# PRE-UNIVERSITY REMEDIAL PROGRAM FOR THE 2014 E.C. ESSLCE EXAMINEES

# MATHEMATICS MODULE

**Wachemo University**

**College of Natural and Computational Sciences**

**Department of Mathematics**

**Compiled Mathematics Module for Social Science Pre-University Remedial students**

**Prepared by**

**Tola Bekene (Ph.D.)**

**16/03/2023, Hosanna, Ethiopia**

Contents

**Chapter 1: Solving Equations and Inequalities (12 hrs)** ........ 4
1.1. Equations involving exponents and radicals ........ 4
**1.2. System of linear equations in two variables** ........ 4
**1.3. Equations Involving Absolute Value** ........ 10
1.4. Inequalities involving absolute values ........ 12
**1.5. System of linear inequalities in two variables** ........ 14
1.6. **Quadratic equations and inequalities** ........ 17
**Chapter 2: Relations and Functions (10 hrs)** ........ 32
2.1. Definition and examples of relations ........ 32
2.2. Definition and examples of functions ........ 34
2.3. Classification of functions (one to one, onto, even and odd) and inverse of functions ........ 36
**2.4. Operations on functions and composition of functions** ........ 38
2.5. Types of functions ........ 39
2.5.1. Polynomial functions ........ 39
2.5.1.1. Theorems on polynomial functions (Division, remainder, factor ........ 41
and location theorems) ........ 41
2.5.1.2. Graphs of polynomial functions ........ 44
2.5.1.3. Rational functions and their graphs ........ 44
2.5.2. Exponential functions and their graphs ........ 47
2.5.3. Logarithmic functions, graphs of logarithmic functions and equations involving logarithms ........ 51
2.5.5. Power function, Absolute value function, Signum function and greatest integer function ........ 54
2.5.6. Trigonometric (Sine, Cosine and Tangent) functions and their graphs. ........ 56
**Chapter 3: Geometry and Measurement (10 hrs)** ........ 58
3.1. Theorems on triangles ........ 58
3.2. Special quadrilaterals ........ 60
3.3. Circles ........ 60
3.4. Regular Polygons ........ 61
3.5. Congruency and Similarity ........ 64
3.6. Areas of Triangle and parallelogram ........ 65
3.7. Surface area and volume of solid figures (Prism, Cylinder, Cone and Sphere) ........ 65
3.8. Frustum of pyramids and cones ........ 65
**Chapter 4: Coordinate Geometry (8 hrs)** ........ 65
4.1.Distance between points in a plane ........ 65
4.2. Division of a line segment ........ 66
4.3. Equation of a straight line /gradient line ........ 69
4.4. Distance between a point and a line and between two lines ........ 71

4.5. Conic Sections 72
**Chapter 5: Statistics and Probability (12 hrs)** 76
5.1. Statistics 76
5.1.1. Types of data 77
5.1.2. Measures of *location* for ungrouped and group data 77
5.1.3. Measures of *Dispersion* for ungrouped and grouped data 77
5.2. Probability 77
5.2.1. Permutation and combination 78
5.2.2. Binomial theorem 80
5.2.3. Random experiment and their outcomes 80
5.2.4. Events 81
**Chapter 6: Limit and Continuity (8 hrs)** 82
6.1. Limit of functions 82
6.2. Continuity of functions 88
**Chapter 7: Derivative and its Application (11 hrs)** 92
7.1. Definition of derivative and its geometric interpretation 92
7.2. Rules of differentiation 93
7.3. Derivatives of special functions (exponential, logarithmic and trigonometric) 94
7.4. Extreme values of functions Reading assignment 97
**Chapter 8: Integration and its Application (11 hrs)** 99
8.1. Integration as a reverse process of derivative 99
8.3. Definite integral and Fundamental theorem of calculus 100
8.4. Area of a region under a curve and between two curves 100
**Chapter 9: Introduction to Linear Programming (6 hrs)** 100
9.1. Graphical solutions of systems of linear inequalities 101
9.2. Maximum and minimum values 101
9.3. Real life linear programming problems 101
**Chapter 10: Mathematical Applications in Business (8 hrs)** 101

**Chapter 1**: **Solving Equations and Inequalities (12 hrs)**

1.1. Equations involving exponents and radicals

Equations are equality of expressions. There are different types of equations that depend on the variable(s) considered. When the variable in use is said to be an equation involving exponents.

Example: Solve each of the following equations.

a) $\sqrt{x} = 3$ b) $x^3 = 8$ c) $2^x = 4$

**Definition**

For $a > 0$, $a^x = a^y$, if and only if $x = y$.

Depending up on this try to solve the following:

**Example 2** Solve $3^{2x+1} = 3^{x-2}$

**Solution:** By using the rule, since $3 > 0$, $3^{2x+1} = 3^{x-2}$, if and only if the exponents $2x + 1 = x - 2$. From this we can see that the solution is $x = -3$.

**Example 3** Solve each of the following equations.

a $8^x = 2^{2x+1}$ b $9^{x-3} = 27^{3x}$ c $\sqrt[3]{3^x} = 3^{2x+5}$

Exercises

1 Solve each of the following equations.

a $3^x = 27$ b $\left(\frac{1}{4}\right)^x = 16$ c $\left(\frac{1}{16}\right)^{3x-1} = 32$

d $81^{5x+2} = \frac{1}{243}$ e $9^{2x} = 27^{2x+1}$ f $16^{x+4} = 2^{3x}$

g $(3x+1)^3 = 64$ h $\sqrt[3]{81^{2x-1}} = 3^x$

2 Solve $(2x+3)^2 = (3x-1)^2$.

3 Solve each of the following equations.

a $9^{2x}27^{1-x} = 81^{2x+1}$ b $9^{2x+2}\left(\frac{1}{81}\right)^{x+2} = 243^{-3x-2}$

c $16^{3x+4} = 2^{3x}64^{-4x+1}$

### 1.2. System of linear equations in two variables

Recall that, for real numbers a and b ,any equation of the form $ax + b = 0$ where $a \neq 0$ is called a linear equation. The number a and b are called coefficients.

Definition

Any equation that can be reduced to the form $ax + b = 0$,and $a \neq 0$, is called a linear equation in in one variable.

Linear equations in two variables

We discussed how we solve equations with one variable that can be reduced to the $ax + b = 0$. What do you think the solution is, if the equation is given as $y = ax + b$?

Question

1. Which of the following are linear equations in two variables? And justify

a) $2x - y = 5$ c) $-x + 7 = y$ d) $\frac{1}{y} + \frac{1}{x} = 6$

b) $2x - y^2 = 7$

2. A house was rented for Birr 2,000 per month plus Birr 2 for water consumption per $m^3$

a) Write an equation for the total cost of $x$-years rent and 200 $m^3$ of water used

b) If the total cost for $x$-years rent and $y$ m3 of water used is Birr 106,000 write an equation.

**System of linear equations and their solutions**

You have discussed solutions to a linear equation in two variables and observed that there are infinite solutions. Now you will see the joint consideration of two or more linear equations in two variables.

**Definition**

A set of two or more linear equations is called a system of linear equations. Systems of two linear equations in two variables are equations that can be represented as

$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$, where $a_1, a_2, b_1, b_2, c_1$ and $c_2$ are the parameters of the system whose specific values characterize the system and $a_1 \neq 0$ or $b_1 \neq 0$, $a_2 \neq 0$ or $b_2 \neq 0$.

Example

The following are examples of systems of linear equations in two variables.

a $\begin{cases} 2x + 3y = 1 \\ x - 2y = 3 \end{cases}$ b $\begin{cases} 3x - 2y = 2 \\ 9x - 6y = 5 \end{cases}$ c $\begin{cases} x + y = 3 \\ 2x + 2y = 6 \end{cases}$

Definition

A solution to a system of linear equations in two variables means the set of ordered pairs $(x, y)$ that satisfy both equations.

Example

Determine the solution of the following system of linear equations.

$$\begin{cases} 2x+3y=8 \\ 5x-2y=1 \end{cases}$$

Solution

The set $\left\{\left(0,\frac{8}{3}\right),(1,2),\left(2,\frac{4}{3}\right),\left(3,\frac{2}{3}\right),(4,0)\right\}$ contains some of the solutions to the linear equation $2x + 3y = 8$.

The set $\left\{\left(0,-\frac{1}{2}\right),(1,2),\left(2,\frac{9}{2}\right),(3,7)\left(4,\frac{19}{2}\right)\right\}$ contains some of the solutions to the linear equation $5x - 2y = 1$.

From the definition given above, the solution to the given system of linear equations should satisfy both equations $2x + 3y = 8$ and $5x - 2y = 1$.

Therefore, the solution is (1, 2) and it satisfies both equations.

**Solution to a system of linear equations in two variables**

You saw in example above that a solution to a system of linear equations is an ordered pair that satisfies both equations in the system. We obtained it by listing some ordered pairs that satisfy each of the component equations and selecting the common one. But it is not easy to list such solutions. So we need to look for another approach to solving systems of linear equations. These include the graphical method, substitution method and elimination method.

When we draw the lines of each of the component equations in a system of two linear equations, we can observe three possibilities.

a. The two lines intersect at one point, in which case the system has one solution.
b. The two lines are parallel and never intersect. In this case, we say the system does not have a solution
c. The two lines coincide (fit one over the other). In this case, there are infinite solutions.

a) Solving system of linear equations by a graphical method

**Remark**

a) If the lines intersect on the same coordinate plane, there is one solution ,that is the point of their intersection. Please give an example(s)
b) If the lines are parallel,then the system has no solution.
c) If the lines are coinciding, then there are infinite solutions to the system of the equations.

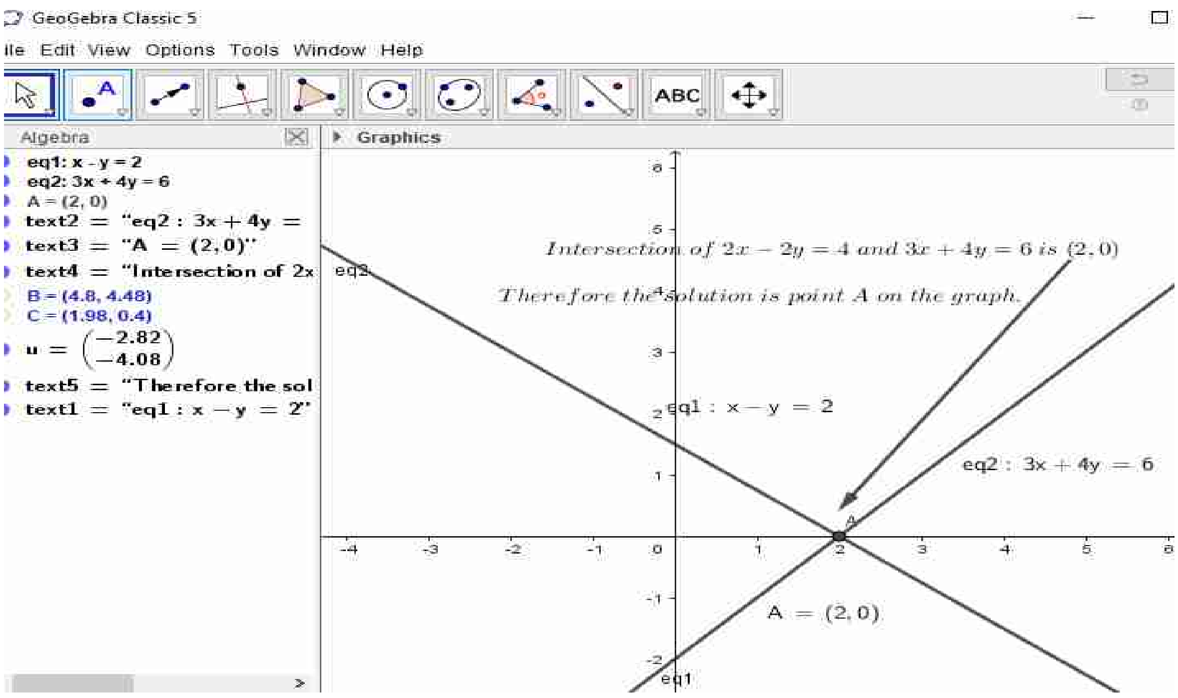
i. What does it mean?

ii. Do you think **that** every point (ordered pair) on the line satisfies both equations in the system?

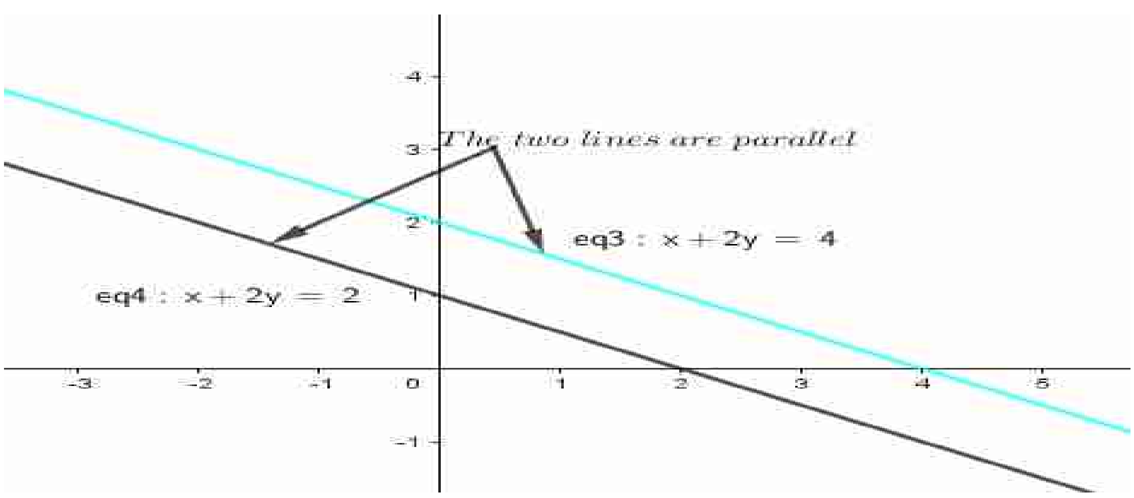
Solve each of the following systems of linear equations

**a** $\begin{cases} 2x-2y=4 \\ 3x+4y=6 \end{cases}$ **b** $\begin{cases} x+2y=4 \\ 3x+6y=6 \end{cases}$ **c** $\begin{cases} 3x-y=5 \\ 6x-2y=10 \end{cases}$

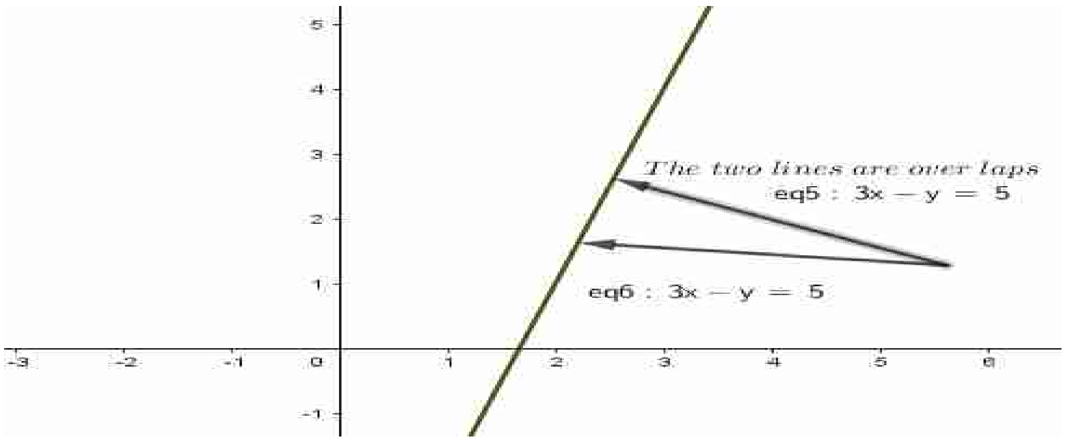
**Solution**

a) 


b) When we draw the line of each equation, we see that the lines are parallel. This means the lines do not intersect. Hence the system does not have a solution.



c) When we draw the line of each component equation, we see that the lines coincide one over the other, which shows that the system has infinite solutions. That is, all points (orderedpairs) on the line are solutions of the system.




b) **Solving systems of linear equations by the substitution method**

To solve a system of two linear equations by the substitution method, you follow the following steps

a) Take one of the linear equations from the system and write one of the variables in terms of the other.

b) Substitute your result into the other equation and solve for the second variable.

c) Substitute this result into one of the equations and solve for the first variable.

Example

Solve the system of linear equations given by $\begin{cases} 2x - 3y = 5 \\ 5x + 3y = 9 \end{cases}$

Activity

Solve each of the following systems of linear equations.

**a** $\begin{cases} 2x - 4y = 5 \\ -6x + 12y = -15 \end{cases}$ **b** $\begin{cases} 2x - y = 1 \\ 3x - 2y = -4 \end{cases}$ **c** $\begin{cases} 4x + 3y = 8 \\ -2x - \frac{3}{2}y = -6 \end{cases}$

c) Solving systems of linear equations by the elimination method

To solve a system of two linear equations by the elimination method, you follow the following steps.

a) Select one of the variables and make the coefficients of the selected variable equal but opposite in sign in the two equations.

b) Add the two equations to eliminate the selected variable and solve for the resulting variable

c) Substitute this result again into one of the equations and solve for the remaining variable.

Example

Solve the system of linear equations given by

**Solution:**

**Step 1** Select one of the variables, say $y$ and make the coefficients of $y$ opposite to one another by multiplying the first equation by 3.

$$\begin{cases} 2x - y = 5 \\ 2x + 3y = 9 \end{cases} \text{ is equivalent with } \begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$

**Step 2** Add the two equations in the system:

$$\begin{cases} 6x - 3y = 15 \\ 2x + 3y = 9 \end{cases}$$ giving $6x - 3y + 2x + 3y = 15 + 9$ which becomes

$8x = 24$.

Therefore $x = 3$.

**Step 3** Substitute $x = 3$ into one of the original equations and solve for $y$.

Choosing $2x - y = 5$ and replacing $x = 3$, get $2(3) - y = 5$ from which

$-y = 5 - 6$

$-y = -1$ which is the same as $y = 1$.

Therefore the solution is $(3, 1)$.

Word problems leading to a system of linear equations

Systems of linear equations have many real life applications. The real life problems need to be constructed in a mathematical form as a system of linear equations which will be solved by the techniques discussed earlier. Here are some examples.

Examples

A) A farmer collected a total of Birr 11,000 by selling 3 cows and 5 sheep. Another farmer collected Birr 7,000 by selling one cow and 10 sheep. What is the price for a cow and a sheep? (Assume all cows have the same price and also the price of every sheep is the same).

Solution

**Solution:** Let $x$ represent the price of a cow and $y$ the price of a sheep.

**Farmer I** sold 3 cows for $3x$ and 5 sheep for $5y$ collecting a total of Birr 11,000. Which means, $3x + 5y = 11{,}000$

**Farmer II** sold 1 cow for $x$ and 10 sheep for $10y$ collecting a total of Birr 7,000. Which means, $x + 10y = 7{,}000$. When we consider these equations simultaneously, we get the following system of equations.

$$\begin{cases} 3x+5y=11{,}000 \\ x+10y=7{,}000 \end{cases}$$

Multiplying the first equation by –2 to make the coefficients of $y$ opposite

$$\begin{cases} -6x-10y=-22{,}000 \\ \quad x+10y=7{,}000 \end{cases}$$

Adding the equations we get $-6x + x - 10y + 10y = -22{,}000 + 7{,}000$

$$-5x = -22{,}000 + 7{,}000$$
$$-5x = -15{,}000$$
$$x = 3{,}000$$

Substituting $x = 3{,}000$ in one of the equations, say $x + 10y = 7{,}000$, we get,

$$3{,}000 + 10y = 7{,}000$$
$$10y = 4{,}000$$
$$y = 400$$

Therefore the solution is (3000, 400) showing that the price for a cow is Birr 3,000 and the price for a sheep is Birr 400.

Activity

1. Simon has twin younger brothers. The sum of the ages of the three brothers is 48 and the difference between his age and the age of one of his younger brothers is 3. How old is Simon? Please take time to solve this in the classroom and present it for your colleages.
2. In a two-digit number, the sum of the digits is 14. Twice the tens digit exceeds the units digit by one. Find the numbers.

### 1.3. Equations Involving Absolute Value

Definition

If $x$ is a point on a number line with coordinate a real number $x$, then the distance of $x$ from the origin is called the absolute value of $x$ and is denoted by $|x|$.

The points represented by numbers 2 and –2 are located on the number line at an equal distance from the origin. Hence, 2 = −2 = 2 .

The absolute value of a number $x$, denoted by $|x|$, is defined as follows.

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Evaluate each of the following.

**a** $|2-5|$ **b** $|-3-4|$ **c** $|8-3|$ **d** $|2-(-5)|$

**Solution:**

**a** $|2-5| = |-3| = 3$ **b** $|-3-4| = |-7| = 7$

**c** $|8-3| = |5| = 5$ **d** $|2-(-5)| = |2+5| = |7| = 7$

Determine the value of the variable _ in each of the following absolute value equations.

**a** $|x| = 4$ **b** $|x-1| = 5$ **c** $|-2x+3| = 4$

**d** $|x| = -5$ **e** $|2x+3| = -3$

**Solution:**

**a** $|x| = 4$ means $x = 4$ or $x = -4$

**b** $|x-1| = 5$ means $x-1 = 5$ or $x-1 = -5$

Therefore $x = 6$ or $x = -4$.

**c** $|-2x+3| = 4$ means $-2x+3 = 4$ or $-2x+3 = -4$

$-2x = 1$ or $-2x = -7$

Therefore $x = \frac{-1}{2}$ or $x = \frac{7}{2}$.

**d** Since $|x|$ is always non-negative, $|x| = -5$ has no solution.

**e** Since $|x|$ is always non-negative, $|2x+3| = -3$ has no solution.

**Note:** For any real number $a$; $|x| = |a|$ means $x = a$ or $x = -a$.

**Example 11** Solve each of the following equations.

**a** $|x-1| = |2x+1|$ **b** $|3x+2| = |2x-1|$

**Solution: a** $|x-1| = |2x+1|$ means $x-1 = 2x+1$ or $x-1 = -(2x+1)$

$x-2x = 1+1$ or $x+2x = -1+1$

$-x = 2$ or $3x = 0$

Therefore $x = -2$ or $x = 0$.

## Properties of absolute value

For any real numbers $x$ and $y$;

1. $x \le |x|$.
2. $|xy| = |x||y|$.
3. $\sqrt{x^2} = |x|$.
4. $|x+y| \le |x| + |y|$ (This is called the triangle inequality).
   a. If $x$ and $y$ are both non-positive or both non-negative, $|x+y| = |x| + |y|$.
   b. If one of $x$ or $y$ is positive and the other is negative, $|x+y| < |x| + |y|$
5. If $y \ne 0$ then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
6. $-|x| \le x \le |x|$.

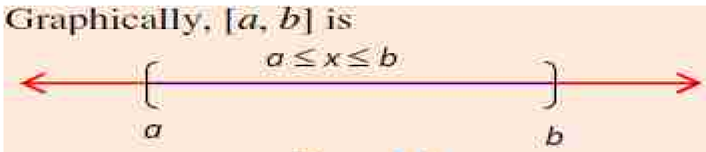
## 1.4. Inequalities involving absolute values

The methods frequently used for describing sets are the complete listing method, the partial listing method and the set-builder method. Sets of real numbers or subsets may be described by using the set-builder method or intervals (*sets of real numbers between any two given real numbers*).

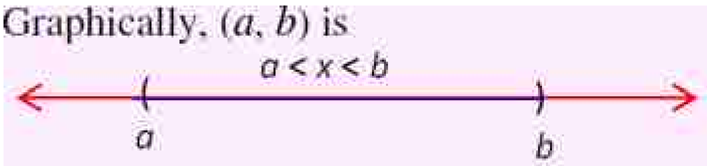
**Notation:** For real numbers $a$ and $b$ where $a < b$,

- ✓ $(a, b)$ is an open interval;
- ✓ $(a, b]$ and $[a, b)$ are half closed or half open intervals; and
- ✓ $[a, b]$ is a closed interval.

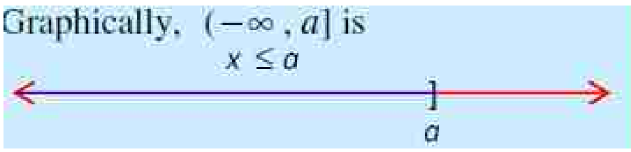
Graphically, $[a, b]$ is



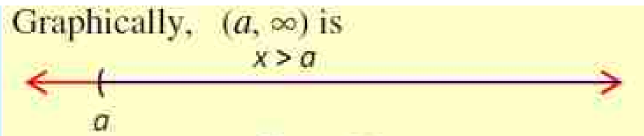

Graphically, $(a, b)$ is




Graphically, $(-\infty, a]$ is




Graphically, $(a, \infty)$ is




Activity

Do you know the notation $\infty$ and $-\infty$ in Mathematics? Please tell your friends sitting near to you in the classroom.

Solution of $x < \boldsymbol{a}$ and $x \leq \boldsymbol{a}$

For any real number $a > 0$,

a) the solution of the inequality $|x| < a$ is $-a < x < a$.
b) the solution of the inequality $|x| \leq a$ is $-a \leq x \leq a$.

Example

Solve each of the following absolute value inequalities:

**a** $|2x - 5| < 3$ **b** $|3 - 5x| \leq 1$

**Solution:**

**a** $|2x - 5| < 3$ is equivalent to $-3 < 2x - 5 < 3$,

$\Rightarrow -3 < 2x - 5$ and $2x - 5 < 3$

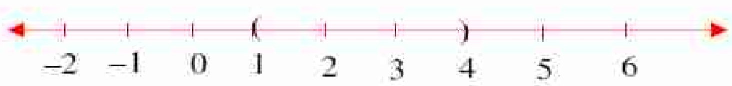
$\Rightarrow -3 + 5 < 2x - 5 + 5$ and $2x - 5 + 5 < 3 + 5$

$\Rightarrow 2 < 2x$ and $2x < 8$

$\Rightarrow 1 < x$ and $x < 4$ that is, $1 < x < 4$

Therefore, the solution set is $\{x: 1 < x < 4\} = (1, 4)$

We can represent the solution set on the number line as follows:




**b** $|3 - 5x| \leq 1$ is equivalent to $-1 \leq 3 - 5x \leq 1$

$\Rightarrow -1 \leq 3 - 5x$ and $3 - 5x \leq 1$

$\Rightarrow -1 - 3 \leq 3 - 3 - 5x$ and $3 - 3 - 5x \leq 1 - 3$

$\Rightarrow -4 \leq -5x$ and $-5x \leq -2$

$\Rightarrow 5x \leq 4$ and $2 \leq 5x$

$\Rightarrow x \leq \frac{4}{5}$ and $x \geq \frac{2}{5}$ that is, $\frac{2}{5} \leq x \leq \frac{4}{5}$

Therefore, the solution set is $\left\{x : \frac{2}{5} \leq x \leq \frac{4}{5}\right\} = \left[\frac{2}{5}, \frac{4}{5}\right]$

**Note:** In $|x| < a$, if $a < 0$ the inequality $|x| < a$ has no solution.

Remark:

**Solution of $|x| > a$ and $|x| \geq a$**

For any real number $a$, if $a > 0$, then

i the solution of the inequality $|x| > a$ is $x < -a$ or $x > a$.

ii the solution of the inequality $|x| \geq a$ is $x \leq -a$ or $x \geq a$.

Solve each of the following inequalities:

**a** $|5 + 2x| > 6$ **b** $\left|\frac{3}{5} - 2x\right| \geq 1$ **c** $|3 - x| > -2$

**a** $|5 + 2x| > 6$ implies $5 + 2x < -6$ or $5 + 2x > 6$

$\Rightarrow 5 - 5 + 2x < -6 - 5$ or $5 - 5 + 2x > 6 - 5$

$\Rightarrow 2x < -11$ or $2x > 1$

$\Rightarrow x < \frac{-11}{2}$ or $x > \frac{1}{2}$

Therefore, the solution set is $\left\{x : x < -\frac{11}{2} \text{ or } x > \frac{1}{2}\right\}$.

