Lecture-18.

Class Note

Real Inner Product:

Let V be a real vector space. A real inner product on V is a mapping $f: V \times V \rightarrow \mathbb{R}$ that assigns to each ordered pair of vectors (α, β) of V a real number $f(\alpha, \beta)$, generally denoted by $\langle \alpha . \beta \rangle$, satisfying the following properties:—

(1) $\langle \alpha . \beta \rangle = \langle \beta . \alpha \rangle \quad \forall \alpha, \beta \in V$ (Symmetry).

(2) $\langle c\alpha + d\beta . \gamma \rangle = c\langle \alpha . \gamma \rangle + d \langle \beta . \gamma \rangle \quad \forall \alpha, \beta, \gamma \in V$, $c, d \in \mathbb{R}$ (linearity)

(3) $\langle \alpha . \alpha \rangle > 0$ if $\alpha \neq \theta$ (positivity)

Complex inner product:

Let V be a complex vector space. A complex inner product is a mapping $f: V \times V \rightarrow \mathbb{C}$ that assigns each ordered pair of vectors (α, β) of V a complex number $f(\alpha, \beta)$, generally denoted by $\langle \alpha . \beta \rangle$, satisfying the following properties:—

(1) $\langle \alpha . \beta \rangle = \overline{\langle \beta . \alpha \rangle}$ where $\overline{\langle \beta . \alpha \rangle}$ is the conjugate of the complex no. $\langle \beta . \alpha \rangle$.

(2) $\langle c\alpha + d\beta . \gamma \rangle = c\langle \alpha . \gamma \rangle + d \langle \beta . \gamma \rangle \quad \forall \alpha, \beta, \gamma \in V$, $c, d \in \mathbb{C}$

(3) $\langle \alpha . \alpha \rangle > 0$ if $\alpha \neq \theta$

Examples:

1) let $V = \mathbb{R}^n$

let $\alpha = (a_1, a_2, \dots a_n)$ & $\beta = (b_1, b_2, \dots b_n) \in \mathbb{R}^n$.

let us define :-

$\langle \alpha . \beta \rangle = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$

Then $\langle \alpha . \beta \rangle$ satisfies all the conditions for a real inner product.

Norm of a vector:

let $\alpha \in (V, \langle . \rangle)$ where $\langle . \rangle$ is the inner product.

Then norm of $\alpha = \|\alpha\| = \sqrt{\langle \alpha . \alpha \rangle}$

Theorem: let $\alpha \in (V, \langle . \rangle)$ and $\|\alpha\|$ be its norm. Then

(i) $\|c\alpha\| = |c| \, \|\alpha\|$, c being a real number.

(ii) $\|\alpha\| > 0$ unless $\alpha = \theta$ and $\|\theta\| = 0$

proof: (i) $\|c\alpha\| = \sqrt{\langle c\alpha . c\alpha \rangle} = \cancel{\sqrt{c}\langle \alpha . \alpha \rangle = |c| \sqrt{\langle \alpha . \alpha \rangle}}$

$= \sqrt{c \langle \alpha . c\alpha \rangle}$

$= \sqrt{c \langle c\alpha . \alpha \rangle}$

$= \sqrt{c^2 \langle \alpha . \alpha \rangle}$

$= |c| \sqrt{\langle \alpha . \alpha \rangle} = |c| \, \|\alpha\|$

(ii) $\|\alpha\| = \sqrt{\langle \alpha . \alpha \rangle} > 0 \quad \because \langle \alpha . \alpha \rangle > 0$ if $\alpha \neq 0$.

$\Rightarrow \|\alpha\| > 0$

Cauchy - Schwartz inequality:

For any any two vectors $\alpha, \beta \in (V, \langle \cdot \rangle)$.

$$|\langle \alpha . \beta \rangle| \leq \|\alpha\| \, \|\beta\|$$

Triangle inequality:

For any two vectors $\alpha, \beta \in (V, \langle \cdot \rangle)$

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$

Proof: $\|\alpha + \beta\| = \sqrt{\langle \alpha + \beta . \alpha + \beta \rangle}$

$$\Rightarrow \|\alpha + \beta\|^2 = \langle \alpha + \beta . \alpha + \beta \rangle$$

$$= \langle \alpha . \alpha + \beta \rangle + \langle \beta . \alpha + \beta \rangle.$$

$$= \langle \alpha + \beta . \alpha \rangle + \langle \alpha + \beta . \beta \rangle.$$

$$= \langle \alpha . \alpha \rangle + \langle \beta . \alpha \rangle + \langle \alpha . \beta \rangle + \langle \beta . \beta \rangle$$

$$= \langle \alpha . \alpha \rangle + \langle \alpha . \beta \rangle + \langle \alpha . \beta \rangle + \langle \beta . \beta \rangle$$

$$= \|\alpha\|^2 + 2 \langle \alpha . \beta \rangle + \|\beta\|^2.$$

$$\leq \|\alpha\|^2 + 2 \|\alpha\| \, \|\beta\| + \|\beta\|^2 \quad \left(\text{C.S inequality}\right)$$

$$= (\|\alpha\| + \|\beta\|)^2$$

$$\Rightarrow \|\alpha + \beta\|^2 \leq \|\alpha\|^2 + \|\beta\|^2$$

Orthogonal:

Let $\alpha, \beta \in (V, \langle \cdot \rangle)$

Then α is said to be orthogonal to a vector β if,

$$\langle \alpha \cdot \beta \rangle = 0$$

We express this by writting $\alpha \perp \beta$.

Orthogonal Set of vectors:

A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is said to be orthogonal if

$$\langle \beta_i \cdot \beta_j \rangle = 0 \quad \text{Whenever } i \neq j$$

Orthonormal set of vectors:

A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is said to be orthonormal if

$$\langle \beta_i \cdot \beta_j \rangle \begin{cases} = 0 & \text{for } i \neq j \\ = 1 & \text{for } i = j \end{cases}$$

Note: An orthogonal set of vectors may contain the null vector θ but an orthonormal set contains only non-null vectors.