Department of Mathematics
Bennett University
EMAT102L: Ordinary Differential Equations
Tutorial Sheet-2 Solutions

1) Solve the following exact/reducible to exact ODEs:

$$
\begin{aligned}
&(a)\quad 2xye^{x^2}dx + e^{x^2}dy = 0, \quad y(0) = 2;\\
&(b)\quad \cos(x+y)dx + (3y^2 + 2y + \cos(x+y))dy = 0;\\
&(c)\quad (1+2x)\cos y dx + \sec y dy = 0;\\
&(d)\quad 3x^2 y dx + 4x^3 dy = 0.
\end{aligned}
$$

Solutions:

(a) Comparing given ODE with $M(x,y)dx + N(x,y)dy = 0$, we get

$$M(x,y) = 2xye^{x^2} \quad \text{and} \quad N(x,y) = e^{x^2}.$$

$$\frac{\partial M}{\partial y} = 2xe^{x^2} = \frac{\partial N}{\partial x}$$

Clearly given ODE is exact. In order to find a solution of the given ODE, we must find a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) = 2xye^{x^2}, \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y) = e^{x^2}.$$

From the first expression we get,

$$F(x,y) = \int 2xye^{x^2}dx + \phi(y) \Rightarrow F(x,y) = ye^{x^2} + \phi(y).$$

From the second expression we get,

$$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow e^{x^2} + \phi'(y) = e^{x^2} \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c_0,$$

where c_0 is an arbitrary constant. So,

$$F(x,y) = ye^{x^2} + c_0.$$

Thus a one-parameter family of solution is given by

$$F(x,y) = c_1 \Rightarrow ye^{x^2} = c_1 - c_0 = c.$$

Use initial Condition

$$y(0) = 2 \Rightarrow c = 2 \Rightarrow y = 2e^{-x^2}.$$

(b) Here,

$$M(x,y) = \cos(x+y) \quad \text{and} \quad N(x,y) = 3y^2 + 2y + \cos(x+y).$$

$$\frac{\partial M}{\partial y} = -\sin(x+y) = \frac{\partial N}{\partial x}$$

Clearly given ODE is exact. In order to find solution of given ODE, we must find function $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = M(x,y) = \cos(x+y), \quad \text{and} \quad \frac{\partial F}{\partial y} = N(x,y) = 3y^2 + 2y + \cos(x+y).$$

From the first expression we get,

$$F(x,y) = \int \cos(x+y)dx + \phi(y) \Rightarrow F(x,y) = \sin(x+y) + \phi(y).$$

From the second expression we get,

$$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow \cos(x+y) + \phi'(y) = 3y^2 + 2y + \cos(x+y)$$
$$\Rightarrow \phi'(y) = 3y^2 + 2y \Rightarrow \phi(y) = y^3 + y^2 + c_0,$$

where c_0 is an arbitrary constant. So,

$$F(x,y) = \sin(x+y) + y^3 + y^2 + c_0.$$

Thus a one-parameter family of solution is given by

$$F(x,y) = c_1 \Rightarrow \sin(x+y) + y^3 + y^2 = c_1 - c_0 = c.$$

$$\Rightarrow \sin(x+y) + y^3 + y^2 = c.$$

(c)

$$M(x,y) = (1+2x)\cos y \quad \text{and} \quad N(x,y) = \sec y$$

$$\frac{\partial M}{\partial y} = -(1+2x)\sin y \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

$$\Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x},$$

hence given ODE is not exact.

$$\frac{N_x - M_y}{M} = \frac{(1+2x)\sin y}{(1+2x)\cos y} = \frac{\sin y}{\cos y},$$

$$\text{so I. F.} = \int e^{\frac{\sin y}{\cos y}dy} = \sec y.$$

Multiply given equation by I. F., we get

$$(1+2x)dx + \sec^2 y dy = 0.$$

Now given ODE is exact. Thus

$$F(x,y) = \int (1+2x)dx + \phi(y) = x + x^2 + \phi(y)$$

$$\text{Now } \frac{\partial F}{\partial y} = N(x,y) \Rightarrow \phi'(y) = \sec^2 y \Rightarrow \phi(y) = \tan y + c_0,$$

where c_0 is an arbitrary constant. So,

$$F(x,y) = c_1 \Rightarrow x + x^2 + \tan y + c_0 = c_1 \Rightarrow \tan y = -x - x^2 + c.$$

(d)

$$M(x,y) = 3x^2 y \quad \text{and} \quad N(x,y) = 4x^3.$$

$$\frac{\partial M}{\partial y} = 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 12x^2.$$