(Try to represent this solution on the number line)

**b** $\left|\frac{3}{5} - 2x\right| \geq 1$ implies $\frac{3}{5} - 2x \leq -1$ or $\frac{3}{5} - 2x \geq 1$

Hence, $\frac{3}{5} - 2x \leq -1$ or $\frac{3}{5} - 2x \geq 1$ gives $\frac{3}{5} - \frac{3}{5} - 2x \leq -1 - \frac{3}{5}$ or $\frac{3}{5} - \frac{3}{5} - 2x \geq 1 - \frac{3}{5}$

$\Rightarrow -2x \leq \frac{-8}{5}$ or $-2x \geq \frac{2}{5}$

$\Rightarrow \frac{8}{5} \leq 2x$ or $-\frac{2}{5} \geq 2x$

$\Rightarrow x \geq \frac{4}{5}$ or $x \leq -\frac{1}{5}$

Therefore, the solution set is $\left\{x : x \leq -\frac{1}{5} \text{ or } x \geq \frac{4}{5}\right\}$.

**c** By definition, $|3 - x| = |x - 3| \geq 0$. So, $|3 - x| > -2$ is true for all real numbers $x$.

Therefore, the solution set is $\mathbb{R}$.

## 1.5. System of linear inequalities in two variables

In a system of equations discussed above, if “=” is replaced by “<”, “>”, “≤” or “≥”, the system becomes a system of linear inequalities.

Activity

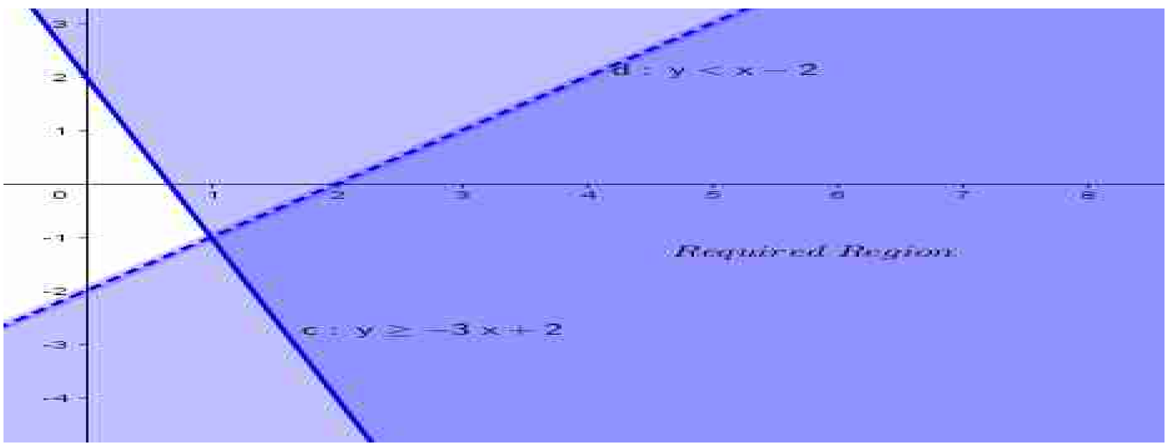
Can you recall the three methods of solving system of linear equation?

Example

Find the solution of the following system of inequalities graphically:

$$\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$$

First draw the graph of one of the boundary lines, $y = -3x + 2$, corresponding to the first inequality.




The solution set of $\begin{cases} y \geq -3x + 2 \\ y < x - 2 \end{cases}$ is shown by the cross-shaded region in the diagram.

Solving $\begin{cases} y = -3x + 2 \\ y = x - 2 \end{cases}$, we get $-3x + 2 = x - 2$

Therefore, $x = 1$ and $y = -1$

So, $x > 1, -3x + 2 \leq y < x - 2$

Hence, the solution set of the system is expressed as

$$\{(x, y): -3x + 2 \leq y < x - 2 \text{ and } 1 < x < \infty\}$$

2. Find the solution of each of the following systems of linear inequalities, graphically:

**a** $\begin{cases} x+y<3 \\ x \geq 0 \\ y \geq 0 \end{cases}$ **b** $\begin{cases} y+x>0 \\ y-x \leq 1 \\ x \leq 2 \end{cases}$

3. A furniture company makes tables and chairs. To produce a table it requires 2 hrs on machine A, and 4 hrs on machine B. To produce a chair it requires 3 hrs on machine A and 2 hrs on machine B. Machine A can operate at most 12 hrs a day and machine B can operate at most 16 hrs a day. If the company makes a profit of Birr 12 on a table and Birr 10 on a chair, how many of each should be produced to maximize its profit?

**Solution:** Let $x$ be the number of tables to be produced and $y$ be the number of chairs to be produced.

Then, if a table is produced in 2 hrs on machine A, $x$ tables require $2x$ hrs. Similarly, $y$ chairs require $3y$ hrs on machine A. On machine B, $x$ tables require $4x$ hrs and $y$ chairs require $2y$ hrs. Since machines A and B can operate at most 12 hrs and 16 hrs, respectively, you have the following system of linear inequalities.

From machine A: $2x + 3y \leq 12$

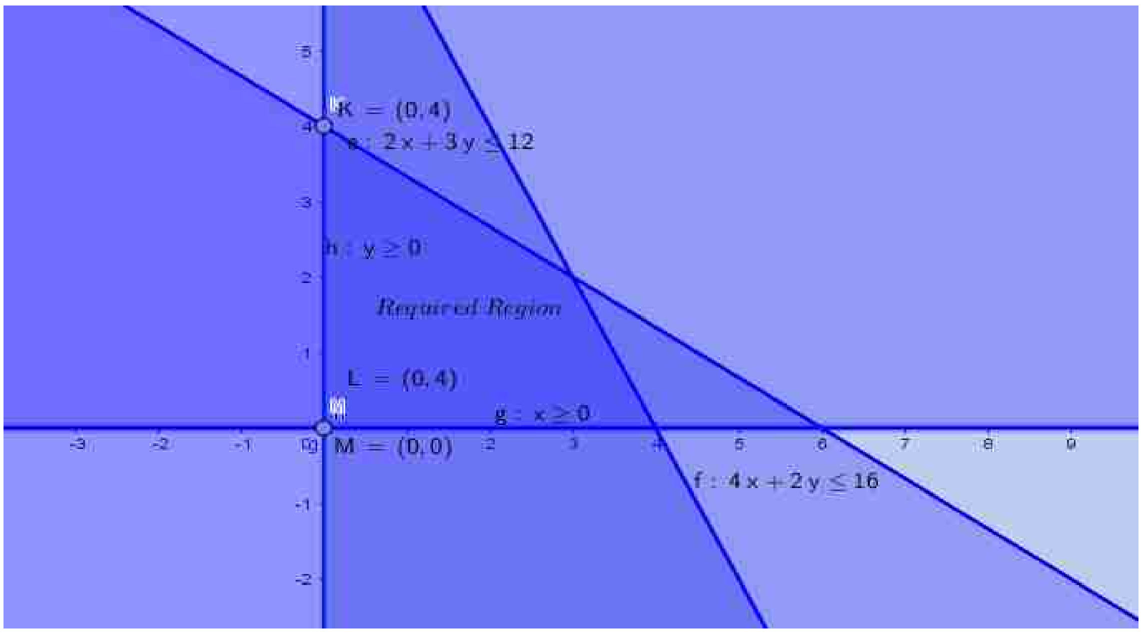
From machine B: $4x + 2y \leq 16$

Also, $x \geq 0$ and $y \geq 0$ since $x$ and $y$ are numbers of tables and chairs.

Hence, you obtain a system of linear inequalities given as follows:

$$\begin{cases} 2x+3y \leq 12 \\ 4x+2y \leq 16 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

Since the inequalities involved in the system are all linear, the boundaries of the graph of the system are straight lines. The region containing the solution to the system is the quadrilateral shown below.




The profit made is Birr 12 on a table, so Birr 12$x$ on $x$ tables and Birr 10 on a chair, so Birr 10$y$ on $y$ chairs. The profit function P is given by P = 12$x$ + 10$y$. The values of $x$ and $y$ which maximize or minimize the profit function on such a system are usually found at vertices of the solution region.

The profit: P = 12$x$ + 10y at each vertex is found to be:

At (0, 0), P = 12 (0) + 10 (0) = 0

At (0, 4), P = 12 (0) + 10 (4) = 40

At (3, 2), P = 12 (3) + 10 (2) = 56

At (4, 0), P = 12 (4) + 10 (0) = 48

Therefore, the profit is maximum at the vertex (3, 2), so the company should produce 3 tables and 2 chairs per day to get the maximum profit of Birr 56.

## 1.6. **Quadratic equations and inequalities**

Recall that for real numbers $a$ and $b$, any equation that can be reduced to the form $ax + b = 0$, where $a \neq 0$ is called a linear equation. Following the same analogy, for real numbers $a$, $b$ and $c$, any equation that can be reduced to the form $ax^2 + bx + c = 0$, where $a \neq 0$ is called a quadratic equation. $x^2 + 3x - 2 = 0, 2x^2 - 5x = 3, 3x^2 - 6x = 0, (x + 3)(x + 2) = 7$ etc, are examples of quadratic equations. In this section, you will study solving quadratic equations. You will discuss three major approaches to solve quadratic equations, namely, the method of factorization, the method of completing the square, and the general formula. Before you proceed to solve quadratic equations, you will first discuss the concept of factorization.

Example

Factorize $2x^2 - 9x$

**Solution:** The two terms in this expression, $2x^2$ and $-9x$, have $x$ as a common factor. Hence $2x^2 - 9x$ can be factorized as $x(2x-9)$.

So $2x^2 - 9x = x(2x-9)$.

Factorizing the difference of two squares

If we multiply $(x + 2)$ and (x-2). we see that $(x + 2)(x - 2) = x^2 - 4$

In general,$a^2 - b^2 = (a + b)(a - b)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example

a) factorize $x^2 - 9$

b) Factorize $x^3 - 8$

## Solving quadratic equations using the method of factorization

Let $ax^2 + bx + c = 0$ be a quadratic equation and let the quadratic polynomial $ax^2 + bx + c$ be expressible as a product of two linear factors, say $(dx + e)$ and $(fx + g)$ where $d, e, f, g$ are real numbers such that $d \neq 0$ and $f \neq 0$.

Then $ax^2 + bx + c = 0$ becomes

$(dx + e)(fx + g) = 0$

So, $dx + e = 0$ or $fx + g = 0$ which gives $x = \frac{-e}{d}$ or $x = \frac{-g}{f}$.

Therefore $x = \frac{-e}{d}$ and $x = \frac{-g}{f}$ are possible roots of the quadratic equation $ax^2 + bx + c = 0$.

For example, the equation $x^2 - 5x + 6 = 0$ can be expressed as:

$$(x-2)(x-3) = 0$$

$$x - 2 = 0 \text{ or } x - 3 = 0$$

$$x = 2 \text{ or } x = 3$$

Therefore the solutions of the equation $x^2 - 5x + 6 = 0$ are $x = 2$ and $x = 3$.

**In order to solve a quadratic equation by factorization, go through the following steps:**

i Clear all fractions and square roots (if any).

ii Write the equation in the form $p(x) = 0$.

**iii** Factorize the left hand side into a product of two linear factors.

**iv** Use the *zero-product rule* to solve the resulting equation.

**Zero-product rule:** If $a$ and $b$ are two numbers or expressions and if $ab = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$.

Example

Solve the following quadratic equations.

**a** $4x^2 - 16 = 0$ **b** $x^2 + 9x + 8 = 0$ **c** $2x^2 - 6x + 7 = 3$

**Solution:**

**a** $4x^2 - 16 = 0$ is the same as $(2x)^2 - 4^2 = 0$

$$(2x - 4)(2x + 4) = 0$$

$$(2x - 4) = 0 \text{ or } (2x + 4) = 0$$

Therefore, $x = 2$ or $x = -2$.

**b**

$$x^2 + 9x + 8 = 0$$

$$x^2 + x + 8x + 8 = 0$$

$$(x^2 + x) + (8x + 8) = 0$$

$$x(x + 1) + 8(x + 1) = 0$$

$$(x + 1)(x + 8) = 0$$

$$(x + 1) = 0 \text{ or } (x + 8) = 0$$

Therefore, $x = -1$ or $x = -8$.

Activity solve C as an exercise.

**Solving quadratic equations by completing the square**

In many cases, it is not convenient to solve a quadratic equation by factorization method. For example, consider the equation $x^2 + 8x + 4 = 0$. If you want to factorize the left hand side of the equation, i.e., the polynomial $x2 + 8x + 4$, using the method of splitting the middle term, you need to find two integers whose sum is 8 and product is 4. But this is not possible. In such cases, an alternative method as demonstrated below is convenient.

$$x^2 + 8x + 4 = 0$$
$$x^2 + 8x = -4$$
$$x^2 + 8x + (4)^2 = -4 + (4)^2 \qquad \left(\text{Adding } \left(\frac{1}{2}\text{Coefficient of } x\right)^2 \text{ on both sides}\right)$$
$$(x + 4)^2 = -4 + 16 = 12 \qquad (x^2 + 8x + 16 = (x + 4)^2)$$
$$x + 4 = \pm\sqrt{12} \qquad \text{(Taking square root of both sides)}$$

Therefore $x = -4 + \sqrt{12}$ and $x = -4 - \sqrt{12}$ are the required solutions.

**In general, go through the following steps in order to solve a quadratic equation by the method of completing the square:**

- **i** Write the given quadratic equation in the standard form.
- **ii** Make the coefficient of $x^2$ unity, if it is not.
- **iii** Shift the constant term to R.H.S.(Right Hand Side)
- **iv** Add $\left(\frac{1}{2}\text{coefficient of } x\right)^2$ on both sides.
- **v** Express L.H.S.(Left Hand Side) as the perfect square of a suitable binomial expression and simplify the R.H.S.
- **vi** Take square root of both the sides.
- **vii** Obtain the values of $x$ by shifting the| constant term from L.H.S. to R.H.S.

**Note:** The number we need to add (or subtract) to construct a perfect square is determined by using the following product formulas:

$$x^2 + 2ax + a^2 = (x + a)^2$$
$$x^2 - 2ax + a^2 = (x - a)^2$$

Note that the last term, $a^2$, on the left side of the formulae is the square of one-half of the coefficient of $x$ and the coefficient of $x^2$ is +1. So, we should add (or subtract) a suitable number to get this form.

Example

Solve $x^2 + 5x - 3 = 0$.

**Solution:** Note that $\left(\frac{5}{2}\right)^2 = \frac{25}{4}$.

Hence, we add this number to get a perfect square.

$$x^2+5x-3=0$$

$$x^2+5x=3$$

$$x^2+5x+\frac{25}{4}=3+\frac{25}{4}$$

$$x^2+5x+\frac{25}{4}=\frac{37}{4}; \quad \left(x^2+5x+\frac{25}{4} \text{ is a perfet squre.}\right)$$

$$\left(x+\frac{5}{2}\right)^2=\frac{37}{4}$$

$$\left(x+\frac{5}{2}\right)=\sqrt{\frac{37}{4}} \text{ or } \left(x+\frac{5}{2}\right)=-\sqrt{\frac{37}{4}}$$

$$x=-\frac{5}{2}+\sqrt{\frac{37}{4}} \text{ or } x=-\frac{5}{2}-\sqrt{\frac{37}{4}}$$

Therefore $x=\frac{-5+\sqrt{37}}{2}$ or $x=\frac{-5-\sqrt{37}}{2}$.

**Solving quadratic equations using the quadratic formula**

Following the method of completing the square, you next develop a general formula that can serve for checking the existence of a solution to a quadratic equation, and for solving quadratic equations.

**Note:** If any quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$ has a solution, then the solution is determined by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and

1 if $b^2-4ac>0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ represents two numbers, namely $x=\frac{-b-\sqrt{b^2-4ac}}{2a}$ and $x=\frac{-b+\sqrt{b^2-4ac}}{2a}$.

Therefore, the equation has two solutions.

2 if $b^2-4ac=0$ then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b}{2a}$ is the only solution.

Therefore, the equation has only one solution.

3 if $b^2-4ac<0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ is not defined in $\mathbb{R}$.

Therefore, the equation does not have any real solution.

The expression $b^2-4ac$ is called the discriminant or discriminator. It helps to determine the existence of solutions.

Example

Using the discriminant, check to see if the following equations have solutions and solve if there is a solution.

**a** $3x^2 - 5x + 2 = 0$ **b** $x^2 - 8x + 16 = 0$ **c** $-2x^2 - 4x - 9 = 0$

**Solutions**

**a** $3x^2 - 5x + 2 = 0$; $a = 3$, $b = -5$ and $c = 2$.

So $b^2 - 4ac = (-5)^2 - 4(3)(2) = 1 > 0$

Therefore, the equation $3x^2 - 5x + 2 = 0$ has two solutions.

Using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-5) - \sqrt{(-5)^2 - 4(3)(2)}}{2(3)} \text{ or } x = \frac{-(-5) + \sqrt{(-5)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{5 - \sqrt{25 - 24}}{6} \text{ or } x = \frac{5 + \sqrt{25 - 24}}{6}$$

$$x = \frac{5 - \sqrt{1}}{6} \text{ or } x = \frac{5 + \sqrt{1}}{6}$$

$$x = \frac{5 - 1}{6} \text{ or } x = \frac{5 + 1}{6}$$

$$x = \frac{4}{6} \text{ or } x = \frac{6}{6}$$

Therefore $x = \frac{2}{3}$ or $x = 1$.

**b** In $x^2 - 8x + 16 = 0$, $a = 1$, $b = -8$ and $c = 16$

So $b^2 - 4ac = (-8)^2 - 4(1)(16) = 0$

Therefore, the equation $x^2 - 8x + 16 = 0$ has only one solution.

Using the quadratic solution formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a}$

$$x = \frac{-(-8)}{2(1)} = 4$$

Therefore the solution is $x = 4$.

**c** In $-2x^2 - 4x - 9 = 0$, $a = -2$, $b = -4$ and $c = -9$

So $b^2 - 4ac = (-4)^2 - 4(-2)(-9) = -56 < 0$

Therefore the equation $-2x^2 - 4x - 9 = 0$ does not have any real solution.

The relationship between the coefficients of a quadratic equation and its roots

You have learned how to solve quadratic equations. The solutions to a quadratic equation are sometimes called roots. The general quadratic equation

$ax^2 + bx + c = 0,\ a \neq 0$ has roots (solutions)

$$r_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \text{ and } r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

**Viete's theorem**

If the roots of $ax^2 + bx + c = 0,\ a \neq 0$ are $r_1 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $r_2 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$, then $\boxed{r_1 + r_2} = \dfrac{-b}{a}$ and $r_1 \times r_2 = \dfrac{c}{a}$

Word problem

a) The sum of two numbers is 11 and their product is 28. Find the numbers.

**Solution:** Let $x$ and $y$ be the numbers.

You are given two conditions, $x + y = 11$ and $xy = 28$

From $xy = 28$ you can express $y$ in terms of $x$, giving $y = \dfrac{28}{x}$

Replace $y = \dfrac{28}{x}$ in $x + y = 11$ to get $x + \dfrac{28}{x} = 11$

Now proceed to solve for $x$ from $x + \dfrac{28}{x} = 11$ which becomes

$$\frac{x^2+28}{x} = 11$$

$$x^2 + 28 = 11x$$

$x^2 - 11x + 28 = 0$, which is a quadratic equation.

Then solving this quadratic equation, you get $x = 4$ or $x = 7$.

If $x = 4$ then from $x + y = 11$ you get $4 + y = 11 \Rightarrow y = 7$

If $x = 7$ then from $x + y = 11$ you get $7 + y = 11 \Rightarrow y = 4$

Therefore, the numbers are 4 and 7.

b. Two different squares have a total area of 274 cm2 and the sum of their perimeters is 88 cm. Find the lengths of the sides of the squares.

Recall, the area of the smaller square is $x^2$ and area of the bigger square is $y^2$.

The perimeter of the smaller square is $4x$ and that of the bigger square is $4y$.

So the total area is $x^2 + y^2 = 274$ and the sum of their perimeters is $4x + 4y = 88$.

From $4x + 4y = 88$ you solve for $y$ and get $y = 22 - x$.

Substitute $y = 22 - x$ in $x^2 + y^2 = 274$ and get $x^2 + (22 - x)^2 = 274$.

This equation is $x^2 + 484 - 44x + x^2 = 274$ which becomes the quadratic equation $2x^2 - 44x + 210 = 0$.

Solving this quadratic equation, you get $x = 15$ or $x = 7$.

Therefore, the side of the smaller square is 7 cm and the side of the bigger square is 15 cm.

Activity

1 The area of a rectangle is 21 cm$^2$. If one side exceeds the other by 4 cm, find the dimensions of the rectangle.

2 The perimeter of an equilateral triangle is numerically equal to its area. Find the length of the side of the equilateral triangle.

3 Divide 29 into two parts so that the sum of the squares of the parts is 425. Find the value of each part.

4 The sum of the squares of two consecutive natural numbers is 313. Find the numbers.

5 A piece of cloth costs Birr 200. If the piece was 5 m longer, and the cost of each metre of cloth was Birr 2 less, the cost of the piece would have remained unchanged. How long is the piece and what is its original price per metre?

6 Birr 6,500 were divided equally among a certain number of persons. Had there been 15 more persons, each would have got Birr 30 less. Find the original number of persons.

7 A person on tour has Birr 360 for his daily expenses. If he extends his tour for 4 days, he has to cut down his daily expense by Birr 3. Find the original duration of the tour.

Quadratic inequality

An inequality that can be reduced to any one of the following forms:

$ax^2 + bx + c \le 0$ or $ax^2 + bx + c < 0$,

$ax^2 + bx + c \ge 0$ or $ax^2 + bx + c > 0$,

where $a$, $b$ and $c$ are constants and $a \neq 0$, is called a quadratic inequality.

Solving Quadratic Inequalities Using Product Properties

**Product properties:**

1. $m.n > 0$, if and only if
   i. $m > 0$ and $n > 0$ or  ii. $m < 0$ and $n < 0$.
2. $m.n < 0$, if and only if
   i. $m > 0$ and $n < 0$ or  ii. $m < 0$ and $n > 0$.

Example

Solve each of the following inequalities:

a $(x + 1)(x - 3) > 0$  b $3x^2 - 2x \geq 0$

c $-2x^2 + 9x + 5 < 0$  d $x^2 - x - 2 \leq 0$

**Solution:**

a By Product property 1, $(x + 1)(x - 3)$ is positive if either both the factors are positive or both are negative.

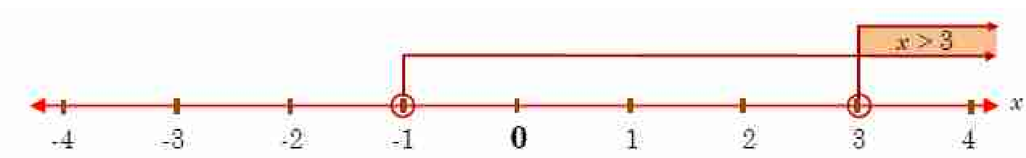
Now, consider case by case as follows:

**Case i** When both the factors are positive

$x + 1 > 0$ and $x - 3 > 0$

$x > -1$ and $x > 3$

The intersection of $x > -1$ and $x > 3$ is $x > 3$.



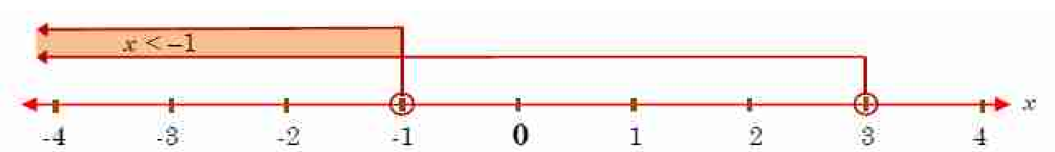

The solution set for this first case is $S_1 = \{x: x > 3\} = (3, \infty)$.

**Case ii** When both the factors are negative

$x + 1 < 0$ and $x - 3 < 0$

$x < -1$ and $x < 3$

The intersection of $x < -1$ and $x < 3$ is $x < -1$.




The solution set for this second case is $S_2 = \{x: x < -1\} = (-\infty, -1)$.

Therefore, the solution set of $(x + 1)(x - 3) > 0$ is:

$$S_1 \cup S_2 = \{x: x < -1 \text{ or } x > 3\} = (-\infty, -1) \cup (3, \infty)$$

Activity

Solve other left exercise accordingly.

Solving Quadratic Inequalities Using the Sign Chart Method

The "sign chart" method allows you to find the sign of $x^2 + 3x - 4$ in each interval.

**Step 1** Factorize $x^2 + 3x - 4 = (x + 4)(x - 1)$

**Step 2** Draw a sign chart, noting the sign of each factor and hence the whole expression as shown below.

| | | -4 | | 1 | |
|---|---|---|---|---|---|
| $x + 4$ | − − − − | 0 | + + + + ++ + + + + + | + | + + + |
| $x - 1$ | − − − − | − | − − − − − − − − − − − | 0 | + + + + |
| $(x + 4)(x - 1)$ | + + + + | 0 | − − − − − − − − − − − | 0 | + + + + + |

**Step 3** Read the solution from the last line of the sign chart

$x^2 + 3x - 4 < 0$ for $x \in (-4, 1)$

Therefore, the solution set is the interval $(-4, 1)$

Solve each of the following inequalities using the sign chart method

**a** $6 + x - x^2 \leq 0$ **b** $2x^2 + 3x - 2 \geq 0$.

For what value(s) of $k$ does the quadratic equation $kx2 - 2x + k = 0$ has

**i** only one real root? **ii** two distinct real roots?

**iii** no real roots?

**Solution:** The quadratic equation $kx^2 - 2x + k = 0$ is equivalent to the quadratic equation $ax^2 + bx + c = 0$ with $a = k$, $b = -2$ and $c = k$

The given quadratic equation has

**i** one real root when $b^2 - 4ac = 0$

So, $(-2)^2 - 4(k)(k) = 0$

$4 - 4k^2 = 0$ equivalently $(2 - 2k)(2 + 2k) = 0$

$2 - 2k = 0$ or $2 + 2k = 0$

$k = 1$ or $k = -1$

Therefore, $kx^2 - 2x + k = 0$ has only one real root if either $k = 1$ or $k = -1$.

Activity

1 Solve each of the following quadratic inequalities using sign charts:

a $x(x+5)>0$ b $(x-3)^2 \geq 0$

c $(4+x)(4-x)<0$ d $\left(1+\frac{x}{3}\right)(5-x)<0$

e $3-x-2x^2>0$ f $-6x^2+2 \leq x$

g $2x^2 \geq -3-5x$ h $4x^2-x-8<3x^2-4x+2$

i $-x^2+3x<4$.

2 Solve each of the following quadratic inequalities using either product properties or sign charts:

a $x^2+x-12>0$ b $x^2-6x+9>0$ c $x^2-3x-4 \leq 0$

d $5x-x^2<6$ e $x^2+2x<-1$ f $x-1 \leq x^2+2$

3 For what value(s) of $k$ does each of the following quadratic equations have

i one real root? ii two distinct real roots? iii no real root?

a $(k+2)x^2-(k+2)x-1=0$

b $x^2+(5-k)x+9=0$

4 For what value (s) of $k$ is

a $kx^2+6x+1>0$ for each real number $x$?

b $x^2-9x+k<0$ only for $x \in (-2, 11)$?

5 A rocket is fired straight upward from ground level with an initial velocity of 480 km/hr. After $t$ seconds, its distance above the ground level is given by

$480t-16t^2$.

For what time interval is the rocket more than 3200km above ground level?

Solving Quadratic Inequalities Graphically

In order to use graphs to solve quadratic inequalities, it is necessary to understand the nature of quadratic functions and their graphs.

a) If $a>0$, then the graph of the quadratic function $f(x)=ax2+bx+c$ is an upward parabola.

b) If $a<0$, then the graph of the quadratic function $f(x)=ax2+bx+c$ is a downward parabola

The graph of a quadratic function has both its ends going upward or downward depending on whether $a$ is positive or negative. From different graphs you can observe that the graph of a quadratic function

$$f(x)=ax^2+bx+c$$

i crosses the $x$-axis twice, if $b^2-4ac>0$.

ii touches the $x$-axis at a point, if $b^2-4ac=0$.

