PROBABILITY & STATISTICS

BS 1402

Contents

- Random Variables
- Discrete Probability Distributions
- Continuous Probability Distributions
- Joint Probability Distribution
- Mean of a Random Variable

Concept of a Random Variable

A **random variable** is a function that associates a real number with each element in the sample space.

Example 1:

Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y, where Y is the number of red balls, are

Sample Space	y
RR	2
RB	1
BR	1
BB	0

Discrete Sample Space

If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a **discrete sample space**.

Continuous Sample Space

If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a **continuous sample space**.

Discrete Probability Distributions

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_{x} f(x) = 1$,
3. $P(X = x) = f(x)$.

Example 2:

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school. Then x can only take the numbers 0, 1, and 2

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95}, \quad f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190},$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}.$$

Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Cumulative Distribution Function

There are many problems where we may wish to compute the probability that the observed value of a random variable *X* will be less than or equal to some real number *x*.

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

Example 3:

Find the cumulative distribution function of the random variable *X* for the following function:

$$f(x) = \frac{1}{16}\binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

$$F(0) = f(0) = \frac{1}{16},$$

$$F(1) = f(0) + f(1) = \frac{5}{16},$$

$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16},$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{15}{16},$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$$

Example 3: (cont.)

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{16}, & \text{for } 0 \le x < 1, \\ \frac{5}{16}, & \text{for } 1 \le x < 2, \\ \frac{11}{16}, & \text{for } 2 \le x < 3, \\ \frac{15}{16}, & \text{for } 3 \le x < 4, \\ 1 & \text{for } x \ge 4. \end{cases}$$

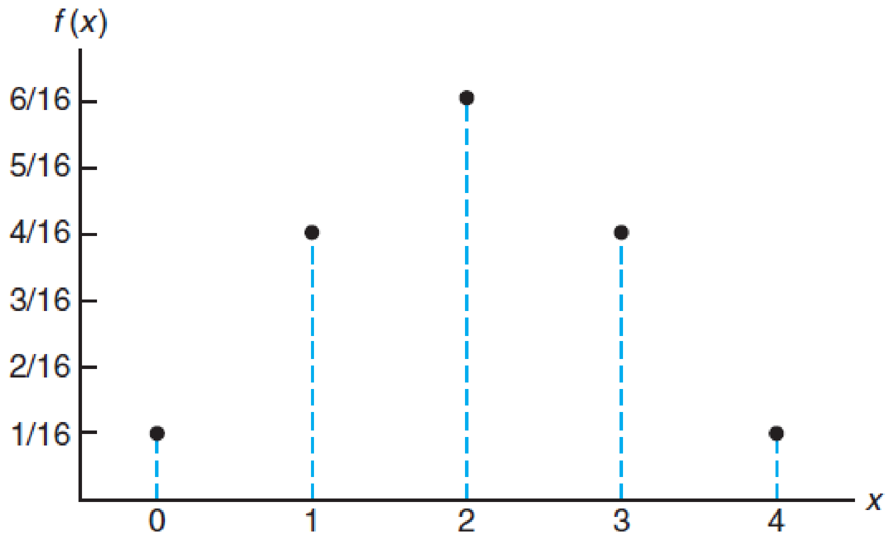

Figure 3.1: Probability mass function plot.

Example 3: (cont.)

