Department of Mathematics
Bennett University
EMAT102L: Ordinary Differential Equations
Tutorial Sheet-1 Solutions

1) Classify each of the following differential equation as linear or nonlinear. Also find the order and degree of differential equation:

$$(a)\quad x^2dy + y^2dx = 0; \quad (b)\quad \frac{d^2y}{dx^2} + x\sin y = 0; \quad (c)\quad \frac{d^6y}{dx^6} + \frac{d^4y}{d^4x}\frac{d^3y}{d^3x} + y = x;$$
$$(d)\quad \left(\frac{dy}{dx}\right)^3 = \sqrt{\frac{d^2y}{d^2x} + 1}.$$

Solution:

(a) Nonlinear, Order one, Degree one.
(b) Nonlinear, Order two, Degree one.
(c) Nonlinear, Order six, Degree one.
(d) Nonlinear, Order two, Degree one.

2) Verify that y is a solution of the ODE. Determine from y the particular solution of the IVP.

$$(a)\quad \frac{dy}{dx} = y - y^2; \quad y = \frac{1}{1+ce^{-x}}, \quad y(0) = \frac{1}{4},$$
$$(b)\quad \frac{dy}{dx} = y + e^x; \quad y = (x+c)e^x, \quad y(0) = \frac{1}{2}.$$

Solution:

(a)

$$y = \frac{1}{1+ce^{-x}} \Rightarrow \frac{dy}{dx} = \frac{ce^{-x}}{(1+ce^{-x})^2}.$$

Also,

$$y - y^2 = \frac{1}{1+ce^{-x}} - \frac{1}{(1+ce^{-x})^2} = \frac{ce^{-x}}{(1+ce^{-x})^2} = \frac{dy}{dx}.$$

Therefore y is a solution of the give IVP. For particular solution,

$$y(0) = \frac{1}{4} \;\Rightarrow\; \frac{1}{4} = \frac{1}{1+c} \;\Rightarrow\; c = 3.$$

Putting the value of c in y we get the particular solution, that is

$$y_p = \frac{1}{1+3e^{-x}}.$$

(b)

$$y = (x+c)e^x \Rightarrow \frac{dy}{dx} = (1+x+c)e^x.$$

Also

$$y + e^x = (x+c)e^x + e^x = (1+x+c)e^x = \frac{dy}{dx}.$$

Therefore y is a solution of the give IVP. For particular solution,

$$y(0) = \frac{1}{2} \ \Rightarrow \ \frac{1}{2} = c.$$

Putting the value of c in y we get the particular solution, that is

$$y_p = (x + \frac{1}{2})e^x.$$

3) Consider the differential equation $\frac{dy}{dx} = y^2 + 4$.

   (a) Show that there exist no constant solutions of the DE.

   (b) Can a solution curve have any relative extrema?

Solution:

(a) The differential equation $\frac{dy}{dx} = y^2 + 4$ implies that $\frac{dy}{dx} > 0$ for all x. Therefore the slope of any solution curve should be increasing. Thats why there exist no constant solutions of the given DE.

(b) No. As mentioned in the answer of part (a), the slope of any solution curve $y = f(x)$ must be strictly increasing. Also since $\frac{dy}{dx}$ can never equal zero, it follows that a solution curve cannot have any relative extrema at any point on it.

4) Solve the following ODEs:

$$(a) \quad \frac{dy}{dx} = (x+1)e^{-x}y^2; \quad (b) \quad \frac{dy}{dx} = \sec^2 y;$$
$$(c) \quad 2xy\frac{dy}{dx} = y^2 - x^2; \quad (d) \quad x\frac{dy}{dx} = y + 3x^4\cos^2(y/x); \ y(1) = 0.$$

Solution:

(a)

$$\frac{dy}{dx} = (x+1)e^{-x}y^2 \ \Rightarrow \ \frac{dy}{y^2} = (x+1)e^{-x}dx,$$

clearly ODE is separable in variables x and y, thus taking the integration we get

$$\begin{aligned}
\int \frac{dy}{y^2} &= \int (x+1)e^{-x}dx + c \\
\Rightarrow -\frac{1}{y} &= -(x+1)e^{-x} + \int e^{-x}dx + c \\
\Rightarrow -\frac{1}{y} &= -(x+1)e^{-x} - e^{-x} + c = -e^{-x}(x+2) + c \\
\Rightarrow y &= \frac{1}{(x+2)e^{-x} - c},
\end{aligned}$$

where c is an arbitrary constant.