Clearly given ODE is not exact.

$$\frac{N_x - M_y}{M} = \frac{12x^2 - 3x^2}{3x^2 y} = \frac{3}{y},$$

$$\text{so I. F. } = \int e^{\frac{3}{y}dy} = y^3.$$

Multiply given equation by I. F., we get

$$3x^2y^4dx + 4x^3y^3dy = 0.$$

This ODE is exact.

$$F(x,y) = \int 3x^2y^4dx + \phi(y) \Rightarrow F(x,y) = x^3y^4 + \phi(y)$$

Now

$$\frac{\partial F}{\partial y} = N(x,y) \Rightarrow 4x^3y^3 + \phi'(y) = 4x^3y^3 \Rightarrow \phi(y) = c_0,$$

where c_0 is an arbitrary constant. So,

$$F(x,y) = c_1 \Rightarrow x^3y^4 + c_0 = c_1 \Rightarrow x^3y^4 = c_1 - c_0 = c.$$

2) Solve the following linear/reducible to linear ODEs:

(a) $\dfrac{dy}{dx} + 3x^2y = x^2, \ y(0) = 2;$

(b) $y^2dx + (3xy - 1)dy = 0;$

(c) $\dfrac{dy}{dx} + y = f(x), \ y(0) = 0, \text{ where } f(x) = \begin{cases} 2 & 0 \le x < 1, \\ 0 & x \ge 1. \end{cases},$

(d) $dy + (4y - 8y^{-3})xdx = 0.$

Solution:

(a) Given ODE is linear in y. Compare with

$$\frac{dy}{dx} + P(x)y = Q(x),$$

we get $P(x) = 3x^2$ and $Q(x) = x^2$.

$$\text{I. F.} = e^{\int p(x)dx} = e^{\int 3x^2 dx} = e^{x^3}.$$

Solution is

$$y \times I.F. = \int (Q(x) \times I.F)\, dx + c,$$

where c is an arbitrary constant.

$$y \times e^{x^3} = \int x^2 e^{x^3}\, dx + c \Rightarrow y \times e^{x^3} = \frac{1}{3} \times e^{x^3} + c$$

Now use initial condition

$$y(0) = 2 \Rightarrow 2 = \frac{1}{3} + c \Rightarrow c = \frac{5}{3}.$$

Solution is

$$y(x) = \frac{5}{3}e^{-x^3} + \frac{1}{3}.$$

(b) This ODE is linear in x. Rewrite as

$$\frac{dx}{dy} + \frac{3}{y}x = \frac{1}{y^2}.$$

$$\text{I. F.} = e^{\int p(y)dy} = e^{\int \frac{3}{y}dy} = y^3.$$

Solution is

$$x \times y^3 = \int \frac{1}{y^2} y^3 dy + c \Rightarrow x \times y^3 = \frac{y^2}{2} + c,$$

where c is an arbitrary constant.

(c) Here,

$$I.F. = e^{\int P(x)dx} = e^{\int 1 dx} = e^x.$$

Solution is

$$y \times I.F. = \int f(x) \times I.F.\, dx + c,$$

where c is an arbitrary constant. $f(x) = 2$ when $0 \le x < 1.$, i.e.,

$$y.e^x = \int 2e^x dx + c \Rightarrow y(x) = 2 + ce^{-x}.$$

Use initial condition

$$y(0) = 0 \Rightarrow 0 = 2 + c \Rightarrow c = -2.$$

That is,

$$y(x) = 2(1 - e^{-x}).$$

Now for $x \geq 1, \quad f(x) = 0$, then solution is

$$y.e^x = c_1 \Rightarrow y = c_1 e^{-x},$$

where c_1 is an arbitrary constant. To determine c_1, use continuity of $y(x)$ at $x = 1$. i.e.,

$$\lim_{x \to 1^+} y(x) = \lim_{x \to 1^-} y(x) \Rightarrow c_1 e^{-1} = 2(1 - e^{-1}) \Rightarrow c = 2(e - 1).$$

So, $y(x) = 2(e-1)e^{-x}$. Hence,

$$y(x) = \begin{cases} 2(1 - e^{-x}) & 0 \leq x < 1, \\ 2(e-1)e^{-x} & x \geq 1. \end{cases}$$

(d)

$$\frac{dy}{dx} + 4xy = 8xy^{-3}$$

Multiply by y^3, we get

$$y^3 \frac{dy}{dx} + 4xy^4 = 8x.$$

Take,

$$y^4 = v \Rightarrow y^3 \frac{dy}{dx} = \frac{1}{4}\frac{dv}{dx}.$$

Now transformed ODE is

$$\frac{1}{4}\frac{dv}{dx} + 4xv = 8x$$

This is linear in v. Solution is

$$v = 2 + ce^{-8x^2} \Rightarrow y^4 = 2 + ce^{-8x^2}.$$

3) Under what conditions for the constants a, b, k, l, is $(ax + by)dx + (kx + ly)dy = 0$ exact? Solve the exact ODE.

