Ordinary Differential Equations

(Lecture-4)

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Learning Outcome of the Lecture

We learn

- Equations reducible to Separable Equations
  - Homogeneous Equations

Equations reducible to Separable Equations

A class of differential equations can be reduced to separable equations by using change of variables.

Definition (Homogeneous Function)

A function F is called homogeneous function of degree n if

$$F(tx, ty) = t^n F(x, y)$$

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Example:

(*i*)
$$F(x, y) = x^2 + y^2$$
$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2 = t^2(x^2 + y^2) = t^2F(x, y)$$

So given function is a homogeneous function of degree 2.

(ii) $F(x, y) = y + x\cos^2\left(\frac{y}{x}\right)$ is homogeneous of degree 1.

Homogenous Equations

Definition

The first order DE

$$M(x,y)dx + N(x,y)dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{dy}{dx} = f(x,y)$$

there exists a function g such that $f(x,y)$ can be expressed in the form $g(y/x)$.

Example-1:

$$(x^2 - 3y^2)dx + 2xydy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3}{2}\frac{y}{x} - \frac{1}{2}\frac{1}{(y/x)} = g(y/x)$$

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Homogenous Equations

Definition

The first order ODE

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is called homogeneous if M and N are homogeneous of equal degree.

Example:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Homogeneous ODE's - Reduction to Variable Separable Form

Definition

If

$$M(x,y)dx + N(x,y)dy = 0$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the DE into a separable equation in variables v and x.

Examples: Homogeneous ODE's

Example-1: Solve the ODE:

$$(y^2 - x^2)\frac{dy}{dx} + 2xy = 0.$$

Solution: Put $y = vx$. Thus, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting this in the given ODE, we get:

$$(v^2x^2 - x^2)(v + x\frac{dv}{dx}) + 2x^2v = 0.$$

For $x \neq 0$,

$$(v^2 - 1)v + (v^2 - 1)x\frac{dv}{dx} + 2v = 0,$$

$$(v^3 + v) + (v^2 - 1)x\frac{dv}{dx} = 0.$$

Thus, we have separable ODE:

$$\frac{(v^2 - 1)}{v(v^2 + 1)}dv + \frac{dx}{x} = 0.$$

Examples: Homogeneous ODE's

Integrating, we get:

$$\int \left(\frac{2v}{v^2+1} - \frac{1}{v} \right) dv + \int \frac{dx}{x} = \ln|c_0|$$

$$\ln(v^2+1) - \ln|v| + \ln|x| = \ln|c_0|$$

$$\Rightarrow \ln \frac{(v^2+1)x}{v} = \ln|c_0| \Rightarrow \frac{(v^2+1)x}{v} = c_0,$$

$$x^2 + y^2 = c_0 y.$$

If we choose $c_0 = 2c$, then,

$$x^2 + (y-c)^2 = c^2.$$

Find the curve through the point $(1, 1)$ in the xy-plane having at each of its points, the slope $-\frac{y}{x}$.

Solution: Slope of the curve is $-\frac{y}{x}$,

$$i.e., \frac{dy}{dx} = -\frac{y}{x}$$

Solving this we get,

$$y = \frac{c}{x}$$

Curve (particular solution of ODE) passes through the the point $(1, 1)$, i.e., $y(1) = 1 \Rightarrow c = 1$.
Hence the particular solution for the above problem is

$$y = \frac{1}{x}$$