Ordinary Differential Equations

(Lecture-6)

Neelam Choudhary

Department of Mathematics
Bennett University
India

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Learning Outcome of the Lecture

We learn

- Exact Differential Equations
  - Necessary and Sufficient Condition for Exactness
  - Examples

Total Differential

The first order differential equation may be expressed in either the derivative form

$$\frac{dy}{dx} = f(x, y)$$

or the differential form

$$M(x, y)dx + N(x, y)dy = 0.$$

Example-1:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$$

Example-2:

$$(\sin x + y)dx + (x + 3y)dy = 0$$

Total Differential

Definition

Let F be a function of two variables such that F has continuous first order partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x,y) = \frac{\partial F(x,y)}{\partial x}dx + \frac{\partial F(x,y)}{\partial y}dy$$

for all $(x,y) \in D$.

Example: $F(x,y) = xy^2 + 2x^3y$

$$\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2y \text{ and } \frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$$

The total differential dF is

$$dF(x,y) = (y^2 + 6x^2y)dx + (2xy + 2x^3)dy$$

for all real (x,y).

Exact Differential Equation

Definition

The expression

$$M(x,y)dx + N(x,y)dy \tag{1}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$.

That means,

expression (1) is an exact differential in D if there exists a function F such that

$$\frac{\partial F}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y) \text{ for all } (x,y) \in D.$$

If $M(x,y)dx + N(x,y)dy$ is an exact differential, then the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an exact differential equation.

Example

Consider the differential equation $y^2dx + 2xydy = 0$, then

$$dF(x, y) = d(xy^2) = y^2dx + 2xydy.$$

This implies that the given differential equation is exact.

Necessary and Sufficient Condition for Exactness

Theorem

Consider the differential equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{2}$$

where M and N have continuous first order partial derivatives at all points (x,y) in a rectangular domain D.

- If the differential equation (2) is exact in D, then

  $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

  for all $(x,y) \in D$.
- Conversely, if

  $$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

  for all $(x,y) \in D$, then the differential equation (2) is exact in D.

Examples

Example-1:

$$y^2dx + 2xydy = 0,$$

is an exact differential equation since $\frac{\partial M(x,y)}{\partial y} = 2y = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$.

Example-2:

$$ydx + 2xdy = 0,$$

is NOT an exact differential equation since $\frac{\partial M(x,y)}{\partial y} = 1 \neq 2 = \frac{\partial N(x,y)}{\partial x}$ for all $(x,y) \in D$.

Example-3: Is

$$(2x\sin y + y^3e^x)dx + (x^2\cos y + 3y^2e^x)dy = 0.$$

exact? Why?

Solution of Exact Differential Equations

Theorem

Suppose the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

satisfies the differentiability requirements of previous theorem and is exact in a rectangular domain D. Then a one-parameter family of solutions of this DE is given by $F(x,y) = c$, where F is a function such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \text{ and } \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$ and c is an arbitrary constant.

Working Rule

For a given exact DE, $M(x,y)dx + N(x,y)dy = 0$, the function $F(x,y)$ can be found either by inspection or by the following method:

- **Step-1**: Integrate $\frac{\partial F(x,y)}{\partial x} = M(x,y)$ with respect to x to obtain
$$F(x,y) = \int M(x,y)dx + \phi(y),$$
where $\phi(y)$ is a constant of integration.
- **Step-2**: To determine the function $\phi(y)$, differentiate the above equation with respect to y, to obtain
$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$
- **Step-3**: Use the condition
$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\left(\int M(x,y)dx\right) + \frac{d\phi(y)}{dy}.$$
Determine $\phi(y)$ and hence the function $F(x,y)$.

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Examples

Example-1: Solve the DE by method of inspection

$$ydx + xdy = 0.$$

Solution:

$$d(xy) = ydx + xdy = 0.$$

$xy = c$ is the solution of the given DE.

Example-2: Solve the DE by method of inspection

$$(2x + y^2)dx + 2xydy = 0.$$

Solution:

$$(2x + y^2)dx + 2xydy = 0 \Rightarrow 2xdx + d(xy^2) = 0 \Rightarrow x^2 + xy^2 = c.$$

$x^2 + xy^2 = c$ is the solution of the given DE.

Examples Continue

Example-3: Solve the DE

$$(y\cos x + 2xe^{y})dx + (\sin x + x^{2}e^{y} - 1)dy = 0.$$

Solution: On comparing with $Mdx + Ndy = 0$, we get

$$M(x,y) = y\cos x + 2xe^{y}, \quad N(x,y) = \sin x + x^{2}e^{y} - 1$$

<u>Check for exactness:</u>

$$\frac{\partial M(x,y)}{\partial y} = \cos x + 2xe^{y} = \frac{\partial N(x,y)}{\partial x},$$

so the given DE is exact.

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Examples Continue

Finding Solution for Exact DE: We need to find $F(x,y)$ such that,

$$\frac{\partial F}{\partial x} = M(x,y) = y\cos x + 2xe^{y}$$

$$\frac{\partial F}{\partial y} = N(x,y) = y\sin x + x^2e^{y} - 1$$

- **Step-1**: Integrate $\frac{\partial F}{\partial x} = M(x,y)$ with respect to x.

$$F(x,y) = \int M(x,y)dx + \phi(y)$$

$$F(x,y) = \int (y\cos x + 2xe^{y})dx + \phi(y) \Rightarrow F(x,y) = y\sin x + x^2e^{y} + \phi(y)$$

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Examples Continue

- **Step-2**: Determine unknown function $\phi(y)$, using condition $\frac{\partial F}{\partial y} = N(x,y)$.

$$\frac{\partial F}{\partial y} = \sin x + x^2 e^y + \phi'(y) = \sin x + x^2 e^y - 1$$

$$\Rightarrow \phi'(y) = -1 \Rightarrow \phi(y) = -y + c_0$$

$$\text{So,} \quad F(x,y) = y \sin x + x^2 e^y - y + c_0$$

- **Step-3**: Solution of ODE:

$$F(x,y) = c_1$$

$$y \sin x + x^2 e^y - y = c_1 - c_0 = c$$

$$y \sin x + x^2 e^y - y = c$$

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