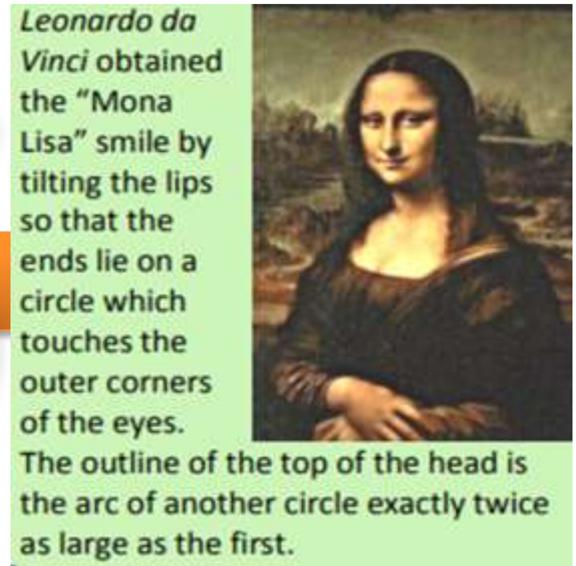


PLANE GEOMETRY

Unit Outcomes:

After completing this unit, you should be able to:

- *Know more theorems special to triangles.*
- *Know basic theorems specific to quadrilaterals.*
- *Know theorems about circles and angles inside, on and outside a circle.*
- *Solve geometrical problems involving quadrilaterals, circles and regular polygons.*

Main Contents:

5.1. Theorems on triangles

5.2. Special quadrilaterals

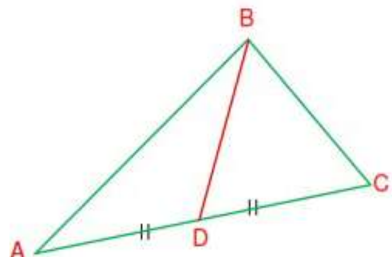
5.3. More on circles

5.4. Regular polygons

5.1. THEOREMS ON TRIANGLES

1. Median of a triangle

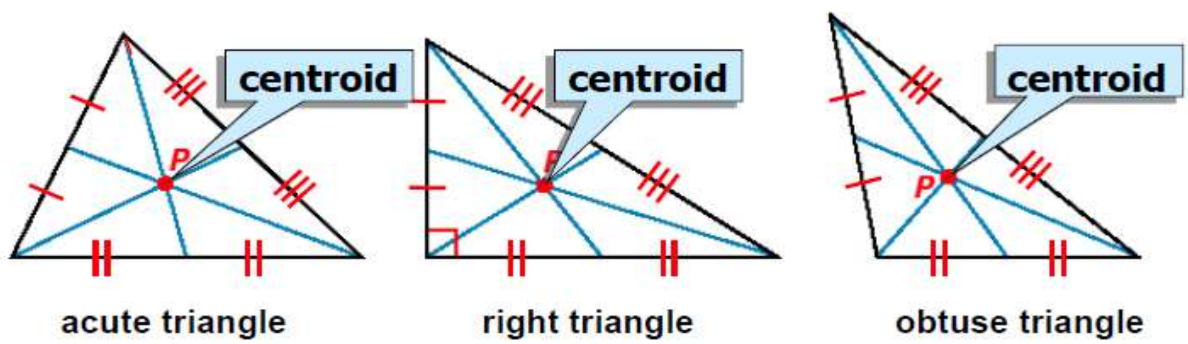
A median of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side.



$\overline{BD}$ is the median of $\Delta ABC \implies \overline{AD} = \overline{CD}$

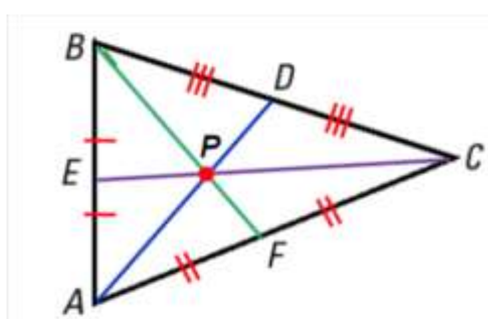
Theorem 5.1
The median of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side

Note : The three medians of a triangle are concurrency is called the **centroid** and is always inside the triangle.



