Tutorial 13 Solution

1. In a simple graph, edges are undirected. To show that *R* is symmetric we must show that if *uRv*, then *vRu.* If *uRv*, then there is an edge associated with {*u*, v}. But {*u*, v} = {*v*, u}, so this edge is associated with {*v*, u} and therefore *vRu.* A simple graph does not allow loops; that is if there is an edge associated with {*u*, v}, then $u \neq v$.
Thus, *uRu* never holds, and so by definition *R* is irreflexive.

2. Because the sum of degrees of the vertices is 6 *10=60, it follows that 2e=60. therefore, e=30.

3. **Answer: (C)** P is true for undirected graph as adding an edge always increases degree of two vertices by 1.
Q is true: If we consider sum of degrees and subtract all even degrees, we get an even number because every edge increases the sum of degrees by 2. So total number of odd degree vertices must be even.

4. Each edge ends at two vertices. If we begin with just the vertices and no edges, every vertex has degree zero, so the sum of those degrees is zero, an even number. Now add edges one at a time, each of which connects one vertex to another, or connects a vertex to itself (if you allow that). Either the degree of two vertices is increased by one (for a total of two) or one vertex's degree is increased by two. In either case, the sum of the degrees is increased by two, so the sum remains even.

5. One way to prove this is by induction on the number of vertices. We will first solve the problem in the case that there are two vertices of odd degree. (If all vertices have even degree, temporarily remove some edge in the graph between vertices a and b and then a and b will have odd degree. Find the path from a to b which we will show how to do below, and then follow the removed edge from b back to a to make a cycle.) Suppose the odd-degree vertices are a and b. Begin at a and follow edges from one vertex to the next, crossing off edges so that you won't use them again until you arrive at vertex b and you have used all the vertices into b. Why is it certain that you will eventually arrive at b? Well, suppose that you don't. How could this happen? After you leave a, if you arrive at a vertex that is not b, there were, before you arrived, an even number of unused edges leading into it. That means that when you arrive, there is guaranteed to be an unused path away from that vertex, so you can continue your route. After entering and leaving a vertex, you reduce the number of edges by 2, so the vertex remains one with an even number (possibly zero) of unused paths. So if you have not yet arrived at vertex b, you can never get stuck at any other vertex, since there's always a way out. Since the graph is finite, you cannot continue forever, so eventually you will have to arrive at vertex b. (And it has to be possible to get to vertex b since the graph is connected.) Note that a similar argument can be used to show that you can wait until you have used all the edges connecting to b. If b has more than one edge, leave each time you arrive until you get stuck at b. Now you have a path something like this: $(a, a_1, a_2, \ldots, a_n, b)$ leading from a to b. If all the edges are used in this path, you are done. If not, imagine that you have erased all the edges that you used. What remains will be a number of components of the graph (perhaps only one) where all the members of each component have even degree. Since b will not be in any of the components, all of them must have fewer vertices than the original graph.

6. The proof is easy and can be done by induction. If n = 1, we simply need to visit each vertex of a two-vertex graph with an edge connecting them. Assume that it's true for n = k. To build a (k + 1)-cube, we take two copies of the k-cube and connect the corresponding edges. Take the Hamiltonian circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the (k + 1)-cube.

8. The path Path1 {A, B, C, D, E, A} and the path Path2 {A, B, C, E, D, A} pass all the vertices but Path1 has a total length of 24 and Path2 has a total length of 31. Therefore, the answer is 24.