(VI) Methods of proving theorems

(a) Direct Proofs

- A direct proof of a conditional statement $\boxed{P \to Q}$ is constructed when the first step is the assumption that P is true; subsequent steps are constructed using rules of inference with the final step showing that Q must also be true.

Def 1: The integer n is even if there exists an integer k such that $\boxed{n = 2k}$ and n is odd if there exists an integer k such that $\boxed{n = 2k+1}$.

Note:- An integer can either be even or odd but not both.

Eg 1:-

Q) Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd".

Soln: The theorem states that $\forall n \, (P(n) \to Q(n))$

where $P(n)$: 'n is an odd integer'.

$Q(n)$ is 'n^2 is an odd integer'.

By def. of an odd integer,

$$n = 2k+1 \quad , \text{ where } k \text{ is some integer}$$

Squaring on both sides,

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$

$$n^2 = 2(2k^2+2k)+1$$

Let $(2k^2+2k)$ be t where t is some integer.

$$\therefore \; n^2 = 2t+1$$

So, n^2 is also an odd integer.

Eg2:-

Q) Give a direct proof that "if m and n are both perfect squares then nm is also a perfect square".

[Given: An integer a is a perfect square if there is an integer b such that $a = b^2$].

Soln.:- We assume that m and n are both perfect squares.

By def. of a perfect square it follows that there are integers s and t such that

$$\boxed{m = s^2} \quad \text{and} \quad \boxed{n = t^2}$$

then, $mn = s^2t^2$

ie $mn = (st)^2$ [using commutivity & associativity of multiplication]

$\therefore$ mn is also a perfect square.

(b) <u>Proof by Contradiction</u>

- In this type of proof, we assume the opposite of what we are trying to prove and get a logical contradiction.
- Hence, our assumption must have been false and therefore what we originally required to prove must be true.
- To prove a statement P is true, we assume that $\sim P$ is true and taking $\sim P$ as premise, we draw a contradiction F as the conclusion.
- Now, if $\sim P$ leads to a contradiction is true then $\sim P$ must be false that is P must be true.
- Steps to be followed :-

(i) Assume that P is false.

(ii) Using this assumption show a contradiction.

<u>Eg1</u>:- Show that $\sqrt{2}$ is an irrational number.

<u>Def. of rational number</u> : A rational number Q can be defined as $Q = p/q$ where p and q are integers and have no common factor (assuming these are the lowest terms) and $q \neq 0$

<u>Soln.</u> Here, P: $\sqrt{2}$ is an irrational number.

Assume $\sim P$ is true or $\sqrt{2}$ is a rational number.

Let $\sqrt{2} = p/q$ such that p and q have no common factor.

$\Rightarrow \sqrt{2}q = p$

$\Rightarrow 2q^2 = p^2$ (Squaring on both sides)

$\Rightarrow p^2$ is an even number

$\Rightarrow p$ is an even number (since if p^2 is even, p must be even).

$\Rightarrow p = 2k$ for some integer k.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow q^2 = \dfrac{p^2}{2} = \dfrac{4k^2}{2} = 2k^2$

(on substituting the value of p^2 in $2q^2 = p^2$).

$\therefore q^2$ is also an even number.

So, q is an even number.

$\therefore$ 2 is a common factor between p and q.

This is a contradiction as they should not have an common factor, if $\sqrt{2}$ is a rational no. Therefore, $\sim P$ is F which means P is true and $\sqrt{2}$ is an irrational no.

Eg 2:- Prove that there is no largest integer that is a multiple of 5 using proof by contradiction.

Soln :- Let P: There is no largest integer that is a multiple of 5.

We assume $\sim P$ to be true ie there is a largest integer that is a multiple of 5 and suppose that the integer is $\boxed{m}$.

Thus, $m = 5k$ for some $k \in Z$

Now, consider the integer $m+5$

$$m+5 = 5k+5 = 5(k+1)$$

This shows that $\boxed{m+5}$ is also a multiple of $\boxed{5}$ and $\boxed{m+5}$ is greater than $\boxed{m}$ as well.

