Lecture No 1: Linear Algebra (EMAT102L)

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Text and references

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What is Linear Algebra

linear algebra is a study of "linearity" which arises in two situations.

- study of linear equations in one and more variables
- study of geometry in the algebraic setup.

But geometry can be studied only in one dimension or two dimensions or three dimensions but you cannot visualize geometry in fourth dimension.

- So how do you do geometry in fourth dimension and high dimensions?
- That is done by linear algebra, making it abstract.

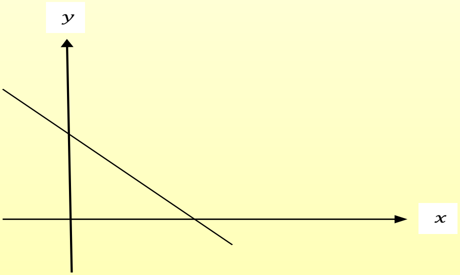
What is linear equation in two variables?

A linear equation in two variables is

$$ax + by = c, \tag{1}$$

where x, y are variables and a, b, c scalars (not all zero simultaneously)

- Geometrically equation (1) represents a straight line and it indicates all the pairs (x, y) which satisfy this equation (1).



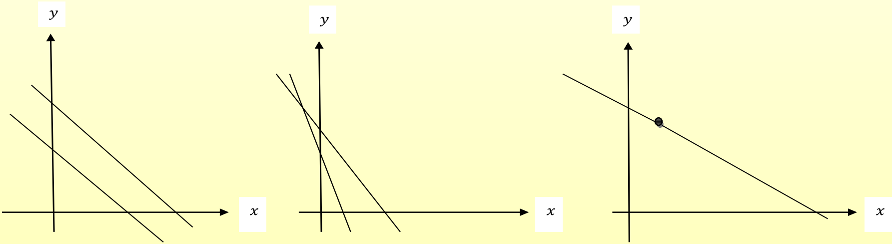
Linear equations in two variables

Let us consider two linear equations

$$a_1x + b_1y = c_1, \tag{2}$$

$$a_2x + b_2y = c_2. \tag{3}$$

Since each equation represents a line, following possibilities arise:



Parallel line but not coincidental

Lines intersect- there is only one solution

Lines are coincidental - there are infinite number solutions

Finding solution for the two intersecting lines

Example:

$$x + y = 3 \tag{4}$$

$$4x + 5y = 6 \tag{5}$$

- Step 1: You try to eliminate one variable. For this, multiply 4 with equation (4) then subtract equation (5). Then we will get

$$y = -6$$

- Step 2: Put the value of y in equation (4), we will get

$$x = 9$$

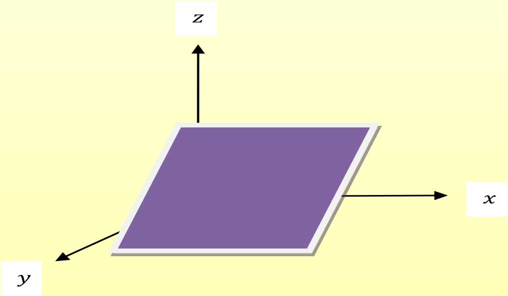
What is linear equation in three variables?

A linear equation in three variables is

$$ax + by + cz = d, \tag{6}$$

where x, y, z are variables and a, b, c, d scalars (not all zero simultaneously)

- Geometrically equation (6) represents a plane and it indicates all the pairs (x, y, z) which satisfy this equation (6).



Linear equations in three variables

Let us consider three linear equations

$$a_1x + b_1y + c_1z = d_1, \tag{7}$$

$$a_2x + b_2y + c_2z = d_2. \tag{8}$$

$$a_3x + b_3y + c_3z = d_3. \tag{9}$$

Since each equation represents a plane, following possibilities arise:

- One solution
- Infinite solution
- No solution

Finding solution for the three intersecting planes

Example:

$$2x + y + z = 5 \tag{10}$$
$$4x - 6y = -2 \tag{11}$$
$$-2x + 7y + 2z = 9 \tag{12}$$

- Step 1: You try to eliminate one variable from equation (11) and (12). For this, (a) substract 2 times the equation (10) from the equation (11). (b) substract -1 times the equation (10) from the equation (12).Then we will get

$$2x + y + z = 5 \tag{13}$$
$$-8y - 2z = -12 \tag{14}$$
$$8y + 3z = 14 \tag{15}$$

- Step 2: Substract -1 times the equation (14) from the equation (15). Then we will get

$$2x + y + z = 5 \tag{16}$$
$$-8y - 2z = -12 \tag{17}$$
$$z = 2 \tag{18}$$

Put the value of z in equations (17) (18), we will get final solution

$$x = 1, y = 1, z = 2$$

What we observe in solving all these things

Observations:

The solution of a system of equation does not change if

- any two equation are interchanged.
- any equation is multiplied by a non-zero scalar.
- one equation is added to another equation.

