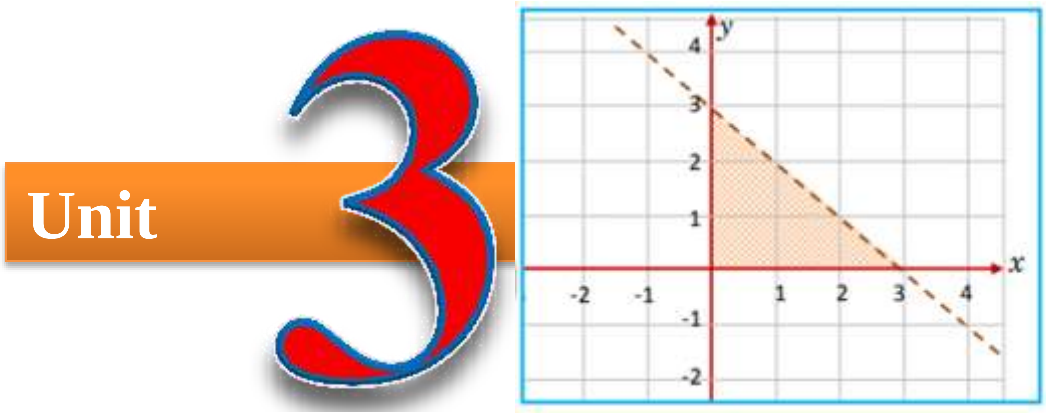


SOLVING INEQUALITIES

Unit Outcomes:

After completing this unit, you should be able to:

- *Know and apply methods and procedure in solving problems on inequalities involving absolute value.*
- *Solve quadratic inequalities.*

Main Contents:

3.1. Solving Linear inequalities in one variuable

3.2. Inequalities involving absolute value

3.3. Quadratic inequalities

3.1. SOLVING LINEAR INEQUALITIES IN ONE VARIABLE

We know that a linear equation in one variable can be expressed as $ax + b = 0$. A linear inequality in one variable can be written in one of the following forms $ax+b<0$, $ax+b>0$, $ax+b\leq 0$ or $ax+b\geq 0$ in each form $a \neq 0$.

Properties of inequalities:

For any three real numbers a, b and c:

1. **The addition property of inequalities**
   - if $a < b$, then $a + c < b + c$.
   - if $a < b$, then $a - c < b - c$.
2. **The positive multiplication property of inequalities**
   - if $a < b$ *and c is positive*, then $a \times c < b \times c$.
   - if $a < b$ *and c is positive*, then $\frac{a}{c} < \frac{b}{c}$.
3. **The negative multiplication property of inequalities**
   - if $a < b$ *and c is negative*, then $a \times c > b \times c$.
   - if $a < b$ *and c is negative*, then $\frac{a}{c} > \frac{b}{c}$.

Example 1: Solve the following inequalities:

a. $4(\mathrm{x}+1)+2 \geq 3\mathrm{x}+6$

b. $8x+3 > 3(2x+1)+x+5$

c. $2\mathrm{x}-11 < -3(\mathrm{x}+2)$

Solution:

a. $4(\mathrm{x}+1)+2 \geq 3\mathrm{x}+6$

$4x+4+2 \geq 3x+6$

$4x+6 \geq 3x+6$

$4x-3x \geq 6-6$

$x \geq 0$

Therefore $s.s = \{x: x \geq 0\} = [0, \infty)$

b. $8x+3 > 3(2x+1)+x+5$

$8x+3 > 6x+3+x+5$

$8x+3 > 7x+8$

$8x-7x > 8-3$

$x > 5$

Hence the $s.s = \{x: x > 5\} = (5, \infty)$

c. $2\mathrm{x}-11 < -3(\mathrm{x}+2)$

$2x-11 < -3x-6$

$2x+3x < -6+11$

$5x < 5$

$x < 1$

Therefore $s.s = \{x: x < 1\} = (-\infty, 1)$

3.2. INEQUALITIES INVOLVING ABSOLUTE VALUE

Definition 3.1:
If x is a real number, then the absolute value of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, if\ x \geq 0 \\ -x, if\ x < 0 \end{cases}$$

Example 2:

a. $|21| = 21$ because $21 > 0$

b. $\left|\frac{-2}{3}\right| = -\left(\frac{-2}{3}\right) = \frac{2}{3}$ because $-\frac{2}{3} < 0$

Theorem 3.1: Solution of the equation $|x| = a$

For any real number a, the equation $|x| = a$ has:

i. Two solutions $x = a$ and $x = -a, if\ a > 0$

ii. One solution ,$x = 0, if\ a = 0$ and

iii. No solution, if $a < 0$.

