Solution of

Assignment 4

1)(i) We see the determinant

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 \neq 0 \;\Bigg|\; \begin{array}{l} \Rightarrow \text{Rank of } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ is } 3 \\ \text{ie} \\ \Rightarrow \text{it has 3 linearly independent rows.} \end{array}$$

$\Rightarrow$ the given three vector in S are linearly indepdent in $\mathbb{R}^3$ ~~(i.e. the dimension of $\mathbb{R}^3$ is 3)~~

Now we know that if V be a vector space of demension n over a field F, Then any linearly independent set of n vectors of V is a basis of V.

Here $V = \mathbb{R}^3$ whose demencion is 3 and linearly independent set S contain 3 vectors.

$\Rightarrow$ S is a basis of $\mathbb{R}^3$.

(ii) We see the determinant

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -4 \neq 0$$

$\Rightarrow$ Rank of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ is 3

$\Rightarrow$ All the rows are linearly independent

$\Rightarrow$ the given three vector in S are linearly independent in $\mathbb{R}^3$.

Again $\dim(\mathbb{R}^3) = 3 =$ No of vector in linearly independent set S

$\Rightarrow$ S is a bases of $\mathbb{R}^3$.

(iii) We see the determinant

$$\begin{vmatrix} 4 & 3 & 2 \\ 2 & 1 & 4 \\ 2 & 3 & -8 \end{vmatrix} = -80 + 72 + 8 = 0$$

$\Rightarrow$ Rank of the matrix $\begin{pmatrix} 4 & 3 & 2 \\ 2 & 1 & 4 \\ 2 & 3 & -8 \end{pmatrix}$ is < 3

$\Rightarrow$ All the rows are not linearly independent i.e they are linearly dependent

$\Rightarrow$ the given vectors in S are linearly dependent. Hence S can not be a basis.

(iv) We see the ~~vectors~~ determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ the three vectors $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ are linearly independent in the vector space $\mathbb{R}^3$.

$\because$ $\dim(\mathbb{R}^3) = 3$, So there three vector will form a basis of $\mathbb{R}^3$.

~~Q) let V is the vector~~

2) (i) Find a basis of the subspace W of $\mathbb{R}^3$, where $W = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$

Sol^n:

$$W = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$$
$$= \{(x,y,z) \in \mathbb{R}^3 \mid z = -x-y\}$$
$$= \{(x, y, -x-y)\} \quad x,y \in \mathbb{R}$$
$$= \{x(1,0,-1) + y(0,1,-1)\} \quad x,y \in \mathbb{R}$$
$$= L(S) \quad \text{where } S = \{(1,0,-1), (0,1,-1)\}$$

$\Rightarrow S = \{(1,0,-1), (0,1,-1)\}$ is generates W.

Now $c(1,0,-1) + d(0,1,-1) = (0,0,0)$

$\Rightarrow c=0, d=0$

$\Rightarrow (1,0,-1)$ and $(0,1,-1)$ are linearly independent

$\therefore S = \{(1,0,-1), (0,1,-1)\}$ is generates W and S is linearly independent

$\Rightarrow S = \{(1,0,-1), (0,1,-1)\}$ is a basis of W ~~$\mathbb{R}^3$~~.

Now $\dim(W)$ = no of element in basis i.e in S

$= 2$.

2)(ii) $W = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x+y-z=0 \\ 2x+y-z=0 \end{array} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = x+y \\ 2x+y-x-y=0 \end{array} \right\}$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid z = x+y \;\&\; x=0 \}$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid z = y \;\&\; x=0 \}$

$= \{ (0,y,y) \} \quad y \in \mathbb{R}$

$= \{ y(0,1,1) \} \quad y \in \mathbb{R}$

$= L(S)$ where $S = \{ (0,1,1) \}$

$\Rightarrow S = \{(0,1,1)\}$ is generates W.

