Solution of

Assignment 6

1) (i) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(\lambda-1)(\lambda-3) = 0.$$

$$\Rightarrow \lambda = 1, 1, 3$$

⊞ eigen vector for $\lambda = 1$:

$AX = 1X \Rightarrow (A - I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow x_1 + x_2 + x_3 = 0 \;\Big\}\; \Rightarrow x_1 = -x_2 - x_3$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

⊞ eigen vector for $\lambda = 3$:

$AX = 3X \Rightarrow (A - 3I)X = 0$

$$\Rightarrow \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_3 = 0,$$

$$-x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = x_2$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

1) (ii) $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)^2 - 1 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 3) = 0$.

$\Rightarrow \lambda = 1, 3$.

Eegen vector for $\lambda = 1$:

$AX = 1X \Rightarrow (A - I)X = 0$.

$\Rightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \left.\begin{array}{l} x_1 - x_2 = 0 \\ -x_1 + x_2 = 0 \end{array}\right\} \Rightarrow x_1 = x_2$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eegen vector for $\lambda = 3$:

$AX = 3X \Rightarrow (A - 3I)X = 0$

$\Rightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

1) (iii) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 = -1 \Rightarrow \lambda = i, -i$$

Eegen vector for $\lambda = i$:

$$AX = iX \Rightarrow (A - iI)X = 0$$

$$\Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} ix_1 + x_2 = 0 \\ x_1 - ix_2 = 0 \end{array}\right\} \Rightarrow x_1 = ix_2$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Eegen vector for $\lambda = -i$:

$$AX = -iX \Rightarrow (A + iI)X = 0.$$

$$\Rightarrow \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \left.\begin{array}{l} ix_1 - x_2 = 0 \\ x_1 + ix_2 = 0 \end{array}\right\} x_1 = -ix_2$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

f) (iv) $A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

$\therefore |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 3 \\ 0 & 3-\lambda & 0 \\ 3 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda - 3)(\lambda - 6) = 0$$

$$\therefore \lambda = 0, 3, 6$$

<u>Eegen rector for $\lambda = 0$:</u>

$AX = 0X \Rightarrow AX = 0$

$$\Rightarrow \begin{pmatrix} 3 & 0 & 3 \\ 0 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{ll} 3x_1 + 3x_3 = 0 & \Rightarrow x_1 = -x_3 \\ 3x_2 = 0 & \Rightarrow x_2 = 0 \end{array}\right\}$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

<u>Eegen rector for $\lambda = 3$:</u>

$AX = 3X \Rightarrow (A - 3I)X = 0$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = 0, \ x_3 = 0$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Eegen rector for $\lambda = 6$

$AX = 6X \Rightarrow (A - 6I)X = 0$

$$\Rightarrow \begin{pmatrix} -3 & 0 & 3 \\ 0 & -3 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \left.\begin{matrix} -3x_1 + 3x_3 = 0 \\ -3x_2 = 0 \\ 3x_1 - 3x_3 = 0 \end{matrix}\right\} \quad \begin{matrix} x_1 = x_3 \\ x_2 = 0 \end{matrix}$$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

2) $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda+2)(\lambda+2)(4-\lambda) = 0$

$\Rightarrow \lambda = -2, -2, 4.$

Eegen rector for $\lambda = -2$:

$AX = -2X \Rightarrow (A + 2I)X = 0$

$$\Rightarrow \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow x_1 - x_2 + x_3 = 0 \Rightarrow x_1 = x_2 - x_3.$

$$\therefore X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$= L(S)$ where $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$

S is L.I.

Eigen vector for $\lambda = 4$:

$AX = 4X \Rightarrow (A - 4I)X = 0$

$$\Rightarrow \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -x_1 - x_2 + x_3 &= 0 \\ x_1 - 3x_2 + x_3 &= 0 \\ x_1 - x_2 &= 0 \end{aligned}\right\} \begin{aligned} &-2x_1 + x_3 = 0 \Rightarrow x_1 = \tfrac{1}{2}x_3 \\ &x_1 = x_2 \end{aligned}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} = 2x_3 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\therefore P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Now
$$|P| = 1(0-1) + 1(2) + 1(1) = -1 + 2 + 1 \neq 0$$

$\therefore P^{-1}AP = \text{diag}(-2, -2, 4).$

$$= \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Thus A is diagonalised.

3) $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda = 2, 3.$

Eegen vector for $\lambda = 2$:

$(A - 2I)X = 0$

$\Rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigen vector for $\lambda = 3$:-

$(A - 3I)X = 0$

$\Rightarrow \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_1 = 0 \end{matrix}$

$\therefore X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \quad (|P| = -1 \neq 0)$

$\therefore P^{-1} A P = \text{diag}(2, 3) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$

$\Rightarrow$ A is diagonalised.