Unit 6

The Pyramids at Giza in Egypt are among the best known pieces of architecture in the world. The Pyramid of Khafre was built as the final resting place of the Pharaoh Khafre and is about 136 m high.

MEASUREMENT

Unit Outcomes:

After completing this unit, you should be able to:

- *solve problems involving surface area and volume of solid figures.*
- *know basic facts about frustums of cones and pyramids.*

Main Contents:

6.1. Revision on Surface Areas and Volumes of Prisms and Cylinders

6.2. Pyramids, Cones and Spheres

6.3. Frustums of Pyramids and Cones

6.4. Surface Areas and Volumes of Composite Solids

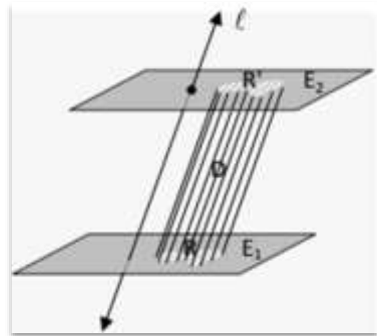
6.5. REVISION ON SURFACE AREAS AND VOLUMES OF PRISMS AND CYLINDERS

NOTE: Some important terms;

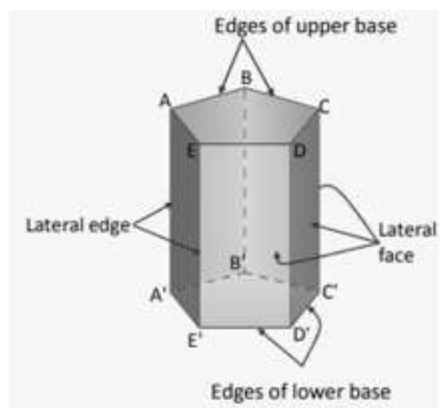
- For the cylinder D, the region R is called its lower base or simply base and R' is its upper base.
- The line ℓ is called its directrix and the perpendicular distance between E_1 and E_2 is the altitude of D.

If ℓ is perpendicular to E_1, then D is called a right cylinder, otherwise it is an oblique cylinder.

If R is a circular region, then D is called a circular cylinder.



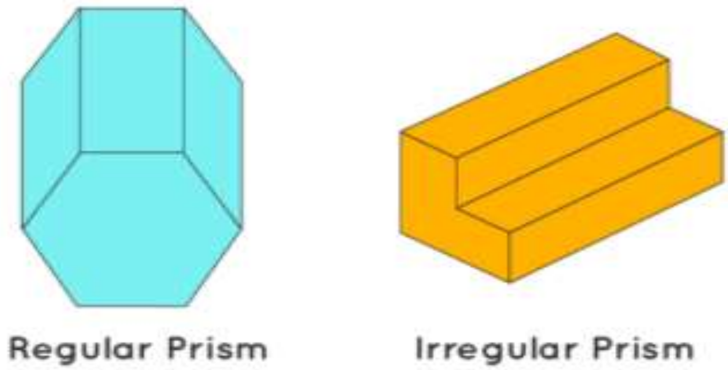
NOTE: In the prism shown below,



1. $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}, \overline{EA}$ are edges of the upper base. $\overline{A'B'}, \overline{B'C'}, \overline{C'D'}, \overline{D'E'}, \overline{E'A'}$ are edges of the lower base.
2. $\overline{AA'}, \overline{BB'}, \overline{CC'}, \overline{DD'}, \overline{EE'}$ are called lateral edges of the prism.
3. The parallelogram regions $ABB'A', BCC'B', AEE'A', DCC'D', EDD'E'$ are called lateral faces of the prism.
4. The union of the lateral faces of a prism is called its lateral surface.
5. The union of its lateral faces and its two bases is called its total surface or simply its surface.

