Unit 1

POLYNOMIAL FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *define polynomial functions.*
- *apply theorems on polynomials to solve related problems.*
- *sketch and analyses the graphs of polynomial functions.*

Main Contents:

1.1. Introduction to polynomial functions

1.2. Theorems on polynomial

1.3. Zeros of polynomial functions

1.4. Graphs of polynomial functions

1.1. INTRODUCTION TO POLYNOMIAL FUNCTIONS

1.1.1. Definition of a Polynomial Function

Definition 1.1

Let n be a non – negative integer and let $a_n, a_{n-1}, \ldots, a_1, a_0$ be real numbers with $a_n \neq 0$.
The function P defined by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is called a **polynomial function** in variable x of degree n.
The expression $a_n x^n + a_{n-1}x^{n-1} + \ldots a_1x^1 + a_0$ is called **polynomial expression** in variable x.

Note: In the definition of a polynomial functions

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 ;$$

i) $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are called the **coefficients** of $P(x)$.
ii) The number a_n, where $a_n \neq 0$, is called the **leading coefficient** and the term $a_n x^n$ is called the **leading term** of $P(x)$.
iii) The number a_0 is called the **constant term** of $P(x)$.
iv) If $a_n \neq 0$, then the number n (the highest exponent of power of x) is called the **degree** of $P(x)$.

Note: The domain of any polynomial function is the set of real number.

Example 1:

a. $(x) = \frac{5}{2}x - 2x^3$ is a polynomial function with degree 3 and constant term 0.
   - ✓ The leading term of f is $-2x^3$ and the leading coefficient is -2.
   - ✓ The coefficient of x^2 is 0 and the coefficient of x is $\frac{5}{2}$.
   - ✓ The domain of f is $\mathbb{R}$.

b. $f(x) = \sqrt{x^4+1}$ has domain $\mathbb{R}$, but $\sqrt{x^4+1}$ cannot be expressed in the standard form of polynomial $a_n x^n + \ldots + a_1x + a_0$.
   Hence, $f(x) = \sqrt{x^4+1}$ is **NOT** polynomial.

c. $f(x) = \sqrt{(x^2-1)^2}$
   - f(x) $= \sqrt{(x^2-1)^2} = |x^2-1|$ which is **not** polynomial function, because $|x^2-1|$ has no the standard form $a_n x^n + a_{n-1}x^{n-1} + \ldots a_1x^1 + a_0$

1.2. THEOREMS ON POLYNOMIALS

Theorem 1.1: Polynomial Division Theorem (Division Algorithm)

If $f(x)$ and $d(x)$ are two polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)q(x) + r(x)$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$.
If the remainder $r(x) = 0$, $f(x)$ divides exactly into $d(x)$ or we say that division of $f(x)$ by $d(x)$ is exact.

Example 2: Divide polynomial $x^3 + 1$ by $x^2 - x + 1$ using long division. Determine the quotient and the remainder. Is the division exact?

Solution: The long division to divide $x^3 + 1$ by $x^2 - x + 1$ is given below.

$$\begin{array}{r|l} & x+1 \\ \hline x^2 - x + 1 & x^3 + 1 \\ & \underline{x^3 - x^2 + x} \\ & \quad x^2 - x + 1 \\ & \quad \underline{x^2 - x + 1} \\ & \qquad 0 \end{array}$$

From the long division we can see that

- the **quotient** is $q(x) = x + 1$, and
- the **remainder** is $r(x) = 0$.

Hence, **the division is exact**.

As written in Division Algorithm,

$$f(x) = q(x)d(x) + r(x)$$
$$x^3 + 1 = (x + 1)(x^2 - x + 1) + 0$$
$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

- Thus, the **quotient** $x + 1$ is the **factor** of the dividend $x^3 + 1$.
- The **divisor** $x^2 - x + 1$ is also the **factor** of the dividend $x^3 + 1$.
  (That is $x + 1$ and $x^2 - x + 1$ are the factors of $x^3 + 1$.)
- The product $(x + 1)(x^2 - x + 1)$ a **factorized form** of $x^3 + 1$.

