Fuzzy Sets

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Overview

Fuzzy Sets

- The word "fuzzy" means "vagueness (ambiguity)".
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Classical set theory allows the membership of the elements in the set in binary terms.
- Fuzzy set theory permits membership function valued in the interval [0,1].

Example:

Words like young, tall, good or high are fuzzy.

- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.
- Age 35 has some possibility of being young and usually depends on the context in which it is being considered.
  Fuzzy set theory is an extension of classical set theory where elements have degree of membership.

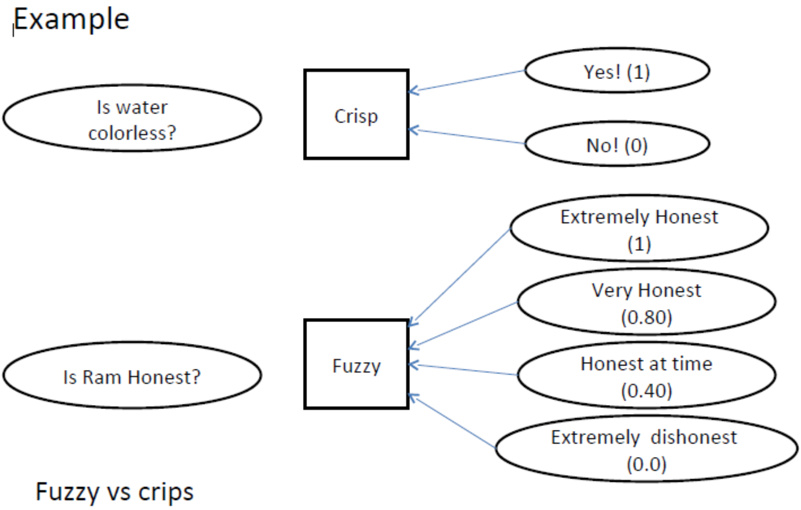

Figure 2: Examples

Fuzzy sets theory is an extension of classical set theory.

• Elements have varying degree of membership. A logic based on two truth values.

• True and False is sometimes insufficient when describing human reasoning.

• Fuzzy Logic uses the whole interval between 0 (false) and 1 (true) to describe human reasoning.

• A Fuzzy Set is any set that allows its members to have different degree of membership, called membership function, having interval [0,1].

Fuzzy Logic is derived from fuzzy set theory
• Many degree of membership (between 0 to 1) are allowed.
• Thus a membership function $\mu_A(x)$ is associated with a fuzzy sets Ã such that the function maps every element of universe of discourse X to the interval [0,1].
• The mapping is written as: $\mu_A(x) : X \implies [0,1]$.
Fuzzy set is defined as follows:
• If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs
$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) : X \implies [0,1]\}$

Example

• Let X = {g1, g2, g3, g4, g5} be the reference set of students.

• Let Ã be the fuzzy set of “smart” students, where “smart” is fuzzy term.

$\tilde{A} = \{(g_1, 0.4), (g_2, 0.5), (g_3, 1), (g_4, 0.9), (g_5, 0.8)\}$

Here Ã indicates that the smartness of g1 is 0.4 and so on

The membership function can be defined as per the suitability of the concept.
Example:
In a certain class, based on the percentage of marks in the final exam, we can define a fuzzy set of brilliant students (B) as follows

$$\mu_B(x) = \begin{cases} 1 & \textit{if} \quad x >= 75 \\ x/75 & \textit{if} \quad x < 75 \end{cases} \tag{1}$$

Student	Marks	Membership
Stud1	79	1
Stud2	74	0.99
Stud3	50	0.67

Table 1: Membership function

The grades of membership shows the belongingness of the element to the set.