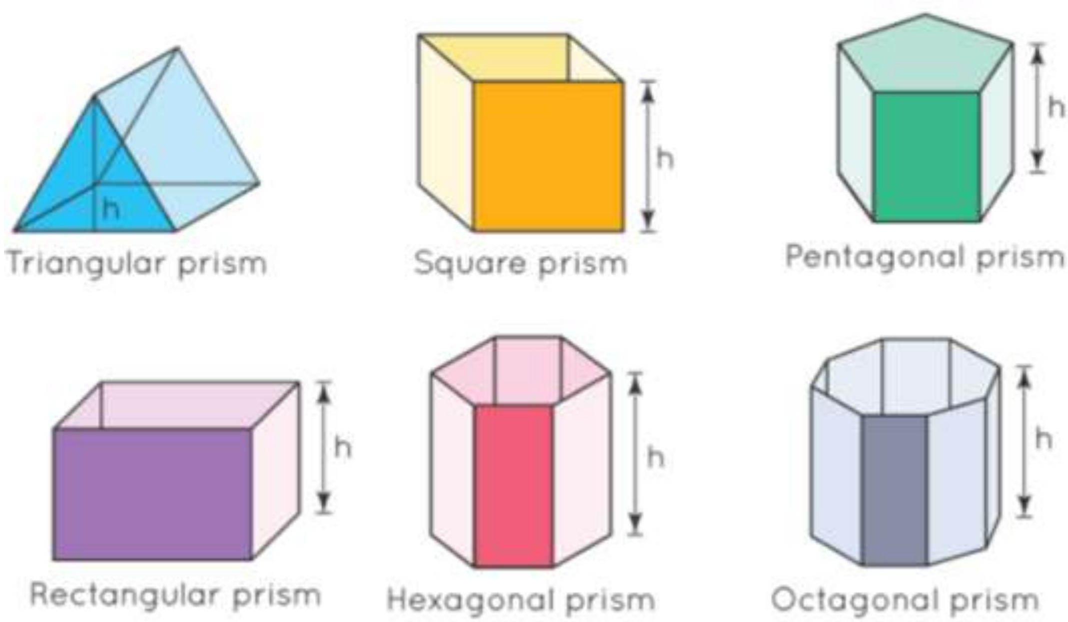
A prism is classified on the basis of the type of polygon base it has. There are two types of prisms in this category named as:

- Regular Prism: If the base of the prism is in the shape of a regular polygon, the prism is a regular prism.
- Irregular Prism: If the base of the prism is in the shape of an irregular polygon, the prism is an irregular prism.



A prism is named on the basis of the shape obtained by the cross-section of the prism. They are further classified as:

- Triangular Prism: A prism whose bases are triangle in shape is considered a triangular prism.
- Square Prism: A prism whose bases are square in shape is considered a square prism.
- Rectangular prism: A prism whose bases are rectangle in shape is considered a rectangular prism (a rectangular prism is cuboidal in shape).
- Pentagonal Prism: A prism whose bases are pentagon in shape is considered a pentagonal prism.
- Hexagonal Prisms: A prism whose bases are hexagon in shape is considered a hexagonal prism.
- Octagonal Prism: A prism whose bases are octagon in shape is considered an octagonal prism.
- Trapezoidal Prism: A prism whose bases are trapezoid in shape is considered a trapezoidal prism.



If we denote the lateral surface area of a prism by A_L, the area of the base by A_B, altitude h and the total surface area by A_T, then:

$A_L = Ph$; where P is the perimeters of the base and h is the height of the prism.

$A_T = 2A_B + A_L$

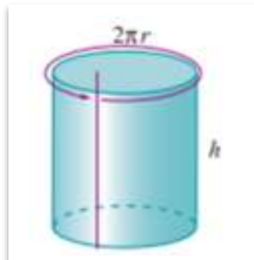
- Similarly, the lateral surface area (A_L) of a right circular cylinder is equal to the product of the circumference of the base and altitude (h) of the cylinder. That is,

  $A_L = 2\pi rh$, where r is the radius of the base of the cylinder.

