Solution of

Assignment 5

$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x+y, x-y)$

1)(i) let $\alpha = (x_1, x_2)$ & $\beta = (y_1, y_2) \in \mathbb{R}^2$

Now, $c, d \in \mathbb{R}$,

$$c\alpha + d\beta = (cx_1 + dy_1, cx_2 + dy_2)$$

$$T(c\alpha + d\beta) = T(cx_1 + dy_1, cx_2 + dy_2)$$

$$= (cx_1 + dy_1 + cx_2 + dy_2, cx_1 + dy_1 - cx_2 - dy_2)$$

$$= (c(x_1 + x_2) + d(y_1 + y_2), c(x_1 - x_2) + d(y_1 - y_2))$$

$$= c((x_1 + x_2), (x_1 - x_2)) + d(y_1 + y_2, y_1 - y_2)$$

$$= cT(\alpha) + dT(\beta)$$

$(\because T(\alpha) = (x_1 + x_2, x_1 - x_2)$ & $T(\beta) = (y_1 + y_2, y_1 - y_2))$

$\Rightarrow$ T is linear.

$$\text{Ker } T = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = (0,0)\}$$

$$= \{(x,y) \in \mathbb{R}^2 \mid (x+y, x-y) = (0,0)\}$$

$$= \left\{(x,y) \in \mathbb{R}^2 \mid \begin{array}{l} x+y=0 \\ x-y=0 \Rightarrow x=y \end{array}\right\}$$

$$= \left\{(x,y) \in \mathbb{R}^2 \mid \begin{array}{l} x+x=0 \\ x=y \end{array}\right\}$$

$$= \left\{(x,y) \in \mathbb{R}^2 \mid \begin{array}{l} 2x=0 \Rightarrow x=0 \\ x=y \Rightarrow y=0 \end{array}\right\}$$

$$= \{(0,0)\}$$

$\therefore$ Ker $T = \{\theta\}$

$\because$ Ker $T = \{\theta\} \Rightarrow \dim(\text{Ker } T) = 0$

$$\operatorname{Im} T = \{T(x,y) \mid (x,y) \in \mathbb{R}^2\}$$

$$= \{(x+y,\, x-y) \mid (x, y \in \mathbb{R}^2\}$$

$$= \{x(1,1) + y(1,-1)\} \quad x, y \in \mathbb{R}$$

$$= L(S) \quad \text{where} \quad S = \{(1,1), (1,-1)\}$$

Now $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2 \neq 0$

$\Rightarrow S = \{(1,1), (1,-1)\}$ is L.I

$\Rightarrow$ S is basis of Im T

$\therefore$ dim (Im T) = no of element in basis of Im T

= no of element in S

= 2.

Rank- Nullity - Theorem of a linear mapping.

Now dim (ker T) + dim (Im T)

= 0 + 2 = 2.

Now dim (domain set)

= dim ($\mathbb{R}^2$) = 2.

$\therefore$ dim (ker T) + dim (Im T) = dim ($\mathbb{R}^2$)

I) (ii) $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x+2y,\ 2x+y,\ x+y)$

let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2) \in \mathbb{R}^2$

$c\alpha + d\beta = (cx_1 + dy_1,\ cx_2 + dy_2)$

$$\therefore\ T(c\alpha + d\beta) = T(cx_1 + dy_1,\ cx_2 + dy_2)$$

$$= \left(cx_1 + dy_1 + 2cx_2 + 2dy_2,\ 2cx_1 + 2dy_1 + cx_2 + dy_2,\ cx_1 + dy_1 + cx_2 + dy_2\right)$$

$$= \left(c(x_1 + 2x_2) + d(y_1 + 2y_2),\ c(2x_1 + x_2) + d(2y_1 + y_2),\ c(x_1 + x_2) + d(y_1 + y_2)\right)$$

$$= c\left(x_1 + 2x_2,\ 2x_1 + x_2,\ x_1 + x_2\right) + d\left(y_1 + 2y_2,\ 2y_1 + y_2,\ y_1 + y_2\right)$$

$$= c\,T(\alpha) + d\,T(\beta)$$

$(\because T(\alpha) = (x_1 + 2x_2,\ 2x_1 + x_2,\ x_1 + x_2)$ & $T(\beta) = (y_1 + 2y_2,\ 2y_1 + y_2,\ y_1 + y_2))$

$\Rightarrow$ T is linear.

$$\text{Ker } T = \{(x, y) \in \mathbb{R}^2 \mid T(x, y) = (0, 0, 0)\}$$

$$= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x + 2y = 0 \\ 2x + y = 0 \\ x + y = 0 \Rightarrow x = -y \end{array}\right\}$$

$$= \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} -y + 2y = 0 \Rightarrow y = 0 \\ 2x + y = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \end{array}\right\}$$

$$= \{(0, 0)\} = \{0\}.$$

$\therefore$ $\dim(\text{Ker } T) = 0$

⊞ $\operatorname{Im} T = \{ T(x, y) \mid (x, y) \in \mathbb{R}^2 \}$.