(b)

$$\frac{dy}{dx} = \sec^2 y \Rightarrow \cos^2 y dy = dx \Rightarrow \frac{1}{2} \cdot (1 + \cos 2y)dy = dx.$$

Taking the integration we get

$$\frac{y}{2} + \frac{\sin 2y}{4} = x + c \Rightarrow 2y + \sin 2y = 4(x + c),$$

where c is an arbitrary constant.

(c)

$$2xy\frac{dy}{dx} = y^2 - x^2 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

This is a homogenous ODE, therefore let $y = vx$, then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Putting this into the give ODE we get

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{v^2x^2 - x^2}{2x^2v} = \frac{v^2 - 1}{2v} \\ \Rightarrow \quad x\frac{dv}{dx} &= \frac{v^2 - 1 - 2v^2}{2v} = -\frac{1 + v^2}{2v} \\ \Rightarrow \quad & \frac{2vdv}{1 + v^2} + \frac{dx}{x} = 0. \end{aligned}$$

Taking the integration we get

$$ln(1 + v^2) + ln|x| = c \;\Rightarrow\; (1 + v^2)x = e^c \;\Rightarrow\; (x^2 + y^2) = Cx,$$

where $C = e^c$ is an arbitrary constant.

(d)

$$x\frac{dy}{dx} = y + 3x^4\cos^2(y/x) \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{x} + 3x^3\cos^2(y/x).$$

Let $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Putting this into the give ODE we get

$$\begin{aligned} v + x\frac{dv}{dx} &= v + 3x^3\cos^2(v) \\ \Rightarrow \quad \frac{dv}{\cos^2 v} &= 3x^2dx \;\Rightarrow\; \sec^2 v dv = 3x^2dx \Rightarrow \quad \tan v = x^3 + c \quad \Rightarrow \tan\frac{y}{x} = x^3 + c, \end{aligned}$$

where c is an arbitrary constant. Use initial condition,

$$y(1) = 0 \Rightarrow 0 = 1^3 + c \Rightarrow c = -1.$$

Therefore the solution is

$$\tan\frac{y}{x} = x^3 - 1.$$

5) Solve the following ODEs:

$(a) \quad (x\tan(y/x) + y)dx - xdy = 0; \quad (b) \quad (5x + 2y + 1)dx + (2x + y + 1)dy = 0.$

Solution:

(a)

$$(x\tan(y/x)+y)dx - xdy = 0 \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}.$$

Let $y = vx$, and follow the steps in question $4(c)$.

The answer is $\sin(y/x) = cx$.

(b)

$$(5x+2y+1)dx + (2x+y+1)dy = 0 \;\Rightarrow\; \frac{dy}{dx} = -\frac{(5x+2y+1)}{(2x+y+1)}$$

. Since $\frac{5}{2} \neq \frac{2}{1}$, this ODE can be transformed into an homogenous equation by using the transformations $x = X + h$ and $y = Y + k$. This gives

$$\frac{dY}{dX} = -\frac{5X + 2Y + (5h + 2k + 1)}{2X + Y + (2h + k + 1)}.$$

Take (h, k) such that $5h+2k+1 = 0$ and $2h+k+1 = 0$, we get $h = 1$ and $k = -3$. Thus again,

$$\frac{dY}{dX} = -\frac{5X + 2Y}{2X + Y}.$$

To solve this homogeneous equation use the transformation $Y = vX$, this gives

$$\begin{aligned} X\frac{dv}{dX} &= -\frac{v^2+4v+5}{2+v} \\ \frac{(2+v)dv}{v^2+4v+5} &= -\frac{dX}{X}. \end{aligned}$$

Solution of this equation is

$$(v^2+4v+5)X^2 = c_0 \;\Rightarrow\; Y^2 + 4XY + 5X^2 = c_0.$$

Put the values $X = x - h = x - 1$ and $Y = y - k = y + 3$.We get

$$\begin{aligned} &(y+3)^2 + 4(x-1)(y+3) + 5(x-1)^2 = c_0, \\ \Rightarrow\quad & 5x^2 + 4xy + y^2 + 2x + 2y = c. \end{aligned}$$