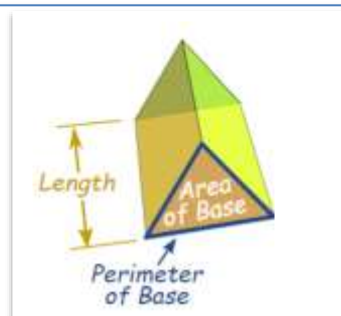
- The total surface area A_T is equal to the sum of the areas of the bases and the lateral surface area. That is,

  $A_T = A_L + 2A_B$

  $A_T = 2\pi rh + 2\pi r^2 = 2\pi r(h+r)$



- The volume (V) of any prism equals the product of its base area (A_B) and altitude (h). That is, $V = A_B h$

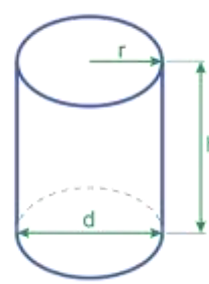


- Volume of a right circular cylinder

  The volume (V) of a circular cylinder is equal to the product of the base area (A_B) and its altitude (h). That is,

  $V = A_B h$

  $V = \pi r^2 h$, where r is the radius of the base.



Example1: Bontu has given a cylinder of surface area 1728π square units. Find the height of the cylinder if the radius of the base of the circle is 24 units.

Solution: The surface of the cylinder, $A_L = 1728\pi$

Using the total surface area, $A_T = 2\pi r(h+r)$:

$$1728\pi = 2\pi \times 24 \times (h+24)$$

$$\Rightarrow h+24 = \frac{1728\pi}{48\pi} = 36$$

$$\Rightarrow h = 12 \text{ units}$$

So, the height of the cylinder is 12 units .

6.2. PYRAMIDS, CONES AND SPHERES

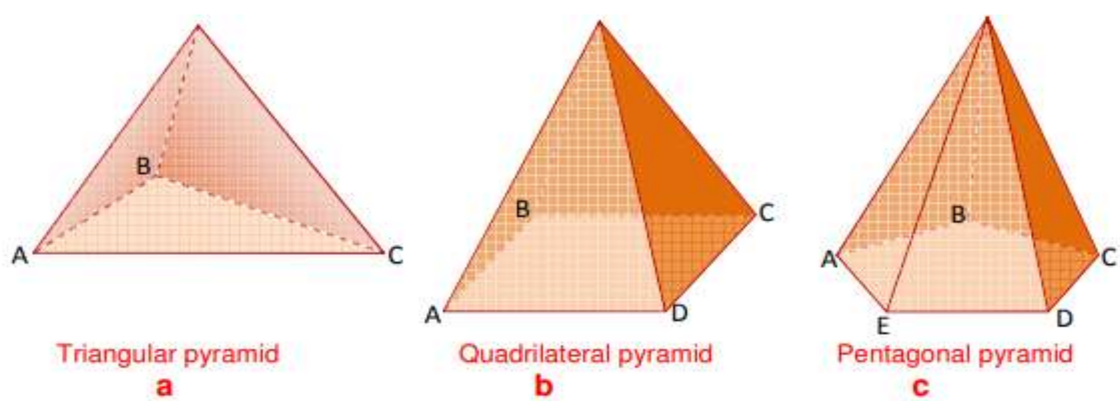
Do you remember what you learnt about pyramids, cones and spheres in your previous grades? Can you give some examples of pyramids, cones and spheres from real life?

1. PYRAMIDS

Definition 6.2:

A pyramid is a solid figure formed when each vertex of a polygon is joined to the same point not in the plane of the polygon.

Examples:



NOTE:

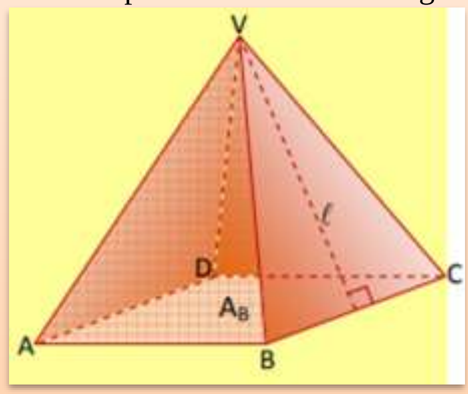
a. The altitude of a pyramid is the length of the perpendicular from the vertex to the plane containing the base.

b. The slant height of a regular pyramid is the altitude of any of its lateral faces.

c. A regular pyramid is a pyramid whose base is a regular polygon and whose altitude passes through the center of the base.

