\operatorname{sh} \omega' t \,(C - D)}_{B} \right]$$

$$\boxed{x = e^{-\delta t} (A \operatorname{ch} \omega' t + B \operatorname{sh} \omega' t)}$$

$t = 0 \quad \Rightarrow \quad x_0 = A$

$x = x_0$

$v = 0$

$$v = -\delta e^{-\delta t} (A \operatorname{ch} \omega' t + B \operatorname{sh} \omega' t) + \omega' e^{-\delta t} (A \operatorname{sh} \omega' t + B \operatorname{ch} \omega' t)$$

$$0 = -\delta A + \omega' B \quad \rightarrow \quad B = \frac{\delta x_0}{\omega'}$$

$$\boxed{x = x_0 e^{-\delta t} \left(\operatorname{ch} \omega' t + \frac{\delta}{\omega'} \operatorname{sh} \omega' t \right)}$$

(DE) $t = 0$, $x = 0$, $v = v_0$