Functions

- → Used in programming languages.
- → What is the mathematical basis of functions?
- → A function may be defined by a formula that tells how to calculate the output for a given input.

$$\text{Input } x \Rightarrow \boxed{\text{function}} \Rightarrow \text{Output } f(x)$$

Eg: $f(x) = x - 1$

(I) Definition of a Function

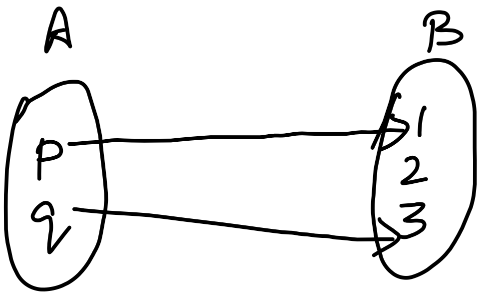
→ It is a special type of relation with following properties:

(a) $\forall x \in$ Domain, there is a mapping.

(b) Unique image $\forall x \in$ Domain.

Eg: $A = \underbrace{\{p, q\}}_{\text{Domain}}$ $B = \underbrace{\{1, 2, 3\}}_{\text{Co-Domain}}$

$F : \underbrace{A \to B}_{\text{Mapping}}$



$\{p, q\}$ → Domain or Pre-image

$\{1, 3\}$ → Range or Image

→ Let X and Y be two non-empty sets.

→ A function $f : X \to Y$ where,

Set $X \to$ domain

Set $Y \to$ Co-domain

→ f maps every element $x \in X$ to the element $y \in Y$ and can be written as

$$\boxed{y = f(x)}$$

→ The element $y \in Y$ } → Image of $x \in X$

→ The element $x \in X$ } → Pre-image of $y \in Y$

→ The set of all image values $\{ f(x) : x \in X \}$ is called the range of f.

Note: Range is always a subset of co-domain

Dom (f) → Domain

Ran (f) → Range

Function denotes a mapping / transformation.

(4) Relation V/S Function

→ A function $f : X \to Y$ is a special kind of relation $R : X \to Y$ if it satisfies the following additional properties :

1. Every element $x \in X$ has an image $y \in Y$.
2. One element of X can have only one image, that is

$$\boxed{\text{if } (x,y) \in f \text{ and } (x,z) \in f \text{ then } y = z}$$

(b) Graphical Determination of a Relation as a Function.

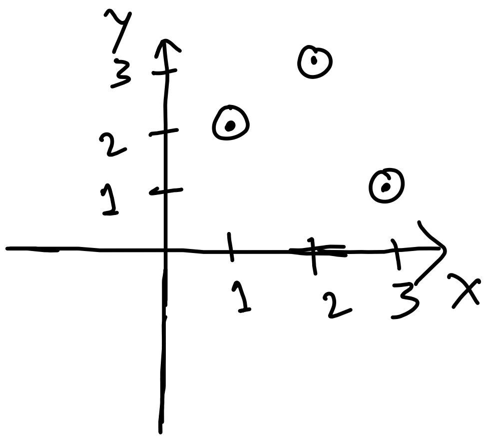
→ Use vertical line test.

→ If the line intersects the graph of the relation at more than one point then not a function else a function.

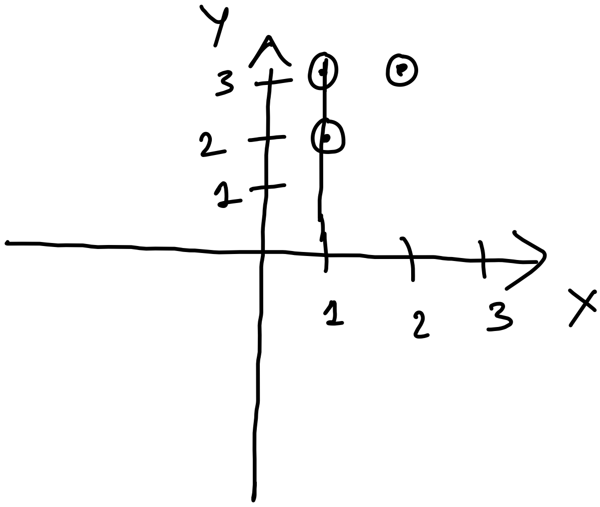
Eg: Let $X = \{1, 2, 3\}$

$R_1 = \{(1,2), (2,3), (3,1)\}$

$R_2 = \{(1,2), (2,3), (1,3)\}$



R₁ is a function



R₂ is not a function

(c) Why do we need a function?

