Lecture 13 and 14

Class Note

Linear transformation: Let V and W be vector spaces over the same field F. A mapping

$$T : V \to W$$

is said to be a linear transformation if it satisfies the following conditions:

1. $T(\alpha + \beta) = T(\alpha) + T(\beta) \; \forall \; \alpha, \beta$ in V.
2. $T(c\alpha) = c\,T(\alpha) \; \forall \; c \in F \;\&\; \forall \; \alpha$ in V

Note: There two conditions can be replaced by the single condition; i.e

$$T(c\alpha + d\beta) = cT(\alpha) + d\,T(\beta)$$

$$\forall \; c, d \in F \;\&\; \forall \; \alpha, \beta \in V$$

Example: Let $T : \mathbb{R}^3 - \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1, x_2, 0) \text{ where } (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Check, is it a linear-transformation or not?

Soln: let $\alpha = (y_1, y_2, y_3) \;\&\; \beta = (z_1, z_2, z_3) \in \mathbb{R}^3$.

$$\therefore c\alpha + d\beta = (cy_1 + dz_1, cy_2 + dz_2, cy_3 + dz_3) \in \mathbb{R}^3.$$

Now $T(c\alpha + d\beta)$

$$= (cy_1 + dz_1, cy_2 + dz_2, 0)$$

$$= c(y_1, y_2, 0) + d(z_1, z_2, 0)$$

$T(c\alpha + d\beta) = cT(\alpha) + dT(\beta) \Rightarrow T$ is linear Transformation

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_1+1,\ x_2+1,\ x_3+1)$$

$$(x_1, x_2, x_3) \in \mathbb{R}^3$$

Check is it a linear-transformation or not

Soln: Let $\alpha = (y_1, y_2, y_3)$ & $\beta = (z_1, z_2, z_3) \in \mathbb{R}^3$.

Now $c\alpha + d\beta = (cy_1 + dz_1,\ cy_2 + dz_2,\ cy_3 + dz_3) \in \mathbb{R}^3$.

Now $T(c\alpha + d\beta) = (cy_1 + dz_1 + 1,\ cy_2 + dz_2 + 1,\ cy_3 + dz_3 + 1)$

$$= c(y_1, y_2, y_3) + d(z_1, z_2, z_3) + (1, 1, 1)$$

Now $cT(\alpha) = cT(y_1, y_2, y_3)$

$$= c(y_1+1,\ y_2+1,\ y_3+1)$$

$$dT(\beta) = d(z_1+1,\ z_2+1,\ z_3+1)$$

$$\Rightarrow cT(\alpha) + dT(\beta) = c(y_1+1,\ y_2+1,\ y_3+1) + d(z_1+1,\ z_2+1,\ z_3+1)$$

$$= c(y_1, y_2, y_3) + d(z_1, z_2, z_3) + (c+d,\ c+d,\ c+d)$$

$$\Rightarrow T(c\alpha + d\beta) \neq cT(\alpha) + dT(\beta)$$

$\Rightarrow$ T is not linear transformation

Note: Another name of a linear transformation is linear map.

Theorem: Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. Then (i) $T(\theta) = \theta'$ where θ, θ' are zero elements in V and W respectively.

(ii) $T(-\alpha) = -T(\alpha) \quad \forall \alpha \in V$.

Kernel of a linear mapping: Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear mapping. Then

Kernel of T = $\operatorname{Ker} T = \{\alpha \in V : T(\alpha) = \theta'\}$

where θ' being the zero vector in W.

Example: A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x_1, x_2, x_3) = (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3)$, where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find $\operatorname{Ker}(T)$?

Soln:

$$\operatorname{Ker} T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid T(x_1, x_2, x_3) = (0,0,0,0)\}$$

$$= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2,\ x_1 + x_2 + x_3) = (0,0,0,0)\}$$

$$= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 \\ x_3 + x_1 = 0 \Rightarrow x_1 = -x_3 \\ x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}\right\}.$$

$$= \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_2 = -x_3 \\ x_1 = -x_3 \\ x_1 + x_2 = 0 \Rightarrow -2x_3 = 0 \Rightarrow x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array}\right\}.$$

$$\text{Ker } T = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x_3 = 0 \\ x_2 = -x_3 = 0 \\ x_1 = -x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{array} \right\}$$

$$= \{(0,0,0)\}$$

$$= \{\theta\}$$

$\therefore$ $\text{Ker } T = \{\theta\}$

<u>Theorem:</u> Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping. Then $\text{Ker } T$ is a subspace of V.

<u>proof:</u> $\text{Ker } T = \{\alpha \in V \mid T(\alpha) = \theta'\}$ where $\theta' \in W$

Now for $\theta \in V$, $T(\theta) = \theta'$ $\Rightarrow$ $\theta \in \text{Ker } T$

$\Rightarrow$ $\text{Ker } T$ is non-empty.

<u>case 1:</u> $\text{Ker } T = \{\theta\}$. Then $\text{Ker } T$ is a subspace of V.