Fuzzy Set Operation

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

Union:

$$\mu_{A \cup B}(x) = max(\mu_A(x), \mu_B(x))$$

Intersection:

$$\mu_{A \cap B}(x) = min(\mu_A(x), \mu_B(x))$$

Complement:

$$\mu_A(x) = 1 - \mu_A(x)$$

Example:

Fuzzy Set $A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$
Fuzzy Set $B = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

Union:

Fuzzy Set, $A \cup B = \{(x_1, 0.8), (x_2, 0.7), (x_3, 1)\}$

Because

$\mu_{A \cup B}(x_1) = max(\mu_A(x_1), \mu_B(x_1))$
$=$ max(0.5,0.8)
$=$ 0.8
$\mu_{A \cup B}(x_2) = 0.7 and \mu_{A \cup B}(x_3) = 1$

Example:

Fuzzy Set $A = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0)\}$
Fuzzy Set $B = \{(x_1, 0.8), (x_2, 0.2), (x_3, 1)\}$

Intersection:

$A \cap B = (x_1, 0.5), (x_2, 0.2), (x_3, 0)$

Because

$\mu_{A \cap B}(x_1) = min(\mu_A(x_1), \mu_B(x_1))$
$=$ min(0.5,0.8)
$= 0.5$
$\mu_{A \cap B}(x_2) = 0.2 and \mu_{A \cap B}(x_3) = 0$

Example:

Fuzzy Set$A = \{(x1, 0.5), (x2, 0.7), (x3, 0)\}$

Complement:

Fuzzy Set $A_c = \{(x1, 0.5), (x2, 0.3), (x3, 1)\}$ Because

$\mu_A(x1) = 1 - \mu_A(x1)$

$= 1 - 0.5$

$= 0.5$

$\mu_A(x2) = 0.3 and \mu_A(x3) = 1$

alpha cut and strong alpha cut

α-cut and Strong α-cut

Given a fuzzy set A, defined on X and any number $\alpha \in$ [0,1], α-cut and strong α-cut are the crisp sets defined as follows :

$\alpha_A = \{x : \mu_A(x) >= \alpha\}$

$\alpha +_A = \{x : \mu_A(x) > \alpha\}$

Example

Let A be a fuzzy set on a set X.

X=$\{$10,20,30,40,50$\}$ whose membership function is defined as :

$\mu_A(x) = x/(x+10)$

Find A_α for α=0.6

Fuzzy Set A=$\{$(10,0.5),(20,0.67),(30,0.75),(40,0.80),(50,0.83)$\}$

$A_{0.6} = \{20, 30, 40, 50\}$

Properties of α-cuts and strong α-cuts

For fuzzy set A, let $\alpha 1$ and $\alpha 2 \in [0,1]$

1. $\alpha 1 < \alpha 2 \implies A_{\alpha 2} \subseteq A_{\alpha 1}$
2. $\alpha 1 < \alpha 2 \implies A+_{\alpha 2} \subseteq A+_{\alpha 1}$

Example

Fuzzy Set A={(10,0.5),(20,0.6),(30,0.7),(40,0.8),(50,0.8)}
Let $\alpha 1$=0.5 and $\alpha 2$=0.6
$A_{0.5} = \{10, 20, 30, 40, 50\}$
$A_{0.6} = \{20, 30, 40, 50\}$
We can see that $A_{0.6} \subseteq A_{0.5}$
$A+_{0.5} = \{20, 30, 40, 50\}$
$A+_{0.6} = \{30, 40, 50\}$
We can see that $A+_{0.6} \subseteq A+_{0.5}$

Support of a Fuzzy Set:
Given a fuzzy set A, defined on X, Support of A is the crisp set that contains all the elements of X that have a non-zero grade of membership in A.
Denoted as Supp(A)
It is a strong α cut of A for $\alpha = 0$
Supp(A)$=A+_0$
Crossover Point:
The element $x \in X$ at which $\mu_A(x) = 0.5$ is called the crossover point.

