B. Tech, Spring-2021

EPHY108L

Problem Set-2: Answer

1. Determine the gradient of the following functions:
   i) $f(x,y,z) = xyz;$
   Ans
   $\vec{\nabla} f = yz\,\hat{\imath} + xz\,\hat{\jmath} + xy\,\hat{k}.$
   ii) $g(x,y,z) = x^4 + y^4 + z^4$
   Ans
   $\vec{\nabla} g = 4(x^3\,\hat{\imath} + y^3\,\hat{\jmath} + z^3\,\hat{k}).$
   iii) $h(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2$
   Ans
   $\vec{\nabla} h = 2x(y^2+z^2)\hat{\imath} + 2y(x^2+z^2)\hat{\jmath} + 2z(x^2+y^2)\hat{k}.$
   iv) $\phi(x,y,z) = 3xy^2z^3 + 2xyz + 4x^2y^2$
   Ans
   $\vec{\nabla}\phi = (3y^2z^3 + 2yz + 8xy^2)\hat{\imath} + (6xyz^3 + 2xz + 8x^2y)\hat{\jmath} + (9xy^2z^2 + 2xy)\hat{k}.$
   v) $\psi(x,y,z) = x^2y^2 + y^2z^2 + z^2x^2 + 2xyz$
   Ans
   $\vec{\nabla}\psi = 2(xy^2 + xz^2 + yz)\hat{\imath} + 2(x^2y + yz^2 + xz)\hat{\jmath} + 2(zx^2 + y^2z + xy)\hat{k}.$
2. If $\vec{A} = xz\hat{\imath} + (2x^2 - y)\hat{\jmath} - yz^2\hat{k}$, then determine $\vec{\nabla}\cdot\vec{A}$.
   Ans
   $\vec{\nabla}\cdot\vec{A} = z - 1 + 2yz.$

3. If $\phi = 3x^2y + y^2z^3$, then determine $\vec{\nabla}\phi$.
   Ans
   $\vec{\nabla}\phi = 6xy\hat{\imath} + (3x^2 + 2yz^3)\hat{\jmath} + 3y^2z^2\hat{k}.$

4. A constant force $\vec{F}$ acting on a particle of mass *m* changes the velocity from $\vec{v}_1$ to $\vec{v}_2$ in time τ. Prove that $\vec{F} = \frac{m(\vec{v}_1 - \vec{v}_2)}{\tau}$.
5. Prove that if $\vec{F}$ is the force acting on a particle and $\vec{v}$ is the (instantaneous) velocity of the particle, then the (instantaneous) power applied to the particle is given by $P = \vec{F}\cdot\vec{v}$.

6. Determine whether the force $\vec{F} = (y^2z^3 - 6xz^2)\hat{\imath} + 2xyz^3\hat{\jmath} + (3xy^2z^2 - 6x^2z)\hat{k}$ is conservative or not.

   Ans

   The given force is conservative.

7. Using known results from the Cartesian coordinate system, and the relation between plane-polar and Cartesian basis vectors, calculate: (a) $\hat{r} \times \hat{\theta}$, (b) $\hat{\theta} \times \hat{k}$, and (c) $\hat{\hat{k}} \times \hat{r}$.

   Ans

   (a) $\hat{k}$; (b) $\hat{r}$; (c) $\hat{\theta}$

8. A particle is moving along a circular path of radius a, with angular velocity given by $\omega(t) = \omega_0 + \alpha t$, where ω_0 and α are constants. Obtain the radial and tangential components of its velocity and acceleration.

   Ans

   $\vec{v}(t) = a(\omega_0 + \alpha t)\hat{\theta}$.

   $\vec{a}(t) = -a(\omega_0 + \alpha t)^2\hat{r} - a\alpha\hat{\theta}$.

9. A particle is moving along the line $y = a$, with the velocity $\vec{v} = u\hat{\imath}$, where u is a constant. Express its velocity in plane polar coordinates.

   Ans

   $\vec{v} = u\cos\theta\,\hat{r} - u\sin\theta\,\hat{\theta}$.

10. A particle moves in such a way that $\dot{\theta} = \omega$ (constant), and $r = r_0e^{\beta t}$, where r_0 and β are constants. Write down its velocity and acceleration in plane polar coordinates. For what values of β will the radial acceleration of the particle by zero?

    Ans

    $\vec{v}(t) = r_0e^{\beta t}(\beta\hat{r} + \omega\hat{\theta})$.

    $\vec{a}(t) = r_0e^{\beta t}(\beta^2 - \omega^2)\hat{r} + 2r_0\omega\beta e^{\beta t}\hat{\theta}$.

    Radial component of acceleration will vanish if $\beta = \pm\omega$.

11. Consider a circle of radius a, with the origin of the plane polar coordinate system placed at a point on the circumference. The particle is moving along the circle with a constant speed u.

    a. What is the equation of the circle in this coordinate system?

       Ans

       $r = 2a\cos\theta$.

    b. What is the value of $\dot{\theta}$ in terms of u and a?

       Ans

       $\dot{\theta} = \frac{u}{2a}$.

c. Write down the velocity of the particle in plane-polar coordinate system.

