AI1103 : Assignment 2

Amaan - EP20BTECH11003

Download all python codes from

https://github.com/amaan28/Assignment2/blob/main/Assignment2/codes/Assignment2.py

and latex-tikz codes from

https://github.com/amaan28/Assignment2/blob/main/Assignment2/Assignment2.tex

GATE 2012 EE Q.47

A fair coin is tossed till head appears for the first time. The probability that the number of required tosses is odd, is,

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) $\frac{2}{3}$
4) $\frac{3}{4}$

GATE 2012 EE Q.47 - SOLUTION

Given, a fair coin is tossed till heads turns up.

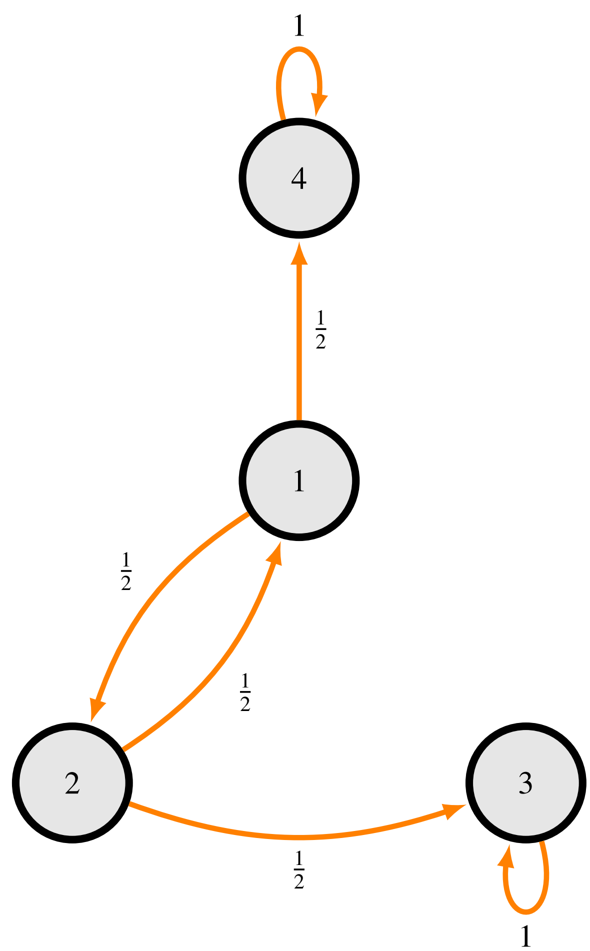
$$p = \frac{1}{2}, q = \frac{1}{2} \tag{47.1}$$

Let us define a Markov chain $\{X_0, X_1, X_2 \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$ where $n \in \{0, 1, 2, \dots\}$,

TABLE 1: States and their notations

Notation	State
$S = 1$	Even numbered toss
$S = 2$	Odd numbered toss
$S = 3$	Head appears in odd number of tosses
$S = 4$	Head appears in even number of tosses

Markov chain diagram



such that the state transition matrix for the Markov chain is,

$$P = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{47.2}$$

Definition 1. *The standard form of a state transition* *matrix is,*

$$P = \begin{array}{c} \\ A \\ N \end{array} \begin{array}{c} \begin{array}{cc} A & N \end{array} \\ \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \end{array} \quad (47.3)$$

where,

TABLE 2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
I	Identity matrix
O	Zero matrix
R, Q	Other submatices

Corollary 0.1. *P in standard form is,*

$$P = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \quad (47.4)$$

From (47.4),

$$R = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, Q = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \quad (47.5)$$

Definition 2. *The limiting matrix for absorbing Markov chain is,*

$$\bar{P} = \begin{bmatrix} I & O \\ FR & O \end{bmatrix} \quad (47.6)$$

where,

$$F = (I - Q)^{-1} \quad (47.7)$$

is called the fundamental matrix of P.

Corollary 0.2. *Limiting Matrix of the Markov chain under observation is,*

$$\bar{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \end{array} \quad (47.8)$$

Definition 3. *A element $\bar{p}_{ij}$ of $\bar{P}$ denotes the absorption probability in state j, starting from state i.*

Corollary 0.3. *The required probability is,*

$$P = \bar{p}_{14} \quad (47.9)$$

From (47.8) *and* (47.9)*,*

$$\bar{p}_{14} = \frac{2}{3} \quad (47.10)$$

Therefore, option 3) is correct.