Lecture - 17

Class Note

Diagonalisation of matrices

Similar matrix:

Let A & B be $n\times n$ matrices. An matrix $A_{n\times n}$ is said to be similar to $B_{n\times n}$ matrix if $\exists$ a non-singular $n\times n$ matrix P s.t

$$B = P^{-1}AP.$$

Diagonalisable matrix $A_{n\times n}$:

An $n\times n$ matrix A is said to be diagonalisable if A is similar to an $n\times n$ diagonal matrix,

$$\therefore \; P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \cdots & & & \cdots & \lambda_n \end{pmatrix}_{n\times n}.$$

i.e (B = diagonal matrix in $B = P^{-1}AP$)

Theorem: Let A be an $n\times n$ matrix over a field F. If the eigen values of A be all distinct and belong to F, then A is diagonalisable.

Ex: $A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}_{2\times 2}$ its eigen values are, $\lambda = -1, 7$

$\therefore$ A is 2×2 matrix and its two eigen values are distinct $\Rightarrow$ A is diagonalisable.

⊞ Now if $A_{n \times n}$ is similar to diagonal matrix, then $\exists$ non-singular P s.t $P^{-1}AP =$ diagonal matrix.

$\therefore$ Question is, How will ~~we~~ find out the matrix P?

Q:- Find the matrix P for a given matrix

$$A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$ s.t $P^{-1}AP$ is a diagonal matrix.

Solution: $|A - \lambda I_3| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(2-\lambda)(-1-\lambda) - 1\} - 1\{1+\lambda - 0\} - 2\{-1-0\} = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 2)(\lambda + 1) = 0$$

$$\therefore \lambda = 1, 2, -1$$

Eigen vector corresponding to eigen value $\lambda = 1$.

$AX = 1X \Rightarrow AX - X = 0 \Rightarrow (A-I)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} x_2 - 2x_3 = 0 \\ -x_1 + x_2 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array}$$

$\Rightarrow x_2 = +2x_3$

$x_1 = x_2 + x_3 = 2x_3 + x_3 = 3x_3$.

$$\therefore \quad X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad x_3 \neq 0$$

Eigen vector corresponding to eigen value 2:

$$AX = 2X \Rightarrow (A - 2I)X = 0.$$

$$\Rightarrow \begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} -x_1 + x_2 - 2x_3 &= 0. \\ -x_1 + x_3 &= 0 \Rightarrow x_1 = x_3. \\ x_2 - 3x_3 &= 0 \Rightarrow x_2 = 3x_3. \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1 &= x_3. \\ x_2 &= 3x_3. \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 3x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \qquad x_3 \neq 0.$$

Eigen vector corresponding to eigen value −1:

$$AX = -1X \Rightarrow (A + I)X = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} 2x_1 + x_2 - 2x_3 &= 0 \\ -x_1 + 3x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &x_2 = 0 \\ &2x_1 + 2x_3 = 0 \Rightarrow x_1 = x_3 \\ &-x_1 + x_3 = 0 \Rightarrow x_1 = x_3 \end{aligned}$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad x_1 \neq 0$$

$\therefore \quad \lambda_1 = 1, \quad X_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2, \quad X_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

$\lambda_3 = -1 \quad X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\because \lambda_1 \neq \lambda_2 \neq \lambda_2 \Rightarrow \{X_1, X_2, X_3\}$ is L.I

$\therefore \left| X_1 \ X_2 \ X_3 \right| \neq 0$

$\Rightarrow \begin{vmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} \neq 0 \Rightarrow \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is non-singular

~~Let~~

$\therefore \ P = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ is non-singular.

In this way we can find out the non-singular matrix P.

So, $P^{-1}AP = \text{diag}(1, 2, -1)$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Ex2: Diagonalize the symmetric matrix A

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Soln: $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 8)(\lambda - 2)(\lambda - 2) = 0 \quad \Rightarrow \quad \lambda = 8, 2, 2$$

Eigen vector corresponding to $\lambda = 8$:

$AX = 8X \quad \Rightarrow \quad (A - 8I)X = 0$

$$\Rightarrow \begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} &\Rightarrow -4x_1 + 2x_2 + 2x_3 = 0 \Rightarrow -2x_1 + x_2 + x_3 = 0 \\ &\Rightarrow 2x_1 - 4x_2 + 2x_3 = 0 \Rightarrow x_1 - 2x_2 + x_3 = 0 \\ &\Rightarrow 2x_1 + 2x_2 - 4x_3 = 0 \Rightarrow x_1 + x_2 - 2x_3 = 0 \end{aligned}\right\}$$

$\therefore \quad x_1 = 2x_2 - x_3 \quad \Rightarrow \quad x_1 = x_2$

$2x_2 - x_3 + x_2 - 2x_3 = 0 \Rightarrow 3x_2 - 3x_3 = 0 \Rightarrow x_3 = x_2$

$$\therefore \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\therefore$ for $\lambda_1 = 8 \quad X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Eegen rector for $\lambda = 2$:

$$AX = 2X \Rightarrow (A - 2I)X = 0$$

$$\Rightarrow \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow 2x_1 + 2x_2 + 2x_3 = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

$$\Rightarrow x_1 = -x_2 - x_3$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Now $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are L.I $\left(\because c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = 0 = c_2 = 0 \right)$

$$\therefore \text{ for } \lambda_2 = 2 \quad X_2 = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{+} \quad P = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{where } |P| \neq 0.$$

$$\therefore P^{-1}AP = \operatorname{diag}(8, 2, 2)$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$