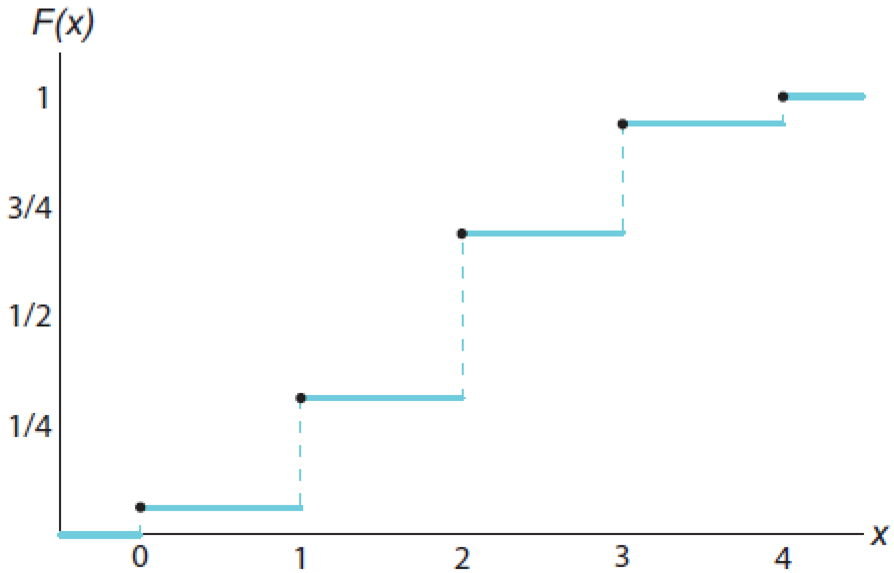

Figure 3.3: Discrete cumulative distribution function.

Continuous Probability Distributions

Probability Density Function of a continuous random variable:

The function $f(x)$ is a **probability density function** (pdf) for the continuous random variable X, defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
3. $P(a < X < b) = \int_a^b f(x)\, dx$.

Example 4:

Suppose that the error in the reaction temperature, in °C, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere}. \end{cases}$$

.

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Solution: We use Definition 3.6.

(a) Obviously, $f(x) \geq 0$. To verify condition 2 in Definition 3.6, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-1}^{2} \frac{x^2}{3} dx = \frac{x^3}{9}\Big|_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1.$$

Example 4: (cont.)

(b) Using formula 3 in Definition 3.6, we obtain

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \left. \frac{x^3}{9} \right|_0^1 = \frac{1}{9}.$$

Continuous Probability Distributions

Cumulative Distribution Function of a continuous random variable:

The **cumulative distribution function** $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt, \quad \text{for } -\infty < x < \infty.$$

$$P(a < X < b) = F(b) - F(a)$$

Example 5:

For the density function of Example 4, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

***Solution*:** For $-1 < x < 2$,

$$F(x) = \int_{-\infty}^{x} f(t)\ dt = \int_{-1}^{x} \frac{t^2}{3} dt = \left.\frac{t^3}{9}\right|_{-1}^{x} = \frac{x^3+1}{9}.$$

Therefore,

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x^3+1}{9}, & -1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

The cumulative distribution function $F(x)$ is expressed in Figure 3.6. Now

$$P(0 < X \leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9},$$

Joint Probability Distribution

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum\limits_{x} \sum\limits_{y} f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum\limits_{A} f(x, y)$.

Example 6:

Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If *X* is the number of blue pens selected and *Y* is the number of red pens selected, find the joint probability function *f* (*x*, *y*)

***Solution*:** The possible pairs of values (x, y) are $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$, and $(2,0)$.

Now, $f(0,1)$, for example, represents the probability that a red and a green pens are selected. The total number of equally likely ways of selecting any 2 pens from the 8 is $\binom{8}{2} = 28$. The number of ways of selecting 1 red from 2 red pens and 1 green from 3 green pens is $\binom{2}{1}\binom{3}{1} = 6$. Hence, $f(0,1) = 6/28$

$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \leq x + y \leq 2$.

Example 6: (cont.)

Table 3.1: Joint Probability Distribution for Example 3.14

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Mean of a Random Variable

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

Example 7:

Assuming that 1 fair coin was tossed twice, we find that the sample space for our experiment is

$$S = \{HH, HT, TH, TT\}.$$

Since the 4 sample points are all equally likely, it follows that

$$P(X = 0) = P(TT) = \frac{1}{4}, \quad P(X = 1) = P(TH) + P(HT) = \frac{1}{2},$$

and

$$P(X = 2) = P(HH) = \frac{1}{4},$$

$$\mu = E(X) = (0)\left(\frac{1}{4}\right) + (1)\left(\frac{1}{2}\right) + (2)\left(\frac{1}{4}\right) = 1.$$

This result means that a person who tosses 2 coins over and over again will, on the average, get 1 head per toss.

Example 8:

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Solution:

Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Example 9:

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20{,}000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

***Solution*:** Using Definition 4.1, we have

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x) f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\ dx$$

if X is continuous.

Example 10:

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution: By Theorem 4.1, the attendant can expect to receive

$$\begin{aligned} E[g(X)] = E(2X-1) &= \sum_{x=4}^{9}(2x-1)f(x) \\ &= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right) + (13)\left(\frac{1}{4}\right) \\ &\quad + (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right) = \$12.67. \end{aligned}$$

Thank You.