Ans

$\vec{v}(t) = -u \sin\left(\frac{ut}{2a}\right) \hat{r} + u \cos\left(\frac{ut}{2a}\right) \hat{\theta}.$

d. What is the acceleration of the particle in plane-polar coordinate system?

Ans

$\vec{a}(t) = -\frac{u^2}{a} \cos\left(\frac{ut}{2a}\right) \hat{r} - \frac{u^2}{a} \sin\left(\frac{ut}{2a}\right) \hat{\theta}.$

12. A particle moves along the curve $r = A\theta$, with $A = 1/\pi$ m/rad, and $\theta = \alpha t^2$, where α is a constant. Obtain the expressions for the velocity and acceleration of this particle in plane polar coordinates.

Ans

$\vec{v}(t) = 2A\alpha t\, \hat{r} + 2A\alpha^2 t^3\, \hat{\theta}.$

$\vec{a}(t) = 2A\alpha(1 - 2\alpha^2 t^4)\, \hat{r} + 6A\alpha^2 t^2\, \hat{\theta}.$

a. Show that the radial acceleration is zero when $\theta = 1/\sqrt{2}$ rad.

b. At what angles do radial and tangential components of the acceleration haveequal magnitude?

Ans

$\theta = \alpha t^2 = \frac{\pm 3 + \sqrt{17}}{4}$ radians

13. Mass m rotates on a frictionless table, held to circular path by a string which passesthrough a hole in the table. The string is slowly pulled through the hole so that theradius of the circle changes from R_0 to R_1. Show that the work done in pulling the string equals the increase in kinetic energy of the mass.

14. A particle of mass m moves in one dimension along the positive x axis. It is acted on by a constant force directed towards the origin with the magnitude B, and inverselaw repulsive force of magnitude A/x^2.

a. Find the potential energy function $V(x)$

Ans

$V(x) = Bx + \frac{A}{x}.$

b. Plot the potential energy as a function of x, and the total energy of the system, assuming that the maximum kinetic energy $K_0 = \frac{1}{2}mv^2$.

c. What is the point of equilibrium, i.e., the point where net force acting on theparticle is zero.

Ans

$x = \sqrt{\frac{A}{B}}.$

15. A particle of mass M is held fixed at the origin. The gravitational potential energy of another particle of mass m, in the field of the first mass, is given by $V(r) = -\frac{GMm}{r}$, where G is the gravitational constant, and r is the distance of mass m from the origin.

(a) What is the force acting on the particle of mass m?

Ans

$\vec{F}(r) = -\frac{GMm\hat{r}}{r}$.

(b) Calculate the curl of this force.

Ans

0

16. Consider a 2D force field $\vec{F} = A(y^2\hat{\imath} + 2x^2\hat{\jmath})$. Calculate the work done by this force in going around a closed path which is a square made up of sides of length a, lying in the xy-plane, with two of its vertices located at the origin, and point (a, a). Find the answer by doing the line integral, as well as by using the Stokes' theorem. The path is traversed in a counter clockwise manner.

Ans

$2Aa^3$.

17. Find the forces for the following potential energies (A, B, and C are constants),

a. $V(x, y, z) = Ax^2 + By^2 + Cz^2$

Ans

$\vec{F} = -2Ax\,\hat{\imath} - 2By\,\hat{\jmath} - 2Cz\,\hat{k}$.

b. $V(x, y, z) = A\ln(x^2 + y^2 + z^2)$

Ans

$\vec{F} = \frac{2A(x\,\hat{\imath}+y\,\hat{\jmath}+z\,\hat{k})}{x^2+y^2+z^2}$.

c. $V(r, \theta) = A\cos\theta/r^2$ (r and θ are plane polar coordinates)

Ans

$\vec{F} = \frac{A(2x^2-y^2)}{(x^2+y^2)^{\frac{5}{2}}}\,\hat{\imath} + \frac{3Axy}{(x^2+y^2)^{\frac{5}{2}}}\,\hat{\jmath}$.

18. Determine whether each of the following forces is conservative. Find the potential energy function if it exists. A, α, β are constants.

a. $\vec{F} = A(3\hat{\imath} + z\hat{\jmath} + y\hat{k})$

Ans

The force is conservative. $V(x, y, z) = -3Ax - Ayz + C$. Here C is a constant.

b. $\overrightarrow{F} = Axyz(\hat{\imath} + \hat{\jmath} + \widehat{k})$

Ans

$\vec{\nabla} \times \vec{F} = A(xz - xy)\hat{\imath} + A(xy - yz)\hat{\jmath} + A(yz - xz)\hat{k}.$

c. $F_x = A \sin \alpha y \cos \beta z,\ F_y = -Ax\alpha \cos \alpha y \cos \beta z\ ,\ F_z = Ax \sin \alpha y \sin \beta z$

Ans

$$\vec{\nabla} \times \vec{F} = A(x\alpha\, cos(\alpha y)\ sin(\beta z)\ -\ x\alpha\beta\ cos(\alpha y)\ sin(\beta z))\hat{\imath}$$
$$+ A\big(-x\beta\ sin(\alpha y) sin(\beta z) -\ x\alpha\ cos(\alpha y) sin(\beta z)\big)\hat{\jmath}$$
$$+ A\big(0\ -\ \alpha\ cos(\alpha y) cos(\beta z)\big)\widehat{k}$$