NOTE:

The lateral surface area of a regular pyramid is equal to half the product of its slant height and the perimeter of the base. That is,



$$A_L = \frac{1}{2}P\ell,$$

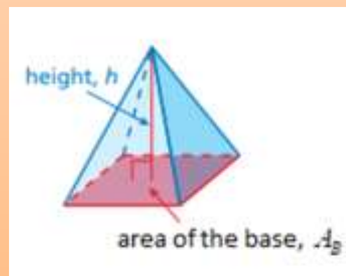
where A_L denotes the lateral surface area;

P denotes of the perimeter of the base;

ℓ denotes the slant height.

The total surface area (A_T) of a pyramid is given by $A_T = A_B + A_L = A_B + \frac{1}{2}P\ell$,

where A_B is area of the base.



The volume V of a pyramid is one-third the product of the area of the base and the height of the pyramid.

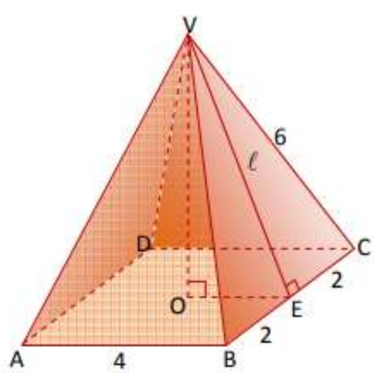
That is, $V = \frac{1}{3}A_B h$

Example 2: A regular pyramid has a square base whose side is $4cm$ long. The lateral edges are $6cm$ each.

a. What is its slant height?
b. What is the lateral surface area?
c. What is the total surface area?
d. What is the volume of the pyramid?

Solution:

Consider the following figure,



a. $(VE)^2 + (EC)^2 = (VC)^2$

$\Rightarrow \ell^2 + 2^2 = 6^2$

$\Rightarrow \ell^2 = 32$

$\Rightarrow \ell = 4\sqrt{2}cm$

Therefore, the slant height is $4\sqrt{2}cm$.

b. There are 4 isosceles triangles.

Therefore,

$$A_L = 4 \times \frac{1}{2} BC \times VE$$

$$= 4 \times \left(\frac{1}{2} \times 4 \times 4\sqrt{2}\right) = 32\sqrt{2}cm^2 \text{ or}$$

$$A_L = \frac{1}{2} P\ell = \frac{1}{2}(4+4+4+4) \times 4\sqrt{2} = 8 \times 4\sqrt{2} = 32\sqrt{2}cm^2$$

c. $A_T = A_B + A_L = 32\sqrt{2} + 4 \times 4 = 16\left(2\sqrt{2}+1\right)cm^2$.

d. $(VO)^2 + (OE)^2 = (VE)^2$

$\Rightarrow h^2 + 2^2 = \left(4\sqrt{2}\right)^2$

$\Rightarrow h^2 + 4 = 32$

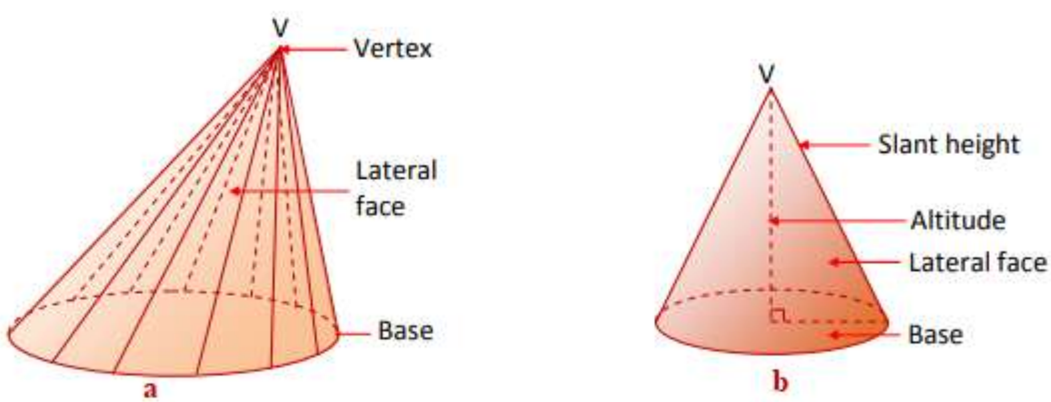
$\Rightarrow h^2 = 28 \Rightarrow h = 2\sqrt{7}cm$