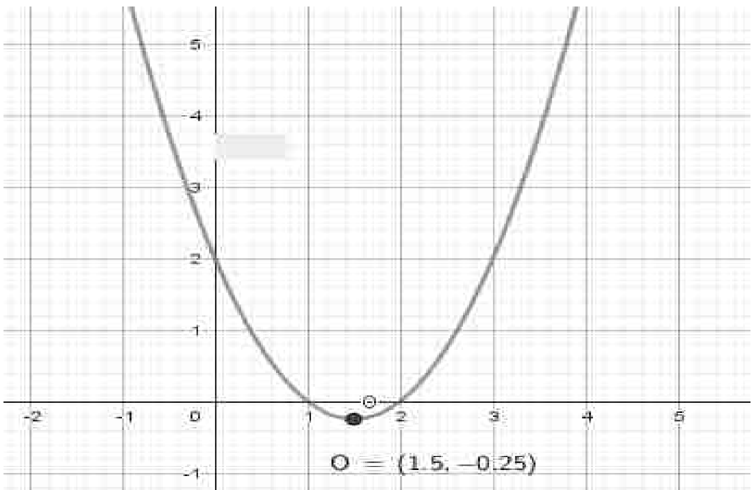
iii does not touch the $x$-axis at all, if $b^2-4ac<0$.

To solve a quadratic inequality graphically, find the values of $x$ for which the part of the graph of the corresponding quadratic function is above the $x$-axis, below the $x$-axis or on the $x$-axis. Consider the following examples.

e.g Solve the quadratic inequality $x^2-3x+2<0$, graphically.

**Solution:** Begin by drawing the graph of $f(x) = x^2 - 3x + 2$. Some values for $x$ and $f(x)$ are given in the table below and the corresponding graph is

| $x$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $f(x)$ | | 12 | | 2 | | 0 | |




The graph of $f(x) = x^2 - 3x + 2$

From the graph, $f(x) = 0$ when $x = 1$ and when $x = 2$. On the other hand, $f(x) > 0$ when $x < 1$ and when $x > 2$ and $f(x) < 0$ when $x$ lies between 1 and 2.

This inequality could be tested by setting $x = \frac{3}{2}$, giving $f\left(\frac{3}{2}\right) = -\frac{1}{4}$. So $f\left(\frac{3}{2}\right) < 0$.

It follows that the solution set of $x^2 - 3x + 2 < 0$ consists of all real numbers greater than 1 and less than 2. That is, S.S $= \{x : 1 < x < 2\} = (1, 2)$.

If the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ has discriminant $b^2 - 4ac < 0$, then the equation has no real roots. Moreover,

**i** the solution set of $ax^2 + bx + c \geq 0$ is the set of all real numbers if $a > 0$ and is empty set if $a < 0$.

**ii** the solution set of $ax^2 + bx + c \leq 0$ is the set of all real numbers if $a < 0$ and is empty set if $a > 0$.

## 1.7. Simplification of rational expressions and solving rational equations

A rational expression is the quotient $\frac{P(x)}{Q(x)}$ of two polynomials $P(x)$ and $Q(x)$, where $Q(x) \neq 0$. $P(x)$ is called the numerator and $Q(x)$ is called the denominator.

**Example 1** Which of the following are rational expressions?

**a** $\dfrac{x-2}{2x^2-3x+4}$ **b** $\dfrac{1}{x^4-1}$ **c** $\dfrac{x^3+3x-6}{4}$ **d** $\sqrt{1-5x}$

**Solution** All except **d** are rational expressions

**Example 2** Evaluate the rational expression $\dfrac{2x-5}{3x+9}$ for the given values of $x$:

**a** $x=5$ **b** $x=-6$

**Solution**

**a** At $x=5$, $\dfrac{2x-5}{3x+9}=\dfrac{2(5)-5}{3(5)+9}=\dfrac{10-5}{15+9}=\dfrac{5}{24}$

**b** At $x=-6$, $\dfrac{2x-5}{3x+9}=\dfrac{2(-6)-5}{3(-6)+9}=\dfrac{-12-5}{-18+9}=\dfrac{-17}{-9}=\dfrac{17}{9}$

**Steps to find the domain of a rational expression:**

1 Set the denominator of the expression equal to zero and solve.

2 The domain is the set of all real numbers except those values found in step 1.

**Example 3** Find the domain of each of the following rational expressions:

**a** $\dfrac{19}{3x}$ **b** $\dfrac{x^2-9}{x^2-7x+10}$

**Solution**

**a** Set the denominator equal to zero and solve: $3x=0 \Rightarrow x=0$.

Thus, the domain is $\{x : x$ is a real number and $x \neq 0\}$ or $\mathbb{R}\setminus\{0\}$.

**b** Set the denominator equal to zero and solve:

$x^2-7x+10=0$ (factor)

$(x-5)(x-2)=0$ (set each factor equal to 0 and solve)

$x-5=0$ or $x-2=0$

$x=5$ or $x=2$

Thus, the domain is $\{x : x$ is a real number and $x \neq 2, x \neq 5\} = \mathbb{R}\setminus\{2, 5\}$

Definition: We say that a rational expression is reduced to lowest terms (or in its lowest terms or in simplest form), if the numerator and denominator do not have any common factor other than 1.

**To simplify a rational expression:**

1 Find the domain.

2 Factorize the numerator and denominator completely.

3 Divide the numerator and denominator by any common factor (i.e. cancel like terms).

**Example 4** Simplify the following.

a $\dfrac{2y^2+6y+4}{4y^2-12y-16}$. b $\dfrac{x^4+18x^2+81}{x^2+9}$. c $\dfrac{1-a}{7a^2-7}$.

**Solution**

a The universal set is $\mathbb{R}\setminus\{-1, 4\}$.

Thus, $\dfrac{2y^2+6y+4}{4y^2-12y-16}=\dfrac{2(y+2)(y+1)}{4(y-4)(y+1)}=\dfrac{y+2}{2(y-4)}$, for $y \neq 4$ and $y \neq -1$.

b $\dfrac{x^4+18x^2+81}{x^2+9}=\dfrac{(x^2+9)(x^2+9)}{x^2+9}=x^2+9$, for all $x \in \mathbb{R}$.

c $\dfrac{1-a}{7a^2-7}=\dfrac{-a+1}{7(a^2-1)}=\dfrac{-(a-1)}{7(a-1)(a+1)}=-\dfrac{1}{7(a+1)}$, for $a \in \mathbb{R}\setminus\{-1, 1\}$.

Operations with Rational Expressions

Remark: Rational expressions obey the same rules as rational numbers, for addition, subtraction, multiplication and division.

Decomposition of Rational Expressions into Partial Fractions

Definition

In a rational expression $\dfrac{P(x)}{Q(x)}$, if the degree of $P(x)$ is less than that of $Q(x)$, then $\dfrac{P(x)}{Q(x)}$ is called a proper rational expression. Otherwise it is called improper.

**Example 9** Express $\dfrac{2x^3+10x^2-3x+1}{x+3}$ as a sum of a polynomial and a proper rational fraction.

**Solution** Using long division,

$2x^3 + 10x^2 - 3x + 1 = (x + 3)(2x^2 + 4x - 15) + 46.$

Thus, $\dfrac{2x^3+10x^2-3x+1}{x+3}=(2x^2+4x-15)+\dfrac{46}{x+3}$.

Moreover, you need to rely on the following definition to do the partial fraction decomposition:

Definition

Two polynomials of equal degree are equal to each other, if and only if the coefficients of terms of like degree are equal.

Remark

$ax^2 + bx + c$ is not reducible in real numbers, if $b^2 - 4ac < 0$.

Linear and quadratic factor theorem

For a polynomial with real coefficients, there always exists a complete factorization involving only linear and/or quadratic factors (raised to some power of natural number $k$ ³ 1), with real coefficients, where the linear and quadratic factors are not reducible relative to real numbers.

| | Factor in the Denominator | Corresponding term in the Partial Fraction |
|---|---|---|
| 1 | $ax+b$ | $\frac{A}{ax+b}$, $A$ constant |
| 2 | $(ax+b)^k$ | $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}$, $A_1, A_2, \cdots, A_k$ are constants |
| 3 | $ax^2+bx+c$ (with $b^2-4ac<0$) | $\frac{Ax+B}{ax^2+bx+c}$, $A$, $B$ are constants |
| 4 | $(ax^2+bx+c)^k$ (with $b^2-4ac<0$) | $\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$, $A_1, A_2, \cdots, A_k$, $B_1, B_2, \cdots, B_k$ are constants. |

**Example 10** Decompose each of the following rational expressions into partial fractions:

a $\frac{5x+7}{x^2+2x-3}$  b $\frac{6x^2-14x-27}{(x+2)(x-3)^2}$  c $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)}$

d $\frac{x^3-4x^2+9x-5}{(x^2-2x+3)^2}$  e $\frac{x^3}{(x+1)(x+2)}$

**Solution**

a The denominator $x^2+2x-3=(x-1)(x+3)$. The two factors $(x-1)$ and $(x+3)$ are distinct. Thus, we apply part 1 of the table to get:

$$\frac{5x+7}{x^2+2x-3}=\frac{5x+7}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$$

To find the constants $A$ and $B$, we combine the fractions on the right side of the above equation to obtain

Zeros under A is x=1, then A=$\frac{5x1+7}{1+3}=3$ and zeros under B is x=-3, then B=$\frac{5x(-3)+7}{-3-1}=\frac{8}{4}=2$

Thus

$$\frac{5x+7}{x^2+2x-3}=\frac{3}{x-1}+\frac{2}{x+3}.$$

**To solve rational equations, you follow the following steps:**

1. Factorize all the denominators and determine their LCM.
2. Restrict the values of the variable that make the LCM equal to 0.
3. Multiply both sides of the rational equation by the LCM and simplify.
4. Solve the resulting equation.
5. Check the answers against the restricted values in step 2. Any such value must be excluded from the solution.

**Example 1** Solve each of the following equations:

**a** $\frac{2}{x+1}=\frac{3}{x-2}$ **b** $\frac{x}{x+4}-\frac{4}{x-4}=\frac{x^2+16}{x^2-16}$

**c** $\frac{10}{x(x-2)}+\frac{4}{x}=\frac{5}{x-2}$ **d** $\frac{3a-5}{a^2+4a+3}+\frac{2a+2}{a+3}=\frac{a-1}{a+1}$

**Solution**

**a** Your restrictions are $x \neq -1$ and $x \neq 2$. Now, multiply both sides of the equation by their LCM $(x+1)(x-2)$:

$$\frac{2}{x+1}=\frac{3}{x-2} \Rightarrow \left(\frac{2}{x+1}\right)\left(\frac{(x+1)(x-2)}{1}\right)=\left(\frac{3}{x-2}\right)\left(\frac{(x+1)(x-2)}{1}\right)$$

$$2(x-2)=3(x+1) \Rightarrow x=-7$$

This does not contradict our restrictions that $x \neq -1$ and $x \neq 2$.

Thus, our solution set is $\{-7\}$.

Activity

a. Be a group and do question above b,c and d and present for the class!!

b. Two planes leave an airport flying at the same rate. If the first plane flies 1.5 hours longer than the second plane and travels 2700 miles while the second plane travels only 2025 miles, for how long was each plane flying?

c. A tree casts a shadow of 34 feet at the time when a 3-foot tall child casts a shadow of 1.7 feet. What is the height of the tree?

**Chapter 2**: **Relations and Functions (10 hrs)**

2.1. Definition and examples of relations

Let A and B be non-empty sets. A relation R from A to B is any subset of $A \times B$. In other words, R is a relation from A to B if and only if $R \subseteq (A \times B)$.

**Example 1** Let A = { 1, 2, 3, 4} and B = { 1, 3, 5}

**i** $R_1 = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$ is a relation from A to B because $R_1 \subseteq (A \times B)$. Is $R_1$ a relation from B to A? Justify.

Notice that we can represent $R_1$ in the set builder method as

$$R_1 = \{ (x, y) \mid x \in A, y \in B, x < y\}$$

**ii** $R_2 = \{(1, 1), (2, 1), (3, 1), (3, 3), (4, 1), (4, 3)\}$ is a relation from A to B because $R_2 \subseteq (A \times B)$.

In the set builder method, $R_2$ is represented by $R_2 = \{ (x, y) \mid x \in A, y \in B, x \geq y\}$

**Example 2** Let A = {1, 2, 3} then observe that

$R_1 = \{(1, 2), (1, 3), (2, 3)\}$, $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$

and $R_3 = \{(x, y) \mid x, y \in A, x + y \text{ is odd}\}$ are relations on A.

## Domain and Range

Definition

Let R be a relation from a set A to a set B. Then

a. Domain of $R = \{x : (x, y)$ belongs to R for some $y\}$

b. Range of $R = \{y : (x, y)$ belongs to R for some $x\}$

**Example 1** Given the relation R = {(1, 3), (2, 5), (7, 1), (4, 3)}, find the domain and range of the relation R.

**Solution:** Since the domain contains the first coordinates, domain = {1, 2, 7, 4} and the range contains the second coordinates, range = {3, 5, 1}

**Example 2** Given A = {1, 2, 4, 6, 7} and B = {5, 12, 7, 9, 8, 3}

Find the domain and range of the relation $R = \{(x, y): x \in A, y \in B, x > y\}$

**Solution:** If we describe R by complete listing method, we will find

R = {(4, 3), (6, 3), (7, 3), (6, 5), (7, 5)}.

This shows that the domain of R = {4, 6, 7} and the range of R = {3, 5}

Examples

1. Find the domain and the range of the relation

$$R = \{(x, y): y \geq x + 2 \text{ and } y > -x;\ x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

From the graph sketched above, since any vertical line meets the graph, the domain of the relation is the set of real numbers, $\mathbb{R}$.

That is, domain of R = $\mathbb{R}$. But not all horizontal lines meet the graph, only those that pass through $y$: $y > 1$. Hence, the range of the relation is the set $\{y: y > 1\}$.

2. Sketch the graph of the following relation and determine its domain and range.

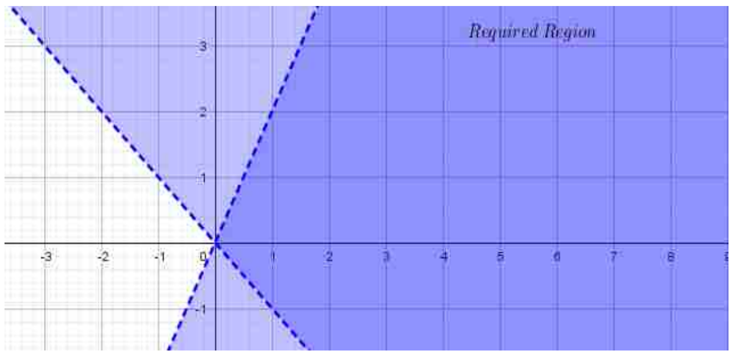
$R = \{(x, y): y < 2x \text{ and } y > -x\}$.

Solution

Sketch the graphs of $y < 2x$ and $y > -x$ on same coordinate system. Note that these two lines divide the coordinate system into four regions. Take any points one from each region and check if they satisfy the relation. Say, $(3, 0)$, $(0, 4)$, $(-1, 0)$ and $(0,-2)$. $(3, 0)$ satisfies both inequalities of the relation. So the graph of the relation is the region that contains $(3, 0)$.

Hence, Domain of R = {xÎℝ: x >0}

Range of R = {y: yÎℝ}.




Definition

Let R be a relation from A to B. The inverse of R, denoted by $R^{-1}$, is a relation from B to A, given by $R^{-1} = \{(b, a): (a, b) \in R\}$.

**Example 1** Let A = {0, –1, 2} and B = {5, 6}.

Give the inverse of R = {(0, 5), (0, 6), (–1, 6)}.

**Solution** $(a, b) \in$ R means $(b, a) \in R^{-1}$. Thus, $R^{-1} = \{(5, 0), (6, 0), (6, -1)\}$

## 2.2. Definition and examples of functions

### Functions

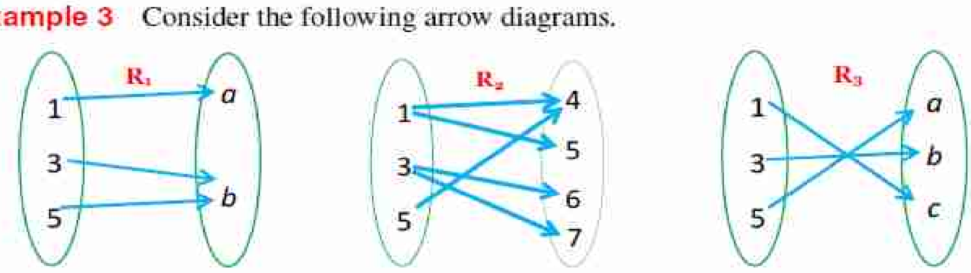
A function is a relation such that no two ordered pairs have the same first-coordinates and different second-coordinates.

**Example 1** Consider the relation R = {(1, 2), (7, 8), (4, 3), (7, 6)}

Since 7 is paired with both 8 and 6 the relation R is not a function.

**Example 2** Let R = {(1, 2), (7, 8), (4, 3)}. This relation is a function because no *first*-coordinate is paired (mapped) with more than one element of the *second*-coordinate.

**Example 3** Consider the following arrow diagrams.



Which of these relations are functions?

Solution:

R1 is a function. (Why?)

R2 is not a function because 1 and 3 are both mapped onto two numbers.

R3 is a function. (Why?)

Domain and range of a function

As a function is a special type of a relation, the domain and range of a function are determined in exactly the same way.

So, find the domain and range of the following function.

**a** F = {(2, –1), (4, 3), (0, 1)} **b** F = {(2, –1), (4, 3), (0, –1), (3, 4)}

Solution:

**a.** Domain D = {0, 2, 4} and range R = {–1, 1, 3 }

**b.** Domain D = {0, 2, 3, 4} and range R = {–1, 3, 4}

**Example 7** Is the relation $R = \{(x, y): x = y^2\}$ a function?

**Solution:** This is not a function because numbers for $x$ are paired with more than one number in $y$. For example, (9, –3) and (9, 3) satisfy the relation with 9 being mapped to both –3 and 3.

**Example 8** Is $R = \{(x, y): y = |x|\}$ a function?

**Solution:** Since for every number there is unique absolute value, each number $x$ is mapped to one and only one number $y$, so the relation $R = \{(x, y): y = |x|\}$ is a function.

**Notation:** If $x$ is an element in the domain of a function $f$, then the element in the range that is associated with $x$ is denoted by $f(x)$ and is called the image of $x$ under the function $f$. This means $f = \{(x, y): y = f(x)\}$

The notation $f(x)$ is called function notation. Read $f(x)$ as "$f$ of $x$".

**Note:** $f$, $g$ and $h$ are the most common letters used to designate a function. But, any letter of the alphabet can be used.

e.g Suppose $f$: A ® B is the function that gives $5x$ – 1for any $x$ÎA. What are the possible ways of writing this function?

Solution

We can write it as $f$ : $x$ ® $5x$ – 1 or $f(x) = 5x - 1$ or $y = 5x - 1$ or $x$¾$f$® $5x$ – 1.

Remark

Vertical line test:

A set of points in the Cartesian plane is the graph of a function, if and only if no vertical line intersects the set more than once.

2.3. Classification of functions (one to one, onto, even and odd) and inverse of functions

A function $f: A \to B$ is said to be

i odd, if and only if, for any $x \in A$, $f(-x) = -f(x)$.

ii even, if and only if, for any $x \in A$, $f(-x) = f(x)$. The evenness or oddness of a function is called its parity.

Example 2

a $f(x) = x^3$ is odd, since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

b $f(x) = x^2$ is even since $f(-x) = (-x)^2 = x^2 = f(x)$.

c $f(x) = x + 1$ is neither even nor odd since $f(-x) = -x + 1 \neq -(x + 1) = -f(x)$ and $f(-x) = -x + 1 \neq x + 1 = f(x)$.

**Note:**

Exponential and Logarithmic Functions

✓ A function $f: \mathbb{R} \to (0,\infty)$ given by $f(x) = a^x$, $a > 0$, $a \neq 1$ is called an exponential function.

✓ A function $f: (0,\infty) \to \mathbb{R}$ given by $f(x) = \log_a x$, $a > 0$, $a \neq 1$ is called a logarithmic function.

✓ If $a > 0$, $a \neq 1$, then, $\log_a a^x = a^{\log_a x} = x$.

### One-to-One Functions

A function *f*: A ® B is said to be one-to-one (an injection), if and only if, each element of the range is paired with exactly one element of the domain, i.e., $f(x1) = f(x2) \Rightarrow x1 = x2$, for any $x1$ and $x2$ ÎA.

Remark

This is the same as saying $x1 \neq x2 \Rightarrow f(x1) \neq f(x2)$.

Example 1 Show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is one-to-one.

Solution Let $x_1, x_2 \in \mathbb{R}$ be any two elements such that $f(x_1) = f(x_2)$.

Then, $2x_1 = 2x_2 \Rightarrow \frac{1}{2}(2x_1) = \frac{1}{2}(2x_2) \Rightarrow x_1 = x_2$

Thus, $f$ is one-to-one.

Example 2 Show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one.

Solution Take $x_1 = 2$ and $x_2 = -2$.

Obviously, $x_1 \neq x_2$ i.e $2 \neq -2$

But $f(x_1) = f(2) = 2^2 = 4 = (-2)^2 = f(-2) = f(x_2)$

This means there are numbers $x_1, x_2 \in \mathbb{R}$ for which $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ does not hold.

Thus, $f$ is not one-to-one.

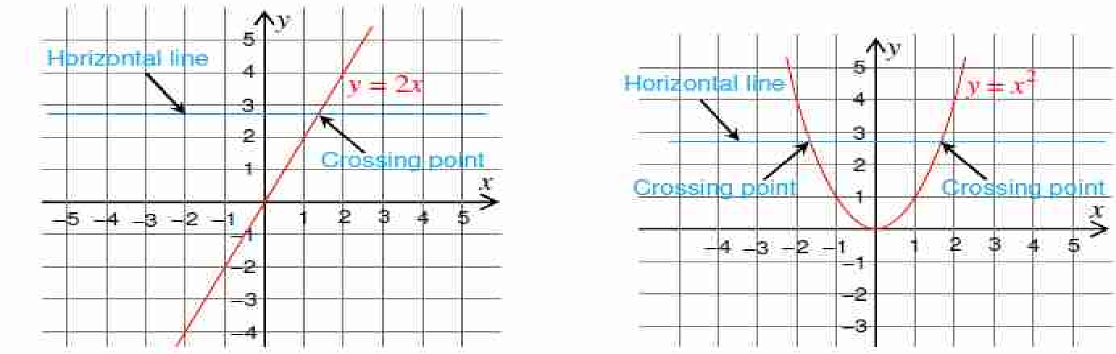
When the graph of $f: \mathbb{R} \to \mathbb{R}$ is given, i.e. $f$ is a graphical function, there is another way of checking its one-to-oneness.

The horizontal line test:

A function $f: A \to B$ is one-to-one, if and only if any horizontal line crosses its graph at most once.

**Example 3** Using the horizontal line test, show that $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is one-to-one.

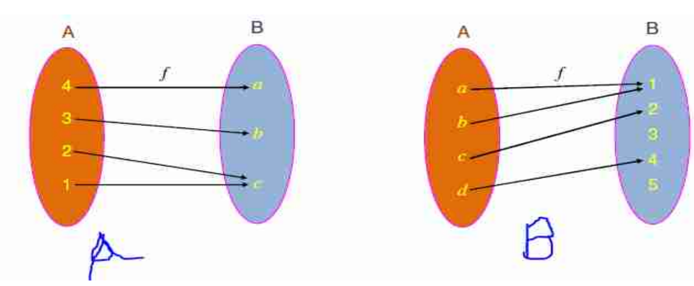
**Solution** From Figure 1.16, it is clear that any horizontal line crosses $y = 2x$ at most once. Hence, $f(x) = 2x$ is a one-to- one function.




Onto Functions

A function $f$ : A ® B is onto (a surjection), if and only if, Range of $f$ = B.

Which of the following diagram shows f is onto function?




Let $f$: $A \to B$ be a function.

Range of $f$ = B means for any $y \in$ B, there is $x \in$ A, such that $y = f(x)$.

So, to show $f$ is onto, if possible, show that for any $y$, there is $x \in$ A such that $f(x) = y$.

To show $f$ is not onto, find $y \in$ B that is not an image of any of the elements of A.

Definition

A function $f$ : A ® B is a one-to-one correspondence (a bijection), if and only if, $f$ is one-to-one and onto.

Examples

**Example 9** Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 5x - 7$. Show that $f$ is a one-to- one correspondence.

**Solution** Let $x_1, x_2 \in \mathbb{R}$, such that $f(x_1) = f(x_2)$

$$\Rightarrow 5x_1 - 7 = 5x_2 - 7 \Rightarrow 5x_1 - 7 + 7 = 5x_2 - 7 + 7$$

$$\Rightarrow 5x_1 = 5x_2 \Rightarrow x_1 = x_2$$

So, $f$ is one-to-one.

Let $y \in \mathbb{R}$. Is there $x \in \mathbb{R}$ such that $y = f(x)$?

If there is, it can be found by solving $y = f(x) = 5x - 7$

$$\Rightarrow y + 7 = 5x \Rightarrow x = \frac{y+7}{5}.$$

So for any $y \in \mathbb{R}$, take $x = \frac{y+7}{5} \in \mathbb{R}$.

Then $f(x) = f\left(\frac{y+7}{5}\right) = 5\left(\frac{y+7}{5}\right) - 7 = y$

So $f$ is onto.

Therefore, $f$ is a one-to-one correspondence.

## 2.4. Operations on functions and composition of functions

Note: we have four mathematical operations in Mathematics such as addition, substruction, division and multiplication.

### A Sum of functions

Suppose $f$ and $g$ are two functions. The sum of these functions is a function which is defined as $f + g$, where $(f + g)(x) = f(x) + g(x)$.

**Example 1** If $f(x) = 2 - x$ and $g(x) = 3x + 2$ then the sum of these functions is given by $(f + g)(x) = (2 - x) + (3x + 2) = 2x + 4$, which is also a function.

The domain of $f = \mathbb{R}$ and the domain of $g = \mathbb{R}$.

The function $(f + g)(x) = 2x + 4$ has also domain $= \mathbb{R}$.

**Example 2** Let $f(x) = 2x$ and $g(x) = \sqrt{2x}$. Determine

**a** the sum $f + g$ **b** the domain of $(f + g)$

**Solution:**

**a** $(f + g)(x) = f(x) + g(x) = 2x + \sqrt{2x}$ **b** Domain of $f + g = \{x: x \geq 0\}$.

### B Difference of functions

Suppose $f$ and $g$ are two functions. The difference of these functions is also a function, defined as $f - g$, where $(f - g)(x) = f(x) - g(x)$.

**Example 3** If $f(x) = 3x + 2$ and $g(x) = x - 4$, then the difference of these functions is $(f - g)(x) = f(x) - g(x) = (3x + 2) - (x - 4) = 2x + 6$ and

the domain of $f - g = \mathbb{R}$.

**Example 4** Let $f(x) = 2x$ and $g(x) = \sqrt{1-x}$. Determine:

**a** the difference $f - g$ **b** the domain of $f - g$

**Solution:**

**a** $(f - g)(x) = f(x) - g(x) = 2x - \sqrt{1-x}$

**b** Domain of $f - g = \{x: x \leq 1\}$.

## C Product of functions

Suppose $f$ and $g$ are two functions. The product of these functions is also a function, defined as $fg$, $(fg)(x) = f(x)g(x)$. Again,

**Example 5** If $f(x) = 2x$ and $g(x) = 3 - x$ then the product of these functions

$$(fg)(x) = f(x)\, g(x) = (2x)(3 - x) = 6x - 2x^2 \text{ and}$$

the domain of $fg = \mathbb{R}$.

**Note:** The domain of the sum, difference and product of functions $f$ and $g$ is the intersection of the domain of $f$ and of the domain of $g$.

## D Quotients of functions

Suppose $f$ and $g$ are two functions with $g \neq 0$. The quotient of these functions is also a function, defined as $\frac{f}{g}$ where $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

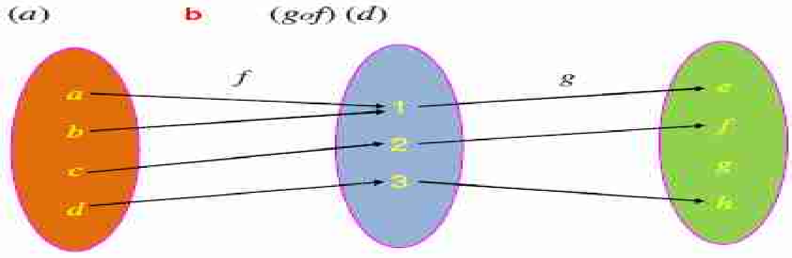
**Example 6** If $f(x) = 3$ and $g(x) = 2 + x$ then the quotient of these functions

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3}{2+x} \text{ and the domain of } \frac{f}{g} = \mathbb{R}\setminus\{-2\}.$$

COMPOSITION OF FUNCTIONS

Let $f: A® B$ and $g: B®C$ be functions. Then, the composition of $g$ by $f$, denoted by $gof$, is given as $(gof)(x) = g(f(x))$.

