Lecture - 4

Class Note

Row - Echelon Matrix: A matrix A is said to be row - echelon matrix if :-

1) All zero rows, if any belongs at the bottom of matrix.
2) As we pass from row to row downwards the number of zeros preceeding the first non-zero element of a row increases.

Example: The following are all row - echelon - matrix.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 8 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of a matrix: Let $A_{m \times n}$ be a given matrix

Then Rank of A = No of non-zero rows in row-echelon matrix

Example: Find the rank of $A = \begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -2 & 2 & 1 & 1 & 0 \end{pmatrix}$

Solution: $A = \begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -2 & 2 & 1 & 1 & 0 \end{pmatrix}$

$$\xrightarrow{R_{12}} \begin{pmatrix} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ -2 & 2 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3' \to R_3 + R_1} \begin{pmatrix} 2 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_{23}} \begin{pmatrix} 2 & 0 & 1 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix} = \text{row-echelon matrix} = B$$

$\therefore$ Rank of A = No of non-zero rows in B

$= 3$

Elementary matrices :

An $n \times n$ matrix obtained by applying a single elementary row operation on identity matrix I_n, is said to be an elementary matrix of order n.

There are three types of elementary matrices,

type 1 : The elementary matrix obtained by applying R_{ij} on I_n is denoted by E_{ij}

Ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = E_{23}$

type 2 : The elementary matrix obtained by applying cR_i on I_n is denoted by $E_i(c)$.

Ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{cR_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_2(c)$

type 3 : The elementary matrix obtained by applying $R_i + cR_j$ on I_n is denoted by $E_{ij}(c)$

Ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 + cR_3} \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_{13}(c)$

Property of Elementary Matrix :

The elementary matrices are non-singular. Furthermore, their inverse is also an elementary matrix.

Note:

When we transform a matrix in row-echelon form ~~using~~ we do it by applying several elementary row operations. Therefore this can be simulated by using elementary matrices. Rather than explaing how this is done in general, we illustrate the technique with a specific example.

Example: Write the matrix $A_{3\times3}$ given below in row-echelon form.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 1 & 5 & 3 \end{pmatrix}$$

Step 1:

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 1 & 5 & 3 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 1 & 5 & 3 \end{pmatrix} = A_1 \text{ say.}$$

Now $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_{21}(-2) = E_1$$

Now $$E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 1 & 5 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 1 & 5 & 3 \end{pmatrix} = A_1$$

$\therefore$ $A_1 = E_1 A$

Step 2:

$$A_1 = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 1 & 5 & 3 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 2 & -2 \end{pmatrix} = A_2$$

Now $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = E_{31}(-1) = E_2$$

Now $$E_2 A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 1 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 2 & -2 \end{pmatrix} = A_2$$

$\therefore$ $A_2 = E_2 A_1 = E_2 E_1 A$ $(\because A_1 = E_1 A)$

Step 3:

$$A_2 = \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & \frac{9}{5} \\ 0 & 2 & -2 \end{pmatrix} = A_3$$

Now $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_2(-1/5) = E_3$

Now, $E_3 A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -9 \\ 0 & 2 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix} = A_3$$

$\therefore A_3 = E_3 A_2 = E_3 E_2 E_1 A \quad (\because A_2 = E_2 E_1 A)$

Step 4:

$$A_3 = \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix} = A_4$$

Now $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - 3R_2} \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E_{12}(-3) = E_4$

Now, $E_4 A_3 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix} = A_4$$

$\therefore A_4 = E_4 A_3 = E_4 E_3 E_2 E_1 A \quad (\because A_3 = E_3 E_2 E_1 A)$

Step 5:

$$A_4 = \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & -\frac{28}{5} \end{pmatrix} = A_5$$

Now $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_3 - 2R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} = E_{32}(-2) = E_5$$

$$E_5 A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & -\frac{28}{5} \end{pmatrix} = A_5$$

$$\therefore \quad A_5 = E_5 A_4 = E_5 E_4 E_2 E_1 A$$

Step 6:

$$A_5 = \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & -28/5 \end{pmatrix} \xrightarrow{-\frac{5}{28} R_3} \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & 1 \end{pmatrix} = A_6$$

