Lecture - 6

Class Note

Solvability of system of linear equations

Homogeneous & Non-Homogeneous systems: A linear equations $A_{m\times n} X_n = B_m$ is called homogeneous if $B_m = O$ (where O = zero matrix) and non-homogeneous if $B_m \neq O$.

Theorem: A necessary and sufficient condition for a non-homogeneous system $A_{m\times n} X_n = B_m$ to be consistent is, Rank of A = Rank of augmented matrix = Rank of $(A|B)$

Existence and number of solutions of the non-homogeneous system $AX = B$, where A is $m\times n$ matr

case 1: $m = n$

The system is consistent if and only if Rank of A = Rank of augmented matrix $(\bar{A})$

For a consistent system, two cases arise

subcase 1: Rank of A = Rank of $\bar{A} = n$

Then the system possesses the unique solution.

subcase 2: Rank of A = Rank of $\bar{A} < n$

Then the system possesses infinitely many solution.

case 2: $m < n$

The system is consistent if and only if Rank of A = Rank of augmented matrix($\bar{A}$) $\leq m$

i.e if consistent, Rank A = Rank $\bar{A}$ $< n$ ($\because m < n$)

Therefore the system possesses infinitely many solutions.

case 3: $m > n$

The system is consistent if and only if Rank of A = Rank of augmented matrix ($\bar{A}$) $\leq n$

For a consistent system two cases arise

subcase 1: if Rank A = Rank $\bar{A}$ $= n$

The system possesses unique solution

subcase 2: if Rank A = Rank $\bar{A}$ $< n$

The system possesses infinitely many solution.

Example: Examine whether the system of equations

$$x_1 + 2x_2 - x_3 = 3$$
$$3x_1 - x_2 + 2x_3 = 1$$
$$2x_1 - 2x_2 + 3x_3 = 2$$
$$x_1 - x_2 + x_3 = -1$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

is consistent. Solve, it, if it is consistent.

Solution: Here the coefficient Matrix is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

and Augmented Matrix is

$$\overline{A} = (A|B) = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right)$$

We see the coefficient matrix A is a submatrix of augmented matrix $\overline{A}$. So if we apply elementary row operation on $\overline{A}$, then A will be automatically get operated.

Let us transform $\overline{A}$ to an Echelon matrix by applying elementary row operations:

$$\overline{A} = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{array}\right) \xrightarrow[\substack{R_3' \to R_3 - 2R_1 \\ R_4' \to R_4 - R_1}]{R_2' \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{array}\right)$$

$$\xrightarrow[R_4' \to R_4 - \frac{3}{7}R_2]{R_3' \to R_3 - \frac{6}{7}R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & \frac{5}{7} & \frac{20}{7} \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right) \xrightarrow[7R_4]{7R_3} \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & -1 & -4 \end{pmatrix}$$ (Just to remove fraction)

$$\xrightarrow{R_1' \to R_1 + \frac{R_3}{5}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right) = C \text{ (say)}$$

Now C is an Echelon matrix. Since it has 3 non-zero rows. So Rank of $C = 3$, and So, Rank of $\overline{A} = 3$ (as $\overline{A}$ and C are row equivalent)

Again, If the last column is deleted from the matrix C, we see the matrix A is row equivalent to the echelon matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{pmatrix} = D \text{ (say)}$$

Since this Echelon matrix has 3 non-zero rows so Rank of $D = 3$, So, Rank of $A = 3$ (as A and D are row equivalent)

Thus, Rank of $A =$ Rank of $\overline{A} = 3$

So, the system of equation is consistent. i.e it has solution.

Since Rank of $A = 3$ and number of unknowns is also 3 so the system has only one solution.

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Let us find that solution:

As the augmented matrix is transformed to

$$C = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

the system of equations is transformed to the system.

$$x_1 + 2x_2 - x_3 = 3$$
$$-7x_2 + 5x_3 = -8$$
$$5x_3 = 26$$

From the last we have $x_3 = 4$.

Putting this in 2nd we get

$$-7x_2 + 5 \times 4 = -8$$

$$\Rightarrow x_2 = 4.$$

Putting these in 1st we get

$$x_1 + 2 \times 4 - 4 = 3.$$

$$\Rightarrow x_1 = -1$$

So, the solution is $(-1, 4, 4)$

This process of computing the solutions of system of equations is known as Gauss-Elémination Process.

Homogeneous systems of linear equations:

A linear system of the form $A_{m\times n} X_n = O_m$ where O_m is a zero matrix, is called homogeneous system.

Note: A homogeneous system is consistent, since $(0, 0, \cdots, 0)$ is always a solution of the system.

Theorem: Let A be an $n\times n$ matrix, then the following statement are equivalent.

- $A_{n\times n}$ is invertible ($\Rightarrow$ determinant of $A \neq 0$)
- $A_{n\times n} X_n = 0$ has only trivial solution and $A_{n\times n} X_n = B_n$ has only one solution.
- The row echelon matrix of $A_{n\times n}$ is the identity matrix.
- $\text{Rank}(A_{n\times n}) = n$

Null Spaces: Given an $m\times n$ matrix A. Let the solution set of the homogeneous system $A_{m\times n} X_n = 0$ be denoted ~~by~~ as

$$N(A) = \{ X_n \in \mathbb{R}^n \mid A_{m\times n} X_n = 0 \}$$ called the null space.

Note: $\because$ the homogeneous system is always solvable by $X_n = O_n = (0, 0, \cdots, 0)$

$\therefore$ Null space is non-empty set.

Note: (i) For the square matrix if $\text{Rank}(A_{n\times n}) = n$, then homogeneous $A_{n\times n} X_n = 0$ system has only zero solution i.e $(0, 0, \cdots 0)$

(ii) If $\text{Rank}(A_{n\times n}) < n$, then homogeous system $A_{n\times n} X_n = 0$ has non-zero solution infinitely many.

Theorem: ~~IOA~~ let A be an $n \times n$ matrix, then

(i) if $|A| \neq 0$

$\Rightarrow$ Rank ~~R~~ of $A = n$ = no of unknows in $AX = B$

$\Rightarrow$ $AX = B$ has unique solution.

(ii) if $|A| = 0$

$\Rightarrow$ Rank of $A < n$ = no of unknown in $AX = B$

$\Rightarrow$ $AX = B$ has many solution.