Lecture 7

Class Note

Cartesian Product of sets:

let A and B be two-non-empty sets. The cartesian product of A and B, denoted by $A \times B$, is the set defined by

$$A \times B = \{ (a,b) : a \in A,\ b \in B \}$$

Mapping: Let A and B be two non-empty sets. A mapping 'f' from A to B is a rule that assigns to each element x of A a unique element y in B.

$$'f' : A \to B$$

Internal composition:

let V be a non-empty set. A mapping '$\oplus$': $V \times V \to V$ is said to be an internal composition. i.e α, β (be any two elements in V), we get result $\alpha \oplus \beta \in V$.

External composition:

let F and V be two non-empty sets. A mapping '$\odot$': $F \times V \to V$ is said to be an external composition of F with V. i.e $c \in F$, $\alpha \in V$, we get result $c \odot \alpha \in V$.

Vector Space: A non-empty set V is said to to form a vector space over a field F if (i) there is a internal operation $\oplus$ on V called 'addition', satisfying the conditions.

(V1). $\alpha \oplus \beta \in V \quad \forall\ \alpha, \beta \in V$

(V2) $\alpha \oplus \beta = \beta \oplus \alpha \quad \forall\ \alpha, \beta \in V$

(V3) $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma) \quad \forall\ \alpha, \beta, \gamma \in V$

(V4.) $\exists$ an element θ in V such that

$$\alpha \oplus \theta = \alpha \quad \forall\ \alpha \in V$$

(V5) For each α in V $\exists$ $-\alpha$ in V

s.t. $\alpha \oplus (-\alpha) = \theta$.

and (ii) there is an external composition of F with V, called 'multiplication by an elements of F' satisfying the conditions.

(V6.) $c \odot \alpha \in V \quad \forall\ c \in F\ \&\ \alpha \in V$

(V7) $c \odot (d \odot \alpha) = (c.d) \odot \alpha \quad \forall\ c, d \in F\ \&\ \alpha \in V$

(V8) $c \odot (\alpha \oplus \beta) = (c \odot \alpha) \oplus (c \odot \beta) \quad \forall\ c \in F$, $\alpha \in V$

(V9) $(c+d) \odot \alpha = (c \odot \alpha) \oplus (d \odot \alpha) \quad \forall\ c, d \in F\ \&\ \alpha \in V$

(V10) $1 \odot \alpha = \alpha \quad \forall\ \alpha \in V\ \&\ 1 \in F$

1 being the identity element in F.

Scanned with CamScanner

Example: (i) Consider the set $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ where R is set of all real number.

Now we define '$\oplus$' and '$\odot$' by the rule.

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $c \odot (x_1, y_1) = (cx_1, cy_1)$ where $c \in \mathbb{R}$.

Then we get if $\alpha = (x_1, y_1) \in \mathbb{R}^2$

and $\beta = (x_2, y_2) \in \mathbb{R}^2$

then $\alpha \oplus \beta = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$

$\Rightarrow$ $\oplus$ is an internal composition in $\mathbb{R}$.

Again $c \odot \alpha = (cx_1, cy_1) \in \mathbb{R}^2$ $c \in \mathbb{R}$.

$\Rightarrow$ $\odot$ is an external composition in $\mathbb{R}^2$

Note: Here $\oplus$ is an usual adition in $\mathbb{R}^2$

and $\odot$ is an usual multiplication in $\mathbb{R}^2$

- The element of V are called vectors.
- The element of F are called scalars.
- In particular, V is said to be real vector space (or a complex vector space) if the F be $\mathbb{R}$ (or $\mathbb{C}$).

<u>Remark</u>: $\oplus$ and $\odot$ are nothing but external and internal composition in V over F

~~When~~ For convenience we shall use the symbol '$+$' and '$\cdot$'

Thus if $\alpha, \beta \in V$, then <u>$\alpha + \beta$ stands for $\alpha \oplus \beta$.</u>

and if $\alpha \in V$, $c \in F$, then <u>$c.\alpha$ stands for $c \odot \alpha$</u>

Example of Vector Space:

1. Real Vector space $\mathbb{R}^n$:

let $V = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$.

let, $\alpha = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$

& $\beta = (y_1, y_2, \ldots y_n) \in \mathbb{R}^n$

then $\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

and $c \cdot \alpha = (cx_1, cx_2, \ldots, cx_n)$

Here, '+' is an internal composition on $V = \mathbb{R}^n$

and '.' is an external " on V over $\mathbb{R}$ ($V = \mathbb{R}^n$)

We shall show $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$.

V1. let $\alpha, \beta \in \mathbb{R}^n$

then $\alpha + \beta \in \mathbb{R}^n$ ($\because x_i + y_i \in \mathbb{R}$ for $x_i, y_i \in \mathbb{R}$)

V2. $\alpha + \beta = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

$\beta + \alpha = (y_1 + x_1, y_2 + x_2, \ldots, y_n + x_n)$

$\Rightarrow \alpha + \beta = \beta + \alpha$.

$\gamma = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$

V3 $(\alpha + \beta) + \gamma = (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_n + y_n + z_n)$

& $\alpha + (\beta + \gamma) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_n + y_n + z_n)$

$\Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

V4 $\theta = (0, 0, \ldots, 0) \in \mathbb{R}^n$

because $\alpha + \theta = (x_1 + 0, x_2 + 0, \ldots, x_n + 0)$

$= (x_1, x_2, \ldots, x_n)$

$= \alpha.$

V5 if $\alpha \in \mathbb{R}^n$ then $-\alpha = (-x_1, -y_1, \ldots, -z_1) \in \mathbb{R}^n$

Now $\alpha + (-\alpha) = (x_1 - x_1, y_1 - y_1, \ldots, z_1 - z_1)$

$= (0, 0, \ldots, 0) = \theta.$

V6 $c\alpha = (cx_1, cx_2, \ldots, cx_n) \in \mathbb{R}^n$

V7 $c(d\alpha) = c(dx_1, dx_2, \ldots, dx_n)$

$= (cdx_1\ cdx_2 \ldots, cdx_n)$

$= cd(x_1, x_2, \ldots, x_n) = cd(\alpha)$

V8 $c(\alpha + \beta) = c(x_1 + y_1, \ldots, x_n + y_n)$

$= (cx_1 + cy_1, \ldots, cx_n + cy_n)$

$= (cx_1, cx_2, \ldots, cx_n) + (cy_1, cy_2, \ldots cy_n)$

$= c\alpha + c\beta.$

V9 $(c+d)\alpha = (cx_1 + dx_1, cx_2 + dx_2, \ldots, cx_n + dx_n)$

$= c(x_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n)$

$= c\alpha + d\alpha$

V10 $1\alpha = 1(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n)$

$\Rightarrow$ $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$.