Solution of

Assignment - 7

(1) (a) For f, g in $C[a,b]$, set

$$\langle f, g \rangle = \int_a^b f(t)\, g(t)\, dt.$$

Show that $(C[a,b], \langle \cdot \rangle)$ is an inner product space.

Soln: (i) $\langle f, f \rangle = \int_a^b f^2(t)\, dt \geq 0$.

The function $f^2(t)$ is continuous & non negative.

Now if $f^2(t) = 0 \Rightarrow f^2(t)$ must be identically zero on $[a,b]$ $\therefore \langle f, f \rangle = 0 \Rightarrow f$ is the zero function.

(ii) $\langle f, g \rangle = \int_a^b f(t) g(t)\, dt = \int_a^b g(t) f(t)\, dt = \langle g, f \rangle$

(iii) $\langle cf + dg, h \rangle = \int_a^b (cf + dg)(t)\, h(t)\, dt$

$= c\int_a^b f(t) h(t)\, dt + d\int_a^b g(t)\, h(t)\, dt$

$= c\langle f, h \rangle + d\langle g, h \rangle$

$\Rightarrow \langle f, g \rangle$ defines a inner product on $C[a,b]$.

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Q(1)

Q2) For the vectors $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in $\mathbb{R}^2$, define:—

$$\langle u.v\rangle = 4u_1v_1 + 5u_2v_2 \rightarrow (1)$$

Show that (1) defines an inner product?

Soln: (i) $\langle u.v\rangle = 4u_1v_1 + 5u_2v_2$

$= 4v_1u_1 + 5v_2u_2$

$= \langle v.u\rangle$

(ii) $\langle cu+dv.w\rangle = 4(cu_1+dv_1)w_1 + 5(cu_2+dv_2)w_2$

$= c(4u_1w_1 + 5u_2w_2) + d(4v_1w_1 + 5v_2w_2)$

$= c\langle u.w\rangle + d\langle v.w\rangle$

(iii) $\langle u.u\rangle = 4u_1^2 + 5u_2^2 \geq 0$

and $4u_1^2 + 5u_2^2 = 0$ iff $u_1=0,\ u_2=0$.

i.e $u=(u_1,u_2)=(0,0)$

$\Rightarrow$ (1) defines inner product space in $\mathbb{R}^2$

(3) Let $(C[0,1], \langle \cdot \rangle)$ is a inner product space with $\langle f \cdot g \rangle = \int_0^1 f(t)\, g(t)\, dt$. and let W be the subspace spanned by the polynomials $\{P_1(t) = 1,\ P_2(t) = 2t-1,\ P_3(t) = 12t^2\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

Solⁿ: Let $v_1 = P_1(t) = 1$.

$$v_2 = P_2(t) - \frac{\langle P_2 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1$$

$$= 2t-1 - \frac{0}{1} 1 = 2t-1.$$

(Now $\langle P_2 \cdot v_1 \rangle = \int_0^1 (2t-1)\cdot 1\, dt = t^2 - t \Big|_0^1 = 0$

$\langle v_1 \cdot v_1 \rangle = \int_0^1 1 \cdot 1\, dt = 1$)

$$v_3 = P_3(t) - \frac{\langle P_3 \cdot v_1 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1 - \frac{\langle P_3 \cdot v_2 \rangle}{\langle v_2 \cdot v_2 \rangle} v_2$$

$$= 12t^2 - \frac{4}{1}\cdot 1 - \frac{2}{1/3}(2t-1) = 12t^2 - 12t + 2$$

(Now $\langle P_3 \cdot v_1 \rangle = \int_0^1 12t^2 \cdot 1\, dt = 4t^3 \Big|_0^1 = 4$.

$\langle v_1 \cdot v_1 \rangle = \int_0^1 1 \cdot 1\, dt = 1$

$\langle P_3 \cdot v_2 \rangle = \int_0^1 12t^2 \cdot (2t-1)\, dt = \int_0^1 (24t^3 - 12t^2)\, dt = 2.$

$\langle v_2 \cdot v_2 \rangle = \int_0^1 (2t-1)^2 dt = \frac{1}{6}(2t-1)^3 \Big|_0^1 = \frac{1}{3}$)

4 let $W = \text{Span}\{x_1, x_2\}$.

where $x_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $x_2 = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}$

construct an orthonormal basis for W.

Solⁿ: let $v_1 = x_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$$v_2 = x_2 - \frac{\langle x_2 \cdot v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$= \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} - \frac{0}{\langle v_1 . v_1\rangle} v_1 = \begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix} = x_2.$$

$\Rightarrow \{v_1, v_2\}$ is orthogonal $\Rightarrow \{x_1, x_2\}$ is orthogonal

$\Rightarrow \{x_1, x_2\}$ is L.I

$\Rightarrow \{v_1, v_2\}$ is L.I.

Now $W = L\{x_1, x_2\}$ & $\{x_1, x_2\}$ is L.I

$\Rightarrow \{x_1, x_2\}$ is a basis of W.

Now $\{x_1, x_2\}$ is a orthogonal basis of W ($\because \{x_1, x_2\}$ is orthogonal)

$\Rightarrow \left\{\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}\right\}$ is a orthonormal basis of W.

