Tutorial Solution – 04

Q1) Let $P(n)$ be the statement that there is a survivor whenever $2n + 1$ people stand in a yard at distinct mutual distances and each person throws a pie at their nearest neighbour. To prove this result, we will show that $P(n)$ is true for all positive integers n. This follows because as n runs through all positive integers, $2n + 1$ runs through all odd integers greater than or equal to 3. Note that one person cannot engage in a pie fight because there is no one else to throw the pie at.

BASIS STEP: When $n = 1$, there are $2n + 1 = 3$ people in the pie fight. Of the three people, suppose that the closest pair are A and B, and C is the third person. Because distances between pairs of people are different, the distance between A and C and the distance between B and C are both different from, and greater than, the distance between A and B. It follows that A and B throw pies at each other, while C throws a pie at either A or B, whichever is closer. Hence, C is not hit by a pie. This shows that at least one of the three people is not hit by a pie, completing the basis step.

INDUCTIVE STEP: For the inductive step, assume that $P(k)$ is true. That is, assume that there is at least one survivor whenever $2k + 1$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbour. We must show that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that there is at least one survivor whenever $2(k + 1) + 1 = 2k + 3$ people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbour, is also true.

So, suppose that we have $2(k + 1) + 1 = 2k + 3$ people in a yard with distinct distance between pairs of people. Let A and B be the closest pair of people in this group of 2k + 3 people. When each person throws a pie at the nearest person, A and B throw pies at each other. If someone else throws a pie at either A or B, then altogether at least three pies are thrown at A and B, and at most $(2k + 3) - 3 = 2k$ pies are thrown at the remaining $2k + 1$ people. This guarantees that at least one person is a survivor, for if each of these $2k + 1$ people were hit by at least one pie, a total of at least $2k + 1$ pies would have to be thrown at them. **(The reasoning used in this last step is an example of the pigeonhole principle which will be discussed further).**

To complete the inductive step, suppose no one else throws a pie at either A or B. Besides A and B, there are $2k + 1$ people. Because the distances between pairs of these people are all different, we can use the inductive hypothesis to conclude that there is at least one survivor S when these $2k + 1$ people each throw pies at their nearest neighbours. Furthermore, S is also not hit by either the pie thrown by A or the pie thrown by B because A and B throw their pies at each other, so S is a survivor because S is not hit by any of the pies thrown by these $2k + 3$ people. This completes the inductive step and proves that $P(n)$ is true for all positive integers n. We have completed both the basis step and the inductive step. So, by mathematical induction it follows that $P(n)$ is true for all positive integers n. We conclude that whenever an odd number of people located in a yard at distinct mutual distances each throw a pie at their nearest neighbours, there is at least one survivor.

Note:- This result is false when there are an even number of people.

Q2) Let $P(n)$ be the proposition that a set with n elements has 2^n subsets.

Basis step :- $P(0)$ is true, because a set with zero elements, the empty set has exactly $2^0 = 1$ subset namely itself.

Inductive Step :- For the inductive hypothesis, we assume $P(k)$ is true for ~~all~~ the non-negative integer k, ie we assume that every set with k elements has 2^k subsets.

Therefore, we need to show that under this assumption. $P(k+1)$ which is the statement that every set with $k+1$ elements has 2^{k+1} subsets, must also be true.

To show this, let T be a set with $k+1$ elements. Then, it is possible to write $T = S \cup \{a\}$, where a is one of the elements of T and $S = T - \{a\}$. (Hence, $|S| = k$)

The subsets of T can be obtained in the following way: For each subset X of S there are exactly two subsets of T, namely X and $X \cup \{a\}$.

