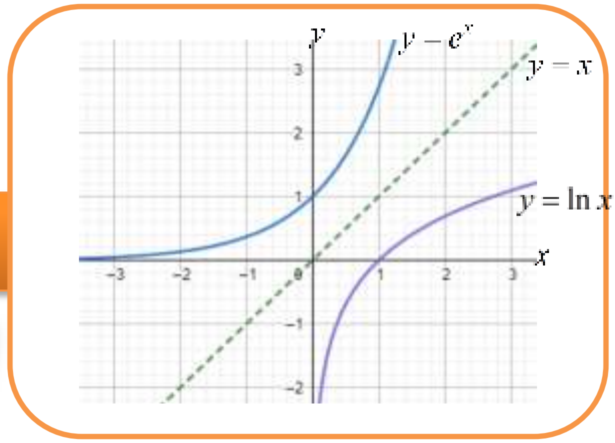
Unit 2



EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Unit Outcomes:

After completing this unit, you should be able to:

- *understand the laws of exponents for real exponents.*
- *know specific facts about logarithms.*
- *know basic concepts about exponential and logarithmic functions.*

Main Contents:

2.1. Exponents.
2.2. Exponential Functions and Their Graphs.
2.3. Logarithms.
2.4. Logarithmic Functions and Their Graphs.
2.5. Equations Involving Exponents and Logarithms.

2.1. EXPONENTS

Laws of Exponents

NOTE: If a and b are non-zero real numbers and the exponents m and n are integers, then:

1. $a^m \cdot a^n = a^{m+n}$
2. $(ab)^n = a^n b^n$
3. $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$
4. $(a^m)^n = a^{mn}$
5. $a^{\frac{1}{n}} = \sqrt[n]{a}$, if n is odd, $a \in \mathbb{R}$ and if n is even, $a \geq 0$
6. $a^m = b^n$ if and only if $m = n,\ a \neq \pm 1$
7. $a^0 = 1$
8. $a^{-n} = \dfrac{1}{a^n}$, $n > 0$
9. $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n}$, for $n > 0$
10. $\dfrac{a^m}{a^n} = a^{m-n}$
11. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 1: Simplify each of the following:

a. $\dfrac{\left[\left(x^2\right)^3\right]^4}{x^5 \times x^{10} \times x^{-2}}$

b. $\left(\dfrac{a^{-\frac{1}{3}} b^{\frac{1}{2}}}{a^{-\frac{1}{4}} b^{\frac{1}{3}}}\right)^6$

Solution:

a. $$\frac{\left[\left(x^2\right)^3\right]^4}{x^5 \times x^{10} \times x^{-2}} = \frac{\left[x^{2\times 3}\right]^4}{x^{5+10+(-2)}} = \frac{\left[x^6\right]^4}{x^{13}} = \frac{x^{6\times 4}}{x^{13}} = x^{24-13} = x^{11}$$

b. $$\left(\frac{a^{-\frac{1}{3}} b^{\frac{1}{2}}}{a^{-\frac{1}{4}} b^{\frac{1}{3}}}\right)^6 = \left(a^{-\frac{1}{3}+\frac{1}{4}} b^{\frac{1}{2}-\frac{1}{3}}\right)^6 = \left(a^{-\frac{1}{3}+\frac{1}{4}} b^{\frac{1}{2}-\frac{1}{3}}\right)^6 = \frac{b}{\sqrt{a}}$$,for $a \geq 0$ and $b > 0$.

2.2. THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

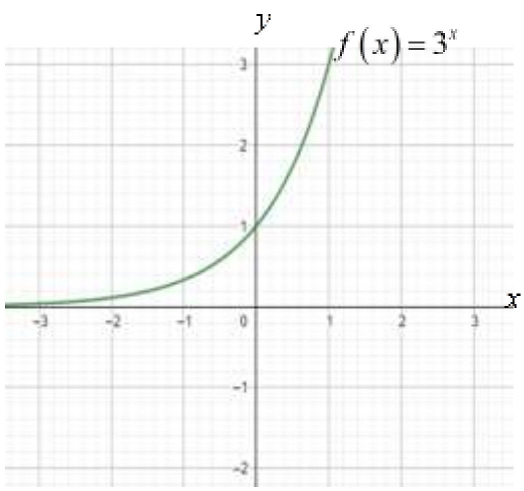
Definition: The exponential function f with the base b is defined by $f(x) = b^x$ where $b > 0,\ b \neq 1$ and x is any real number.