**a** $(gof)(a)$ **b** $(gof)(d)$




Answer

$(gof)(a) = g(f(a)) = g(1) = e$ and $(gof)(d) = g(f(d)) = g(3) = h$

2.5. Types of functions

2.5.1. Polynomial functions

Let $n$ be a non-negative integer and let $a_n, a_{n-1}, \ldots, a_1, a_o$ be real numbers with $a_n \neq 0$. The function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$$

is called a polynomial function in variable $x$ of degree $n$.

Note that in the definition of a polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$$

**i** $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_o$ are called the coefficients of the polynomial function (or simply the polynomial).

**ii** The number $a_n$ is called the leading coefficient of the polynomial function and $a_n x^n$ is the leading term.

**iii** The number $a_0$ is called the constant term of the polynomial.

**iv** The number $n$ (the exponent of the highest power of $x$), is the degree of the polynomial.

Note that the domain of a polynomial function is $\mathbb{R}$.

**Example 1** Which of the following are polynomial functions? For those which are polynomials, find the degree, leading coefficient, and constant term.

**a** $f(x) = \frac{2}{3}x^4 - 12x^2 + x + \frac{7}{8}$ **b** $f(x) = \frac{x}{x}$

**c** $g(x) = \sqrt{(x+1)^2}$ **d** $f(x) = 2x^{-4} + x^2 + 8x + 1$

**e** $k(x) = \frac{x^2+1}{x^2+1}$ **f** $g(x) = \frac{8}{5}x^{15}$

**g** $f(x) = (1 - \sqrt{2}x)(1 + \sqrt{2}x)$ **h** $k(y) = \frac{6}{y}$

**Solution:**

**a** It is a polynomial function of degree 4 with leading coefficient $\frac{2}{3}$ and constant term $\frac{7}{8}$.

**b** It is not a polynomial function because its domain is not $\mathbb{R}$.

Remark

**Based on the types of coefficients it has, a polynomial function $p$ is said to be:**

- ✓ a polynomial function over integers, if the coefficients of $p(x)$ are all integers.
- ✓ a polynomial function over rational numbers, if the coefficients of $p(x)$ are all rational numbers.
- ✓ a polynomial function over real numbers, if the coefficients of $p(x)$ are all real numbers.

**Remark:** Every polynomial function that we will consider in this unit is a polynomial function over the real numbers.

For example, if $g(x) = \frac{2}{3}x^4 - 13x^2 + \frac{7}{8}$, then $g$ is a polynomial function over rational and real numbers, but not over integers.

If $p(x)$ can be written in the form, $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o$, then different expressions can define the same polynomial function.

Activity

Student we can apply four mathematical operations on the polynomial functions. So, you need to discuss them and present it for the classroom!!

2.5.1.1. Theorems on polynomial functions (Division, remainder, factor and location theorems)

**Polynomial Division Theorem**

Recall that, when we divided one polynomial by another, we apply the long division procedure, until the remainder was either the zero polynomial or a polynomial of lower degree than the divisor.

For example, if we divide $x^2 + 3x + 7$ by $x + 1$, we obtain the following.

$$
\begin{array}{rrll}
 & x + 2 & \leftarrow & \text{quotient} \\
\text{Divisor} \rightarrow \; x + 1 \,\big) & x^2 + 3x + 7 & \leftarrow & \text{dividend} \\
 & \underline{x^2 + x} & & \\
 & 2x + 7 & & \\
 & \underline{2x + 2} & & \\
 & 5 & \leftarrow & \text{remainder}
\end{array}
$$

In fractional form, we can write this result as follows:

$$\underbrace{\overbrace{\frac{x^2 + 3x + 7}{x + 1}}^{\text{dividend}}}_{\text{divisor: } x+1} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overset{\text{remainder}}{5}}{\underbrace{x+1}_{\text{divisor}}}$$

This implies that $x^2 + 3x + 7 = (x + 1)(x + 2) + 5$ which illustrates the theorem called the polynomial division theorem.

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underbrace{f(x)}_{\text{Dividend}} = \underbrace{d(x)}_{\text{Divisor}} \, \underbrace{q(x)}_{\text{Quotient}} + \underbrace{r(x)}_{\text{Remainder}}$$

where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, $f(x)$ divides exactly into $d(x)$.

**Example 1** In each of the following pairs of polynomials, find polynomials $q(x)$ and $r(x)$ such that $f(x) = d(x)\,q(x) + r(x)$.

**a** $f(x) = 2x^3 - 3x + 1$; $d(x) = x + 2$

**b** $f(x) = x^3 - 2x^2 + x + 5$; $d(x) = x^2 + 1$

**c** $f(x) = x^4 + x^2 - 2$; $d(x) = x^2 - x + 3$

**Solution:**

**a** $\dfrac{f(x)}{d(x)} = \dfrac{2x^3 - 3x + 1}{x+2} = 2x^2 - 4x + 5 - \dfrac{9}{x+2}$

$\Rightarrow 2x^3 - 3x + 1 = (x+2)(2x^2 - 4x + 5) - 9$

Therefore $q(x) = 2x^2 - 4x + 5$ and $r(x) = -9$.

**b** $\dfrac{f(x)}{d(x)} = \dfrac{x^3 - 2x^2 + x + 5}{x^2 + 1} = x - 2 + \dfrac{7}{x^2+1}$

$\Rightarrow x^3 - 2x^2 + x + 5 = (x^2 + 1)(x - 2) + 7$

Therefore $q(x) = x - 2$ and $r(x) = 7$.

**c** $\dfrac{f(x)}{d(x)} = \dfrac{x^4 + x^2 - 2}{x^2 - x + 3} = x^2 + x - 1 + \dfrac{-4x+1}{x^2 - x + 3}$

$\Rightarrow x^4 + x^2 - 2 = (x^2 - x + 3)(x^2 + x - 1) + (-4x + 1)$

giving us $q(x) = x^2 + x - 1$ and $r(x) = -4x + 1$.

Definition **Remainder theorem**

Let $f(x)$ be a polynomial of degree greater than or equal to 1 and let c be any real number. If $f(x)$ is divided by the linear polynomial $(x - c)$, then the remainder is $f(c)$.

Example

1. Find the remainder by dividing $f(x)$ by $d(x)$ in each of the following pairs of polynomials, using the polynomial division theorem and the remainder theorem:
   a. $f(x) = x^3 - x^2 + 8x - 1$; $d(x) = x + 2$
   b. $f(x) = x^4 + x^2 + 2x + 5$; $d(x) = x - 1$

**Solution:**

**a** Polynomial division theorem

$\dfrac{x^3 - x^2 + 8x - 1}{x+2} = x^2 - 3x + 14 - \dfrac{29}{x+2}$

Remainder theorem

$f(-2) = (-2)^3 - (-2)^2 + 8(-2) - 1$

$= -8 - 4 - 16 - 1 = -29$

Therefore, the remainder is $-29$.

**b** Polynomial division theorem

$\dfrac{x^4 + x^2 + 2x + 5}{x - 1} = x^3 + x^2 + 2x + 4 + \dfrac{9}{x - 1}$

Remainder theorem

$f(1) = (1)^4 + (1)^2 + 2(1) + 5$

$= 1 + 1 + 2 + 5 = 9$

Therefore, the remainder is 9.

2. When $x^3 - 2x^2 + 3bx + 10$ is divided by $x - 3$ the remainder is 37. Find the value of $b$.

Solution:

Let $f(x) = x3 - 2x2 + 3bx + 10$.

$f(3) = 37$. (By the remainder theorem)

$\Rightarrow (3)3 - 2(3)2 + 3b(3) + 10 = 37$

$27 - 18 + 9b + 10 = 37 \Rightarrow 9b + 19 = 37 \Rightarrow b = 2$.

**Factor theorem**

Let *f(x)* be a polynomial of degree greater than or equal to one, and let c be any real number, then

a. $x - c$ is a factor of *f(x)*, if $f(c) = 0$, and
b. $f(c) = 0$, if $x - c$ is a factor of $f(x)$.

Examples

1) Let $f(x) = x3 + 2x2 - 5x - 6$. Use the factor theorem to determine whether:
   a. $x + 1$ is a factor of $f(x)$
   b. $x + 2$ is a factor of $f(x)$.

**Solution:**

a. Since $x + 1 = x - (-1)$, it has the form $x - c$ with $c = -1$.
   $f(-1) = (-1)3 + 2(-1)2 - 5(-1) - 6 = -1 + 2 + 5 - 6 = 0$.
   So, by the factor theorem, $x + 1$ is a factor of $f(x)$.
b. $f(-2) = (-2)3 + 2(-2)2 - 5(-2) - 6 = -8 + 8 + 10 - 6 = 4$ ¹ 0.
   By the factor theorem, $x + 2$ is not a factor of $f(x)$.

Definition

For a polynomial function $f$ and a real number $c$, if $f(c) = 0$, then $c$ is a zero of $f$.

If $(x - c)^k$ is a factor of $f(x)$, but $(x - c)^{k+1}$ is not, then $c$ is said to be a zero of multiplicity $k$ of $f$.

Example

Given that $-1$ and 2 are zeros of $f(x) = x4 + x3 - 3x2 - 5x - 2$, determine their multiplicity.

**Solution:** By the factor theorem, $(x + 1)$ and $(x - 2)$ are factors of $f(x)$

Hence, $f(x)$ can be divided by $(x + 1)(x - 2) = x^2 - x - 2$, giving you

$$f(x) = (x^2 - x - 2)(x^2 + 2x + 1) = (x + 1)(x - 2)(x + 1)^2 = (x + 1)^3(x - 2)$$

Therefore, $-1$ is a zero of multiplicity 3 and 2 is a zero of multiplicity 1.

Location theorem

Let $a$ and $b$ be real numbers such that $a < b$. If $f$ is a polynomial function such that $f(a)$ and $f(b)$ have opposite signs, then there is at least one zero of $f$ between $a$ and $b$.

Example

Let $f(x) = x4 - 6x3 + x2 + 12x - 6$. Construct a table of values and use the location theorem to locate the zeros of $f$ between successive integers.

**Solution:** Construct a table and look for changes in sign as follows:

| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 210 | 38 | –10 | –6 | 2 | –10 | –42 | –70 | –44 | 102 |

Since $f(-2) = 38 > 0$ and $f(-1) = -10 < 0$, we see that the value of $f(x)$ changes from positive to negative between $-2$ and $-1$. Hence, by the location theorem, there is a zero of $f(x)$ between $x = -2$ and $x = -1$.

Since $f(0) = -6 < 0$ and $f(1) = 2 > 0$, there is also one zero between $x = 0$ and $x = 1$.

Similarly, there are zeros between $x = 1$ and $x = 2$ and between $x = 5$ and $x = 6$.

**Rational root test**

If the rational number $\frac{p}{q}$, in its lowest terms, is a zero of the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

with integer coefficients, then $p$ must be a factor of $a_0$ and $q$ must be a factor of $a_n$.

Examples

In each of the following, find all the rational zeros of the polynomial:

**a** $f(x) = x^3 - x + 1$ **b** $g(x) = 2x^3 + 9x^2 + 7x - 6$

**c** $g(x) = \frac{1}{2}x^4 - 2x^3 - \frac{1}{2}x^2 + 2x$

**Solution:**

**a** The leading coefficient is 1 and the constant term is 1. Hence, as these are factors of the constant term, the possible rational zeros are $\pm 1$.

Using the remainder theorem, test these possible zeros.

$$f(1) = (1)^3 - 1 + 1 = 1 - 1 + 1 = 1$$

$$f(-1) = (-1)^3 - (-1) + 1 = -1 + 1 + 1 = 1$$

So, we can conclude that the given polynomial has no rational zeros.

2.5.1.2. Graphs of polynomial functions

The graph of a polynomial function of degree $n$ has at most $n - 1$ turning points and intersects the $x$-axis at most $n$ times. The graph of every polynomial function has no sharp corners; it is a smooth and continuous curve.

Student if you know the steps to sketch the graph of polynomial function, please try to tell your friends and give examples.

2.5.1.3. Rational functions and their graphs

A rational function is a function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials in $x$ and $q(x)\neq 0$.

1 The line $x=a$ is called a vertical asymptote of the graph of $f$ if $f(x)\to\infty$ or $f(x)\to-\infty$ as $x\to a$, either from the left or from the right.

2 The line $y=b$ is called a horizontal asymptote of the graph of $f$ if $f(x)\to b$ as $x\to\infty$ or as $x\to-\infty$.

**Rules for asymptotes and holes**

Once the domain is established and the restrictions are identified, here are the pertinent facts.

Let $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{ax^n+...+a_0}{bx^m+...+b_0}$, be a rational function, where $n$ is the largest exponent in the numerator and $m$ is the largest exponent in the denominator.

1 The graph will have a vertical asymptote at $x=a$ if $q(a)=0$ and $p(a)\neq 0$. In case $p(a)=q(a)=0$, the function has either a hole at $x=a$ or requires further simplification to decide.

2 If $n<m$, then the $x$-axis is the horizontal asymptote.

3 If $n=m$, then the line $y=\dfrac{a}{b}$ is a horizontal asymptote.

4 If $n=m+1$, the graph has an oblique asymptote and we can find it by long division.

5 If $n>m+1$, the graph has neither an oblique nor a horizontal asymptote.

Example

Give the vertical and horizontal asymptotes, if they exist:

a $f(x)=\dfrac{1}{x+2}$ b $f(x)=\dfrac{x-2}{x^2-4}$

c $f(x)=\dfrac{x^2-1}{x^2+3x+2}$ d $f(x)=\dfrac{(x-1)(x+1)}{(x+1)^2(x+2)}$

Solution

a $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{1}{x+2}$. The domain of $f$ is $\{x : x\neq -2\}$.

Since $p(-2)\neq 0$ and $q(-2)=0$, $x=-2$ is a vertical asymptote.
Besides degree of $p(x)<$ degree of $q(x)$. Thus, $y=0$ is a horizontal asymptote..

b Consider the rational function $f(x)=\dfrac{p(x)}{q(x)}=\dfrac{x-2}{x^2-4}$. The domain of $f$ is all real numbers except $x=-2$ and $x=2$.

$p(-2)\neq 0$ and $q(-2)=0$. Thus, $x=-2$ is a vertical asymptote.

$p(2)=q(2)=0$. Thus, $f$ has a hole at $\left(2, \dfrac{1}{4}\right)$

Activity. Please do c and d above left questions.

The zeros of a rational function

Let $f(x)=\dfrac{p(x)}{q(x)}$ be a rational function. An element $a$ in the domain of $f$ is called a zero of $f$, if and only if $p(a)=0$.

Example

Find the zeros of the following rational functions:

**a** $f(x)=\dfrac{x^2+3x+2}{x^2-2x-3}$ **b** $f(x)=\dfrac{x^2-6x+9}{x^2-9}$

**Solution**

**a** We first factorize both numerator and denominator.

$f(x)=\dfrac{(x+1)(x+2)}{(x+1)(x-3)}$ The domain is $\mathbb{R}\setminus\{-1, 3\}$. Now for any $x$ in the domain $f(x)=0$ means the numerator $x^2+3x+2=0$. i.e. $x=-1$ or $x=-2$. But, since $x=-1$ is not in the domain of $f$, the only zero of $f$ is $x=-2$.

**b** Factorize both numerator and denominator: $f(x)=\dfrac{(x-3)^2}{(x+3)(x-3)}$

The domain is $\mathbb{R}\setminus\{-3, 3\}$. The numerator is zero at $x=3$. But since 3 is not in the domain $f$, $f$ has no zero.

Graphs of Rational Functions

**Steps to sketch the graph of a rational function:**

1 Reduce the rational function to lowest terms and check for any open holes in the graph.
2 Find $x$-intercept(s) by setting the numerator equal to zero.
3 Find the $y$-intercept (if there is one) by setting $x=0$ in the function.
4 Find all its asymptotes (if any).
5 Determine the parity (i.e. whether it is even or odd or neither).

6 Use the $x$-intercepts and vertical asymptote(s) to divide the $x$-axis into intervals. Choose a test point in each interval to determine if the function is positive or negative there. This will tell you whether the graph approaches the vertical asymptote in an upward or downward direction.

7 Sketch the graph! Except for breaks at the vertical asymptotes or cusps, the graph should be a nice smooth curve with no sharp corners.

To draw the graph of $f(x)=\dfrac{p(x)}{q(x)}$,

| We need to find | Criteria |
|---|---|
| Domain | $\mathbb{R} \setminus \{x: q(x)=0\}$ |
| $x$ - intercept | Zero of $f$ |
| $y$ - intercept | $x=0$ and $0 \in$ domain of $f$ |
| Vertical asymptote | $P(x) \neq 0$ and $q(x)=0$ |
| Horizontal asymptote | Degree of $p(x) \leq$ Degree of $q(x)$ |
| Oblique asymptote | Degree of $p(x)=$ Degree $q(x)+1$ |
| Parity | $f$ is odd or even or neither |

Sketch the graph of each of the following functions:

a $f(x)=-\dfrac{1}{x^2}$.

b $f(x)=\dfrac{3x^2}{(x-2)(x+1)}$

c $f(x)=\dfrac{x+1}{(x-2)(x+3)^2}$

d $f(x)=\dfrac{x^2+5x+6}{x+1}$.

e $f(x)=\dfrac{x-2}{x^2-4}$.

## 2.5.2. Exponential functions and their graphs

If the bases $a$ and $b$ are non-zero real numbers and the exponents $m$ and $n$ are integers, then

| | | |
|---|---|---|
| 1 | $a^m \times a^n = a^{m+n}$ | To multiply powers of the same base, keep the base and add the exponents. |
| 2 | $\dfrac{a^m}{a^n} = a^{m-n}$ | To divide powers of the same base, keep the base and subtract the exponents. |
| 3 | $(a^m)^n = a^{m\times n} = a^{mn}$ | To take a power of a power, keep the base and multiply the exponents. |
| 4 | $(a \times b)^n = a^n \times b^n$ | The power of a product is the product of the powers. |
| 5 | $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ | The power of a quotient is the quotient of the powers. |

Simplify each of the following:

| | | | | | |
|---|---|---|---|---|---|
| a | $(4t)^2 \times (4t)^7$ | b | $r^8 \times r^{-3}$ | c | $\dfrac{10^3}{10^5}$ |
| d | $(x^2)^m$ | e | $16 \times 4^{3t}$ | f | $\left(\dfrac{2y}{25}\right)^2$ |

**Solution:**

| | | | |
|---|---|---|---|
| a | $(4t)^2 \times (4t)^7 = (4t)^{2+7} = (4t)^9$ | b | $r^8 \times r^{-3} = r^{8+(-3)} = r^5$ |
| c | $\dfrac{10^3}{10^5} = 10^{3-5} = 10^{-2}$ | d | $(x^2)^m = x^{2\times m} = x^{2m}$ |
| e | $16 \times 4^{3t} = 2^4 \times (2^2)^{3t} = 2^4 \times 2^{6t} = 2^{4+6t}$ | f | $\left(\dfrac{2y}{25}\right)^2 = \dfrac{2^2 \times y^2}{25^2} = \dfrac{4y^2}{625}$ |

If $x$ is an irrational number and $a > 0$, then $a^x$ is the real number between $a^{x_1}$ and $a^{x_2}$ for all possible choices of rational numbers $x_1$ and $x_2$ such that $x_1 < x < x_2$

Activities

Simplify each of the following expressions using one or more of the laws of exponents:

| | | | | | |
|---|---|---|---|---|---|
| a | $a^2 \times a \times a^3$ | b | $(2^{-3} + 3^{-2})^{-1}$ | c | $\left(\sqrt[3]{343}\right)^{-2}$ |
| d | $(2a^{-3} \times b^2)^{-2}$ | e | $\dfrac{(3a)^4}{(3a)^3}$ | f | $\left(\dfrac{a^2}{b}\right)^3$ |
| g | $\left(\dfrac{a^3}{b^5}\right)^{-2}$ | h | $\dfrac{(n^2)^4 \times (n^3)^{-2}}{n^{-1}}$ | i | $\left(\dfrac{m^{-3}m^3}{n^{-2}}\right)^{-2}$ |
| j | $\left(\dfrac{m^{\frac{-2}{3}}}{n^{\frac{-1}{2}}}\right)^{-6}$ | k | $\left(\dfrac{a^{\frac{-1}{3}} b^{\frac{1}{2}}}{a^{\frac{-1}{4}} b^{\frac{1}{3}}}\right)^6$ | l | $\dfrac{\left(3^{\sqrt{2}}\right)^2 \times 9^{-\sqrt{3}}}{3^{-\sqrt{12}}}$ |
| m | $\left(2^{\sqrt{3}}\right)^2 \div \left(4^{\sqrt{3}}\right)^{-2}$ | n | $\left(\dfrac{2^{\sqrt{5}} \times 2^{-\sqrt{5}}}{\sqrt{2}}\right)^2$ | | |
| o | $\dfrac{2^{\sqrt{7}} \times 2^{-\sqrt{7}} \times 32^{\sqrt{2}}}{4^{\sqrt{8}}}$ | p | $\sqrt[6]{64a^6b^{-2}}$ | | |

The function $f(x) = b^x$, $b > 0$ and $b \neq 1$ defines an exponential function.

Activity

Give examples of exponential functions

Graphs of exponential functions

**Example 1** Draw the graph of $f(x)=2^x$.

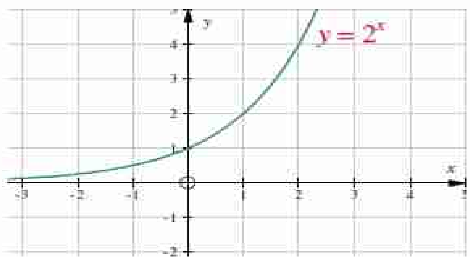
**Solution:** Evaluate $y=2^x$ for some integral values of $x$ and prepare a table of values.

For example: $f(-3)=2^{-3}=\frac{1}{8}$; $f(-2)=2^{-2}=\frac{1}{4}$; $f(-1)=2^{-1}=\frac{1}{2}$;

$f(0)=2^0=1$; $f(1)=2^1=2$; $f(2)=2^2=4$; $f(3)=2^3=8$.

| $x$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)=2^x$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |

Now plot these points on the co-ordinate system and join them by a smooth curve to obtain the graph of $f(x)=2^x$




The graph of $f(x)=b^x$, $b>1$ *has the following basic properties:*

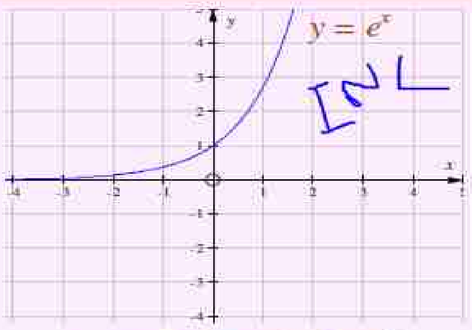
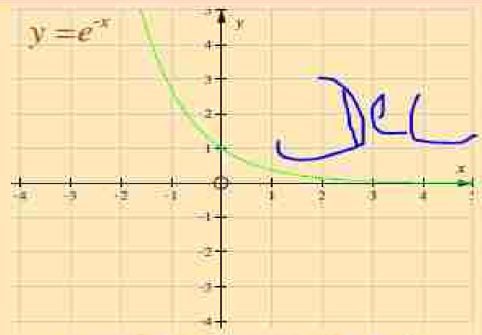
1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point (0, 1), i.e. the $y$-intercept is 1.
4. The function is increasing.
5. The values of the function are greater than 1 for $x>0$ and between 0 and 1 for $x<0$.
6. The graph approaches the $x$ – axis as an asymptote on the left and increases without bound on the right.

The graph of $f(x)=b^x$, $0<b<1$ *has the following basic properties:*

1. The domain is the set of all real numbers.
2. The range is the set of all positive real numbers.
3. The graph includes the point (0, 1), i.e. the $y$ - intercept is 1.
4. The function is decreasing.
5. The values of the function are greater than 1 for $x<0$ and between 0 and 1 for $x>0$.
6. The graph approaches the $x$-axis as an asymptote on the right and increases without bound on the left.

**The Natural Exponential Function**

For any real number $x$, the function, $f(x) = e^x$ defines the exponential function with base $e$, usually called the natural exponential function.

| | |
|---|---|
|   [redacted] The graph of $y = e^x$. |   [redacted] The graph of $y = e^{-x}$. |
| The domain of $y = e^x$ is $\mathbb{R}$. | The domain of $y = e^{-x}$ is $\mathbb{R}$. |
| The range is $(0, \infty)$ | The range is $(0, \infty)$ |
| $y = e^x$ is an increasing function. | $y = e^{-x}$ is a decreasing function. |
| The graph of $y = e^x$ intersects the $y$-axis at (0, 1). | The graph of $y = e^{-x}$ intersects the $y$-axis at (0, 1). |
| $e^x > 1$, if $x > 0$ | $e^{-x} > 1$, if $x < 0$ |
| $0 < e^x < 1$, if $x < 0$ | $0 < e^{-x} < 1$, if $x > 0$ |

Solving exponential equations

**Example 1** Solve for $x$.

**a** $3^x = 81$ **b** $2^x = \frac{1}{32}$ **c** $\left(\frac{2}{3}\right)^{2x+1} = \left(\frac{9}{4}\right)^x$ **d** $4^x = \left(\frac{1}{2}\right)^{x-3}$

**Solution:**

**a** $3^x = 81 = 3^4$ … *look for a common base*

$\Rightarrow x = 4$ … *property of equality of bases*

**b** $2^x = \frac{1}{2^5} = 2^{-5}$ … *look for a common base*

$\Rightarrow x = -5$ … *property of equality of bases*

Activity

**1** Solve for $x$:

**a** $5^x = 625$ **b** $2^{3-x} = 16$ **c** $4^{3x-8} = 2^{3x+9}$

**d** $\frac{1}{27} = \left(\frac{1}{9}\right)^{2x}$ **e** $3^{-x} = 81$ **f** $2^{x^2-2} = 4$

**g** $7^{x^2+x} = 49$ **h** $3^{6(x+2)} = 9^{x+2}$ **i** $3\left(\frac{27}{8}\right)^{\frac{2}{3}x-1} = 2\left(\frac{32}{243}\right)^{2x}$

**2** Solve for $x$ by taking the logarithm of each side:

**a** $2^x = 15$ **b** $10^x = 14.3$ **c** $10^{3x+1} = 92$ **d** $1.05^x = 2$

**e** $6^{3x} = 5$ **f** $4^{2x} = 61$ **g** $10^{5x-2} = 348$ **h** $2^{-x} = 0.238$

2.5.3. Logarithmic functions, graphs of logarithmic functions and equations involving logarithms

For a fixed positive number $b \neq 1$, and for each $a > 0$

$b^c = a$, if and only if $c = \log_b a$.

Hence, the function $y = \log_b x$, where $x > 0$, $b > 0$ and $b \neq 1$ is called a logarithmic function with base $b$.

The following functions are all logarithmic:

a $f(x) = \log_2 x$ b $g(x) = \log_{\frac{3}{2}} x$ c $h(x) = \log_3 x$

d $k(x) = \log_{10} x$ e $f(x) = \log_{\frac{1}{10}} x$ f $g(x) = \log_{\frac{1}{3}} x$

g $h(x) = \log_{\frac{1}{2}} x$ h $k(x) = \log_{\frac{2}{3}} x$

Example

Write an equivalent logarithmic statement for:

a $3^4 = 81$ b $4^3 = 64$ c $8^{\frac{1}{3}} = 2$

**Solution:**

a From $3^4 = 81$, we deduce that $\log_3 81 = 4$

b From $4^3 = 64$, we have $\log_4 64 = 3$

c Since $8^{\frac{1}{3}} = 2$, $\log_8 2 = \frac{1}{3}$

**Example 2** Write an equivalent exponential statement for:

a $\log_{12} 144 = 2$ b $\log_4 \left(\frac{1}{64}\right) = -3$ c $\log_{10} \sqrt{10} = \frac{1}{2}$

**Solution:**

a From $\log_{12} 144 = 2$, we deduce that $12^2 = 144$.

b $\log_4 \frac{1}{64} = -3$ is the same as saying $4^{-3} = \frac{1}{64}$.

c $\log_{10} \sqrt{10} = \frac{1}{2}$ can be written in exponential form as $10^{\frac{1}{2}} = \sqrt{10}$.