$= \{ (x+2y, 2x+y, x+y) \} \quad x, y \in \mathbb{R}$

$= \{ x(1, 2, 1) + y(2, 1, 1) \}$.

$= L(S)$ where $S = \{(1,2,1), (2,1,1)\}$

Now $c(1,2,1) + d(2,1,1) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} c + 2d &= 0 \\ 2c + d &= 0 \\ c + d &= 0 \end{aligned}\right\} \Rightarrow c = 0, \; d = 0.$$

$\Rightarrow$ $S = \{(1,2,1), (2,1,1)\}$ is L.I

$\Rightarrow$ S is a bases of $\operatorname{Im} T$

$\therefore$ $\dim(\operatorname{Im} T) = 2$.

⊞ $\dim(\ker T) + \dim(\operatorname{Im} T)$

$= 0 + 2 = 2$.

$\dim$ (domain set)

$= \dim(\mathbb{R}^2)$

$= 2$.

$\therefore$ $\dim(\ker T) + \dim(\operatorname{Im} T) = \dim(\mathbb{R}^2)$

(iii) $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (yz, zx, xy)$.

let $\alpha \in (x_1, x_2, x_3) \in \mathbb{R}^3$.

$\therefore \quad c\alpha = (cx_1, cx_2, cx_3)$.

Now $T(c\alpha) = T(cx_1, cx_2, cx_3)$

$= (c^2 x_2 x_3, c^2 x_3 x_1, c^2 x_1 x_2)$

$= c^2 (x_2 x_3, x_3 x_1, x_1 x_2)$

$= c^2 T(\alpha)$

$\neq c\, T(\alpha)$

$\Rightarrow$ T is not linear.

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2) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps the bases vectors $(0,1,1), (1,0,1), (1,1,0)$ of $\mathbb{R}^3$ to $(1,1,1), (1,1,1), (1,1,1)$ respectively. Verify that $\dim(\operatorname{Ker} T) + \dim(\operatorname{Im} T) = 3$.

Sol^{n}: let $v = (x,y,z)$ be an arbitrary vector of the domain space $\mathbb{R}^3$.

$\because \{(0,1,1), (1,0,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

$$\Rightarrow v = c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0) \qquad c_i \in \mathbb{R}.$$

$$\Rightarrow (x,y,z) = (c_2+c_3,\ c_1+c_3,\ c_1+c_2)$$

$\therefore$

$c_2 + c_3 = x \Rightarrow c_2 = x - c_3$

$c_1 + c_3 = y \Rightarrow c_1 = y - c_3$

$c_1 + c_2 = z \Rightarrow y - c_3 + x - c_3 = z$

$$\Rightarrow y + x - z = 2c_3$$

$$\Rightarrow c_3 = \frac{x+y-z}{2}.$$

$$c_2 = x - \frac{x+y-z}{2} = \frac{x-y+z}{2} = \frac{x+z-y}{2}.$$

$$c_1 = y - \frac{x+y-z}{2} = \frac{y+z-x}{2}$$

$\because T$ is linear.

$$\Rightarrow T(x) = T\left(c_1(0,1,1) + c_2(1,0,1) + c_3(1,1,0)\right)$$

$$= c_1 T(0,1,1) + c_2 T(1,0,1) + c_3 T(1,1,0)$$

$$= c_1(1,1,1) + c_2(1,1,1) + c_3(1,1,1)$$

$$= (c_1+c_2+c_3,\ c_1+c_2+c_3,\ c_1+c_2+c_3)$$

$$= \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)$$

$$\therefore T(x,y,z) = \left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)$$

田 $\text{Ker } T = \{(x,y,z) \in \mathbb{R}^3 \mid T(x,y,z) = (0,0,0)\}$

$$= \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \frac{x+y+z}{2} = 0\right\}$$