In this section you will draw graphs and investigate the major properties of functions ofthe form $f(x) = 2^x, g(x) = \left(\frac{1}{2}\right)^x$, $h(x) = \left(\frac{2}{3}\right)^x$, $k(x) = 3^{5x}$, etc.

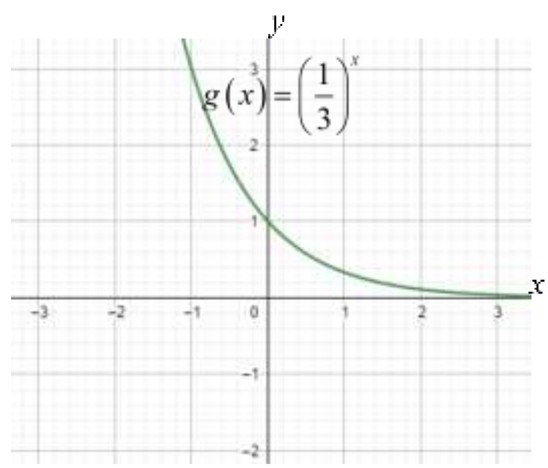
Example 2: Draw the graphs of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$.

Solution: We begin by calculating values of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ for integer values of x as shown in the following table.

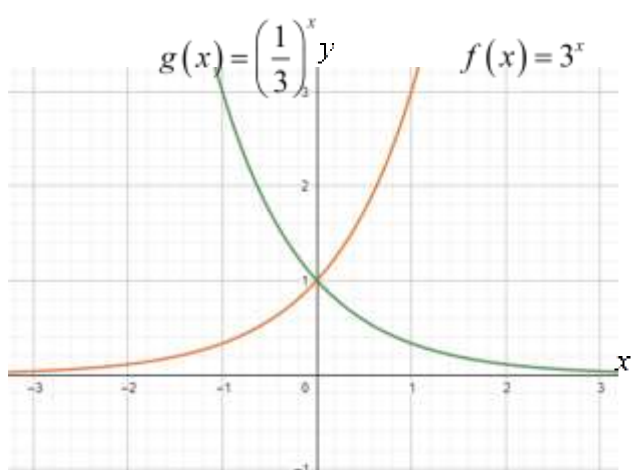
x	-3	-2	-1	0	1	2	3
$f(x)=3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	8	27
$g(x)=\left(\frac{1}{3}\right)^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$



The graph of $f(x)=3^x$



The graph of $g(x)=\left(\frac{1}{3}\right)^x$

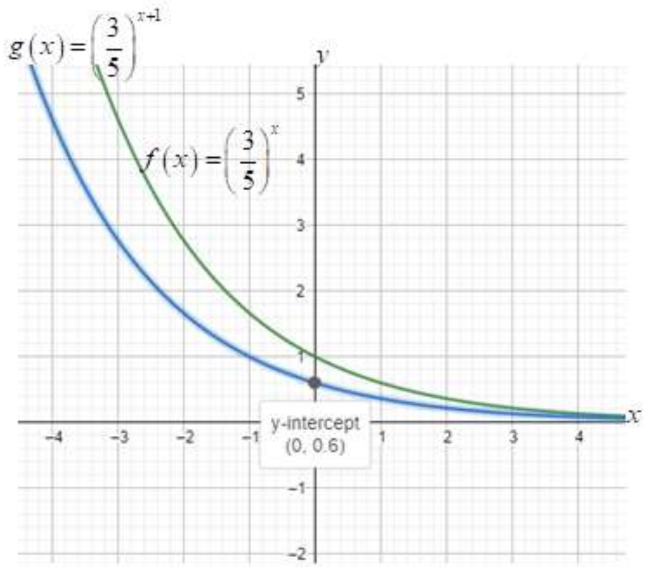


Graphs of $f(x)=3^x$ and $g(x)=\left(\frac{1}{3}\right)^x$ drawn using the same coordinate axes

Example 3: Sketch the graphs of $f(x)=\left(\frac{3}{5}\right)^{x}$ and $f(x)=\left(\frac{3}{5}\right)^{x+1}$.

