Ordinary Differential Equations

(Lecture-2)

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4^{th} June, 2021

Learning Outcome of the Lecture

We learn

- Solution
- Initial Value Problems

Can we solve it?

Given an equation, you would like to solve it. At least, try to solve it.

Questions:

1. What is a solution?
2. Does an equation always have a solution? If so, how many?
3. Can the solutions be expressed in a nice form? If not, how to get a feel for it? (for example, properties of the solution)
4. How much can we proceed in a systematic manner? (analytical ? numerical?)
   order - first, second, $\cdots$, n-th, $\cdots$, linear or non-linear?

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Nature of Solutions

Consider the DE

$$F\left[x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right] = 0,$$

where F is a real function of its (n+2) arguments $x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}$.

Definition

Explicit Solution: An explicit solution of the ODE

$$F\left[x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right] = 0,$$

on the real interval I is a function $f(x)$ such that $f', f'', \dots, f^n$ exist and satisfy

$$F\left[x, f(x), f'(x), \dots, f^n(x)\right] = 0,$$

which is defined for all $x \in I$.

Nature of Solutions

Definition

Implicit Solution: A relation $g(x, y) = 0$ is called an implicit solution of

$$F\left[x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right] = 0,$$

if this relation defines at least one real function $f(x)$ on the interval I such that this function is an explicit solution of above defined DE on interval I.

Examples:

$$x^2 + y^2 - 25 = 0$$

is an implicit solution of

$$x + y\frac{dy}{dx} = 0$$

on the interval I defined by $-5 < x < 5$, because it defines two functions

$$f_1(x) = \sqrt{25 - x^2} \text{ and } f_2(x) = -\sqrt{25 - x^2}$$

for all $x \in I$. Both of these solutions are explicit solutions of the DE.

First order ODE & Initial Value Problem for first order ODE

We now consider first order ODE of the form $F(x, y, \frac{dy}{dx}) = 0$ or $\frac{dy}{dx} = f(x, y)$.

Consider a linear first order ODE of the form

$$\frac{dy}{dx} + a(x)y = b(x). \tag{1}$$

If $b(x) = 0$, then we say that the equation is homogeneous.

Think: Do the solutions of a homogeneous differential equation form a vector space under usual addition and scalar multiplication?

Initial Value Problems

Recall that first order ODE can be expressed as

$$F(x,y,y') = 0 \qquad \text{or} \qquad \frac{dy}{dx} = f(x,y).$$

Definition

Initial value problem (IVP): A differential equation along with an initial condition is called an initial value problem (IVP), i.e.,

$$\frac{dy}{dx} = f(x,y); \qquad y(x_0) = y_0.$$

Geometrically, the IVP is to find an integral curve of the DE that passes through the point (x_0, y_0).

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Example: Radioactivity - Exponential Decay

Problem: Given an amount of a radioactive substance, say $0.5gm$, find the amount present at any later time.

Solution: We solve the given problem in three steps:

- Step-1: Mathematical modeling of the problem Suppose $y(t)$ denotes the amount of substance still present at any time t. We know the time rate of change dy/dt is proportional to $y(t)$, i.e.,

$$\frac{dy}{dt} \propto y(t)$$

$$\frac{dy}{dt} = -Ky(t); \ K > 0$$

negative sign due to decay. Initially at time $t = 0$, amount is $0.5gm$, i.e., $y(0) = 0.5gm$.

Step-2: Mathematical solution of the problem Find solution of IVP:

$$\frac{dy}{dt} = -Ky(t);\ y(0) = 0.5gm.$$

Solution is $y(t) = ce^{-Kt},\ y(0) = 0.5 \Rightarrow c = 0.5$ i.e.,$y(t) = 0.5e^{-Kt}$.

Step-3: Interpretation of the solution The function $y(t)$ gives the amount of radioactive substance at time t. It is easy to see that $y(t) = 0.5e^{-Kt} \Rightarrow y(0) = 0.5and\frac{dy}{dt} = -K(0.5e^{-Kt}) = -Ky$. It starts from the correct initial amount and decreases with time because k is positive. The limit of y as $t \to \infty$ is zero.