<u>case 2:</u> $\text{Ker } T \neq \{\theta\}$

Let $\alpha, \beta \in \text{Ker } T$ $\Rightarrow$ $T(\alpha) = \theta'$, & $T(\beta) = \theta'$

Now $T(c\alpha + d\beta) = cT(\alpha) + dT(\beta)$ ($\because$ T is linear mapping)

$= c\theta' + d\theta'$

$= \theta'$

$\Rightarrow$ $c\alpha + d\beta \in \text{Ker } T$.

$\Rightarrow$ $\text{Ker } T$ is a subspace of V.

Note: Ker T is also called the null space of T. and is denoted by N(T).

Theorem: let V and W be vector spaces over a field F. let $T: V \to W$ be a linear mapping such that $\ker T = \{\theta\}$. Then the images of a linearly independent set of vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ in V are linearly independent in W.

i.e if $\{\alpha_1, \ldots \alpha_r\}$ is L.I in V

then $\{T(\alpha_1), \ldots, T(\alpha_r)\}$ is L.I in W.

proof: To prove $\{T(\alpha_1), \ldots, T(\alpha_r)\}$ is L.I in W,

let us consider the relation.

$$c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_r T(\alpha_r) = \theta'$$

$$\Rightarrow T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r) = \theta' \quad (\because T \text{ is linear})$$

$$\Rightarrow c_1\alpha_1 + c_2\alpha_2 + \cdots + c_r\alpha_r = \theta \quad (\because \ker T = \{\theta\})$$

$$\Rightarrow c_1 = c_2 = \cdots = c_r = 0 \quad (\because \{\alpha_1, \alpha_2, \ldots \alpha_r\} \text{ is L.I})$$

$$\Rightarrow \{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_r)\} \text{ is L.I.}$$

Image of a linear mapping :

let V and W be vector spaces over a field F. Then let $T: V \to W$ be a linear mapping. Then

$$\text{image of } T = \operatorname{Im} T = \{T(\alpha) : \alpha \in V\}$$

Theorem : $\operatorname{Im} T$ is a subspace of W, where $T: V \to W$ is a linear mapping.

proof : let θ, θ' be the null/zero element in V & W respectively

$\because T(\theta) = \theta' \Rightarrow \theta' \in \operatorname{Im} T \Rightarrow \operatorname{Im} T$ is non-empty.

case 1 : $\operatorname{Im} T = \{\theta'\}$. Then $\operatorname{Im} T$ is a subspace of W.

case 2 : $\operatorname{Im} T \neq \{\theta'\}$.

let $\xi, \eta \in \operatorname{Im} T \Rightarrow \exists\ \alpha, \beta \in V$ s.t

$$\xi = T(\alpha) \ \& \ \eta = T(\beta)$$

Now
$$\begin{aligned} c\xi + d\eta &= cT(\alpha) + dT(\beta) \\ &= T(c\alpha) + T(d\beta) \quad (\because T \text{ is linear}) \\ &= T(c\alpha + d\beta) \quad (\because T \text{ is linear}) \end{aligned}$$

$\because \alpha, \beta \in V$ & V is vector space

$\Rightarrow c\alpha + d\beta \in V$

$\therefore c\xi + d\eta = T(c\alpha + d\beta)$ where $c\alpha + d\beta \in V$

$\Rightarrow c\xi + d\eta \in \operatorname{Im} T \Rightarrow \operatorname{Im} T$ is a subspace of W

Theorem: Let V and W be vector spaces over a field F. Let $T: V \to W$ be a linear mapping and $\ker T = \{\theta\}$. Then if $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V, then $\{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)\}$ becomes a basis of $\operatorname{Im} T$

Proof: I want to prove $\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a basis of $\operatorname{Im} T$.

It is given that (i) T is linear
(ii) $\ker T = \{\theta\}$.
(iii) $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V.

Now $\because \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V

$\Rightarrow \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is L.I

$\Rightarrow \{T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n)\}$ is L.I (From previous Theorem).

Let $\xi \in \operatorname{Im} T$, then $\exists\ \alpha \in V$

s.t $\xi = T(\alpha) \to (1)$

Now $\alpha \in V$ & $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V

$\Rightarrow \alpha = c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n$ $c_i \in F$

$\therefore \xi = T(\alpha)$
$= T(c_1\alpha_1 + c_2\alpha_2 + \dots + c_n\alpha_n)$
$= c_1 T(\alpha_1) + \dots + c_n T(\alpha_n)$ ($\because T$ is linear)

$\because$ each $T(\alpha_i) \in \operatorname{Im} T$.

$\Rightarrow$ $\operatorname{Im} T$ is generated by the set $\{T(\alpha_1), \dots, T(\alpha_n)\}$.

$\therefore$ $\{T(\alpha_1), \dots T(\alpha_n)\}$ is L.I & it generate $\operatorname{Im} T$.

$\Rightarrow$ $\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a baries of $\operatorname{Im} T$.

Example: A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)$

Find $\operatorname{Im} T$?