Solution: For exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow b = k.$$

For solution of exact DE

$$F(x, y) = \int (ax + by)dx + \phi(y) \Rightarrow F(x, y) = \frac{ax^2}{2} + bxy + \phi(y).$$

Now to determine $\phi(y)$, use

$$\frac{\partial F}{\partial y} = N(x, y) \Rightarrow bx + \phi'(y) = kx + ly \Rightarrow \phi(y) = \frac{ly^2}{2} + c_0.$$

$$\text{Thus } F(x, y) = c_1 \Rightarrow \frac{ax^2}{2} + bxy + \frac{ly^2}{2} = c.$$

4) Find the orthogonal trajectories of the family of circles which are tangent to the y axis at the origin.

Solution: Given family of circles is

$$x^2 + y^2 = 2ax.$$

On differentiating

$$2x + 2y\frac{dy}{dx} = 2a \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

For differential equation of orthogonal trajectories, replace $\frac{dy}{dx}$ by $\frac{-1}{dy/dx}$, we get

$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2},$$

Which is a homogeneous DE. Use transformation $y = vx$. On solving we get

$$x^2 + y^2 = my.$$

5) Find the value of n such that the curves $x^n + y^n = c$ are orthogonal trajectories of the family $y = \frac{x}{1-c_1 x}$.
Solution:

$$x^n + y^n = c$$

On differentiating

$$nx^{n-1} + ny^{n-1}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^{n-1}}{y^{n-1}}$$

Differential equation of orthogonal trajectories

$$\frac{dy}{dx} = \frac{y^{n-1}}{x^{n-1}} \Rightarrow y^{1-n}dy = x^{1-n}dx$$

On integrating we get,

$$y^{2-n} = x^{2-n} + (2-n)c_1,$$

where c_1 is an arbitrary constant. For $n = 3$,

$$y^{2-3} = x^{2-3} + (2-3)c_1 \Rightarrow \frac{1}{y} = \frac{1}{x} - c_1 \Rightarrow y = \frac{x}{1 - c_1 x}.$$

6) Does the IVP $(x-2)\frac{dy}{dx} = y$; $y(2) = 1$ have a solution? Justify your answer.
Solution: No solution. Do yourself.

7) Show that existence and uniqueness theorem guarantees the existence of a unique solution of the IVP-

(a) $\dfrac{dy}{dx} = e^{2y}$; $y(0) = 0$.

(b) $\dfrac{dy}{dx} = y^{4/3}$; $y(x_0) = y_0$.

Solutions:

(a) Consider a rectangle $R : |x| \leq a,\ |y| \leq b$. Clearly the function e^{2y} is continuous in R and $|e^{2y}| \leq e^{2b} = k$ in R, so by existence theorem there exists a solution of given IVP in the interval

$$|x| \leq \alpha, \text{ where } \alpha = \min\left(a, \frac{b}{k}\right) = \min\left(a, \frac{b}{e^{2b}}\right).$$

Also,

$$\frac{\partial f}{\partial y} = \frac{\partial e^{2y}}{\partial y} = 2e^{2y},$$

is bounded in R, so by uniqueness theorem, there exists a unique solution of IVP in above defined interval, $|x| \leq \alpha$. where,

$$\alpha = \min\left(a, \frac{b}{e^{2b}}\right).$$

Consider $F(b) = \frac{b}{e^{2b}} \Rightarrow F'(b) = \frac{1-2b}{e^{2b}}$. The maximum value of $F(b)$ occurs at $b = \frac{1}{2}$ and $F(1/2) = \frac{1}{2e}$. Now if $a \geq F(b)$, then $\alpha = \min(a, F(b)) = F(b) \leq \frac{1}{2e}$ for all $b > 0$. If $a < \frac{1}{2e} \Rightarrow \alpha < \frac{1}{2e}$. For $b = \frac{1}{2}$, $a \geq \frac{1}{2e}$, $\alpha = \frac{1}{2e}$ Thus, in any case $\alpha \leq \frac{1}{2e}$, i.e., $|x| \leq \frac{1}{2e}$.

(b) Do yourself.