Concurrency of medians of a triangle
The medians of a triangle intersect at a point that is two-thirds of the distance from each vertex to the mid-point of the opposite side

Illustration
If p is the centroid of $\triangle ABC$, then



- $AP = \frac{2}{3}AD$
- $BP = \frac{2}{3}BF$
- $CP = \frac{2}{3}CE$
- $DP = \frac{1}{3}AD, EP = \frac{1}{3}CE, FP = \frac{1}{3}BF$

Example 1: In the figure 6.7, $\overline{AN}, \overline{CM}$ and $\overline{BL}$ are medians of ΔABC. If $AN = 12cm, OM = 5cm$ and $BO = 6cm$, find BL, ON and OL.

Solution: By theorem 6.1 $BO = \frac{2}{3}BL$ and $AO = \frac{2}{3}AN$

Substituting $6 = \frac{2}{3}BL$ and $AO = \frac{2}{3} \times 12$

So $BL = 9cm$ and $AO = 8cm$

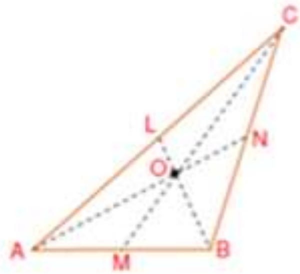
Since $BL = BO + OL$

$\Rightarrow OL = BL - BO = 9 - 6 = 3cm$

Now $AN = AO + ON$

$\Rightarrow ON = AN - AO = 12 - 8 = 4cm$

$\therefore BL = 9cm, OL = 3cm$ and $ON = 4cm$



Theorem 5.2
The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

Theorem 6.3
The altitudes of a triangle are concurrent.

2. Angle bisector of a triangle

Theorem 5.4
The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

3. Altitude theorem

The altitude theorem is stated here for a right angled triangle. It relates the length of the altitude to the hypotenuse of a right angled triangle, to the lengths of the segments of the hypotenuse.

Theorem 5.5 Altitude theorem
In a right angled triangle ABC with altitude $\overline{CD}$ to the hypotenuse $\overline{AB}$, $\frac{AD}{DC} = \frac{CD}{DB} \Rightarrow (CD)^2 = (AD)(DB)$.

Proof: consider ΔABC as shown in the figure 6.13 $\triangle ABC \sim \triangle ACD$ . . . . AA similarity

So $\angle ABC \cong \angle ACD$

Similarly, $\triangle ABC \sim \triangle CBD$ . . . . AA similarity.

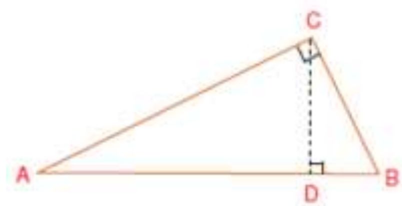
So $\angle ABC \cong \angle CBD$

It follows that $\angle ACD \cong \angle CBD$

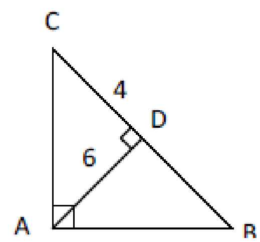
By AA similarity , $\triangle ACD \sim \triangle CBD$

Hence $\frac{AD}{CD} = \frac{CD}{BD}$ . . . .*

Equivalently ,$\frac{AD}{DC} = \frac{CD}{DB}$



The following are some forms of the altitude theorem from $*$,
$(CD)^2 = (AD)(BD)$ *or* $(CD)(CD) = (AD)(DB)$
This can be stated as the square of the length of the altitude is the product of the length of the segments of the hypotenuse.
Example 2: Find the lengths of all sides that are not given in the following figures



