Lecture - 9

Class Note

Theorem: The intersection of two subspaces of a vector space V over a field F is a subspace of V.

Proof: Let W_1 and W_2 be two subspaces of V.

Then $W_1 \cap W_2$ is not empty.

because $\theta \in W_1$ & $\theta \in W_2$ $\Rightarrow$ $\theta \in W_1 \cap W_2$

Case 1: Let $W_1 \cap W_2 = \{\theta\}$. Then $W_1 \cap W_2$ is a subspace of V ($\because$ The set containing only of the null vector θ of V forms a subspace of V. This subspace is called the trivial subspace of V)

$\underset{\text{zero vector}}{\text{null vector}}$

Case 2: Let $W_1 \cap W_2 \neq \{\theta\}$ and let

$\alpha_1 \in W_1 \cap W_2$, $\alpha_2 \in W_1 \cap W_2$

Then, $\alpha_1 \in W_1$, $\alpha_1 \in W_2$, and $\alpha_2 \in W_1$, $\alpha_2 \in W_2$

$\because$ W_1 is a subspace of V $\Rightarrow$ $c\alpha_1 + d\alpha_2 \in W_1$ $\forall\, c, d \in F$

$\because$ W_2 is a " " " $\Rightarrow$ $c\alpha_1 + d\alpha_2 \in W_2$ $\forall\, c, d \in F$

$\Rightarrow$ $c\alpha_1 + d\alpha_2 \in W_1 \cap W_2$ $\Rightarrow$ $W_1 \cap W_2$ is a sub-space of V

Note: The union of two sub-spaces of V is not, in general, a subspace of V

~~Examples~~ For example, let $S = \{(x, y, z) \in \mathbb{R}^3;\ y = 0, z = 0\}$ and $T = \{(x, y, z) \in \mathbb{R}^3;\ x = 0, z = 0\}$.

Then S and T is a two subspace in $\mathbb{R}^3$

Let $\alpha = (1,0,0) \in S$ and $\beta = (0,1,0) \in T$

Then $\alpha + \beta = (1,1,0) \notin S \cup T$ $\left(\because \alpha+\beta \notin S \text{ \& } \alpha+\beta \notin T\right)$

$\Rightarrow$ $S \cup T$ is not a subspace of $\mathbb{R}^3$.

Linear sum of two subspaces:

Let U and W be two subspaces of a vector space V over a field F. Then the subset $\{u+w : u \in U, w \in W\}$ is said to be the linear sum of the subspaces U and W.

~~Proof~~

Theorem: Let U and W be two subspaces of a vector space V over a field F. Then the linear sum $U+W$ is a sub-space of V.

Proof: Let $S = U + W = \{u+w : u \in U, w \in W\}$.

$\theta \in U, \theta \in W, \Rightarrow \theta \in S$ and therefore S is non-empty.

Let $\alpha_1, \alpha_2 \in S$, Then $\alpha_1 = u_1 + w_1$ for some $u_1 \in U, w_1 \in W$

$\alpha_2 = u_2 + w_2$ for some $u_2 \in U$, $w_2 \in W$.

Now,
$$c\alpha_1 + d\alpha_2 = c(u_1+w_1) + d(u_2+w_2)$$
$$= (cu_1 + du_2) + (cw_1 + dw_2) \in S.$$
$$\left(\because cu_1 + du_2 \in U,\ cw_1 + dw_2 \in W\right)$$
$$\forall\ c, d \in F$$

This proves that $S = U + W$ is a subspace of V.

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Linear ~~span~~ combination and linear span:

Let $v_1, v_2, \ldots, v_k \in V$ and $c_1, c_2, \ldots c_k \in F$

Then $c_1v_1 + c_2v_2 + \cdots + c_kv_k$ is called a finite linear combination of $v_1, v_2, \ldots, v_k$. (Here V = vector sp, F = Field)

- Let $S = \{v_1, v_2, \ldots, v_k\} \subseteq V$ (vector space)

let $L(S) = \left\{ c_1v_1 + c_2v_2 + \cdots + c_kv_k \mid c_1, c_2, \ldots, c_k \in F \right\}$

Then $L(S)$ is the set of all finite linear combinations of elements of the set $S \subseteq V$ and is called linear span of S.

Theorem: $L(S)$ is a sub-space in ~~vector~~ space a vector space V. and this is the smallest subspace that contain S.

Example: In $\mathbb{R}^3$, $\alpha = (4, 3, 5)$, $\beta = (0, 1, 3)$,

$\gamma = (2, 1, 1)$

Examine if (i) α is a linear combination of β and γ.

Solution: Let $\alpha = c\beta + d\gamma$ where $c, d \in \mathbb{R}$

Then $(4, 3, 5) = c(0, 1, 3) + d(2, 1, 1)$.

$= (0, c, 3c) + (2d, d, d)$

$\Rightarrow (4, 3, 5) = (2d, c + d, 3c + d)$

$\Rightarrow 2d = 4, \quad c + d = 3, \quad 3c + d = 5$

$\Rightarrow d = 2, \quad c = 1$

Hence $\alpha = \beta + 2\gamma$ and α is a linear combination of β and γ.