1)

ample: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$, then find A^{-1} using determinant method.

olution: Now, $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

$$= 1 \begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} - 0 \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= (16-15) - 0 + (9-8) = 1 + 1 = 2 \neq 0$$

We know that $A^{-1} = \dfrac{adj(A)}{|A|} = \dfrac{adj(A)}{2}$.

Now co-factor matrix $= \begin{pmatrix} +\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}$$

$\therefore$ adj A = transpose of a co-factor matrix

$$= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}^{t} = \begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{2} \, adj \, A = \frac{1}{2} \begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$$

Example:

$$\begin{aligned} 3x + y + z &= 4 \\ x - y + 2z &= 6 \\ x + 2y - z &= -3 \end{aligned} \Rightarrow \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$$

(1) Here the coefficient determinant

$$D = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= -3 \neq 0.$$

So Cramer's rule can be applied.

Now,

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = -3.$$

$$D_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3.$$

$$D_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = -6$$

So by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{-3}{3} = 1$$

$$y = \frac{D_2}{D} = \frac{3}{-3} = -1$$

$$z = \frac{D_3}{D} = \frac{-6}{-3} = 2.$$

$\therefore$ the solution is $x = 1$, $y = -1$, $z = 2$.

Elementary Row operation:

An elementary row operation on a matrix $A_{m\times n}$ is an operation of the following three types:

type 1: The interchange of the ith and jth row is denoted by R_{ij}

type 2: Multiplication of the ith row by a non-zero scalar c is denoted by cR_i

type 3: Addition of c times the jth row to the ith row is denoted by $R_i + cR_j$

Example: let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$

Now example of type 1:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Now example of type 2:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{pmatrix}$$

Now example of type 3:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$