Ordinary Differential Equations

(Lecture-7)

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23^{rd} June, 2021

Learning Outcome of the Lecture

We learn

- Integrating Factors
- Linear Equations

Integrating Factor

Question: For what value of α the equation $\alpha y dx + \alpha 2x dy = 0$ is exact ?

Answer: $\alpha = y$.

Integrating Factor

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Definition

If the differential equation

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact in a domain D but the differential equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact in D, then $\mu(x,y)$ is called an integrating factor (IF) of the differential equation

Continuation of Previous Slide

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, that means

$$\frac{d(\mu M)}{dy} = \frac{d(\mu N)}{dx}.$$

Thus,

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

That is, $\mu(x,y)$ satisfies the DE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0. \tag{1}$$

a function $\mu(x,y)$ that solves the DE (1) is called an integrating factor of the given ODE $M(x,y)dx + N(x,y)dy = 0$.

Example: $(3y + 4xy^2)dx + (2x + 3x^2y)dy = O$

IF: $\mu(x,y) = x^2y$.

Finding Integrating Factor - Function of x alone

In practice, we start by looking for an IF which depends only on one variable x or y, because it may be difficult to solve the DE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0.$$

Case-1: Suppose μ is a function of x alone. That is, $\mu = \mu(x)$, $\mu_y = 0$. Then, the DE above reduces to

$$\mu_x N = (M_y - Nx)\mu.$$

Thus,

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu.$$

If further, $\frac{M_y - N_x}{N}$ is a function of x then the above DE is separable we try to solve it to find $\mu(x)$.

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

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Finding Integrating Factor - Function of y alone

Case-2: Suppose μ is a function of y alone in the DE

$$\mu_y M - \mu_x N + (M_y - N_x)\mu = 0.$$

That is, $\mu = \mu(y)$, $\mu_x = 0$. Then, we have

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu.$$

If further, $\frac{N_x - M_y}{M}$ is a function of y, then the above DE is separable we try to solve it to find $\mu(y)$.

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}} dy.$$

Example

Solve the differential equation $(2x^2 + y)dx + (x^2y - x)dy = 0$.

Answer: Here $M = 2x^2 + y \Rightarrow M_y = 1$ and $N = x^2y - x \Rightarrow N_x = 2xy - 1$. Clearly given ODE is not exact. We see that,

$$\frac{M_y - N_x}{N} = \frac{1 - (2xy - 1)}{x(xy - 1)} = \frac{2(1 - xy)}{x(xy - 1)} = \frac{-2}{x},$$

which depends upon x only, so integrating factor is

$$I.F. = e^{\int \frac{M_y - N_x}{N} dx} = e^{\int \frac{-2}{x} dx} = \frac{1}{x^2}.$$

Multiplying ODE by I.F., we get

$$(2 + \frac{y}{x^2})dx + (y - \frac{1}{x})dy = 0,$$

which is an exact ODE.

Example Cont.

Solution of exact ODE

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2}, \quad \frac{\partial F}{\partial y} = y - \frac{1}{x}$$

$$\frac{\partial F}{\partial x} = 2 + \frac{y}{x^2} \Rightarrow F(x,y) = 2x - \frac{y}{x} + \phi(y),$$

To determine unknown function $\phi(y)$, use condition $F_y = N$.

$$\frac{\partial F}{\partial y} = y - \frac{1}{x} \Rightarrow \frac{-1}{x} + \phi'(y) = y - \frac{1}{x} \Rightarrow \phi(y) = \frac{y^2}{2} + c_0$$

Solution of exact ODE is

$$2x - \frac{y}{x} + \frac{y^2}{2} = c.$$

Example

Solve the differential equation $xydx + (2x^2 + 3y^2 - 20)dy = 0$.

Answer:

$$M = xy \Rightarrow M_y = x, \quad N = 2x^2 + 3y^2 - 20 \Rightarrow N_x = 4x$$

Clearly given ODE is not exact. We see that,

$$\frac{N_x - M_y}{M} = \frac{4x - x}{xy} = \frac{3x}{xy} = \frac{3}{y},$$

which depends upon y only, so integrating factor is

$$I.F. = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{3}{y} dy} = y^3.$$

Multiplying ODE by I.F., we get

$$xy^4 dx + (2x^2y^3 + 3y^5 - 20y^3)dy = 0,$$

which is an exact ODE. Solution of exact ODE is

$$\frac{1}{2}x^2y^4 + \frac{1}{2}y^6 - 5y^4 = c.$$

Linear Equations

Definition

A first-order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Note that a linear ODE can be converted in an exact ODE by using integrating factor

$$\mu(x) = e^{\int p(x)dx}. \quad (\textit{Exercise})$$

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Solving Linear ODE

Theorem

The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu(x) = e^{\int p(x)dx}$$

A one-parameter family of solutions of this equation is

$$ye^{\int p(x)dx} = \int e^{\int p(x)dx} Q(x)d(x) + c,$$

or,

$$y = e^{-\int p(x)dx} \left(\int e^{\int p(x)dx} Q(x)d(x) + c \right).$$

Example - Linear ODE

Solve the ODE

$$\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}.$$

Answer: On comparing with $\frac{dy}{dx} + P(x)y = Q(x)$,
we get $P(x) = \frac{2x+1}{x}$ and $Q(x) = e^{-2x}$
Step-1: Find integrating factor.
$I.F. = e^{\int P(x)dx} = e^{\int \left(\frac{2x+1}{x}\right)dx} = e^{\int \left(2+\frac{1}{x}\right)dx} = xe^{2x}$
Step-2: Solution is given by

$$y \times I.F. = \int Q(x) \times I.F.dx + c$$

$$y \times xe^{2x} = \int e^{-2x} \times xe^{2x} dx + c$$

$$y = \frac{1}{2} xe^{-2x} + \frac{c}{x} e^{-2x},$$

where c is an arbitrary constant.