Therefore, $V = \frac{1}{3} A_B h = \frac{1}{3} \times (4 \times 4) \times 2\sqrt{7} = \frac{32}{3}\sqrt{7}cm^3$

2. CONES

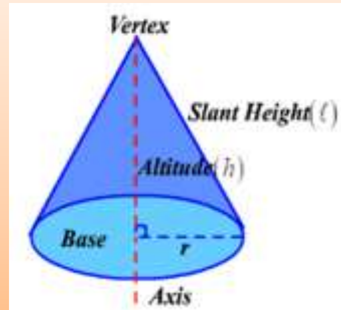
Definition 6.3:

The solid figure formed by joining all points of a circle to a point not on the plane of the circle is called a cone.



NOTE:

- The lateral surface area of a right circular cone is equal to half the product of its slant height and the circumference of the base. That is,



$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(2\pi r)\ell = \pi r\ell\ ;$$

$$\ell = \sqrt{h^2 + r^2}$$

where P denotes the lateral surface area;

r stands for the base radius;

l denotes the slant height;

h for the altitude.

- The total surface area is equal to the sum of the area of the base and the lateral surface area. That is, $A_T = A_B + A_L = \pi r\ell + \pi r^2 = \pi r(\ell + r)$, where A_B is area of the base.
- The volume V of a circular cone is one-third the product of its base area and its altitude. That is, $V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$

Example 3: The base radius and height of a right circular cone is 7cm and 24cm. Find its curved surface area, total surface area and volume.

Solution:

Here, $r = 7cm$ and $h = 24cm$

- ✓ So, slant height $\ell = \sqrt{r^2 + h^2}$

$\Rightarrow \ell = \sqrt{7^2 + 24^2} = 25cm$

- ✓ Thus, curved surface area $A_L = \pi r\ell = \pi \times 7cm \times 24cm = 168\pi cm^2$
- ✓ Total surface area $A_T = \pi r(\ell + r)$

$$\Rightarrow A_T = \pi \times 7cm \times (25cm + 7cm) = 224\pi cm^2$$

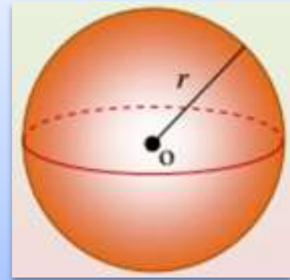
- ✓ The volume $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \times (7cm)^2 \times 24cm = 392\pi cm^3$

3. SPHERES

Definition 6.4:

A sphere is a closed surface, all points of which are equidistant from a point called the centre.

Most familiar examples of a sphere are baseball, tennis ball, bowling, and so forth.

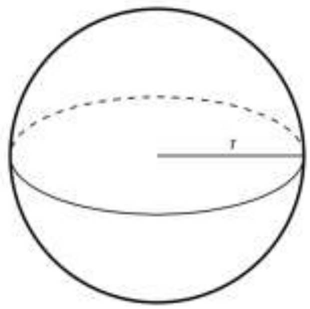


The surface area (A) and the volume (V) of

a sphere of radius r are given by

$$A = 4\pi r^2$$

$$V = \frac{4}{3}\pi r^3$$



Example 4: The diameter of a sphere is 13.5m. Find its surface area and volume.

Solution:

Here, $d = 13.5m$

- Surface area $A = 4\pi r^2 = 4\pi\left(\frac{d}{2}\right)^2$

  $$\Rightarrow A = \pi d^2 = \pi(13.5m)^2 = 182.25\pi m^2$$

- Volume of sphere $V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3$, **(Why?)**

  $$\Rightarrow V = \frac{\pi}{6}(13.5m)^3 = 410.0625\pi m^3$$

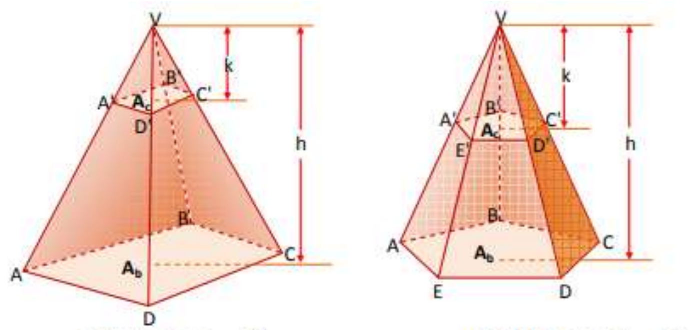
6.3. FRUSTUMS OF PYRAMIDS AND CONES

Definition 6.5:

If a pyramid or a cone is cut by a plane parallel to the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross section of the pyramid (or the cone).

