Problem Set-1

Thursday, March 18, 2021 9:27 AM

$$\bar{A} = 2\hat{i} - \hat{j} + 3\hat{k}$$
$$\bar{B} = \hat{i} + \hat{j} - 2\hat{k}$$

Angle between $\bar{A}$ & $\bar{B}$

$$\theta = \cos^{-1}\left(\frac{\bar{A}\cdot\bar{B}}{\|A\|\|B\|}\right)$$

$$\bar{A}\cdot\bar{B} = \|A\|\|B\|\cos\theta$$

$\cos\theta =$

$$\bar{A}\cdot(\bar{B}\times\bar{A}) = \bar{B}\cdot(\underbrace{\bar{A}\times\bar{A}}_{0}) = 0$$

$$\bar{A}\cdot(\bar{B}\times\bar{C}) = \bar{B}\cdot(\bar{A}\times\bar{C}) = \bar{C}\cdot(\bar{B}\times\bar{A})$$

1. Find out a unit vector which lies in the xy plane and which is perpendicular to A of previous problem

$$\bar{A} = 2\hat{i} - \hat{j} + 3\hat{k}$$
$$C = x\hat{i} + y\hat{j}$$
$$\bar{A}\cdot\bar{C} = 0$$
$$\Rightarrow 2x - y = 0$$
$$\bar{C} = x(\hat{i} + 2\hat{j})$$

$$\bar{C} = C\,\underline{\underline{\hat{C}}} \rightarrow \frac{\bar{C}}{\|C\|} = \frac{x(\hat{i}+2\hat{j})}{\sqrt{x^2+4x^2}}$$

$$= \frac{1}{\sqrt{5}}(\hat{i} + 2\hat{j})$$

$$\cancel{A^2}+\cancel{B^2}+2\bar{A}\cdot\bar{B} = \cancel{A^2}+\cancel{B^2}-2\bar{A}\cdot\bar{B}$$

$$4\bar{A}\cdot\bar{B}=0$$

$$\bar{A}\ \bar{B}=0 \Rightarrow \bar{A}\perp\bar{B}$$

$$\bar{j}(t) = a\hat{i}+bt\hat{j}+ct^2\hat{k} = \frac{d\bar{a}}{dt}$$

$$j_x = \frac{da_x}{dt} = a$$

$$a_x = at + C_1$$

$$j_y = \frac{da_y}{dt} = bt$$

$$a_y = \frac{bt^2}{2} + C_2$$

$$j_z = \frac{da_z}{dt} = ct^2$$

$$a_z = \frac{c}{3}t^3 + C_3$$

$$C_1 = C_2 = C_3 = 0$$

$$\bar{a}(t) = at\hat{i} + \frac{b}{2}t^2\hat{j} + \frac{c}{3}t^3\hat{k}$$

$$\bar{a}(t) = -\omega^2\bar{r}(t)$$

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 x \rightarrow \frac{dv_x}{dt} = -\omega^2 x$$

$$a_y = \frac{d^2y}{dt^2} = -\omega^2 y$$

$$\bar{a} = \frac{d\bar{v}}{dt} = -\omega^2\bar{r}(t)$$

$$\left(\frac{d^2x}{dt^2} = -\omega^2 x\right) \times 2\frac{dx}{dt}$$

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -\omega^2\left(2x\frac{dx}{dt}\right) \rightarrow \frac{d}{dt}(x^2)$$

$$2x\frac{dx}{dt}$$

$$\cancel{\frac{d}{dt}}\left(\frac{dx}{dt}\right)^2 = -\omega^2\cancel{\frac{d}{dt}}(x^2)$$

$$\cancel{\frac{d}{dt}}\left(\frac{dx}{dt}\right)^2 = -\omega^2\cancel{\frac{d}{dt}}\left(x^2\right) \qquad 2x\frac{dx}{dt}$$

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + C^2$$

$$v = \frac{dx}{dt} = \sqrt{C^2 - \omega^2 x^2} = f(x)$$

$$\frac{dx}{f(x)} = dt$$

$$\frac{dx}{\omega\sqrt{\alpha^2 - x^2}} \Rightarrow dt$$

$$\frac{dx}{\sqrt{\alpha^2 - x^2}} = \omega\, dt$$

$$\sin^{-1}\left(\frac{x}{\alpha}\right) = \omega t + C_1$$

$$x = \alpha \sin(\omega t + C_1)$$

$$= A\sin\omega t + B\cos\omega t$$

$$y = C\sin\omega t + D\cos\omega t$$

$$m\ddot{x} = -kx$$

$$x(0) = 0, \quad y(0) = a, \quad v_x(0) = a\omega, \quad v_y(0) = 0$$

$$\vec{r}(0) = a\hat{j}$$

$$x(0)\hat{i} + y(0)\hat{j} = a\hat{j}$$

$$v_x(0) = A\omega\cos\omega t - B\omega\sin\omega t$$

$$v_y(0) = C\omega\cos\omega t - D\omega\sin\omega t$$

$$x(0) = 0 \Rightarrow \underline{B = 0}$$

$$x(t) = A\sin\omega t + B\cos\omega t$$

$$x(t=0) = A\sin 0 + B\cos 0$$

$$= 0 + B = 0$$

$$v_x(t) = A\omega\cos\omega t - \underset{\uparrow 0}{B}\omega\cancel{\sin\omega t}$$

