Lecture - 16

Class Note

Theorem: The eigen values of a diagonal matrix are its diagonal elements.

Ex: Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ –

$\therefore |A - \lambda I_3| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(6-\lambda) = 0$

$\Rightarrow \lambda = 1, 2, 6$

Theorem: The eigen values of a real symmetric matrix are all real.

Ex:- Let $A = \begin{pmatrix} 1 & 3 \\ 3 & 2 \end{pmatrix}$ $\therefore$ A is real symmetric matrix

Now $|A - \lambda I_2| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(2-\lambda) - 9 = 0$

$\Rightarrow 2 - 2\lambda - \lambda + \lambda^2 - 9 = 0$

$\Rightarrow \lambda^2 - 3\lambda - 7 = 0$

$\therefore \lambda = \dfrac{3 \pm \sqrt{9 - 4.(-7)}}{2} = \dfrac{3 \pm \sqrt{9+28}}{2}$

$\Rightarrow \lambda = \dfrac{3+\sqrt{37}}{2}$ & $\lambda = \dfrac{3-\sqrt{37}}{2}$

both real.

Theorem: The eigen values of a real skew symmetric matrix are purely imaginary or zero

Ex: $A = \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix}$ A is real skew-symmetric

$|A - \lambda I_3| = 0$

$\Rightarrow \begin{vmatrix} 0-\lambda & 1 & -2 \\ -1 & 0-\lambda & 3 \\ 2 & -3 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda(\lambda^2+9) - 1(\lambda - 6) - 2(3+2\lambda) = 0$

$\Rightarrow -\lambda^3 - 9\lambda - \lambda + \cancel{6} - \cancel{6} - 4\lambda = 0$

$\Rightarrow -\lambda^3 - 14\lambda = 0$ $\quad (\because \sqrt{-1} = i)$ complex

$\Rightarrow \lambda(\lambda^2 + 14) = 0$

$\therefore \lambda = 0, \quad \lambda^2 = -14 \Rightarrow \lambda = \pm\sqrt{-14} = \pm\sqrt{14}\,i$

$\therefore \lambda = 0, \quad \lambda = +\sqrt{14}\,i, \quad \lambda = -\sqrt{14}\,i$

are zero & complex no.

Theorem: Each eigen value of a real orthogonal matrix has unit modulus.

Ex: $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \quad A^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \quad AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$\Rightarrow$ A is orthogonal.

Now $\quad |A - \lambda I_2| = 0$

$\Rightarrow \begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (-1-\lambda)(1-\lambda) = 0$

$\Rightarrow \quad \lambda = 1, \quad \lambda = -1.$

$\therefore \quad |\lambda| = |1| = 1,$

$\quad |\lambda| = |-1| = 1.$

$\Rightarrow$ real orthogonal matrix has unit modulus.

Theorem: If A and P be both $n \times n$ matrix and P be non-singular. Then A & $P^{-1}AP$ have the same eigen values.

Soln:- $|P^{-1}AP - \lambda I|$

$= |P^{-1}AP - P^{-1}(\lambda I)P|$ $\quad$ $\left(\because P^{-1}(\lambda I)P = P^{-1}\lambda IP = P^{-1}\lambda P = \lambda P^{-1}P = \lambda I\right)$

$= |P^{-1}(A - \lambda I)P|$

$= |P^{-1}|\,|A - \lambda I|\,|P|$ $\quad$ $(\because |AB| = |A|\,|B|)$

$= |A - \lambda I|\,|P^{-1}|\,|P|$

$= |A - \lambda I|\,|P^{-1}P|$

$= |A - \lambda I|\,|I|$

$= |A - \lambda I|$

$\therefore$ A & $P^{-1}AP$ have same characteristic polynomial and so they have the same eigen values.

Theorem: If $X_1, X_2, \cdots, X_r$ be r eigen vectors of an an $n \times n$ matrix $A_{n\times n}$ corresponding to r distinct eigen values $\lambda_1, \lambda_2, \cdots, \lambda_r$ respectively, then $\{X_1, X_2, \cdots, X_r\}$ are linearly independent.

Ex: $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}$

$\therefore$ Its eigen values are $\lambda_1 = -1$ & $\lambda_2 = 7$.

and eigen vector are, $\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ corresponding to λ_1 & λ_2 respectively.

Then, $\left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}$ is L.I

Now $c_1 \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \left.\begin{aligned} -3/2\, c_1 + \tfrac{1}{2} c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \begin{aligned} &\Rightarrow -3c_1 + c_2 = 0 \\ &\Rightarrow c_1 = -c_2 \end{aligned}$$

$$\therefore -3c_1 - c_1 = 0 \Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = -c_1 = 0$$

$\therefore c_1 = c_2 = 0$

$\Rightarrow \left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \right\}$ is L.I

Diagonalisation of matrices

Similar matrix:

Let A & B be $n \times n$ matrices. An matrix $A_{n\times n}$ is said to be similar to $B_{n\times n}$ matrix if $\exists$ a non-singular $n\times n$ matrix P s.t

$$\underline{B = P^{-1}AP}$$

Diagonalisable matrix $A_{n\times n}$:

An $n\times n$ matrix A is said to be diagonalisable if A is similar to an $n\times n$ diagonal matrix.

$$\therefore \quad P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \cdots & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}_{n\times n}.$$

i.e (B = diagonal matrix in $B = P^{-1}AP$)

Theorem: Let A be an $n\times n$ matrix (over a field F). If the eigen values of A be all distinct and belong to F, then A is diagonalisable.

Ex: $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}_{2\times 2}$ its eigen values are $\lambda = -1, 7$

$\therefore$ A is 2×2 matrix and its two eigen values are distinct $\Rightarrow$ A is diagonalisable.