Theorem 6.1:

In any pyramid, the ratio of the area of a cross-section to the area of the base is $\frac{k^2}{h^2}$, where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross-section.



$$\frac{A_c}{A_b} = \frac{\text{area}(A'B'C'D')}{\text{area}(ABCD)} = \frac{k^2}{h^2} \qquad \frac{\text{area}(A'B'C'D'E')}{\text{area}(ABCD)} = \frac{k^2}{h^2}$$

Example 5: The area of the base of a pyramid is $90cm^2$. The altitude of the pyramid is $12cm$. What is the area of a horizontal cross-section $4cm$ from the vertex?

Solution:

Let A_c be the area of the cross-section, and A_b the base area.

Then, $\frac{A_c}{A_b} = \frac{k^2}{h^2} \Rightarrow \frac{A_c}{90} = \frac{4^2}{12^2}$

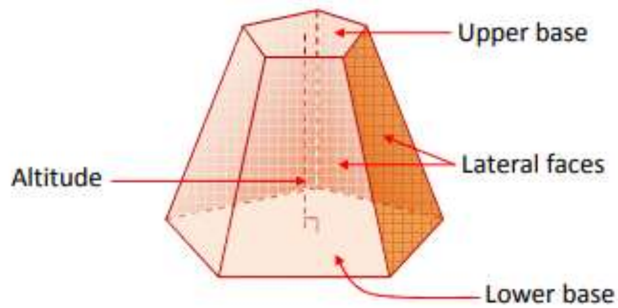
$$\therefore A_c = \frac{90 \times 16}{144} cm^2 = 10cm^2$$

6.3.1. Frustum of a pyramid

Definition 6.6:

A frustum of a pyramid is a part of the pyramid included between the base and a plane parallel to the base.

The altitude of a frustum of a pyramid is the perpendicular distance between the bases.



NOTE:

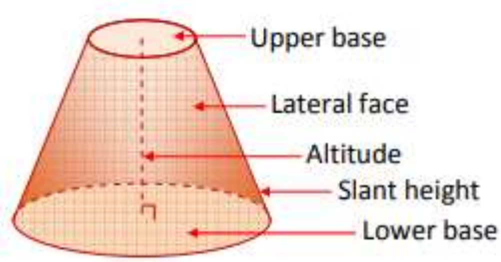
1. The altitude of a frustum of a pyramid is the perpendicular distance between the bases.
2. The lateral faces of a frustum of a pyramid are trapeziums.
3. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
4. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
5. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces.

6.3.2. Frustum of a cone

Definition 6.7:

A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.

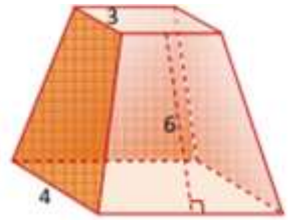
A frustum of a cone is a part of the cone included between the base and a horizontal cross-section made by a plane parallel to the base.



Example 6: The lower base of the frustum of a regular pyramid is a square $4cm$ long, the upper base is $3cm$ long. If the slant height is $6cm$, find its lateral surface area.

Solution:

As shown in figure below, each lateral face is a trapezium,



the area of each lateral face is

$$A_L = \frac{1}{2}\times h(b_1+b_2) = \frac{1}{2}\times 6(3+4) = 21cm^2$$

Theorem6.2:

The lateral surface area (A_L) of a frustum of a regular pyramid is equal to half the product of the slant height (ℓ) and the sum of the perimeter (P) of the lower base and the perimeter (P') of the upper base.