Laws of logarithms:

If $b$, $x$ and $y$ are positive numbers and $b \neq 1$, then

i $\log_b xy = \log_b x + \log_b y$ ii $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$

iii For any real number $k$, $\log_b (x^k) = k \log_b x$

If $a$, $b$ and $c$ are positive real numbers, and $a \neq 1$, $b \neq 1$, then

i $\log_a c = \frac{\log_b c}{\log_b a}$ ("change of base law") ii $b^{\log_b c} = c$

**Note:** If $b > 0$ and $b \neq 1$, then

**i** $\log_b b = 1$ **ii** $\log_b 1 = 0$

Example

Use the laws of logarithms to find:

**a** $\log_2 16 + \log_2 4$ **b** $\log_4 \sqrt{16} - \log_4 4$

**c** $2((\log_{10} 100) - 1)$ **d** $\log_{10} \sqrt[4]{0.01}$

**Solution:**

**a** $\log_2 16 + \log_2 4 = \log_2 (16 \times 4) = \log_2 64 = 6$

$\ldots$ *using the law* $\log_b xy = \log_b x + \log_b y$

**b** $\log_4 \sqrt{16} - \log_4 4 = \log_4 \dfrac{\sqrt{16}}{4} = \log_4 \dfrac{4}{4} = \log_4 1 = 0$

$\ldots$ *using the law* $\log_b (\frac{x}{y}) = \log_b x - \log_b y$

**Remark**

$\log 526 = \log(5.26 \times 10^2) = \underset{\text{Mantissa}}{\log 5.26} + \underset{\text{Characteristic}}{\log 10^{2}} = \log 5.26 + 2 = \underbrace{0.7210}_{M} + \underbrace{2}_{C} = 2.7210$

Please refer this calculation from grade 10th at the end of your textbook.

**Example 1** Draw the graph of each of the following using:

**i** different coordinate systems **ii** the same coordinate system.

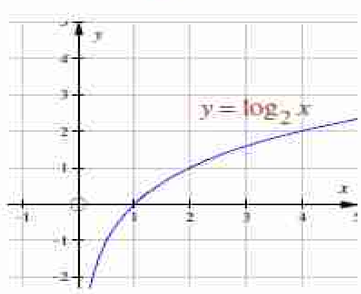
**a** $f(x) = \log_2 x$ **b** $g(x) = \log_{\frac{3}{2}} x$.

**Solution:** The tables below indicate some values for $f(x)$ and $g(x)$. Plot the corresponding points on the co-ordinate system. Join these points by smooth curves to get the required graphs as indicated in

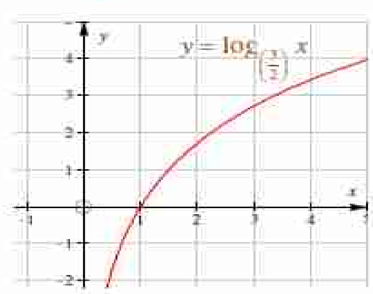
| $x$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 |
|---|---|---|---|---|---|
| $f(x) = \log_2 x$ | $-2$ | $-1$ | 0 | 1 | 2 |

| $x$ | $\frac{4}{9}$ | $\frac{2}{3}$ | 1 | $\frac{3}{2}$ | $\frac{9}{4}$ |
|---|---|---|---|---|---|
| $g(x) = \log_{\frac{3}{2}} x$ | $-2$ | $-1$ | 0 | 1 | 2 |

**a**




**b**



The graph of $y = \log_b x, (b > 1)$ has the following properties.

1 The domain is the set of all positive real numbers.
2 The range is the set of all real numbers.
3 The graph includes the point (1, 0) i.e. the $x$-intercept of the graph is 1.
4 The function increases, as $x$ increases.
5 The $y$-axis is a vertical asymptote of the graph.
6 The values of the function are negative for $0 < x < 1$ and they are positive for $x > 1$.

The graph of $y = \log_b x, (0 < b < 1)$ has the following properties.

1 The domain is the set of all positive real numbers.
2 The range is the set of all real numbers.
3 The graph has its $x$-intercept at (1, 0) i.e. its $x$-intercept is 1.
4 The function decreases as $x$ increases.
5 The $y$-axis is an asymptote of the graph.
6 The values of the function are positive when $0 < x < 1$ and they are negative when $x > 1$.

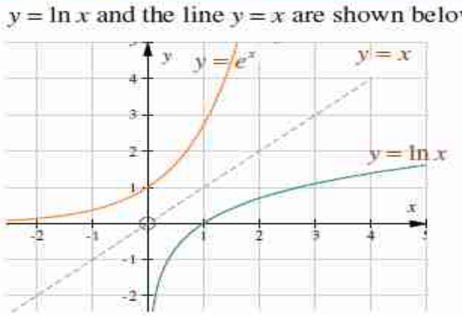
## Natural logarithm

If we start with natural exponential function $y = e^x$ and interchange $x$ and $y$, we obtain $x = e^y$ which is the same as $y = \log_e x$.

$y = \log_e x$ is the mirror image of $y = e^x$ along the line $y = x$.

**Notation:** $\ln x$ is used to represent $\log_e x$.
$\ln x$ is called the natural logarithm of $x$.

The graphs of $y = e^x$, $y = \ln x$ and the line $y = x$ are shown below:



**Example 1** Solve each of the following for $x$, checking that your solutions are valid.

**a** $\log_2 (x-3) = 5$ **b** $\log_4 (5x-1) = 3$

**c** $\log (x+3) + \log x = 1$ **d** $\log_3 (x+1) - \log_3 (x+3) = 1$

**e** $\log 8x + \log (x-20) = 3$

**Solution:**

**a** $\log_2 (x-3) = 5 \Rightarrow 2^5 = x-3$ … *changing to exponential form*

Hence, $32 = x-3$

Therefore, $x = 35$

Check!

From the definition of logarithms, we know that $\log_2 (x-3)$ is valid only when $x-3 > 0$, i.e. When $x > 3$. So $\{x \mid x > 3\} = (3, \infty)$ is known as the universe for $\log_2(x-3)$. Since $x = 35$ is an element of the universe, $x = 35$ is the solution of the given equation.

A universe is the largest set in $\mathbb{R}$ for which the given expression is defined.

---

**1** State the universe and solve each of the following for $x$:

**a** $\log_3 (2x-1) = 5$ **b** $\log_{\sqrt{2}} x = -6$

**c** $\log_3 (x^2 - 2x) = 1$ **d** $\log_2 (x^2 + 3x + 2) = 1$

**e** $\log_2 (1 + \frac{1}{x}) = 3$ **f** $\log_2 (x-1) + \log_2 3 = 3$

**g** $\log (x^2 - 121) - \log (x+11) = 1$ **h** $\log_3 (x+4) - \log_3 (x-1)$

**i** $\log (6x+5) - \log 3 = \log 2 - \log x$ **j** $\log x - \log 3 = \log 4 - \log (x+4)$

**k** $\log_3 (x+1) + \log_3(x+3) = 1$ **l** $\log_2 2 + \log_2(x+2) - \log_2(3x-5) = 3$

**m** $\log_x (x+6) = 2$

**2** Apply the property of Equality for Logarithmic Equations to solve the following equations (Check that your solutions are valid):

**a** $\log_3 x + \log_3 5 = 0$ **b** $\log_3 25 - 2\log_3 x = 0$

**c** $\log_5 x + \log_5 (x+1) = \log_5 2$ **d** $\log 2^x - \log 16 = 0$

**e** $\log_4(3^{6(x+2)}) - \log_4(9^{x+2}) = 0$ **f** $\log_2(x^2 - 9) - \log_2(3+x) = 2$

Application of exponential and logarithmic function

Activity

**1** Suppose you are observing the behaviour of cell duplication in a laboratory. In one experiment, you start with one cell and the cell population is tripling every minute.

**a** Write a formula to determine the number of cells after $t$ minutes.

**b** Use your formula to calculate the number of cells after an hour.

**c** Determine how long it would take the number of cells to reach 100,000.

**2** Suppose in an experiment you started with 100,000 cells and observed that the cell population decreased by one half every minute.

**a** Write a formula for the number of cells after $t$ minutes.

**b** Determine the number of cells after 10 minutes.

**c** Determine how long it would take the cell population to reach 10.

2.5.5. Power function, Absolute value function, Signum function and greatest integer function

A power function is a function which can be written in the form $f(x) = ax^r$, where $r$ is a rational number and $a$ in $\mathbb{R}$, is a fixed number.

Definition

For any real number $a$, the absolute value or modulus of $a$, is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0 \end{cases}$$

The modulus (Absolute value) function is the function given by $f(x) = |x|$.

**Signum Function**

The signum function, read as signum $x$, is written as sgn $x$ and is defined by

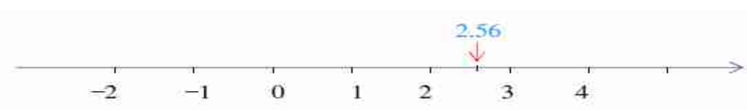
$$y = f(x) = \operatorname{sgn} x = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

**Greatest Integer (Floor) Function**

The greatest integer function, denoted by

$f(x) = \lfloor x \rfloor$, is defined as the greatest integer $\leq x$.

What is the largest among the integers that is less than or equal to 2.56?




You can see that it is 2.

Thus, $\lfloor 2.56 \rfloor = 2$

Example

Find $\lfloor x \rfloor$ when

**a** $x = -4.6$ **b** $x = 3$ **c** $x = 7.2143\ldots$

**Solution**

**a** $-5$ is the largest integer $\leq -4.6$. i.e., $\lfloor -4.6 \rfloor = -5$.

**b** 3 is the largest integer $\leq 3$. i.e. $\lfloor 3 \rfloor = 3$.

**c** $7.2143\ldots$ is between 7 and 8. Thus, $\lfloor 7.2143 \rfloor = 7$.

Activity

1 What is the value of each of the following?

a $\lfloor \pi \rfloor$ b $\lfloor -21.01 \rfloor$ c $\lfloor 21.01 \rfloor$ d $\lfloor 0 \rfloor$

2 Given $f(x) = \lfloor x \rfloor$,

i verify that if $k \in \mathbb{Z}, x \in \mathbb{R}$, then $f(x+k) = f(x)+k$ by taking

a $x = 4.25, k = 6$ b $x = -3.21, k = 7$ c $x = 8, k = -11$

ii verify that $f(x) + f(y) \leq f(x + y) \leq x + y$, using

a $x = 4.25, y = 6.32$ b $x = -2.01, y = \pi$ c $x = 4, y = -6.24$

iii verify that $f(x) \leq x < f(x)+1$ by taking

a $x = 2.5$ b $x = -3.54$ c $x = 4$

3 Let $a = x - \lfloor x \rfloor$.

a Using Question 2iii above, show that $0 \leq a < 1$.

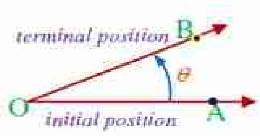
b Show that $x = \lfloor x \rfloor + a, \quad 0 \leq a < 1$.

c Show that $f(x + k) = f(x) + k$, when $k \in \mathbb{Z}, x \in \mathbb{R}$ using 3b.

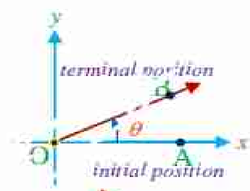
2.5.6. Trigonometric (Sine, Cosine and Tangent) functions and their graphs.

**The Sine, Cosine and Tangent Functions**

If a given ray $OA$ (written as $\overline{OA}$) rotates around a point O from its initial position to a new position, it forms an angle $\theta$ as shown below.



a



b

$\overline{OA}$ (initial position) is called the initial side of $\theta$.

$\overline{OB}$ (terminal position) is called the terminal side of $\theta$.

The angle formed by a ray rotating anticlockwise is taken to be a positive angle.

An angle formed by a ray rotating clockwise is taken to be a negative angle.

Angles in standard position

An angle in the coordinate plane is said to be in standard position, if

a. its vertex is at the origin, and

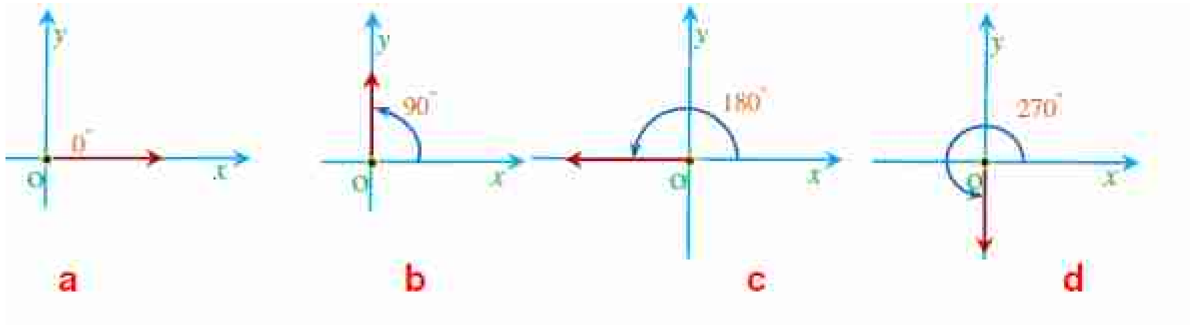
b. its initial side lies on the positive $x$-axis.

Quadrantal angles

a. If the terminal side of an angle in standard position lies along the $x$-axis or the $y$-axis,

b. then the angle is called a quadrantal angle.

The following all are quadrantal angles.




The Sine, Cosine and Tangent Functions are the three basic trigonometric functions.

Trigonometric functions were originally used to relate the angles of a triangle to the lengths of the sides of a triangle. It is from this practice of measuring the sides of a triangle with the help of its angles (or vice versa) that the name trigonometry was coined.


Let us consider the right angled triangles in

You already know that, for a given right angled triangle, the hypotenuse (HYP) is the side which is opposite the right angle and is the longest side of the triangle.

For the angle marked by $\theta$

- ✓ $\overline{BC}$ is the side opposite (OPP) angle $\theta$.
- ✓ $\overline{AC}$ is the side adjacent (ADJ) angle $\theta$.

Similarly, for the angle marked by $\phi$

- ✓ $\overline{AC}$ is the side opposite (OPP) angle $\phi$.
- ✓ $\overline{BC}$ is the side adjacent (ADJ) angle $\phi$.

Remark

Trigonometric functions can be considered in the same way as any general function, linear, quadratic, exponential or logarithmic. The input value for a trigonometric function is an angle. That angle could be measured in degrees or radians. The output value for a trigonometric function is a pure number with no unit.

Co–terminal angles are angles in standard position that have a common terminal side.

**a** The three angles with measures $30^\circ$, $-330^\circ$ and $390^\circ$ are co-terminal angles


**b** The three angles with measures $55^\circ$, $-305^\circ$ and $415^\circ$ are also co-terminal.

Graphs of the Sine, Cosine and Tangent Functions

This part need detail discussion!!!!

**Chapter 3**: **Geometry and Measurement (10 hrs)**

3.1. Theorems on triangles

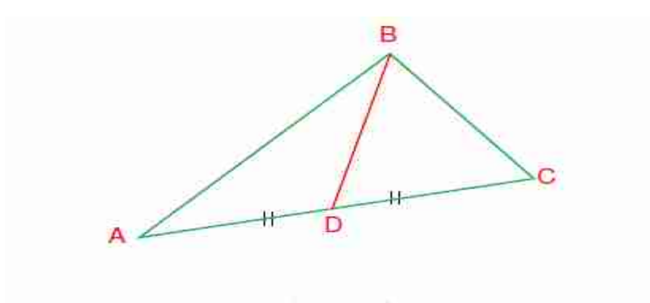
Definition

Triangle is a polygon with three sides and is the simplest type of polygon

Three or more points that lie on one line are called collinear points. Three or more lines that pass through one point are called concurrent lines.


Definition

A median of a triangle is a line segment drawn from any vertex to the mid-point of the opposite side.




$\overline{BD}$ is a median of triangle ABC.

**Theorem**

The medians of a triangle are concurrent at a point2/3 of the distance from each vertex to the mid-point of the opposite side.

**Definition**

The **altitude** of a triangle is a line segment drawn from a vertex, perpendicular to the opposite side, or to the opposite side produced.

**Definition**

The perpendicular bisectors of the sides of any triangle are concurrent at a point which is equidistant from the vertices of the triangle.

Remark

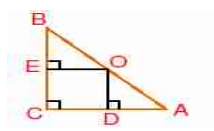
The point of intersection of the perpendicular bisectors of a triangle is called circumcentre of the triangle.

Theorem

The angle bisectors of any triangle are concurrent at a point which is equidistant from the sides of the triangle.

In a right angle triangle *ABC*, Ð*C* is a right angle, *AB* = 8 cm and *CA* = 6 cm. Find the length of *CO* where *O* is the point of intersection of the perpendicular bisectors of D*ABC*.

**Solution:** The perpendicular bisector of *CA*is parallel to *CB*.




Hence, *O* is on *AB* .

Therefore, *AO* = 4. *AO* = *BO*

*O* is equidistant from *A*, *B* and *C*

Therefore, *CO* = *AO* = 4 cm.

Altitude theorem

In a right angled triangle $ABC$ with altitude $\overline{CD}$ to the hypotenuse $AB$,

$$\frac{AD}{DC} = \frac{CD}{DB}$$

Remark

The square of the length of the altitude is the product of the lengths of the segments of the hypotenuse.

Example

In D*ABC*, *CD* is the altitude to the hypotenuse *AB*, *AD*= 9 cm and *BD* = 4 cm. How long is the altitude *CD* ?

**Solution** Let $h = CD$. From the Altitude Theorem, $(CD)^2 = (AD)\,(BD)$

Substituting, $h^2 = 9 \times 4 = 36$ cm $^2$

So, $h = 6$ cm.

The length of the altitude is 6 cm.

Menelaus' theorem

If points $D$, $E$ and $F$ on the sides $BC$, $\overline{CA}$ and $\overline{AB}$ respectively of $\Delta ABC$ (or their extensions) are collinear, then $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$. Conversely, if $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$, then the points $D$, $E$ and $F$ are collinear.

3.2. Special quadrilaterals

The special quadrilaterals are trapezium, parallelogram, rectangle, rhombus and square.

Activity

Students try to discuss all special quadrilaterals. Refer grade 10th textbook.

3.3. Circles

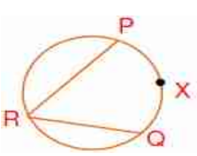
In this section, you are going to study circles and the lines and angles associated with them. Of all simple geometric figures, a circle is perhaps the most appealing. Have you ever considered how useful a circle is? Without circles there would be no watches, wagons, automobiles, steamships, electricity or many other modern conveniences. Recall that a circle is a plane figure, all points of which are equidistant from a given point called the centre of the circle.

Theorem

The measure of an angle inscribed in a circle is half the measure of the arc subtending it.

Example

Let $m\ arc(PXQ) = 110°$. Find the measure of Ð $PRQ$ of the figure.




$$m(\angle PRQ) = \frac{1}{2}\, m(\widehat{PXQ}) = \frac{1}{2}(110°) = 55°$$

Remark

a. An angle inscribed in a semi-circle is a right angle.
b. An angle inscribed in an arc less than a semi-circle is obtuse
c. An angle inscribed in an arc greater than a semi-circle is acute.

d. An angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.

e. The measure of an angle formed by two chords intersecting inside a circle is half the sum of the measures of the arc subtending the angle and its vertically opposite angle.

Example

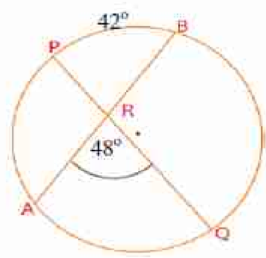
1. An angle formed by two chords intersecting within a circle is 48°, and one of the intercepted arcs measures 42°. Find the measures of the other intercepted arc.

$$m(\angle PRB) = \frac{1}{2}\, m(\overset{\frown}{PB}) + \frac{1}{2}\, m(\overset{\frown}{AQ})$$

$$48^\circ = \frac{1}{2}(42^\circ) + \frac{1}{2}(\overset{\frown}{AQ})$$

$$\Rightarrow 96^\circ = 42^\circ + m(\overset{\frown}{AQ})$$

$$\therefore 54^\circ = m(\overset{\frown}{AQ})$$




Please student there are more left examples that are find on your grade 10th textbooks. So, try to read them from the library.

## 3.4. Regular Polygons

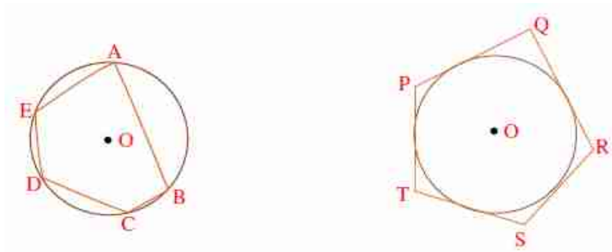
A polygon is a simple closed curve, formed by the union of three or more line segments, no two of which in succession are collinear. The line segments are called the sides of the polygon and the end points of the sides are called the vertices.

In other words, a polygon is a simple closed plane shape consisting of straight-line segments such that no two successive line segments are collinear.

A polygon whose vertices are on a circle is said to be inscribed in the circle. The circle is circumscribed about the polygon.

a. The polygon *ABCDE* is inscribed in the circle, or the circle is circumscribed about the polygon.

b. The pentagon *PQRST* is circumscribed about the circle. The circle is inscribed in the pentagon.




**Perimeter of a Regular Polygon**

You have studied how to find the length of a side (s) and perimeter (P) of a regular polygon with radius "$r$" and the number of sides "$n$" in grade 9. The following example is given to refresh your memory.

**Example 1** The perimeter of a regular polygon with 9 sides is given by:

$$P = 9 \times 2r \sin\frac{180^\circ}{9} = 9d \sin\frac{180^\circ}{9}, \text{ where } d = 2r \text{ is diameter}$$

$$= 9d \sin 20^\circ \approx 3.0782d$$

**Example 2** Find the length of a side and the perimeter of a regular quadrilateral with radius 5 units.

**Solution:** $s = 2r \sin\frac{180^\circ}{n}$ $\qquad$ $P = 2nr \sin\frac{180^\circ}{n}$

$s = 2 \times 5 \sin\frac{180^\circ}{4} = 10 \sin 45^\circ$ $\qquad$ $P = 2 \times 4 \times 5 \sin\frac{180^\circ}{4} = 40 \sin 45^\circ$

$= 10 \times \frac{\sqrt{2}}{2}$ $\qquad$ $= 40 \times \frac{\sqrt{2}}{2}$

$\therefore s = 5\sqrt{2}$ units. $\qquad$ $\therefore P = 20\sqrt{2}$ units.

**<u>Interior and exterior angles of a polygon</u>**

An angle at a vertex of a polygon that is supplementary to the interior angle at that vertex is called an exterior angle. It is formed between one side of the polygon and the extended adjacent side.

Note that the number of *vertices*, *angles* and *sides* of a polygon are the same.

| Number of sides | Number of interior angles | Name of polygon |
|---|---|---|
| 3 | 3 | Triangle |
| 4 | 4 | Quadrilateral |
| 5 | 5 | Pentagon |
| 6 | 6 | Hexagon |
| 7 | 7 | Heptagon |
| 8 | 8 | Octagon |
| 9 | 9 | Nonagon |
| 10 | 10 | Decagon |

Remark

If the number of sides of a polygon is $n$, then the sum of the measures of all its interior angles is equal to $(n-2) \times 180^\circ$.

A regular polygon is a convex polygon in which the lengths of all of its sides are equal and the measures of all of its angles are equal.

The measure of each interior angle of a regular $n$-sided polygon is $\frac{(n-2)180^\circ}{n}$.

A polygon is said to be inscribed in a circle if all of its vertices lie on the circle.

**Example 1**

i Find the measure of each interior angle and each central angle of a regular polygon with:

a 3 sides b 5 sides

ii Find the measure of each exterior angle of a regular $n$-sided polygon.

**Solution:**

i a Since the sum of interior angles of a triangle is $180^\circ$, each interior angle is $\frac{180^\circ}{3} = 60^\circ$.

Recall that a 3-sided regular polygon is an equilateral triangle.

To find the measure of a central angle in a regular $n$-sided polygon, recall that the sum of the measures of angles at a point is $360^\circ$. Hence, the sum of the measures of the central angles is $360^\circ$. ([illegible]). So, the measure of each central angle in an $n$-sided regular polygon is $\frac{360^\circ}{n}$. From this, we conclude that the measure of each central angle of an equilateral triangle is $\frac{360^\circ}{3} = 120^\circ$.

For any regular $n$-sided polygon:

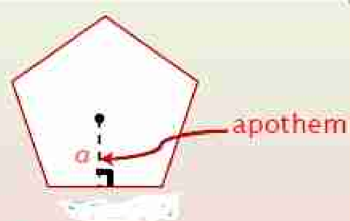
i Measure of each interior angle = $\frac{(n-2)180^\circ}{n}$.

ii Measure of each central angle = $\frac{360^\circ}{n}$.

iii Measure of each exterior angle = $\frac{360^\circ}{n}$.

**Definition**

The distance $a$ from the centre of a regular polygon to a side of the polygon is called the apothem of the polygon. That is, the apothem $a$ of a regular polygon is the length of the line segment drawn from the centre of the polygon perpendicular to the side of the polygon.



Regular Pentagon

Theorem

The area $A$ of a regular polygon with $n$ sides and radius $r$ is

$$A = \frac{1}{2}nr^2 \sin\frac{360^\circ}{n}.$$

Example

Show that the area $A$ of a regular hexagon inscribed in a circle with radius r is $\frac{3\sqrt{3}}{2}r^2$.

**Solution:** $A = \frac{1}{2}nr^2 \sin\frac{360°}{n} = \frac{1}{2}\times 6\times r^2 \sin\frac{360°}{6} = 3r^2 \sin 60°$

$A = 3r^2 \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}r^2}{2}$ sq units.

Exercises

1. Find the area of a regular nine-sided polygon with radius 5 units.
2. Find the area of a regular twelve-sided polygon with radius 3 units.
3. Prove that the area $A$ of an equilateral triangle inscribed in a circle with radius $r$ is $A = \frac{3\sqrt{3}r^2}{4}$. Use this formula to find the area of an equilateral triangle inscribed in a circle with radius:

   a 2 cm　　b 3 cm　　c $\sqrt{2}$ cm　　d $2\sqrt{3}$ cm.
4. Prove that the area A of a square inscribed in a circle with radius r is $A = 2r^2$. Use this formula to find the area of a square inscribed in a circle with radius:

   a 3 cm　　b 2 cm　　c $\sqrt{3}$ cm　　d 4 cm.
5. Show that all the distances from the centre of a regular polygon to the sides are equal.
6. $ABXY$ is a parallelogram of area 18 cm², $AB = 6$ cm, $AY = 4$ cm and $C$ is a point on $\overline{YX}$ or extended such that $BC = 5$ cm. Find:

   a the area of $\Delta ABC$　　b the distance from $B$ to $\overline{AY}$.

   c the distance from $A$ to $\overline{CB}$

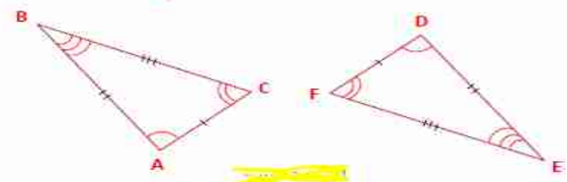
3.5. Congruency and Similarity

## Congruency of Triangles

Triangles that have the same size and shape are called congruent triangles. That is, the six parts of the triangles (three sides and three angles) are correspondingly congruent. If two triangles, $\Delta ABC$ and $\Delta DEF$ are congruent like those given below, then we denote this as

$$\Delta ABC \cong \Delta DEF.$$

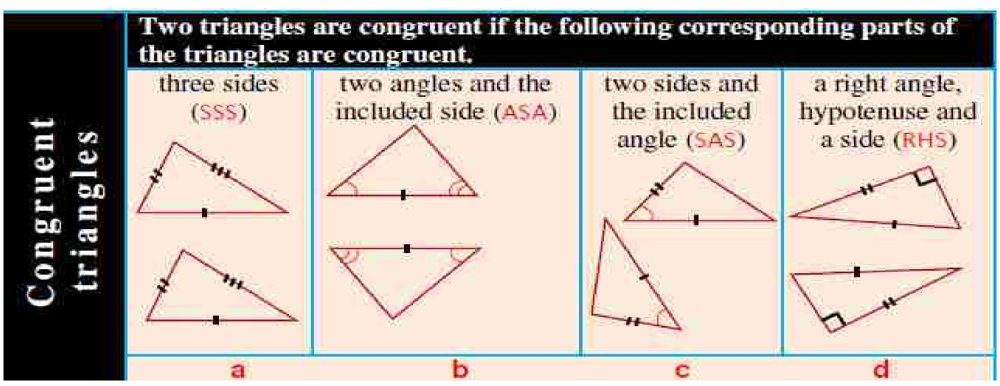
The notation "$\cong$" means "is congruent to".




| Congruent angles | Congruent sides |
|---|---|
| $\angle A \cong \angle D$; $\angle B \cong \angle E$; $\angle C \cong \angle F$ | $\overline{AB} \cong \overline{DE}$; $\overline{BC} \cong \overline{EF}$; $\overline{AC} \cong \overline{DF}$. |

Parts of congruent triangles that "match" are called corresponding parts. For example, in the triangles above, $\angle B$ corresponds to $\angle E$ and $\overline{AC}$ corresponds to $\overline{DF}$.

