(VI) Methods of proving theorems

(a) Direct Proofs

- A direct proof of a conditional statement $\boxed{P \to Q}$ is constructed when the first step is the assumption that P is true; subsequent steps are constructed using rules of inference with the final step showing that Q must also be true.

Def 1: The integer n is even if there exists an integer k such that $\boxed{n = 2k}$ and n is odd if there exists an integer k such that $\boxed{n = 2k+1}$.

Note:- An integer can either be even or odd but not both.

Eg1:-

Q) Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd".

Soln: The theorem states that $\forall n \, (P(n) \to Q(n))$

where $P(n)$: 'n is an odd integer'.

$Q(n)$ is 'n^2 is an odd integer'.

By def. of an odd integer,

$$n = 2k+1 \quad , \text{ where } k \text{ is some integer}$$

Squaring on both sides,

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$

$$n^2 = 2(2k^2 + 2k) + 1$$

Let $(2k^2 + 2k)$ be t where t is some integer.

$\therefore$ $n^2 = 2t + 1$

So, n^2 is also an odd integer.

Eg2:-

Q) Give a direct proof that "if m and n are both perfect squares then nm is also a perfect square".

[Given: An integer a is a perfect square if there is an integer b such that $a = b^2$].

Soln.: - We assume that m and n are both perfect squares.

By def. of a perfect square it follows that there are integers s and t such that

$$\boxed{m = s^2} \quad \text{and} \quad \boxed{n = t^2}$$

then, $mn = s^2 t^2$

ie $mn = (st)^2$ [using commutivity & associativity of multiplication]

$\therefore$ mn is also a perfect square.

(b) <u>Proof by Contradiction</u>

- In this type of proof, we assume the opposite of what we are trying to prove and get a logical contradiction.
- Hence, our assumption must have been false and therefore what we originally required to prove must be true.
- To prove a statement P is true, we assume that $\sim P$ is true and taking $\sim P$ as premise, we draw a contradiction F as the conclusion.
- Now, if $\sim P$ leads to a contradiction is true then $\sim P$ must be false that is P must be true.
- Steps to be followed :-
  (i) Assume that P is false.
  (ii) Using this assumption show a contradiction.

Eg1:- Show that $\sqrt{2}$ is an irrational number.

Def. of rational number : A rational number Q can be defined as $Q = p/q$ where p and q are integers and have no common factor (assuming these are the lowest terms) and $q \neq 0$

Soln. Here, P: $\sqrt{2}$ is an irrational number.

Assume $\sim P$ is true or $\sqrt{2}$ is a rational number.

Let $\sqrt{2} = p/q$ such that p and q have no common factor.

$\Rightarrow \sqrt{2}q = p$

$\Rightarrow 2q^2 = p^2$ (Squaring on both sides)

$\Rightarrow p^2$ is an even number

$\Rightarrow p$ is an even number (since if p^2 is even, p must be even).

$\Rightarrow p = 2k$ for some integer k.

$\Rightarrow p^2 = 4k^2$

$\Rightarrow q^2 = \frac{p^2}{2} = \frac{4k^2}{2} = 2k^2$

(on substituting the value of p^2 in $2q^2 = p^2$).

$\therefore q^2$ is also an even number.

So, q is an even number.

$\therefore$ 2 is a common factor between p and q.

This is a contradiction as they should not have an common factor, if $\sqrt{2}$ is a rational no. Therefore, $\sim P$ is F which means P is true and $\sqrt{2}$ is an irrational no.