$$v_x(t=0) = A\omega \times 1 = a\omega$$

$$A = a$$

$$y(t) = C\sin\omega t + D\cos\omega t$$

$$y(0) = a = D$$

$$C = 0$$

$$x(t) = a\sin\omega t$$

$$y(t) = a\cos\omega t$$

$$\bar{r}(t) = a(\sin\omega t\,\hat{i} + \cos\omega t\,\hat{j})$$

Problem Set-II

Thursday, April 8, 2021 10:45 AM

PQ6, $\bar{F}$, $\bar{\nabla}\times\bar{F}=0$

$$\bar{F}=(y^2z^3-6xz^2)\hat{i}+2xyz^3\hat{j}+(3xy^2z^2-6x^2z)\hat{k}$$

$\Rightarrow$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3-6xz^2 & 2xyz^3 & 3xy^2z^2-6x^2z \end{vmatrix}$$

$$= (6xyz^2-6xyz^2)\hat{i}+(3y^2z^2-12xz-3y^2z^2+12xz)\hat{j} + (2yz^3-2yz^3)\hat{k} = 0$$

$$\hat{r}\times\hat{\theta}=(\cos\theta\,\hat{i}+\sin\theta\,\hat{j})\times(-\sin\theta\,\hat{i}+\cos\theta\,\hat{j})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = (\cos^2\theta+\sin^2\theta)\hat{k}$$

$$=\hat{k}$$

$\hat{\theta}\times\hat{k}=\hat{r}$

$\hat{r},\hat{\theta},\hat{k}\rightarrow$ cylindrical co-ordinate system

$\hat{i}\times\hat{j}=\hat{k}$

$\hat{r}\times\hat{\theta}=\hat{k}$

$\hat{\theta}\times\hat{k}=\hat{r}$

$\hat{k}\times\hat{r}=\hat{\theta}$

$\hat{r},\hat{\theta},\hat{\varphi}$ $\quad \hat{r}\times\hat{\theta}=\hat{\varphi}$

Q8

$\bar{r} = a$

$\dot{r} = 0, \quad \ddot{r} = 0$

$\dot{\theta} = \omega_0 + \alpha t$

$\ddot{\theta} = \alpha$

$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = a(\omega_0 + \alpha t)\hat{\theta}$

$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= -a(\omega_0 + \alpha t)^2\hat{r} + a\alpha\hat{\theta}$

9// $\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

10// $\dot{\theta} = \omega, \quad \ddot{\theta} = 0$

$r(t) = r_0 e^{\beta t}$

$\dot{r} = r_0 \beta e^{\beta t}$

$\ddot{r} = r_0 \beta^2 e^{\beta t}$

$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = r_0\beta e^{\beta t}\hat{r} + r_0 e^{\beta t}\omega\hat{\theta}$

$= r_0 e^{\beta t}(\beta\hat{r} + \omega\hat{\theta})$

$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$= (\underbrace{r_0\beta^2 e^{\beta t} - r_0 e^{\beta t}\omega^2}_{\to 0})\hat{r}$

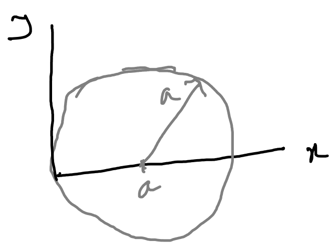
$+ (2r_0\beta e^{\beta t}\omega\hat{\theta})$

$\beta^2 = \omega^2 \to \beta = \pm\omega$

$x = r\cos\theta$

11//

$r^2 = x^2 + y^2$



$(x-a)^2 + y^2 = a^2$

$x^2 + \cancel{a^2} - 2ax + y^2 = \cancel{a^2}$

$x^2 + y^2 = 2ax$

$r^{\cancel{2}} = 2a\cancel{r}\cos\theta$

$r = 2a\cos\theta$

12// $r = A\theta, \quad A = 1/\pi, \quad \theta = \alpha t^2$

$r = A\alpha t^2 \qquad \dot{\theta} = 2\alpha t$

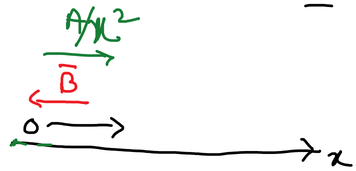
$$\dot{r} = 2A\alpha t \qquad \dot{\theta} = 2\alpha$$
$$\ddot{r} = 2A\alpha$$

$$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$
$$= \underbrace{2A\alpha(1-2\alpha^2 t^4)}\hat{r} + \underbrace{6A\alpha^2 t^2}\hat{\theta}$$

$$\cancel{2}\cancel{A}\cancel{\alpha}(1-2\alpha^2 t^4) = \overset{3}{\cancel{6}}\cancel{A}\cancel{\alpha}^2 t^2$$
$$2\alpha^2 t^4 + 3\alpha t^2 - 1 = 0$$
$$t^2 = \frac{-3 \pm \sqrt{17}}{4\alpha}$$