Example

Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
Fuzzy Set $A =$
$\{(1, 0), (2, 0), (3, 0.2), (4, 0.5), (5, 0.3), (6, 0.4), (7, 0), (8, 0), (9, 0), (10, 0)\}$
$Supp(A) = \{3, 4, 5, 6\}$
Since $Supp(A) = \{x \in X | \mu_A(x) > 0\}$
x=4 is the Crossover Point

Core of a Fuzzy Set:
The core of a fuzzy set is the α cut of A for which $\alpha = 1$.
Denoted as Core(A)

$Core(A) = A_1$

$Core(A) = \{x \in X | \mu_A(x) = 1\}$

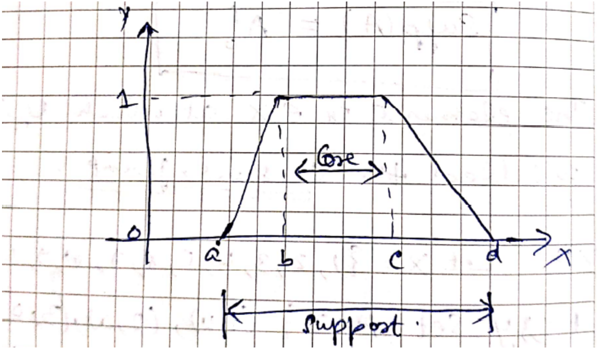

Figure 3: Support and Core of a Fuzzy Set

Height of a Fuzzy Set:
The height of a fuzzy set is the highest grade of membership of any element in the set and it is denoted by h(A)
$h(A) = max_{x \in X} \mu_A(x)$
Normal Fuzzy Set:
A fuzzy set is called Normal if h(A)=1 and subnormal if $h(A) < 1$
Cardinality of a Fuzzy Set:
It is the sum of the membership values of all the elements of the fuzzy set.
Denoted by n(A) where A is a fuzzy set.

Example

1) Consider two fuzzy subsets of the set $X = \{a, b, c, d, e\}$ referred to as $\tilde{A}$ and $\tilde{B}$ such that $\tilde{A} = \{(1, a), (0.3, b), (0.2, c), (0.8, d), (0, e)\}$ and $\tilde{B} = \{(0.6, a), (0.9, b), (0.1, c), (0.3, d), (0.2, e)\}$. Compute the following:

a) $supp(\tilde{A})$ *and* $supp(\tilde{B})$

b) $core(\tilde{A})$ *and* $core(\tilde{B})$

c) $n(\tilde{A})$ *and* $n(\tilde{B})$

d) $\neg(\tilde{A})$ *and* $\neg(\tilde{B})$

e) $\tilde{A} \cup \tilde{B}$

f) $\tilde{A} \cap \tilde{B}$

g) $a\tilde{A}$ *and* $a\tilde{B}$ when $a = 0.5$

h) $\tilde{A}^a$ *and* $\tilde{B}^a$ when $a = 2$

i) $\alpha - cuts$ of $\tilde{A}$ *and* $\tilde{B}$ for $\alpha = 0.3$ *and* $\alpha = 0.9$

j) $h(\tilde{A})$ *and* $h(\tilde{B})$

k) Which one is a normal fuzzy set?

Fuzzy Sets in Decision Making

Example:
Someone is looking for a two-bedroom apartment having low rent(around Rs 30000) and is near (within 1 Km) his/her office.
The decision making can be done through fuzzy set theory:

$$\mu_A(x) = \begin{cases} 1 & for \quad x <= 30000 \\ 30000/x & for \quad x > 30000 \end{cases} \tag{2}$$

$$\mu_B(x) = \begin{cases} 1 & for \quad x <= 1 \\ 1/x & for \quad x > 1 \end{cases} \tag{3}$$

Each apartment in the area will have two grades of membership.
We need to compute $\mu_{A \cap B}(x)$ for every flat x.
The highest grade of membership shows the best choice.
Other Application areas of Fuzzy Logic
Machine Learning, Artificial Intelligence, Knowledge Aquisition.