AI1103 : Assignment 2

Amaan - EP20BTECH11003

Download all python codes from

https://github.com/amaan28/Assignment2/blob/main/Assignment2/codes/Assignment2.py

and latex-tikz codes from

https://github.com/amaan28/Assignment2/blob/main/Assignment2/Assignment2.tex

GATE 2012 EE Q.47

A fair coin is tossed till head appears for the first time. The probability that the number of required tosses is odd, is,

1) $\frac{1}{3}$
2) $\frac{1}{2}$
3) $\frac{2}{3}$
4) $\frac{3}{4}$

GATE 2012 EE Q.47 - SOLUTION

Given, a fair coin is tossed till heads turns up.

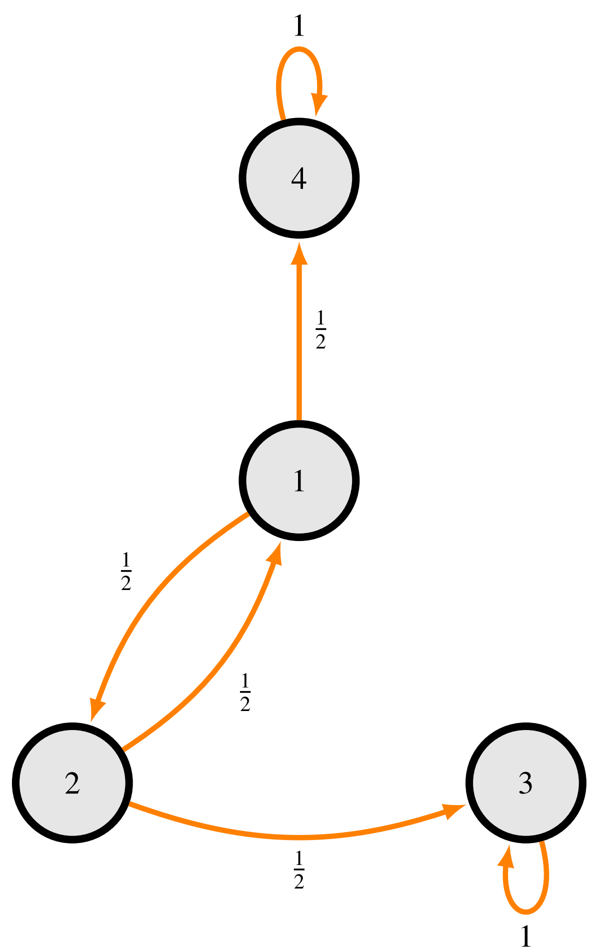
$$p = \frac{1}{2}, q = \frac{1}{2} \tag{47.1}$$

Let us define a Markov chain $\{X_0, X_1, X_2 \dots\}$, where $X_n \in S = \{1, 2, 3, 4\}$ where $n \in \{0, 1, 2, \dots\}$,

TABLE 1: (X, Y) represents a state in which X tells whether the number of tosses done till now is even($X = 0$) or odd($X = 1$) and Y tells what does the coin shows right now, Tails($Y = 1$) and Heads($Y = 0$)

Notation	State
$S = 1$	(0,1)
$S = 2$	(1,1)
$S = 3$	(0,0)
$S = 4$	(1,0)

Markov chain diagram



Definition 1. *The standard form of a state transition matrix is,*

$$\mathbf{P} = \begin{array}{c c} & \begin{array}{cc} A & N \end{array} \\ \begin{array}{c} A \\ N \end{array} & \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix} \end{array} \tag{47.2}$$

where,

TABLE 2: Notations and their meanings

Notation	Meaning
A	All absorbing states
N	All non-absorbing states
$\mathbf{I}$	Identity matrix
$\mathbf{O}$	Zero matrix
$\mathbf{R}, \mathbf{Q}$	Other submatrices

Corollary 0.1. *The state transition matrix for the above Markov chain, in which, the states, 3,4 are Absorbing states and 1,2 are Non-absorbing states, is,*

$$\mathbf{P} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{bmatrix} \end{array} \tag{47.3}$$

From (47.3),

$$\mathbf{R} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix} \tag{47.4}$$

Definition 2. *The limiting matrix for absorbing Markov chain is,*

$$\bar{\mathbf{P}} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{FR} & \mathbf{O} \end{bmatrix} \tag{47.5}$$

where,

$$\mathbf{F} = (\mathbf{I} - \mathbf{Q})^{-1} \tag{47.6}$$

is called the fundamental matrix of $\mathbf{P}$.

Corollary 0.2. *Limiting Matrix of the Markov chain under observation is,*

$$\bar{\mathbf{P}} = \begin{array}{c} \\ 3 \\ 4 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix} \end{array} \tag{47.7}$$

Definition 3. *A element $\bar{p}_{ij}$ of $\bar{\mathbf{P}}$ denotes the absorption probability in state j, starting from state i.*

Corollary 0.3. *The required probability is,*

$$P = \bar{p}_{14} \tag{47.8}$$

From (47.7) *and* (47.8),

$$P = \frac{2}{3} \tag{47.9}$$

Therefore, option 3) is correct.