Solution:

x	-3	-2	-1	0	1	2	3
$f(x)=\left(\frac{3}{5}\right)^{x}$	$\frac{125}{27}$	$\frac{25}{9}$	$\frac{5}{3}$	1	$\frac{3}{5}$	$\frac{9}{25}$	$\frac{27}{125}$
$g(x)=\left(\frac{3}{5}\right)^{x+1}$	$\frac{25}{9}$	$\frac{5}{3}$	1	$\frac{3}{5}$	$\frac{9}{25}$	$\frac{27}{125}$	$\frac{81}{625}$



Graphs of $g(x)=\left(\frac{3}{5}\right)^{x}$ and $g(x)=\left(\frac{3}{5}\right)^{x+1}$ drawn using the same coordinate axes

Note: Basic properties

The graph of $f(x)=b^x$, $b>1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point $(0, 1)$, i.e. the $y-$intercept is 1.
4. The function is increasing.
5. The values of the function are greater than 1 for $x>0$ and between 0 and 1 for $x<0$.
6. The graph approaches the $x-$as an asymptote on the left and increases without bound on the right.

The graph of $f(x) = b^x$, $0 < b < 1$ has the following basic properties:

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point $(0, 1)$, i.e. the $y-$intercept is 1.
4. The function is decreasing.
5. The values of the function are greater than 1 for $x < 0$ and between 0 and 1 for $x > 0$.
6. The graph approaches the $x-$as an asymptote on the right and increases without bound on the left.

Theorem 2.1 Let $a > 0$ and $x, y \in R$.

a. If $a > 1$, then :
   i. $a^x > a^y$ if and only if $x > y$.
   ii. $a^x = a^y$ if and only if $x = y$.

b. If $0 < a < 1$, then :
   iii. $a^x < a^y$ if and only if $x > y$.
   iv. $a^x = a^y$ if and only if $x = y$.

Theorem 2.2 Let $a > 0$, $b > 0$ and $x \in R$.

i. If $x > 0$, then $a^x > b^x$ if and only if $a > b$.
ii. If $x < 0$, then $a^x > b^x$ if and only if $a < b$.
iii. If $x = 0$, then $a^x = b^x$.

2.3. LOGARITHMS

NOTE:

1. For a fixed positive number $b \neq 1$ and for each $a > 0$, $b^c = a$ if and only if $c = \log_b a$. The value of $\log_b a$ is the answer of the question "to what power must b be raised to produce a"?
2. The equations $y = \log_a x$ and $a^y = x$ are equivalent.

Example 4: Write an equivalent logarithmic statement for:

a. $8^{\frac{1}{3}} = 2$

b. $2^{-5} = \frac{1}{32}$

Solution:

a. From $8^{\frac{1}{3}} = 2$, we have $\log_8 2 = \frac{1}{3}$.

b. Since $2^{-5} = \frac{1}{32}$, $\log_2 \frac{1}{32} = -5$.

Laws of Logarithms

If $a, b,\ x$ and y are positive numbers and $a, b \neq 1$ then:

1. $\log_b xy = \log_b x + \log_b y$ .... Product formula.
2. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$ ....quotient formula.
3. $\log_b x^k = k\log_b x$, k is any real number............... Power formula.
4. $\log_b x = \frac{\log_a x}{\log_a b}$ .... Change of base formula.
5. $b^{\log_b c} = c$
6. $\log_a a^x = x$
7. $\log_{\frac{1}{b}} a = -\log_b a$
8. $\log_a x = \frac{1}{\log_x a}$, for $x > 0, x \neq 1$, $a > 0$ and $a \neq 1$ Thus $(\log_a x)\left(\frac{1}{\log_x a}\right) = 1$
9. $\log_{a^k} x = \frac{1}{k}\log_a x$, for $a > 0$ and $a \neq 1$
10. $\log_{a^{\frac{1}{k}}} x = k\log_a x$, for $a > 0$ and $a \neq 1$.