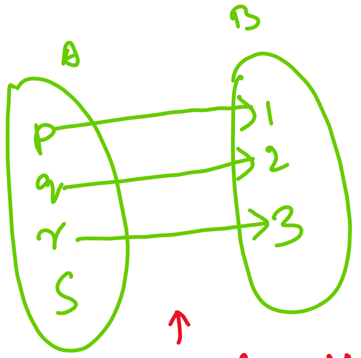
ADD(2,3) → Guaranteed Result
→ Unambiguous Output

Result will always be 5 and not 1, 10, 20 etc.

8) Determine which of the following are functions:

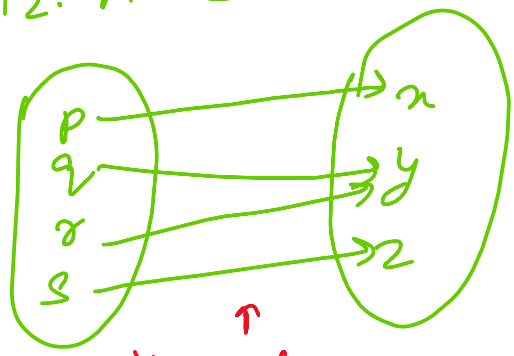
Let $A = \{p, q, r, s\}$, $B = \{1, 2, 3\}$, $C = \{x, y, z\}$

$F_1 : A \to B$



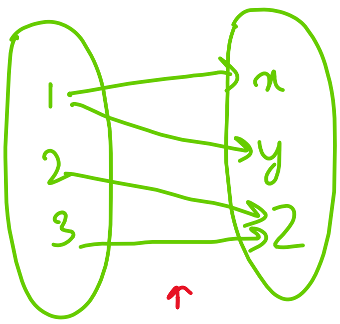
Not a function

$F_2 : A \to C$



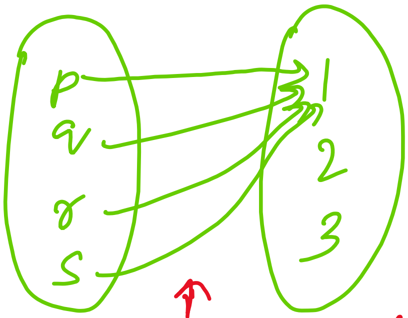
Yes, a function

$F_3 : B \to C$



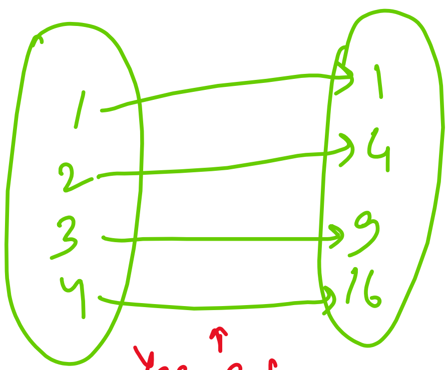
Not a function

$F_4 : A \to B$



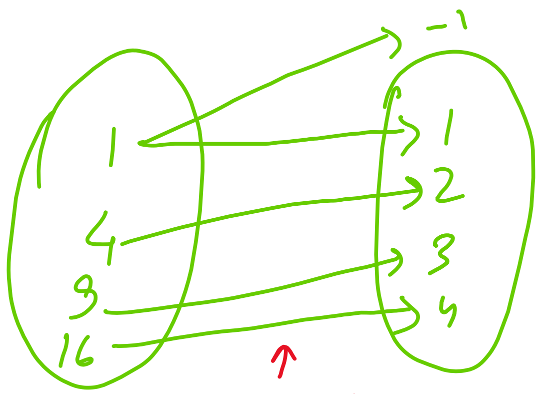
Yes, a function (Constant fn)

$F = \{(x, x^2) \mid x \in \mathbb{R}\}$



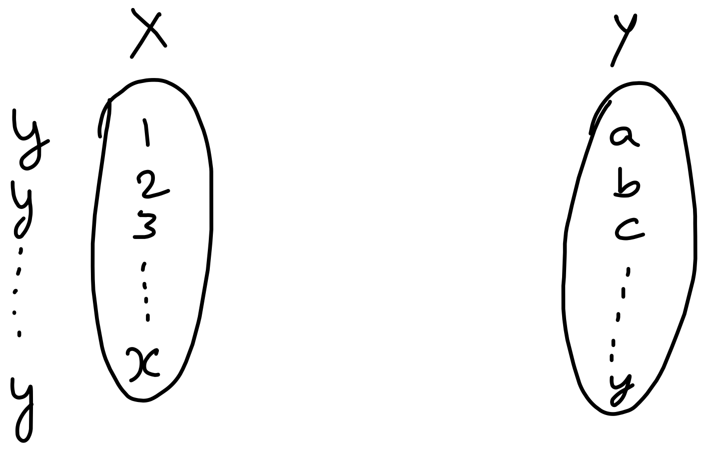
Yes, a fn

$G = \{(x^2, x) \mid x \in \mathbb{R}\}$



Not a fn

Q) How many functions are possible from a set of 'x' elements to a set of 'y' elements?