$$\theta = \alpha t^2$$

14

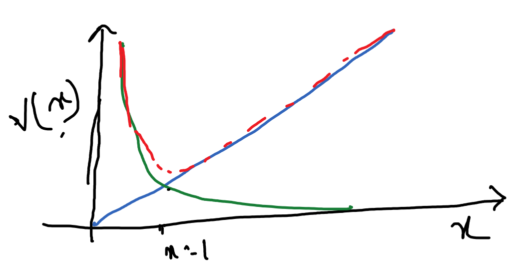


$$\bar{F} = \left(-B + \frac{A}{x^2}\right)\hat{i}, \quad x > 0$$

$$-\frac{dV}{dx} = -B + \frac{A}{x^2}$$
$$V(x) = Bx + \frac{A}{x} + C$$

C is a constant, and I can set

$C = 0$



$V(x) = Bx + \frac{A}{x}$

$B = A = 1$

$x = 0.1, 0.2, 0.5, 1, 2, 5$

$V(x) = 0.1 + 10$
$= 0.2 + 5$
$= 0.5 + 2$
$= 1 + 1$
$= 2 + 0.5$
$= 5 + 0.2$

$x < 1$, $\frac{A}{x}$ dominate

$x > 1$, Bx dominate

$$\frac{dV}{dx} = 0 \rightsquigarrow B - \frac{A}{x^2} = 0$$

$$\Rightarrow x_0 = \sqrt{\frac{A}{B}}$$

$$V(x)\Big|_{x=x_0} = + Bx\Big|_{x=x_0} + \frac{A}{x}\Big|_{x=x_0}$$

$$= + B \times \sqrt{\frac{A}{B}} + A\sqrt{\frac{B}{A}}$$

$$= 2\sqrt{AB}$$

15

$$V = -\frac{GMm}{r} = -\frac{GMm}{\sqrt{x^2+y^2+z^2}}$$

$$\bar{F}(\bar{r}) = -\bar{\nabla} V(r) = GMm\left(\frac{\partial}{\partial x}\frac{\hat{i}}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial y}\frac{\hat{j}}{\sqrt{x^2+y^2+z^2}} + \frac{\partial}{\partial z}\frac{\hat{k}}{\sqrt{x^2+y^2+z^2}}\right)$$

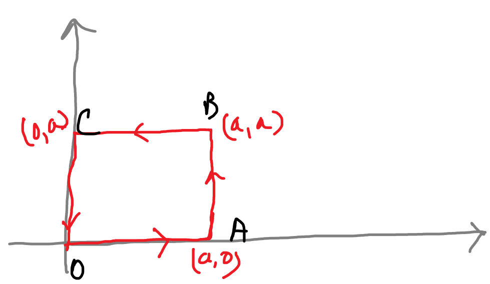
$$= -GMm\left(\frac{\cancel{2}x\,\hat{i}}{\cancel{2}(x^2+y^2+z^2)^{3/2}} + \frac{y\,\hat{j}}{(x^2+y^2+z^2)^{3/2}} + \frac{z\hat{k}}{(\;)}\right)$$

$$= -GMm\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}\right) = -\frac{GMm\,\bar{r}}{r^3}$$

$$= -\frac{GMm}{r^2}\hat{r}$$

$$\bar{\nabla}\times\bar{F} = 0$$

16 $\bar{F} = A(y^2\hat{i} + 2x^2\hat{j})$



$$\oint \bar{F}\cdot d\bar{l} \qquad \int(\bar{\nabla}\times\bar{F})\cdot d\bar{S}$$

$\oint \bar{F}\cdot d\bar{l}$, $\int (\bar{\nabla}\times\bar{F})\cdot d\bar{S}$

$\Updownarrow$

$\int_C \bar{F}\cdot d\bar{l}$

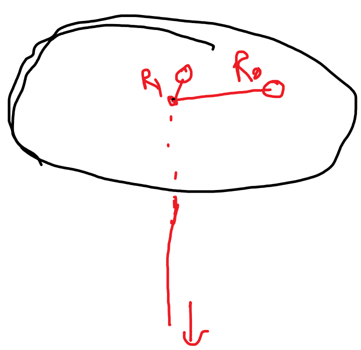
$$\oint \bar{F}\cdot d\bar{r} = \int_{OA} \bar{F}\, d\bar{r} + \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\, d\bar{r} + \int_{CO} \bar{F}\cdot d\bar{r}$$

0 F is conservative

$\bar{\nabla}\times\bar{F} \neq 0$, $W = \oint \bar{F}\cdot d\bar{r}$

$\int_S (\bar{\nabla}\times\bar{F})\cdot d\bar{S}$

13



$a_r = \ddot{r} - r\dot{\theta}^2$

$\ddot{r} \simeq 0$

$a_r = -r\dot{\theta}^2 = -r\omega^2$

$F = T = -m\omega^2 r$

$W = \int_{R_0}^{R_1} F\, dr$