Ordinary Differential Equations

(Lecture-10)

Neelam Choudhary

Department of Mathematics
Bennett University
India

28^{th} June, 2021

Learning Outcome of the Lecture

We learn

- Example - Existence and Uniqueness Theorem
- Picard's Iteration Method
- Summary of First Order ODE

Existence-Uniqueness Theorem

Theorem

Let R be a rectangle and (x_0, y_0) be an interior point of R, let

- $f(x,y)$ be continuous at all points (x,y) in

$$R \ : \ |x - x_0| \leq a, \ |y - y_0| \leq b \quad \text{and}$$

- Bounded in R, that is, $|f(x,y)| \leq K$ for all $(x,y) \in R$.

Then, the IVP $\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$ has at least one solution $y(x)$ defined for all x in the interval $|x - x_0| \leq \alpha$; where

$$\alpha = \min\{a, \frac{b}{K}\}.$$

In addition, if f satisfy the Lipschitz condition with respect to y in R, that is,

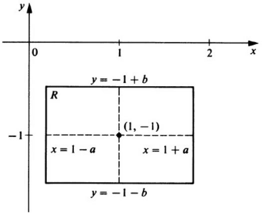
$$|f(x,y_1) - f(x,y_2)| \leq M|y_1 - y_2|, \text{ for all } (x,y_1),(x,y_2) \in R.$$

then, the solution $y(x)$ defined at least for all x in the interval $|x - x_0| \leq \alpha$, with α defined above is unique.

Example

Consider $\frac{dy}{dx} = y^2, \; y(1) = -1$. Find α in the existence and uniqueness theorem.

- $f(x, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are both continuous for all (x, y). Thus f satisfies the hypothesis of existence-uniqueness theorem in every rectangle R, $|x-1| \leq a, \quad |y+1| \leq b,$



- $|y+1| \leq b$, equivalently, $-b < y+1 < b$, this implies

$$|f(x, y)| = |y|^2 \leq |(-b-1)|^2 \leq (b+1)^2.$$

This implies $\alpha = \min\{a, \frac{b}{(b+1)^2}\}$.

Example Continue..

- Consider, $F(b) = \frac{b}{(b+1)^2}$.

  $F'(b) = \frac{1-b}{(b+1)^2} \Rightarrow$ the maximum value of $F(b)$ for $b > 0$ occurs at $b = 1$,and we find $F(1) = \frac{1}{4}$.
- Hence, if $a \geq \frac{1}{4}$, then $F(b) = \frac{b}{(b+1)^2} \leq a$ for all b.

  Thus $\alpha = \min\{a, F(b)\} = F(b) \leq \frac{1}{4}$, whatever be a.

  If $a < \frac{1}{4}$, then certainly $\alpha < \frac{1}{4}$.

  For $b = 1$, $a \geq \frac{1}{4}$, $\alpha = \min\{a, \frac{1}{4}\} = \frac{1}{4}$.

  Thus in any case, $\alpha \leq \frac{1}{4}$.

This $\alpha = \frac{1}{4}$ is the best possible, according to the theorem, the IVP has a unique solution in

$$|x - 1| \leq \frac{1}{4} \quad \Rightarrow \quad \frac{3}{4} \leq x \leq \frac{5}{4}.$$

Picard's Iteration Method

Objective: To solve

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{1}$$

Procedure:

1. Integrate both side of (1) to obtain

$$\begin{aligned} y(x) - y(x_0) &= \int_{x_0}^{x} f(t, y(t))dt \\ y(x) &= y_0 + \int_{x_0}^{x} f(t, y(t))dt. \end{aligned} \tag{2}$$

BENNETT UNIVERSITY
A TIMES GROUP INITIATIVE

Continuation of Previous Slide

2. Solve (2) by iteration:

$$
\begin{aligned}
y_1(x) &= y_0 + \int_{x_0}^{x} f(t, y_0)\,dt \\
y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1)\,dt \\
&\;\;\vdots \\
y_n(x) &= y_0 + \int_{x_0}^{x} f(t, y_{n-1})\,dt.
\end{aligned}
$$

3. Under the assumptions of existence-uniqueness theorem, the sequence of approximations converges to the solution $y(x)$ of (1). That is,

$$y(x) = \lim_{n \to \infty} y_n(x).$$

Example - Picard Method

Solve : $\frac{dy}{dx} = xy, \quad y(0) = 1,$ using Picard's iteration method.

1. The integral equation is

$$y(x) = 1 + \int_{x_0}^{x} ty(t)\,dt.$$

2. The successive approximations are :

$$\begin{aligned} y_1(x) &= 1 + \int_0^x t \cdot 1 dt = 1 + \frac{x^2}{2} \\ y_2(x) &= 1 + \int_0^x t \cdot (1 + \frac{t^2}{2}) dt = 1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} \\ y_n(x) &= 1 + \frac{x^2}{2} + \frac{1}{2!}(\frac{x^2}{2})^2 + \cdots + \frac{1}{n!}(\frac{x^2}{2})^n. \text{ (By induction)} \end{aligned}$$

3. $y(x) = \lim_{n\to\infty} y_n(x) = e^{\frac{x^2}{2}}.$

Summary of First Order ODE

- Linear Equations - Solution
  - Reducible to linear - Bernoulli
- Non-linear equations
  - Variable separable
  - Reducible to variable separable
  - Exact equations - Integrating factors
  - Reducible to Exact
- Existence and Uniqueness theorem for IVP :

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0.$$

- Picard's iteration method