Solution:

By altitude theorem

$(AD)^2 = (CD)(BD)$

$\Rightarrow (6)^2 = (4)(BD)$

$\Rightarrow BD = \frac{36}{4} = 9\ units$

By Pythagoras theorem $(for\ \triangle ADC)$

$(AC)^2 = (AD)^2 + (CD)^2$

$\Rightarrow (AC)^2 = (6)^2 + (4)^2$ = 36 + 14 = 52

$\Rightarrow (AC)^2$ =52 $\Rightarrow$ $AC = \sqrt{52}$ = $2\sqrt{13}$ Units

Again by Pythagoras theorem $(for\ \triangle ABC)$

$(BC)^2 = (AC)^2 + (AB)^2$

$\Rightarrow (BD + DC)^2 = \left(\sqrt{52}\right)^2 + (AB)^2$

$\Rightarrow (9 + 4)^2 = 52 + (AB)^2$

$\Rightarrow (13)^2 = 52 + (AB)^2$

$\Rightarrow 169 - 52 = (AB)^2$

$\Rightarrow (AB)^2 = 117$

$\therefore AB = \sqrt{117}$ = $\sqrt{9 \times 13}$ = $3\sqrt{13}$ units

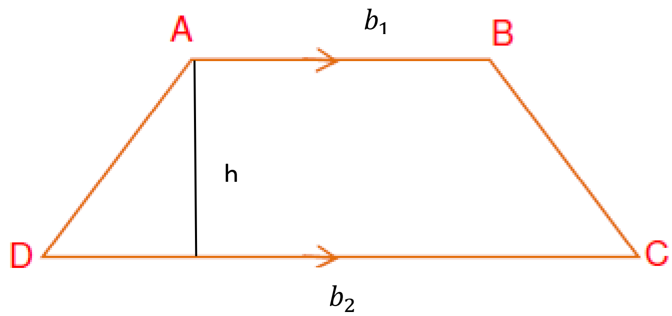
4. Menelaus' theorem

Theorem 5.6 Menelaus' theorem
If points D, E and F on the sides $\overline{BC}, \overline{CA}\ and\ \overline{AB}$ respectively of $\triangle ABC$ (on their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$,then the points D, E and F are collinear.

5.2. SPECIAL QUADRILATERALS

1. TRAPEZIUM

Definition 5.1: A trapezium is a quadrilateral where only two of the sides are parallel.



$\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \nparallel \overline{BC}$

$\overline{AB}$ *and* $\overline{DC}$ are its bases

If the two non- parallel sides of a trapezium are congruent is called isosceles trapezium.

In the above isosceles trapezium $\overline{AD} \cong \overline{BC}$ and $\angle D \cong \angle C$

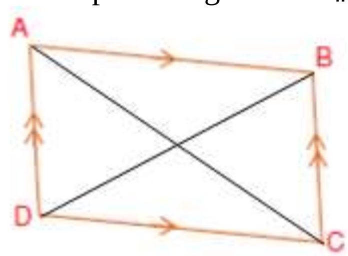
Area of trapezium

The area of a trapezium with bases b_1 *and* b_2 and altitude h is given by $A = \frac{1}{2}(b_1 + b_2)h$.

2. PARALLELOGRAM

Definition 5.2: A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel

In Figure 6.23, the quadrilateral *ABCD* is a parallelogram. *AB* ∥ *DC* and *AD*∥ *BC*



Properties of a parallelogram and tests for a quadrilateral to be a parallelogram are stated in the following theorem:

Theorem 5.7

a. The opposite sides of a parallelogram are congruent i.e $\overline{AB} \cong \overline{DC}$ and $\overline{AD} \cong \overline{BC}$

b. The opposite angles of a parallelogram are congruent i.e $\angle A \cong \angle C$ and $\angle B \cong \angle D$

c. The diagonals of a parallelogram bisect each other i.e $\overline{AO} \cong \overline{CO}$ *and* $\overline{BO} \cong \overline{DO}$

d. If the opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

e. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

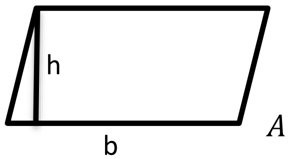
f. If the opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.

g. consecutive angles in a parallelogram are supplementary

i.e $\angle A + \angle B = 180^\circ, \angle B + \angle C = 180^\circ, \angle C + \angle D = 180^\circ$ *and* $\angle D + \angle A = 180^\circ$

Area of a parallelogram

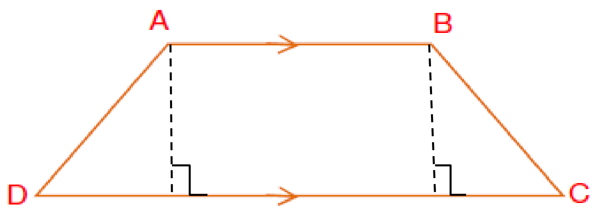
The area of a parallelogram with base b and altitude h is given by



$A = bh$

Example 3: The shorter base of an isosceles trapezium is 12cm long and the non-parallel bases are each 10cm. Find the area of this trapezium if its altitude is 6cm

Solution:



In ΔDEA ,By Pythagoras theorem

$(AD)^2 = (AE)^2 + (DE)^2$

$\Rightarrow (10)^2 = (6)^2 + (DE)^2$

$\Rightarrow 100 - 36 = (DE)^2$

$\Rightarrow 64 = (DE)^2$

$\Rightarrow DE = \sqrt{64}$ = 8cm

$DE \cong FC$, and $AB \cong EF$

$\Rightarrow DC = DE + EF + FC$

$= 8cm + 12cm + 8cm = 28cm$

$A = \frac{1}{2}(b_1 + b_2)h$

$= \frac{1}{2}(12cm + 28cm) \times 6cm$

$= 120cm^2$

Example 4: One of the sides of the parallelogram is 8cm long and the perimeter of this parallelogram is 28cm.if the altitude to the longer base is 4cm, what the altitude to the shorter base is and what is the area of the parallelogram

Solution:

Perimeter = $2(s_1 + s_2)$ where $s_1 and\ s_2$ are the sides of parallelogram.

$28cm = 2(8cm + s_2)$

$\Rightarrow 14cm = 8cm + s_2$

$\Rightarrow s_2 = 6cm$

Hence the longer base is 8 cm and the area of the parallelogram becomes

Area $= bh = 8cm \times 4cm = 32\ cm^2$

Now the altitude to the shorter base can be found from:

$Area = bh$

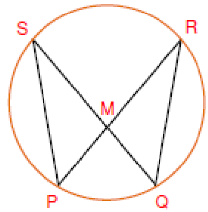
$\Rightarrow 32cm^2 = 6cm \times h$

$$\Rightarrow h = \frac{32cm^2}{6cm} = \frac{16}{3}cm$$

5.3. MORE ON CIRCLES MORE ON CIRCLES

Theorem 5.8
The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Example 5: In the figure beow, $m(\angle MRQ) = 30^\circ$, and $m(\angle MQR) = 40^\circ$.Write down the measure of all the other angles in the two triangles, $\triangle PSM$ and $\triangle QMR$. What do you notice about the two triangles?



Solution: $m(\angle QMR) = 180^\circ - (30^\circ + 40^\circ)$(why?)

