Example: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$, then find A^{-1} using determinant method.

Solution: Now, $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix}$

$$= 1\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} + 1\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix}$$

$$= (16-15) - 0 + (9-8) = 1 + 1 = 2 \neq 0$$

We know that $A^{-1} = \dfrac{\operatorname{adj}(A)}{|A|} = \dfrac{\operatorname{adj}(A)}{2}$.

Now co-factor matrix $= \begin{pmatrix} +\begin{vmatrix} 4 & 5 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}$$

$\therefore$ adj A = tanspose of a co-factor matrix

$$= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 2 & -3 \\ -4 & -2 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{2} \operatorname{adj} A = \frac{1}{2}\begin{pmatrix} 1 & 3 & -4 \\ -2 & 2 & -2 \\ 1 & -3 & 4 \end{pmatrix}$$

Example:

$3x + y + z = 4$
$x - y + 2z = 6$
$x + 2y - z = -3$

$$\Rightarrow \begin{pmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 3 \end{pmatrix}$$

Here the coefficient determinant

$$D = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= -3 \neq 0.$$

So Cramer's rule can be applied.

Now,

$$D_1 = \begin{vmatrix} 4 & 1 & 1 \\ 6 & -1 & 2 \\ -3 & 2 & -1 \end{vmatrix} = -3.$$

$$D_2 = \begin{vmatrix} 3 & 4 & 1 \\ 1 & 6 & 2 \\ 1 & -3 & -1 \end{vmatrix} = 3.$$

$$D_3 = \begin{vmatrix} 3 & 1 & 4 \\ 1 & -1 & 6 \\ 1 & 2 & -3 \end{vmatrix} = -6$$

So by Cramer's Rule

$$x = \frac{D_1}{D} = \frac{-3}{3} = 1$$

$$y = \frac{D_2}{D} = \frac{3}{-3} = -1$$

$$z = \frac{D_3}{D} = \frac{-6}{-3} = 2.$$

$\therefore$ the solution is $x = 1$, $y = -1$, $z = 2$.

Elementary Row operation:

An elementary row operation on a matrix $A_{m\times n}$ is an operation of the following three types:

type 1: The interchange of the ith and jth row is denoted by R_{ij}

type 2: Multiplication of the ith row by a non-zero scalar c is denoted by cR_i

type 3: Addition of c times the jth row to the ith row is denoted by $R_i + cR_j$

Example: let $A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}$

Now example of type 1:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_{23}} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Now example of type 2:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{2R_3} \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{pmatrix}$$

Now example of type 3:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

Row Equivalence :

Two $m \times n$ matrices A & B are said to be row equivalent if B can be obtained from A by a finite sequence of elementary row operations.

If $A_{m\times n}$ and $B_{m\times n}$ are row equivalent, we write

$$A_{m\times n} \cong B_{m\times n}$$

Example : Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$

Show that $A \cong B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix}$

Sol^n :

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix}$$

$$\xrightarrow[R_3 + R_1]{R_2 - 2R_1} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 8 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 + R_2} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} = B$$

$\therefore \ A \cong B$

Theorem: If A and B are row-equivalent $m \times n$ matrices, then the homogeneous system of linear equation $AX = 0$ and $BX = 0$ have exactly the same solution.

Example:

$$2x + y + z = 0$$
$$4x - 6y = 0$$
$$-2x + 7y + 2z = 0$$

Now

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad AX = 0$$

Where $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{pmatrix} \cong \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} = B$

(See previous page)

$\because \; A \cong B$.

$\Rightarrow \; AX = 0$ & $BX = 0$ have exacty the same solution.

So now,

$$AX = BX = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 2x + y + z = 0$$
$$-8y - 2z = 0$$
$$z = 0$$

$$\Rightarrow \quad z = 0, \; y = 0, \; x = 0.$$

Theorem: A matrix A can be made now equivalent to a row reduced echelon matrix B by elementary row operations.

