Lecture - 15

Class Note

Eigen Value of a matrix :

let A be a $n \times n$ matrix. The scalar λ is called an eigen value of a matrix $A_{n \times n}$, if there exists a non-zero vector X such that

$$AX = \lambda X \quad , \quad X \neq 0$$

Eigen vector of a matrix : let A be a $n \times n$ matrix. The non-zero vector X is said to be an eigen vector of a matrix $A_{n \times n}$, if there exists a scalar λ such that

$$AX = \lambda X \quad , \quad X \neq 0$$

Eigen value Problem : (EVP) Given a matrix $A_{n \times n}$, find its eigen values and eigen vectors corresponding to them.

Finding the eigen values :

Rewrite the EVP as

$$AX - \lambda X = 0 \quad \text{where } X \neq 0$$

$$\Rightarrow (A - \lambda I_n) X = 0$$

This is a homogeneous equations.

$\therefore$ A non-zero solution exit if

$$\det (A - \lambda I_n) = 0$$

$$\Rightarrow |A - \lambda I_n| = 0$$

Let $A = (a_{ij})_{n\times n}$

then $|A - \lambda I_n| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$

$= c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n$

$\therefore |A - \lambda I_n|$ is a polynomial of degree n and $|A - \lambda I_n|$ is called the characteristic polynomial of the matrix A.

and $|A - \lambda I_n| = 0$ is said to be the characteristic equation of the matrix A.

characteristic roots / eigenvalues of the matrix

The roots of the characteristic equation $|A - \lambda I_n| = 0$ are called the characteristic roots or the eigenvalues of A.

Ex: Find the eigenvalues and eigenvectors of the matrix $A_{2\times2}=\begin{pmatrix}1 & 3\\ 4 & 5\end{pmatrix}$

Sol^n:
$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix}1-\lambda & 3\\ 4 & 5-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(5-\lambda)-12 = 0$$

$$\Rightarrow 5-5\lambda-\lambda+\lambda^2-12 = 0$$

$$\Rightarrow \lambda^2-6\lambda-7 = 0$$

$$\Rightarrow \lambda^2-7\lambda+\lambda-7 = 0$$

$$\Rightarrow \lambda(\lambda-7)+(\lambda-7) = 0$$

$$\Rightarrow (\lambda-7)(\lambda+1) = 0$$

$\therefore$ $\lambda = 7, -1$

$\therefore$ The eigen values of A are -1 and 7

Let $X=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ be an eigen vector corresponding to -1.

$$\therefore \left\{X=\begin{pmatrix}x_1\\ x_2\end{pmatrix} \middle| AX=-1X\right\} \quad (\because \lambda=-1)$$

$$\Rightarrow \left\{X=\begin{pmatrix}x_1\\ x_2\end{pmatrix} \middle| (A+1I)X=0\right\}$$

$$\Rightarrow \left\{X=\begin{pmatrix}x_1\\ x_2\end{pmatrix} \middle| \begin{pmatrix}1+1 & 3\\ 4 & 5+1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \begin{matrix} 2x_1 + 3x_2 = 0 \\ 4x_1 + 6x_2 = 0 \end{matrix} \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \begin{matrix} 2x_1 + 3x_2 = 0 \\ 2(2x_1 + 3x_2) = 0 \end{matrix} \right\}.$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; 2x_1 + 3x_2 = 0 \right\}.$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 = -\frac{3}{2} x_2 \right\}.$$

$$\Rightarrow \left\{ X = \begin{pmatrix} -\frac{3}{2} x_2 \\ x_2 \end{pmatrix} \right\}.$$

$$\Rightarrow \left\{ X = x_2 \begin{pmatrix} -\frac{3}{2} \\ 1 \end{pmatrix} \right\} \quad \text{where } x_2 \neq 0$$

$$\Rightarrow \cancel{X = L(S), \text{ where } S = \left\{ \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\}}$$

let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to eigen value 7.

$$\therefore \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; AX = 7X \right\}.$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; (A - 7I_2)\,X = 0 \right\}.$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \begin{pmatrix} 1-7 & 3 \\ 4 & 5-7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \begin{matrix} -6x_1 + 3x_2 = 0 \\ 4x_1 - 2x_2 = 0 \end{matrix} \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; \begin{matrix} -6(x_1 - \frac{1}{2}x_2) = 0 \\ 4(x_1 - \frac{1}{2}x_2) = 0 \end{matrix} \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 - \frac{1}{2}x_2 = 0 \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 = \frac{1}{2}x_2 \right\}$$

$$\Rightarrow \left\{ X = \begin{pmatrix} \frac{1}{2}x_2 \\ x_2 \end{pmatrix} \right\}$$

$$\Rightarrow \left\{ X = x_2 \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right\} \quad x_2 \neq 0.$$