Ordinary Differential Equations

(Lecture-9)

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Learning Outcome of the Lecture

We learn

- Lipschitz Condition
- Checking Criteria for Lipschitz Condition
- Examples
- Existence Uniqueness Theorem
- Examples

Definition

Definition (Bounded function in xy-plane)

Let f be a real function defined on D, where D is either a domain or a closed domain of the xy-plane. The function f is said to be bounded in D if there exists a positive real number M such that

$$|f(x,y)| \leq M$$

for all (x,y) in D.

Result: Let f be defined and continuous on a closed rectangle $R : a \leq x \leq b, c \leq y \leq d$. Then, f is bounded in R.

Proof: Exercise.

Lipschitz Continuity

Definition

Let f be defined on D, where D is either a domain or a closed domain of the xy- plane. The function f is said to satisfy Lipschitz Condition (with respect to y) in D if there exists a constant $M > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

for every (x, y_1), (x, y_2) which belong to D. The constant M is called the Lipschitz constant. We say f is Lipschitz continuous in D with respect to y.

Example: The function x^2 is Lipschitz continuous in $[0, 2]$.

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| = |(x_1 + x_2)(x_1 - x_2)| \\ &\leq \max_{x_1, x_2 \in [0,2]} |(x_1 + x_2)||(x_1 - x_2)| \\ &= 4|x_1 - x_2|. \end{aligned}$$

Lipschitz Condition $\Rightarrow$ Continuity ?

Result: If f satisfies Lipschitz condition with respect to y in D, then for each fixed x, the resulting function of y is a continuous function of y, for all (x, y) in D.

Example: Let $f(x, y) = y + [x]$, where $[x]$ is the greatest integer function in x (Recall that for all real numbers x, the greatest integer function returns the largest integer less than or equal to x).
For fixed x,

$$f(x, y_1) - f(x, y_2) = y_1 + [x] - y_2 - [x] = y_1 - y_2.$$

That is,

$$|f(x, y_1) - f(x, y_2)| = |y_1 - y_2| \leq 1|y_1 - y_2|.$$

This implies that for any fixed x, f is continuous w.r.t y.
But note that f is discontinuous w.r.t. x for every integer value of x.

Remark: Note that the condition of Lipschitz continuity implies nothing about the continuity of f with respect to x.

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Does Continuity w.r.t. $y \Rightarrow$ Lipschitz condition w.r.t. y ?

Result: Continuity w.r.t. second variable $\not\Rightarrow$ Lipschitz condition w.r.t. second variable.

Example: $f(x,y) = \sqrt{|y|}$.

Check that f is continuous for all y. (Exercise ?),

But f doesn't satisfy Lipschitz condition in any region that includes $y = 0$ as for $y_1 = 0,\ y_2 > 0$, we have

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{\sqrt{y_2}}{|y_2|} = \frac{1}{\sqrt{y_2}}$$

which can be made as large as we want by making y_2 smaller.

The Lipschitz condition requires that the quotient should be bounded by a fixed constant K.

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Sufficiency for Lipschitz condition

Result: If f is such that $\frac{\partial f}{\partial y}$ exists and is bounded for all $(x, y) \in D$, then f satisfies Lipschitz condition w.r.t. y in D, where the Lipschitz constant is given by

$$M = \sup_{(x,y)\in D} \left| \frac{\partial f}{\partial y} \right|.$$

Proof: Using mean value theorem

$$f(x, y_1) - f(x, y_2) = (y_1 - y_2)\frac{\partial f}{\partial y}(x, t), \quad t \in (y_1, y_2).$$

$$\begin{aligned} |f(x, y_1) - f(x, y_2)| &= |y_1 - y_2| \left| \frac{\partial f}{\partial y}(x, t) \right| \\ &\leq |y_1 - y_2| \sup_{(x,y)\in D} \left| \frac{\partial f}{\partial y} \right|. \end{aligned}$$

This implies f satisfies Lipschitz condition.

Example

Show that $f(x,y) = x^2 + y^2$ satisfies a Lipschitz condition in rectangle D defined by $D: \ |x| \leq a, \ |y| \leq b$.