Worked Examples:

Apply elementary row operation to reduce the following matrix A_{4M} to a row echelon matrix.

$$A_{4\times4} = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

Solution:

$$\begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_3-5R_1]{R_2-3R_1} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 3 & 2 & 5 \end{pmatrix}$$

$$\xrightarrow[R_4-3R_2]{R_3-2R_2} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

$$\xrightarrow{R_{34}} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_3} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1-2R_3} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ This is the answer

Working procedure : for the ~~for~~ previous page problem

The element in (1, 1) position is 2

Step 1 : Multiply the 1st row by $\frac{1}{2}$. The leading element in the first row becomes 1 in the (1, 1) position.

Step 2 : To reduce all other elements in the first column to zero, perform the operations $R_2 - 3R_1$, $R_3 - 5R_1$.

Step 3 : Multipy the second row by $\frac{1}{2}$. The leading element in the second row becomes 1 in the (2, 2) position.

Step 4 : To reduce all other elements in the second column to zero, perform the operations $R_3 - 2R_2$, $R_4 - 3R_2$.

Step 5 : ∵ the third row becomes a zero row. Perform R_{34} to bring the zero row to the last.

Step 6 : Multiply the third row by $\frac{1}{2}$. The leading element in the third row becomes 1 in the (3, 3) position.

Step 7 : To reduce all other elements in the third column to zero, perform the operation $R_1 - 2R_3$.

Process terminates

Rank of a Matrix:

Let A be a matrix of order $m \times n$. Then, the maximum number of linearly independent rows in a matrix $A_{m\times n}$ is called rank of $A_{m\times n}$.

$\therefore$ Rank of $A_{m\times n} \leq \min(m, n)$

Method of finding Rank of Matrix $A_{m\times n}$:

If a matrix $A_{m\times n}$ is ~~row~~ equivalent to row reduced echelon matrix $B_{m\times n}$, then number of non-zero rows in a row-reduced echelon matrix $B_{m\times n}$ is the rank of matrix $A_{m\times n}$.

Example: Find the rank of the following matrix:-

$$A_{4\times 5} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

Solution:

$$A_{4\times 5} = \begin{pmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

$$\xrightarrow{R_{12}} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{pmatrix}$$

$$\xrightarrow[R_4 - 3R_1]{R_3 - 2R_1} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{pmatrix}$$

$$\xrightarrow[\substack{R_3 + 2R_2 \\ R_4 + 5R_2}]{R_1 - 2R_2} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

$$\xrightarrow{R_{34}} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{pmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[R_2 - R_3]{R_2 - 2R_3} \begin{pmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B_{4\times 5}$$

In $B_{4\times5}$ has 3 non-zero rows.

$\therefore$ Rank of $A_{4\times5}$ = No of non-zero rows in $B_{4\times5}$

$= 3$

Working procedure for the above problem:

Step 1: The first column is a non-zero column. The element in the (1,1) position is zero. Perform R_{12} to bring a non-zero element to (1,1) position. The leading 1 in the first row occurs in the first column.

Step 2: To reduce the other element in the first column to zero, perform the operations $R_3 - 2R_1$, $R_4 - 3R_1$

Step 3: Observe that the element in the (2,2) position is zero. So that multiple the second row by $\frac{1}{2}$ ($\because$ the element in (2,3) position is 2). So that the leading element in (2,3) position becomes 1.

Step 4: Reduce all other elements in the 3rd column to 0 by performing the operations $R_1 - 2R_2$, $R_3 + 2R_2$, $R_4 + 5R_2$.

Step 5: Perform R_{34} to bring the zero row to the last.

Step 6: Multiply the 3rd row by $1/3$. The leading element in the row becomes 1 in the 4th column.

Step 7: Reduce all other element in the 4th column to zero by performing the operation $R_1 - 2R_3$, $R_2 - R_3$.

Step 8: Find the no of non-zero row