Useful identities for logarithms

1. $\log_a a = 1$, for all $a > 0$
2. $\log_a 1 = 0$, for all $a > 0$

2.4. THE LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

2.4.1. Graphs of logarithmic functions

Example 5: Draw the graph of each of the following using:

i. Different coordinate system
ii. The same coordinate system

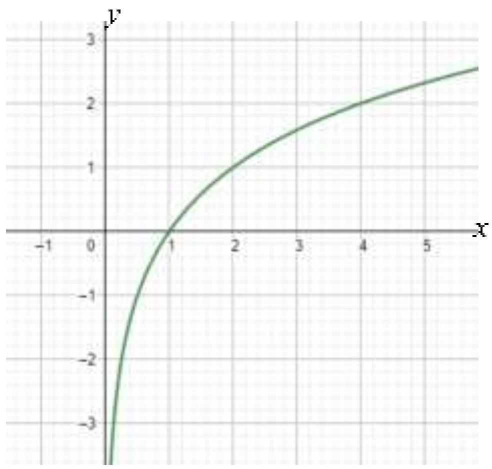
a. $f(x) = \log_2 x$ b. $g(x) = \log_{\frac{1}{2}} x$

Solution:

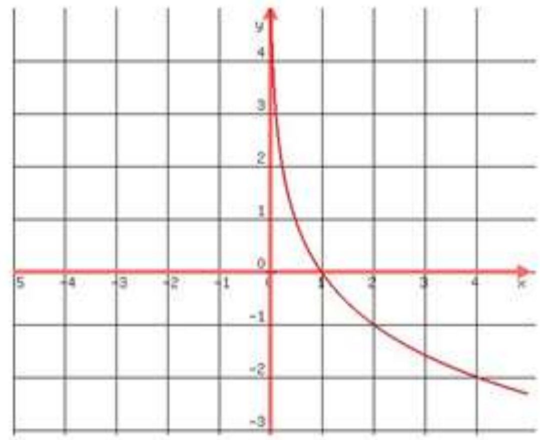
We begin by calculating the values of $f(x) = \log_2 x$ and $g(x) = \log_{\frac{1}{2}} x$ for positive values of x.

x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
$f(x) = \log_2 x$	-4	-3	-2	-1	0	1	2	3	4
$g(x) = \log_{\frac{1}{2}} x$	4	3	2	1	0	-1	-2	-3	-4

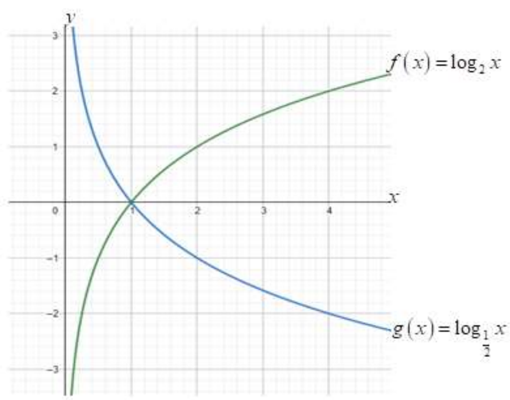
Then we plot the points corresponding to the pairs we have found and connect the points with smooth curves to obtain the graphs as shown below.



The graph of $f(x) = \log_2 x$



The graph of $f(x) = \log_{\frac{1}{2}} x$



Example 6: Draw the graph of each of the following using:

a. Different coordinate system
ii. The same coordinate system

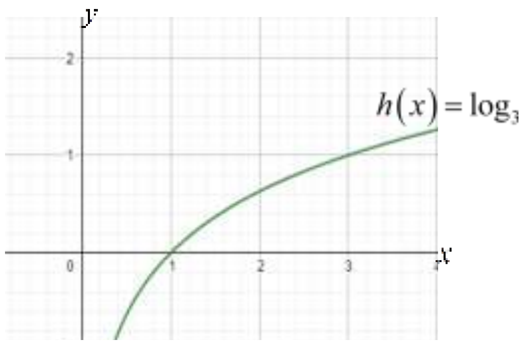
 a. $h(x) = \log_3 x$
 b. $k(x) = \log_{\frac{3}{2}} x$

