Lecture 12

Class Note

Nullity of a matrix $A_{m\times n}$:

If A is any $m\times n$ real matrix, then the dimension of the null space $\mathcal{N}(A)$ of $A_{m\times n}$ is called the nullity of A and is denoted by null (A)

Problem: Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Sol^n: $\mathcal{N}(A) = \{X \mid AX = 0\}$

$$= \left\{(x_1, x_2, x_3, x_4) \mid \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

$$= \left\{(x_1, x_2, x_3, x_4) \;\middle|\; \begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 + 3x_2 + 4x_3 + 5x_5 = 0 \end{matrix}\right\} \longrightarrow (1)$$

Now $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

$$\xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$\therefore$ System of eq^n (1) equavalent to the following system

$$x_1 - x_3 - 2x_4 = 0 \Rightarrow x_1 = x_3 + 2x_4$$

$$x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_2 = -2x_3 - 3x_4.$$

$\therefore$ Solution (x_1, x_2, x_3, x_4)

$$= (x_3 + 2x_4, -2x_3 - 3x_4, x_3, x_4)$$

$$= x_3(1, -2, 1, 0) + x_4(2, -3, 0, 1)$$

$\therefore \mathcal{N}(A) = \{(x_1, x_2, x_3, x_4)\}$

$= \{x_3(1, -2, 1, 0) + x_4(2, -3, 0, 1)\}$

$= \{L(S) \mid S = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$

$\therefore \mathcal{N}(A) = L(S)$ where $S = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$

$\longrightarrow (1)$

Now $c_1(1, -2, 1, 0) + c_2(2, -3, 0, 1) = 0$

$\Rightarrow \quad c_1 + 2c_2 = 0 \quad \Rightarrow \quad c_1 = c_2 = 0$

$-2c_1 - 3c_2 = 0$

$c_1 = 0$

$c_2 = 0$

$\Rightarrow S = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$ is linearly independent. $\longrightarrow (2)$

$\therefore$ From (1), (2), we get that $S = \{(1, -2, 1, 0), (2, -3, 0, 1)\}$ is a basis of null space $\mathcal{N}(A)$.

$\therefore$ dimension of null space

$=$ no of element in basis S

$= 2. =$ nullity of the matrix A.

Rank and nullity theorem of matrix $A_{m\times n}$:

Let A be any $m\times n$ real matrix. Let null (A) and rank (A) be respectively the nullity and rank of A. Then

$$\text{rank}(A) + \text{null}(A) = n = \text{number of columns of } A.$$

3) (i)

~~Example~~ Verify rank-nullity theorem the matrix.

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{pmatrix}.$$

Solⁿ: $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & -1 & 2 \\ -1 & -2 & 5 & 4 \end{pmatrix}$

$$\xrightarrow[R_3+R_1]{R_2-3R_1} \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & -1 & 7 & 7 \end{pmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-R_2} \begin{pmatrix} 1 & 0 & 9 & 10 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B$$

Rank (A) = no of non zero rows in row-~~reduced~~ echelon form $= 2$.

Null Space $= \mathcal{N}(A) = \{X \mid AX = 0\}$

$= \{(x_1, x_2, x_3, x_4) \mid BX = 0\}$ $(\because A \cong B)$

$= \left\{(x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{l} x_1 + 9x_3 + 10x_4 = 0 \\ x_2 - 7x_3 - 7x_4 = 0 \end{array}\right\}.$

$= \left\{(x_1, x_2, x_3, x_4) \,\middle|\, \begin{array}{l} x_1 = -9x_3 - 10x_4 \\ x_2 = +7x_3 + 7x_4 \end{array}\right\}$

$$N(A) = \{(-9x_3 - 10x_4,\ 7x_3 + 7x_4,\ x_3,\ x_4)\}$$
$$= \{x_3(-9, 7, 1, 0) + x_4(-10, 7, 0, 1)\}$$
$$= L(S) \text{ where } S = \{(-9, 7, 1, 0), (-10, 7, 0, 1)\}$$
$$\hookrightarrow (1)$$

Again $c_1(-9, 7, 1, 0) + c_2(-10, 7, 0, 1) = 0$

$\Rightarrow c_1 = c_2 = 0$

$\Rightarrow$ S is linearly independent. $\rightarrow$ (2)

From (1), (2) we get, S is a basis of $N(A)$

$\therefore$ nullity of A = dim $(N(A))$

= no of element in basis S

= 2.

$\therefore$ Rank (A) = 2

Null (A) = 2.

$\therefore$ Rank(A) + Null (A) = 2+2 = 4 = no of column of A.