Remark: In division of a polynomial (dividend) with degree n by a polynomial (divisor) of degree m, where $n \geq m$,

i) The remainder is a zero polynomial or a polynomial of degree less than the degree of the divisor.

ii) The degree of the quotient = $n - m$ = degree of f − degree of g .

1.2.2 The Remainder Theorem

Theorem 1.2: (Remainder Theorem)

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let $c \in \mathbb{R}$. If $f(x)$ is divided by the linear polynomial$x - c$, then the remained is $f(c)$.

Example 3: Using Remainder Theorem, find the remainder if:

a) $f(x) = x^3 + 2x^2 - 6$ divided by $d(x) = x - 3$

b) $g(x) = 1 + 5x + x^2 - 2x^5$ divided by $d(x) = x + 2$

c) $f(x) = 2 + 7x^{25}$ divided by $d(x) = x - 1$

Solution:

a) $f(x) = x^3 + 2x^2 - 6$. Here, $x - c = x - 3$ implies $c = 3$.
Then by Division Algorithm, the remainder is
$$R = f(3) = (3)^3 + 2(3)^2 - 6 = 27 + 18 - 6 = 39$$

b) $g(x) = 1 + 5x + x^2 - 2x^5$. Here, $x - c = x + 2$ implies $c = -2$.
Then by Division Algorithm, the remainder is
$$R = g(-2) = 1 + 5(-2) + (-2)^2 - 2(-2)^5 = 1 - 10 + 4 + 64 = 59$$

c) $f(x) = 2 + 7x^{25}$. Here, $x - c = x - 1$ implies $c = 1$.
Then by Division Algorithm, the remainder is
$$R = f(1) = 2 + 7(-1)^{25} = 2 + 7(-1) = 2 - 7 = -5$$

Example 4: When the polynomial $f(x) = x^7 - kx^6 + 5x^3 - x + 11$ is divided by $x + 1$, the remainder is 15, what is the values of k.

Solution: $x - c = x + 1$ implies $c = -1$

By Remainder Theorem, the remainder is

$$R = f(c) \quad \Rightarrow \quad f(-1) = 15$$

$\Rightarrow \quad (-1)^7 - k(-1)^6 + 5(-1)^3 - (-1) + 11 = 15$

$\Rightarrow \quad -1 - k - 5 + 1 + 11 = 1 \;\Rightarrow\; k = -11$

1.2.3 The Factor Theorem

Theorem 1.3: Factor Theorem

Let $f(x)$ be a polynomial of degree greater than or equal to one, and let c be any real number. Then $x - c$ is a factor of $f(x)$, if and only if $f(c) = 0$.

Example 5: In each of the following, use the factor theorem to determine whether or not $g(x)$ is a factor of $f(x)$.

a. $f(x) = x^{15} + 1$; $g(x) = x + 1$,

b. $f(x) = 3x^4 + 7x^2 - x - 2$; $g(x) = x - 1$

Solution:

a. $x + 1 = x - c$ implies $c = -1$. Then $f(-1) = (-1)^{15} + 1 = -1 + 1 = 0$.

Thus, by Factor Theorem, $x + 1$ is a factor of $x^{15} + 1$.

b. $x - 1 = x - c$ implies $c = 1$. Then

$f(1) = 3(1)^4 + 7(1)^2 - (1) - 2 = 3 + 7 - 1 - 2 = 7 \neq 0$.

Therefore, $x - 1$ is **NOT** a factor of $f(x) = 3x^4 + 7x^2 - x - 2$

Example 6: In each of the following find a number k such that:

a) $x + 2$ is a factor of $3x^4 - 8x^2 - kx + 6$.

b) $3x - 2$ is a factor of $6x^3 - 4x^2 + 2kx - k - 3$.

Solution:

a) Let $f(x) = 3x^4 - 8x^2 - kx + 6$ then $f(2) = 0$.

$f(-2) = 3(-2)^4 - 8(-2)^2 - k(-2) + 6 = 3(16) - 8(4) + 2k + 6 = 0$

$\Rightarrow \quad 48 - 32 + 2k + 6 = 22 + 2k = 0$

$\Rightarrow \quad k = -11$

b) $3x - 2 = ax + b \;\Rightarrow\; -\frac{b}{a} = \frac{2}{3}$.

