Predicates and Quantifiers

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Overview

Limitations of propositional logic:

Proposition logic cannot adequately express the meaning of statements

- Suppose we know "Every computer connected to the university network is functioning property"
  No rules of propositional logic allow us to conclude
  "MATH3 is functioning property"
  where MATH3 is one of the computers connected to the university network
- Cannot use the rules of propositional logic to conclude from "CS2 is under attack by an intruder" where CS2 is a computer on the university network to conclude the truth
  "There is a computer on the university network that is under attack by an intruder"

Predicates

- Predicate and quantifiers can be used to express the meaning of a wide range of statements
- Allow us to reason and explore relationship between objects
- **Predicates: statements involving variables**, e.g., $x > 3, x = y + 3, x + y = z$, computer x is under attack by an intruder, computer x is functioning properly

Predicates

Example: $x > 3$

• The variable x is the **subject** of the statement

• **Predicate** "is greater than 3" refers to a property that the subject of the statement can have

• Can denote the statement by p(x) where p denotes the predicate "is greater than 3" and x is the variable

• p(x): also called the value of the propositional function p at x

• Once a value is assigned to the variable x, p(x) becomes a **proposition** and has a truth value. Replacing x by the value of 4 (or by 2) is a way to quantify the propositional function P(x). Quantify means to make it true or false.

Examples

- Let p(x) denote the statement $x > 3$

– p(4): setting x=4, thus p(4) is true

– p(2): setting x=2, thus p(2) is false

- Let a(x) denote the statement "computer x is under attack by an intruder". Suppose that only CS2 and MATH1 are currently under attack

– a(CS1)? : false

– a(CS2)? : true

– a(MATH1)?: true

Universe of Discourse

Predicates become propositions once every variable is bound- by assigning it a value from the Universe of Discourse U or quantifying it.
(The collection of values that a variable x can take is called x's Universe of Discourse.)

n-place predicate

Predicate logic generalizes the grammatical notion of a predicate to also include propositional functions of any number of arguments.

Example :

Let P(x,y,z) = "x gave y the grade z", then if x="Mike", y="Mary", z="A", then P(x,y,z) = "Mike gave Mary the grade A."

Example:

Let R be the three-variable predicate $R(x, y, z)$: $x + y = z$

Find the truth values of

$R(2, -1, 5)$, $R(3, 4, 7)$

Quantifiers

Express the extent to which a predicate is true

- In English, all, some, many, none, few
- Focus on two types:

– Universal: a predicate is true for every element under consideration

– Existential: a predicate is true for there is one or more elements under consideration

- Predicate calculus: the area of logic that deals with predicates and quantifiers

Universal Quantifier

- "p(x) for all values of x in the domain"
- Read it as "for all x p(x)" or "for every x p(x)"
- A statement is false if and only if p(x) is not always true
- An element for which p(x) is false is called a counterexample
- A single counterexample is all we need to establish that is not true

Example

Let the u.d. of x be parking spaces at BU.
Let P(x) be the predicate "x is full."
Then the universal quantification of P(x), $\forall x P(x)$, is the proposition:
– "All parking spaces at BU are full."
– i.e., "Every parking space at BU is full."
– i.e., "For each parking space at BU, that space is full."

The power of distinguishing objects from predicates is that it lets you state things about many objects at once.
• E.g., let $P(x) = x + 1 > x$. We can then say, "For any number x, P(x) is true $(\forall x, p(x))$" instead of
$(0 + 1 > 0) \wedge (1 + 1 > 1) \wedge (2 + 1 > 2) \wedge \ldots$

Existential Quantifier

Existential Quantifier of a proposition: there exists an element x in the universe of discourse such that P(x) is true
– That is, there is an x, or at least ONE x, such that P(x) is true
– In this case, one would use the backwards E to denote this type quantifier rather than the all inclusive upside down A:

Example 1

$\exists x P(x)$
For example, if P(x) was the statement $x > 89$, and your data set included test scores of 65, 72, 85, 88, and 95 what would be the existential quantification of P(x)?
– TRUE

Example 2

Let the u.d. of x be parking spaces at BU. Let P(x) be the predicate “x is full.”
Then the existential quantification of P(x), $\exists x P(x)$, is the proposition:

– “Some parking space at BU is full.”
– “There is a parking space at BU that is full.”
– “At least one parking space at BU is full.”

Statement	When true?	When false?
$\forall x P(x)$	P(x) is true for every x	there is an x for which P(x) is false
$\exists x P(x)$	There is an x for which P(x) is true	P(x) is false for every x

Table 1: Quantifiers

Free and Bound Variables

A variable in a predicate is bound by a Quantifier.
A variable is free if it is not bounded.
Example
$\forall x P(x, y)$ and $\exists x P(x, y)$
In both cases, x is a bound variable and y is a free variable.

