Tutorial 14 Solution

1.

a) Vertex *a* is the root, since it is drawn at the top.
b) The internal vertices are the vertices with children, namely *a, b, c, d, f, h , j , q,* and *t.*
c) The leaves are the vertices without children, namely e, *g,* i, *k, l,* m, n, o, p, *r, s,* and *u.*
d) The children of *j* are the vertices adjacent to *j* and below *j,* namely *q* and *r.*
e) The parent of *h* is the vertex adjacent to *h* and above *h,* namely c.
f) Vertex *o* has only one sibling, namely *p,* which is the other child of o's parent, *h.*
g) The ancestors of *m* are all the vertices on the unique simple path from *m* back to the root, namely *f, b,*
and *a.*
h) The descendants of *b* are all the vertices that have *b* as an ancestor, namely e, *f, l,* m, and *n.*

2. Let *P* be a person sending out the letter. Then 10 people receive a letter with P's name at the bottom of the list (in the sixth position). Later 100 people receive a letter with P's name in the fifth position. Similarly, 1000 people receive a letter with *P's* name in the fourth position, and so on, until 1,000,000 people receive the letter with *P's* name in the first position. Therefore *P* should receive $1,000,000. The model here is a full 10-ary tree.

3. Suppose that $n = 2^k$, where k is a positive integer. We want to show how to add n numbers in log n steps using a tree-connected network of n - 1 processors (recall that log n means $\log_2 n$). Let us prove this by mathematical induction on k. If k = 1 there is nothing to prove, since then n = 2 and n - 1 = 1, and certainly in log 2 = 1 step we can add 2 numbers with 1 processor. Assume the inductive hypothesis, that we can add $n = 2^k$ numbers in log n steps using a tree-connected network of n - 1 processors. Suppose now that we have $2n = 2^{k+1}$ numbers to add, $x_1, x_2, \ldots, x_{2n}$.. The tree-connected network of 2n - 1 processors consists of the tree-connected network of n - 1 processors together with two new processors as children of each leaf in the (n - 1)-processor network. In one step we can use the leaves of the larger network to add $x_1 + x_2$, $x_3 + x_4$, ... , $x_{2n-1} + x_{2n}$. This gives us n numbers. By the inductive hypothesis we can now use the rest of the network to add these numbers using log n steps. In all, then, we used 1 + (log n) steps, and, just as desired, log(2n) = log 2 + log n = 1 +log n. This completes the proof.

4. There are of course two things to prove here. First let us assume that G is a tree. We must show that G contains no simple circuits (which is immediate by definition) and that the addition of an edge connecting two nonadjacent vertices produces a graph that has exactly one simple circuit. Clearly the addition of such an edge e = { u, v} produces a graph with a simple circuit, namely u, e, v, P, u, where P is the unique simple path joining v to u in G. Since P is unique, moreover, this is the only simple circuit that can be formed. To prove the converse, suppose that G satisfies the given conditions; we want to prove that G is a tree, in other words, that G is connected (since one of the conditions is already that G has no simple circuits). If G is not connected, then let u and v lie in separate components of G. Then edge { u, v} can be added to G without the formation of any simple circuits, in contradiction to the assumed condition. Therefore, G is indeed a tree.