Example 3: Solve each of the following absolute value equation

a. $|2x + 1| = x + 5$

b. $3|x + 4| - 2 = 7$

Solution:

a. $2x + 1 = x + 5$ or $2x + 1 = -(x + 5)$

$\Rightarrow 2x - x = 5 - 1$ or $2x + 1 = -x - 5$

$\Rightarrow x = 4$ or $2x + x = -5 - 1$

$\Rightarrow x = 4$ or $3x = -6 \Rightarrow x = -2$

Therefore $s.s = \{4, -2\}$

b. $3|x + 4| - 2 = 7$

$\Rightarrow 3|x + 4| - 2 + 2 = 7 + 2$ . . . .add 2 to each side

$\Rightarrow 3|x + 4| = 9$

$\Rightarrow \frac{3|x+4|}{3} = \frac{9}{3}$ . . . . .divide both sides by 3

$\Rightarrow |x + 4| = 3$

$\Rightarrow x + 4 = 3$ or $x + 4 = -3$

$\Rightarrow x = 3 - 4$ or $x = -3 - 4$

$\Rightarrow x = -1$ or $x = -7$

Therefore $s.s = \{-1, -7\}$

3.3. QUADRATIC INEQUALITIES

In grade 9 mathematics, you have learned how to solve quadratic equations of the form $ax^2+bx+c=0$, $a\neq 0$ and a, b and $c\in \mathrm{R}$.

Definition 3.2:
An inequality that can be reduced to any one of the following forms
$ax^2+bx+c\leq 0$ or $ax^2+bx+c<0$ or
$ax^2+bx+c\geq$ or $ax^2+bx+c>0$
where a, b and c are constants and $a\neq 0$, is called a quadratic inequality.

Solving quadratic inequalities using product properties

Product properties:

1. $m\times n>0$ *if and only if*
   i. $m>0$ *and* $n>0$ or ii. $m<0$ *and* $n<0$
2. $m\times n<0,$ *if and only if*
   i. $m>0$ *and* $n<0$ or ii. $m<0$ *and* $n>0$

Examples 4: Solve each of the following inequalities

a. $(2x+7)(3x-2)>0$ b. $3x^2+4x+1\geq 0$

Solution:

a. $(2x+7)(3x-2)>0$

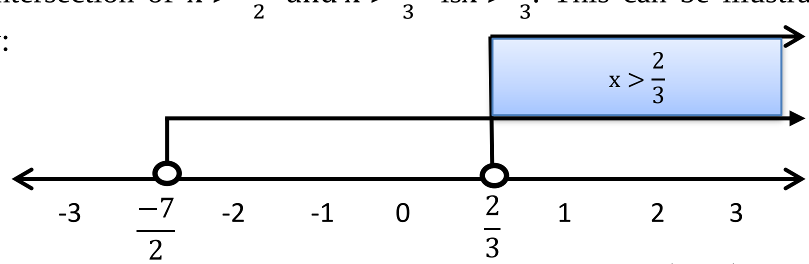
By product property, $(2x+7)(3x-2)>0$ is positive if either both factors are positive or both factors are negative.

Now, consider the following cases:

Case 1: When both factors are positive

$\Rightarrow 2x+7>0$ and $3x-2>0$
$\Rightarrow 2x>-7$ and $3x>2$
$\Rightarrow x>\frac{-7}{2}$ and $x>\frac{2}{3}$

The intersection of $x>\frac{-7}{2}$ and $x>\frac{2}{3}$ is $x>\frac{2}{3}$. This can be illustrated on the number line as shown below:



The solution set for this first case is $s.s_1=\{x: x>\frac{2}{3}\}=\left(\frac{2}{3},\infty\right)$

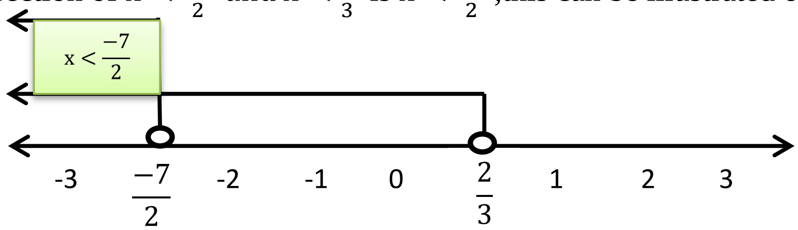
Case 2: When both factors are negative

$\Rightarrow 2x + 7 < 0$ and $3x - 2 < 0$

$\Rightarrow 2x < -7$ and $3x < 2$

$\Rightarrow x < \frac{-7}{2}$ and $x < \frac{2}{3}$

The intersection of $x < \frac{-7}{2}$ and $x < \frac{2}{3}$ is $x < \frac{-7}{2}$,this can be illustrated on the number line as below:



The solution set for the second case is $s.s_2 = \{x: x < -\frac{7}{2}\} = \left(-\infty, -\frac{7}{2}\right)$

Therefore, the solution set of $(2x + 7)(3x - 2) > 0$ is $s.s = s.s_1 \cup s.s_2 = \left(-\infty, -\frac{7}{2}\right) \cup \left(\frac{2}{3}, \infty\right)$

b. $3x^2 + 4x + 1 \geq 0$

First factorize $3x^2 + 4x + 1$ as$(3x + 1)(x + 1)$. So $3x^2 + 4x + 1 = (3x+1)(x+1) \geq 0$.

