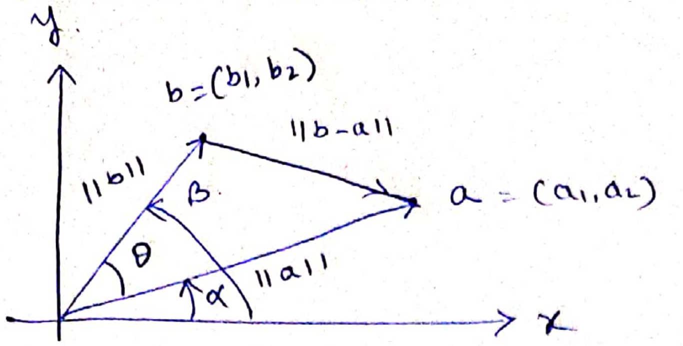
Lecture - 20

Class Note

Angle between two vectors:



$$\therefore \quad \sin\alpha = \frac{a_2}{\|a\|} \qquad \cos\alpha = \frac{a_1}{\|a\|}$$

$$\sin\beta = \frac{b_2}{\|b\|} \qquad \cos\beta = \frac{b_1}{\|b\|}$$

Now $\cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$

$$= \frac{a_1 b_1 + a_2 b_2}{\|a\|\,\|b\|}$$

$$\boxed{\cos\theta = \frac{\langle a . b\rangle}{\|a\|\,\|b\|}}$$

Law of cosines:

$$\|b-a\|^2 = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta .$$

$$\Rightarrow \langle b-a . b-a\rangle = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta .$$

$$\Rightarrow \langle b . b-a\rangle - \langle a . b-a\rangle = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta .$$

$$\Rightarrow \langle b-a . b\rangle - \langle b-a . a\rangle = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow \langle b . b\rangle - \langle a . b\rangle - \langle b . a\rangle + \langle a . a\rangle = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow \cancel{\|b\|^2} - 2\langle a . b\rangle + \cancel{\|a\|^2} = \cancel{\|b\|^2} + \cancel{\|a\|^2} - 2\|b\|\|a\|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\langle a . b\rangle}{\|a\|\,\|b\|}$$

Scanned with CamScanner

The Gram - Schmidt Orthogonalization process:

Any basis $x_1, x_2, ---, x_n$	$\longrightarrow$	Orthogonal basis $v_1, v_2, ---, v_n$.

let V be a vector space with an inner product

Suppose $\{x_1, x_2, ---, x_n\}$ is a basis for V.

let.
$$v_1 = x_1$$
$$v_2 = x_2 - \frac{\langle x_2 . v_1 \rangle}{\langle v_1 . v_1 \rangle} v_1$$
$$v_3 = x_3 - \frac{\langle x_3 . v_1 \rangle}{\langle v_1 . v_1 \rangle} v_1 - \frac{\langle x_3 . v_2 \rangle}{\langle v_2 . v_2 \rangle} v_2 .$$

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$$v_n = x_n - \frac{\langle x_n . v_1 \rangle}{\langle v_1 . v_1 \rangle} v_1 - - - - - - \frac{\langle x_n . v_{n-1} \rangle}{\langle v_{n-1} . v_{n-1} \rangle} v_{n-1}$$

Then $\{v_1, v_2, ---- v_n\}$ is an orthogonal.

basis for V.

Gram-Schmidt process combines Orthogonalization with normalization.

Suppose $\{x_1, x_2, \ldots, x_n\}$ is a basis for an inner product space V. Let

$$v_1 = x_1, \quad w_1 = \frac{v_1}{\|v_1\|}$$

$$v_2 = x_2 - \langle x_2 \cdot w_1 \rangle w_1, \quad w_2 = \frac{v_2}{\|v_2\|}$$

$$v_3 = x_3 - \langle x_3 \cdot w_1 \rangle w_1 - \langle x_3 \cdot w_2 \rangle w_2, \quad w_3 = \frac{v_3}{\|v_3\|}$$

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$$v_n = x_n - \langle x_n \cdot w_1 \rangle w_1 - \langle x_n \cdot w_2 \rangle w_2 - \cdots - \langle x_n \cdot w_{n-1} \rangle w_{n-1}, \quad w_n = \frac{v_n}{\|v_n\|}$$

Then $\{w_1, w_2, \ldots, w_n\}$ is a orthonormal basis for V.

⊞ Another Process:

Suppose $\{x_1, x_2, \ldots x_n\}$ is a basis for V

Then $\{v_1, v_2, \ldots v_n\}$ is orthogonal basis for V (using Gram-Schmidt process)

Then $\left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots, \frac{v_n}{\|v_n\|}\right\}$ is orthonormal basis for V.

Problem: Let $\{(1,2,2), (-1,0,2), (0,0,1)\}$ is a basis of $\mathbb{R}^3$ (i) Find an orthogonal basis for $\mathbb{R}^3$.

(ii) " " orthonormal " " $\mathbb{R}^3$.

Solⁿ: Let $x_1 = (1,2,2)$, $x_2 = (-1,0,2)$, $x_3 = (0,0,1)$.

Let $v_1 = x_1 = (1,2,2)$

$$v_2 = x_2 - \frac{\langle x_2 . v_1\rangle}{\langle v_1 . v_1\rangle} v_1$$

$$= (-1,0,2) - \frac{3}{9}(1,2,2)$$

$$= (-4/3, -2/3, 4/3)$$

$$v_3 = x_3 - \frac{\langle x_3 . v_1\rangle}{\langle v_1 . v_1\rangle} v_1 - \frac{\langle x_3 . v_2\rangle}{\langle v_2 . v_2\rangle} v_2$$

$$= (0,0,1) - \frac{2}{9}(1,2,2) - \frac{4/3}{4}\left(-\frac{4}{3}, -\frac{2}{3}, \frac{4}{3}\right).$$

$$= (2/9, -2/9, 1/9)$$

$\therefore$ $\{v_1, v_2, v_3\}$ is orthogonal basis for $\mathbb{R}^3$

Now $\|v_1\| = \sqrt{\langle v_1 . v_1\rangle} = 3$

$\|v_2\| = \sqrt{\langle v_2 . v_2\rangle} = 2.$

$\|v_3\| = \sqrt{\langle v_3 . v_3\rangle} = 1/3.$

$\therefore$ $w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{3}(1,2,2)$,

$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{3}(-2,-1,2)$

$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{3}(2,-2,1)$

$\therefore$ $\{w_1, w_2, w_3\}$ is orthonormal basis for $\mathbb{R}^3$.