Department of Mathematics
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EMAT102L: Ordinary Differential Equations
Tutorial Sheet-3 Solutions

1) Consider the linear differential equation

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0.$$

(a) Show that x^3 and $|x^3|$ are two linearly independent solutions of the differential equation on $x \in (-\infty, \infty)$.
Solution: Suppose $c_1, c_2 \in \mathbb{R}$ and

$$c_1y_1(x) + c_2y_2(x) = 0, \qquad \forall\, x \in (-\infty, \infty),$$

then

$$c_1x^3 + c_2|x^3| = 0, \qquad \forall\, x \in (-\infty, \infty).$$

Now take $x = 1$, we get

$$c_1 + c_2 = 0.$$

For $x = -1$, we get

$$-c_1 + c_2 = 0.$$

Solving these two we get, $c_1 = c_2 = 0$. Hence x^3 and $|x^3|$ are two linearly independent.

(b) x^3 and $|x^3|$ are two linearly independent solutions of the differential equation but $W(x^3, |x^3|) = 0,\ \forall x \in \mathbb{R}$. Does it violate any result? Explain.
Solution: We know that

$$W(y_1, y_2)(x) = \det\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right).$$

Thus,

$$W(x^3, |x^3|) = \begin{cases} x^3(3x^2) - 3x^2(x^3) & x \geq 0, \\ -x^3(3x^2) - x^3(-3x^2) & x < 0. \end{cases}$$

$$\Rightarrow W(x^3, |x^3|) = 0,\ \forall\, x \in \mathbb{R}.$$

Note that $a_0(x) = x^2$ is zero at $x = 0$. The theorem stating that wronskian of linearly independent solutions is non-zero, is not applicable here.

(c) x^2 and x^3 are also two linearly independent solutions of the differential equation. Can we write general solution of the differential equation in terms of these solutions?
Solution: No. None of the linear combinations $c_1x^3 + c_2|x^3|$ or $C_1x^2 + C_2x^3$ is a general solution of the given DE in the interval $(-\infty, \infty)$ because if $C_1x^2+C_2x^3$ is a general solution on the interval $(-\infty, \infty)$ we should be able to find constants C_1 and C_2 such that

$$x^3 + |x^3| = C_1x^2 + C_2x^3 \quad \forall\, x \in (-\infty, \infty), \tag{1}$$

which is not possible. Because if we take $x = 1$ in equation (1), we get $C_1 + C_2 = 2$. Taking $x = -1$ in equation (1), we get $C_1 - C_2 = 0$. Solving these two for C_1 and C_2, we get $C_1 = C_2 = 1$, i.e.,

$$x^3 + |x^3| = x^2 + x^3 \quad \forall\, x \in (-\infty, \infty) \Rightarrow |x^3| = x^2 \quad \forall\, x \in (-\infty, \infty),$$

which is not possible. Similarly we can show that $c_1x^3 + c_2|x^3|$ cannot be a general solution.

2) Use reduction of order method to find the second linearly independent solution of the following differential equations

(a) $(x-1)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0; \quad y_1(x) = x.$
Solution: Let $y_2(x) = y_1(x)v(x)$ is the second linearly independent solution for a suitable function $v(x)$. Then $y_1(x) = x$ implies

$$y_2 = xv \Rightarrow \frac{dy_2}{dx} = x\frac{dv}{dx} + v \Rightarrow \frac{d^2y_2}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}.$$

Substituting the expressions for y_2, $\frac{dy_2}{dx}$ and $\frac{d^2y_2}{dx^2}$ into the given ODE, we get

$$(x-1)\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - x\left(x\frac{dv}{dx} + v\right) + vx = 0$$

$$\Rightarrow x(x-1)\frac{d^2v}{dx^2} + \{2(x-1) - x^2\}\frac{dv}{dx} = 0$$

$$\text{Taking } \frac{dv}{dx} = w \Rightarrow \frac{d^2v}{dx^2} = \frac{dw}{dx}.$$

Using this we get a first order ODE

$$x(x-1)\frac{dw}{dx} + \{2(x-1) - x^2\}w = 0$$

$$\Rightarrow \frac{dw}{w} + \frac{\{2(x-1) - x^2\}}{x(x-1)}dx = 0$$

$$\Rightarrow \frac{dw}{w} + \frac{2}{x}dx - \frac{1}{x-1}dx - dx = 0$$

On integrating,

$$\ln|w| + 2\ln|x| - \ln|x-1| - x = \ln|c| \ \Rightarrow \ \text{for } c = 1,\ w = \frac{e^x(x-1)}{x^2}.$$

$$\int dv = \int \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) dx \ \Rightarrow \ v = \frac{e^x}{x} + c_1,$$

where c_1 is an arbitrary constant. Thus for $c_1 = 0$,

$$y_2 = vx = e^x,$$

is a second linearly independent solution. The general solution is

$$y(x) = Ax + Be^x,$$

where A and B are two arbitrary constants.

