Fuzzy Sets

→ The sets that we have studied so far are classical sets termed as crisp sets.

→ In a crisp set A, an element of a given universal set X is either a member of A or it is not a member of A which can be defined as

$$\mu_A : X \to \{0, 1\}$$

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Ex :- Let us consider a set of brilliant students of a class.

Criteria : A score greater than or equal to 75 in the previous examination.

∴ The set will include all those students who have scored 75% or more, whereas a student who has scored 74.9% will be excluded from the set and will be considered as not brilliant.

→ Terms such as brilliant, slow, fast, low, high, cold, hot, etc. are vague terms and crisp sets are not suitable for defining vague terms or concepts.

→ We can overcome this problem in fuzzy sets by associating a grade of membership with every x of set A.

- Def:- A fuzzy set A for a given universal set X is a set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) : X \to [0,1]\}$$

where $\mu_A(x)$ → membership function.

assigns each element of X a real no. in $[0,1]$ called the degree of membership.

Note: The membership function can be defined as per the suitability of the concept.

Ex:- In a certain class, based on the percentage of marks in the final exam, we can define a fuzzy set of brilliant students (B) as follows:

$$\mu_B(x) = \begin{cases} 1 & \text{if } x \geq 75 \\ \frac{x}{75} & \text{if } x < 75 \end{cases}$$

Therefore, the membership degree can be computed as:

Student	Marks	Membership Degree
Stud 1	79	1
Stud 2	74	0.99
Stud 3	50	0.67

→ The grades of membership show the belongingness of the element to the set.

(I) Operations on Fuzzy Sets

Let A and B be two fuzzy sets with respect to a universal set X. Then, the standard union and intersection of A and B are defined as follows:

(a) $A \cup B = \{(x, \mu_{A\cup B}(x)) : x \in X,$

$\mu_{A\cup B}(x) = \text{Max}(\mu_A(x), \mu_B(x))\}$

(b) $A \cap B = \{(x, \mu_{A\cap B}(x)) : x \in X,$

$\mu_{A\cap B}(x) = \text{Min}(\mu_A(x), \mu_B(x))\}$

(c) $\bar{A} = \{(x, \mu_{\bar{A}}(x)) : x \in X, \mu_{\bar{A}}(x) = 1 - \mu_A(x)\}$

Ex:- Let A and B be two fuzzy sets defined on a set $X = \{a, b, c, d\}$ and

$A = \{(a, 0.2), (b, 0.4), (c, 0.3), (d, 0.7)\}$

$B = \{(a, 0.5), (b, 0.3), (c, 0.4), (d, 0.5)\}$

(a) $\bar{A} = \{(a, 0.8), (b, 0.6), (c, 0.7), (d, 0.3)\}$

(b) $A \cup B = \{(a, 0.5), (b, 0.4), (c, 0.4), (d, 0.7)\}$

(c) $A \cap B = \{(a, 0.2), (b, 0.3), (c, 0.3), (d, 0.5)\}$

(II) α-Cut and strong α-cut

Def: Given a fuzzy set A, defined on X and any number $\alpha \in [0, 1]$,

α-Cut and strong α-cut are the crisp sets defined as follows:

$${}^{\alpha}A \text{ or } A_{\alpha} = \{x : \mu_A(x) \geq \alpha\}$$

$${}^{\alpha+}A \text{ or } A^{+}_{\alpha} = \{x : \mu_A(x) > \alpha\}$$

Ex:- Let A be a fuzzy set on a set X

$X = \{10, 20, 30, 40, 50\}$ whose membership function is defined as $\mu_A(x) = \dfrac{x}{x+10}$

Find A_{α} for $\alpha = 0.6$.