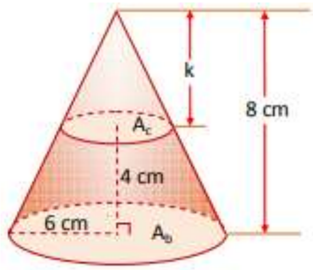
That is, $A_L = \frac{1}{2}\ell(P+P')$

Theorem 6.3:

For a frustum of a right circular cone with altitude h and slant height ℓ, if the circumferences of the bases are c and c', then the lateral surface area of the frustum is given by

$$A_L = \frac{1}{2}\ell(c+c') = \frac{1}{2}\ell(2\pi r+2\pi r') = \ell\pi(r+r')$$

Example 7: A frustum of height $4cm$ is formed from a right circular cone of height $8cm$ and base radius $6cm$ as shown in below. Calculate the lateral surface area of the frustum.



Solution:

Let A_b, A_c and A_L stand for area of the base of the cone, area of the cross-section and lateral surface area of the frustum, respectively.

$$\frac{\text{Area of cross-section}}{\text{Area of the base}} = \left(\frac{k}{h}\right)^2$$

$$\Rightarrow \frac{A_c}{A_b} = \left(\frac{4}{8}\right)^2, \text{ since } k = 8cm - 4cm = 4cm$$

$$\Rightarrow \frac{A_c}{36\pi} = \frac{1}{4} \left(\text{area of the base} = \pi r^2 = \pi \times 6^{\,2} = 36\pi\right)$$

$$\Rightarrow A_c = \frac{1}{4} \times 36\pi = 9\pi cm^2$$

$$\Rightarrow A_c = \pi (r')^2$$

$$\Rightarrow \pi (r')^2 = 9\pi cm^2 \Rightarrow r' = 3cm$$

Slant height of the bigger cone is: $\ell = \sqrt{h^2 + r^2} = \sqrt{8^2 + 6^2} = 10cm$

Slant height of the smaller cone is: $\ell = \sqrt{k^2 + (r)^2} = \sqrt{4^2 + 3^2} = 5cm$

Now the lateral surface area of:

the smaller cone $= \pi r'\ell' = \pi(3cm) \times 5cm = 15\pi cm^2$

the bigger cone $= \pi r\ell = \pi(6cm) \times 10cm = 60\pi cm^2$

Hence, the area of the lateral surface of the frustum is:

$A_L = 60\pi cm^2 - 15\pi cm^2 = 45\pi cm^2$.

The lateral surface (curved surface) of a frustum of a circular cone is a trapezium whose parallel sides (bases) have lengths equal to the circumference of the bases of the frustum and whose height is equal to the height of the frustum.