$$n(f) = y \times y \times y \times \cdots\cdots x \text{ times}$$

$$\boxed{n(f) = y^x}$$

(II) Types of Functions

(a) One - One Function

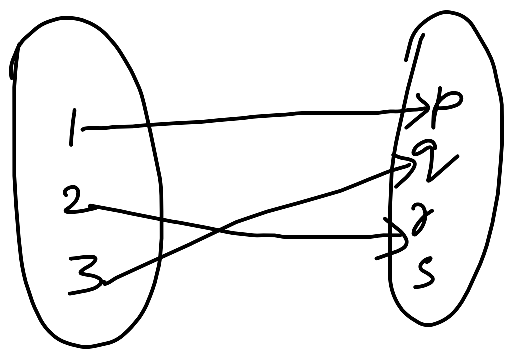
$f : X \to Y$ is one to one if

$$f(x_1) = f(x_2) \Rightarrow (x_1) = (x_2)$$

or

$$(x_1) \neq (x_2) \Rightarrow f(x_1) \neq f(x_2)$$

Eg:-



One - one

(Injection)

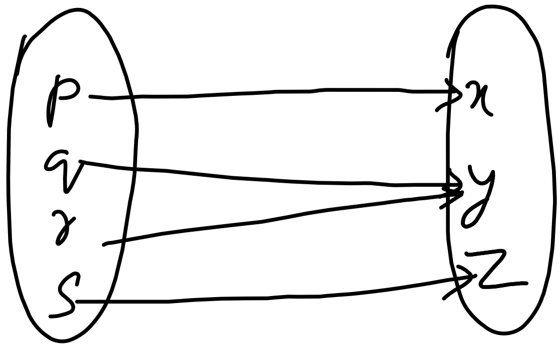
(b) Onto Function

$f : X \to Y$ is an onto function if

Ran(f) = Y ie for

each $y \in Y$, there is an $x \in X$ such that $\boxed{f(x) = y}$

Eg:-



onto

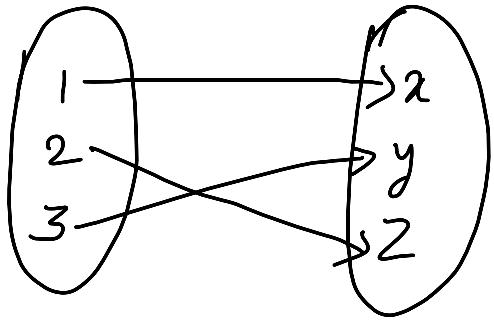
(Surjection)

Note : $\boxed{\text{Range} = \text{Codomain}}$

(c) <u>One - One Onto Function</u>

- If it is both one-one as well as onto.
- Also called <u>bijection</u>.

Eg:-



<u>Bijection</u>

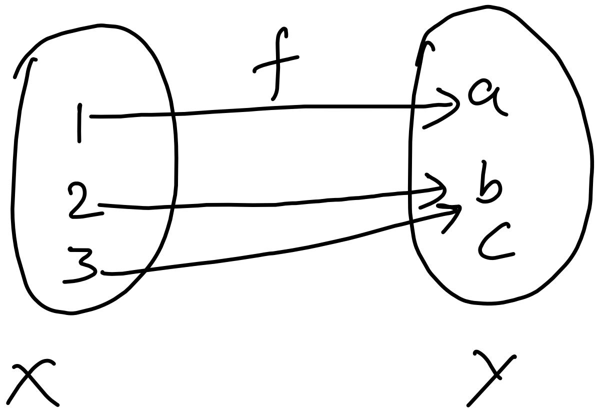
(d) Many - One Function

$f : X \to Y$ is a many to one if

$\exists\, x_1, x_2 \in X$ such that

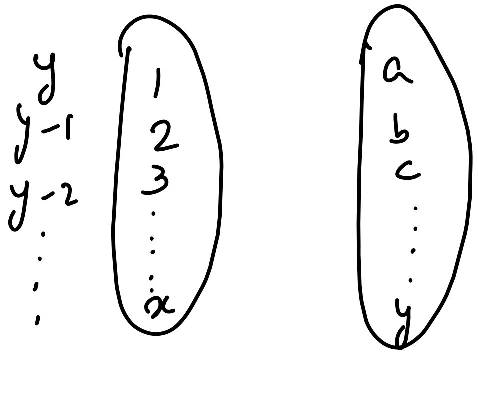
$x_1 \neq x_2 \Rightarrow f(x_1) = f(x_2)$

Eg :-



Many - One

Q) How many one - one functions are possible from a set of 'x' elements to a set of 'y' elements ?