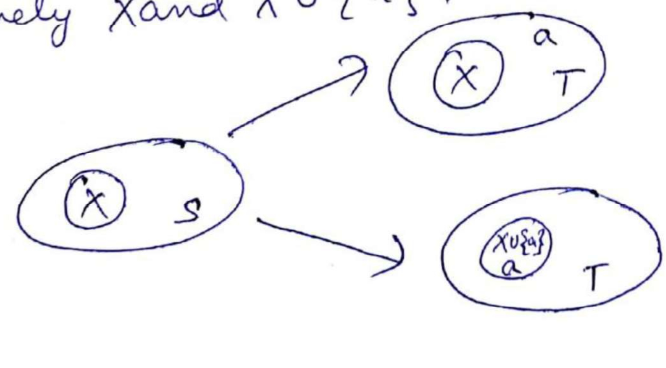

Fig 1.: Generating subsets of a set with $k+1$ elements. Here, $T = S \cup \{a\}$

These constitute all the subsets of T and are all distinct. Because there are 2^k subsets of S, there are $2 \cdot 2^k = 2^{k+1}$ subsets of T. Here we finish the inductive ~~agreement~~ argument.

Because we have completed the basis step and the inductive step, by mathematical induction it follows that $P(n)$ is true for all nonnegative integers n.

Q3) Let $P(n)$ be the statement that $2^n < n!$

Here, the base value is 4

For $n = 4$

$L.H.S = 2^4 = 16$

$R.H.S = 4! = 24$

$16 < 24$, hence $P(4)$ is true.

Let the statement $P(n)$ be true for $n = k$ $(k \geq 4)$,

then, $2^k < k!$

Now, for $n = k+1$

$$2^{k+1} = 2 \cdot 2^k < 2 \cdot k! < (k+1) k! \quad (\because k \geq 4) < (k+1)!$$

$P(n)$ is true for $n = k+1$.

Therefore, by principle of mathematical induction, $P(n)$ is true for all positive integers $n \geq 4$.

Q4) Let $P(n)$ be the statement that $1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$

For $n = 0$,

$L.H.S = 2^0 = 1$

$R.H.S = 2^{0+1} - 1 = 2^1 - 1 = 2 - 1 = 1$

$L.H.S = R.H.S$

Assuming true for $n = k$

$1 + 2 + \dots + 2^k = 2^{k+1} - 1$

Show true for $n = k+1$

$$\underbrace{1 + 2 + \cdots + 2^k}_{} + 2^{k+1} = 2^{k+1+1} - 1$$

$$\Rightarrow \quad 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$$

$$\Rightarrow \quad 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1$$

$$\Rightarrow \quad 2^{k+2} - 1 = 2^{k+2} - 1$$

Hence Proved

Q5) Let P be '$3n+2$ is odd' and Q be 'n is odd'.
We assume P and $\sim Q$ to be true.
Therefore, $\sim Q$ is 'n is even'.
So, $n = 2k$ where k is some integer.

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k+1)$$

Let $3k + 1 = t$

Then, $3n + 2 = 2t$ where t is some integer.
The above equation shows that $3n+2$ is even which $\sim P$.

$\because$ Both P and $\sim P$ becomes true, therefore, by proof of contradiction, we can say that

'If $3n+2$ is odd, then n is odd'.

Hence Proved

Q6) Here $P: x^2+y^2=z^2$ and $Q: x+y \geq z$

We shall assume that P is true and $\sim Q$ is true.

Thus, $x^2+y^2 = z^2$ and $x+y<z$

$x+y<z \Rightarrow (x+y)^2 < z^2$ (since all are non negative real nos)

$\Rightarrow x^2+y^2+2xy < z^2$

$\Rightarrow x^2+y^2 < z^2$ (since $2xy$ is also a non negative real number)

This is a contradiction to the assumption $x^2+y^2=z^2$

thus $x+y<z$ is not true.

Hence, for all non negative real numbers x, y and z

if $x^2+y^2=z^2$, then $x+y \geq z$.

Q7) Suppose that $0 \leq \frac{a+b}{2} < \sqrt{ab}$.

Then, $\left(\frac{a+b}{2}\right)^2 < (\sqrt{ab})^2$

$\Rightarrow \frac{(a+b)^2}{4} < ab$

$\Rightarrow (a+b)^2 < 4ab$

$\Rightarrow a^2+2ab+b^2 < 4ab$

$\Rightarrow a^2+b^2 < 2ab$

$\Rightarrow a^2-2ab+b^2 < 0$

$\Rightarrow (a-b)^2 < 0$

This is a contradiction since no square can be a negative no.

2

Q8) This theorem has the form 'p if and only if q' where p is 'n is odd' and q is 'n^2 is odd'.

