Ordinary Differential Equations

(Lecture-8)

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Learning Outcome of the Lecture

We learn

- Bernoulli Equation
- Equations Reducible to Linear Equations
- Orthogonal Trajectories

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Bernoulli Equation - (Non-Linear Reducible to Linear)

Definition

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Proof If we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

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Continuation of Previous Slide

Equation (3) transforms into

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x),$$

or, equivalently,

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x).$$

Letting

$$P_1(x) = (1-n)P(x) \quad \text{and} \quad Q_l(x) = (1-n)Q(x),$$

this may be written

$$\frac{dv}{dx} + P_l(x)v = Q_l(x).$$

which is linear in v.

Example - Bernoulli Equation

Example: Solve the ODE $\frac{dy}{dx} + y = xy^3$.

Answer: **Step-1:** Rewrite ODE as $y^{-3}\frac{dy}{dx} + y^{-2} = x$.
Take the transformation
$v = y^{1-n} = y^{-2} \Rightarrow \frac{dv}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow y^{-3}\frac{dy}{dx} = \frac{-1}{2}\frac{dv}{dx}$
Now ODE is transformed into

$$\frac{-1}{2}\frac{dv}{dx} + v = x \Rightarrow \frac{dv}{dx} - 2v = -2x,$$

which is a linear ODE in v and x.
Step-2: Solve linear ODE. Integration factor=$e^{\int -2dx} = e^{-2x}$ Solution is given by

$$v \times I.F. = \int (-2x) \times I.F.dx$$

$$v \times e^{-2x} = \int (-2x) \times e^{-2x} dx$$

$$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x},$$

where c is an arbitrary constant.

Equations Reducible to Linear Equations

Consider

$$\frac{d}{dy}(f(y))\frac{dy}{dx} + P(x)f(y) = Q(x),$$

where f is an unknown function of y.
Set $v = f(y)$. Then,

$$\frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{d}{dy}(f(y))\frac{dy}{dx}.$$

Hence the given equation is

$$\frac{dv}{dx} + P(x)v = Q(x),$$

which is linear in v:

Remark: Bernoulli DE is a special case when $f(y) = y^{1-n}$.

Orthogonal Trajectories

Definition

Let

$$F(x, y, c) = 0 \tag{2}$$

be a given one-parameter family of curves in the xy-plane. A curve that intersects the curves of the family (2) at right angles is called an orthogonal trajectory of the given family.

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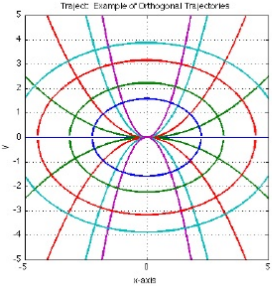

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- Find the DE of the family (3).

$$\frac{dy}{dx} = f(x, y),$$

by first differentiating (3) implicitly with respect to x and then eliminating the parameter c between the derived equation so obtained and the given equation (3) itself.

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- Obtain a one parameter family of curves $G(x, y, c) = 0$ as solutions of the above DE (4), gives the family of orthogonal trajectories.

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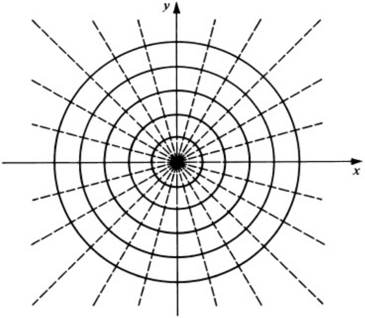

Example - Orthogonal Trajectories

Step-1: Find DE of given family of curves $x^2 + y^2 = c^2$.

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Step-2: Find slope of the orthogonal trajectories

$$\frac{dy}{dx} = \frac{-1}{f(x,y)} = \frac{y}{x}.$$

Step-3: Solve DE of orthogonal trajectories

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow y = kx.$$

$y = kx$ gives the family of orthogonal trajectories.