Tutorial -12 Solution

Q1) Let $a, b \in G$, then $a \circ b \in G$ (closure)
Hence, by the given conditions, we have

$$a \circ b = (a \circ b)^{-1}$$
$$= b^{-1} \circ a^{-1}$$
$$= b \circ a \quad \text{since } a^{-1} = a \text{ and } b^{-1} = b.$$

Thus, $a \circ b = b \circ a$, for every $a, b \in G$.
Therefore, it is an abelian group.
The converse is not true, For example, $(R, +)$, where R is the set of all real numbers, is an abelian group but no element except 0 is its own inverse.

Q2) (a) Semi Group. (b) Monoid, Identity 1.

Q4) (a) Let E denotes the set of all even integers (including zero). Then it can be verified that the set E is a ring under addition and multiplication binary operations. Also, the multiplication is a commutative operation and hence E is a commutative ring.
E is without zero divisors because the product of two non-zero even integers cannot be equal to zero which is the zero element of this ring.
But the integer $1 \notin E$. So, E is a commutative ring without zero-divisors and without unity. Therefore, $(E, +, \cdot)$ is not an integral domain and hence it is not a field also.

(b) Let N denotes the set of all positive integers. Now N does not contain the additive identity since $0 \notin N$. So, N is not a ring nor a field.

Hence (N, +, ·) is neither an integral domain nor a field.

Q5) (A) (i) and (iv) only.

Q6) (a) $(R, \leq)$ – The relation less than or equal to defined over a set of real numbers is a partial order relation, and the set $(R, \leq)$ is a poset.

(b) $(Z, >)$ – This is not a poset because the relation $>$ is not reflexive.

Q7)

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	a	a
c	c	e	a	b

→ 1st column should have the elements in same order so,

e a b c.

Since $a * c = e$ So, c is the inverse of a.

$\therefore c * a = e$.

(d) c e a b Ans.

1) The set given is $\{1, 2, 4, 7, 8, 11, 13, 14\}$ is a group under modulo 15.

$4 \times_{15} 4 = 16 \bmod 15 = 1$

$7 \times_{15} 13 = 91 \bmod 15 = 1$

$\therefore$ (c) 4 and 13 Ans.

9) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $G = \langle \{A, B, C, D.\}, \times \rangle$

$$A \times A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A.$$

Similarly, $A \times B = B$, $A \times C = C$, $A \times D = D$, $B \times B = A$ etc.

Now, we can draw the composition table as

$\times$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

(i) Closure Property :- We can see that all entries in the composition table are the elements of G and hence G is closed w.r.t. matrix multiplication.

(ii) Associative Law:- Multiplication is associative in G. Since associative law holds, in case of matrix multiplication, i.e,

$$(AB)C = A(BC)$$

(iii) Commutative Law:- The entries in the first, second and third and fourth row.

This shows that G is commutative.

(iv) Existence of Identity:- From the composition table it can be seen that $A \times A = A$, $A \times B = B$, $A \times C = C$, $A \times D = D$.
Here A is the identity.

(v) Existence of Inverse:- $A \times A = A$, $B \times B = A$, $C \times C = A$,

$$D \times D = A.$$

Thus, every element is its own inverse.

Therefore, $(G, \times)$ is an abelian group.

Q10) Composition Table of the cyclic group ~~are~~ is as follows:-

*	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	b	d	a
d	d	c	a	b

Checking for generators:-

$a^n = \underline{a * a} * a * \dots$ n times. ✗

$\underline{a * a}$

a

$b^n = \underline{b * b} * b \dots\dots$ n times ✗.

$\underline{a * b}$

$\underline{b * b}$

a

Therefore, (c) is the Ans.

Q 11)

a) $(R, *)$ $a * b = a + b$

It represents an Abelian Group as it satisfies the following properties →

1. Closure: Addition of two real number is a real numbers.

i.e $\forall a, b \in R$,

$a + b \in R$

2. Associative: $a + (b + c) = (a + b) + c$

$\forall a, b, c \in R$

3. Identity: $\boxed{e = 0}$

$a + 0 = 0 + a = a$

i.e $[a + e = e + a \Rightarrow e = 0 \in R]$

4. Inverse: $a * a^{-1} = 0 = a^{-1} * a$

$a + a^{-1} = 0 = a^{-1} + a$

$\boxed{a^{-1} = -a \in R}$

5. Commutative: $a + b = b + a$

$\forall a, b \in R$

(b) $(R, *)$ $a * b = \min(a, b)$

It represents a Semi-Group because the following properties are satisfied →

1) Closure $\forall a, b \in R$

$\min(a, b) \in R$

2. Associative $\min\left[[a], \min[b, c]\right] = \min\left[\min[a, b], c\right]$

3. Identity Not satisfied

$a * e = a = e * a$

$\mathrm{Min}(a, e) = a = \mathrm{Min}(e, a)$

$\Rightarrow e = +\infty \notin R$

(c) $(R, *)$ $a * b = a^b$

It represents a Groupoid because it satisfied only the Closure property.

1) Closure $\forall a, b \in R$

$a^b \in R$

2) Associativity Not Satisfied

$a^{(b^c)} \neq (a^b)^c$

Q 12)

$(\{1, \omega, \omega^2\}, *)$

Yes it represents a Cyclic Group. [from the Composition table]
$\because$ it satisfies the given properties $\rightarrow$

1) Closure   2) Associative   5. Commutative

3) Identity $e = 1$ 4) Inverse   6. $\exists g \in \{1, \omega, \omega^2\}$

$*$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$$\left.\begin{array}{l} 1^{-1} = 1 \\ \omega^{-1} = \omega^2 \\ (\omega^2)^{-1} = \omega \end{array}\right\}$$ from the Composition Table

Generators ω and ω^2 are Generators.

$\omega^0 = e = 1$

$\omega^1 = \omega$

$\omega^2 = \omega^2$

$\therefore$ ω is a Generator

If g is generator, g^{-1} is also generator

$\omega^{-1} = \omega^2$

$\therefore$ ω^2 is also Generator

Q 13)

a) True

i.e $(\{1,2,3,\ldots,p-1\}, \times_p)$ is an Abelian Group

eg.

$(\{1,2,3,4\}, \times_5)$

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

It satisfies the foll. properties →
[We can see from the Composition table]

1. Closure i.e $\forall a,b \in \{1,2,3,4\}$, $a \times_5 b \in \{1,2,3,4\}$
2. Associative ie $a * (b*c) = (a*b)*c$
3. Identity $a*e = e*a = a$

   $e = 1$
4. Inverse $a * a^{-1} = e = a^{-1} * a$

   $1^{-1} = 1$

   $2^{-1} = 3$

   $3^{-1} = 2$

   $4^{-1} = 4$
5. Commutative $a*b = b*a \ \forall a,b \in \{1,2,3,4\}$