Lecture - 5

Class Note

Gauss ellimination method for a system of m linear equations in n variables:

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_2 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A_{m\times n} X_n = B_m$$

Consistent system: A system of equation is said to consistent if it has a solution.

Inconsistent system: A system of eqvation is said to be inconsistent if it has no solution.

Augmented matrix:

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} & b_i \\ \vdots & & \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} & b_m \end{array}\right)$$

$\Rightarrow \bar{A} = \left(A_{m\times n} \mid B_m \right)$ is said to to be the augmented matrix.

> Note:
> If Rank of A = Rank of augmented matrix
> Then the system is consistent

Example: System of equations:-

$$x_1 + x_2 = 4$$

$$x_2 - x_3 = 1$$

$$2x_1 + x_2 + 4x_3 = 7$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}.$$

Now the augmented matrix $\bar{A} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{array}\right)$

Gauss - Elimination method:

Let $A_{m\times n} X_n = B_m$ be a given system.

Step 1: Apply elementary row operation on augmented matrix $= \bar{A} = (A \mid B)$ to get a row echelon form of $\bar{A}$.

Step 2: If row-echelon form of $\bar{A} = (A \mid B)$ is $(A_1 \mid B_1)$

Then, if Rank of $\bar{A} \{= (A \mid B)\}$ = Rank of A

then system is consistent.

(Note: Rank of $\bar{A}$ = no of nonzero rows in $(A_1 \mid B_1)$

and Rank of A = no of non zero rows in A_1)

Step 3: Then Solve the system $A_1 X = B_1$

Example: Solve the system of equations using Gauss-ellimination method;

$$\left.\begin{aligned} x_1 + x_2 &= 4 \\ x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 4x_3 &= 7 \end{aligned}\right\} \overset{①}{\Rightarrow} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}$$

$\Rightarrow AX = B$

Soln: Augmented matrix $= \bar{A} = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 2 & 1 & 4 & 7 \end{array}\right) = (A \mid B)$

$$\xrightarrow{R_3' \to R_3 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & -1 \end{array}\right) \xrightarrow{R_3' \to R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right) = (A_1 \mid B_1)$$

Now Rank of $\bar{A} = 3$ $\Rightarrow$ System is consistent

Rank of $A = 3$

Hence the system $AX = B$ equivalent to $A_1 X = B_1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow \quad x_1 + x_2 = 4$

$$x_2 - x_3 = 1$$

$$3x_3 = 0$$

$\Rightarrow \quad x_3 = 0, \quad x_2 = 1, \quad x_1 = 3.$

$\therefore \quad (x_1, x_2, x_3) = (3, 1, 0)$

Solvability of system of linear equations

Homogeneous & Non-Homogeneous systems: A linear equations $A_{m\times n} X_n = B_m$ is called homogeneous if $B_m = O$ (where O = zero matrix) and non-homogeneous if $B_m \neq O$.

Theorem: A necessary and sufficient condition for a non-homogeneous system $A_{m\times n} X_n = B_m$ to be consistent is, Rank of A = Rank of augmented matrix = Rank of $(A|B)$

Existence and number of solutions of the non-homogeneous system $AX = B$, where A is $m\times n$ matrix

case 1: $m = n$

The system is consistent if and only if Rank of A = Rank of augmented matrix $(\bar{A})$

For a consistent system, two cases arise

subcase 1: Rank of A = Rank of $\bar{A} = n$

Then the system possesses the unique solution.

subcase 2: Rank of A = Rank of $\bar{A} < n$

Then the system possesses infinitely many solution.

case 2: $m < n$

The system is consistent if and only if Rank of A = Rank of augmented matrix $(\bar{A}) \leq m$

i.e if consistent, Rank A = Rank $\bar{A} < n$ ($\because m<n$)

Therefore the system possesses infinitely many solutions.

case 3: $m > n$

The system is consistent if and only if Rank of A = Rank of augmented matrix $(\bar{A}) \leq n$

For a consistent system two cases arise.

Subcase 1: if Rank A = Rank $\bar{A}$ $= n$

The system possesses unique solution

Subcase 2: if Rank A = Rank $\bar{A}$ $< n$

The system possesses infinitely many solution.