$= 180^\circ - 70^\circ = 110^\circ$

$m(\angle RQS) = \frac{1}{2} m(\widehat{RS})$

So, $40^\circ = \frac{1}{2} m(\widehat{RS})$

$\therefore\ m(\widehat{RS}) = 80^\circ$

$m(\angle PRQ) = \frac{1}{2} m(\widehat{PQ})$

Hence $30^\circ = \frac{1}{2} m(\widehat{PQ})$

$\therefore\ m(\widehat{PQ}) = 60^\circ$

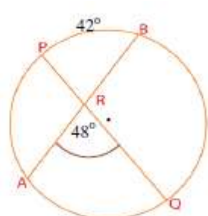
$m(\angle PSQ) = \frac{1}{2} m(\widehat{PQ}) = \frac{1}{2}(60^\circ) = 30^\circ$

$m(\angle RPS) = \frac{1}{2} m(\widehat{RS}) = \frac{1}{2}(80^\circ) = 40^\circ$

The two triangles are similar by **AA** similarity theorem.

Example 6: An angle formed by two chords intersecting within a circle is 48°, and one of the intercepted arcs measures 42°. Find the measures of the other intercepted arc.

Solution: Consider the the following figure.



$m(\angle PRB) = \frac{1}{2} m(\widehat{PB}) + \frac{1}{2} m(\widehat{AQ})$ (by theorem 6.11)

$\Rightarrow 48^\circ = \frac{1}{2}(42^\circ) + \frac{1}{2} m(\widehat{AQ}) = \frac{42^\circ + m(\widehat{AQ})}{2}$

$\Rightarrow 48^\circ \times 2 = 42^\circ + m(\widehat{AQ})$

$\Rightarrow 96^\circ - 42^\circ = m(\widehat{AQ})$

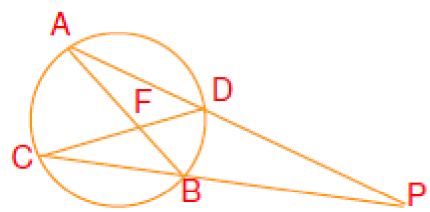
$\therefore m(AQ) = 54^\circ$

ANGLES AND ARCS DETERMINED BY LINES INTERSECTING OUTSIDE A CIRCLE

Theorem 5.9
The measure of the angle formed by the lines of two chords intersecting outside a circle is half the difference of the measure of the arcs they intercept

Theorem 5.10
The measure of an angle formed by a tangent and a secant drawn to a circle from a point outside the circle is equal to one-half the difference of the measures of the intercepted arcs.

Example 7: In figure below , from P secants $\overline{PA}$ and $\overline{PC}$ are drawn so that $m\,(\angle APC) = 30^{\circ}$; chords $\overline{AB}$ and $\overline{CD}$ intersect at F such that $m\,(A\angle FC\,) = 85^{\circ}$. Find the measure of arc AC, measure of arc BD and measure of $A\angle BC$.



Solution: Let $m(\widehat{AC}) = x \text{ and } m(\widehat{BD}) = y.$

Since $m(\angle AFC) = \frac{1}{2}m(\widehat{AC}) + \frac{1}{2}m(\widehat{BD})$

$\Rightarrow 85^{\circ} = \frac{1}{2}(x + y)$

$\Rightarrow x + y = 170^{\circ}. \;.\;.\;.\;.(1)$

Again as $m(\angle APC) = \frac{1}{2}m(\widehat{AC}) - \frac{1}{2}m(\widehat{BD})$