Let $f(x) = 6x^3 - 4x^2 + 2kx - k - 3$. Then

$f\left(\frac{2}{3}\right) = 0 \;\Rightarrow\; 6\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right)^2 + 2k\left(\frac{2}{3}\right) - k - 3 = 0$

$\Rightarrow \; \frac{16}{9} - \frac{16}{9} + \frac{4}{3}k - k - 3 = 0 \;\Rightarrow\; \frac{1}{3}k = 3.$

Therefore, $k = 9$

1.3. ZEROS OF A POLYNOMIAL FUNCTION

Definition 1.2: For a polynomial function P and a real number c, if $P(c) = 0$, then c is a **zero** of P.

For instance, for a polynomial function $f(x) = 2x^3 - x^2 + x - 2$,

$$f(1) = 2(1)^3 - 1^2 + 1 - 2 = 2 - 1 + 1 - 2 = 0$$

Therefore, $x = 1$ is a zero of f.

1.3.1 Zeros and Their Multiplicities

Definition 1.3: If $f(x-c)^k$ is a factor of f(x), but $(x-c)^{k+1}$ is not, then c is said to be a zero of multiplicity k of f .

Example 7: Given that -1 and 2 are zeros of $f(x) = x^4 + x^3 - 3x^2 - 5x - 2$, determine their multiplicity.

Solution: By the factor theorem (x+1) and (x-2) are factors of f(x) hence f(x) can be divided by $(x+1)(x-2) = x^2 - x - 2$ gives;

$$\begin{aligned} f(x) &= (x^2 - x - 2)(x^2 + 2x + 1) \\ &= (x+1)(x-2)(x+1)^2 \\ &= (x+1)^3(x-2) \end{aligned}$$

$\therefore -1$ is a zero of multiplicity 3 and 2 is a zero of multiplicity 1

1.4. GRAPHS OF POLYNOMIAL FUNCTIONS

Properties of $f(x) = ax + b,\ a \neq 0$ graph

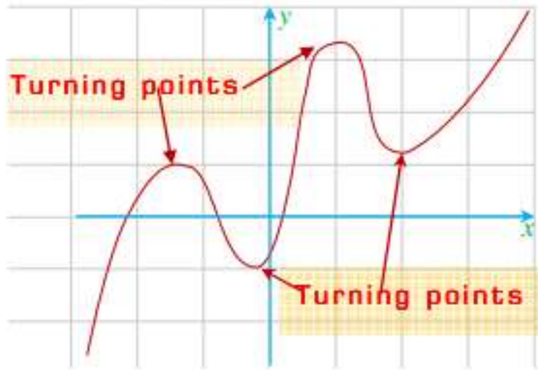
- its graph is a straight line
- the domain of f is real number
- the range of f is real number
- $x-$ intercept of f is $\left(\frac{-b}{a}, 0\right)$
- y −intercept of f is (0,b)
- slope of the graph is a.
- if $a > 0$, then the function is increasing.
- if $a < 0$, then the function is decreasing.

Graph of quadratic function $f(x) = ax^2 + bx + c$, $a, b,$ and c $\in R$, $a \neq 0$

- ✓ Graph of quadratic function is a curve known as parabola
- ✓ If $a > 0$ the parabola opens up ward
- ✓ If $a < 0$ the parabola opens down ward
- ✓ Vertex of the parabola (turning point) $v = \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$
- ✓ The domain of the function is real number
- ✓ The range of the function is $y \geq \frac{4ac-b^2}{4a}$, $a > 0$, $y \leq \frac{4ac-b^2}{4a}$, $a < 0$

Note: The graph of polynomial function of degree n meets the $x -$ axis at most n times.

- Every polynomial function of degree n has at most n zeros.
- The graph of polynomial function f(x) = $a_n x^n + a_{n-1}x^{n-1} + ... a_1 x + a_0$ eventually rises or falls



Note:

1.