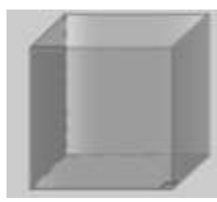
Summary

1. Prism

$A_L = Ph$

$A_T = 2A_b + A_L$

$V = A_b h$

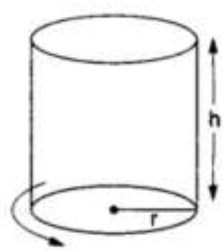


2. Right circular cylinder

$A_L = 2\pi rh$

$A_T = 2\pi r^2 + 2\pi rh = 2\pi r(r + h)$

$V = \pi r^2 h$

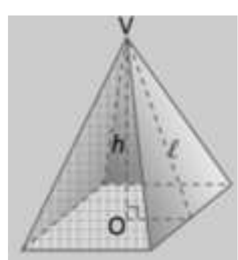


3. Regular pyramid

$A_L = \frac{1}{2} P\ell$

$A_T = A_b + \frac{1}{2} P\ell$

$V = \frac{1}{3} A_b h$

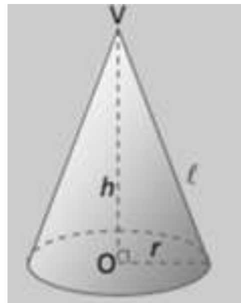


4. Right circular cone

$$A_L = \pi r \ell$$

$$A_T = \pi r^2 + \pi r \ell = \pi r \left(r + \ell \right)$$

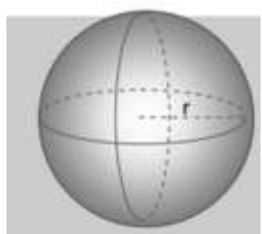
$$V = \frac{1}{3} \pi r^2 h$$



5. Sphere

$$A = 4\pi r^2$$

$$V = \frac{4}{3} \pi r^3$$

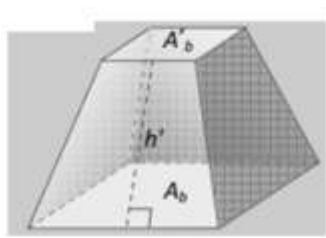


6. Frustum of a pyramid

$$A_L = \frac{1}{2} \ell \left(P + P' \right)$$

$$A_T = \frac{1}{2} \ell = \pi r \left(P + P' \right) + A_b + A'_b$$

$$V = \frac{1}{3} h' \left(A_b + A'_b + \sqrt{A_b A'_b} \right)$$



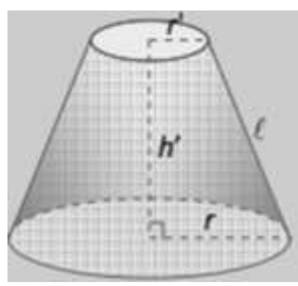
7. Frustum of a pyramid

$$A_L = \frac{1}{2} \ell \left(2\pi r + 2\pi r' \right) = \ell \pi \left(r + r' \right)$$

$$A_T = \frac{1}{2} \ell \left(2\pi r + 2\pi r' \right) + \pi r^2 + \pi \left(r' \right)^2$$

$$= \ell \pi \left(r + r' \right) + \pi \left(r^2 + \left(r' \right)^2 \right)$$

$$V = \frac{1}{3} h' \pi \left(r^2 + \left(r' \right)^2 + r r' \right)$$



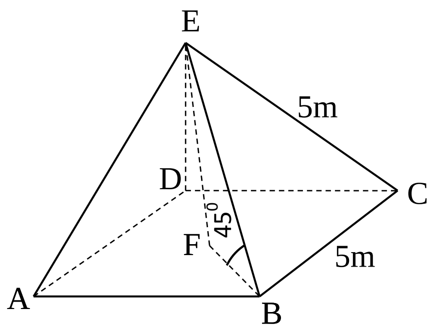
PRACTICE QUESTIONS ON UNIT 6

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. The slant height of a right circular cone is $8cm$. If the angle between the slant height and the height at the vertex of the cone is 30^0, then what is the volume of the cone?

   A. $\frac{128\pi}{3}cm^3$ B. $64\pi cm^3$ $\frac{64\sqrt{3}\pi}{3}cm^3$ C. $128\pi cm^3$ D. $\frac{64\sqrt{3}\pi}{3}cm^3$

2. The right pyramid in the figure below has square base of dimension 5m by 5m. If the edge $\overline{EB}$ makes angle 45^0 with the diagonal $\overline{DB}$, what s the volume of the pyramid?



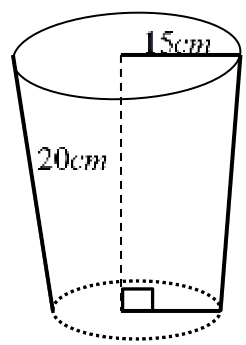
   A. $\frac{125\sqrt{3}}{6}$ m^3

   B. $\frac{125\sqrt{2}}{6}$ m^3

   C. 125 m^3

   D. 25 m^3

3. The figure shown below is a container made from an inverted frustum a right circular cone. The radius of its lower base is 10 cm and that of the upper base is 15 cm.



   If the height of this container is 20 cm, then which one of the following is its volume?

   A. 5000π cm^3 B. $\frac{9500}{3}\pi$ cm^3 C. 7500π cm^3 D. $\frac{2500}{3}\pi$ cm^3

4. The volume of a pyramid that has height of 8 in and a rectangular base of dimension 6 in by 4in is

   A. 576 in^3 B. 192 in^3 C. 96 in^3 D. 64 in^3

5. The diameter of the base and the height of a circular cone are found to be *a* and *2b* units long respectively. What is the formula for the volume V of the cone?

   A. $V = \frac{2}{3}\pi a^2 b$ B. $V = \frac{1}{3}\pi a^2 b$ C. $V = \frac{1}{6}\pi a^2 b$ D. $V = \frac{4}{3}\pi a^2 b$