Therefore, this is a contradiction that $\boxed{m}$ is the largest integer that is a multiple of 5 and our assumption is not true.

Hence, there is no largest integer that is a multiple of 5.

→ To prove the conditional statement $P \to Q$ we assume both P and $\sim Q$ are true.

→ Then considering $\sim Q$ as a premise, we draw the conclusion $\sim P$.

→ Thus, we get the contradiction $\boxed{P \wedge \sim P}$.

→ Therefore, we say that our initial assumption is not true ie $\sim Q$ is false as P is assumed to be true.

→ Finally, $\sim Q$ is false implies that Q is true and hence $P \to Q$.

→ Steps are as follows:

(a) Assume both P and $\sim Q$ are true.

(b) Use $\sim Q$ and show that P is false, which is a contradiction.

Eg 3:- Prove the statement :-

'If $3n+1$ is even, then n is odd'

by using the method of proof by contradiction.

<u>Soln.</u> :- Here, P : $3n+1$ is even

Q : n is odd

We shall assume that P is true and $\sim Q$ is true.

$\therefore$ Let $3n+1$ is even and n is even.

We can say, $\boxed{n = 2k}$ where k is some integer then

then, $3n+1 = 3(2k)+1 = 6k+1$

since, $6k = 2(3k)$, $\therefore$ $6k$ is an even no.

$\Rightarrow$ $6k+1$ is an odd number.

$\Rightarrow$ $3n+1$ is an odd number.

So, this is a contradiction to the assumption that $3n+1$ is even.

Hence, n is not even ie n is odd.

This proves the statement 'if $3n+1$ is even, then n is odd'.

Eg 4:- Prove that the sum of two consecutive integers is odd.

Soln: Let a and b be two integers.

Here P: a and b are two consecutive integers.

Q: $a+b$ is odd.

We shall assume that P and $\sim Q$ is true.

Thus, a and b are consecutive integers and

$\boxed{a+b \text{ is even}}$

$\therefore$ $a = k$ and $b = k+1$ for some integers k.

Thus $a+b = k+k+1 = 2k+1$ which is an odd no.

Contradiction

$\therefore$ "sum of 2 consecutive integers is odd" by contradiction

(c) Proof by Mathematical Induction

→ Following are the steps:-

(i) Show true for the first term mostly $n=1$ } Base Case

(ii) Assume true for $n=k$ } Induction Hypothesis

(iii) Prove/Show true for $n=k+1$ } Conclusion

(iv) Restate ∴ by the process of mathematical induction given statement.

Eg 1:- Prove: $3+6+9+12+\dots\dots+3n = \frac{3n(n+1)}{2}$

(i) Show true for $\boxed{n=1}$

$$3(1) = \frac{3(1)(1+1)}{2}$$

$$\Rightarrow 3 = \frac{3(\cancel{2})}{\cancel{2}}$$

$$\Rightarrow 3 = 3 \checkmark$$

(ii) Assume true for $\boxed{n=k}$

$$3+6+9+12+\dots\dots+3k = \frac{3k(k+1)}{2}$$

(iii) Show true for $\boxed{n=k+1}$

$$\underbrace{3+6+9+12+\dots\dots+3k}_{\downarrow} + 3(k+1) = \frac{3(k+1)(k+1+1)}{2}$$

$\frac{3k(k+1)}{2}$ (as shown in step(ii))

$$\Rightarrow \frac{3k(k+1)}{2} + 3(k+1) = \frac{3(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{3k(k+1) + 6(k+1)}{2} = \frac{3(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{3(k+1)[k+2]}{2} = \frac{3(k+1)(k+2)}{2}$$