System of m equations in n variables

A m linear equations in n variables is

$$\begin{aligned}
a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{1n}\ x_n &= b_1 \\
&\vdots \\
a_{i1}\ x_1 + a_{i2}\ x_2 + \cdots + a_{in}\ x_n &= b_i \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}$$

Now, we can write the above system of equations in matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{pmatrix} \tag{19}$$

Review of Matrices

Matrix: A matrix is a rectangular arrangement of numbers (real or complex) which may be represented as

$$A_{mn} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

matrix is enclosed by [ ] or ( ). Compact form the above matrix is represented by $[a_{ij}]_{mn}$ or $A = [a_{ij}]$.

- Element of a Matrix: The numbers a_{11}, a_{12}etc., in the above matrix are known as the element of the matrix, generally represented as a_{ij} , which denotes element in ith row and jth column.
- Order of a Matrix: In above matrix has m rows and n columns, then A is of order mn.

Types of Matrices

Row Matrix A matrix having only one row and any number of columns is called a row matrix. It can be represented as

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$$

Column Matrix A matrix having only one column and any number of rows is called column matrix. It can be represented as

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

Rectangular Matrix A matrix of order mn, such that $m \neq n$, is called rectangular matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Square Matrix A matrix of order mn, such that $m = n$, is called square matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Types of Matrices

Null/Zero Matrix A matrix of any order, having all its elements are zero, is called a null/zero matrix. i.e., $a_{ij} = 0, i, j$

Diagonal Matrix A square matrix $A = [a_{ij}]_{nn}$, is called a diagonal matrix, if all the elements except those in the leading diagonals are zero, i.e., $a_{ij} = 0$ for ij. It can be represented as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Scalar Matrix A square matrix in which every non-diagonal element is zero and all diagonal elements are equal, is called scalar matrix. i.e., in scalar matrix $a_{ij} = 0$, for ij and $a_{ij} = k$, for $i = j$. It can be represented as

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Types of Matrices

Unit/Identity Matrix A square matrix, in which every non-diagonal element is zero and every diagonal element is 1, is called, unit matrix or an identity matrix. i.e., in Identity matrix $a_{ij} = 0$, for ij and $a_{ij} = 1$, for $i = j$. It can be represented as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Upper Triangular Matrix A square matrix $A = [a_{ij}]_{nn}$ is called a upper triangular matrix, if $a_{ij} = 0, i > j$. It can be represented as

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$$

Lower Triangular Matrix A square matrix $A = [a_{ij}]_{nn}$ is called a upper triangular matrix, if $a_{ij} = 0, i < j$. It can be represented as

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{pmatrix}$$

Algebra of Matrices

Addition of Matrices

Let A and B be two matrices each of order mn. Then, the sum of matrices $A+B$ is defined only if matrices A and B are of same order.
If $A=[a_{ij}]_{mn}, B=[b_{ij}]_{mn}$ Then, $A+B=[a_{ij}+b_{ij}]_{mn}$

Properties of Addition of Matrices

If A, B and C are three matrices of order mn, then

1. Commutative Law $A+B=B+A$
2. Associative Law $(A+B)+C=A+(B+C)$

Subtraction of Matrices

Let A and B be two matrices each of order mn. Then, the subtraction of matrices $A-B$ is defined only if matrices A and B are of same order.
If $A=[a_{ij}]_{mn}, B=[b_{ij}]_{mn}$ Then, $A-B=[a_{ij}-b_{ij}]_{mn}$

Algebra of Matrices

Multiplication of a Matrix by a Scalar
Let $A = [a_{ij}]_{mn}$ be a matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA, given as $kA = [ka_{ij}]_{mn}$

Multiplication of Matrices Let $A = [a_{ij}]_{mn}$ and $B = [b_{ij}]_{np}$ are two matrices such that the number of columns of A is equal to the number of rows of B, then multiplication of A and B is denoted by AB, is given by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

where c_{ij} is the element of matrix C_{mp} and $C_{mp} = A_{mn} B_{np}$

The rule of multiplication of matrices

The rule of multiplication of matrices is row column wise (or$\longrightarrow\downarrow$ wise). The ijth element of the product AB is obtained by multiplying the corresponding element of ith row of A and jth column of B and adding the product.

Example: $A_{42}B_{23}$= $\begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 6 & 1 \\ 3 & 8 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1\times 0+0\times 3 & 1\times 6+0\times 8 & 1\times 1+0\times -2 \\ -2\times 0+3\times 3 & -2\times 6+3\times 8 & -2\times 1+3\times -2 \\ 5\times 0+4\times 3 & 5\times 6+4\times 8 & 5\times 1+4\times -2 \\ 0\times 0+1\times 3 & 0\times 6+1\times 8 & 0\times 1+1\times -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 6 & 1 \\ 9 & 12 & -8 \\ 12 & 62 & -3 \\ 3 & 8 & -2 \end{pmatrix}$$

Matrix multiplication is not commutative i.e $AB \neq BA$

Example: Let A_{22}=$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and B_{22}=$\begin{pmatrix} 5 & 6 \\ 4 & 2 \end{pmatrix}$

Then $A_{22}B_{22}$=$\begin{pmatrix} 13 & 10 \\ 22 & 18 \end{pmatrix}$

$B_{22}A_{22}$=$\begin{pmatrix} 17 & 28 \\ 8 & 14 \end{pmatrix}$

This imply $AB \neq BA$

Transpose of a Matrix

Let $A = (a_{ij})_{mn}$, be a matrix of order mn. Then, transpose of A is obtained by interchanging the rows and columns of A and is denoted by $A^t = (a_{ji})_{nm}$ of order nm.