$n(f) = y \times (y-1) \times (y-2) \times \cdots \cdots \times (y-(x-1))$

$= {}^{y}P_{x}$ or $P(y, x)$

$$\boxed{{}^{n}P_{r} \text{ or } P(n, r) = \frac{n!}{(n-r)!}}$$

Q) How many one-one function are there from a set A with 'n' elements onto itself?

$f : X \to Y$ $\quad n(f) = {}^{y}P_{x}$

$\underbrace{(x \to y)}_{\text{no. of elements}}$

$\therefore f : A \to A$ $\quad n(f) = {}^{n}P_{n} = \boxed{n!}$

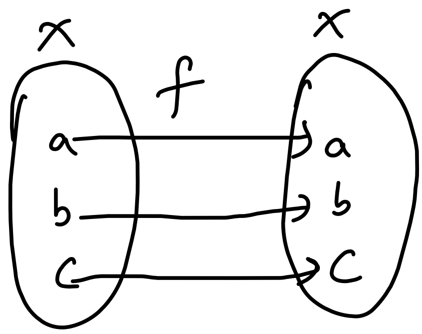
$\underbrace{(n \to n)}_{\text{no. of elements}}$

(e) <u>Identity Function</u>

Let X be a non-empty set.

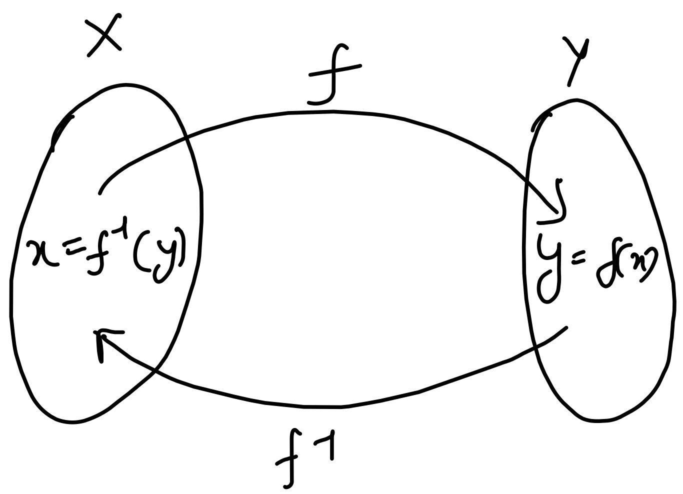
$I_x : X \to X$ is called an identity function

if $I_x(x) = x \quad \forall x \in X$



(f) Invertible Functions

$f: X \to Y$ will be invertible if its inverse f^{-1} is a function from Y to X such that $\forall y \in Y$, f^{-1} assigns a unique value of X.



Q) Check for $\boxed{f(x) = x^2}$ where $f: Z \to Z$ is invertible or not?

Note: (1) Inverse can be defined only for functions which are bijection.

(2) Not every function may have its inverse possible

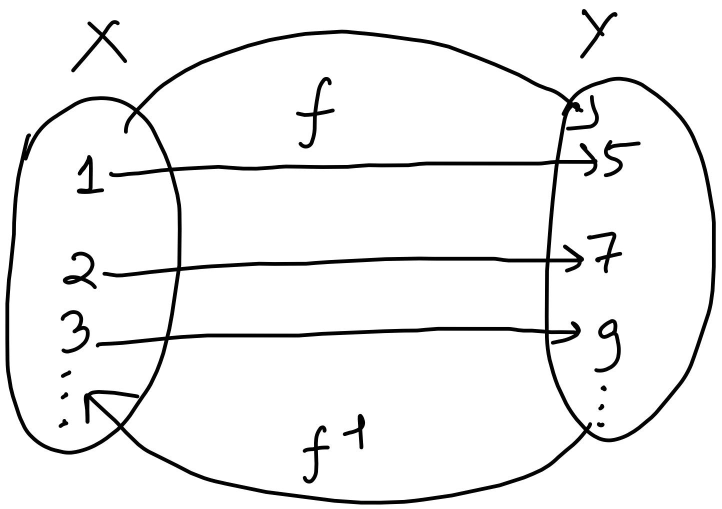
Q) Check whether $f(x) = (2x+3)$ is invertible or not for $f: R \to R$.