" " "

$\Rightarrow \left\{\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \frac{3}{\sqrt{6}}\begin{pmatrix}1/3\\1/3\\-2/3\end{pmatrix}\right\}$ is a "

(5) Let W be the subspace of $\mathbb{R}^2$ spanned by $x = (2/3, 1)$. Find a unit vector that is a basis for W.

Solⁿ: W consists of all multiples of x. ~~and~~

Any non-zero vector in W is a basis for W.

To simplify the calculation, "scale" x to eliminate fractions. That is multiple x by 3 to get $y = 3x = 3(2/3, 1) = (2, 3)$

$\therefore \|y\|^2 = \sqrt{2^2 + 3^2} = 13$

$\Rightarrow \|y\| = \sqrt{13}$

$\therefore$ Unit vector $= \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(6) The set $S = \{u_1, u_2, u_3\}$ is an orthogonal basis for $\mathbb{R}^3$, where $u_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, $u_3 = \begin{pmatrix} -1/2 \\ -2 \\ 7/2 \end{pmatrix}$.

Express the vector $y = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix}$ as a linear combination of the vectors in S.

Solⁿ:- Let $y = \begin{pmatrix} 6 \\ 1 \\ -8 \end{pmatrix} = c_1 u_1 + c_2 u_2 + c_3 u_3$

$\Rightarrow y = c_1 u_1 + c_2 u_2 + c_3 u_3$

$\Rightarrow \langle y . u_1 \rangle = c_1 \langle u_1 . u_1 \rangle + c_2 \langle u_1 . u_2 \rangle + c_3 \langle u_3 . u_1 \rangle$

$\Rightarrow \langle y . u_1 \rangle = c_1 \langle u_1 . u_1 \rangle \quad \left(\because \langle u_1 . u_2 \rangle = 0, \ \langle u_3 . u_1 \rangle = 0 \right)$

$\Rightarrow c_1 = \frac{\langle y . u_1 \rangle}{\langle u_1 . u_1 \rangle}$ Similarly $c_2 = \frac{\langle y . u_2 \rangle}{\langle u_2 . u_2 \rangle}$

& $c_3 = \frac{\langle y . u_3 \rangle}{\langle u_3 . u_3 \rangle}$

$\therefore\ c_1 = \dfrac{\langle y \cdot u_1\rangle}{\langle u_1 \cdot u_1\rangle} = \dfrac{11}{11} = 1.$

$c_2 = \dfrac{\langle y \cdot u_2\rangle}{\langle u_2 \cdot u_2\rangle} = \dfrac{-12}{6} = -2$

$c_3 = \dfrac{\langle y \cdot u_3\rangle}{\langle u_3 \cdot u_3\rangle} = \dfrac{-33}{33/2} = -2.$

$\Rightarrow\ y = c_1 u_1 + c_2 u_2 + c_3 u_3$

$= u_1 - 2u_2 - 2u_3.$

(7) Let $y = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ and $u = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$. Find the orthogonal projection of y onto u. ~~Then write y as the sum of two orthogonal vectors, one in span$\{u\}$ and one orthogonal to u.~~

Solⁿ: $\langle y \cdot u\rangle = 40$

$\langle u \cdot u\rangle = 20$

$\therefore$ The orthogonal projection of y onto u is

$= \dfrac{\langle y \cdot u\rangle}{\langle u \cdot u\rangle} u = \dfrac{40}{20}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$

(8) Show that $\{v_1, v_2, v_3\}$ is an orthonormal bais of $\mathbb{R}^3$, where

$$v_1 = \begin{pmatrix} \frac{3}{\sqrt{11}} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix} \quad v_2 = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \quad v_3 = \begin{pmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{pmatrix}$$

Sol^n:

$$\langle v_1 . v_2 \rangle = -3/\sqrt{66} + 2/\sqrt{66} + \frac{1}{\sqrt{66}} = 0$$

$$\langle v_1 . v_3 \rangle = \frac{-3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0.$$

$$\langle v_2 . v_3 \rangle = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0.$$

$\Rightarrow$ $\{v_1, v_2, v_3\}$ is an orthogonal.

Now $\|v_1\| = \sqrt{\langle v_1 . v_1 \rangle} = 1$

$\|v_2\| = \sqrt{\langle v_2 . v_2 \rangle} = 1.$

$\|v_3\| = \sqrt{\langle v_3 . v_3 \rangle} = 1.$

$\Rightarrow$ $\{v_1, v_2, v_3\}$ is an orthonormal set.

$\therefore$ they are L.I set in $\mathbb{R}^3$.

$\Rightarrow$ $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$.

$\therefore$ $\{v_1, v_2, v_3\}$ is an orthonormal bais of $\mathbb{R}^3$.

9) Find a least-squares solution of $Ax = b$ for

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$$

Soln:-

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 2 \\ 6 \end{pmatrix}$$

The augmented matrix for $A^T A x = A^T b$ is.

$$\left(\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right) \xrightarrow[\text{Reduced form}]{\text{Row -}} \begin{pmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ The general solution is.

$$x_1 = 3 - x_4$$

$$x_2 = -5 + x_4.$$

$$x_3 = -2 + x_4.$$

$$\hat{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 - x_4 \\ -5 + x_4 \\ -2 + x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ -2 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$