Now we know that a set containing a single non-zero ~~element~~ vector in a vector space is linearly independent

$\because$ S contain a single non-zero vector

$\Rightarrow$ S is linearly indepent

$\therefore$ $S = \{(0,1,1)\}$ is generates W and S is linearly indendent

$\Rightarrow S = \{(0,1,1)\}$ is basis of W

$\therefore$ dim (W) = No of element in basis i.e in S

$= 1$

2)(iii) $W = \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x+2y+z=0 \\ 2x+y+3z=0 \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x = -2y-z \\ 2(-2y-z)+y+3z=0 \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x = -2y-z \\ -3y+z=0 \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x = -2y-z \\ z=-3y \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x = -2y-3y=-5y \\ z=-3y \end{matrix} \right\}$

$= \{(-5y, y, -3y)\} \quad y \in \mathbb{R}$

$= \{y(-5, 1, -3)\}$

$= L(S) \quad$ where $S = \{(-5, 1, -3)\}$

$\Rightarrow$ S generates W

Now S contain only one vector

$\Rightarrow$ S is Linearly independent

$\therefore$ S generates W and S is linearly indepe

$\Rightarrow$ S is a basis of W.

Now dim (W) = No of element in basis i.e in S

$= 1$.

Scanned with CamScanner

Rank and nullity theorem of matrix $A_{m\times n}$:

Let A be any $m\times n$ real matrix. Let null (A) and rank (A) be respectively the nullity and rank of A. Then

$$\text{rank}(A) + \text{null}(A) = n = \text{number of columns of } A.$$

3) (i) ~~Example~~ Verify rank-nullity theorem the matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{pmatrix}.$$

Sol^{n}: $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{pmatrix}$

$$\xrightarrow[R_3+R_1]{R_2-3R_1} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & -1 & 7 & 7 \end{pmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 9 & 10 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Rank (A) = no of non zero rows in row-~~reduced~~ echelon form $= 2$.

Null Space $= N(A) = \{X \mid AX = 0\}$

$= \{(x_1, x_2, x_3, x_4) \mid BX = 0\}$ $(\because A \cong B)$

$= \left\{(x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{l} x_1 + 9x_3 + 10x_4 = 0 \\ x_2 - 7x_3 - 7x_4 = 0 \end{array}\right\}.$

$= \left\{(x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{l} x_1 = -9x_3 - 10x_4 \\ x_2 = +7x_3 + 7x_4 \end{array}\right\}$

$N(A) = \{(-9x_3 - 10x_4, 7x_3 + 7x_4, x_3, x_4)\}$

$= \{x_3(-9, 7, 1, 0) + x_4(-10, 7, 0, 1)\}$

$= L(S)$ where $S = \{(-9, 7, 1, 0), (-10, 7, 0, 1)\}$

$\hookrightarrow$ ①

Again $c_1(-9, 7, 1, 0) + c_2(-10, 7, 0, 1) = 0$

$\Rightarrow c_1 = c_2 = 0$

$\Rightarrow$ S is linearly independent $\to$ ②

From ①, ② we get, S is a basis of $N(A)$

$\therefore$ nullity of A = dim $(N(A))$

= no of element in basis S

= 2.

$\therefore$ Rank (A) = 2

Null (A) = 2.

$\therefore$ Rank(A) + Null (A) = 2+2 = 4 = no of column of A.

3) (ii) $A = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & 6 \end{pmatrix}$

$\xrightarrow{R_2 + 2R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 12 \end{pmatrix} = B.$

$\therefore$ Rank of $A = 2$.

Null space $= \mathcal{N}(A) = \{ X \in \mathbb{R}^3 \mid AX = 0 \}$

$= \{ (x,y,z) \in \mathbb{R}^3 \mid BX = 0 \}$ $(\because A \simeq B)$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 12 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{matrix} x - 2y + 3z = 0 \\ 12z = 0 \end{matrix} \right\}$

$= \left\{ (x,y,z) \in \mathbb{R}^3 \mid \begin{matrix} z = 0 \\ x = 2y \end{matrix} \right\}$

$= \{ (2y, y, 0) \}$ $y \in \mathbb{R}$

$= \{ y(2,1,0) \}$ $y \in F\ \mathbb{R}$

$= L(S)$ where $S = \{(2,1,0)\}$

$\Rightarrow$ $S = \{(2,1,0)\}$ generates $\mathcal{N}(A)$

Now S contain only one vector

$\Rightarrow$ S is L.I

$\therefore$ S is a bases of $\mathcal{N}(A)$

$\therefore$ $\dim(\mathcal{N}(A))$ = no of ~~...~~ vectors in basis i.e in S

$= 1$

$\therefore$ Rank of A + dim of null space $= 2 + 1 = 3$

= no of columns in A