GATE EE EXAM (2012), Question 47

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Prerequisites

State Space

The set of all possible states the system/object can be in. Without loss in generality, the state space can be identified with the set $S = \{1, 2, \ldots, \ell\}$, $\ell \in N$.

Random Variables

Let $\{X_0, X_1, X_2 \ldots\}$ be a sequence of discrete random variables, where each $X_t \in S$, and X_t represents the state of the system at time t.

Prerequisites Contd.

Markov Processes and Markov Chains

- A process is said to be a Markov Process if the probability of transitioning to any particular state is dependent solely on the current state and does not depend on how the current state was reached.
- Mathematically, $\{X_0, X_1, \ldots\}$ is called a Markov chain if

$$\Pr\left(\left(X_n = i_n \mid X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right) = \Pr\left(X_n = i_n \mid X_{n-1} = i_{n-1}\right) \tag{1}$$

Prerequisites Contd.

One-Step Transition Matrix

- For each pair $i, j \in S$, consider the (conditional) probability $p_{ij} \in [0, 1]$ for the transition of the object or system from state i to j within ONE TIME STEP.
- The $\ell \times \ell$ matrix $\mathbf{P} = (p_{ij})_{i,j=1,\ldots,\ell}$ of the transition probabilities p_{ij} where

$$p_{ij} \geq 0\,, \qquad (\mathbf{P})_{ij} = \Pr\left(X_{n+1} = j \mid X_n = i\right) \tag{2}$$

is called *One-Step Transition matrix* or simply, the Transition Matrix of the Markov chain.

Absorbing and Non-absorbing states

A state is said to be absorbing, if the Markov chain, after attaining that state, remains in it forever. i.e,

$$p_{ii} = 1 \Leftrightarrow \text{State i is absorbing} \tag{3}$$

Any state which is not absorbing is Non-absorbing, i.e, the Markov Chain may perform a transition to another state, i.e,

$$p_{ii} \neq 1 \Leftrightarrow \text{State i is Non-absorbing} \tag{4}$$

Definition 1

The standard form of a state transition matrix is,

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix} \end{array} \tag{5}$$

where,

Table: Notations and their meanings

Notation	Meaning
A	All Absorbing states
N	All Non-absorbing states
$\mathbf{I}$	Identity matrix
$\mathbf{O}$	Zero matrix
$\mathbf{R}, \mathbf{Q}$	Other sub-matrices

Prerequisites Contd.

2-Step Transition Matrix

The 2-step transition matrix is $\mathbf{P^2}$, and hence, the 2-step Transition probability from state i to state j is given by,

$$\Pr\left(X_{n+2} = j | X_n = i\right) = \left(\mathbf{P}^2\right)_{ij} \tag{6}$$

Proof.

- Recall matrix multiplication. Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be $N \times N$ matrices.The product matrix is $\mathbf{A} \times \mathbf{B} = \mathbf{AB}$, with elements

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj} \tag{7}$$

□

Proof Contd.

$$\Pr\left(X_2 = j | X_0 = i\right) = \sum_{k=1}^{\ell} \Pr\left(X_2 = j | X_1 = k, X_0 = i\right) \Pr\left(X_1 = k | X_0 = i\right) \tag{8}$$

$$= \sum_{k=1}^{\ell} \Pr\left(X_2 = j | X_1 = k\right) \Pr\left(X_1 = k | X_0 = i\right) \tag{9}$$

$$= \sum_{k=1}^{\ell} p_{kj} p_{ik} \tag{10}$$

$$= \sum_{k=1}^{\ell} p_{ik} p_{kj} = \left(\mathbf{P}^2\right)_{ij} \tag{11}$$

$$\implies \Pr\left(X_2 = j | X_0 = i\right) = \Pr\left(X_{n+2} = j | X_n = i\right) = \left(\mathbf{P}^2\right)_{ij} \tag{12}$$

- Similarly, it can be proved that the n-step Transition Matrix is equal to $\mathbf{P}^n$.

Prerequisites Contd.

Definition 2

The limiting matrix for absorbing Markov chain is,

$$\bar{\mathbf{P}} = \lim_{n \to \infty} \mathbf{P}^n = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{FR} & \mathbf{O} \end{bmatrix} \tag{13}$$

where,

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} \tag{14}$$

is called the fundamental matrix of $\mathbf{P}$.

Definition 3

A element $\bar{p}_{ij}$ of $\bar{\mathbf{P}}$ denotes the absorption probability in state j, starting from state i, when Markov Chain is continued for sufficiently large amount of time.

Question

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A fair coin is tossed till head appears for the first time. The probability that the number of required tosses is odd, is,

1. $\frac{1}{3}$
2. $\frac{1}{2}$
3. $\frac{2}{3}$
4. $\frac{3}{4}$

Solution

Given, a fair coin is tossed till heads turns up,

$$p = \frac{1}{2}, q = \frac{1}{2} \tag{15}$$

Let us define a Markov chain $\{X_0, X_1, X_2 \dots\}$, where $X_n \in S$ $\forall$n$\in \{0, 1, 2, \dots\}$, and $S = \{1, 2, 3, 4\}$ is the state space.

Table: Definition of Random Variables

R.V	Value=0	Value=1
X	$N_{tosses} = 2k$	$N_{tosses} = 2k - 1$
Y	H	T

Solution Contd.

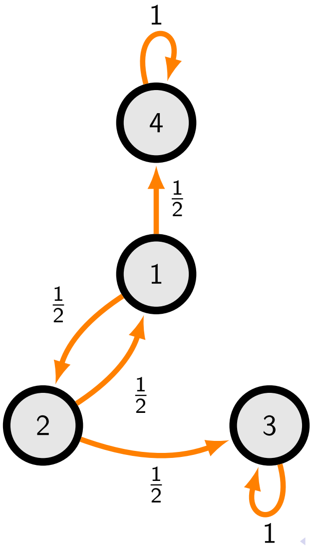
The definition of state space S,

Table: Markov states and Notations

Notation	State
$S = 1$	$(X, Y) = (0, 1)$
$S = 2$	$(X, Y) = (1, 1)$
$S = 3$	$(X, Y) = (0, 0)$
$S = 4$	$(X, Y) = (1, 0)$

Solution Contd.

Markov chain diagram



Solution Contd.

For the above discussed Markov Chain, the states 3 and 4 are the Absorbing states, and, 1 and 2 are Non-absorbing states,

Corollary 1

The state transition matrix for the above Markov chain is,

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{16}$$

Solution Contd.

From (5) and (16), we get,

$$\mathbf{R} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{17}$$

Corollary 2

Limiting Matrix of the Markov chain under observation is,

$$\bar{\mathbf{P}} = \begin{array}{c c} & \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \end{array} \tag{18}$$

Solution Contd.

Corollary 3

The required probability is,

$$P = \bar{p}_{14} \tag{19}$$

From (18) and (19),

$$P = \frac{2}{3} \tag{20}$$

$\therefore$ option 3 is correct.