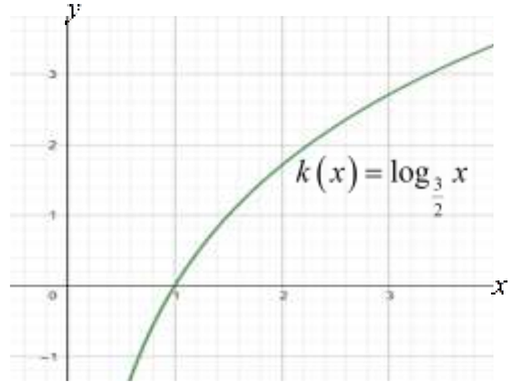
Solution:

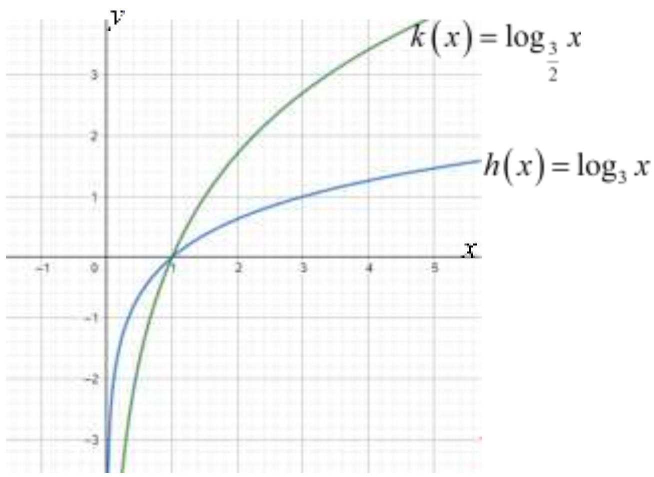
We begin by calculating the values of $f(x) = \log_2 x$ and $g(x) = \log_{\frac{1}{2}} x$ for positive values of x.

x	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9		$\frac{4}{9}$	$\frac{2}{3}$	1	$\frac{3}{2}$	$\frac{9}{4}$
$h(x) = \log_3 x$	-2	-1	0	1	2	$k(x) = \log_{\frac{3}{2}} x$	-2	-1	0	1	2

Then we plot the points corresponding to the pairs we have found and connect the points with smooth curves to obtain the graphs as shown below.







In general, the graph of $f(x) = \log_b x$, for any $b > 1$ has the following shape

Basic properties of the graph of $y = \log_b x, (b > 1)$

1. The domain is the set of positive real numbers
2. The range is the set of all real numbers.
3. The graph includes the points $(1, 0)$ i.e. the $x-$intercept of the graph is 1.
4. The value of the function increases as x increases.
5. The $y-$axis is a vertical asymptote of the graph.
6. The values of the function are negative for $0 < x < 1$ and they are positive for $x > 1$.

Example 7: Draw the graph of each of the following using:

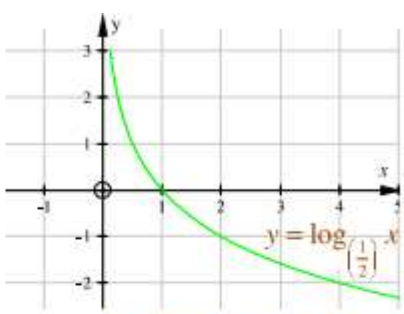
i. Different coordinate system
ii. The same coordinate system

a. $h(x) = \log_{\frac{1}{2}} x$ b. $k(x) = \log_{\frac{2}{3}} x$

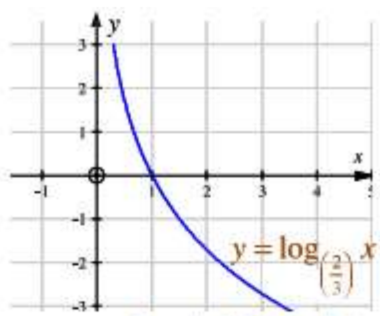
Solution: calculate the values of the given functions for some values of x as shown in the table below. The plot the corresponding points on the co-ordinate system.