(b) $x^2\frac{d^2y}{dx^2} - (2a-1)x\frac{dy}{dx} + a^2y = 0;\ a \neq 0,\ x > 0, \quad y_1(x) = x^a.$
Solution: Proceed like in part (a). Second solution is $y_2 = x^a \log x$.

(c) $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = 0; \quad y_1(x) = x\sin(\log x).$
Solution: Proceed like in part (a). Second solution is $y_2 = -x\cos(\log x)$.

3) Find the second order differential equation corresponding to given linearly independent solutions

(a) $y_1 = \cos 2\pi x$, $y_2 = \sin 2\pi x$.
Solution: The general solution is

$$y(x) = c_1y_1 + c_2y_2 = c_1 \cos 2\pi x + c_2 \sin 2\pi x$$

Clearly the roots of auxiliary equation (A. E.) of the given differential equation are imaginary, i.e., when $m = a \pm ib$, solution is

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx).$$

On comparing $a = 0$ and $b = 2\pi$., i.e., roots are $\pm 2\pi i$. Therefore the A. E. is

$$(m - 2\pi i)(m + 2\pi i) = 0 \Rightarrow m^2 + 4\pi^2 = 0$$

Thus, the differential equation is

$$\frac{d^2y}{dx^2} + 4\pi^2 y = 0.$$

(b) $y_1 = e^{-\sqrt{2}x}$, $y_2 = xe^{-\sqrt{2}x}$.
Solution: The general solution is

$$y(x) = c_1y_1 + c_2y_2 = c_1e^{-\sqrt{2}x} + c_2xe^{-\sqrt{2}x} = (c_1 + c_2x)e^{-\sqrt{2}x}$$

Clearly the roots of A. E. of differential equation are repeated and here $m = -\sqrt{2}$. Therefore the A. E. is

$$(m + \sqrt{2})(m + \sqrt{2}) = 0 \Rightarrow m^2 + 2\sqrt{2}m + 2 = 0.$$

Thus, the differential equation is

$$\frac{d^2y}{dx^2} + 2\sqrt{2}\frac{dy}{dx} + 2y = 0.$$

(c) $y_1 = e^{(-1+i\sqrt{2})x}$, $y_2 = e^{(-1-i\sqrt{2})x}$.
Solution:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0.$$

4) Solve the IVP's

(a) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 3y = 0; \quad y(-1) = e,\ y'(-1) = -\frac{e}{4}.$
Solution: Auxiliary equation is

$$m^2 - 2m - 3 = 0 \Rightarrow (m - 3)(m + 1) = 0 \Rightarrow m = -1,\ 3.$$

Thus, the general solution is

$$y(x) = c_1e^{-x} + c_2e^{3x}.$$

Now use the initial conditions:

$$y(-1) = e \Rightarrow c_1e + c_2e^{-3} = e,$$

and

$$y'(-1) = -\frac{e}{4} \Rightarrow -c_1 e + 3c_2 e^{-3} = \frac{-e}{4}.$$

Solving for c_1 and c_2, we get $c_1 = \frac{13}{16}$ and $c_2 = \frac{3e^4}{16}$. Hence the solution is

$$y(x) = \frac{13}{16} e^{-x} + \frac{3e^4}{16} e^{3x}.$$

(b) $\frac{d^2y}{dx^2} - k^2 y = 0;\ k \neq 0, \quad y(0) = 1,\ y'(0) = 1.$
Solution: Auxiliary equation is

$$m^2 - k^2 = 0 \Rightarrow (m-k)(m+k) = 0 \Rightarrow m = k,\ -k.$$

Thus, the general solution is

$$y(x) = c_1 e^{kx} + c_2 e^{-kx}.$$

Now use the initial conditions:

$$y(0) = 1 \Rightarrow c_1 + c_2 = 1,$$

and

$$y'(0) = 1 \Rightarrow kc_1 - kc_2 = 1.$$

Solving for c_1 and c_2, we get $c_1 = \frac{k+1}{2k}$ and $c_2 = \frac{k-1}{2k}$. Hence the solution is

$$y(x) = \frac{k+1}{2k} e^{kx} + \frac{k-1}{2k} e^{-kx}.$$

(c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0; \quad y(0) = 3,\ y'(0) = -1.$
Solution: Do yourself.