$\therefore$ Fuzzy Set $A = \{(10, 0.5), (20, 0.67), (30, 0.75), (40, 0.80), (50, 0.83)\}$

Now, $A_{0.6} = \{20, 30, 40, 50\}$

$\rightarrow$ Properties of α-cuts and strong α-cuts

For a fuzzy set A and $\alpha_1, \alpha_2 \in [0, 1]$

(i) $\alpha_1 < \alpha_2 \Rightarrow A_{\alpha_2} \subseteq A_{\alpha_1}$

(ii) $\alpha_1 < \alpha_2 \Rightarrow A^+_{\alpha_2} \subseteq A^+_{\alpha_1}$

(III) <u>Some more terms</u>

Given a fuzzy set A, defined on X

(a) <u>Support</u> of A: It is the crisp set that contains all the elements of X that have a non-zero grade of membership in A.

Denoted as $\boxed{\text{Supp}(A)}$

Note: It is a strong α- cut of A for $\alpha=0$

$$\boxed{Supp(A) = A_0^+}$$

The element $x \in X$ at which $\mu_A(x) = 0.5$ is called the crossover point.

Ex:- Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Fuzzy Set $A = \{(1,0), (2,0), (3, 0.2), (4, 0.5), (5, 0.3), (6, 0.4), (7, 0), (8, 0), (9, 0), (10, 0)\}$

$= \{(3, 0.2), (4, 0.5), (5, 0.3), (6, 0.4)\}$

where $\mu_A(x) > 0$

$$Supp(A) = \{x \in X \mid \mu_A(x) > 0\}$$

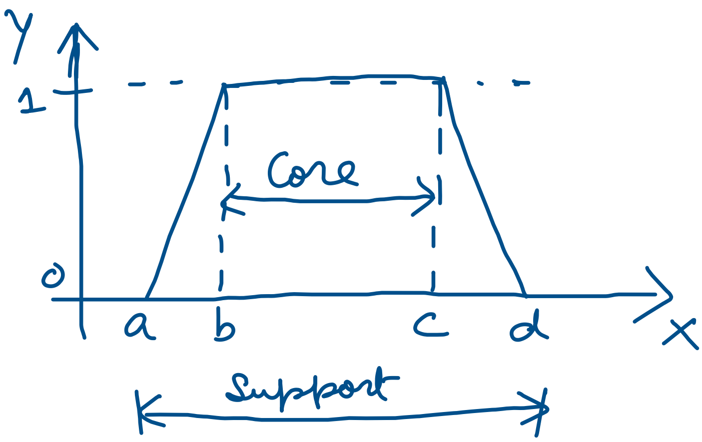
$\therefore$ $Supp(A) = \{3, 4, 5, 6\}$

$\boxed{x = 4}$ $\rightarrow$ Crossover point

(b) The core of a fuzzy set is the α-cut of A for $\alpha = 1$.

Denoted as $\boxed{Core(A) = A_1}$

$$Core(A) = \{ x \in X \mid \mu_A(x) = 1 \}$$



(c) The height of a fuzzy set is the highest grade of membership of any element in the set and it is denoted by $\boxed{h(A)}$.

$$\boxed{h(A) = \max_{x \in X} \mu_A(x)}$$

(d) A fuzzy set is called Normal if $\boxed{h(A) = 1}$ and subnormal if

$$\boxed{h(A) < 1}$$

(IV) Fuzzy Sets in decision making

Ex:- Rahul is looking for a two-bedroom apartment having low rent (around Rs. 30,000) and is near (within 1Km distance) his/her office, then the decision making can be done through fuzzy set theory.

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \leq 30{,}000 \\ \dfrac{30{,}000}{x} & \text{for } x > 30{,}000 \end{cases}$$

$$\mu_B(x) = \begin{cases} 1 & \text{for } x \leq 1 \\ \dfrac{1}{x} & \text{for } x > 1 \end{cases}$$

→ Therefore, each apartment in the area will have two grades of membership.

→ We need to compute $\mu_{A \cap B}(x)$ for every flat (x).

→ The highest grade of membership shows the best choice.

→ Applications of fuzzy set includes AI, ML, knowledge acquisition, decision analysis etc.