Two triangles are congruent when all of the corresponding parts are congruent. However, you do not need to know all of the six corresponding parts to conclude that the triangles are congruent. Each of the following Theorems states that three corresponding parts determine the congruence of two triangles.




3.6. Areas of Triangle and parallelogram

3.7. Surface area and volume of solid figures (Prism, Cylinder, Cone and Sphere)

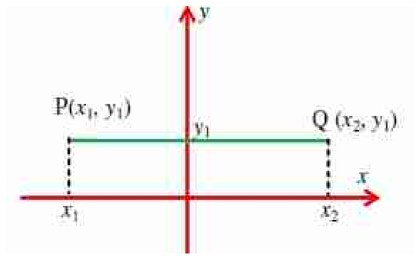
### 3.8. Frustum of pyramids and cones

Note 3.6 ,3.7 and 3.8 are reading assignment with presentation!!

## Chapter 4: Coordinate Geometry (8 hrs)

4.1.Distance between points in a plane

Suppose P ($x1$, $y1$) and Q ($x2$, $y2$) are two distinct points on the $xy$-coordinate plane. We can find the distance between the two points $P$ and $Q$. One option is when $P$ and $Q$ are on a line parallel to the $x$-axis (that is, $PQ$ is a horizontal segment) as shown in the figure.




Since the two points *P* and *Q* have the same *y*-coordinate (ordinate), the distance between *P* and *Q* is

$$PQ = | x_2 - x_1 |$$

Students we have also other way of finding a distance between the points. Please refer this part from your grade 10th textbook on page 145.

In general, the distance *d* between any two points P ($x_1$, $y_1$) and Q ($x_2$, $y_2$) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is called the **distance formula.**

**Example 1** Find the distance between the given points.

**a** A $(1, \sqrt{2})$ and B $(1, -\sqrt{2})$ **b** P $\left(\frac{17}{4}, -2\right)$ and Q $\left(\frac{1}{4}, -2\right)$

**c** R $\left(-\sqrt{2}, -1\right)$ and S $\left(\sqrt{2}, -\sqrt{2}\right)$ **d** A $(a, -b)$ and B $(-b, a)$

**Solution:**

**a** $AB = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(1-1)^2 + (-\sqrt{2} - \sqrt{2})^2}$

$= \sqrt{(0)^2 + (-2\sqrt{2})^2} = 2\sqrt{2}.$

Or, more simply

$AB = |y_2 - y_1| = |-\sqrt{2} - \sqrt{2}|$

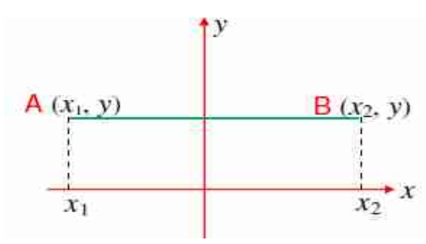
$= 2\sqrt{2}$ units

Activity

Please try to do b,c,and d.

4.2. Division of a line segment

a line segment passing through two points *A* and *B* is horizontal if the two points have the same *y*-coordinate. i.e., a line segment whose end-points are A $(x_1, y)$ and $B$ $(x_2, y)$ is a horizontal line segment as shown in the figure




Consider the horizontal line segment with end-points A ($x1$, $y$) and B ($x2$, $y$) as shown the above figure.

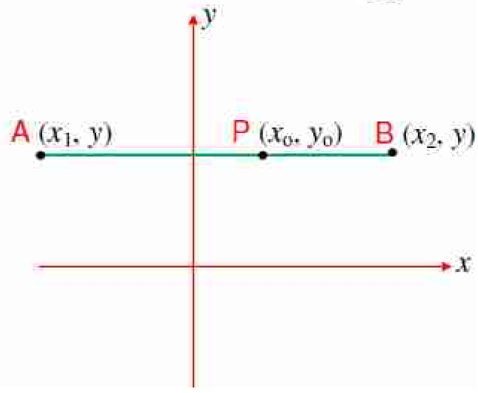
In terms of the coordinates of $A$ and $B$, determine the coordinates of the point P ($x_o$, $y_o$) that divides $AB$ internally in the ratio $m$:$n$.

Clearly, the ratio of the line segment $AP$ to the line segment $PB$ is given by $\frac{AP}{PB}$

The distance between $A$ and $P$ is $AP = x_o - x_1$.

The distance between $P$ and $B$ is $PB = x_2 - x_o$.

Therefore, $\frac{AP}{PB} = \frac{m}{n}$ i.e., $\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n}$.




Solving this equation for $x_o$:

$\Rightarrow n\,(x_o - x_1) = m\,(x_2 - x_o)$

$\Rightarrow nx_o - nx_1 = mx_2 - mx_o$

$\Rightarrow nx_o + mx_o = nx_1 + mx_2$

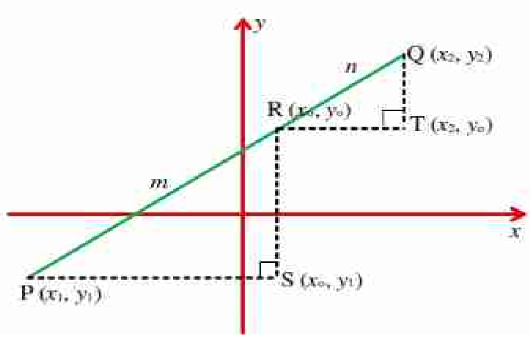
$\Rightarrow x_o\,(n + m) = nx_1 + mx_2$

$$\Rightarrow x_o = \frac{nx_1 + mx_2}{n+m}$$

Since $\overline{AB}$ is parallel to the $x$-axis ( $\overline{AB}$ is a horizontal line segment) and obviously, $y_o = y$, therefore, the point P ($x_o$, $y_o$) is $\left(\frac{nx_1 + mx_2}{n+m},\ y\right)$.

Given a line segment $PQ$ with end point coordinates P ($x_1$, $y_1$) and Q ($x_2$, $y_2$), let us find the coordinates of the point $R$ dividing the line segment $PQ$ internally in the ratio $m:n$.

i.e., $\frac{PR}{RQ} = \frac{m}{n}$, Where $m$ and $n$ are given positive real numbers.




Let the coordinates of $R$ be ($x_o$, $y_o$). Assume that $x1$ ¹ $x2$ and $y1$ ¹ $y2$. If you draw lines through the points $P$, $Q$ and $R$ parallel to the axes as shown in figure. The points $S$ and $T$ have the coordinates ($x_o$, $y_1$) and ($x_2$, $y_o$), respectively.

$PS = x_o - x_1$, $RT = x_2 - x_o$, $SR = y_o - y_1$ and $TQ = y_2 - y_o$

Since triangles *PSR* and *RTQ* are similar (Why?),

$$\frac{PS}{RT} = \frac{PR}{RQ} \text{ and } \frac{SR}{TQ} = \frac{PR}{RQ}$$

$$\frac{x_o - x_1}{x_2 - x_o} = \frac{m}{n} \text{ and } \frac{y_o - y_1}{y_2 - y_o} = \frac{m}{n}$$

Solving for $x_o$ and $y_o$,

$\Rightarrow \quad n(x_o - x_1) = m(x_2 - x_o)$ and $n(y_o - y_1) = m(y_2 - y_o)$

$\Rightarrow \quad nx_o - nx_1 = mx_2 - mx_o$ and $ny_o - ny_1 = my_2 - my_o$

$\Rightarrow \quad nx_o + mx_o = nx_1 + mx_2$ and $ny_o + my_o = ny_1 + my_2$

$\Rightarrow \quad x_o(n + m) = nx_1 + mx_2$ and $y_o(n + m) = ny_1 + my_2$

$\Rightarrow \quad x_o = \dfrac{nx_1 + mx_2}{n+m}$ and $y_o = \dfrac{ny_1 + my_2}{n+m}$

The point R $(x_o, y_o)$ dividing the line segment *PQ* internally in the ratio $m$: $n$ is given by

$$R(x_o, y_o) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

This is called the **section formula**.

**Example 1** Find the coordinates of the point *R* that divides the line segment with end- points A (6, 2) and B (1, – 4) in the ratio 2:3.

**Solution:** Put $(x_1, y_1) = (6, 2)$, $(x_2, y_2) = (1, -4)$, $m = 2$ and $n = 3$. Using the section formula, you have

$$R(x_o, y_o) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right) = \left(\frac{3\times6+2\times1}{3+2}, \frac{3\times2+2\times(-4)}{3+2}\right)$$

$$= \left(\frac{18+2}{5}, \frac{6-8}{5}\right) = \left(4, -\frac{2}{5}\right)$$

Therefore, R is $\left(4, -\frac{2}{5}\right)$.

**Example 2** A line segment has end-points (–2, –3) and (7, 12) and it is divided into three equal parts. Find the coordinates of the points that trisect the segment.

Student: Try to take time to do this example!!!

Let P ($x_1$, $y_1$) and Q ($x_2$, $y_2$) be the end-points of $\overline{PQ}$.

If $PR = RQ$ (the case where $m = n$), then $R$ is the mid-point of the line segment $PQ$.

Now let us derive the mid-point formula.

$$R(x_o, y_o) = \left(\frac{nx_1 + mx_2}{n+m}, \frac{ny_1 + my_2}{n+m}\right)$$

$$= \left(\frac{nx_1 + nx_2}{n+n}, \frac{ny_1 + ny_2}{n+n}\right) = \left(\frac{n(x_1 + x_2)}{2n}, \frac{n(y_1 + y_2)}{2n}\right) \text{(as } m = n\text{)}$$

$$= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

This is the formula used to find the mid-point of the line segment $PQ$ whose end points are P ($x_1$, $y_1$) and Q ($x_2$, $y_2$).

The mid-point of the line segment joining the points ($x_1$, $y_1$) and ($x_2$, $y_2$) is given by

$$M(x_o, y_o) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

**Example 3** Find the coordinates of the mid-point of the line segments with end-points:

**a** P (–3, 2) and Q (5, –4)

**Solution:**

**a** $M(x_o, y_o) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

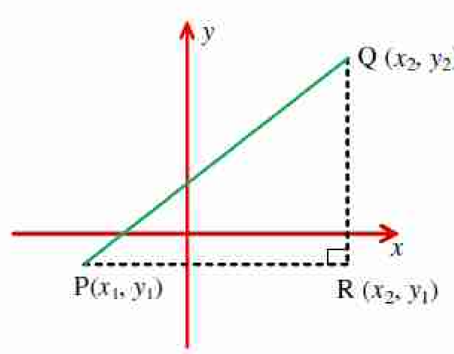
$x_o = \frac{x_1 + x_2}{2}$ and $y_o = \frac{y_1 + y_2}{2}$

$x_o = \frac{-3+5}{2} = 1$ and $y_o = \frac{2-4}{2} = -1$

Therefore M ($x_o$, $y_o$) = (1, –1).

## 4.3. Equation of a straight line /gradient line

In coordinate geometry, the gradient of a non-vertical straight line is the ratio of "change in $y$-coordinates" to the corresponding "change in $x$-coordinates". That is, the slope of a line through $P$ and $Q$ is the ratio of the vertical distance from $P$ to $Q$ to the horizontal distance from $P$ to $Q$.




If we denote the gradient of a line by the letter $m$, then

$$m = \frac{\textit{change in y-coordinates}}{\textit{change in x-coordinates}} = \frac{y_2 - y_1}{x_2 - x_1};\ x_1 \neq x_2$$

Definition

If $(x_1, y_1)$ and $(x_2, y_2)$ are points on a line with $x_1 \neq x_2$, then the **gradient** of the line, denoted by $m$, is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

**Example 1** Find the gradient of the line passing through each of the following pairs of points:

**a** P $(-7, 2)$ and Q $(4, 3)$ **b** A $(\sqrt{2}, 1)$ and B $(-\sqrt{2}, -3)$

**c** P $(2, -3)$ and Q $(5, -3)$ **d** A$\left(-\frac{1}{2}, -2\right)$ and B$\left(-\frac{1}{2}, 2\right)$.

**Solution:**

**a** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{4-(-7)} = \frac{1}{11}$

**b** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-1}{-\sqrt{2}-\sqrt{2}} = \frac{-4}{-2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

**c** $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3-(-3)}{5-2} = \frac{-3+3}{3} = \frac{0}{3} = 0$

So, $m = 0$. Is the line horizontal? What is its equation?

**d** $x_1 = -\frac{1}{2}$ and $x_2 = -\frac{1}{2}$

The line is vertical. So it has no measurable gradient.

The equation of the line is $x = x_1 = x_2 = -\frac{1}{2}$ or simply $x = -\frac{1}{2}$

Remark

- ✓ Any two points determine a straight line.
- ✓ If P$(x_1, y_1)$ and Q$(x_2, y_2)$ are points on a line with $x_1 \neq x_2$, then $y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ is the equation of the straight line and the ratio $m = \frac{y_2 - y_1}{x_2 - x_1}$ is the slope of the line.
- ✓ If $x_2 = x_1$, then the line is vertical and its equation is given by $x = x_1$; in this case the line has no slope.
- ✓ If two lines $\ell_1$ and $\ell_2$ have the same slope, then the two lines are parallel.
- ✓ If the product of the slopes of two lines $\ell_1$ and $\ell_2$ is $-1$, then the two lines are perpendicular.
- ✓ If the equation of a line is given by $y = mx + b$, then $m$ is the slope of the line and $b$ is its $y$ - intercept.

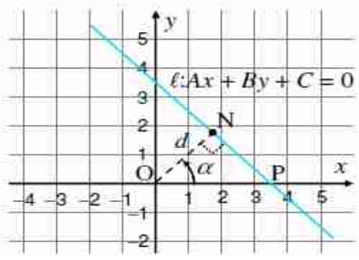
Activity

Write down Different Forms of Equations of a Line!!

4.4. Distance between a point and a line and between two lines.

Suppose a line ℓ and a point P($x$, $y$) are given. If P does not lie on ℓ, then we define the distance $d$ from P to ℓ as the perpendicular distance between P and ℓ. If P is on ℓ,the distance is taken to be zero.

Let a line $\ell : Ax + By + C = 0$ with $A$, $B$ and $C$ all non-zero be given. To find the distance from the origin to the line $Ax + By + C = 0$, you can do the following:




Draw $\overline{ON}$ perpendicular to $Ax + By + C = 0$. $\Delta$ONP is right angled triangle. Thus

$$|\cos\alpha| = \frac{d}{\text{OP}} \Rightarrow d = \text{OP}\,|\cos\alpha|.$$

The $x$-intercept of $Ax + By + C = 0$ is $-\frac{C}{A}$.

Thus, $d = \frac{|C|}{|A|}|\cos\alpha|$

Again $\overline{ON}$ being $\perp$ to the line $Ax + By + C = 0$ gives: slope of $\overline{ON}$ $= \tan\alpha = \frac{B}{A}$

$\left(\text{because slope of } Ax + By + C = 0 \text{ is } \frac{-A}{B}\right)$

This gives $|\cos\alpha| = \frac{|A|}{\sqrt{A^2 + B^2}}$

Hence, the distance from the origin to any line $Ax + By + C = 0$ with $A \neq 0$, $B \neq 0$ and $C \neq 0$ is given by $\frac{|C|}{\sqrt{A^2 + B^2}}$

Example

a. Find the distance from the origin to the line $5x - 2y - 7 = 0$.

Solution

$$\text{The distance } d = \frac{|-7|}{\sqrt{5^2 + (-2)^2}} = \frac{7}{\sqrt{29}}$$

b. Find the distance between P(–4, 2) and $\ell : 2x + 9y - 3 = 0$

Solution

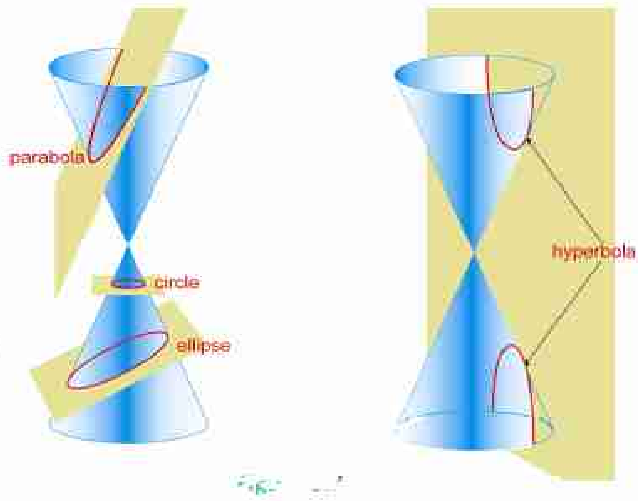
$$d = \frac{|2(-4) + 9(2) - 3|}{\sqrt{2^2 + 9^2}} = \frac{|-8 + 18 - 3|}{\sqrt{85}} = \frac{7}{\sqrt{85}}$$

## 4.5. Conic Sections

Definition

A locus is a system of points, lines or curves on a plane which satisfy one or more given conditions.

1. If a horizontal plane intersects /slices through one of the cones, the section formed is a circle.
2. If a slanted plane intersects /slices through one of the cones, then the section formed is either an ellipse or a parabola.
3. If a vertical plane intersects /slices through the pair of cones, then the section formed is a hyperbola.




Since each of these plane curves are formed by intersecting a pair of cones with a plane, they are called conic sections.

Let us see a quick definition and equation of each conic section shown in the above figure taken from your textbooks.

Definition (circle).

A circle is the locus of a point that moves in a plane with a fixed distance from a fixed point. The fixed distance is called the radius of the circle and the fixed point is called the centre of the circle.

From the above definition, for any point P($x$, $y$) on a circle with centre C ($h$, $k$) and radius $r$, PC = $r$ and by the distance formula you have,

$$\sqrt{(x-h)^2+(y-k)^2}=r$$

Example

a. Write down the standard form of the equation of a circle with the given centre and radius.

**a** C (0, 0), $r = 8$ **b** C (2, –7), $r = 9$

**Solution**

**a** $h = k = 0$ and $r = 8$

Therefore, the equation of the circle is $(x-0)^2 + (y-0)^2 = 8^2$.

That is, $x^2 + y^2 = 64$.

**b** $h = 2$, $k = -7$ and $r = 9$.

Therefore, the equation of the circle is $(x-2)^2 + (y+7)^2 = 9^2$.

That is, $(x-2)^2 + (y+7)^2 = 81$.

b. Write the standard form of the equation of the circle with centre at C (2, 3) and that passes through the point P (7, –3).

**Solution** Let $r$ be the radius of the circle. Then the equation of the circle is

$$(x-2)^2+(y-3)^2=r^2$$

Since the point P (7, –3) is on the circle, you have

$$(7-2)^2+(-3-3)^2=r^2.$$

This implies, $5^2+(-6)^2=r^2$.

So, $r^2=61$.

Therefore, the equation of the circle is

$$(x-2)^2+(y-3)^2=61.$$

**Example 4** Give the centre and radius of the circle,

**a** $(x-5)^2+(y+7)^2=64$ **b** $x^2+y^2+6x-8y=0.$

**Solution**

**a** The equation is $(x-5)^2+(y+7)^2=8^2$. Therefore, the centre C of the circle is C (5, –7) and the radius $r$ of the circle is $r=8$.

Remark

A line with equation $Ax+By+C=0$ intersects a circle with equation $(x-h)^2+(y-k)^2=r^2$, if and only if,

$$\frac{|Ah+Bk+C|}{\sqrt{A^2+B^2}}\le r.$$

If a line with equation $Ax+By+C=0$ intersects a circle with equation $(x-h)^2+(y-k)^2=r^2$, then $(x-h)^2+\left(-\frac{A}{B}x-\frac{C}{B}-k\right)^2=r^2$ is a quadratic equation in $x$. If $B=0$, then $x=-\frac{C}{A}$ is a vertical line.

$(y-k)^2=r^2-\left(-\frac{C}{A}-h\right)^2=r^2-\left(\frac{C+hA}{A}\right)^2$, which is a quadratic in $y$.

Solving this equation, you can get point(s) of intersection of the line and the circle.

Example

1. Find the intersection of the circle with equation $(x-1)^2+2)^2=25$ with each of the following lines.

**a** $4x - 3y - 7 = 0$ **b** $x = 4$

**Solution**

**a** $4x - 3y - 7 = 0 \iff y = \frac{4x - 7}{3}$

So $(x-1)^2 + \left(\frac{4x - 7}{3} + 1\right)^2 = 25$

$$\Rightarrow (x-1)^2 + \left(\frac{4x-4}{3}\right)^2 = 25$$
$$\Rightarrow 9(x-1)^2 + (4x-4)^2 = 225$$
$$\Rightarrow 9(x^2 - 2x + 1) + (16x^2 - 32x + 16) = 225$$
$$\Rightarrow 9x^2 - 18x + 9 + 16x^2 - 32x + 16 = 225$$
$$\Rightarrow 25x^2 - 50x - 200 = 0$$
$$\Rightarrow x^2 - 2x - 8 = 0$$
$$\Rightarrow (x+2)(x-4) = 0$$
$$\Rightarrow x = -2 \text{ or } x = 4$$

This gives $y = -5$ and $y = 3$, respectively.

Hence the line and the circle intersect at the points P(–2, –5) and Q(4, 3).

If a line $\ell$ is tangent to a circle $(x - h)^2 + (y - k)^2 = r^2$ at a point $T(x_0, y_0)$, then the equation of $\ell$ is given by

$$\frac{y - y_o}{x - x_o} = -\frac{x_o - h}{y_o - k}$$

Example

1. Find the equation of the circle with centre at O(2, 5) and the line with equation $x - y = 1$ is a tangent line to the circle.

**Solution** The distance from the centre O(2, 5) of the circle to the line with equation $x - y - 1 = 0$ is the radius.

Thus, $r = \frac{|2 - 5 - 1|}{\sqrt{1^2 + (-1)^2}} = 2\sqrt{2}$

Hence, the equation of the circle is $(x-2)^2 + (y-5)^2 = \left(2\sqrt{2}\right)^2 = 8$

Parabola

The equation

$$(y-k)^2 = \pm 4p\,(x-h)$$

represents a parabola with:

- ✓ vertex V $(h, k)$
- ✓ focus $(h \pm p, k)$.
- ✓ directrix : $x = h \pm p$.
- ✓ axis of symmetry $y = k$.
- ✓ If the sign in front of $p$ is positive, then the parabola opens to the right.
- ✓ If the sign in front of $p$ is negative, then the parabola opens to the left.

Example

Find the equation of the directrix, the focus of the parabola, the length of the latus rectum and draw the graph of the parabola $y^2 = 4x$.

**Solution** The vertex is at $(0,0)$ and $4p = 4$. Hence $p = 1$.

The parabola opens to the right with focus $(h + p, k) = (0 + 1, 0) = (1, 0)$ and the directrix $x = h - p = 0 - 1 = -1$. The axis of the parabola is the $x$-axis.

The latus rectum passes through the focus F$(1, 0)$ and is perpendicular to the axis, that is the $x$-axis.

Therefore, the equation of the line containing the latus rectum is $x = 1$.

To find the endpoints of the latus rectum, you have to find the intersection point of the line $x = 1$ and the parabola. That is, $y^2 = 4\times 1 = 4 \Leftrightarrow y = \pm 2$.

Therefore, the end points of the latus rectum are $(1, -2)$ and $(1, 2)$ and the length of the latus rectum is:

$$\sqrt{(1-1)^2 + (-2-2)^2} = \sqrt{16} = 4.$$

Exercises

1 Write the equation of each parabola given below.

a Vertex $(-2, 5)$; focus $(-2, -8)$ b Vertex $(-3, 4)$; focus $(-3, 12)$

c Vertex $(4, 6)$; focus $(-8, 6)$ d Vertex $(-1, 8)$; focus $(6, 8)$

2 Name the vertex, focus and directrix of the parabola whose equation is given and sketch the graph of each of the following.

a $x^2 = 2y$ b $(x + 2)^2 = 4(y - 6)$

c $(y + 2)^2 = -16(x - 3)$ d $(x - 3)^2 = 4y$

3 Write the equation of each parabola described below.

a Focus $(3, 5)$; directrix $y = 3$ b Vertex $(-2, 1)$; axis $y = 1$; $p = 1$

c Vertex $(4, 3)$; passes through $(5, 2)$, vertical axis

d Focus $(5, 0)$; $p = 4$; vertical axis

4 Write the equation of each parabola described below.

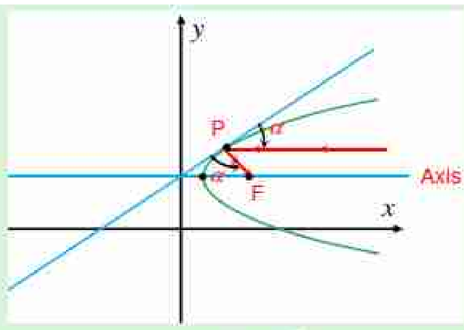
a Vertex at the origin, axis along the $x$-axis, passing through A $(3, 6)$

b Vertex at $(4, 2)$, axis parallel to the $x$-axis, passing through A $(8, 7)$

c Vertex at $(5, -3)$, axis parallel to the $y$-axis, passing through B $(1, 2)$

5 The parabola has a multitude of scientific applications. A reflecting telescope is designed by using the property of a parabola:

*If the axis of a parabolic mirror is pointed toward a star, the rays from the star, upon striking the mirror, will be reflected to the focus.*




Answer the following questions

a A parabolic reflector is designed so that its diameter is 12 m when its depth is 4 m. Locate the focus.

b A parabolic head light lamp is designed in such a way that when it is 16 cm wide it has 6 cm depth. How wide is it at the focus?

6 Find the equation of the parabola determined by the given data.

a The vertex is at $(1,2)$, the axis is parallel to the $x$-axis and the parabola passes through $(6,3)$.

b The focus is at $(3,4)$, the directrix is at $x = 8$.

Activities

Discuss both Ellipse and hyperbola in classroom and present it for your friend.

To do this, please refer grade $11^{th}$ your textbook on pages 95 through 103.

## Chapter 5: Statistics and Probability (12 hrs)

5.1. Statistics

Definition

Statistics is the science of *collecting*, *organizing*, *presenting*, *analyzing* and *interpreting data* (quantitative information) in order to draw conclusions.

5.1.1. Types of data

Discus both primary and secondary data, please give examples to both.

5.1.2. Measures of ***location*** for ungrouped and group data

Quantitative variables contained in raw data or in frequency tables can also be summarized by means of a few numerical values. A key element of this summary is called the measure of average or measure of location. The three commonly used measures of location are the arithmetic mean (or the mean), the median and the mode(s).

5.1.3. Measures of ***Dispersion*** for ungrouped and grouped data

Dispersion or Variation is the scatter (or spread) of data values from a measure of central tendency.There are several measures of dispersion that can be calculated for a set of data. In this section, we will consider only three of them, namely, the range, variance and the standard deviation.

5.2. Probability

An experiment is a trial by which an observation is obtained but whose outcome cannot be predicted in advance.

