Solution of

Assignment 1

Solution 1. ~~Theorem~~ A real square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Proof: Let A be a given matrix. Then A can be expressed as:-

$$A = \underbrace{\frac{1}{2}(A + A^t)} + \underbrace{\frac{1}{2}(A - A^t)}.$$

Now
$$\left[\frac{1}{2}(A + A^t)\right]^t = \frac{1}{2}\left(A^t + (A^t)^t\right)$$
$$= \frac{1}{2}(A^t + A)$$
$$= \frac{1}{2}(A + A^t)$$

So it is symmetric.

Now
$$\left[\frac{1}{2}(A - A^t)\right]^t = \frac{1}{2}\left(A^t - A^{t^t}\right)$$
$$= \frac{1}{2}(A^t - A)$$
$$= -\frac{1}{2}(A - A^t)$$

So it is skew symmetric

Therefore A is expressed as a sum of symmetric matric and a skew symmetric.

Now we show that this decomposition is unique.

Let $A = P + Q$ where P is symmetric and Q is skew symmetric.

Then $A^t = P^t + Q^t = P - Q$ $\left(\because \begin{array}{l} P^t = P \\ Q^t = -Q \end{array}\right)$

We have $A + A^t = 2P$, $A - A^t = 2Q$.

So $P = \frac{1}{2}(A + A^t)$ & $Q = \frac{1}{2}(A - A^t)$ and.

this proves the theorem

2. $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}_{4\times 2}$ $\quad B = \begin{pmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{pmatrix}_{2\times 3}$

$$\therefore AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 0 + 2\times 3 & 1\times 6 + 2\times 8 & 1\times 1 + 2\times -2 \\ 3\times 0 + 4\times 3 & 3\times 6 + 4\times 8 & 3\times 1 + 4\times -2 \\ 5\times 0 + 6\times 3 & 5\times 6 + 6\times 8 & 5\times 1 + 6\times -2 \\ 7\times 6 + 8\times 3 & 7\times 6 + 8\times 8 & 7\times 1 + 8\times -2 \end{pmatrix}_{4\times 3}$$

$$= \begin{pmatrix} 6 & 22 & -3 \\ 12 & 50 & -5 \\ 18 & 78 & -7 \\ 24 & 106 & -9 \end{pmatrix}_{4\times 3}$$

3. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ (applying property 6) $R_1 - R_2 \rightarrow R_1'$

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b+c)(b-c) \\ 1 & c & c^2 \end{vmatrix}$$ (applying property 6) $R_2 - R_3 \rightarrow R_2'$

$$= \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b) \begin{vmatrix} 0 & 1 & a+b \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$ applying property 4

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$ applying property 4

$$= (a-b)(b-c) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$ applying property 1 (transposing)

$$= (a-b)(b-c) \left\{ 0 \times \begin{vmatrix} 1 & c \\ b+c & c^2 \end{vmatrix} - 0 \begin{vmatrix} 1 & c \\ a+b & c^2 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix} \right\}$$

$$= (a-b)(b-c)\{\cancel{b}+c-a-\cancel{b}\}$$

$$= (a-b)(b-c)(c-a)$$

4)
$$D = \begin{vmatrix} 1 & a & a^2 & a^3 + bcd \\ 1 & b & b^2 & b^3 + cda \\ 1 & c & c^2 & c^3 + adb \\ 1 & d & d^2 & d^3 + abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & abd \\ 1 & d & d^2 & abc \end{vmatrix} \quad \text{(Property 5)}$$

$$= D_1 + D_2$$

Now, $$D_2 = \begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & abd \\ 1 & d & d^2 & abc \end{vmatrix}$$

$$= \frac{1}{abcd} \begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix} \quad \text{(Property 4.)}$$

$$= \frac{\cancel{abcd}}{\cancel{abcd}} \begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix} \quad \text{(Proper 4)}$$

$$= - \begin{vmatrix} a & a^2 & 1 & a^3 \\ b & b^2 & 1 & b^3 \\ c & c^2 & 1 & c^3 \\ d & d^2 & 1 & d^3 \end{vmatrix}$$ interchanging c_4 and c_5 (Property 2)

$$= \begin{vmatrix} a & 1 & a^2 & a^3 \\ b & 1 & b^2 & b^3 \\ c & 1 & c^2 & c^3 \\ d & 1 & d^2 & d^3 \end{vmatrix}$$ interchanging c_2 & c_3 (Property 2)

$$= - \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix},$$ interchanging c_1 & c_2 (Property 2)

$= -D_1$

$\therefore \quad D_2 = -D_1 \Rightarrow D_1 + D_2 = 0$

$\Rightarrow D = 0$

5. The co-factor of $7 \ (= a_{13})$ is

$$= (-1)^{1+3} \begin{vmatrix} 6 & 9 \\ 4 & 6 \end{vmatrix}$$

$= 36 - 36 = 0$

6) Co-factor of 1 $(= a_{11}) = (-1)^{1+1} \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$

Co-factor of 2 $(= a_{12}) = (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$

Co-factor of 1 $(= a_{13}) = (-1)^{1+3} \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 5$

Similarly,

Co-factor of 1 $(= a_{21}) = 5$

Co-factor " -1 $(= a_{22}) = -3$

co- " " 1 $(= a_{23}) = 1$

co- " " 2 $(= a_{31}) = 3.$

co- " " 3 $(= a_{32}) = 0$

co- " " -1 $(= a_{33}) = -3.$

$$\therefore \quad adj(A) = \begin{bmatrix} -2 & 3 & 5 \\ 5 & -3 & 1 \\ 3 & 0 & -3 \end{bmatrix}^t$$

$$= \begin{bmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}$$

Inverse of A using determinant method

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

Now $\operatorname{adj}(A) = \begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}$ (evaluated in a previous page)

$$\det(A) = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= -2 + 6 + 5 = 9 \neq 0$$

$$\therefore \quad A^{-1} = \frac{\operatorname{adj}(A)}{|A|}$$

$$= \frac{1}{9} \begin{pmatrix} -2 & 5 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{pmatrix}$$

7) Here the coefficient determinant

$$D = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= -3 \neq 0$$

So Cramer's rule can be applied.

Now,

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = -3.$$

$$D_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3.$$

$$D_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = -6$$

So by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{-3}{3} = 1$$

$$y = \frac{D_2}{D} = \frac{3}{-3} = -1$$

$$z = \frac{D_3}{D} = \frac{-6}{-3} = 2.$$

$\therefore$ the solution is $x = 1$, $y = -1$, $z = 2$.

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