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$h(x) = \log_{\frac{1}{2}} x$	−3	−2	−1	0	1	2	3

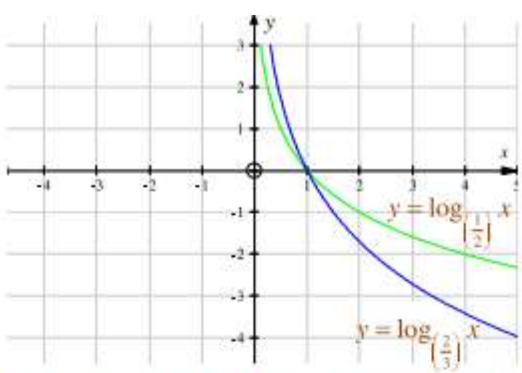
x	$\frac{27}{8}$	$\frac{9}{4}$	$\frac{3}{2}$	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{8}{27}$
$k(x) = \log_{\frac{2}{3}} x$	−3	−2	−1	0	1	2	3



Graph of $h(x) = \log_{\left(\frac{1}{2}\right)} x$

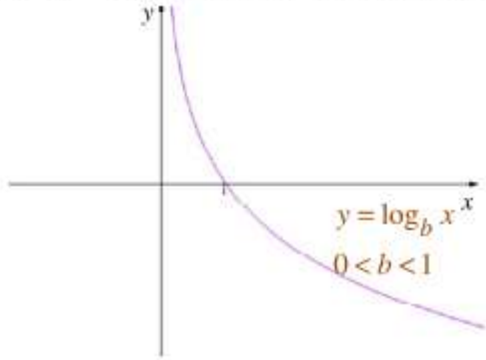


Graph of $k(x) = \log_{\left(\frac{2}{3}\right)} x$



Graphs of $y = \log_{\left(\frac{1}{2}\right)} x$ and $y = \log_{\left(\frac{2}{3}\right)} x$ drawn using the same coordinate axes

In general, the graph of $f(x)=\log_b x$ for $0<b<1$ looks like the one given below .



Basic properties of the graph of $y=\log_b x, (0<b<1)$

1. The domain is the set of positive real numbers
2. The range is the set of all real numbers.
3. The graph includes the points $(1, 0)$ i.e. the $x-$intercept of the graph is 1.
4. The value of the function decreases as x increases.
5. The $y-$axis is a vertical asymptote of the graph.
6. The values of the function are positive for $0<x<1$ and they are negative for $x>1$.

2.5. EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

2.5.1. Solving Exponential Equations

NOTE: property of equality of exponential equations

For $b>0, b\neq 1, x$ and y real numbers,

1. $b^x=b^y$, if and only if $x=y$.
2. $a^x=b^x, (x\neq 0)$, if and only if $a=b$.

Examples 8: Solve the following exponential equations

a. $\left(\frac{2}{3}\right)^{2x+1}=\left(\frac{9}{4}\right)^x$

b. $4^x=\left(\frac{1}{2}\right)^{x-3}$

c. $7^{x^2+x}=49$

d. $\frac{10^{x+2}}{2^{x-3}}=5^{x+1}$

Solution:

a. $\left(\frac{2}{3}\right)^{2x+1}=\left(\frac{9}{4}\right)^x \Rightarrow \left(\frac{2}{3}\right)^{2x+1}=\left(\frac{3}{2}\right)^{2x}=\left(\frac{2}{3}\right)^{-2x}$