To prove this theorem, we need to show that both $p \to q$ and $q \to p$ is true.

(i) To show $p \to q$ is true i.e 'if n is odd then n^2 is odd'.

$\because$ n is an odd integer, therefore

$n = 2k+1$ where k is some integer.

$\Rightarrow (n)^2 = (2k+1)^2$

$\Rightarrow n^2 = 4k^2 + 4k + 1$

$\Rightarrow n^2 = 2(2k^2+2k) + 1$

Let $2k^2+2k$ be t.

$\Rightarrow n^2 = 2t+1$

$\therefore$ n^2 is an odd integer.

(ii) To show $q \to p$ is true i.e 'if n is an integer and n^2 is odd then n is odd.' q: n^2 is odd, p: n is odd.

Let us take a hypothesis statement that n is not odd.

Because every integer is odd or even, this means that n is even.

$\therefore$ we can say $n = 2k-0$ where k is some integer.

To prove the theorem, we need to show that this hypothesis implies the conclusion that n^2 is not odd that is n^2 is even.

Squaring on both sides of eq. (1)

$$(n)^2 = (2k)^2$$

$$\Rightarrow n^2 = 4k^2 = 2(2k^2)$$

Let $2k^2 = t$

$\therefore n^2 = 2t$ So, n^2 is also an even no.

Therefore, we have proved that if n is an integer and n^2 is odd then n is odd.

This is ~~per~~ proof by contraposition.

Note: In proof by contraposition we use the fact that $P \to Q$ is equivalent to its contrapositive $\sim Q \to \sim P$.
So, in order to show that $P \to Q$, we can also prove $\sim Q \to \sim P$

9. Proof: By induction on n. Base
case: n =4. 2 two's, done.
Induction step: suppose the machine can already handle n>= 4 dollars. To produce n dollars, we proceed as follows.
Case 1: The n dollar output contains a five. Then we can replace the five by 3 two's to get n +1 dollars.
Case 2: The n dollar output contains only two's.
Since n >=4, there must be at least 2 two's. Remove 2, and replace them by 1 five.

10.

Let P: Atleast 10 of any 64 days chosen must fall on the same day of the week.

By Method of Contradiction,

Let $\sim P$

ie It is not true that At least 10 of any 64 days chosen must fall on same day of the week.

If there were 9 or fewer days on same day of the week, then we would have $9 \times 7 = \underline{63}$ (Atmost 63 days)

But we have chosen 64 days

$\therefore$ There is a Contradiction

$\therefore \underline{P}$

11.

Q. If n is a positive integer, then n is odd if and only if $5n+6$ is odd.

$p \to$ n is odd

$q \to$ $5n+6$ is odd

To Prove $p \leftrightarrow q$

i.e $p \to q$ and $q \to p$

1) $p \to q$

Let p is true

i.e n is odd

$n = 2k+1$ [By def^n of odd integer]

$5n+6 = 5(2k+1)+6$

$= 10k+11 = 10k+10+1$

$= 2(5k+5)+1$

$\Rightarrow$ $5n+6$ is odd

$\therefore$ $p \to q$ [By Direct Method] (1)

2) $q \to p$

(or) $\sim p \to \sim q$ [Taking Contrapositive]

$\sim p \to$ n is even

$\sim q \to$ $5n+6$ is even

$n = 2k$ [By def^n of even integer]

$5n+6 = 5(2k)+6 \Rightarrow 10k+6 = 2(5k+3) =$ even

12. Let us assume, that n is an even number. Then we can write $n = 2k$, where $k \in Z$.

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

This $\Rightarrow$ n^2 is an even number.

Hence, Proved using Direct Proof.

13. Using direct proof, to show that if

$a|b$ and $a|c$ then $a|b+c$ where a, b and c

are integers such that $a \neq 0$

Since $a|b$, we know that there is an integer k

such that $\boxed{b = ak}$.

Similarly, since $a|c$, we know that there is an

integer m such that $\boxed{c = am}$.

Therefore, $\boxed{b+c = ak+am = a(k+m)}$

We know that $\boxed{k+m}$ is an integer, so the

equation shows that $b+c$ is an integer multiple

of a. Therefore, $b+c$ is divisible by a.

Hence Proved using Direct Proof.

r