Probability determined using data collected from repeated experiments is called experimental probability.

**Example 1** The numbers 1 to 20 are each written on one of 20 identical cards. One card is chosen at random.

**a** List the set of all possible outcomes.

**b** List the elements of the following events:

- **i** The number is less than 5.
- **ii** The number is greater than 15.
- **iii** The number is greater than 21.
- **iv** The number is divisible by 5.
- **v** The number is a prime.

**Solution:**

**a** $S = \{1, 2, 3, \ldots, 19, 20\}$

**b**
- **i** $\{1, 2, 3, 4\}$
- **ii** $\{16, 17, 18, 19, 20\}$
- **iii** $\{\ \}$ or $\varnothing$, since no card has a number greater than 20.
- **iv** $\{5, 10, 15, 20\}$
- **v** $\{2, 3, 5, 7, 11, 13, 17, 19\}$

If an event E can happen in $m$ ways out of $n$ equally likely possibilities, the probability of the occurrence of an event E is given by

$$P(E)=\frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}=\frac{n(E)}{n(S)}=\frac{m}{n}$$

**Example 2** A box contains 4 red and 5 black balls. If one ball is drawn at random, what is the probability of getting a

**a** red ball? **b** black ball?

**Solution** Let event R = a red ball appears and event B = a black ball appears. Then,

**a** $P(R)=\frac{n(R)}{n(S)}=\frac{4}{9}$ **b** $P(B)=\frac{n(B)}{n(S)}=\frac{5}{9}$

**Example 3** If a number is to be selected at random from the integers 1 through to10, what is the probability that the number is

**a** odd? **b** divisible by 3?

**Solution** $S=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

**a** odd is the event $E=\{1, 3, 5, 7, 9\} \Rightarrow P(\text{odd})=\frac{\text{number of odds}}{\text{total numbers}}=\frac{5}{10}=\frac{1}{2}$

**b** divisible by 3 is the event $E=\{3, 6, 9\} \Rightarrow P(\text{divisible by }3)=\frac{3}{10}$

5.2.1. Permutation and combination

There are two fundamental principles that are helpful for counting. These are the multiplication principle and the addition principle.

## Multiplication principle

Before we state the principle, let us consider the following example.

**Example 4** Suppose Nuria wants to go from Harrar via Dire Dawa to Addis Ababa. There are two minibuses from Harrar to Dire Dawa and 3 buses from Dire Dawa to Addis Ababa. How many ways are there for Nuria to travel from Harrar to Addis Ababa?

**Solution:** Let M stand for Minibus and B stand for Bus.


There are $(2 \times 3) = 6$ possible ways.

These are $M_1B_1$, $M_1B_2$, $M_1B_3$, $M_2B_1$, $M_2B_2$ , $M_2B_3$.

The example above illustrates the Multiplication Principle of Counting.

If an event can occur in $m$ different ways, and for every such choice another event can occur in $n$ different ways, then both the events can occur in the given order in $m \times n$ different ways. That is, the number of ways in which a series of successive things can occur is found by multiplying the number of ways each thing can occur.

In the above illustration, Nuria has two possible choices to go from Harar to Dire Dawa and three alternatives from Dire Dawa to Addis Ababa.

The total number of ways is $2 \times 3 = 6$.

**Example 5** Suppose there are 5 seats arranged in a row. In how many different ways can five people be seated on them?

**Solution:** The first man has 5 choices, the 2nd man has 4 choices, the 3rd man has 3 choices, the 4th has two choices, and the 5th has only one choice. Therefore, the total number of possible seating arrangements is

$$5 \times 4 \times 3 \times 2 \times 1 = 120.$$

**Example 6** Suppose that you have 3 coats, 8 shirts and 6 different trousers. In how many different ways can you dress?

**Solution:** $3 \times 8 \times 6 = 144$ ways.

**Addition principle**

If an event $E1$ can occur in $m$ ways and another event $E2$ can happen in $n$ ways, then either of the events can occur in $n + m$ ways. This is true when $E1$ and $E2$ are mutually exclusive events.

In tossing a coin, Head and Tail are mutually exclusive events because they cannot appear at the same time.

**Example 7** A question paper has two parts where one part contains 4 questions and the other 3 questions. If a student has to choose only one question, from either part, in how many ways can the student do it?

**Solution:** The student can choose one question in $4 + 3 = 7$ ways.

**Combined counting principles**

The fundamental counting principles can be extended to any number of sequences of events.

**Example 8** A question paper has three parts: language, arithmetic and aptitude tests. The language part has 3 questions, the arithmetic part has 6 questions and the aptitude part has 5 questions. If a student is expected to answer one question from each of two of the three parts, with arithmetic being compulsory, in how many ways can the student take the examination?

**Solution:** The student can either take language and arithmetic or arithmetic and aptitude. This gives $3 \times 6 + 5 \times 6 = 48$ possibilities.

Definition

The number of ways $r$ objects can be chosen from a set of $n$ objects without considering the order of selection is called the number of combinations of $n$ objects taking $r$ of them at a time, denoted by

$$C(n,r) = \binom{n}{r} = C_r^n \text{. and defined by } C(n,r) = \frac{n!}{(n-r)!r!}, 0 < r \le n$$

5.2.2. Binomial theorem

For a non – negative integer $n$, the binomial expansion of $(x + y)^n$ is given by

$$(x+y)^n = C(n,0)x^n + C(n,1)x^{n-1}y + C(n,2)x^{n-2}y^2 + \ldots + C(n,r)x^{n-r}y^r + \ldots + C(n,n)y^n$$

**Example 15** Expand $(x + y)^4$.

**Solution:** $(x + y)^4 = C(4,0)\, x^4 + C(4,1)\, x^3y + C(4,2)\, x^2y^2 + C(4,3)\, xy^3 + C(4,4)\, y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

**Exemple 16** Find the coefficient of $x^2y^3$ in the expansion of $(x + y)^5$.

**Solution :** $(x + y)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + \binom{5}{5}y^5.$

Thus, the coefficient of $x^2y^3$ is $\binom{5}{3} = \frac{5!}{3!2!} = \frac{5\times 4}{2} = 10.$

5.2.3. Random experiment and their outcomes

A random experiment is an experiment (activity) which produces some well defined results. If the experiment is repeated under identical conditions it does not necessarily produce the same results.

Example

Give the outcomes for each of the following experiments a Tossing a coin    b Tossing a pair of coins

c Rolling a die    d Rolling a pair of dice

Solution:

a {H, T}    b {HH, HT, TH, TT}    c {1, 2, 3, 4, 5, 6}

d

| | | | | | |
|---|---|---|---|---|---|
| (1 ,1) | (1, 2) | (1, 3) | (1, 4) | (1, 5) | (1, 6) |
| (2, 1) | (2, 2) | (2, 3) | (2, 4) | (2, 5) | (2, 6) |
| (3, 1) | (3, 2) | (3, 3) | (3, 4) | (3, 5) | (3, 6) |
| (4, 1) | (4, 2) | (4, 3) | (4, 4) | (4, 5) | (4, 6) |
| (5, 1) | (5, 2) | (5, 3) | (5, 4) | (5, 5) | (5, 6) |
| (6, 1) | (6, 2) | (6, 3) | (6, 4) | (6, 5) | (6, 6) |

5.2.4. Events

Recall that any subset of a sample space is called an event and is usually denoted by *E*. An event is a collection of sample points.

Example

The four faces of a regular tetrahedron are numbered 1, 2, 3 and 4. If it is thrown and the number on the bottom face (on which it stands) is registered, then list the events of this experiment.

Solution

The sample space = {1, 2, 3, 4}.

The possible events are {1}, {2}, {3} and {4}.

**Types of events**

a. **Simple Event (Elementary Event)** is an event containing exactly one sample point.

**Example** In a toss of one coin, the occurrence of tail is a simple event.

b. **Compound Event** When two or more events occur simultaneously, their joint occurrence is known as a compound event, an event that has more than one sample point.

**Example**

When a die is rolled, if you are interested in the event "getting even number", then the event will be a compound event, i.e. { 2, 4, 6}.

We can determine the possible number of events that can be associated with an experiment whose sample space is S. As events are subsets of a sample space, and any set with m elements has 2m subsets, the number of events associated with a sample space with m elements is 2m. (Sometimes this is called the *exhaustive number of events*).

Example

Suppose our experiment is tossing a fair coin. The sample space for this experiment is S = {H, T}. Thus, this sample space has a total of four possible events that are subsets of S. The list of the possible events is { }, {H}, {T}, and {H, T}.

Please

Read your grade 9th and 11th on these topics!!

5.2.5. Probability of an event

Please

Read your grade 9th and 11th on these topics!!

**Chapter 6**: **Limit and Continuity (8 hrs)**

6.1. Limit of functions

Definition(optional)

Let { *an* } be a sequence and *m*, *M* Îℝ . Then

- **a)** *M* is said to be an upper bound of {*an*}, if *M* ³ *ai* for all *ai* Î { *an*}.
- **b)** *m* is said to be a lower bound of { *an* }, if *m* £ *ai* for all *ai* Î {*an*}
- **c)** A sequence is said to be bounded, if it has an upper bound (is bounded above) and if it has a lower bound (is bounded below).

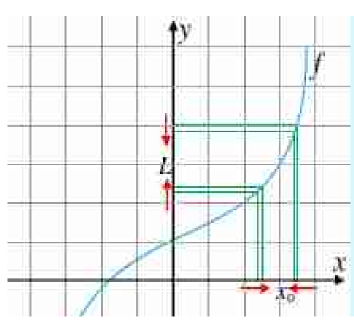
In this topic, you will use functions such as polynomial, rational, exponential, logarithmic, absolute value, trigonometric and other piece-wise defined functions in order to introduce the concept "limit of a function".

We will see different techniques of finding the limit of a function at a point such as cancelling common factors in rational expressions, like $\frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(x+1)}$, for $x \neq 2$, rationalization, like $\frac{(\sqrt{x}-1)}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1}$, graphs, tables of values and other properties.

Let $y = f(x)$ be a function defined on an interval surrounding $x_0 \in \mathbb{R}$ (but f need not be defined at $x = x_0$). If $f(x)$ gets closer and closer to a single real number $L$ as $x$ gets closer and closer to (but not equal to) $x_0$, then we say that the limit of $f(x)$ as $x$ approaches $x_0$ is $L$.

Symbolically, this is written as

$$\lim_{x \to x_0} f(x) = L$$



**Example 1** Let $f(x) = x$. Then $\lim_{x \to x_0} f(x) = x_o$

**Example 2** Let $f(x) = \dfrac{x^2 - 4}{x - 2}$. Evaluate $\lim_{x \to 2} f(x)$

**Solution** Look at the graph of

$$f(x) = \frac{x^2-4}{x-2} = \begin{cases} x+2, & \text{if } x \neq 2 \\ \nexists, & \text{if } x = 2 \end{cases}$$

As $x$ gets closer and closer to 2, $f(x)$ gets closer and closer to 4.

$$\Rightarrow \quad \lim_{x \to 2} f(x) = \lim_{x \to 2}(x+2) = 4$$

Remark

If $f(x)$ approaches to different numbers as $x$ approaches to $x_0$ from the right and from the left, then we conclude that $\lim_{x \to x_0} f(x)$ does not exist.

**Example 3** Evaluate each of the following limits.

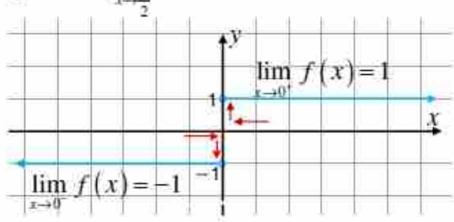
**a** $\lim_{x \to 2}(2x-1)$ **b** $\lim_{x \to 0} \dfrac{|x|}{x}$ **c** $\lim_{x \to 3} \dfrac{x^2 - 5x + 2}{x+4}$

**d** $\lim_{x \to -2} \dfrac{x^2 + x - 2}{x+2}$ **e** $\lim_{x \to 1} \dfrac{x}{x^2-1}$ **f** $\lim_{x \to \frac{\pi}{2}^+} \tan x$

**Solution**

**a** $\lim_{x \to 2}(2x-1) = 2(2) - 1 = 3$

**b** $\dfrac{|x|}{x} = \begin{cases} 1, \text{if } x > 0 \\ \nexists, \text{ if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$




$\Rightarrow \quad \lim_{x \to 0} f(x)$ doesn't exist.

## Basic limit theorems

Suppose $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and $k \in \mathbb{R}$.

Then, $\lim_{x \to a}(f(x) + g(x))$, $\lim_{x \to a}(f(x) - g(x))$, $\lim_{x \to a} kf(x)$, $\lim_{x \to a}(fg)(x)$, $\lim_{x \to a}\left(\dfrac{f}{g}\right)(x)$, provided that $\lim_{x \to a} g(x) \neq 0$, exist and

1. $\lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}(f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$
4. $\lim_{x \to a}(fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
6. $\lim_{x \to a}\sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}$, provided that $f(x) \geq 0$ for $x$ near $a$.

Examples

$$\lim_{x\to2}\left(x^3+4x^2-\frac{1}{x}+7x+11\right)$$
$$= \lim_{x\to2}x^3+4\lim_{x\to2}x^2-\lim_{x\to2}\left(\frac{1}{x}\right)+7\lim_{x\to2}x+\lim_{x\to2}(11)$$
$$= (2)^3+4\left(\lim_{x\to2}x\right)^2-\frac{\lim_{x\to2}(1)}{\lim_{x\to2}(x)}+7(2)+11$$
$$= 2^3+4\times2^2-\frac{1}{2}+25 = 48.5$$

**The limit of a polynomial function**

Suppose $p(x)$ is a polynomial, then $\lim_{x\to c} p(x) = p(c)$.

Theorem

Let $f$ and $g$ be functions. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and $f(x) = g(x)$, $\forall x \neq a$. Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

Find $\lim_{x\to1}\frac{x^2-1}{x-1}$. Find $\lim_{x\to1}\frac{x-1}{\sqrt{x}-1}$. Evaluate $\lim_{x\to-3}\frac{x^3+3x^2-x-3}{4x^3+12x^2-x-3}$

In order to do these questions we need to use simplification ,rationalization and factoring method as follows:

Solution $\frac{x^2-1}{x-1} = x+1$; for $x \neq 1$. Let $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x+1$.

$f(x) = g(x)$, $\forall x \neq 1$. Then $\lim_{x\to1} f(x) = \lim_{x\to1} g(x) \Rightarrow \lim_{x\to1}\frac{x^2-1}{x-1} = \lim_{x\to1}(x+1) = 2$.

Solution What happens to $\lim_{x\to1}\frac{x-1}{\sqrt{x}-1}$ when $x=1$? Is the result defined?

Rewrite the expression by rationalizing the denominator.

$$\frac{x-1}{\sqrt{x}-1} = \frac{(x-1)\left(\sqrt{x}+1\right)}{x-1}$$
$$\Rightarrow \lim_{x\to1}\frac{x-1}{\sqrt{x}-1} = \lim_{x\to1}\left(\sqrt{x}+1\right) = 2$$

Solution $x^3+3x^2-x-3 = x^2(x+3)-(x+3) = (x^2-1)(x+3)$

$$4x^3+12x^2-x-3 = 4x^2(x+3)-(x+3) = (4x^2-1)(x+3)$$
$$\Rightarrow \lim_{x\to-3}\frac{x^3+3x^2-x-3}{4x^3+12x^2-x-3} = \lim_{x\to-3}\frac{(x^2-1)(x+3)}{(4x^2-1)(x+3)} = \lim_{x\to-3}\frac{x^2-1}{4x^2-1} = \frac{8}{35}$$

Examples

Let $f(x) = \sqrt{x}$ . Find $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$.

**Solution** $\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{\sqrt{4+h} - 2}{h} = \lim_{h \to 0} \left[ \frac{\sqrt{4+h} - 2}{h} . \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \right]$

$= \lim_{h \to 0} \left( \frac{1}{\sqrt{4+h} + 2} \right) = \frac{1}{4}$

**Evaluate** $\lim_{x \to -3} \frac{\sqrt{1+\sqrt{4+x}} - \sqrt{2}}{x+3}$.

**Solution** $\lim_{x \to -3} \frac{\sqrt{1+\sqrt{4+x}} - \sqrt{2}}{x+3} = \frac{\sqrt{1+\sqrt{4+x}} - \sqrt{2}}{x+3} . \frac{\sqrt{1+\sqrt{4+x}} + \sqrt{2}}{\sqrt{1+\sqrt{4+x}} + \sqrt{2}}$

$= \frac{\sqrt{4+x} - 1}{x+3} . \frac{1}{\sqrt{1+\sqrt{4+x}} + \sqrt{2}}$. (*Explain!*)

$= \frac{x+3}{x+3} . \frac{1}{\left(\sqrt{4+x} + 1\right)\left(\sqrt{1+\sqrt{4+x}} + 2\right)}$ (*Explain!*)

$\Rightarrow \lim_{x \to -3} \frac{\sqrt{1+\sqrt{4+x}} - \sqrt{2}}{x+3} = \frac{\sqrt{2}}{8}$. (*Explain!*)

Exercises

**1** Use the following graph of the function $f$ to determine each of the limits.

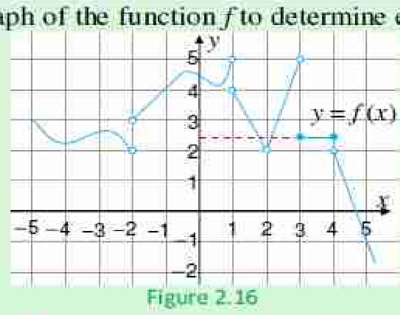



Figure 2.16

**a** $\lim\limits_{x\to 1^-} f(x)$ **b** $\lim\limits_{x\to 2} f(x)$ **c** $\lim\limits_{x\to -2} f(x)$

**d** $\lim\limits_{x\to 1^+} f(x)$ **e** $\lim\limits_{x\to 4^-} f(x)$ **f** $\lim\limits_{x\to 3} f(x)$

**2** Let $f(x)=\begin{cases} 1-x^2, if \ -1<x<2 \\ -3 \ if \ x=-1 \\ -x-1, if \ x<-1 \\ x-5, \ if \ x\geq 2 \end{cases}$

Sketch the graph of $f$ and determine each of the following limits.

**a** $\lim\limits_{x\to -1} f(x)$ **b** $\lim\limits_{x\to 2} f(x)$ **c** $\lim\limits_{x\to 5} f(x)$ **d** $\lim\limits_{x\to 3} f(x)$

**3** Suppose that $f$, $g$ and $h$ are functions with $\lim\limits_{x\to 2} f(x)=7$, $\lim\limits_{x\to 2} g(x)=-4$ and $\lim\limits_{x\to 2} h(x)=\frac{3}{5}$, evaluate

**a** $\lim\limits_{x\to 2}(f(x)+g(x))$ **b** $\lim\limits_{x\to 2}((fg)(x)-3h(x))$

**c** $\lim\limits_{x\to 2}\frac{f(x)g(x)h(x)}{f(x)+g(x)-5h(x)}$

**4** Determine each of the following limits.

**a** $\lim\limits_{x\to 3}\frac{x-3}{\sqrt{x^2-6x+9}}$ **b** $\lim\limits_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}$ **c** $\lim\limits_{x\to \frac{1}{3}}\frac{x+1}{3x-1}$

**d** $\lim\limits_{x\to -2}\frac{x^3+8}{x+2}$ **e** $\lim\limits_{x\to 0}\frac{x^3}{|x|+x}$ **f** $\lim\limits_{x\to -5}\frac{x^2+x-20}{x^2+4x-5}$

**g** $\lim\limits_{x\to 0}\frac{\sin x+1}{x+\cos x}$ **h** $\lim\limits_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}$ **i** $\lim\limits_{x\to 2}\frac{\sqrt{x-2\sqrt{x}+1}-1}{\sqrt{x}-2}$

**j** $\lim\limits_{x\to 1}\frac{\sqrt{x-1}+\sqrt{x}-1}{\sqrt{x^2-1}}$

Limit at infinity

Let $f$ be a function and $L$ be a real number.
If $f(x)$ gets closer to $L$ as $x$ increases without bound, then $L$ is said to be the limit of $f(x)$ as $x$ approaches to infinity.
This statement is expressed symbolically by $\lim\limits_{x\to\infty} f(x)=L$

Now to evaluate such like limit we need to consider three things depending up on the degree of both denominator and denominator involved in the limit.

Please student try to look at these examples with your teacher as it is very simple in manipulation!!

Let $f$ be a function defined on an open interval about $a$, except possibly at $a$ itself. Then, $\lim_{x \to a} f(x)$ exists, if both $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist and are equal: That is, $\lim_{x \to a} f(x)$ exists, if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

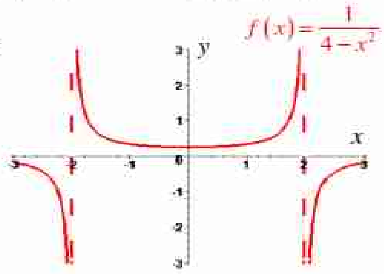
In this case, $\lim_{x \to a} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$.

Examples

Evaluate each of the following limits.

**a** $\lim_{x \to 2^-} \frac{1}{4-x^2}$ **b** $\lim_{x \to 2^+} \frac{1}{4-x^2}$ **c** $\lim_{x \to -2^+} \frac{1}{4-x^2}$ **d** $\lim_{x \to -2^-} \frac{1}{4-x^2}$

**Solution** Sketch the graph of $f(x) = \frac{1}{4-x^2}$ in order to determine each limit at the same time.




[redacted]

If you try to substitute $x = 2$, the denominator equals 0.

**a** $\lim_{x \to 2^-} \frac{1}{4-x^2} = \infty$. The graph is going up indefinitely to $\infty$.

**b** $\lim_{x \to 2^+} \frac{1}{4-x^2} = -\infty$. The graph is going indefinitely down to $-\infty$.

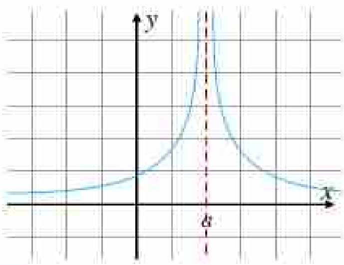
**c** $\lim_{x \to -2^+} \frac{1}{4-x^2} = \infty$ **d** $\lim_{x \to -2^-} \frac{1}{4-x^2} = -\infty$

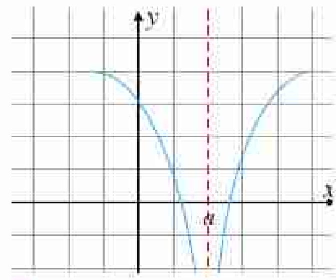
Recall that the lines $x = 2$ and $x = -2$ are vertical asymptotes of the rational function $f(x) = \frac{1}{4-x^2}$.

## Vertical asymptotes

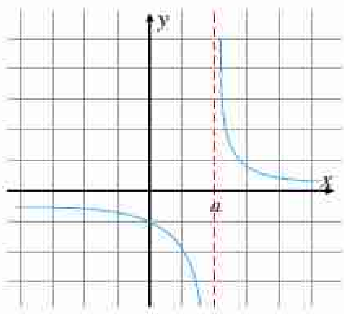
The vertical line $x = a$ is a vertical asymptote to the graph of $y = f(x)$, if one of the following is true.

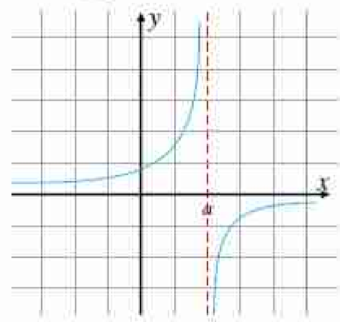
**1** $\lim\limits_{x \to a} f(x) = \infty$ **2** $\lim\limits_{x \to a^+} f(x) = \infty$ **3** $\lim\limits_{x \to a} f(x) = -\infty$ **4** $\lim\limits_{x \to a^+} f(x) = -\infty$




**a** $\lim\limits_{x \to a} f(x) = \infty$ **b** $\lim\limits_{x \to a} f(x) = -\infty$




**c** $\lim\limits_{x \to a^+} f(x) = \infty;\ \lim\limits_{x \to a^-} f(x) = -\infty$ **d** $\lim\limits_{x \to a^+} f(x) = -\infty;\ \lim\limits_{x \to a^-} f(x) = \infty$

In each of the following functions, determine whether the graph has a hole or a vertical asymptote at the given point. Determine the one side limits at the given points.

**a** $f(x) = \dfrac{x}{x+5}; x = -5$ **b** $f(x) = \dfrac{x^3+1}{x+1}; x = -1$

**c** $f(x) = \dfrac{|x^2-1|}{x-1}, x = 1$ **d** $f(x) = \dfrac{(x-3)^3}{|x-3|}; x = 3$

**e** $f(x) = \dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}; x = 0$ **f** $f(x) = \dfrac{x}{\sin x}; x = \pi$

## 6.2. Continuity of functions

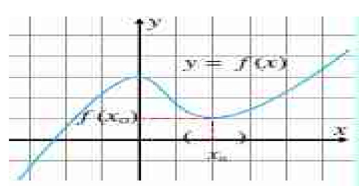
*Continuous function at a point*

A function $f$ is said to be continuous at $x_0$, if

**i** $x_0 \in D_f$ (domain of $f$)

**ii** $\lim\limits_{x \to x_0} f(x)$ exists and

**iii** $\lim\limits_{x \to x_0} f(x) = f(x_o)$



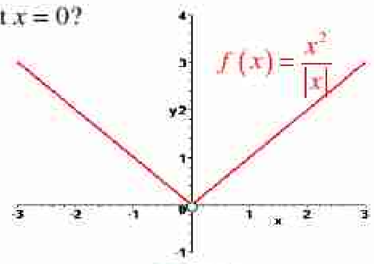

$\lim\limits_{x \to x_o} f(x) = f(x_o)$

A function $f$ is said to be discontinuous at $x_0$, if $f$ is defined on an open interval containing $x_0$ (except possibly at $x_0$) and $f$ is not continuous at $x_0$.

**Example 1** Let $f(x)=\dfrac{|x|}{x}$. Is $f$ continuous at $x=-3$? , $x=0$? and $x=1$?

**Solution** $f(x)=\dfrac{|x|}{x} \Rightarrow f(x)=\begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \\ \nexists, & \text{if } x=0 \end{cases}$

**Example 2** Let $f(x)=\dfrac{x^2}{|x|}$. Is $f$ continuous at $x=0$?




[redacted]

**Solution** $\dfrac{x^2}{|x|}=\begin{cases} x, & \text{if } x>0 \\ \nexists, & \text{if } x=0 \\ -x, & \text{if } x<0 \end{cases}$

$f(0)$ is undefined. But $\lim\limits_{x\to 0} f(x)=0$.

$\Rightarrow$ $f$ is not continuous at $x=0$

**Example 3** Find out the condition that makes $f(x)=\dfrac{1}{x-3}$ discontinuous at $x=3$?

**Solution** $f$ is discontinuous at $x=3$ because

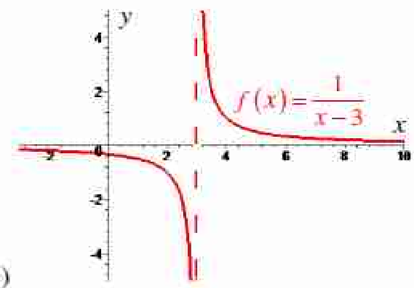
i $f(3)$ is undefined

ii $\lim\limits_{x\to 3^+} f(x)=\infty$

$\lim\limits_{x\to 3^-} f(x)=-\infty$

$\Rightarrow$ $\lim\limits_{x\to 3} f(x)$ doesn't exist.

Note that $f$ is unbroken on the interval $(3, \infty)$ and on $(-\infty, 3)$.




[redacted]

### *Continuity of a function on an interval.*

**1** ***Open interval***

A function $f$ is continuous on an open interval $(a, b)$, if

$\lim\limits_{x\to c} f(x)=f(c)\ \forall c\in(a,b)$.

**2** ***Closed interval***

A function $f$ is continuous on the closed interval $[a, b]$ provided that

i $f$ is continuous on $(a, b)$

ii $f$ is continuous from the right at $a$, and

iii $f$ is continuous from the left at $b$.

A function $f$ is continuous, if it is continuous over its domain.