$\Rightarrow 30^{\circ} = \frac{1}{2}(x - y)$

$\Rightarrow x - y = 60^{\circ}. \;.\;.\;.\;.(2)$

Solving equation 1 and equation 2 simultaneously, we get

$$\begin{cases} x + y = 170^{\circ} \\ x - y = 60^{\circ} \end{cases}$$

$\Rightarrow 2x = 230^{\circ} \Longrightarrow x = 115^{\circ}$

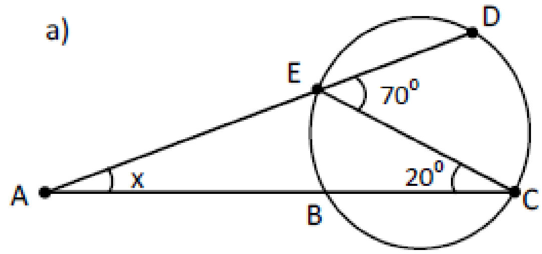
Substituting for x in equation 2

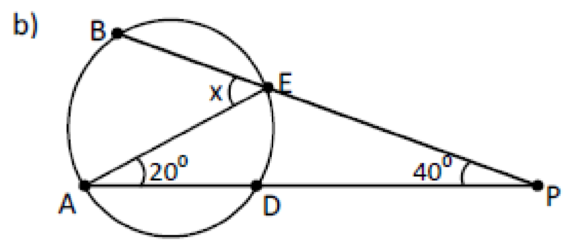
$\Rightarrow 115^{\circ} - y = 60^{\circ} \Longrightarrow y = 55^{\circ}$

Therefore, $m(\widehat{AC}) = 115^{\circ}$ and $m(\widehat{DB}) = 55^{\circ}$

$m(\angle ABC) = \frac{1}{2}m(\widehat{AC}) = \frac{1}{2}(115^{\circ}) = 57.5^{\circ}$

Example 8: Find the measures of the marked angles





Solution:

a. $70^\circ = \frac{1}{2}\left(m(\widehat{DC})\right) \Rightarrow 70^\circ \times 2 = \left(m(\widehat{DC})\right)$

$\Rightarrow 140^\circ = \left(m(\widehat{DC})\right)$

$\Rightarrow \angle ECB = \frac{1}{2}\left(m(\widehat{EB})\right)$

$\Rightarrow 20^\circ = \frac{1}{2}\left(m(\widehat{EB})\right)$

$\Rightarrow 20^\circ \times 2 = \left(m(\widehat{EB})\right)$

$\Rightarrow 40^\circ = \left(m(\widehat{EB})\right)$

$\therefore \angle DAC = \frac{1}{2}\left(m(\widehat{DC}) - m(\widehat{EB})\right)$ and $\angle x = \frac{1}{2}(140^\circ - 40^\circ) = 50^\circ$

b. $\angle EAD = \frac{1}{2}\left(m(\widehat{ED})\right) \Rightarrow 20^\circ = \frac{1}{2}\left(m(\widehat{ED})\right)$

$\Rightarrow 20^\circ \times 2 = \left(m(\widehat{ED})\right)$

$\Rightarrow 40^\circ = \left(m(\widehat{ED})\right)$

$\Rightarrow \angle BPA = \frac{1}{2}\left(m(\widehat{AB}) - m(\widehat{ED})\right)$

$\Rightarrow 40^\circ = \frac{1}{2}\left(m(\widehat{AB}) - 40^\circ\right)$

$\Rightarrow 40^\circ \times 2 = \left(m(\widehat{AB}) - 40^\circ\right)$

$\Rightarrow 80^\circ = \left(m(\widehat{AB}) - 40^\circ\right)$

$\Rightarrow 80^\circ + 40^\circ = m(\widehat{AB})$

$\Rightarrow \left(m(\widehat{AB})\right) = 120^\circ$

$\therefore \angle AEB = \frac{1}{2}\left(m(\widehat{AB})\right) = \frac{1}{2}(120^\circ)$ and $\angle x = \frac{1}{2}(120^\circ) = 60^\circ$.

5.4. REGULAR POLYGON

You may recall that a polygon all whose angles have equal measure and all of whose sides have equal length is called a regular polygon. i.e. a regular polygon is both equiangular and equilateral. In this section, we will study regular polygons by relating them to circles.

5.4.1. Perimeter of a Regular Polygon

You have studied how to find the length of a side (s) and perimeter (P) of a regular polygon with radius "*r*" and the number of sides "*n*" in Grade 9.

