GATE EE EXAM (2012), Question 47

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Prerequisites

State Space

The set of all possible states the system/object can be in. Without loss in generality, the state space can be identified with the set $S = \{1, 2, \ldots, \ell\}$, where ℓ is a fixed arbitrary natural number.

Random Variables

Let $\{X_0, X_1, X_2 \ldots\}$ be a sequence of discrete random variables, where each $X_t \in S$, and X_t represents the state of the system at time t.

Prerequisites Contd.

Markov Processes and Markov Chains

- A process is said to be a Markov Process if the probability of transitioning to any particular state is dependent solely on the current state and does not depend on how the current state was reached.
- Mathematically, $\{X_0, X_1, \dots\}$ is called a Markov chain if

$$\Pr\left(\left(X_n = i_n \mid X_{n-1} = i_{n-1}, \dots, X_0 = i_0\right) = \Pr\left(X_n = i_n \mid X_{n-1} = i_{n-1}\right) \tag{1}$$

Prerequisites Contd.

Transition Matrix

- For each pair $i, j \in S$, consider the (conditional) probability $p_{ij} \in [0, 1]$ for the transition of the object or system from state i to j within one time step.
- The $\ell \times \ell$ matrix $\mathbf{P} = (p_{ij})_{i,j=1,\ldots,\ell}$ of the transition probabilities p_{ij} where

$$p_{ij} \geq 0\,, \qquad p_{ij} = \Pr\left(X_{n+1} = j \mid X_n = i\right) \tag{2}$$

is called one-step *transition matrix* or simply transition matrix of the Markov chain.

Prerequisites Contd.

Definition 1

The standard form of a state transition matrix is,

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix} \end{array} \tag{3}$$

where, A and N are Absorbing and Non-absorbing states.$\mathbf{I}$ is the Identity matrix, $\mathbf{O}$ being the Null matrix, and $\mathbf{R}, \mathbf{Q}$ are other sub-matrices.

Prerequisites Contd.

Definition 2

The limiting matrix for absorbing Markov chain is,

$$\bar{\mathbf{P}} = \lim_{n \to \infty} \mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{FR} & \mathbf{O} \end{bmatrix} \tag{4}$$

where,

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} \tag{5}$$

is called the fundamental matrix of $\mathbf{P}$.

Prerequisites Contd.

Definition 3

A element $\bar{p}_{ij}$ of $\bar{\mathbf{P}}$ denotes the absorption probability in state j, starting from state i.

Question

GATE EE EXAM (Dec 2012), Question 47

A fair coin is tossed till head appears for the first time. The probability that the number of required tosses is odd, is,

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{2}{3}$
4. $\frac{3}{4}$

Solution

Given, a fair coin is tossed till heads turns up,

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{6}$$

Let us define a Markov chain $\{X_0, X_1, X_2 \dots\}$, where $X_n \in S$ $\vee$n$\in \{0, 1, 2, \dots\}$, and $S = \{1, 2, 3, 4\}$ is the state space.

Table: Definition of Random Variables

R.V	Value=0	Value=1
X	$N_{tosses} = 2k$	$N_{tosses} = 2k - 1$
Y	H	T

Solution Contd.

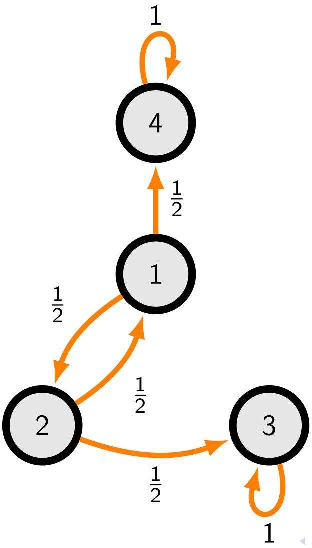
The definition of state space S,

Table: Markov states and Notations

Notation	State
$S = 1$	$(X, Y) = (0, 1)$
$S = 2$	$(X, Y) = (1, 1)$
$S = 3$	$(X, Y) = (0, 0)$
$S = 4$	$(X, Y) = (1, 0)$

Solution Contd.

Markov chain diagram



Solution Contd.

For the above discussed Markov Chain, the states 3 and 4 are the Absorbing states, and, 1 and 2 are Non-absorbing states,

Corollary 1

The state transition matrix for the above Markov chain is,

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{7}$$

Solution Contd.

From (3) and (7), we get,

$$\mathbf{R} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{8}$$

Corollary 2

Limiting Matrix of the Markov chain under observation is,

$$\bar{\mathbf{P}} = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \end{array} \tag{9}$$

Solution Contd.

Corollary 3

The required probability is,

$$P = \bar{p}_{14} \tag{10}$$

From (9) and (10),

$$P = \frac{2}{3} \tag{11}$$

$\therefore$ option 3 is correct.