Ordinary Differential Equations

(Lecture-3)

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Learning Outcome of the Lecture

We learn

- Solution
- Initial Value Problems
- General, Particular and Singular Solutions
- Separable Equations
- Equations reducible to Separable Equations

Initial Value Problems

Recall that first order ODE can be expressed as

$$F(x,y,y')=0 \qquad \text{or} \qquad \frac{dy}{dx}=f(x,y).$$

Definition

Initial value problem (IVP): A differential equation along with an initial condition is called an initial value problem (IVP), i.e.,

$$\frac{dy}{dx}=f(x,y); \qquad y(x_0)=y_0.$$

Geometrically, the IVP is to find an integral curve of the DE that passes through the point (x_0,y_0).

Solution of IVP

What can we say about the solutions of the following IVPs?

1. $$\frac{dy}{dx} = 2x, \quad y(0) = 0; \; y = x^2, \text{Unique Solution}$$

2. $$x\frac{dy}{dx} = y - 1, \quad y(0) = 1; \; y = 1 + cx, \text{Infinitely Many Solutions}$$

3. $$\left|\frac{dy}{dx}\right| + |y| = 0, \quad y(0) = 1; \text{No Solution}$$

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Observations: IVPs can have unique solution, infinitely many solutions or no solution.

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Observations: IVPs can have unique solution, infinitely many solutions or no solution.

Question: What can we say about the existence and uniqueness of a solution for an IVP ?.

General, Particular and Singular Solutions

Consider a first order ODE $F(x, y, y') = 0$. Then

One parameter family of solutions is given by $g(x, y, c) = 0$.

This one parameter family of solutions is called general solution of given ODE.

- **General Solution:** Solution containing an arbitrary constant is called a general Solution of ODE.
- **Particular Solution:** Solution corresponding to a particular value of constant is called a particular solution of ODE.
- **Singular Solution:** Solution which cannot be obtained from a general solution is called a singular solution of ODE.

Example

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$

Solution:

- **General Solution:** $y = cx - c^2$.
- **Particular Solution:** put $c = 1$, then $y = x - 1$.
- **Singular Solution:** $\frac{x^2}{4}$.

Separable Equations

A first order ODE $F(x, y, y') = 0$ may be expressed in either derivative form

$$\frac{dy}{dx} = f(x, y)$$

or, in differential form

$$M(x, y)dx + N(x, y)dy = 0.$$

Definition

An ODE of the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{1}$$

is called a separable ODE.

After integration equation (1), we get

$$\int M(x)dx + \int N(y)dy = c.$$

$$\int M(x)dx + \int N(y)dy = c$$

leads to the general solution of ODE as

$$G(x) + H(y) = c,$$

where c is an arbitrary constant.

Examples: Solve the following ODEs:

$$(i) \quad \frac{dy}{dx} = -2xy \qquad y(x_0) = y_0.$$

$$(ii) \quad \frac{dy}{dx} = 1 + y^2.$$

$$(iii) \quad (x-4)y^4dx - x^3(y^2-3)dy = 0.$$

$$(iv) \quad x\sin y dx + (x^2+1)\cos y dy = 0 \qquad y(1) = \frac{\pi}{2}.$$

Separable Equations Cont.

Solutions:

$$(i) \qquad \frac{dy}{dx} = -2xy; \qquad y(x_0) = y_0.$$

$$\frac{dy}{y} = -2xdx \Rightarrow \int \frac{dy}{y} = \int -2xdx + \ln|c|,$$

$$y = ce^{-x^2},$$

$$y(x_0) = y_0 \Rightarrow c = y_0 e^{x_0^2},$$

$$y(x) = y_0 e^{x_0^2 - x^2}.$$

$$(ii) \qquad \frac{dy}{dx} = 1 + y^2,$$

$$\frac{dy}{1+y^2} = dx \Rightarrow \int \frac{dy}{1+y^2} = \int dx + c,$$

$$\arctan y = x + c \Rightarrow y = \tan(x+c).$$

Separable Equations Cont.

Solutions:

(iii) (x-4)y$^4 dx - x^3(y^2-3)dy = 0 \Rightarrow \frac{(x-4)}{x^3}dx - \frac{(y^2-3)}{y^4}dy = 0,$

$\int \frac{(x-4)}{x^3}dx - \int \frac{(y^2-3)}{y^4}dy = c \Rightarrow \frac{-1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c.$

(iv) $\quad x \sin y dx + (x^2+1)\cos y dy = 0; \qquad y(1) = \frac{\pi}{2},$

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0 \Rightarrow \int \frac{x}{x^2+1}dx + \int \frac{\cos y}{\sin y}dy = c_0,$$

$$\frac{1}{2}\ln(x^2+1) + \ln|\sin y| = c_0 \Rightarrow \ln(x^2+1) + 2\ln|\sin y| = 2c_0,$$

$$(x^2+1)\sin^2 y = e^{2c_0} = c,$$

$$(x^2+1)\sin^2 y = c, \;\; y(1) = \pi/2 \Rightarrow c = 2,$$

$$(x^2+1)\sin^2 y = 2.$$

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Method of separation of variables doesn't yield all solutions!

Example: Find the solution of the following IVP

$$\frac{dy}{dx} = 3y^{2/3}, \qquad y(0) = 0.$$

Solution: Given

$$\frac{dy}{dx} = 3y^{2/3}$$

If $y \neq 0$, then

$$\frac{dy}{y^{2/3}} = 3dx \;\Rightarrow\; 3y^{1/3} = 3(x+c) \;\Rightarrow\; y = (x+c)^3.$$

- Using initial condition we get, $c = 0$, i.e., $y = x^3$.
- Observe that $y = 0$ is also a solution.

Method of separation of variables doesn't yield all solutions!

Example-2: Find the solution of the following ODE

$$\frac{dy}{dx} = y^2 - 4.$$

Solution:

$$\frac{dy}{dx} = y^2 - 4 \Rightarrow \frac{dy}{y^2 - 4} = dx \Rightarrow \int \frac{dy}{y^2 - 4} = \int dx$$

$$\Rightarrow \int \frac{dy}{(y-2)(y+2)} = \int dx \Rightarrow \frac{1}{4} \int \left[\frac{1}{y-2} + \frac{-1}{y+2} \right] dy = \int dx$$

$$\Rightarrow \ln|y-2| - \ln|y+2| = 4x + c \Rightarrow \frac{y-2}{y+2} = c_1 e^{4x}$$

$$\Rightarrow y = \frac{2 + 2c_1 e^{4x}}{1 - c_1 e^{4x}}$$

where c_1 is a arbitrary constant. Note that $y = \pm 2$ are also solutions of DE. $c_1 = 0 \Rightarrow y = 2$, is a solution obtained from general solution. But no value of c_1 gives $y = -2$ solution. Thus, $y = -2$ is a singular solution.