$\Leftrightarrow 2x+1=-2x$

$$\Rightarrow 2x+2x=-1$$
$$\Rightarrow x=-\frac{1}{4}$$

b. $4^x=\left(\frac{1}{2}\right)^{x-3} \Rightarrow 4^x=\left(2^{-1}\right)^{x-3}=2^{(-x+3)}$

$$\Rightarrow \left(2^2\right)^x=2^{(-x+3)}$$
$$\Rightarrow 2^{2x}=2^{(-x+3)}$$
$$\Leftrightarrow 2x=-x+3$$
$$\Rightarrow 2x=-x+3$$
$$\Rightarrow x=1$$

c. $7^{x^2+x}=49 \Rightarrow 7^{x^2+x}=7^2$

$$\Leftrightarrow x^2+x=2$$
$$\Rightarrow x^2+x-2=0$$
$$\Rightarrow (x-1)(x+2)=0$$
$$\Rightarrow x=1 \text{ or } x=-2$$

d. $\frac{10^{x+2}}{2^{x-3}}=5^{x+1} \Rightarrow \frac{10^x\times 10^2}{2^x\times 2^{-3}}=5^x\times 5$

$$\Rightarrow \frac{100\left(10^x\right)}{\frac{2^x}{8}}=5^x\times 5$$
$$\Rightarrow \frac{160\left(10^x\right)}{2^x}=5^x$$
$$\Rightarrow 160\left(10^x\right)=10^x$$
$$\Rightarrow 160=1^x \Rightarrow \text{No solution. Why?}$$

2.5.2. Solving Logarithmic Equations

Logarithmic equations can be solved by changing them to equivalent exponential form. However, it is necessary first to state the **universe** and to use the basic properties of logarithms to simplify one side of an equation.

A **universe** is the largest set in $\mathbb{R}$ for which the given expression is defined.

Examples 9: Sate the universe and solve the following equations:

a. $\log_2(x-3)=5$

b. $\log(x+3)+\log x=1$

c. $\log_3(x+1)-\log_3(x+3)=1$

d. $\log\left(x^2-121\right)-\log(x+11)=1$

Solution:

a. First find domain of $\log_2(x-3)=5$ i.e. $x-3>0 \Rightarrow x>3$

So the domain is $x \in (3, \infty)$.

Now by changing $\log_2(x-3)=5$ to exponential equation we get $2^5 = x-3$

$\Rightarrow 32 = x-3$

$\Rightarrow x = 35$

Check! For $x=35$, $\log_2(x-3)=\log_2(35-3)=\log_2 32=5$ is true.

b. $\log(x+3)+\log x$ is valid for $x+3>0$ and $x>0$ i.e. $x>-3$ and $x>0$

Therefore the universe $\text{U}=(0, \infty)$

$\log(x+3)+\log x=1 \Rightarrow \log(x+3)(x)=1$ by the law $\log x+\log y=\log xy$

$\Rightarrow \log(x^2+3x)=1$

$\Rightarrow 10^1 = x^2+3x$

$\Rightarrow x^2+3x-10=0$

$\Rightarrow (x-2)(x+5)=0$

$\Rightarrow x-2=0$ or $x+5=0$

$\Rightarrow x=2$ or $x=-5$ but $x=-5 \notin$ Domain

Therefore the only solution set is $\{2\}$.

c. $\log_3(x+1)-\log_3(x+3)$ is valid for $x+1>0$ and $x+3>0$ i.e. $x>-1$ and $x>-3$

Therefore the universe $\text{U}=(-1, \infty)$

$\log_3(x+1)-\log_3(x+3)=1 \Rightarrow \log_3\left(\frac{x+1}{x+3}\right)=1$ since $\log x-\log y=\log\left(\frac{x}{y}\right)$

$\Rightarrow 3^1 = \frac{x+1}{x+3}$

$\Rightarrow x+1=3(x+3)=3x+9$

Therefore $-2x=8$ and $x=-4$ but -4 is **not** in the universe.

Hence there no x that satisfies the given equation and the solution set is empty set.

d. $\log(x^2-121)-\log(x+11)$ is valid for $x^2-121>0$ and $x+11>0$ i.e. $|x|>11$ and $x>-11$

Therefore the universe $\text{U}=(11, \infty)$

$\log(x^2-121)-\log(x+11)=1 \Rightarrow \log\left(\frac{x^2-121}{x+11}\right)=1$ since $\log x-\log y=\log\left(\frac{x}{y}\right)$

$\Rightarrow 10^1 = \frac{x^2-121}{x+11}$

$\Rightarrow x^2-121=10(x+11)=10x+110$

$\Rightarrow x^2-10x-231=0$

$$\Rightarrow (x-21)(x+11)=0$$

$$\Rightarrow x-21=0 \text{ or } x+11=0$$

$\Rightarrow x=21$ or $x=-11$ but -11 is **not** in the universe.