Theorem 5.11: Formulae for the length of side *s*, apothem *a*, perimeter *P* and area *A* of a regular polygon with *n* sides and radius *r*

1. $s = 2r\sin\frac{180^\circ}{n}$
2. $a = r\cos\frac{180^\circ}{n}$
3. $p = 2nr\sin\frac{180^\circ}{n}$
4. $A = \frac{1}{2}ap$

Example 9: Find the length of a side and perimeter of a regular quadrilateral with radius 6 units.

Solution: Given: $n = 4, \quad r = 6units$

Length of a side s:

$$s = 2r\sin\frac{180^\circ}{n}$$
$$\Rightarrow s = 2 \times 6 \times \sin\frac{180^\circ}{4}$$
$$\Rightarrow s = 12 \times \sin 45^\circ$$
$$\therefore s = 6\sqrt{2}\ units$$

perimeter p:

$$\Rightarrow p = 2nr\sin\frac{180^\circ}{n}$$
$$\Rightarrow p = 2 \times 4 \times 6\sin\frac{180^\circ}{4}$$
$$\Rightarrow p = 48 \times \sin 45^\circ$$
$$\therefore p = 24\sqrt{2}\ units$$

5.4.2. Area of a Regular Polygon

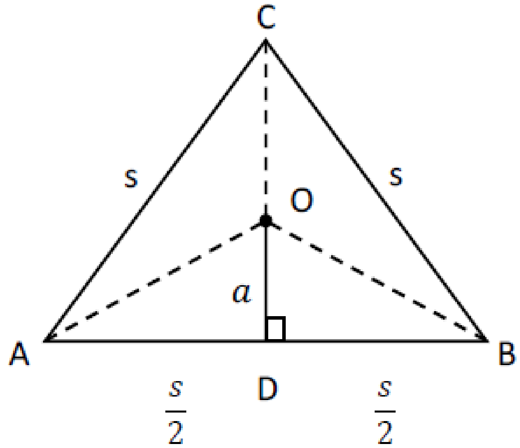
Theorem 5.12

The area *A* of a regular polygon with *n* sides and radius *r* is $A = \frac{1}{2}nr^2\sin\frac{360^\circ}{n}$.

This formula for the area of a regular polygon can be used to find the area of a circle. As the number of sides increases, the area of the polygon becomes closer to the area of the circle.

Example 10: Find the side, apothem, perimeter and area of an equilateral triangle of radius 6 units.

Solution: Given $n = 3 \quad and\ r = 6units$



1. $s = 2r \sin \frac{180^\circ}{n}$

   $\Rightarrow s = 2 \times 6 \times \sin \frac{180^\circ}{3}$

   $\Rightarrow s = 12 \times \sin 60^\circ$

   $\Rightarrow s = 6 \times \frac{\sqrt{3}}{2}\ units$

   $\Rightarrow s = 3\sqrt{3}\ units$

2. $p = 2nr \sin \frac{180^\circ}{n}$

   $\Rightarrow p = 2 \times 3 \times 6 \sin \frac{180^\circ}{3}$

   $\Rightarrow p = 36 \times \sin 60^\circ$

   $\Rightarrow p = 36 \times \frac{\sqrt{3}}{2}\ units = 18\sqrt{3}\ units$

3. $A = \frac{1}{2} nr^2 \sin \frac{3600^\circ}{n}$

   $\Rightarrow A = \frac{1}{2} \times 3 \times 6^2 \times \sin \frac{360^\circ}{3}$

   $\Rightarrow A = \frac{1}{2} \times 3 \times 36 \times \sin 120^\circ$

   $\Rightarrow A = 3 \times 18 \times \frac{\sqrt{3}}{2}\ units$

   $\Rightarrow A = 27\sqrt{3}\ units^2$

4. $a = r \cos \frac{180^\circ}{n}$

   $\Rightarrow a = 6 \times \sin \frac{180^\circ}{3}$

   $\Rightarrow a = 6 \times \sin 60^\circ$

   $\Rightarrow a = 6 \times \frac{1}{2}\ units = 3units$