Tutorial 11-Solution

1. We can construct a bit string of length eight that either starts with a 1 bit or ends with the two bits 00, by constructing a bit string of length eight beginning with a 1 bit or by constructing a bit string of length eight that ends with the two bits 00. We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways. This follows by the product rule, because the first bit can be chosen in only one way and each of the other seven bits can be chosen in two ways. Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways. This follows by the product rule, because each of the first six bits can be chosen in two ways and the last two bits can be chosen in only one way.
Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00. There are $2^5 = 32$ ways to construct such a string. This follows by the product rule, because the first bit can be chosen in only one way, each of the second through the sixth bits can be chosen in two ways, and the last two bits can be chosen in one way. Consequently, the number of bit strings of length eight that begin with a 1 or end with a 00, which equals the number of ways to construct a bit string of length eight that begin with a 1 or that ends with 00, equals 128 + 64 - 32 = l60

2. a) Suppose there are four boxes, one for each suit, and as cards are selected they are placed in the box reserved for cards of that suit. Using the generalized pigeonhole principle, we see that if N cards are selected, there is at least one box containing at least $\lceil N/4 \rceil$ cards. Consequently, we know that at least three cards of one suit are selected if $\lceil N/4 \rceil \geq 3$. The smallest integer N such that $\lceil N/4 \rceil \geq 3$ is $N = 2 \cdot 4 + 1 = 9$, so nine cards suffice. Note that if eight cards are selected, it is possible to have two cards of each suit, so more than eight cards are needed. Consequently, nine cards must be selected to guarantee that at least three cards of one suit are chosen. One good way to think about this is to note that after the eighth card is chosen, there is no way to avoid having a third card of some suit.
b) We do not use the generalized pigeonhole principle to answer this question, because we want to make sure that there are three hearts, not just three cards of one suit. Note that in the worst case, we can select all the clubs, diamonds, and spades, 39 cards in all, before we select a single heart. The next three cards will be all hearts, so we may need to select 42 cards to get three hearts.

3. There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

4. Suppose that we label the workstations W_1 , W_2 , …., W_{15} and the servers S_1 , S_2 , . . . , S_{10}. Furthermore, suppose that we connect W_k to S_k for k = 1 , 2, . . . , 10 and each of W_{11} , W_{12} , W_{13} , W_{14}, and W_{15} to all 1 0 servers. We have a total of 60 direct connections. Clearly any set of 10 or fewer workstations can simultaneously access different servers. We see this by noting that if workstation W_j is included with $1 \leq j \leq 10$, it can access server Sj' and for each workstation W_k with $k \geq 11$ included, there must be a corresponding workstation W_j with $1 \leq j \leq 10$ not included, so W_k can access server S_j . (This follows because there are at least as many available servers S_j as there are workstations W_j with $1 \leq j \leq 10$ not included.) Now suppose there are fewer than 60 direct connections between workstations and servers. Then some server would be connected to at most $\lfloor 59/10 \rfloor = 5$ workstations. (lf all servers were connected to at least six workstations, there would be at least 6 * 10 = 60 direct connections.) This means that the remaining nine servers are not enough to allow the other 10

workstations to simultaneously access different servers. Consequently, at least 60 direct connections are needed. It follows that 60 is the answer.

5. There are eight million different phone numbers of the form NXX -XXXX. Hence, by the generalized pigeonhole principle, among 25 million telephones, at least $\lceil 25,000,000/8,000,000 \rceil$ of them must have identical phone numbers. Hence, at least four area codes are required to ensure that all 1 0-digit numbers are different.

6. Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is P(l00, 3) = 100 ·99 · 98 = 970,200.

7. The number of possible paths between the cities is the number of permutations of seven elements, because the first city is determined, but the remaining seven can be ordered arbitrarily. Consequently, there are 7! = 7 · 6 · 5 · 4 · 3 · 2 · 1 = 5040 ways for the saleswoman to choose her tour. If, for instance, the saleswoman wishes to find the path between the cities with minimum distance, and she computes the total distance for each possible path, she must consider a total of 5040 paths!

8. Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are 6! = 720 permutations of the letters ABCDEFGH in which ABC occurs as a block.

9. The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of set with 30 elements, because the order in which these people are chosen doesn't matter. The number of such combinations is:

$$C(30,6) = \frac{30!}{6!\,24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775.$$

10. By the product rule, the answer is the product of the number of 3 – combinations of set with nine elements and the number of 4- combinations of set with 11 elements. The number of ways to select the committee is:

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27,720.$$

11. Solution:
Total seats are 8
5people are to be seated.
2 in front row
3 in back row
First, we have to choose two people who will be seated in the front row. This can be done in C(5,2) ways.
After 2 people have been chosen, they can be seated in the front row in P(4,2) ways and the remaining three people in the back row in P(4,3)ways.
Thus, total number of ways=C(5,2)*P(4,2)*P(4,3)=2880 ways

12.

<u>Soln</u>;

$$x_1 = y_1 + 2$$
$$x_2 = y_2 + 4$$
$$x_3 = y_3 + 5$$
$$x_4 = y_4 + 6$$

where ($y_i \geq 0$, $1 \leq i \leq 4$)

Now, the given equation is equivalent to

$$y_1 + 2 + y_2 + 4 + y_3 + 5 + y_4 + 6 = 30$$

or

$$y_1 + y_2 + y_3 + y_4 = 13$$

$n = 4$, $r = 13$

$\therefore$ Total no. of combinations $= C(4+13-1, 13)$
$= 560$ Ans.