The graph of a polynomail function with leading term $a_n x^n$ has the following right and left beahviour:		
	$n-$even	$n-$odd
$a_n > 0$	Up to left and up to right	Down to left and up to right
$a_n < 0$	Downto left and down to right	Up to left and down to right

2. If the multiplicity of the root c is an **odd** number, then the graph of the function **crosses** the $x-$ axis at $x = c$.
3. If the multiplicity of the root c is an **even** number, then the graph of the function **touches**(**but does not cross**) the $x-$ axis at $x = c$.

PRACTICE QUESTIONS ON UNIT 1

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. Which of the following is **NOT** a polynomial?

A. $P(x) = |x-2| + 4$ C. $P(x) = \dfrac{x^3 - 5x^2 + x - 5}{x^2 + 1}$

B. $P(x) = (-3x)\left(1 - \dfrac{2x}{5}\right)$ D. $P(x) = \dfrac{x^{50}}{3} + \sqrt{17}x^5 + \pi$

2. When divide the polynomial $f(x) = 3x^3 + 2x^2 - 19x + 6$ by $x - 1$, the remainder that you will get is__________

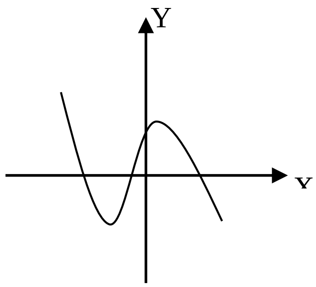
A. 0 B. -8 C. 24 D. 8

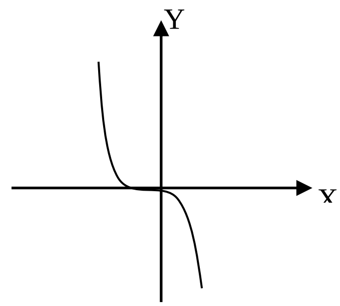
3. If $f(x)$ a polynomial of degree is n, where $n \geq 1$ and if c any real number, then which one of the following statements is true?
   A. If $f(c)=0$ and $n=1$, then $f(x)=k(x-c)$, for some non-zero real number k.
   B. If $f(c)=0$ and $n=2$, then $f(x)=(x-c)q(x)+r(x)$, where both $q(x)$ and $r(x)$ are polynomials of degree 1.
   C. If $f(c)=0$ and $n>1$, then $f(x)=(x-c)q(x)+r(x)$, where $q(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is 1.
   D. If $f(c)=0$ and $n>2$, then $f(x)=(x-c)q(x)$, where $q(x)$ is a polynomial of degree 1.

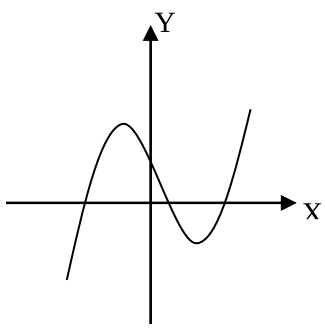
4. What number must be added to x^3+5x^2+6x+4 so that $x+1$ is a factor?
   A. 2 B. -19 C. 19 D. -2

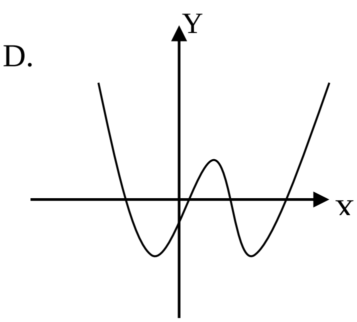
5. Which of the following function is neither even nor odd?
   A. $k(x)=x^5$ C. $h(x)=2x^2-3|x|$
   B. $g(x)=x-1$ D. $f(x)=x^4$

6.Which of the following can be the graph of a polynomial function of degree three with positive leading coefficient?

A.


C.


B.


D.


7. Let $f(x)=ax^{10}+bx^5-1$. If $x+1$ is a factor of $f(x)$ and when $f(x)$ is divided by $x-1$ the remainder is 4, then a and b respectively equal to:
   A. -1, 1 B. 3, 2 C. 1, 5 D. 3, 1