$\therefore$ L.H.S = R.H.S

Hence, by the process of mathematical induction,

$$3+6+9+12+\ldots\ldots+3n = \frac{3n(n+1)}{2}$$

Eg 2:- Using mathematical induction, prove that for every natural number

$$1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$$

(i) Show true for $\boxed{n=1}$

$$\Rightarrow 1 = \frac{1(1+1)}{2}$$

$$\Rightarrow 1 = \frac{1(\cancel{2})}{\cancel{2}}$$

$$\Rightarrow 1 = 1 \checkmark$$

(ii) Assume true for $\boxed{n=k}$

$$1+2+3+\ldots+k = \frac{k(k+1)}{2}$$

(iii) Show true for $\boxed{n=k+1}$

$$\underbrace{1+2+3+\ldots+k}_{k(k+1)/2} + k+1 = \frac{(k+1)(k+1+1)}{2}$$

$$\Rightarrow \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\Rightarrow \frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

$\therefore$ By the process of mathematical induction

$$1+2+3+\ldots\ldots+n = \frac{n(n+1)}{2}$$

(VII) Mechanization of Reasoning

→ Every X is Y
A is X

Therefore, A is Y

→ Every politician is clever
Manoj is a politician

Therefore, Manoj is clever

→ This is an effort of mechanizing the rules of inferences or simply reasoning.

→ Mechanization of reasoning leads to automated deduction.

(a) Satisfiable :- A set of formula is called satisfiable if for a set of truth values of the variables in the formula, all the formula are true.

Eg:- 1. $\{P, Q\}$ is satisfiable as both the formula are true when P and Q is true.

2. $\{P, \sim P\}$ is not satisfiable.

3. $\{P, \sim P \vee Q\}$ is satisfiable.

(b) Consistence :- A set of formula is called consistent if we cannot derive a contradiction from the set.

or

A set of formula is called consistent if there is no formula P such that both P and $\sim P$ can be proved from the given premises and deductive system of formula.

(c) Applications of propositional logic

(i) Excel, (ii) Programming languages,

(iii) Digital logic, (iv) Artificial Intelligence

(v) Web search engines (vi) relational calculus.

(a) Russell's Paradox

Let X be a set containing all sets that do not contain themselves,

$$X = \{x : x \notin X\}$$

Now, consider two cases:

(i) If $X \in X$, then the set X contains itself.
$\hookrightarrow$ Contradiction

(ii) If $X \notin X$, then the set X does not contain itself, but according to the definition X must contain all the sets that do not contain themselves.
$\hookrightarrow$ Contradiction

This is a paradox.

Eg:- There is a city X, where a barber does the shave for all those men in the city who do shave themselves.
Now, the question is who does the shave for the barber?

① <u>Trivial Proof</u>

$p \to q$

If q = true

$p \to q$ [$p = 0$ or 1]

<u>e.g</u> If $x = 2$, then $x^2 = 4$

p : $x = 2$

q : $x^2 = 4$

If q = true given

then $x = 2 \to x^2 = 4$ is true

② <u>Vacous Proof</u> $p \to q$

$p = 0$

then $p \to q$ is true

<u>eg</u> Prove that $P(0)$ where $P(n)$ is "If n is a positive integer greater than 1, then $n^2 > n$"

$P(0)$: $n = 0$

i.e p is false

$\therefore$ $p \to q$

Scanned with CamScanner

(2) Indirect Proof

$$p \to q$$
$$\sim q \to \sim p$$

e.g. Prove if n is an integer and n^3+5 is odd, then n is even

$P \to$ n^3+5 is odd

$Q \to$ n is even

$\underbrace{P \to Q}_{\text{To Prove}}$

$\sim Q$ = n is odd

$$n = 2k+1$$
$$n^3+5 = (2k+1)^3+5$$
$$= 2[4k^3+6k^2+3k+3]$$
$$= \text{even}$$

$\therefore \sim P$

$\sim Q \to \sim P$

$\therefore P \to Q$

Q1 Prove that the sum of a rational and irrational number is irrational.

Q2 Show that at least 10 of any 64 days chosen must fall on the same day of the week.

Q3 P(n): " If a and b are positive real numbers, then $(a+b)^n \geq a^n + b^n$ "
Prove P(1) is true.

Q4 Use Mathematical Induction to prove $2^n < n!$ for every positive integer n with $n \geq 4$.