An expression with zero free variables is a bona-fide (actual) proposition.

• An expression with one or more free variables is still only a predicate: $\exists x P(x, y)$.

REMEMBER! A predicate is not a proposition until all variables have been bound either by quantification or assignment of a value.

Examples

Example 1

Let P(x) : x is even number, and the universe of discourse for x is the set 1,2,3,4.

1. Find the truth value of $\forall x P(x)$.
2. Find the truth value of $\exists x P(x)$

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.
For example, $\forall x P(x) \vee Q(x)$ is the disjunction of $\forall x P(x) and Q(x)$.
It means $(\forall x P(x)) \vee Q(x)$ rather than $\forall x (P(x) \vee Q(x))$

Negation of Quantifiers

Example
Every politician is clever
P(x) : x is clever
u.d. is set of politicians
Symbolized as- $\forall x P(x)$
Negation
It is not the case that every politician is clever.
Alternatively, it can be written as
There is a politician who is not clever
$\exists x \neg P(x)$

Example

Write the negation of following statements-

1. All states in India are highly populated.
2. Some states in India are highly populated.

Laws of Equivalence

$$\neg \forall x P(x) \iff \exists x \neg P(x)$$
$$\neg \exists x P(x) \iff \forall x \neg P(x)$$

$$\begin{aligned}
\neg \forall x P(x) &\iff \neg(P(x_1)) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\
&\iff \neg P(x_1) \vee \neg P(x_2) \vee \neg \ldots \vee \neg P(x_n) \\
&\iff \exists x \neg P(x)
\end{aligned}$$

Negation of Quantified statements with more than one variable

Quantified statements with more than one variable may be obtained by successively applying the rule for negating a statement with only one quantifier.
Thus each $\forall$ is changed to $\exists$ and $\exists$ is changed to $\forall$ as the negation symbol passes through the statement from left to right.

Examples

1. $\neg[\forall x \exists y P(x, y)]$

   $\equiv \exists x[\neg \exists y P(x, y)]$

   $\equiv \exists x \forall y[\neg P(x, y)]$

2. $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$

   $\equiv \forall x \forall y P(x, y) \vee (\exists x \exists y \neg Q(x, y))$

Removing Quantifiers from predicates

Let P(x) be the predicate on x and let the universe of discourse of x be the set $x_1, x_2, ...x_n$.
Then,
$\forall x P(x) \iff P(x_1) \wedge P(x_2) \wedge ... \wedge P(x_n)$
$\exists x P(x) \iff P(x_1) \vee P(x_2) \vee ... \vee P(x_n)$

Examples

Assume that P(x) and Q(x) are two predicates on x, where $x \in 1, 2, 3$.
Remove the quantifiers from the following:-

1. $\exists x P(x)$
2. $\forall x P(x)$
3. $\exists x P(x) \wedge \forall x Q(x)$

Nested Quantifiers

$\forall x \exists y P(x, y)$: The proposition is same as $\forall x Q(x)$, where Q(x) is $\exists y P(x, y)$
Let the universe of discourse for the set x and y be the set of positive integers and let P(x,y): $x^2 = y$. Then translate $\forall x \exists y P(x, y)$ in english.

Example

Translate the given statement into English-
$\forall x \forall y((x > 0) \wedge (y < 0) \implies (xy < 0))$,
where the domain of both variables consist of all real numbers.

Examples

Let P(x,y) be the statement "x+y=y+x". What are the truth values of the quantifications $\forall x \forall y P(x, y) and \forall y \forall x P(x, y)$, where the domain for all variables consist of all real numbers.

Examples

Let Q(x,y) denote "x+y=0". What are the truth values of $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain of all variables consist of all real numbers.

Table 1 *Quantifications of two variables.*

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Example

Example: Let Q(x,y) be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.

1. $\exists x \forall y Q(x, y)$ There is some student in your class who has sent a message to every student in your class.
2. $\exists y \forall x Q(x, y)$ There is a student in your class who has been sent a message by every student in your class.
3. $\forall x \exists y Q(x, y)$ Every student in your class has sent a message to at least one student in your class.
4. $\forall y \exists x Q(x, y)$ Every student in your class has been sent a message from at least one student in your class.

English to logic conversion

1. Every student in the class has studied calculus.
2. All humming birds are small
3. Some students in the class have studied calculus
4. Every car is fast and dangerous
5. All fast cars are dangerous
6. Everybody is honest politician
7. Every female parent is someones mother
8. All Diamond and Pearls are precious
9. All tigers and lions attack when hungry or threatened