Solution: Apply the sufficient condition,
$\frac{\partial f}{\partial y} = 2y$ is bounded in D. The Lipschitz contant is

$$M = \sup_{(x,y)\in D} \left| \frac{\partial f}{\partial y} \right| = \sup_{(x,y)\in D} |2y| = 2b.$$

(**Exercise**: Verify Lipschitz condition by definition)

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Bounded $\frac{\partial f}{\partial y}$ is sufficient but not necessary for Lipschitz condition

Example: Consider the function $f(x, y) = x|y|$, where D is the rectangle defined by $|x| \leq a,\ |y| \leq b$.

- f satisfies

$$|f(x, y_1) - f(x, y_2)| = |x|y_1| - x|y_2|| \leq |x||y_1 - y_2| \leq a|y_1 - y_2|$$

  for all $(x, y_1),\ (x, y_2) \in D$.

- Therefore f satisfies a Lipschitz condition (with respect to y) in D.

However, the partial derivative $\frac{\partial f}{\partial y}$ does not exist at any point $(x, 0) \in D$ for which $x \neq 0$.

Existence Theorem

Theorem

Let R be a rectangle and (x_0, y_0) be an interior point of R, let

- $f(x,y)$ be continuous at all points (x,y) in
$$R \; : \; |x - x_0| \leq a, \; |y - y_0| \leq b \quad \text{and}$$
- Bounded in R, that is, $|f(x,y)| \leq K$ for all $(x,y) \in R$.

Then, the IVP
$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$
has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| \leq \alpha$; where
$$\alpha = \min\{a, \frac{b}{K}\}.$$

Uniqueness Theorem

Theorem

Let R be a rectangle and (x_0, y_0) be an interior point of R,

- $f(x,y)$ be continuous at all points (x,y) in
$$R \; : \; |x - x_0| \le a, \; |y - y_0| \le b \quad \text{and}$$
- Bounded in R, that is, $|f(x,y)| \le K$ for all $(x,y) \in R$.
- f satisfies the Lipschitz condition with respect to y in R, that is,
$$|f(x,y_1) - f(x,y_2)| \le M|y_1 - y_2|, \text{ for all } (x,y_1),(x,y_2) \in R.$$

Then, the IVP
$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$
has a unique solution $y(x)$.
This solution is defined at least for all x in the interval $|x - x_0| \le \alpha$; where
$$\alpha = \min\{a, \frac{b}{K}\}.$$

Example

Consider

$$\frac{dy}{dx} = 1 + y^2, \quad y(0) = 0.$$

- Take the Rectangle $R : |x-0| \leq 5, \ |y-0| \leq 3 \ (x_0 = 0, \ y_0 = 0, \ a = 5, \ b = 3)$.f(x,y)=$y^2+1$ is continuous and $|f(x,y)| = |1+y^2| \leq 1 + |y|^2 \leq 10$, thus existence of a solution is guaranteed.

- For uniqueness of the solution we check for Lipschitz condition.

$$\begin{aligned} |f(x,y_1) - f(x,y_2)| &= |1 + y_1^2 - 1 - y_2^2| = |y_1^2 - y_2^2| \\ &= |(y_1 + y_2)(y_1 - y_2)| \leq 6|(y_1 - y_2)|, \end{aligned}$$

thus uniqueness of solution is guaranteed.
of existence of unique solution is $|x-0| \leq \alpha$; where

$$\alpha = \min\{a, \frac{b}{K}\} = \min\{5, \frac{3}{10}\} = 0.3$$

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Example

Consider

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 0.$$

- $f(x,y)$ is continuous thus existence of a solution is guaranteed.
- for uniqueness of the solution we check for Lipschitz condition.

$$\frac{f(x,y_1) - f(x,y_2)}{|y_1 - y_2|} = \frac{y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}}{|y_1 - y_2|}$$

if we choose $y_1 = \delta > 0$ and $y_2 = -\delta$, this becomes

$$\frac{\delta^{1/3} - (-\delta)^{1/3}}{\delta - (-\delta)} = \frac{1}{\delta^{2/3}}.$$

- This becomes unbounded as δ approaches zero, which shows that f does not satisfy a Lipschitz condition in D.

Exercise: Find two different solution of the above IVP.