Soln: $\operatorname{Im} T = \{T(\alpha) \mid \alpha \in \mathbb{R}^3\}$,

$$= \{T(x_1, x_2, x_3) \mid (x_1, x_2, x_3) \in \mathbb{R}^3\}$$

$$= \{(x_2 + x_3, x_3 + x_1, x_1 + x_2, x_1 + x_2 + x_3)\}$$

$$= \{x_1(0,1,1,1) + x_2(1,0,1,1) + x_3(1,1,0,1)\}$$

$$= L\{S\}$$

where $S = \{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$

Nullity of a linear mapping:

let V and W be vector spaces over a field F and $T: V \to W$ be a linear mapping.
Then Ker T is a subspace of V.
$\therefore$ Nullity of $T = \dim(\text{Ker } T)$

Rank of a linear mapping:

let V and W be vector spaces over a field F
let $T: V \to W$ be a linear mapping.
Then Im T is a subspace of W.
$\therefore$ Rank of $T = \dim(\text{Im } T)$

Rank and Nullity theorem for linear mapping:

let V and W be vector spaces over a field F and V is finite dimensional. If $T: V \to W$ be a linear mapping then,

nullity of T + Rank of T = dim V.

$\dim \ker T + \dim \text{Im } T = \dim V$

$\Rightarrow$

Example: For the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^4$
$T(x_1, x_2, x_3) = (x_2+x_3, x_3+x_1, x_1+x_2, x_1+x_2+x_3)$
varify that $\dim \ker T + \dim \operatorname{Im} T = \dim(\mathbb{R}^3)$
$= 3$.

Soln: For the the given linear mapping already we determined that

$$\ker T = \{\theta\}$$

and $\operatorname{Im} T = L(S)$ where $S = \{(0,1,1,1), (1,0,1,1), (1,1,0,1)\}$.

Now $\dim \ker T = 0$ $\quad (\because \ker T = \{\theta\})$

and $\operatorname{Im} T = L(S) = S$ generate $\operatorname{Im} T$

Now $c_1(0,1,1,1) + c_2(1,0,1,1) + c_3(1,1,0,1) = (0,0,0,0)$

$\Rightarrow$ $c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$.

$c_1 + c_3 = 0 \Rightarrow c_1 = -c_3$.

$c_1 + c_2 = 0 \Rightarrow c_1 + c_2 = -c_3 - c_3 = -2c_3 = 0$

$\Rightarrow c_3 = 0$.

$c_1 + c_2 + c_3 = 0$

$\therefore$ $c_3 = 0 \Rightarrow c_1 = 0$ & $c_2 = 0$.

$\Rightarrow$ S is L.I

$\therefore$ S is basis of $\operatorname{Im} T$

Now $\operatorname{Im} T =$ No of element in basis i.e S

$= 3$.

$\therefore$ $\dim \ker T + \dim \operatorname{Im} T = 0 + 3 = 3$

$= \dim(\mathbb{R}^3)$

(varified)

Matrix representation of a linear mapping:

let V and W be finite dimensional vector space over a field F with dim V = n and dim W = m.
let $T : V \to W$ be a linear mapping
let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered bases of V
and $\{\beta_1, \beta_2, \dots, \beta_m\}$ be an " " of W.

Then for $\alpha_1, \alpha_2, \dots, \alpha_n \in V$

$T(\alpha_1), T(\alpha_2), \dots, T(\alpha_n) \in W$

But $\{\beta_1, \beta_2, \dots, \beta_m\}$ is a bases of W

$$\Rightarrow \left.\begin{aligned} T(\alpha_1) &= a_{11}\beta_1 + a_{21}\beta_2 + \dots + a_{m1}\beta_m \\ T(\alpha_2) &= a_{12}\beta_1 + a_{22}\beta_2 + \dots + a_{m2}\beta_m \\ &\vdots \\ T(\alpha_n) &= a_{1n}\beta_1 + a_{2n}\beta_2 + \dots + a_{mn}\beta_m \end{aligned}\right\}$$

where a_{ij} are unique scalars in F.

$$\Rightarrow \begin{pmatrix} T(\alpha_1) \\ T(\alpha_2) \\ \vdots \\ T(\alpha_n) \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}}_{\text{say this is } A \text{ matrix.}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}$$

Then matrix representation of linear mapping T is $= A^T$ = transpose of A $= \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$

Example: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ is a linear mapping defined by: $T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3)$. $(x_1, x_2, x_3) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered bases

$\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and

$\{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

Soln: $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$

and $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$

Now $T(1,0,0) = (3,1)$

$$= 3(1,0) + 1(0,1)$$

$$T(0,1,0) = (-2,-3)$$

$$= -2(1,0) - 3(0,1)$$

$$T(0,0,1) = (1,-2)$$

$$= 1(1,0) - 2(0,1).$$

Now
$$\begin{pmatrix} T(1,0,0) \\ T(0,1,0) \\ T(0,0,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ -2 & -3 \\ 1 & -2 \end{pmatrix}}_{A} \begin{pmatrix} (1,0) \\ (0,1) \end{pmatrix}$$

$\therefore$ Matrix of T = transpose of A

$$= A^T = \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

Theorem: let V and W be rector spaces of finite demensioral over a field F. and $T: V \to W$ be a linear mapping. Then

rank of T = rank of matrix of T