(d) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0; \quad y(0) = 2,\ y'(0) = -3.$
Solution: Do yourself.

(e) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0; \quad y(0) = 1,\ y'(0) = -8,\ y''(0) = -4.$
Solution: Do yourself.

5) Use method of Undetermined Coefficients to find the particular integral of the following differential equations

(a) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x^3 + x + e^{-2x}.$
Solution: The corresponding homogeneous differential equation is

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0.$$

The complementary function is

$$y_c = c_1 e^{2x} + c_2 e^{4x}.$$

The UC set of x^3 is $\{x^3, x^2, x, 1\}$. The UC set of x is $\{x, 1\}$, and the UC set of e^{-2x} is $\{e^{-2x}\}$. Clearly the UC set of x is contained in UC set of x^3, so omit the UC set of x. Now, the choice of particular integral y_p is

$$y_p = Ax^3 + Bx^2 + Cx + D + Ee^{-2x}.$$

Substitute in the given nonhomogeneous differential equation and determine the values of coefficients A, B, C, D and E. The general solution of given ODE is

$$y = y_c + y_p.$$

(b) $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$.
Solution: The corresponding homogeneous differential equation is

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

The A. E. is

$$m^4 + 2m^2 + 1 = 0 \Rightarrow (m^2+1)^2 = 0 \Rightarrow m = \pm i,\ \pm i.$$

The complementary function is

$$y_c = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x.$$

In order to find general solution of given ODE, we must find a particular solution of the given nonhomogeneous ODE. Note that here nonhomogeneous term is a product of two UC functions. The UC set of x^2 is, say $S_1 = \{x^2,\ x,\ 1\}$, and the UC set of $\cos x$ is, say $S_2 = \{\cos x,\ \sin x\}$. Find $S_1 \times S_2$,

$$S_1 \times S_2 = \{x^2\cos x,\ x^2\sin x,\ x\cos x,\ x\sin x,\ \cos x,\ \sin x\}.$$

Now we need to check that whether there is any term in this set which is contained in the complementary function. Clearly $x\cos x$, $x\sin x$, $\cos x$, $\sin x$ are in complementary function. Therefore we need to multiply this set by the lowest power of x so that there is no function which is a solution of corresponding the homogeneous DE. Here we need to multiply by x^2. Now the modified set is

$$x^2 \times S_1 \times S_2 = \{x^4\cos x,\ x^4\sin x,\ x^3\cos x,\ x^3\sin x,\ x^2\cos x,\ x^2\sin x\}.$$

Now, the choice of particular integral y_p is

$$y_p = Ax^4\cos x + Bx^4\sin x + Cx^3\cos x + Dx^3\sin x + Ex^2\cos x + Fx^2\sin x,$$

where A, B, C, D, E and F are undetermined coefficients. Substitute in the given nonhomogeneous differential equation and determine the values of coefficients. The general solution of given ODE is

$$y = y_c + y_p.$$

6) Solve the following non-homogeneous differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = \frac{e^{-3x}}{x^3}$.
Solution: Here we need to use variation of parameters as method of undetermined coefficients is not applicable. The complementary function is

$$y_c = c_1e^{-3x} + c_2xe^{-3x}.$$

Take $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$, then

$$W(y_1, y_2) = e^{-6x}.$$

This implies that

$$v_1 = -\int \frac{f(x)y_2}{a_0(x)W(y_1, y_2)}dx = -\int \frac{e^{-3x}}{x^3} \times \frac{xe^{-3x}}{e^{-6x}}dx = \frac{1}{x},$$

similarly

$$v_2 = \int \frac{f(x)y_1}{a_0(x)W(y_1, y_2)}dx = \int \frac{e^{-3x}}{x^3} \times \frac{e^{-3x}}{e^{-6x}}dx = -\frac{1}{2x^2}.$$

Thus, the particular solution is

$$y_p = v_1y_1 + v_2y_2 = \frac{1}{x} \times e^{-3x} - \frac{1}{2x^2}xe^{-3x} = \frac{e^{-3x}}{2x}.$$

Hence the general solution is

$$y = y_c + y_p = c_1e^{-3x} + c_2xe^{-3x} + \frac{e^{-3x}}{2x}.$$