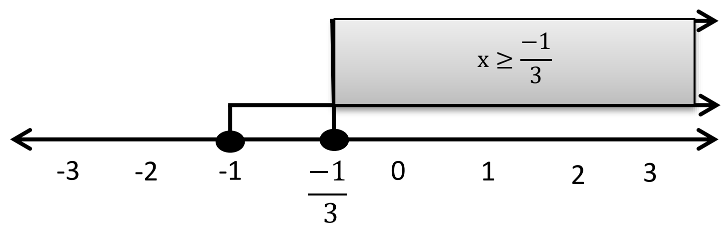
Case 1: When both factors are positive

$\Rightarrow 3x + 1 \geq 0$ and $x + 1 \geq 0$

$\Rightarrow 3x \geq -1$ and $x \geq -1$

$\Rightarrow x \geq \frac{-1}{3}$ and $x \geq -1$

The intersection of $x \geq \frac{-1}{3}$ and $x \geq -1$ is $x \geq -\frac{1}{3}$.this can be illustrated on the number line as below.



The solution set for this first case ıs $s.s_1 = \{x: x \geq -\frac{1}{3}\} = [\frac{-1}{3}, \infty)$.

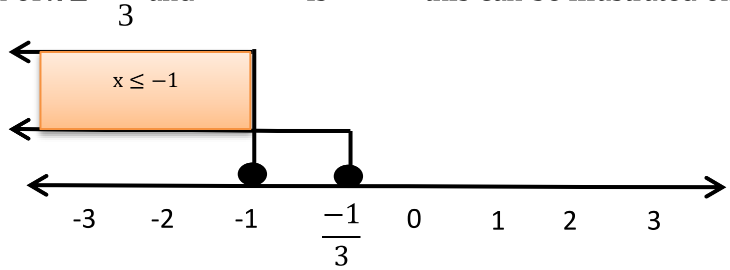
Case 2: When both factors are negative

$\Rightarrow 3x + 1 \leq 0$ and $x + 1 \leq 0$

$\Rightarrow 3x \leq -1$ and $x \leq -1$

$\Rightarrow x \leq \frac{-1}{3}$ and $x \leq -1$

The intersection of $x \leq -\frac{1}{3}$ and $x < -1$ is $x \leq -1$ this can be illustrated on the number line as below.



The solution set for the second case is $s.s_2 = \{x: x \leq -1\} = (-\infty, -1]$.

Therefore, the solution set of $3x^2 + 4x + 1 \geq 0$ is $s.s = s.s_1 \cup s.s_2 = (-\infty, -1] \cup [-\frac{1}{3}, \infty)$.

PRACTICE QUESTIONS ON UNIT 3

CHOOSE THE BEST ANSWER FROM THE GIVEN ALTERNATIVES

1. The solution of the inequality $|2x-5|>3$?

   A. $\{x:1\le x\le 4\}$ C. $\{x:x<1 \text{ or } x>4\}$
   B. $\{x:2\le x\le 8\}$ D. $\{x:x<2 \text{ or } x>8\}$

2. Which of the following is the solution set of the inequality $\frac{37-2x}{3}+x\le\frac{3x-8}{4}-9$?

   A. $\{x\in\Re : x\le 56\}$ C. ∅
   B. $\{x\in\Re : x\ge -56\}$ D. $\{x\in\Re : 56\le x\}$

3. The solution set of $-6<11x+3\le 3$ is

   A. $\left\{x:-\frac{9}{11}\le x<\frac{6}{11}\right\}$ C. $\left\{x:-\frac{9}{11}\le x\le 0\right\}$
   B. $\left\{x:-\frac{9}{11}\le x<0\right\}$ D. $\left\{x:-\frac{9}{11}\le x\le\frac{6}{11}\right\}$

4. The solution set of the inequality $x^2+7x+12\ge 0$ is:

   A. $\{x:-3\le x\le -4\}$ C. $\{x:x\le 3 \text{ or } x\ge 4\}$
   B. $\{x:3\le x\le 4\}$ D. $\{x:x\le -4 \text{ or } x\ge -3\}$

5. What is the solution set of the inequality $4x^2+4x+1>0$?

   A. $\left(-\frac{1}{2},\frac{1}{2}\right)$ B. $\mathbb{R}\setminus\left(-\frac{1}{2}\right)$ C. $\mathbb{R}$ D. $\left(-\infty,-\frac{1}{2}\right)\cup\left(\frac{1}{2},\infty\right)$

6. If $|3x+2|<1$, then x belongs to the interval:

   A. $\left(-1,-\frac{1}{3}\right)$ B. $(-\infty,-1)$ C. $\left[-1,-\frac{1}{3}\right]$ D. $\left(-\frac{1}{3},\infty\right)$

7. Which of the following is the least integral value of k such that $(k-2)x^2+8x+k+4\ge 0$ for all $x\in\mathbb{R}$?

   A. 5 B. 4 C. 3 D. k