### Some continuous functions

- ✓ Polynomial functions
- ✓ Absolute value of continuous functions
- ✓ The sine and cosine functions
- ✓ Exponential functions
- ✓ Logarithmic functions

Determine the value of $a$ so that the piecewise defined function

$$f(x)=\begin{cases} x+3, & \text{if } x>2 \\ ax-1, & \text{if } x\le 2 \end{cases}$$ is continuous on $(-\infty, \infty)$.

**Solution** If $f$ is continuous on $(-\infty, \infty)$, then $f$ must be continuous at $x = 2$.

$$\Rightarrow \lim_{x\to 2^-} f(x) = f(2) \Rightarrow \lim_{x\to 2^+}(x+3) = a(2)-1 \Rightarrow 5 = 2a-1 \Rightarrow a = 3$$

$$\Rightarrow f(x)=\begin{cases} x+3, \text{if } x>2 \\ 3x-1, \text{if } x\le 2 \end{cases}$$

Properties of continous function

If $f$ and $g$ are continuous at $x = x_0$, then the following functions are continuous at $x = x_0$.

1. $f+g$
2. $f-g$
3. $kg,\ k\in\mathbb{R}$
4. $fg$
5. $\frac{f}{g}$, provided that $g(x_0) \neq 0$.

Theorem Continuity of compositions of functions

If a function $f$ is continuous at $x = x_0$ and the function $g$ is continuous at $y = f(x_0)$, then the composition function $gof$ is continuous at $x = x_0$.

i.e., $\lim_{x\to x_0} g(f(x)) = \lim_{y\to f(x_0)} g(y) = g(f(x_0)) = (gof)(x_0)$.

## 6.3. Intermediate value theorem

### Theorem The intermediate value theorem

Suppose $f$ is a continuous function on the closed interval $[a, b]$ and $k$ is any real number with either $f(a) \le k \le f(b)$ or $f(b) \le k \le f(a)$, then there exists $c$ in $[a, b]$ such that $f(c) = k$.

Please student!!

Try to do Exercises included in this compiled Module and your textbook.

**1** Evaluate each of the following limits.

**a** $\lim\limits_{x \to 0}(2x-1)$ **b** $\lim\limits_{x \to -1} \dfrac{x+1}{x^2+7x+6}$ **c** $\lim\limits_{x \to 9} \dfrac{\sqrt{x}-3}{x^2-81}$

**d** $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4}-2}{x}$ **e** $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$

**2** Let $f(x) = \dfrac{x\,|x-5|}{x^2-25}$, evaluate

**a** $\lim\limits_{x \to 5^+} f(x)$ **b** $\lim\limits_{x \to -5^+} f(x)$ **c** $\lim\limits_{x \to 5^-} f(x)$ **d** $\lim\limits_{x \to -5^-} f(x)$

**3** Let $f(x) = \begin{cases} 3, \text{ if } x = -5 \\ -0.6, \text{ if } -5 < x \leq -2 \\ x^2-4, \text{ if } -2 < x < 3 \\ x+2, \text{ if } x \geq 3 \end{cases}$

Sketch the graph of $f$ and evaluate each of the following limits.

**a** $\lim\limits_{x \to -5} f(x)$ **b** $\lim\limits_{x \to -2} f(x)$ **c** $\lim\limits_{x \to 3} f(x)$

**4** Evaluate each of the following limits.

**a** $\lim\limits_{x \to 3} \left(x^3-4x^2+5x-11\right)$ **b** $\lim\limits_{x \to 2} \sqrt{x^2-5x}$

**5** Test whether or not each of the given functions is continuous at the indicated number.

**a** $f(x) = \begin{cases} x^2-x, \text{ if } x \geq 1 \\ x+1, \text{ if } x < 1 \end{cases}; x = 1$ **b** $f(x) = \dfrac{x^2\left|9-x^2\right|}{3-x}; x = 3$

**c** $f(x) = \begin{cases} \dfrac{\sin x}{x}, \text{if } x \neq 0 \\ 1, \text{if } x = 0 \end{cases}; x = 0$ **d** $f(x) = \begin{cases} \dfrac{1}{4}, \text{ if } x \in \mathbb{Z} \\ 4^x, \text{ if } x \in \mathbb{Z} \end{cases}; x = \dfrac{1}{2}$

**e** $f(x) = \begin{cases} \dfrac{\cos x}{e^x}, \text{ if } x > 0 \\ e^x, \text{ if } x \leq 0 \end{cases}; x = 0$

**6** Determine the values of the constants so that each of the given functions is continuous.

**a** $f(x) = \begin{cases} ax-1, \text{if } x \leq 2 \\ x^2+3x, \text{ if } x > 2 \end{cases}$ **b** $f(x) = \begin{cases} \dfrac{x^2-ax}{x-a}, \text{ if } x \neq a \\ 2, \text{ if } x = a \end{cases}$

**c** $f(x) = f(x) = \begin{cases} \sin(k+x), \text{ if } x \leq 0 \\ 1, \text{ if } x > 0 \end{cases}$ **d** $f(x) = \begin{cases} x^2+1, \text{ if } x < a \\ 15-5x, \text{ if } a \leq x \leq b \\ 5x-25, \text{ if } x > b \end{cases}$

**7** Evaluate each of the following limits.

**a** $\lim\limits_{x \to \infty} \dfrac{3x^3+5x^2-11}{2x^3-1}$ **b** $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+1}-10}{\sqrt{x^2+1}+9}$

**8** Evaluate each of the following one side limits.

**a** $\lim\limits_{x \to 0^+} |x|-3x$ **b** $\lim\limits_{x \to 3^+} \sqrt{3-x}$ **c** $\lim\limits_{x \to 3^+} \sqrt{3x-9}$

**d** $\lim\limits_{x \to 0^+} \ln x$ **e** $\lim\limits_{x \to 5^+} \dfrac{x}{(x-5)^3}$ **f** $\lim\limits_{x \to 2^+} \sqrt{1-\sqrt{x-1}}$

9 Determine the largest interval on which each of the given functions is continuous.

a $f(x)=\sqrt{\frac{1-x}{x}}$ b $f(x)=\sqrt{\ln\sqrt{x}}$

c $f(x)=\ln\left(\frac{x}{x^4-1}\right)$ d $f(x)=\sqrt{\frac{4x-3}{x-4}}$

10 Determine the maximum and minimum values of each of the functions defined on the indicated closed interval.

a $f(x)=3x+5$ ; $[-3, 2]$ b $g(x)=1-x^2$ ; $[-2, 3]$

c $h(x)=x^4-x^2$ ; $[-2, 2]$ d $f(x)=\frac{1}{x}$ ; $[-2, 2]$

e $h(x)=4x^2-5x+1$ ; $[-1.5, 1.5]$ f $f(x)=\begin{cases} x^2, & \text{if } |x|\le 1 \\ 2-|x|, & \text{if } |x|>1 \end{cases}$; $[-3,2]$

11 Locate the zeros of each of the following functions using the intermediate value theorem.

a $f(x)=x^2-x-1$ b $g(x)=x^3+2x^2-5$

c $h(x)=x^3-x+2$ d $f(x)=x^4-2x^3-x^2+3x-2$

e $g(x)=x^4-9x^2+14$

12 Evaluate each of the following limits.

a $\lim_{x\to 0}\frac{\sin\left(\frac{x}{\pi}\right)}{\tan x}$ b $\lim_{x\to 0}\frac{\sin(x^3)}{x^3}$ c $\lim_{x\to\infty} x\tan\left(\frac{1}{x}\right)$

d $\lim_{x\to 0}\frac{x-\tan x}{x}$ e $\lim_{x\to\infty}\left(1+\frac{3}{x+11}\right)^{x+6}$

13 In a certain country, the life expectancy for males $x$-years from now is given by the formula $f(x)=\frac{210x+116}{3x+4}$ years. What will be the life expectancy of males in this country as time passes? Discuss whether or not the life expectancy in the country is increasing.

**Chapter 7**: **Derivative and its Application (11 hrs)**

7.1. Definition of derivative and its geometric interpretation

Activity Give definition Secant line and tangent line

Definition

Let $x_o$ be in the domain of a function $f$.

If $\lim_{x\to x_o}\frac{f(x)-f(x_o)}{x-x_o}$ exists, then we call this limit the derivative of $f$ at $x_o$.

**Example 9** Find the derivative of each of the following functions at the given number.

**a** $f(x) = 4x + 5;\ x_0 = 2$ **b** $f(x) = \frac{1}{4}x^2 + x;\ x_0 = -1$

**c** $f(x) = x^3 - 9x;\ x_0 = \frac{1}{3}$ **d** $f(x) = \sqrt{x};\ x_0 = 4$

**Solution** Using the Definition,

$f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$, you obtain,

**a** $f'(2) = \lim\limits_{x\to 2}\frac{(4x+5)-(4(2)+5)}{x-2} = \lim\limits_{x\to 2}\frac{4x-8}{x-2} = \lim\limits_{x\to 2}\frac{4(x-2)}{x-2} = 4.$

**b** $f'(-1) = \lim\limits_{x\to -1}\frac{\frac{1}{4}x^2+x-\left(\frac{1}{4}(-1)^2-1\right)}{x-(-1)} = \lim\limits_{x\to -1}\frac{\frac{1}{4}x^2+x+\frac{3}{4}}{x+1}$

$= \lim\limits_{x\to -1}\frac{\frac{1}{4}(x+3)(x+1)}{x+1} = \frac{1}{4}\lim\limits_{x\to -1}(x+3) = \frac{1}{2}.$

**c** $f'\left(\frac{1}{3}\right) = \lim\limits_{x\to \frac{1}{3}}\frac{x^3-9x-\left(\left(\frac{1}{3}\right)^3-9\left(\frac{1}{3}\right)\right)}{x-\frac{1}{3}} = \lim\limits_{x\to\frac{1}{3}}\frac{\left(x^2+\frac{1}{3}x-\frac{80}{9}\right)\left(x-\frac{1}{3}\right)}{x-\frac{1}{3}}$

$= \left(\frac{1}{3}\right)^2 + \frac{1}{3}\times\frac{1}{3} - \frac{80}{9} = \frac{-26}{3}.$

**Example 10** Find the equation of the line tangent to the graph of $f(x) = x^2$ at

**a** $x = 1$, **b** $x = 0$, **c** $x = -5$

**Solution** $f(x) = x^2 \Rightarrow f'(x) = 2x$

**a** $f(1) = 1$ and $f'(1) = 2$

$\Rightarrow$ The equation of the tangent line is:

$y - f(1) = f'(1)(x-1) \Rightarrow y - 1 = 2(x-1) \Rightarrow y = 2x - 1$

**b** $y - f(0) = f'(0)(x-0) \Rightarrow y = 0$

**c** $y - f(-5) = f'(-5)(x-(-5))$

$\Rightarrow y - 25 = -10(x+5) \Rightarrow y = -10x - 25$

7.2. Rules of differentiation

a) **Derivative of a power function**

**Theorem** Power rule for differentiation

Let $f(x) = x^n$, where $n$ is a positive integer. Then $f'(x) = nx^{n-1}$

Theorem Derivatives of sine and cosine functions

**1** If $f(x) = \sin x$, then $f'(x) = \cos x$. **2** If $f(x) = \cos x$, then $f'(x) = -\sin x$.

Exercises

1. Find the derivatives of each of the following functions with respect to the appropriate variable.

   a. $f(x) = -\sin x$  b. $g(\theta) = -\cos\theta$

   c. $f(x) = \sec x$  d. $g(t) = \csc t$

2. Find the equation of the tangent line to the graph of $f$ at the given point.

   a. $f(x) = \sin x; \left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$  b. $g(x) = \cos x; \left(\frac{\pi}{2}, 0\right)$

   c. $h(x) = \tan x; \left(\frac{\pi}{4}, 1\right)$

3. If the line tangent to the graph of $f(x) = \sin x$ at $x = a$ has $y$-intercept $\frac{\sqrt{3}}{6} - \frac{\pi}{3}$, find the $x$-intercept of the line when $0 < a < \frac{\pi}{2}$.

## 7.3. Derivatives of special functions (exponential, logarithmic and trigonometric)

### Theorem Derivatives of exponential functions

If $f(x) = a^x; a > 0$, then $f'(x) = a^x \ln a$.

**Example** Find the derivative of each of the following exponential functions.

a. $f(x) = 4^x$  b. $f(x) = \sqrt{5^x}$  c. $f(x) = \pi^x$

d. $f(x) = e^{x+3}$  e. $f(x) = \sqrt[3]{e^x}$  f. $f(x) = 2^{3x+5}$

**Solution**

a. $f(x) = 4^x \Rightarrow f'(x) = 4^x \ln 4$

b. $f(x) = \sqrt{5^x} \Rightarrow f'(x) = \sqrt{5^x} \ln\sqrt{5} = \frac{5^{\frac{1}{2}x}}{2} \ln 5$

c. $f(x) = \pi^x \Rightarrow f'(x) = \pi^x \ln\pi$

d. $f(x) = e^{x+3} \Rightarrow f(x) = e^3.e^x \Rightarrow f'(x) = e^3 e^x = e^{x+3}$

e. $f(x) = \sqrt[3]{e^x} \Rightarrow f'(x) = \sqrt[3]{e^x} \ln\sqrt[3]{e} = \frac{1}{3}\sqrt[3]{e^x} \ln e = \frac{1}{3}e^{\frac{x}{3}}$

f. $f(x) = 2^{3x+5} \Rightarrow f'(x) = 2^5 \times 8^x \ln 8$

$\Rightarrow f'(x) = 2^5 \times 2^{3x} (3\ln 2) \quad \Rightarrow f'(x) = 96\,(2^{3x}) \ln 2.$

### Theorem Derivatives of logarithmic functions

If $f(x) = \ln x, x > 0$, then $f'(x) = \frac{1}{x}$.

**Example** Find the derivatives of each of the following logarithmic functions

a $f(x) = \log_2 x$ b $f(x) = \log x$ c $f(x) = \log_{\frac{1}{5}} x$

d $f(x) = \log(x^3)$ e $f(x) = \ln\sqrt[5]{x}$ f $f(x) = \log_5 \sqrt{x^3}$

**Solution**

a $f(x) = \log_2 x \Rightarrow f'(x) = \dfrac{1}{x \ln 2}$

b $f(x) = \log x \Rightarrow f'(x) = \dfrac{1}{x \ln 10}$

c $f(x) = \log_{\frac{1}{5}} x \Rightarrow f'(x) = \dfrac{1}{x \ln\left(\frac{1}{5}\right)} = -\dfrac{1}{x \ln 5}$

d $f(x) = \log(x^3) \Rightarrow f(x) = 3\log x \Rightarrow f'(x) = \dfrac{3}{x \ln 10}$

e $f(x) = \ln\sqrt[5]{x} \Rightarrow f(x) = \dfrac{1}{5}\ln x \Rightarrow f'(x) = \dfrac{1}{5x}$

f $f(x) = \log_5 \sqrt{x^3} \Rightarrow f(x) = \dfrac{3}{2}\log_5 x \Rightarrow f'(x) = \dfrac{3}{2x \ln 5}$

## DERIVATIVES OF COMBINATIONS AND COMPOSITIONS OF FUNCTIONS

**Please read this part from your textbook.**

### Derivatives Of Product Of Functions

If $f$ and $g$ are differentiable functions at $x_0$, then the product $fg$ is differentiable at $x_0$ and its derivative is given as follows:

$$(fg)'(x_0) = f'(x_0)\, g(x_0) + g'(x_0)\, f(x_0).$$

### The quotient rule

If $f$ and $g$ are differentiable functions and $g(x) \neq 0$, then $\dfrac{f}{g}$ is differentiable for all $x$ at which $f$ and $g$ are differentiable with

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

**1** Differentiate each of the following functions using the appropriate rules.

**a** $f(x) = 1 - x - x^2 + x^3$ **b** $g(x) = 7\sqrt{x} + e^x - \sin x$

**c** $h(x) = \dfrac{x}{x+5}$ **d** $l(x) = x + \sin x - e^x$

**e** $k(x) = \dfrac{x \sin x}{x - e^x}$ **f** $f(x) = \dfrac{\sqrt{x}}{x \cos x}$

**g** $g(x) = \csc x \sec x$ **h** $h(x) = \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{\sec x}{x^2}$

**i** $k(x) = \dfrac{4x+5}{x^2+1}$ **j** $f(x) = x^2 \ln x$

**2** For each of the following functions, find $\dfrac{dy}{dx}$.

**a** $y = \ln x + e^x$ **b** $y = (x^2 - 2x - 3)\,e^x$ **c** $y = \dfrac{1 - \ln x}{x^2}$

**d** $y = \dfrac{x^2+1}{\cos x}$ **e** $y = \dfrac{e^x + x - 1}{x+1}$ **f** $y = \dfrac{\sin x}{1 - \cos x}$

**g** $y = \dfrac{1 - \sin x}{x + \cos x}$ **h** $y = \dfrac{e^x \sin x}{e^x + 1}$ **i** $y = \dfrac{x^2}{x + \ln x}$

**j** $y = e^x(1 + x^2)\tan x$ **k** $y = \dfrac{\left(1 + \dfrac{1}{x^2}\right)}{1 - \dfrac{1}{x^2}}$ **l** $y = \left(e^x - \sqrt{x}\right)^2$

**3** In each of the following, find the equation of the tangent line to the graph of $f$ at $(a, f(a))$.

**a** $f(x) = \dfrac{x-1}{x+1};\ a=0$ **b** $f(x) = \dfrac{3x+1}{4-x^2};\ a=1$

**c** $f(x) = e^x \sin x;\ a=0$ **d** $f(x) = \dfrac{x^2-4x}{e^x+1};\ a=0$

## Chain rule

Let $g$ be differentiable at $x_0$ and $f$ be differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at $x_0$ and $(f \circ g)'(x_0) = f'(g(x_0)).g'(x_0)$

## Exercises

1 Use the chain rule and any other appropriate rule to differentiate each of the following functions.

a $f(x)=e^{x^2+6}$ b $f(x)=(x+5)^{10}$ c $f(x)=(4x+5)^{12}$

d $f(x)=\sin(3x)$ e $f(x)=\cos(x^2+1))$ f $f(x)=\dfrac{e^{(x+2)}}{xe^x-1}$

g $f(x)=e^{-3x}\sin(4x^2+5x+1)$ h $f(x)=\sqrt{x^2+2x+3}$

i $f(x)=\log_3(x^2+4)$ j $f(x)=\dfrac{x^2}{x+\ln(x^2+9)}$

k $f(x)=\dfrac{\sin x}{\sqrt{2x+1}}$ l $f(x)=\sin(x^2)+\cos(x^2)$

m $f(x)=\ln\left(\dfrac{1}{x^2+1}\right)$ n $f(x)=\ln\sqrt{x^2+1}$

o $f(x)=\sin\sqrt{\ln(x^2+7)}$ p $\log_a x$

q $f(x)=e^{-\sqrt{x^2+1}}\sin\left(\sqrt{x^2+1}\right)$ r $f(x)=\ln\sqrt{\cos(x^2+3)}$

2 Find the equation of the line tangent to the graph of $f$ at $(a, f(a))$, if

a $f(x)=xe^{-\sqrt{x+1}}$ at $(0, 0)$ b $f(x)=e^{2-x^2}$ at $(1, e)$

c $f(x)=\ln\left(\dfrac{x+1}{\cos x}\right)$ at $(0, 0)$ d $f(x)=\dfrac{e^{3x+2}}{1-2x}$ at $\left(-1, \dfrac{1}{3e}\right)$

e $f(x)=(8-x^3)\sqrt{2-x}$ at $(-2, 32)$

3 Find $\dfrac{dy}{dx}$.

a $y=\sqrt{1+x^8}$ b $y=\sqrt{1+3x^2}\,e^x$ c $y=\dfrac{2x^3}{\sqrt{1+x^4}}$

d $y=\sqrt{\dfrac{x^2}{x^3+1}}$ e $y=\left(\dfrac{2x-1}{3-4x}\right)^n$ f $y=\cos(\ln\sqrt{e^x})$

g $y=(ax+b)^r$; where $r$ is a real number.

## Higher Order Derivatives of a Function

If $y=f(x)$, then, we write $f^{(n)}(x)=\dfrac{d^n y}{dx^n}=\dfrac{d^n}{dx^n}f(x)\equiv D^n f(x)$

Find a formula for the $n^{th}$ derivative of each of the following functions for the given values of $n$.

a $f(x)=e^{(3x+1)}$; $n\in\mathbb{N}$ b $f(x)=e^{x^2}$; $n=6$

c $f(x)=\ln\left(\dfrac{1}{x^2+1}\right)$; $n=4$ d $f(x)=e^{-x^2+3x-5}$; $n=4$

7.4. Extreme values of functions Reading assignment

7.5. Minimization and Maximization problems Reading assignment

**1 Critical number**

Suppose $f$ is defined at $c$ and either $f'(c)=0$ or $f'(c)$ does not exist. Then the number $c$ is called a critical number of $f$ and the point with coordinates $(c, f(c))$ on the graph of $f$ is called a critical point; this critical point is either a valley or a peak of the graph.

**2 Absolute maximum and absolute minimum**

Let $f$ be a function defined on some set $S$ that contains $c$. Then

$f(c)$ is an absolute maximum of $f$ on S if $f(c) \geq f(x)$ for all $x$ in $S$.

$f(c)$ is an absolute minimum of $f$ on S if $f(c) \leq f(x)$ for all $x$ in $S$.

**3 Relative maximum and relative minimum**

The function $f$ is said to have a relative maximum at $c$, if $f(c) \geq f(x)$ for all $x$ in an open interval containing $c$.

The function $f$ is said to have a relative minimum at $c$, if $f(c) \leq f(x)$ for all $x$ in an open interval containing $c$.

**4 First derivative test**

Let $f$ be a function which is continuous and differentiable on an interval $I$. Then

**a** *First derivative test for local extreme values*

If $f'$ changes sign from positive to negative at $c$ then $f$ has a local maximum value at $c$ for some critical number $c$.

If $f'$ changes sign from negative to positive at $c$ then $f$ has a local minimum value at $c$ for some critical number $c$.

**b** *First derivative test for intervals of monotonicity*

If $f'(x)>0$ on $I$, then $f$ is strictly increasing on $I$; if $f'(x)<0$ on $I$, then $f$ is strictly decreasing on $I$.

**5 Second derivative test**

Let $f$ be a function such that $f'(c)=0$ and the second derivative exists on an open interval $I$ containing $c$. Then

**a** *Second derivative test for local extreme values*

If $f''(c)>0$ then $f(c)$ is a local minimum value on $I$.

If $f''(c)<0$ then $f(c)$ is a local maximum value on $I$.

If $f''(c)=0$, then the test fails.

**b** *Second derivative test for intervals of concavity*

If $f''(x)>0$ for all $x$ in $I$ then the graph of $f$ is concave upward on $I$.

If $f''(x)<0$ for all $x$ in $I$ then the graph of $f$ is concave downward on $I$.

**6 Inflection point**

The point at which concavity changes, either from concave up to concave down, or from concave down to concave up is called an inflection point.

Exercises

1 For each of the following functions find: critical numbers, local extreme values, intervals of monotonicity, intervals of concavity and inflection points.

a $f(x)=x^4-8x^2+6$  b $f(x)=x^3+3x^2-9x+5$

c $f(x)=\dfrac{2x}{x^2+1}$  d $f(x)=\dfrac{x^2-2x+4}{x-2}$

2 Find the absolute maximum and minimum values of each of the following functions on the indicated intervals.

a $f(x)=x^4-8x^2+6$; $[-3, 3]$  b $f(x)=x^3+3x^2-9x+5$; $[-2, 2]$

c $f(x)=\dfrac{2x}{x^2+1}$; $[1, 2]$  d $f(x)=\dfrac{x^2-2x+4}{x-2}$; $[-3, 1]$

3 A box is to have a square base, an open top, and volume of 32 $m^3$. Find the dimensions of the box that use the least amount of material.

4 Determine the point(s) $f(x)=x^2+1$ that are closest to the point (0, 2).

5 Find the maximum and minimum of the function $f(x) = \cos 2x - 2\sin x$ for $0 \le x \le 2\pi$.

6 A window whose bottom is a rectangle and top is a semicircle being built. It there is 12m of framing materials, then what must be the dimension of the window?

7 Determine the area of the largest rectangle that can be inscribed in a circle of radius 9 m.

8 Water is being poured into a conical vase at a rate of 18 $cm^3$/sec. The diameter of the cone is 30 cm and its height is 25 cm. At what rate is the water level rising when its depth is 20 cm?

9 Two poles, one 6 m tall and the other 15 m tall, are 20 m apart. A wire is attached to the top of each pole and is also staked to the ground somewhere between the two poles. Where should the wire be staked so that the minimum amount of wire is used?

10 A car travelling north at 48 km/hr is approaching an intersection. A truck, travelling East at 60 km/hr is moving away from the same intersection. How is the distance between the car and the truck changing when the car is 9 m from the intersection and the truck is 40 m from the intersection?

## Chapter 8: Integration and its Application (11 hrs)

### 8.1. Integration as a reverse process of derivative

The process of finding $f(x)$ from its derivative $f'(x)$ is said to be anti differentiation or integration. $f(x)$ is said to be the anti derivative of $f'(x)$.

$$f(x) \xrightarrow{\text{differentiation}} f'(x)$$
$$f(x) \xleftarrow[\text{anti differentiation}]{} f'(x)$$

Integration is the reverse operation of differentiation.

The set of all anti derivatives of a function $f(x)$ is called the indefinite integral of $f(x)$. The indefinite integral of $f(x)$ is denoted by $\int f(x)dx$ read as "the integral of $f(x)$ with respect to $x$".

- ✓ The symbol $\int$ is said to be the integral sign.
- ✓ The function $f(x)$ is said to be the integrand of the integral.
- ✓ $dx$ denotes that the variable of integration is $x$.
- ✓ If a function has an integral, then it is said to be integrable.
- ✓ If $F'(x) = f(x)$, then $\int f(x)dx = F(x) + c$
- ✓ $\int f(x)dx$ is read as, " the integral of $f(x)$ with respect to $x$".
- ✓ $c$ is said to be the constant of integration.

Example

Example 1 $\int x\,dx = \frac{x^2}{2} + c$ Because $\frac{d}{dx}\left(\frac{x^2}{2} + c\right) = \frac{2x}{2} + 0 = x$

A detail discussion should be there with your teacher.

8.2. Techniques of integration (Substitution, by partial fractions and by parts)

Evaluate each of the following integrals.

1 $\int \frac{d}{dx}(x^3)dx$

2 $\frac{d}{dx}\int x^3 dx$

3 $\int\left(x^8 + x^{\frac{1}{3}} - x^{-4} + x^{-\frac{3}{2}}\right)dx$

4 $\int\left(\sqrt{x} - 3x^3 + x^{-2} + 2\right)dx$

5 $\int \frac{x^3 + x^2 + x + 1}{x^4}dx$

6 $\int \frac{(x+1)^2}{\sqrt{x}}dx$

7 $\int \frac{(z^4 + z^3 - 2z^2 + z + 1)}{z^2}dz$

8 $\int (x-1)(x^2 + x + 1)\,dx$

9 $\int \frac{(t^2 - 3t + 4)}{t}dt$

10 $\int\left(\frac{x+1}{x^2}\right)dx$

11 $\int\left(e^x - e^{-x} + \frac{1}{x}\right)dx$

12 $\int \frac{(e^x - 1)(e^x - 2)}{\sqrt{e^x}}dx$

13 $\int\left(2x^3 + e^{2x} - \frac{1}{2x}\right)dx$

14 $\int e^x(1 - e^x)^2\,dx$

15 $\int\left(3^{1-2x} + \frac{1}{\sqrt{2^x}} + \frac{1}{e^{2x}}\right)dx$

8.3. Definite integral and Fundamental theorem of calculus

8.4. Area of a region under a curve and between two curves

Unit Summary

Review Exercises

**Chapter 9**: **Introduction to Linear Programming (6 hrs)**

9.1. Graphical solutions of systems of linear inequalities

9.2. Maximum and minimum values

9.3. Real life linear programming problems

Unit Summary

Review Exercises

**Chapter 10**: **Mathematical Applications in Business (8 hrs)**

10.1. Basic mathematical concepts in business

10.2. Interest (simple and compound) and depreciation

10.3. Saving, investing and borrowing money

10.4. Taxation

10.5. Purchasing

10.6. Percent increase and percent decrease

10.7. Real state expenses

10.8 Wages