Now $$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{-5/28\, R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{5}{28} \end{pmatrix} = E_3\left(-\tfrac{5}{28}\right) = E_6$$

$$\therefore \quad E_6 A_5 = A_6$$

$$\Rightarrow \quad A_6 = E_6 E_5 E_4 E_3 E_2 E_1 A$$

Step 7:

$$A_6 = \begin{pmatrix} 1 & 0 & -2/5 \\ 0 & 1 & 9/5 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 + R_3 \, 2/5]{R_2 - R_3 \times \frac{9}{5}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = A_7 \; (= I_3)$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 + R_3 \, 2/5]{R_2 - R_3 \times \frac{9}{5}} \begin{pmatrix} 1 & 0 & 2/5 \\ 0 & 1 & -9/5 \\ 0 & 0 & 1 \end{pmatrix} = E_{13}(2/5)\, E_{23}\left(-\tfrac{9}{5}\right) = E_7$$

$$\therefore \quad E_7 A_6 = A_7 = I_3$$

$$\Rightarrow \quad \boxed{E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I}$$ (Identity matrix)

$\therefore E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = I_n$ (Identity matrix)

$\Rightarrow E_7 E_6 E_5 E_4 E_3 E_2 E_1 I_n = A^{-1}$ $\quad (AA^{-1} = I_n)$

Therefore: If a sequence of elementary row operations applied successively on A reduces A to I_n, the same sequence of operations applied successively on I_n will reduce to A^{-1}.

Gauss-Jordan method for finding inverse of a matrix A_{n}:

From the $n \times 2n$ matrix $(A_{n\times n} \mid I_n)$.

Apply elementary row operation successively on the matrix $(A_{n\times n} \mid I_n)$, which will reduce $A_{n\times n}$ to I_n. ~~and~~ Then I_n will be reduced by the operations to A^{-1}. So the matrix $(A \mid I_n)$ will be reduced to $(I_n \mid A^{-1})$

~~Examples~~

Theorem: If A is an $n \times n$ matrix, the following are equivalent.

(i) A is invertible

(ii) A is row-equivalent to the $n \times n$ ~~matrix~~ Identity matrix

(iii) A is a product of elementary matrices.

Example: Using Gauss-Jordan method, find the inverse of matrix

$$A = \begin{pmatrix} 3 & 12 & 9 \\ 2 & 10 & 12 \\ 1 & 12 & 2 \end{pmatrix}$$

Sol^n:

$$(A \mid I_3) = \left(\begin{array}{ccc|ccc} 3 & 12 & 9 & 1 & 0 & 0 \\ 2 & 10 & 12 & 0 & 1 & 0 \\ 1 & 12 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1' \to 1/3 R_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & \frac{1}{3} & 0 & 0 \\ 2 & 10 & 12 & 0 & 1 & 0 \\ 1 & 12 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3' \to R_3 - R_1]{R_2' \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & \frac{1}{3} & 0 & 0 \\ 0 & 2 & 6 & -2/3 & 1 & 0 \\ 0 & 8 & -1 & -1/3 & 0 & 0 \end{array}\right).$$

$$\xrightarrow{R_2' \to 1/2 R_2} \left(\begin{array}{ccc|ccc} 1 & 4 & 3 & 1/3 & 0 & 0 \\ 0 & 1 & 3 & -1/3 & 1/2 & 0 \\ 0 & 8 & -1 & -1/3 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3' \to R_3 - 8R_2]{R_1' \to R_1 - 4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -9 & \frac{5}{3} & -2 & 0 \\ 0 & 1 & 3 & -1/3 & 1/2 & 0 \\ 0 & 0 & -25 & 7/3 & -4 & 1 \end{array}\right)$$

$$\xrightarrow{R_3' \to -1/2 R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -9 & 5/3 & -2 & 0 \\ 0 & 1 & 3 & -1/3 & 1/2 & 0 \\ 0 & 0 & 1 & -\frac{7}{75} & \frac{4}{25} & -\frac{1}{25} \end{array}\right)$$

$$\xrightarrow[R_2' \to R_2 - 3R_3]{R_1' \to R_1 + 9R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{62}{75} & \frac{14}{25} & \frac{9}{25} \\ 0 & 1 & 0 & \frac{1}{75} & \frac{1}{50} & \frac{3}{25} \\ 0 & 0 & 1 & \frac{7}{75} & \frac{4}{25} & \frac{1}{25} \end{array}\right) = (I_3 \mid A^{-1})$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{62}{75} & \frac{14}{25} & \frac{9}{25} \\ \frac{1}{75} & \frac{1}{50} & \frac{3}{25} \\ \frac{7}{75} & \frac{4}{25} & 1/25 \end{pmatrix}$$