Lecture - 19

Class Note

Orthogonal:

Let $\alpha, \beta \in (V, \langle \cdot \rangle)$

Then α is said to be orthogonal to a vector β if,

$$\langle \alpha . \beta \rangle = 0$$

We express this by writing $\alpha \perp \beta$.

Orthogonal Set of vectors:

A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is said to be orthogonal if

$$\langle \beta_i . \beta_j \rangle = 0 \quad \text{whenever } i \neq j$$

Orthonormal set of vectors:

A set of vectors $\{\beta_1, \beta_2, \ldots, \beta_n\}$ is said to be orthonormal if

$$\langle \beta_i . \beta_j \rangle \left.\begin{array}{l} = 0 \\ = 1 \end{array}\right\} \begin{array}{l} \text{for } i \neq j \\ \text{for } i = j \end{array}$$

(Note: when $i = j$, $\langle \beta_i . \beta_i \rangle = \|\beta_i\|^2 = 1 \Rightarrow \|\beta_i\| = 1$)

Note: An orthogonal set of vectors may contain the null vector θ but an orthonormal set contains only non-null vectors.

Normalized Vector:

If $||u|| = 1$, we called u a unit vector and u is said to be normalized.

For any non-zero vector $v \in V$, we have the unit vector $\hat{v} = \frac{v}{||v||}$

This process is called ~~...~~ normalizing v.

The angle between two vector u & v:

From Cauchy-Schwarz Inequality

$$|\langle u . v \rangle| \leq ||u|| \, ||v||$$

Now if $u, v \in V$ are two non-zero vectors, then

$$\frac{|\langle u . v \rangle|}{||u|| \, ||v||} \leq 1$$

$$\Rightarrow \quad -1 \leq \frac{\langle u . v \rangle}{||u|| \, ||v||} \leq 1$$

$$\Rightarrow \quad \cos\theta = \frac{\langle u . v \rangle}{||u|| \, ||v||}$$

Theorem: A orthogonal set of non-null vectors in a inner product space V is linearly independent

proof: let $\{\beta_1, \beta_2, \cdots, \beta_r\}$ be an orthogonal set of non-null vectors.

let us consider the relation: -

$$c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r = \theta \; , \; c_i \in \mathbb{R} \;\&\; \theta = \text{null vector}$$

Then $\langle c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r \cdot \beta_i \rangle = \langle \theta . \beta_i \rangle = 0$ for $i = 1, 2, \cdots, r$

$\Rightarrow c_1\langle \beta_1 . \beta_i \rangle + c_2\langle \beta_2 . \beta_i \rangle + \cdots + c_i\langle \beta_i . \beta_i \rangle + \cdots + c_r\langle \beta_i . \beta_r \rangle = 0$

$\Rightarrow c_i\langle \beta_i . \beta_i \rangle = 0$ $\left(\because \langle \beta_i . \beta_j \rangle = 0 \text{ for } i \neq j \right.$ defn of orthogonal set$\left.\right)$

$\because \beta_i \neq \theta \Rightarrow \langle \beta_i . \beta_i \rangle > 0$

$\Rightarrow c_i = 0$

$\Rightarrow \{\beta_1, \beta_2, \cdots, \beta_r\}$ is linearly independent.

Corollary: An orthonormal set of vectors in a inner product space is linearly independent

Orthogonal Basis:

An orthogonal basis for an inner product space V is a basis for V whose vectors are mutually orthogonal.

Orthonormal basis: If the vectors of an orthogonal basis are normalized the resulting basis is an orthonormal basis.

Example:

Let V be a vector space with an inner product. Suppose $\{v_1, v_2, \dots, v_n\}$ is an orthogonal basis for V.

Let $w_1 = \frac{v_1}{\|v_1\|}$, $w_2 = \frac{v_2}{\|v_2\|}$, ..., $w_n = \frac{v_n}{\|v_n\|}$

Then $\{w_1, w_2, \dots, w_n\}$ is an orthonormal basis for V.

Example: Prove that $\{(1,2,2), (2,-2,1), (2,1,-2)\}$ is an orthogonal basis of the vector space $\mathbb{R}^3$.

Solⁿ: Let $\beta_1 = (1,2,2)$, $\beta_2 = (2,-2,1)$, $\beta_3 = (2,1,-2)$

Now $\langle \beta_1 \cdot \beta_2 \rangle = 1\times2 + 2\times-2 + 2\times1 = 0$

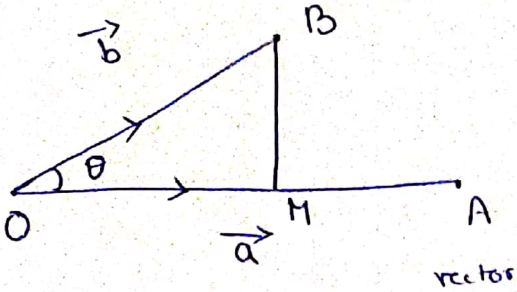
$\langle \beta_2 \cdot \beta_3 \rangle = 2\times2 - 2\times1 + 1\times-2 = 0$

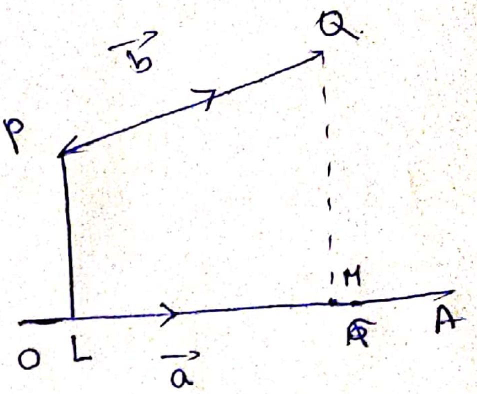
$\langle \beta_3 \cdot \beta_1 \rangle = 2\times1 + 1\times2 + 2\times-2 = 0$

$\Rightarrow \{\beta_1, \beta_2, \beta_3\}$ is an orthogonal set of vector of non-null vector $\Rightarrow$ They are linearly independent.

$\because$ Vector space $\mathbb{R}^3$ dimension is 3 $\Rightarrow \{\beta_1, \beta_2, \beta_3\}$ is basis

Orthogonal Projection:





$\overrightarrow{OM}$ → component vector of $\vec{b}$ on $\vec{a}$

$\overrightarrow{MB}$ → component vector $\vec{b} \perp \vec{a}$

$\overrightarrow{LM}$ = orthogonal projection $\vec{b}$ on $\vec{a}$

We know $\cos\theta = \dfrac{\langle a, b\rangle}{\|a\|\,\|b\|}$

$$\Rightarrow \|b\| \cos\theta = \frac{\langle a, b\rangle}{\|a\|}$$

$$\Rightarrow \|\overrightarrow{OM}\| = \frac{\langle a . b\rangle}{\|a\|} \qquad \left(\because \|b\| \cos\theta = \|\overrightarrow{OM}\|\right.$$

$$\Rightarrow \boxed{\text{Projection } \vec{b} \text{ on } \vec{a} = \frac{\langle a . b\rangle}{\|a\|} \frac{\vec{a}}{\|a\|} = \frac{\langle b . a\rangle}{\|a\|^2} \vec{a}} \qquad \left(\because \frac{\vec{a}}{\|a\|} = \text{unit vector}\right)$$