Given: $y = (2x+3)$ [Let $f(x) = y$]

$\Rightarrow y - 3 = 2x$

$\Rightarrow x = \left(\frac{y-3}{2}\right)$

$\therefore f^{-1}(x) = \frac{(y-3)}{2}$



When $x=1$, $y = 2x+3 = 5$

$x=2$, $y = 2x+3 = 7$

$x=3$, $y = 2x+3 = 9$

⋮

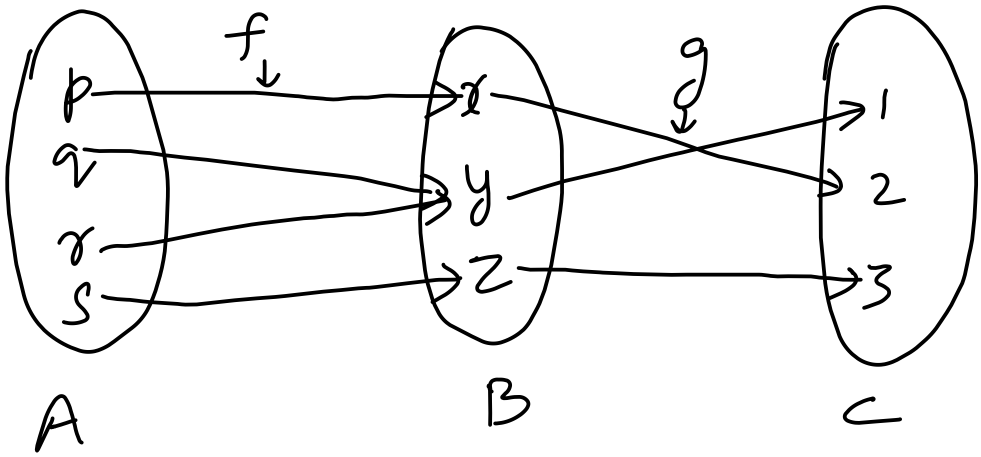
Yes, $f(x)$ is invertible.

(III) Composition of Functions

→ Composition means function of a function

$f : A \to B$ $\qquad\qquad$ $g : B \to C$

↳ 1st Codomain = Domain of 2nd



$B = f(A)$, $\quad C = g(B)$

$g \circ f = g[f(A)]$

Therefore, if $f : X \to Y$ and $g : Y \to Z$

then, $g \circ f = g(f(x)) \quad \forall x \in X$

Q) Let $f: R \to R$ and $g: R \to R$ be two functions defined as $f(x) = x^2$ and $g(x) = 3x+1$.

Find $g \circ f(x)$ and $f \circ g(x)$.

Also, determine whether $g \circ f(x) = f \circ g(x)$?

$$g \circ f(x) = g(x^2) = 3(x^2)+1 = 3x^2+1$$

$$f \circ g(x) = f(3x+1) = (3x+1)^2 = 9x^2+6x+1$$

$$\boxed{g \circ f(x) \neq f \circ g(x)}$$

Q) If $g(x) = (1-x)$ and $h(x) = \frac{x}{x-1}$ then,

$\dfrac{g[h(x)]}{h[g(x)]}$ is

(a) $\dfrac{h(x)}{g(x)}$ ⟸ Ans.

(b) $-1/x$

(c) $g(x)/h(x)$

(d) $x/(1-x)^2$

(IV) Sum and Product of Functions

Let f_1 and f_2 be functions from a set X to R. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from X to R.

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) * f_2(x)$$

Q) Let f_1 and f_2 be a function from R to R such that $f_1(x) = x+2$ and $f_2(x) = x-3$. Find $f_1 + f_2$ and $f_1 f_2$.

$$f_1 + f_2(x) = f_1(x) + f_2(x) = x + 2 + x - 3$$
$$= 2x - 1$$

$$f_1 f_2(x) = f_1(x) * f_2(x) = (x+2)(x-3)$$
$$= x^2 + 2x - 3x - 6$$
$$= (x^2 - x - 6)$$

Q) Let f_1 and f_2 be functions from a set R to R such that $f_1(x) = x^2$ and $f_2(x) = x+1$. Find $(f_1+f_2)(2)$ and $(f_1f_2)(1)$.

(i) $(f_1+f_2)(x) = x^2+x+1$

if $x=2$ then $2^2+2+1 = 7$ Ans.

(ii) $(f_1f_2)(x) = (x^2)(x+1) = x^3+x^2$

if $x=1$ then $1^3+1^2 = 1+1 = 2$ Ans.