Therefore the solution set is only $\{21\}$.

PRACTICE QUESTIONS ON UNIT 2

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. Which of the following is **NOT** true?

   A. $\log_{\frac{1}{a}} x > \log_{\frac{1}{a}} y$, for $a>1, x>0, y>0$ and $x>y$

   B. $\log_a x > \log_a y$, for $a>1, x>0, y>0$ and $x>y$

   C. $\log_{\frac{1}{3}} \sqrt[3]{9} > \log_{\frac{1}{3}} 27$

   D. $\log_{3.5} 7.6 = 0.3357$

2. Which of the following is **NOT** true?

   A. If $\log_2 4x + \log_4 2x = 2x - \frac{1}{2}$, then $x=1$

   B. $(\log_b 4)(\log_8 b^3 = 2)$, for each $b>1$

   C. $2^{\log_2 3} = 3$

   D. $\log_2(\log_4(\log_8 64)) = -1$

3. If $0<x<y<1$, then which of the following is **NOT** true?

   A. $x^{-t} > 1$, for $t<0$.

   B. $\left(\frac{x}{y}\right)^t > 1$, for $t<0$.

   C. $(xy)^t > 1$, for $t<0$.

   D. $\left(\frac{y}{x}\right)^t < 1$, for $t<0$.

4. If $\log_2 9 = x$ and $2^y = \frac{\left(2^{\sqrt{3}}\right)^2 \sqrt{12}}{\sqrt[3]{16}}$, then y equals:

   A. $\frac{32+3x}{12}$ B. $\frac{64+16x}{12}$ C. $\frac{26+3x}{6}$ D. None of the above

5. If $\log 3.54 = 0.549$, then the mantissa of $\log 35{,}400$ is

   A. 0.549 B. 4.549 C. 0.051 D. 4

6. From the following statements, which one is true about an exponential expression a^x?

   A. If $a>0$, then for a positive real number x the value of $a^{\sqrt{x}}$ cannot be a real number.

   B. If $a>0$ and $x, y \in \mathbb{Q}$, then $a^x \in \mathbb{R}$ is found between a^m and a^n, where m and n are integers and $m<x<n$.

   C. If $a>0$ and $x=\frac{m}{n}$ is a positive rational number with $n>0$, then there is $b \in \mathbb{R}$, such that $b=a^x$ and $a^n = b^m$.

   D. For $a \neq 0$ and $x, y \in \mathbb{Q}$, if $a^x = b^y$ and $x \neq y$, then a can be any real number that is different fro 1.

7. Given $\log 5 = b,$ then $\log_5 4$ is equal to

   A. $2\left(\frac{b}{1-b}\right)$ B. $2\left(\frac{b}{b-1}\right)$ C. $2\left(\frac{1}{b}-1\right)$ D. $2\left(1-\frac{1}{b}\right)$

8. Which of the following is the solution of the equation $\log_2(x-2)+\log_2 x = 3$?

   A. -2 and 4 B. 4 C. $1+\sqrt{7}$ D. $1-\sqrt{7}$ and $1+\sqrt{7}$

9. The solution set of the equation $\log_x 5+\frac{1}{2}\log_{\sqrt{5}} x+8\log_{25}\left(\frac{1}{x}\right)=2$ is:

   A. $\{5\}$ B. $\left\{\frac{1}{5},\sqrt[3]{5}\right\}$ C. $\{125\}$ D. $\left\{\frac{1}{3},-1\right\}$

10. Suppose $\log_3 2 = a$ and $\log_3 5 = b.$ Then $\log_3 0.0002$ is equal to:

   A. $-3a-4b$ B. $3a+4b$ C. $a-b$ D. $b-3a$