Ordinary Differential Equations

(Lecture-11)

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Learning Outcome of the Lecture

We learn

- Second Order Differential Equations
  - Homogeneous Linear Second Order DE
  - Existence and Uniqueness Result
- Linear Dependent and Independent Functions
  - Wronskian Determinant
  - Checking Criteria for Linearly Independent Functions

Recall that a general second order linear ODE is of the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Definition

An ODE of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

is called a second order linear ODE in standard form.

Note that: Though there is no formula to find all the solutions of such an ODE, we study the existence, uniqueness and number of solutions of such ODE's.

Homogeneous Linear Second Order DE

Consider a second order linear ODE in the standard form.

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

If $r(x) = 0$ for all x, (that means $r(x)$ is identically zero), then the above ODE is called homogeneous.

If $r(x)$ is not identically zero, then it is called non-homogeneous.

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \qquad \text{homogeneous ODE}$$

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x) \qquad \text{non-homogeneous ODE}$$

Second Order IVP - Existence and Uniqueness

An initial value problem (IVP) of a second order homogeneous linear ODE is of the form:

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad y(x_0) = a, \quad y'(x_0) = b.$$

Theorem

If $p(x)$ and $q(x)$ are assumed to be continuous on an open interval I with $x_0 \in I$, then the IVP has a unique solution $y(x)$ in the interval I.

Important Remark

If the DE is not in standard form, that is, if we consider

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

where the coefficient functions are continuous, then the condition that $a_2(x) \neq 0$ for every $x \in I$ is important.

Example: Consider

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 6, \quad y(0) = 3, \quad y'(0) = 1.$$

Then $y = cx^2 + x + 3$ is a solution of the IVP for any arbitrary constant c.

Linearly Dependent Independent Functions

Definition

The functions $f_1(x)$ and $f_2(x)$ are said to be linearly independent on an open interval I if

$$c_1f_1(x) + c_2f_2(x) = 0 \text{ for all } x \in I \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Otherwise, they are called linearly dependent

Example:

- The functions $\sin 2x$ and $\sin x \cos x$ are linearly dependent on $(-\infty, \infty)$.

$$\sin 2x - 2\sin x \cos x = 0 \text{ for all } x \in (-\infty, \infty).$$

- The functions x and $|x|$ are linearly dependent on $(0, \infty)$ but are linearly independent on $(-\infty, \infty)$.

$$x - |x| = 0 \text{ for all } x \in (0, \infty).$$

Wronskian Determinant

Definition

The Wronskian Determinant of two differentiable functions $y_1(x)$ and $y_2(x)$ is defined by

$$W(y_1, y_2) := W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix}.$$

That mean,

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x).$$

We simply refer Wronskian Determinant by just Wronskian.

Checking Criteria for Linearly Independent Functions

Theorem

Suppose that

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

has continuous coefficients on an open interval I . Then

1. Two solutions y_1 and y_2 of the DE on I are linearly dependent if and only if their Wronskian is 0 at some $x_0 \in I$.
2. Wronskian $W(y_1, y_2)(x) = 0$ for some $x = x_0 \Rightarrow \; W(y_1, y_2)(x) = 0$ for every $x \in I$.
3. If there exists an $x_1 \in I$ at which $W(y_1, y_2)(x) \neq 0$, then y_1 and y_2 are linearly independent on I.

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Proof

1. Let $y_1,\ y_2$ be linearly dependent. Then, $y_1(x) = ky_2(x)$, for some constant k. This implies that $W(y_1, y_2) = W(ky_2, y_2) = 0$.
Conversely, let $W(y_1, y_2)(x_0) = 0$ for some $x_0 \in I$.That is,

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y'_1(x_0) & y'_2(x_0) \end{vmatrix} = 0.$$

Consider the linear system of equations :

$$\begin{aligned} k_1y_1(x_0) + k_2y_2(x_0) &= 0 \\ k_1y_1'(x_0) + k_2y_2'(x_0) &= 0. \end{aligned}$$

$W(y_1, y_2)(x_0) = 0$ implies that there exists a non-trivial solution $[k_1\ \ k_2]^t$ of the above linear system.
Set $y(x) = k_1y_1(x) + k_2y_2(x)$. Then $y(x_0) = 0$ and $y'(x_0) = 0$. By existence-uniqueness theorem, $y(x) = 0\ \forall x$, is the unique solution of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0.$$

This implies $k_1y_1(x) + k_2y_2(x) = 0$ with k_1 and k_2 not both zero.
Hence y_1 and y_2 are linearly dependent.

Example

Show that the solutioin $y_1 = \sin x$ and $y_2 = \cos x$ of

$$\frac{d^2y}{dx^2} + y = 0$$

are linearly independent.

Answer:

$$W(y_1, y_2)(x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & \text{-}\sin x \end{vmatrix} = -\sin x^2 - \cos x^2 = -1 \neq 0$$

for all real x. Thus y_1 and y_2 are linearly independent solutions of the give DE.