$\Rightarrow x+y+z=0$

$\Rightarrow x = -y-z$

$$= \{(-y-z, y, z)\} \quad y,z \in \mathbb{R}.$$

$$= \{y(-1,1,0) + z(-1,0,1)\}$$

$$= L(S)$$

where $S = \{(-1,1,0), (-1,0,1)\}$

Now $c_1(-1,1,0) + c_2(-1,0,1) = (0,0,0)$

$\Rightarrow -c_1 - c_2 = 0,\ c_1 = 0,\ c_2 = 0$

$\Rightarrow S = \{(-1,1,0), (-1,0,1)\}$ is L.I

$\therefore$ S is a basis

$\Rightarrow$ dim (Ker T) = no of element in basis

= 2.

⊞ $Im\,T = \{T(x,y,z) \mid (x,y,z) \in \mathbb{R}^3\}$

$= \left\{\left(\frac{x+y+z}{2}, \frac{x+y+z}{2}, \frac{x+y+z}{2}\right)\right\}$

$= \left\{\frac{x}{2}(1,1,1) + \frac{y}{2}(1,1,1) + \frac{z}{2}(1,1,1)\right\}$

$= L\{(1,1,1), (1,1,1), (1,1,1)\}$

$= L\{(1,1,1)\}$

$= L(S)$ where $S = \{(1,1,1)\}$

$\because$ S is singleton set $\Rightarrow$ S is L.I

$\therefore$ S is a basis

$\therefore$ dim (Im T) = 1.

⊞ dim (Ker T) + dim (Im T)

= 2 + 1 = 3

dim (domain set) = dim ($\mathbb{R}^3$) = 3.

$\therefore$ dim (ker T) + dim (Im T) = dim ($\mathbb{R}^3$)

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3) A linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T(x_1, x_2, x_3) = (3x_1 - 2x_2 + x_3,\ x_1 - 3x_2 - 2x_3) \quad (x_1, x_2, x_3) \in \mathbb{R}^3$$

Find the matrix of T relative to the ordered bases

(i) $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$

(ii) $\{(0,1,0), (1,0,0), (0,0,1)\}$ of $\mathbb{R}^3$ and $\{(0,1), (1,0)\}$ of $\mathbb{R}^2$

(iii) $\{(0,1,1), (1,0,1), (1,1,0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (0,1)\}$ of $\mathbb{R}^2$

Solⁿ: (i) $T(1,0,0) = (3,1) = 3(1,0) + 1(0,1)$

$T(0,1,0) = (-2,-3) = -2(1,0) - 3(0,1)$

$T(0,0,1) = (1,-2) = 1(1,0) - 2(0,1)$.

$$\Rightarrow \begin{pmatrix} T(1,0,0) \\ T(0,1,0) \\ T(0,0,1) \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ -2 & -3 \\ 1 & -2 \end{pmatrix}}_{=A} \begin{pmatrix} (1,0) \\ (0,1) \end{pmatrix}$$

$\therefore$ The matrix of $T = A^T$ = transpose of A

$$= \begin{pmatrix} 3 & -2 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

(ii) $T(0,1,0) = (-2,-3) = -3(0,1) - 2(1,0)$.

$T(1,0,0) = (3,1) = 1(0,1) + 3(1,0)$

$T(0,0,1) = (1,-2) = -2(0,1) + 1(1,0)$

$$\Rightarrow \begin{pmatrix} T(0,1,0) \\ T(1,0,0) \\ T(0,0,1) \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & -2 \\ 1 & 3 \\ -2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} (0,1) \\ (1,0) \end{pmatrix}$$

$\therefore$ The matrix of $T = A^T$

$$= \begin{pmatrix} -3 & 1 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

(iii) $T(0,1,1) = (-1,-5) = -1(1,0) + 5(0,1)$

$T(1,0,1) = (4,-1) = 4(1,0) - 1(0,1)$

$T(1,1,0) = (1,-2) = 1(1,0) - 2(0,1)$.

$$\Rightarrow \begin{pmatrix} T(0,1,1) \\ T(1,0,1) \\ T(1,1,0) \end{pmatrix} = \begin{pmatrix} -1 & -5 \\ 4 & -1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} (1,0) \\ (0,1) \end{pmatrix}$$

$\Rightarrow$ The matrix of $T = A^T$

$$= \begin{pmatrix} -1 & 4 & 1 \\ -5 & -1 & -2 \end{pmatrix}$$