Example: Let

A_{42}=$\begin{pmatrix} 1 & 0 \\ -2 & 3 \\ 5 & 4 \\ 0 & 1 \end{pmatrix}$ Therefore A^t_{24}=$\begin{pmatrix} 1 & -2 & 5 & 0 \\ 0 & 3 & 4 & 1 \end{pmatrix}$

Properties of transpose of a matrix

- $(A^t)^t = A$
- $(A + B)^t = A^t + B^t$
- $(AB)^t = B^t A^t$

Symmetric and Skew-Symmetric Matrices

Symmetric Matrices A square matrix $A_n = (a_{ij})_n$, is said to be symmetric, if $A_n = A_n^t$. i.e., $a_{ij} = a_{ji}$, i and j.

Skew-Symmetric Matrices A square matrix $A_n = (a_{ij})_n$, is said to be skew-symmetric, if $A_n = -A_n^t$. i.e., $a_{ij} = -a_{ji}$, i and j.

Note: Elements of principal diagonals of a skew-symmetric matrix are all zero. i.e., $a_{ii} = a_{ii}$ implies $a_{ii} = 0$, for all values of i.

Trace of a Matrix The sum of the diagonal elements of a square matrix A_n is called the trace of A_n, denoted by trace (A_n) or tr (A_n). Therefore for the matrix A_3=$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$,

trace (A_3)=1+5+9=15

DETERMINANTS

Definition of the Determinant: In linear algebra,the determinant is a scalar value that can be computed from the elements of a square matrix. The determinants of a matrix A is denoted by $det(A)$ or $|A|$.

Let $A = [a_{ij}]_{nn}$ be an $n \times n$ matrix.

(1) If $n = 1$, that is $A = [a_{11}]$, then we define $det(A) = |a_{11}| = a_{11}$.

(2) If $n = 2$, that is $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then we define

$$det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

(2) If $n = 3$, that is $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then we define

$$det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{1+1}a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ (-1)^{1+3}a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Properties of Determinant

Property 1: The value of a determinant is unaltered if the determinant is transposed, i.e, if rows and columns are interchanged.

Example: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Property 2: The value of a determinant is unaltered but sign is altered if two adjacent rows/columns are interchanged.

Example: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = (-1) \begin{vmatrix} b & a \\ d & c \end{vmatrix}$

Property 3: If two rows/columns of a determinant are identical then the value of the determinant is zero.

Example: $\begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0$

Property 4: If all the elements of one row/column are multiplied by a number then the value of the determinant is multiplied by that number.

Example: $\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = k \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Properties of Determinant

Property 5: If each element of a row/column is expressed as the sum of two numbers then the determinant can be expressed as sum of two determinants.

Example: $\begin{vmatrix} a+x & b \\ c+y & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & b \\ y & d \end{vmatrix}$

Property 6: The value of a determinant is unaltered by adding to the elements of any row/column the same multiple of the corresponding elements of any other row/column

Example: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix}$ $(R_1 + kR_2 \to R_1^{'})$

Minor and Co-factor of an element in a square matrix

Minor of an element in a Matrix of order n: Minor of an element a_{ij} in a Matrix of order n is the determinant value of the square sub-matrix of order $n-1$ obtained by deleting i^{th} row and j^{th} column. It is denoted by M_{ij}

Example: Let $A=\begin{pmatrix} 2 & 5 & 7 \\ 8 & 0 & 9 \\ 1 & 3 & 4 \end{pmatrix}$. Then Minor of 8 = $\begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix}$

Co-factor of an element in a Matrix of order n: Co-factor of an element a_{ij} in a Matrix of order n =$(-1)^{i+j} \times M_{ij}$ (Minor of an element a_{ij})

Example: Let $A=\begin{pmatrix} 2 & 5 & 7 \\ 8 & 0 & 9 \\ 1 & 3 & 4 \end{pmatrix}$. Then Co-factor of 8 =$(-1)^{1+2} \begin{vmatrix} 5 & 7 \\ 3 & 4 \end{vmatrix}$

Adjoint of a square matrix

Adjoint of a square matrix: Let $A = (a_{ij})_{nn}$. Let C_{ij} be the co-factor of a_{ij} in a Matrix $A = (a_{ij})_{nn}$. Then the transpose of the matrix (C_{ij}) is said to be the adjoint of a square matrix A. It is denoted by $adj A$.

Therefore $adj A = (C_{ij})^t$= transpose of the co-factor matrix.

Example: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$

Co-factor matrix = $\begin{pmatrix} +\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{pmatrix}$ = $\begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}$

$adj(A)$=$\begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$


Thank you!