Lecture - 10

Class Note

Linear independence: Given a set of of vectors $v_1, v_2, \dots, v_k \in V$, we look at their linear combinations $c_1v_1 + c_2v_2 + \dots + c_kv_k$, $c_i \in F$.

Suppose $c_1v_1 + c_2v_2 + \dots + c_kv_k = 0 \longrightarrow (1)$ only happens when $c_1 = c_2 = \dots = c_k = 0$.

Then the vectors $v_1, v_2, \dots, v_k$ are linearly independent in (1)

Linear dependence: If $\exists$ any c's are non-zero not all zero, i.e. $\exists$ scalars $c_1, c_2, \dots, c_n$ not all zero, then the vectors $v_1, v_2, \dots, v_k$ are linearly dependent.

Example: Prove that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is linearly independent in $\mathbb{R}^3$

Solution: let $\alpha = (1,2,2)$, $\beta = (2,1,2)$, $\gamma = (2,2,1)$

let $c_1\alpha + c_2\beta + c_3\gamma = 0$ where $c_1, c_2, c_3 \in \mathbb{R}$.

$$\Rightarrow c_1(1,2,2) + c_2(2,1,2) + c_3(2,2,1) = (0,0,0)$$

$$\Rightarrow \left.\begin{aligned} c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 + c_2 + 2c_3 &= 0 \\ 2c_1 + 2c_2 + c_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now
$$\begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-4) + 2(4-2)$$
$$= -3 + 4 + 4 = 5 \neq 0$$

$\Rightarrow$ The co-efficient determinant $= 5 \neq 0$.

$\Rightarrow$ Rank of the matrix $= 3 =$ no of variable (n)

$\Rightarrow$ (1) has unique solution.

But system of equation (1) is a homogeneous system whose coefficient matrix rank $= 3 =$ no of unknowns

$\Rightarrow$ It has only zero solution.

$\Rightarrow (c_1, c_2, c_3) = (0, 0, 0)$

$\Rightarrow c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$.

This proves that the set of vectors α, β, γ is linearly independent.

Example: Examine if the set of vectors $\{(2,1,1), (1,2,2), (1,1,1)\}$ is linearly dependent in $\mathbb{R}^3$.

Solution: let $\alpha = (2,1,1)$, $\beta = (1,2,2)$, $\gamma = (1,1,1)$.

Then $c_1\alpha + c_2\beta + c_3\gamma = 0$

$\Rightarrow c_1(2,1,1) + c_2(1,2,2) + c_3(1,1,1) = (0,0,0)$

$$\Rightarrow \begin{aligned} 2c_1 + c_2 + c_3 &= 0 \\ c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow$$ This is a homogeneous system of three equations in c_1, c_2, c_3

Now $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 2(2-2) - 1(1-1) + 1(2-2)$
$= 0$

$\Rightarrow$ co-efficient matrix determinant $= 0$

$\Rightarrow$ Rank of co-efficient matrix < 3 (no of variables)

$\Rightarrow$ The homogeneous system has non-zero soln.

$\Rightarrow$ all c_i are not zero $\Rightarrow \alpha, \beta, \gamma$ are linearly dependent.

Theorem: A set of vectors containing the zero vector in a vector space V is linearly dependent.

proof: let $S = \{\theta\}$ θ = zero vector.

The set S is linearly dependent

$\because$ $c\theta = \theta$ holds for non-zero scalar c.

$\Rightarrow$ S is linearly dependent.

Theorem: The set containing a single non-zero vector in a vector space V is linearly independent.

Proof: let $S = \{\alpha\}$, $\alpha \neq 0 \in V$

$\therefore$ $c\alpha = \theta$ where θ = zero vector

$\Rightarrow c = 0$ $(\because \alpha \neq 0)$

$\Rightarrow$ S is linearly independent.

linear span of a vector space V: (generating set of vector space V)

subset's of vector space 'V'

A non-empty S is said to be that S generates V if every element in V can be written as finite linear combination of elements of S.

i.e $V = L(S) = \{c_1v_1 + c_2v_2 + \cdots + c_kv_k \mid c_i \in F, v_i \in S\}$

Example: let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

Then $L(S) = \mathbb{R}^3$, $\because$ If we take $(x_1, y_1, z_1) \in \mathbb{R}^3$

then $(x_1, y_1, z_1) = x_1(1,0,0) + y_1(0,1,0) + z_1(0,0,1)$

$\Rightarrow$ every element of $\mathbb{R}^3$ can be written as a linear combination of $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ $\Rightarrow \mathbb{R}^3 = L(S)$

~~Example~~

Result: In the vector space $\mathbb{R}^n$, one of the generating set is $S=\{(1,0,0,\dots,0),(0,1,0,\dots,0),\dots,(0,0,0,\dots,1)\}$

Basis: A set S of vectors in V is said to be a basis of V if

(i) S is linearly independent in V and

(ii) S generates V i.e $L(S)=V$

(Here V = vector space)

Example: Prove that $S=\{(1,0),(0,1)\}$ is a basis of the vector space $\mathbb{R}^2$.

Sol: Consider $c_1(1,0)+c_2(0,1)=(0,0)$

$\Rightarrow c_1=0$ & $c_2=0$

$\Rightarrow$ The set S is linearly independent.

Now let $\xi=(a,b)\in\mathbb{R}^2$

then $\xi=(a,b)=a(1,0)+b(0,1)$

$\Rightarrow \xi\in L(S)$

$\Rightarrow \mathbb{R}^2\subset L(S) \rightarrow (1)$

Again, we know that $L(S)$ is vector space in $\mathbb{R}^2$

$\Rightarrow L(S)\subset\mathbb{R}^2 \rightarrow (2)$

From (1), (2) we get $\mathbb{R}^2=L(S)$.

Thus the set $S=\{(1,0),(0,1)\}$ fulfils both the conditions for a basis in $\mathbb{R}^2$.